China Mathematical Olympiad Problems

1986 - 2025

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2025 China National Olympiad

www.artofproblemsolving.com/community/c4110027

by Photaesthesia

– Day 1 (November 27, 2024)

1 Let $\alpha > 1$ be an irrational number and L be a integer such that $L > \frac{\alpha^2}{\alpha-1}$. A sequence $x_1, x_2, \cdots$ satisfies that $x_1 > L$ and for all positive integers n,

$$x_{n+1} = \begin{cases} \lfloor \alpha x_n \rfloor & \text{if } x_n \leqslant L \\ \lfloor \frac{x_n}{\alpha} \rfloor & \text{if } x_n > L \end{cases}.$$

Prove that
(i) $\{x_n\}$ is eventually periodic.
(ii) The eventual fundamental period of $\{x_n\}$ is an odd integer which doesn't depend on the choice of x_1.

2 Let ABC be a triangle with incenter I. Denote the midpoints of AI, AC and CI by L, M and N respectively. Point D lies on segment AM such that $BC = BD$. Let the incircle of triangle ABD be tangent to AD and BD at E and F respectively. Denote the circumcenter of triangle AIC by J, and the circumcircle of triangle JMD by ω. Lines MN and JL meet ω again at P and Q respectively. Prove that PQ, LN and EF are concurrent.

3 Let $a_1, a_2, \ldots, a_n$ be integers such that $a_1 > a_2 > \cdots > a_n > 1$. Let $M = \operatorname{lcm}(a_1, a_2, \ldots, a_n)$. For any finite nonempty set X of positive integers, define

$$f(X) = \min_{1 \leqslant i \leqslant n} \sum_{x \in X} \left\{ \frac{x}{a_i} \right\}.$$

Such a set X is called *minimal* if for every proper subset Y of it, $f(Y) < f(X)$ always holds.

Suppose X is minimal and $f(X) \geqslant \frac{2}{a_n}$. Prove that

$$|X| \leqslant f(X) \cdot M.$$

– Day 2 (November 28, 2024)

4 The *fractional distance* between two points (x_1, y_1) and (x_2, y_2) is defined as

$$\sqrt{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2},$$

where $\|x\|$ denotes the distance between x and its nearest integer. Find the largest real r such that there exists four points on the plane whose pairwise fractional distance are all at least r.

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5 Let p be a prime number and f be a bijection from $\{0, 1, \ldots, p-1\}$ to itself. Suppose that for integers $a, b \in \{0, 1, \ldots, p-1\}$, $|f(a) - f(b)| \leqslant 2024$ if $p \mid a^2 - b$. Prove that there exists infinite many p such that there exists such an f and there also exists infinite many p such that there doesn't exist such an f.

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6 Let $a_1, a_2, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i = n$, $\sum_{i=1}^n a_i^2 = 2n$, $\sum_{i=1}^n a_i^3 = 3n$.
(i) Find the largest constant C, such that for all $n \geqslant 4$,

$$\max\{a_1, a_2, \ldots, a_n\} - \min\{a_1, a_2, \ldots, a_n\} \geqslant C.$$

(ii) Prove that there exists a positive constant C_2, such that

$$\max\{a_1, a_2, \ldots, a_n\} - \min\{a_1, a_2, \ldots, a_n\} \geqslant C + C_2 n^{-\frac{3}{2}},$$

where C is the constant determined in (i).

2024 China National Olympiad

www.artofproblemsolving.com/community/c3631007

by Photaesthesia, sqing, LoloChen

– Day 1 (November 28th)

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1 Find the smallest $\lambda \in \mathbb{R}$ such that for all $n \in \mathbb{N}_+$, there exists $x_1, x_2, \dots, x_n$ satisfying $n = x_1 x_2 \dots x_{2023}$, where x_i is either a prime or a positive integer not exceeding n^λ for all $i \in \{1, 2, \dots, 2023\}$.

Proposed by Yinghua Ai

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2 Find the largest real number c such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(n-|i-j|)x_ix_j \geq c\sum_{j=1}^{n}x_i^2$$

for any positive integer n and any real numbers $x_1, x_2, \dots, x_n$.

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3 Let $p \geqslant 5$ be a prime and $S = \{1, 2, \dots, p\}$. Define $r(x, y)$ as follows:

$$r(x,y) = \begin{cases} y-x & y \geqslant x \\ y-x+p & y < x \end{cases}.$$

For a nonempty proper subset A of S, let

$$f(A) = \sum_{x\in A}\sum_{y\in A}(r(x,y))^2.$$

A *good* subset of S is a nonempty proper subset A satisfying that for all subsets $B \subseteq S$ of the same size as A, $f(B) \geqslant f(A)$. Find the largest integer L such that there exists distinct good subsets $A_1 \subseteq A_2 \subseteq \dots \subseteq A_L$.

Proposed by Bin Wang

– Day 2 (November 29th)

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4 Let $a_1, a_2, \dots, a_{2023}$ be nonnegative real numbers such that $a_1 + a_2 + \dots + a_{2023} = 100$. Let $A = \{(i,j) \mid 1 \leqslant i \leqslant j \leqslant 2023,\ a_ia_j \geqslant 1\}$. Prove that $|A| \leqslant 5050$ and determine when the equality holds.

Proposed by Yunhao Fu

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5 In acute $\triangle ABC$, K is on the extention of segment BC. P, Q are two points such that $KP \parallel AB, BK = BP$ and $KQ \parallel AC, CK = CQ$. The circumcircle of $\triangle KPQ$ intersects AK again at T. Prove that:
(1) $\angle BTC + \angle APB = \angle CQA$.
(2) $AP \cdot BT \cdot CQ = AQ \cdot CT \cdot BP$.

Proposed by *Yijie He* and *Yijuan Yao*

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6 Let P be a regular 99-gon. Assign integers between 1 and 99 to the vertices of P such that each integer appears exactly once. (If two assignments coincide under rotation, treat them as the same.) An *operation* is a swap of the integers assigned to a pair of adjacent vertices of P. Find the smallest integer n such that one can achieve every other assignment from a given one with no more than n operations.

Proposed by Zhenhua Qu

2023 China National Olympiad

www.artofproblemsolving.com/community/c3238312

by CHN_Lucas, JG666, CANBANKAN, David-Vieta

– Day 1 (December 29th)

1 Define the sequences $(a_n),(b_n)$ by

$$\begin{aligned} &a_n,b_n>0,\forall n\in\mathbb{N}_+\\ &a_{n+1}=a_n-\frac{1}{1+\sum_{i=1}^n\frac{1}{a_i}}\\ &b_{n+1}=b_n+\frac{1}{1+\sum_{i=1}^n\frac{1}{b_i}}\end{aligned}$$

1) If $a_{100}b_{100}=a_{101}b_{101}$, find the value of a_1-b_1;
2) If $a_{100}=b_{99}$, determine which is larger between $a_{100}+b_{100}$ and $a_{101}+b_{101}$.

2 Let $\triangle ABC$ be an equilateral triangle of side length 1. Let D,E,F be points on BC,AC,AB respectively, such that $\frac{DE}{20}=\frac{EF}{22}=\frac{FD}{38}$. Let X,Y,Z be on lines BC,CA,AB respectively, such that $XY\perp DE, YZ\perp EF, ZX\perp FD$. Find all possible values of $\frac{1}{[DEF]}+\frac{1}{[XYZ]}$.

3 Given positive integer m,n, color the points of the regular $(2m+2n)$-gon in black and white, $2m$ in black and $2n$ in white.
The *coloring distance* $d(B,C)$ of two black points B,C is defined as the smaller number of white points in the two paths linking the two black points.
The *coloring distance* $d(W,X)$ of two white points W,X is defined as the smaller number of black points in the two paths linking the two white points.
We define the matching of black points $\mathcal{B}$: label the $2m$ black points with $A_1,\cdots,A_m,B_1,\cdots,B_m$ satisfying no A_iB_i intersects inside the gon.
We define the matching of white points $\mathcal{W}$: label the $2n$ white points with $C_1,\cdots,C_n,D_1,\cdots,D_n$ satisfying no C_iD_i intersects inside the gon.
We define $P(\mathcal{B})=\sum_{i=1}^m d(A_i,B_i), P(\mathcal{W})=\sum_{j=1}^n d(C_j,D_j)$.
Prove that: $\max_{\mathcal{B}}P(\mathcal{B})=\max_{\mathcal{W}}P(\mathcal{W})$

– Day 2 (December 30th)

4 Find the minimum positive integer $n\ge 3$, such that there exist n points $A_1,A_2,\cdots,A_n$ satisfying no three points are collinear and for any $1\le i\le n$, there exist $1\le j\le n (j\neq i)$, segment A_jA_{j+1} pass through the midpoint of segment A_iA_{i+1}, where $A_{n+1}=A_1$

5 Prove that there exist $C > 0$, which satisfies the following conclusion:
For any infinite positive arithmetic integer sequence $a_1, a_2, a_3, \cdots$, if the greatest common divisor of a_1 and a_2 is squarefree, then there exists a positive integer $m \le C \cdot {a_2}^2$, such that a_m is squarefree.
Note: A positive integer N is squarefree if it is not divisible by any square number greater than 1.

Proposed by Qu Zhenhua

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6 There are $n (n \ge 8)$ airports, some of which have one-way direct routes between them. For any two airports a and b, there is at most one one-way direct route from a to b (there may be both one-way direct routes from a to b and from b to a). For any set A composed of airports $(1 \le |A| \le n-1)$, there are at least $4 \cdot \min\{|A|, n-|A|\}$ one-way direct routes from the airport in A to the airport not in A.
Prove that: For any airport x, we can start from x and return to the airport by no more than $\sqrt{2n}$ one-way direct routes.

China National Olympiad 22

www.artofproblemsolving.com/community/c2742261

by parmenides51, mofumofu

–	Day 1
1	Let a and b be two positive real numbers, and AB a segment of length a on a plane. Let C, D be two variable points on the plane such that $ABCD$ is a non-degenerate convex quadrilateral with $BC = CD = b$ and $DA = a$. It is easy to see that there is a circle tangent to all four sides of the quadrilateral $ABCD$. Find the precise locus of the point I.
2	Find the largest real number λ with the following property: for any positive real numbers p, q, r, s there exists a complex number $z = a + bi(a, b \in \mathbb{R})$ such that $$\lvert b\rvert \geq \lambda\lvert a\rvert \quad \text{and} \quad (pz^3 + 2qz^2 + 2rz + s) \cdot (qz^3 + 2pz^2 + 2sz + r) = 0.$$
3	Find all positive integers a such that there exists a set X of 6 integers satisfying the following conditions: for every $k = 1, 2, \ldots, 36$ there exist $x, y \in X$ such that $ax + y - k$ is divisible by 37.
–	Day 2
4	A conference is attended by $n(n \geq 3)$ scientists. Each scientist has some friends in this conference (friendship is mutual and no one is a friend of him/herself). Suppose that no matter how we partition the scientists into two nonempty groups, there always exist two scientists in the same group who are friends, and there always exist two scientists in different groups who are friends. A proposal is introduced on the first day of the conference. Each of the scientists' opinion on the proposal can be expressed as a non-negative integer. Everyday from the second day onwards, each scientists' opinion is changed to the integer part of the average of his/her friends' opinions from the previous day. Prove that after a period of time, all scientists have the same opinion on the proposal.
5	On a blank piece of paper, two points with distance 1 is given. Prove that one can use (only) straightedge and compass to construct on this paper a straight line, and two points on it whose distance is $\sqrt{2021}$ such that, in the process of constructing it, the total number of circles or straight lines drawn is at most 10. Remark: Explicit steps of the construction should be given. Label the circles and straight lines

in the order that they appear. Partial credit may be awarded depending on the total number of circles/lines.

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6 For integers $0 \leq a \leq n$, let $f(n,a)$ denote the number of coefficients in the expansion of $(x+1)^a(x+2)^{n-a}$ that is divisible by 3. For example, $(x+1)^3(x+2)^1 = x^4+5x^3+9x^2+7x+2$, so $f(4,3)=1$. For each positive integer n, let $F(n)$ be the minimum of $f(n,0), f(n,1), \ldots, f(n,n)$.

(1) Prove that there exist infinitely many positive integer n such that $F(n) \geq \frac{n-1}{3}$.

(2) Prove that for any positive integer n, $F(n) \leq \frac{n-1}{3}$.

2021 China National Olympiad

www.artofproblemsolving.com/community/c1617667

by Tintarn, Rickyminer, mofumofu, sqing, SpecialBeing2017, NTssu

– Day 1

1 Let $\{z_n\}_{n\ge 1}$ be a sequence of complex numbers, whose odd terms are real, even terms are purely imaginary, and for every positive integer k, $|z_k z_{k+1}|=2^k$. Denote $f_n=|z_1+z_2+\cdots+z_n|$, for $n=1,2,\cdots$
(1) Find the minimum of f_{2020}.
(2) Find the minimum of $f_{2020}\cdot f_{2021}$.

2 Let $m>1$ be an integer. Find the smallest positive integer n, such that for any integers $a_1,a_2,\ldots,a_n;b_1,b_2,\ldots,b_n$ there exists integers $x_1,x_2,\ldots,x_n$ satisfying the following two conditions:

i) There exists $i\in\{1,2,\ldots,n\}$ such that x_i and m are coprime

ii) $\sum_{i=1}^n a_ix_i \equiv \sum_{i=1}^n b_ix_i \equiv 0 \pmod m$

3 Let n be positive integer such that there are exactly 36 different prime numbers that divides n. For $k=1,2,3,4,5$, c_n be the number of integers that are mutually prime numbers to n in the interval $[\frac{(k-1)n}{5},\frac{kn}{5}]$. c_1,c_2,c_3,c_4,c_5 is not exactly the same.Prove that

$$\sum_{1\le i<j\le 5}(c_i-c_j)^2\ge 2^{36}.$$

– Day 2

4 In acute triangle $ABC (AB>AC)$, M is the midpoint of minor arc BC, O is the circumcenter of (ABC) and AK is its diameter. The line parallel to AM through O meets segment AB at D, and CA extended at E. Lines BM and CK meet at P, lines BK and CM meet at Q. Prove that $\angle OPB+\angle OEB=\angle OQC+\angle ODC$.

5 P is a convex polyhedron such that:

(1) every vertex belongs to exactly 3 faces.

(1) For every natural number n, there are even number of faces with n vertices.

An ant walks along the edges of P and forms a non-self-intersecting cycle, which divides the faces of this polyhedron into two sides, such that for every natural number n, the number of faces with n vertices on each side are the same. (assume this is possible)

Show that the number of times the ant turns left is the same as the number of times the ant turn right.

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6 Find $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, such that for any $x, y \in \mathbb{Z}_+$,

$$f(f(x) + y) \mid x + f(y).$$

2020 China National Olympiad

www.artofproblemsolving.com/community/c1018560

by Henry_2001, WypHxr, mofumofu

Day 1 Nov. 25th, 2019

1 Let $a_1,a_2,\cdots,a_{41}\in\mathbb{R}$, such that $a_{41}=a_1, \sum_{i=1}^{40}a_i=0$, and for any $i=1,2,\cdots,40, |a_i-a_{i+1}|\leq 1$. Determine the greatest possible value of $(1)a_{10}+a_{20}+a_{30}+a_{40}$; $(2)a_{10}\cdot a_{20}+a_{30}\cdot a_{40}$.

2 In triangle ABC, $AB>AC$. The bisector of $\angle BAC$ meets BC at D. P is on line DA, such that A lies between P and D. PQ is tangent to $\odot(ABD)$ at Q. PR is tangent to $\odot(ACD)$ at R. CQ meets BR at K. The line parallel to BC and passing through K meets QD, AD, RD at E, L, F, respectively. Prove that $EL=KF$.

3 Let S be a set, $|S|=35$. A set F of mappings from S to itself is called to be satisfying property $P(k)$, if for any $x,y\in S$, there exist $f_1, \cdots, f_k \in F$ (not necessarily different), such that $f_k(f_{k-1}(\cdots (f_1(x))))=f_k(f_{k-1}(\cdots (f_1(y))))$.
Find the least positive integer m, such that if F satisfies property $P(2019)$, then it also satisfies property $P(m)$.

Day 2 Nov. 26th, 2019

4 Find the largest positive constant C such that the following is satisfied: Given n arcs (containing their endpoints) $A_1,A_2,\ldots ,A_n$ on the circumference of a circle, where among all sets of three arcs (A_i,A_j,A_k) $(1\le i< j< k\le n)$, at least half of them has $A_i\cap A_j\cap A_k$ nonempty, then there exists $l>Cn$, such that we can choose l arcs among $A_1,A_2,\ldots ,A_n$, whose intersection is nonempty.

5 Given any positive integer c, denote $p(c)$ as the largest prime factor of c. A sequence $\{a_n\}$ of positive integers satisfies $a_1>1$ and $a_{n+1}=a_n+p(a_n)$ for all $n\ge 1$. Prove that there must exist at least one perfect square in sequence $\{a_n\}$.

6 Does there exist positive reals $a_0,a_1,\ldots ,a_{19}$, such that the polynomial $P(x)=x^{20}+a_{19}x^{19}+\ldots +a_1x+a_0$ does not have any real roots, yet all polynomials formed from swapping any two coefficients a_i,a_j has at least one real root?

2019 China National Olympiad

www.artofproblemsolving.com/community/c789151

by lifeisgood03, sqing, mofumofu, 61plus

–	Day 1
1	Let $a,b,c,d,e \geq -1$ and $a+b+c+d+e=5$. Find the maximum and minimum value of $S=(a+b)(b+c)(c+d)(d+e)(e+a)$.
2	Call a set of 3 positive integers $\{a,b,c\}$ a *Pythagorean* set if a,b,c are the lengths of the 3 sides of a right-angled triangle. Prove that for any 2 Pythagorean sets P,Q, there exists a positive integer $m\geq 2$ and Pythagorean sets $P_1,P_2,\ldots,P_m$ such that $P=P_1,Q=P_m$, and $\forall 1\leq i\leq m-1$, $P_i\cap P_{i+1}\neq\emptyset$.
3	Let O be the circumcenter of $\triangle ABC$($AB<AC$), and D be a point on the internal angle bisector of $\angle BAC$. Point E lies on BC, satisfying $OE \parallel AD$, $DE\perp BC$. Point K lies on EB extended such that $EK=EA$. The circumcircle of $\triangle ADK$ meets BC at $P\neq K$, and meets the circumcircle of $\triangle ABC$ at $Q\neq A$. Prove that PQ is tangent to the circumcircle of $\triangle ABC$.
–	Day 2
4	Given an ellipse that is not a circle. (1) Prove that the rhombus tangent to the ellipse at all four of its sides with minimum area is unique. (2) Construct this rhombus using a compass and a straight edge.
5	Given is an $n\times n$ board, with an integer written in each grid. For each move, I can choose any grid, and add 1 to all $2n-1$ numbers in its row and column. Find the largest $N(n)$, such that for any initial choice of integers, I can make a finite number of moves so that there are at least $N(n)$ even numbers on the board.
6	The point $P_1,P_2,\cdots,P_{2018}$ is placed inside or on the boundary of a given regular pentagon. Find all placement methods are made so that $$S=\sum_{1\leq i<j\leq 2018}\|P_iP_j\|^2$$ takes the maximum value.

2018 China National Olympiad

www.artofproblemsolving.com/community/c562721

by YanYau, fattypiggy123, mofumofu

Day 1 November 15, 2017

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1 Let n be a positive integer. Let A_n denote the set of primes p such that there exists positive integers a, b satisfying
$$\frac{a+b}{p} \text{ and } \frac{a^n+b^n}{p^2}$$
are both integers that are relatively prime to p. If A_n is finite, let $f(n)$ denote $|A_n|$.

a) Prove that A_n is finite if and only if $n \neq 2$.

b) Let m, k be odd positive integers and let d be their gcd. Show that
$$f(d) \leq f(k) + f(m) - f(km) \leq 2f(d).$$

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2 Let n and k be positive integers and let
$$T = \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x, y, z \leq n\}$$
be the length n lattice cube. Suppose that $3n^2 - 3n + 1 + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points.

Prove that there exists at least k unit cubes of length 1, all of whose vertices are colored red.

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3 Let q be a positive integer which is not a perfect cube. Prove that there exists a positive constant C such that for all natural numbers n, one has
$$\{nq^{\frac{1}{3}}\} + \{nq^{\frac{2}{3}}\} \geq Cn^{-\frac{1}{2}}$$
where $\{x\}$ denotes the fractional part of x.

Day 2 November 16, 2017

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4 $ABCD$ is a cyclic quadrilateral whose diagonals intersect at P. The circumcircle of $\triangle APD$ meets segment AB at points A and E. The circumcircle of $\triangle BPC$ meets segment AB at points B and F. Let I and J be the incenters of $\triangle ADE$ and $\triangle BCF$, respectively. Segments IJ and AC meet at K. Prove that the points A, I, K, E are cyclic.

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5 Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \ldots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.

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6 Let $n > k$ be two natural numbers and let $a_1, \ldots, a_n$ be real numbers in the open interval $(k-1, k)$. Let $x_1, \ldots, x_n$ be positive reals such that for any subset $I \subset \{1, \ldots, n\}$ satisfying $|I| = k$, one has

$$\sum_{i \in I} x_i \leq \sum_{i \in I} a_i.$$

Find the largest possible value of $x_1 x_2 \cdots x_n$.

China National Olympiad 2017

www.artofproblemsolving.com/community/c372140

by buzzychaoz, sqing, fattypiggy123

Day 1 November 23rd

1 The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_0 = u_1 = 1$,$u_n = 2u_{n-1} - 3u_{n-2}$ $(n \geq 2)$, $v_0 = a, v_1 = b, v_2 = c$,$v_n = v_{n-1} - 3v_{n-2} + 27v_{n-3}$ $(n \geq 3)$. There exists a positive integer N such that when $n > N$, we have $u_n \mid v_n$. Prove that $3a = 2b + c$.

2 In acute triangle ABC, let $\odot O$ be its circumcircle, $\odot I$ be its incircle. Tangents at B, C to $\odot O$ meet at L, $\odot I$ touches BC at D. AY is perpendicular to BC at Y, AO meets BC at X, and OI meets $\odot O$ at P, Q. Prove that P, Q, X, Y are concyclic if and only if A, D, L are collinear.

3 Consider a rectangle R partitioned into 2016 smaller rectangles such that the sides of each smaller rectangle is parallel to one of the sides of the original rectangle. Call the corners of each rectangle a vertex. For any segment joining two vertices, call it basic if no other vertex lie on it. (The segments must be part of the partitioning.) Find the maximum/minimum possible number of basic segments over all possible partitions of R.

Day 2 November 24th

4 Let $n \geq 2$ be a natural number. For any two permutations of $(1, 2, \cdots, n)$, say $\alpha = (a_1, a_2, \cdots, a_n)$ and $\beta = (b_1, b_2, \cdots, b_n)$, if there exists a natural number $k \leq n$ such that

$$b_i = \begin{cases} a_{k+1-i}, & 1 \leq i \leq k; \\ a_i, & k < i \leq n, \end{cases}$$

we call α a friendly permutation of β.

Prove that it is possible to enumerate all possible permutations of $(1, 2, \cdots, n)$ as $P_1, P_2, \cdots, P_m$ such that for all $i = 1, 2, \cdots, m$, P_{i+1} is a friendly permutation of P_i where $m = n!$ and $P_{m+1} = P_1$.

5 Let D_n be the set of divisors of n. Find all natural n such that it is possible to split D_n into two disjoint sets A and G, both containing at least three elements each, such that the elements in A form an arithmetic progression while the elements in G form a geometric progression.

6 Given an integer $n \geq 2$ and real numbers a, b such that $0 < a < b$. Let $x_1, x_2, \dots, x_n \in [a, b]$ be real numbers. Find the maximum value of

$$\frac{\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1}}{x_1 + x_2 + \cdots + x_{n-1} + x_n}.$$

China National Olympiad 2016

www.artofproblemsolving.com/community/c255106

by rkm0959, sqing, fattypiggy123

Day 1 December 16th

1 Let $a_1, a_2, \cdots, a_{31}; b_1, b_2, \cdots, b_{31}$ be positive integers such that $a_1 < a_2 < \cdots < a_{31} \leq 2015$, $b_1 < b_2 < \cdots < b_{31} \leq 2015$ and $a_1 + a_2 + \cdots + a_{31} = b_1 + b_2 + \cdots + b_{31}$.
Find the maximum value of $S = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{31} - b_{31}|$.

2 In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively.
Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

3 Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod p$ for all $1 \leq i \leq p$

2) For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod p$$

where indices are taken $\pmod p$

Day 2 December 17th

4 Let $n \geq 2$ be a positive integer and define k to be the number of primes $\leq n$. Let A be a subset of $S = \{2, ..., n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B such that $|B| = k$, $A \subseteq B \subseteq S$ and no two elements in B divide each other.

5 Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (Vertices maybe ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

6 Let G be a complete directed graph with 100 vertices such that for any two vertices x, y one can find a directed path from x to y.

a) Show that for any such G, one can find a m such that for any two vertices x, y one can find a directed path of length m from x to y (Vertices can be repeated in the path)

b) For any graph G with the properties above, define $m(G)$ to be smallest possible m as defined in part a). Find the minimim value of $m(G)$ over all such possible G's.

China National Olympiad 2015
www.artofproblemsolving.com/community/c5238
by fattypiggy123, TelvCohl

Day 1

1 Let $z_1, z_2, ..., z_n$ be complex numbers satisfying $|z_i - 1| \leq r$ for some r in $(0, 1)$. Show that

$$\left|\sum_{i=1}^{n} z_i\right| \cdot \left|\sum_{i=1}^{n} \frac{1}{z_i}\right| \geq n^2(1-r^2).$$

2 Let A, B, D, E, F, C be six points lie on a circle (in order) satisfy $AB = AC$.
Let $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$.
Let K be a point lie on ST satisfy $\angle QKS = \angle ECA$.

Prove that $\frac{SK}{KT} = \frac{PQ}{QR}$

3 Let $n \geq 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

i) $|A| = n$, $|B| = m$ and A is a subset of B
ii) For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

Day 2

1 Determine all integers k such that there exists infinitely many positive integers n **not** satisfying

$$n + k | \binom{2n}{n}$$

2 Given 30 students such that each student has at most 5 friends and for every 5 students there is a pair of students that are not friends, determine the maximum k such that for all such possible configurations, there exists k students who are all not friends.

3 Let $a_1, a_2, ...$ be a sequence of non-negative integers such that for any m, n

$$\sum_{i=1}^{2m} a_{in} \leq m.$$

Show that there exist k, d such that

$$\sum_{i=1}^{2k} a_{id} = k - 2014.$$

China National Olympiad 2014

www.artofproblemsolving.com/community/c5237

by 61plus, sqing

Day 1

1	Let ABC be a triangle with $AB > AC$. Let D be the foot of the internal angle bisector of A. Points F and E are on AC, AB respectively such that B, C, F, E are concyclic. Prove that the circumcentre of DEF is the incentre of ABC if and only if $BE + CF = BC$.
2	For the integer $n > 1$, define $D(n) = \{a - b \mid ab = n, a > b > 0, a, b \in \mathbb{N}\}$. Prove that for any integer $k > 1$, there exists pairwise distinct positive integers $n_1, n_2, \ldots, n_k$ such that $n_1, \ldots, n_k > 1$ and $\lvert D(n_1) \cap D(n_2) \cap \cdots \cap D(n_k)\rvert \geq 2$.
3	Prove that: there exists only one function $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying: i) $f(1) = f(2) = 1$; ii)$f(n) = f(f(n-1)) + f(n - f(n-1))$ for $n \geq 3$. For each integer $m \geq 2$, find the value of $f(2^m)$.

Day 2

1	Let $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime factorisation of n. Define $\omega(n) = t$ and $\Omega(n) = a_1 + a_2 + \ldots + a_t$. Prove or disprove: For any fixed positive integer k and positive reals α, β, there exists a positive integer $n > 1$ such that i) $\frac{\omega(n+k)}{\omega(n)} > \alpha$ ii) $\frac{\Omega(n+k)}{\Omega(n)} < \beta$.
2	Let $f : X \to X$, where $X = \{1, 2, \ldots, 100\}$, be a function satisfying: 1) $f(x) \neq x$ for all $x = 1, 2, \ldots, 100$; 2) for any subset A of X such that $\lvert A\rvert = 40$, we have $A \cap f(A) \neq \emptyset$. Find the minimum k such that for any such function f, there exist a subset B of X, where $\lvert B\rvert = k$, such that $B \cup f(B) = X$.
3	For non-empty number sets S, T, define the sets $S + T = \{s + t \mid s \in S, t \in T\}$ and $2S = \{2s \mid s \in S\}$. Let n be a positive integer, and A, B be two non-empty subsets of $\{1, 2 \ldots, n\}$. Show that there exists a subset D of $A + B$ such that 1) $D + D \subseteq 2(A + B)$, 2) $\lvert D\rvert \geq \frac{\lvert A\rvert \cdot \lvert B\rvert}{2n}$,

where $|X|$ is the number of elements of the finite set X.

China National Olympiad 2013

www.artofproblemsolving.com/community/c5236

by yunxiu, sqing

Day 1

1 Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.
i) Show that l_1 and l_2 have a unique common point (denoted by P).
ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

2 Find all nonempty sets S of integers such that $3m-2n \in S$ for all (not necessarily distinct) $m, n \in S$.

3 Find all positive real numbers t with the following property: there exists an infinite set X of real numbers such that the inequality

$$\max\{|x-(a-d)|, |y-a|, |z-(a+d)|\} > td$$

holds for all (not necessarily distinct) $x, y, z \in X$, all real numbers a and all positive real numbers d.

Day 2

1 Let $n \geqslant 2$ be an integer. There are n finite sets $A_1, A_2, \ldots, A_n$ which satisfy the condition

$$|A_i \Delta A_j| = |i-j| \quad \forall i, j \in \{1, 2, ..., n\}.$$

Find the minimum of $\sum\limits_{i=1}^{n} |A_i|$.

2 For any positive integer n and $0 \leqslant i \leqslant n$, denote $C_n^i \equiv c(n,i) \pmod 2$, where $c(n,i) \in \{0,1\}$. Define

$$f(n,q) = \sum_{i=0}^{n} c(n,i) q^i$$

where m, n, q are positive integers and $q+1 \neq 2^\alpha$ for any $\alpha \in \mathbb{N}$. Prove that if $f(m,q) \,|\, f(n,q)$, then $f(m,r) \,|\, f(n,r)$ for any positive integer r.

3 Let m, n be positive integers. Find the minimum positive integer N which satisfies the following condition. If there exists a set S of integers that contains a complete residue system module m such that $|S| = N$, then there exists a nonempty set $A \subseteq S$ so that $n \mid \sum\limits_{x \in A} x$.

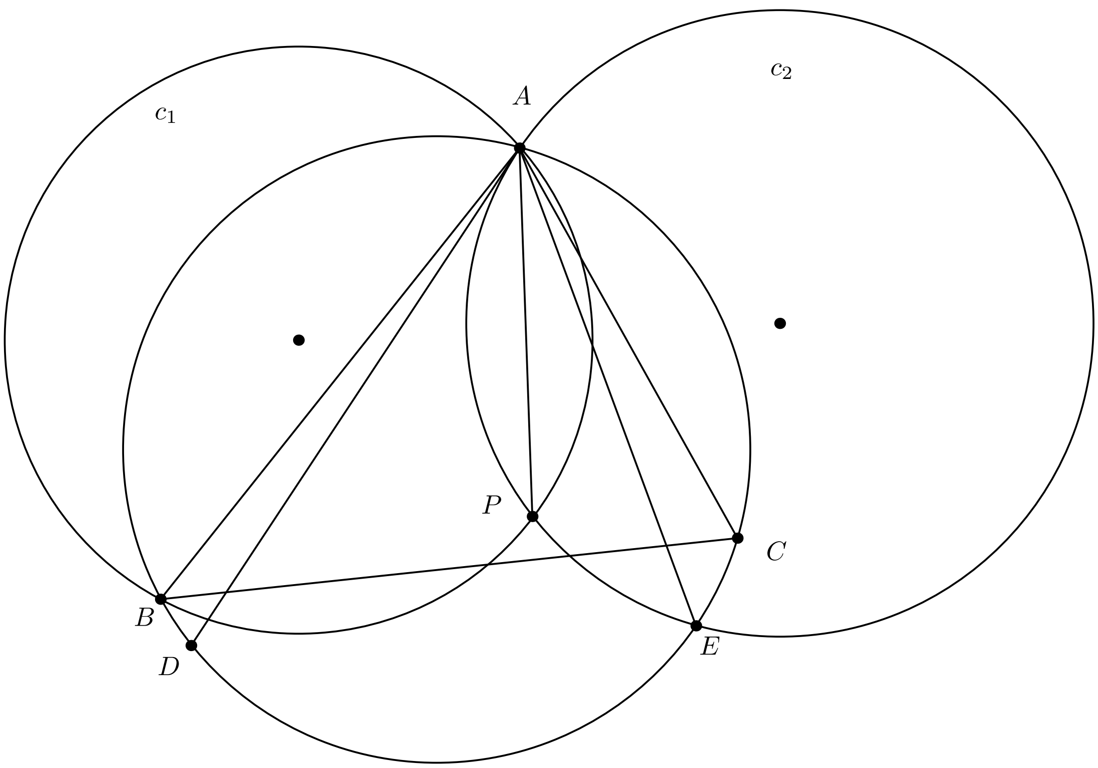
China National Olympiad 2012

www.artofproblemsolving.com/community/c5235

by yunxiu, littletush

Day 1

1 In the triangle ABC, $\angle A$ is biggest. On the circumcircle of $\triangle ABC$, let D be the midpoint of $\widehat{ABC}$ and E be the midpoint of $\widehat{ACB}$. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.



2 Let p be a prime. We arrange the numbers in $\{1, 2, \ldots, p^2\}$ as a $p \times p$ matrix $A = (a_{ij})$. Next we can select any row or column and add 1 to every number in it, or subtract 1 from every number in it. We call the arrangement *good* if we can change every number of the matrix to 0 in a finite number of such moves. How many good arrangements are there?

3 Prove for any $M > 2$, there exists an increasing sequence of positive integers $a_1 < a_2 < \ldots$

satisfying:
1) $a_i > M^i$ for any i;
2) There exists a positive integer m and $b_1, b_2, \ldots, b_m \in \{-1, 1\}$, satisfying $n = a_1b_1 + a_2b_2 + \ldots + a_mb_m$ if and only if $n \in \mathbb{Z}/\{0\}$.

Day 2

1 Let $f(x) = (x+a)(x+b)$ where $a, b > 0$. For any reals $x_1, x_2, \ldots, x_n \geqslant 0$ satisfying $x_1 + x_2 + \ldots + x_n = 1$, find the maximum of $F = \sum\limits_{1 \leqslant i < j \leqslant n} \min\{f(x_i), f(x_j)\}$.

2 Consider a square-free even integer n and a prime p, such that
1) $(n, p) = 1$;
2) $p \le 2\sqrt{n}$;
3) There exists an integer k such that $p | n + k^2$.
Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Proposed by Hongbing Yu

3 Find the smallest positive integer k such that, for any subset A of $S = \{1, 2, \ldots, 2012\}$ with $|A| = k$, there exist three elements x, y, z in A such that $x = a + b$, $y = b + c$, $z = c + a$, where a, b, c are in S and are distinct integers.

Proposed by Huawei Zhu

China National Olympiad 2011

www.artofproblemsolving.com/community/c5234

by zhaobin

Day 1

1 Let $a_1,a_2,\ldots,a_n$ are real numbers, prove that;

$$\sum_{i=1}^{n}a_i^2-\sum_{i=1}^{n}a_ia_{i+1}\le\left\lfloor\frac{n}{2}\right\rfloor(M-m)^2.$$

where $a_{n+1}=a_1, M=\max_{1\le i\le n}a_i, m=\min_{1\le i\le n}a_i$.

2 On the circumcircle of the acute triangle ABC, D is the midpoint of $\overset{\frown}{BC}$. Let X be a point on $\overset{\frown}{BD}$, E the midpoint of $\overset{\frown}{AX}$, and let S lie on $\overset{\frown}{AC}$. The lines SD and BC have intersection R, and the lines SE and AX have intersection T. If $RT\parallel DE$, prove that the incenter of the triangle ABC is on the line RT.

3 Let A be a set consist of finite real numbers,$A_1,A_2,\cdots,A_n$ be nonempty sets of A, such that

(a) The sum of the elements of A is 0,

(b) For all $x_i\in A_i(i=1,2,\cdots,n)$,we have $x_1+x_2+\cdots+x_n>0$.

Prove that there exist $1\le k\le n$, and $1\le i_1<i_2<\cdots<i_k\le n$, such that

$$|A_{i_1}\bigcup A_{i_2}\bigcup\cdots\bigcup A_{i_k}|<\frac{k}{n}|A|.$$

Where $|X|$ denote the numbers of the elements in set X.

Day 2

1 Let n be an given positive integer, the set $S=\{1,2,\cdots,n\}$.For any nonempty set A and B, find the minimum of $|A\Delta S|+|B\Delta S|+|C\Delta S|$, where $C=\{a+b|a\in A,b\in B\}, X\Delta Y=X\cup Y-X\cap Y$.

2 Let $a_i,b_i,i=1,\cdots,n$ are nonnegitive numbers,and $n\ge 4$,such that $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n>0$.

Find the maximum of $\frac{\sum_{i=1}^n a_i(a_i+b_i)}{\sum_{i=1}^n b_i(a_i+b_i)}$

3 Let m,n be positive integer numbers. Prove that there exist infinite many couples of positive integer nubmers (a,b) such that

$$a+b|am^a+bn^b,\quad \gcd(a,b)=1.$$

China National Olympiad 2010
www.artofproblemsolving.com/community/c5233
by Lei Lei, hxy09

Day 1

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1 Two circles Γ_1 and Γ_2 meet at A and B. A line through B meets Γ_1 and Γ_2 again at C and D repsectively. Another line through B meets Γ_1 and Γ_2 again at E and F repsectively. Line CF meets Γ_1 and Γ_2 again at P and Q respectively. M and N are midpoints of arc PB and arc QB repsectively. Show that if $CD = EF$, then C, F, M, N are concyclic.

---

2 Let k be an integer ≥ 3. Sequence $\{a_n\}$ satisfies that $a_k = 2k$ and for all $n > k$, we have

$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } (a_{n-1}, n) = 1 \\ 2n & \text{if } (a_{n-1}, n) > 1 \end{cases}$$

Prove that there are infinitely many primes in the sequence $\{a_n - a_{n-1}\}$.

---

3 Given complex numbers a, b, c, we have that $|az^2 + bz + c| \leq 1$ holds true for any complex number $z, |z| \leq 1$. Find the maximum value of $|bc|$.

Day 2

---

1 Let $m, n \geq 1$ and $a_1 < a_2 < \ldots < a_n$ be integers. Prove that there exists a subset T of $\mathbb{N}$ such that

$$|T| \leq 1 + \frac{a_n - a_1}{2n + 1}$$

and for every $i \in \{1, 2, \ldots, m\}$, there exists $t \in T$ and $s \in [-n, n]$, such that $a_i = t + s$.

---

2 There is a deck of cards placed at every points $A_1, A_2, \ldots, A_n$ and O, where $n \geq 3$. We can do one of the following two operations at each step: $1)$ If there are more than 2 cards at some points A_i, we can withdraw three cards from that deck and place one each at A_{i-1}, A_{i+1} and O. (Here $A_0 = A_n$ and $A_{n+1} = A_1$); $2)$ If there are more than or equal to n cards at point O, we can withdraw n cards from that deck and place one each at $A_1, A_2, \ldots, A_n$.
Show that if the total number of cards is more than or equal to $n^2 + 3n + 1$, we can make the number of cards at every points more than or equal to $n + 1$ after finitely many steps.

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3 Suppose $a_1, a_2, a_3, b_1, b_2, b_3$ are distinct positive integers such that

$$(n+1)a_1^n + na_2^n + (n-1)a_3^n | (n+1)b_1^n + nb_2^n + (n-1)b_3^n$$

holds for all positive integers n. Prove that there exists $k \in N$ such that $b_i = ka_i$ for $i = 1, 2, 3$.

China National Olympiad 2009

www.artofproblemsolving.com/community/c5232

by Fang-jh

Day 1

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1 Given an acute triangle PBC with $PB \neq PC$. Points A, D lie on PB, PC, respectively. AC intersects BD at point O. Let E, F be the feet of perpendiculars from O to AB, CD, respectively. Denote by M, N the midpoints of BC, AD. (1): If four points A, B, C, D lie on one circle, then $EM \cdot FN = EN \cdot FM$. (2): Determine whether the converse of (1) is true or not, justify your answer.

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2 Find all the pairs of prime numbers (p, q) such that $pq|5^p + 5^q$.

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3 Given two integers m, n satisfying $4 < m < n$. Let $A_1A_2\cdots A_{2n+1}$ be a regular $2n+1$ polygon. Denote by P the set of its vertices. Find the number of convex m polygon whose vertices belongs to P and exactly has two acute angles.

Day 2

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1 Given an integer $n > 3$. Let $a_1, a_2, \cdots, a_n$ be real numbers satisfying $min|a_i - a_j| = 1, 1 \le i \le j \le n$. Find the minimum value of $\sum_{k=1}^n |a_k|^3$.

---

2 Let P be a convex n polygon each of which sides and diagnoals is colored with one of n distinct colors. For which n does: there exists a coloring method such that for any three of n colors, we can always find one triangle whose vertices is of P' and whose sides is colored by the three colors respectively.

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3 Given an integer $n > 3$. Prove that there exists a set S consisting of n pairwisely distinct positive integers such that for any two different non-empty subset of S:A, B, $\frac{\sum_{x\in A} x}{|A|}$ and $\frac{\sum_{x\in B} x}{|B|}$ are two composites which share no common divisors.

China National Olympiad 2008

www.artofproblemsolving.com/community/c5231

by Lei Lei, jgnr

Day 1

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1 Suppose $\triangle ABC$ is scalene. O is the circumcenter and A' is a point on the extension of segment AO such that $\angle BA'A = \angle CA'A$. Let point A_1 and A_2 be foot of perpendicular from A' onto AB and AC. H_A is the foot of perpendicular from A onto BC. Denote R_A to be the radius of circumcircle of $\triangle H_A A_1 A_2$. Similiarly we can define R_B and R_C. Show that:

$$\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C}=\frac{2}{R}$$

where R is the radius of circumcircle of $\triangle ABC$.

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2 Given an integer $n \geq 3$, prove that the set $X = \{1, 2, 3, \ldots, n^2 - n\}$ can be divided into two non-intersecting subsets such that neither of them contains n elements $a_1, a_2, \ldots, a_n$ with $a_1 < a_2 < \ldots < a_n$ and $a_k \leq \frac{a_{k-1}+a_{k+1}}{2}$ for all $k = 2, \ldots, n-1$.

---

3 Given a positive integer n and $x_1 \leq x_2 \leq \ldots \leq x_n, y_1 \geq y_2 \geq \ldots \geq y_n$, satisfying

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i$$

Show that for any real number α, we have

$$\sum_{i=1}^{n} x_i [i\alpha] \geq \sum_{i=1}^{n} y_i [i\alpha]$$

Here $[\beta]$ denotes the greastest integer not larger than β.

Day 2

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1 Let A be an infinite subset of $\mathbb{N}$, and n a fixed integer. For any prime p not dividing n, There are infinitely many elements of A not divisible by p. Show that for any integer $m > 1, (m, n) = 1$, There exist finitely many elements of A, such that their sum is congruent to 1 modulo m and congruent to 0 modulo n.

---

2 Find the smallest integer n satisfying the following condition: regardless of how one colour the vertices of a regular n-gon with either red, yellow or blue, one can always find an isosceles trapezoid whose vertices are of the same colour.

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3 Find all triples (p, q, n) that satisfy

$$q^{n+2} \equiv 3^{n+2} \pmod{p^n}, \quad p^{n+2} \equiv 3^{n+2} \pmod{q^n}$$

where p, q are odd primes and n is an positive integer.

China National Olympiad 2007

www.artofproblemsolving.com/community/c5230

by Lei Lei

Day 1

1 Given complex numbers a, b, c, let $|a+b| = m, |a-b| = n$. If $mn \neq 0$, Show that

$$\max\{|ac+b|, |a+bc|\} \geq \frac{mn}{\sqrt{m^2+n^2}}$$

2 Show that:
1) If $2n-1$ is a prime number, then for any n pairwise distinct positive integers $a_1, a_2, \dots, a_n$, there exists $i, j \in \{1, 2, \dots, n\}$ such that

$$\frac{a_i + a_j}{(a_i, a_j)} \geq 2n - 1$$

2) If $2n-1$ is a composite number, then there exists n pairwise distinct positive integers $a_1, a_2, \dots, a_n$, such that for any $i, j \in \{1, 2, \dots, n\}$ we have

$$\frac{a_i + a_j}{(a_i, a_j)} < 2n - 1$$

Here (x, y) denotes the greatest common divisor of x, y.

3 Let $a_1, a_2, \dots, a_{11}$ be 11 pairwise distinct positive integer with sum less than 2007. Let S be the sequence of $1, 2, \dots, 2007$. Define an **operation** to be 22 consecutive applications of the following steps on the sequence S: on i-th step, choose a number from the sequense S at random, say x. If $1 \leq i \leq 11$, replace x with $x + a_i$; if $12 \leq i \leq 22$, replace x with $x - a_{i-11}$. If the result of **operation** on the sequence S is an odd permutation of $\{1, 2, \dots, 2007\}$, it is an **odd operation**; if the result of **operation** on the sequence S is an even permutation of $\{1, 2, \dots, 2007\}$, it is an **even operation**. Which is larger, the number of odd operation or the number of even permutation? And by how many?

Here $\{x_1, x_2, \dots, x_{2007}\}$ is an even permutation of $\{1, 2, \dots, 2007\}$ if the product $\prod_{i>j}(x_i - x_j)$ is positive, and an odd one otherwise.

Day 2

1 Let O, I be the circumcenter and incenter of triangle ABC. The incircle of $\triangle ABC$ touches BC, CA, AB at points D, E, F repsectively. FD meets CA at P, ED meets AB at Q. M and N are midpoints of PE and QF respectively. Show that $OI \perp MN$.

2 Let $\{a_n\}_{n\geq 1}$ be a bounded sequence satisfying

$$a_n < \sum_{k=a}^{2n+2006} \frac{a_k}{k+1} + \frac{1}{2n+2007} \quad \forall \quad n = 1, 2, 3, \ldots$$

Show that

$$a_n < \frac{1}{n} \quad \forall \quad n = 1, 2, 3, \ldots$$

---

3 Find a number $n \geq 9$ such that for any n numbers, not necessarily distinct, $a_1, a_2, \ldots, a_n$, there exists 9 numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_9}, (1 \leq i_1 < i_2 < \ldots < i_9 \leq n)$ and $b_i \in 4, 7, i = 1, 2, \ldots, 9$ such that $b_1 a_{i_1} + b_2 a_{i_2} + \ldots + b_9 a_{i_9}$ is a multiple of 9.

China National Olympiad 2006

www.artofproblemsolving.com/community/c5229

by wenxin, Valentin Vornicu, shobber, zhaobin, dzeta

Day 1 January 12th

1 Let $a_1, a_2, \ldots, a_k$ be real numbers and $a_1 + a_2 + \ldots + a_k = 0$. Prove that

$$\max_{1 \le i \le k} a_i^2 \le \frac{k}{3}\left((a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{k-1} - a_k)^2\right).$$

2 For positive integers $a_1, a_2, \ldots, a_{2006}$ such that $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{2005}}{a_{2006}}$ are pairwise distinct, find the minimum possible amount of distinct positive integers in the set $\{a_1, a_2, ..., a_{2006}\}$.

3 Positive integers k, m, n satisfy $mn = k^2 + k + 3$, prove that at least one of the equations $x^2 + 11y^2 = 4m$ and $x^2 + 11y^2 = 4n$ has an odd solution.

Day 2 January 13th

4 In a right angled-triangle ABC, $\angle ACB = 90^o$. Its incircle O meets BC, AC, AB at D,E,F respectively. AD cuts O at P. If $\angle BPC = 90^o$, prove $AE + AP = PD$.

5 Let $\{a_n\}$ be a sequence such that: $a_1 = \frac{1}{2}$, $a_{k+1} = -a_k + \frac{1}{2 - a_k}$ for all $k = 1, 2, \ldots$. Prove that

$$\left(\frac{n}{2(a_1 + a_2 + \cdots + a_n)} - 1\right)^n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \left(\frac{1}{a_1} - 1\right)\left(\frac{1}{a_2} - 1\right) \cdots \left(\frac{1}{a_n} - 1\right).$$

6 Suppose X is a set with $|X| = 56$. Find the minimum value of n, so that for any 15 subsets of X, if the cardinality of the union of any 7 of them is greater or equal to n, then there exists 3 of them whose intersection is nonempty.

China National Olympiad 2005
www.artofproblemsolving.com/community/c5228
by jackhui, mecrazywong, Soarer, orl

Day 1

1 Suppose $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}), i = 1,2,3,4$. Prove that, there exist $x \in \mathbb{R}$, satisfying two inequalities

$$\begin{aligned} \cos^2\theta_1\cos^2\theta_2 - (\sin\theta\sin\theta_2 - x)^2 &\geq 0, \\ \cos^2\theta_3\cos^2\theta_4 - (\sin\theta_3\sin\theta_4 - x)^2 &\geq 0 \end{aligned}$$

if and only if

$$\sum_{i=1}^{4}\sin^2\theta_i \leq 2(1+\prod_{i=1}^{4}\sin\theta_i + \prod_{i=1}^{4}\cos\theta_i).$$

2 A circle meets the three sides BC, CA, AB of a triangle ABC at points $D_1, D_2; E_1, E_2; F_1, F_2$ respectively. Furthermore, line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the lines AL, BM, CN are concurrent.

3 As the graph, a pond is divided into 2n (n $\geq$ 5) parts. Two parts are called neighborhood if they have a common side or arc. Thus every part has three neighborhoods. Now there are 4n+1 frogs at the pond. If there are three or more frogs at one part, then three of the frogs of the part will jump to the three neighborhoods repsectively. Prove that for some time later, the frogs at the pond will uniformily distribute. That is, for any part either there are frogs at the part or there are frogs at the each of its neighborhoods.

`http://www.mathlinks.ro/Forum/files/china2005_2_214.gif`

Day 2

4 The sequence $\{a_n\}$ is defined by: $a_1 = \frac{21}{16}$, and for $n \geq 2$,

$$2a_n - 3a_{n-1} = \frac{3}{2^{n+1}}.$$

Let m be an integer with $m \geq 2$. Prove that: for $n \leq m$, we have

$$\left(a_n + \frac{3}{2^{n+3}}\right)^{\frac{1}{m}}\left(m - \left(\frac{2}{3}\right)^{\frac{n(m-1)}{m}}\right) < \frac{m^2-1}{m-n+1}.$$

5 There are 5 points in a rectangle (including its boundary) with area 1, no three of them are in the same line. Find the minimum number of triangles with the area not more than $\frac{1}{4}$, vertex of which are three of the five points.

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6 Find all nonnegative integer solutions (x,y,z,w) of the equation

$$2^x\cdot 3^y-5^z\cdot 7^w=1.$$

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China National Olympiad 2004
www.artofproblemsolving.com/community/c5227
by WakeUp, jcc0107

Day 1

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1 Let $EFGH, ABCD$ and $E_1F_1G_1H_1$ be three convex quadrilaterals satisfying:

i) The points E, F, G and H lie on the sides AB, BC, CD and DA respectively, and $\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CG}{GD} \cdot \frac{DH}{HA} = 1$;
ii) The points A, B, C and D lie on sides H_1E_1, E_1F_1, F_1, G_1 and G_1H_1 respectively, and $E_1F_1||EF, F_1G_1||FG$

Suppose that $\frac{E_1A}{AH_1} = \lambda$. Find an expression for $\frac{F_1C}{CG_1}$ in terms of λ.

Xiong Bin

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2 Let c be a positive integer. Consider the sequence $x_1, x_2, \dots$ which satisfies $x_1 = c$ and, for $n \geq 2$,

$$x_n = x_{n-1} + \left\lfloor \frac{2x_{n-1} - (n+2)}{n} \right\rfloor + 1$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Determine an expression for x_n in terms of n and c.

Huang Yumin

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3 Let M be a set consisting of n points in the plane, satisfying:
i) there exist 7 points in M which constitute the vertices of a convex heptagon;
ii) if for any 5 points in M which constitute the vertices of a convex pentagon, then there is a point in M which lies in the interior of the pentagon.
Find the minimum value of n.

Leng Gangsong

Day 2

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1 For a given real number a and a positive integer n, prove that:
i) there exists exactly one sequence of real numbers $x_0, x_1, \dots, x_n, x_{n+1}$ such that

$$\begin{cases} x_0 = x_{n+1} = 0, \\ \frac{1}{2}(x_i + x_{i+1}) = x_i + x_i^3 - a^3, \ i = 1, 2, \dots, n. \end{cases}$$

ii) the sequence $x_0, x_1, \dots, x_n, x_{n+1}$ in i) satisfies $|x_i| \leq |a|$ where $i = 0, 1, \dots, n+1$.

Liang Yengde

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2 For a given positive integer $n \geq 2$, suppose positive integers a_i where $1 \leq i \leq n$ satisfy $a_1 < a_2 < \ldots < a_n$ and $\sum_{i=1}^n \frac{1}{a_i} \leq 1$. Prove that, for any real number x, the following inequality holds

$$\left(\sum_{i=1}^n \frac{1}{a_i^2 + x^2}\right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2}$$

Li Shenghong

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3 Prove that every positive integer n, except a finite number of them, can be represented as a sum of 2004 positive integers: $n = a_1 + a_2 + \cdots + a_{2004}$, where $1 \leq a_1 < a_2 < \cdots < a_{2004}$, and $a_i \mid a_{i+1}$ for all $1 \leq i \leq 2003$.

Chen Yonggao

China National Olympiad 2003

www.artofproblemsolving.com/community/c5226

by littletush

Day 1 January 15th

1 Let I and H be the incentre and orthocentre of triangle ABC respectively. Let P, Q be the midpoints of AB, AC. The rays PI, QI intersect AC, AB at R, S respectively. Suppose that T is the circumcentre of triangle BHC. Let RS intersect BC at K. Prove that A, I and T are collinear if and only if $[BKS] = [CKR]$.

Shen Wunxuan

2 Determine the maximal size of the set S such that:
i) all elements of S are natural numbers not exceeding 100;
ii) for any two elements a, b in S, there exists c in S such that $(a, c) = (b, c) = 1$;
iii) for any two elements a, b in S, there exists d in S such that $(a, d) > 1, (b, d) > 1$.

Yao Jiangang

3 Given a positive integer n, find the least $\lambda > 0$ such that for any $x_1, \dots x_n \in \left(0, \frac{\pi}{2}\right)$, the condition $\prod_{i=1}^{n} \tan x_i = 2^{\frac{n}{2}}$ implies $\sum_{i=1}^{n} \cos x_i \le \lambda$.

Huang Yumin

Day 2 January 16th

1 Find all integer triples (a, m, n) such that $a^m + 1 | a^n + 203$ where $a, m > 1$.

Chen Yonggao

2 Ten people apply for a job. The manager decides to interview the candidates one by one according to the following conditions:
i) the first three candidates will not be employed;
ii) from the fourth candidates onwards, if a candidate's comptence surpasses the competence of all those who preceded him, then that candidate is employed;
iii) if the first nine candidates are not employed, then the tenth candidate will be employed.
We assume that none of the 10 applicants have the same competence, and these competences can be ranked from the first to tenth. Let P_k represent the probability that the kth-ranked applicant in competence is employed. Prove that:
i) $P_1 > P_2 > \dots > P_8 = P_9 = P_{10}$;
ii) $P_1 + P_2 + P_3 > 0.7$
iii) $P_8 + P_9 + P_{10} \le 0.1$.

Su Chun

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3 Suppose a, b, c, d are positive reals such that $ab + cd = 1$ and x_i, y_i are real numbers such that $x_i^2 + y_i^2 = 1$ for $i = 1, 2, 3, 4$. Prove that

$$(ax_1 + bx_2 + cx_3 + dx_4)^2 + (ay_4 + by_3 + cy_2 + dy_1)^2 \leq 2\left(\frac{a^2+b^2}{ab} + \frac{c^2+d^2}{cd}\right).$$

Li Shenghong

China National Olympiad 2002

www.artofproblemsolving.com/community/c5225

by horizon

Day 1

1 the edges of triangle ABC are a,b,c respectively,$b<c$,AD is the bisector of $\angle A$,point D is on segment BC.
(1)find the property $\angle A$,$\angle B$,$\angle C$ have,so that there exists point E,F on AB,AC satisfy $BE=CF$,and $\angle NDE=\angle CDF$
(2)when such E,F exist,express BE with a,b,c

2 Given the polynomial sequence $(p_n(x))$ satisfying $p_1(x)=x^2-1$, $p_2(x)=2x(x^2-1)$, and $p_{n+1}(x)p_{n-1}(x)=(p_n(x))^2-(x^2-1)^2$, for $n\geq 2$, let s_n be the sum of the absolute values of the coefficients of $p_n(x)$. For each n, find a non-negative integer k_n such that $2^{-k_n}s_n$ is odd.

3 In a competition there are 18 teams and in each round 18 teams are divided into 9 pairs where the 9 matches are played coincidentally. There are 17 rounds, so that each pair of teams play each other exactly once. After n rounds, there always exists 4 teams such that there was exactly one match played between these teams in those n rounds. Find the maximum value of n.

Day 2

1 For every four points P_1,P_2,P_3,P_4 on the plane, find the minimum value of $\frac{\sum_{1\le\ i<j\le\ 4}P_iP_j}{\min_{1\le\ i<j\le\ 4}(P_iP_j)}$.

2 Suppose that a point in the plane is called *good* if it has rational coordinates. Prove that all good points can be divided into three sets satisfying:
1) If the centre of the circle is good, then there are three points in the circle from each of the three sets.
2) There are no three collinear points that are from each of the three sets.

3 Suppose that $c\in\left(\frac{1}{2},1\right)$. Find the least M such that for every integer $n\ge 2$ and real numbers $0<a_1\le a_2\le\ldots \le a_n$, if $\frac{1}{n}\sum_{k=1}^{n}ka_{k}=c\sum_{k=1}^{n}a_{k}$, then we always have that $\sum_{k=1}^{n}a_{k}\le M\sum_{k=1}^{m}a_{k}$ where $m=[cn]$

China National Olympiad 2001
www.artofproblemsolving.com/community/c5224
by horizon

Day 1

1 Let a be real number with $\sqrt{2} < a < 2$, and let $ABCD$ be a convex cyclic quadrilateral whose circumcentre O lies in its interior. The quadrilateral's circumcircle ω has radius 1, and the longest and shortest sides of the quadrilateral have length a and $\sqrt{4-a^2}$, respectively. Lines L_A, L_B, L_C, L_D are tangent to ω at A, B, C, D, respectively.

Let lines L_A and L_B, L_B and L_C,L_C and L_D,L_D and L_A intersect at A', B', C', D' respectively. Determine the minimum value of $\frac{S_{A'B'C'D'}}{S_{ABCD}}$.

2 Let $X = \{1, 2, \dots, 2001\}$. Find the least positive integer m such that for each subset $W \subset X$ with m elements, there exist $u, v \in W$ (not necessarily distinct) such that $u + v$ is of the form 2^k, where k is a positive integer.

3 Let P be a regular n-gon $A_1A_2\dots A_n$. Find all positive integers n such that for each permutation $\sigma(1), \sigma(2), \dots, \sigma(n)$ there exists $1 \le i, j, k \le n$ such that the triangles $A_iA_jA_k$ and $A_{\sigma(i)}A_{\sigma(j)}A_{\sigma(k)}$ are both acute, both right or both obtuse.

Day 2

1 Let a, b, c be positive integers such that $a, b, c, a+b-c, a+c-b, b+c-a, a+b+c$ are 7 distinct primes. The sum of two of a, b, c is 800. If d be the difference of the largest prime and the least prime among those 7 primes, find the maximum value of d.

2 Let $P_1P_2\dots P_{24}$ be a regular 24-sided polygon inscribed in a circle ω with circumference 24. Determine the number of ways to choose sets of eight distinct vertices from these 24 such that none of the arcs has length 3 or 8.

3 Let $a = 2001$. Consider the set A of all pairs of integers (m, n) with $n \neq 0$ such that
(i) $m < 2a$;
(ii) $2n|(2am - m^2 + n^2)$;
(iii) $n^2 - m^2 + 2mn \le 2a(n - m)$.
For $(m, n) \in A$, let

$$f(m,n) = \frac{2am - m^2 - mn}{n}.$$

Determine the maximum and minimum values of f.

China National Olympiad 2000

www.artofproblemsolving.com/community/c5223

by jred

Day 1

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1 The sides a, b, c of triangle ABC satisfy $a \le b \le c$. The circumradius and inradius of triangle ABC are R and r respectively. Let $f = a + b - 2R - 2r$. Determine the sign of f by the measure of angle C.

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2 A sequence (a_n) is defined recursively by $a_1 = 0, a_2 = 1$ and for $n \ge 3$,

$$a_n = \frac{1}{2}na_{n-1} + \frac{1}{2}n(n-1)a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right).$$

Find a closed-form expression for $f_n = a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \ldots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1$.

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3 A table tennis club hosts a series of doubles matches following several rules:
(i) each player belongs to two pairs at most;
(ii) every two distinct pairs play one game against each other at most;
(iii) players in the same pair do not play against each other when they pair with others respectively.
Every player plays a certain number of games in this series. All these distinct numbers make up a set called the *set of games*. Consider a set $A = \{a_1, a_2, \ldots, a_k\}$ of positive integers such that every element in A is divisible by 6. Determine the minimum number of players needed to participate in this series so that a schedule for which the corresponding *set of games* is equal to set A exists.

Day 2

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1 Given an ordered n-tuple $A = (a_1, a_2, \cdots, a_n)$ of real numbers, where $n \ge 2$, we define $b_k = \max a_1, \ldots a_k$ for each k. We define $B = (b_1, b_2, \cdots, b_n)$ to be the *innovated tuple* of A. The number of distinct elements in B is called the *innovated degree* of A.
Consider all permutations of $1, 2, \ldots, n$ as an ordered n-tuple. Find the arithmetic mean of the first term of the permutations whose innovated degrees are all equal to 2

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2 Find all positive integers n such that there exists integers $n_1, \ldots, n_k \ge 3$, for some integer k, satisfying

$$n = n_1 n_2 \cdots n_k = 2^{\frac{1}{2^k}(n_1-1)\cdots(n_k-1)} - 1.$$

---

3 A test contains 5 multiple choice questions which have 4 options in each. Suppose each examinee chose one option for each question. There exists a number n, such that for any n sheets among 2000 sheets of answer papers, there are 4 sheets of answer papers such that any two of them have at most 3 questions with the same answers. Find the minimum value of n.

China National Olympiad 1999

www.artofproblemsolving.com/community/c5222

by WakeUp, quangdung_toank13

Day 1

1 Let ABC be an acute triangle with $\angle C > \angle B$. Let D be a point on BC such that $\angle ADB$ is obtuse, and let H be the orthocentre of triangle ABD. Suppose that F is a point inside triangle ABC that is on the circumcircle of triangle ABD. Prove that F is the orthocenter of triangle ABC if and only if $HD||CF$ and H is on the circumcircle of triangle ABC.

2 Let a be a real number. Let $(f_n(x))_{n\ge 0}$ be a sequence of polynomials such that $f_0(x)=1$ and $f_{n+1}(x)=xf_n(x)+f_n(ax)$ for all non-negative integers n.
a) Prove that $f_n(x)=x^nf_n\left(x^{-1}\right)$ for all non-negative integers n.
b) Find an explicit expression for $f_n(x)$.

3 There are 99 space stations. Each pair of space stations is connected by a tunnel. There are 99 two-way main tunnels, and all the other tunnels are strictly one-way tunnels. A group of 4 space stations is called *connected* if one can reach each station in the group from every other station in the group without using any tunnels other than the 6 tunnels which connect them. Determine the maximum number of connected groups.

Day 2

1 Let m be a positive integer. Prove that there are integers a, b, k, such that both a and b are odd, $k\ge 0$ and

$$2m=a^{19}+b^{99}+k\cdot 2^{1999}$$

2 Determine the maximum value of λ such that if $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all its roots nonnegative, then

$$f(x)\geq\lambda(x - a)^3$$

for all $x\geq 0$. Find the equality condition.

3 A $4\times 4\times 4$ cube is composed of 64 unit cubes. The faces of 16 unit cubes are to be coloured red. A colouring is called interesting if there is exactly 1 red unit cube in every $1\times 1\times 4$ rectangular box composed of 4 unit cubes. Determine the number of interesting colourings.

China National Olympiad 1998

www.artofproblemsolving.com/community/c5221

by jred

Day 1

1 Let ABC be a non-obtuse triangle satisfying $AB > AC$ and $\angle B = 45^\circ$. The circumcentre O and incentre I of triangle ABC are such that $\sqrt{2}\,OI = AB - AC$. Find the value of $\sin A$.

2 Given a positive integer $n > 1$, determine with proof if there exist $2n$ pairwise different positive integers $a_1, \ldots, a_n, b_1, \ldots b_n$ such that $a_1 + \ldots + a_n = b_1 + \ldots + b_n$ and

$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}.$$

3 Let $S = \{1, 2, \ldots, 98\}$. Find the least natural number n such that we can pick out 10 numbers in any n-element subset of S satisfying the following condition: no matter how we equally divide the 10 numbers into two groups, there exists a number in one group such that it is coprime to the other numbers in that group, meanwhile there also exists a number in the other group such that it is not coprime to any of the other numbers in the same group.

Day 2

1 Find all natural numbers $n > 3$, such that 2^{2000} is divisible by $1 + C_n^1 + C_n^2 + C_n^3$.

2 Let D be a point inside acute triangle ABC satisfying the condition

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

Determine (with proof) the geometric position of point D.

3 Let $x_1, x_2, \ldots, x_n$ be real numbers, where $n \geq 2$, satisfying $\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1} = 1$. For each k, find the maximal value of $|x_k|$.

China National Olympiad 1997
www.artofproblemsolving.com/community/c5220
by jred

Day 1

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1 Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following conditions:
i) $-\frac{1}{\sqrt{3}} \le x_i \le \sqrt{3}$ for $i = 1, 2, \ldots, 1997$;
ii) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.
Determine (with proof) the maximum value of $x_1^{12} + x_2^{12} + \ldots + x_{1997}^{12}$.

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2 Let $A_1B_1C_1D_1$ be an arbitrary convex quadrilateral. P is a point inside the quadrilateral such that each angle enclosed by one edge and one ray which starts at one vertex on that edge and passes through point P is acute. We recursively define points A_k, B_k, C_k, D_k symmetric to P with respect to lines $A_{k-1}B_{k-1}, B_{k-1}C_{k-1}, C_{k-1}D_{k-1}, D_{k-1}A_{k-1}$ respectively for $k \ge 2$.
Consider the sequence of quadrilaterals $A_iB_iC_iD_i$.
i) Among the first 12 quadrilaterals, which are similar to the 1997th quadrilateral and which are not?
ii) Suppose the 1997th quadrilateral is cyclic. Among the first 12 quadrilaterals, which are cyclic and which are not?

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3 Prove that there are infinitely many natural numbers n such that we can divide $1, 2, \ldots, 3n$ into three sequences (a_n), (b_n) and (c_n), with n terms in each, satisfying the following conditions:
i) $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = \ldots = a_n + b_n + c_n$ and $a_1 + b_1 + c_1$ is divisible by 6;
ii) $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n = c_1 + c_2 + \ldots + c_n$, and $a_1 + a_2 + \ldots + a_n$ is divisible by 6.

Day 2

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1 Consider a cyclic quadrilateral $ABCD$. The extensions of its sides AB, DC meet at the point P and the extensions of its sides AD, BC meet at the point Q. Suppose QE, QF are tangents to the circumcircle of quadrilateral $ABCD$ at E, F respectively. Show that P, E, F are collinear.

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2 Let $A = \{1, 2, 3, \cdots, 17\}$. A mapping $f : A \to A$ is defined as follows: $f^{[1]}(x) = f(x)$, $f^{[k+1]}(x) = f(f^{[k]}(x))$ for $k \in \mathbb{N}$. Suppose that f is bijective and that there exists a natural number M such that:
i) when $m < M$ and $1 \le i \le 16$, we have $f^{[m]}(i+1) - f^{[m]}(i) \not\equiv \pm 1 \pmod{17}$ and $f^{[m]}(1) - f^{[m]}(17) \not\equiv \pm 1 \pmod{17}$;
ii) when $1 \le i \le 16$, we have $f^{[M]}(i+1) - f^{[M]}(i) \equiv \pm 1 \pmod{17}$ and $f^{[M]}(1) - f^{[M]}(17) \equiv \pm 1$

$\pmod{17}$.
Find the maximal value of M.

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3 Let (a_n) be a sequence of non-negative real numbers satisfying $a_{n+m} \leq a_n + a_m$ for all non-negative integers m, n.
Prove that if $n \geq m$ then $a_n \leq ma_1 + \left(\frac{n}{m} - 1\right) a_m$ holds.

China National Olympiad 1996

www.artofproblemsolving.com/community/c5219

by vladimir92, moldovan, Rijul saini, shobber

Day 1

1 Let $\triangle ABC$ be a triangle with orthocentre H. The tangent lines from A to the circle with diameter BC touch this circle at P and Q. Prove that H, P and Q are collinear.

2 Find the smallest positive integer K such that every K-element subset of $\{1, 2, ..., 50\}$ contains two distinct elements a, b such that $a + b$ divides ab.

3 Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2)$$

for all $x, y \in \mathbb{R}$.
Prove that $f(1996x) = 1996f(x)$ for all $x \in \mathbb{R}$.

Day 2

1 8 singers take part in a festival. The organiser wants to plan m concerts. For every concert there are 4 singers who go on stage, with the restriction that the times of which every two singers go on stage in a concert are all equal. Find a schedule that minimises m.

2 Let n be a natural number. Suppose that $x_0 = 0$ and that $x_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. If $\sum_{i=1}^{n} x_i = 1$, prove that

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \ldots + x_{i-1}}\sqrt{x_i + \ldots + x_n}} < \frac{\pi}{2}$$

3 In the triangle ABC, $\angle C = 90^\circ, \angle A = 30^\circ$ and $BC = 1$. Find the minimum value of the longest side of all inscribed triangles (i.e. triangles with vertices on each of three sides) of the triangle ABC.

China National Olympiad 1995

www.artofproblemsolving.com/community/c5218

by jred

Day 1

1 Let $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n (n \geq 3)$ be real numbers satisfying the following conditions:
(1) $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$;
(2) $0 < a_1 = a_2, a_i + a_{i+1} = a_{i+2}$ $(i = 1, 2, \cdots, n-2)$;
(3) $0 < b_1 \leq b_2, b_i + b_{i+1} \leq b_{i+2}$ $(i = 1, 2, \cdots, n-2)$.
Prove that $a_{n-1} + a_n \leq b_{n-1} + b_n$.

2 Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:
(1) $f(1) = 1$;
(2) $\forall n \in \mathbb{N}$, $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$;
(3) $\forall n \in \mathbb{N}$, $f(2n) < 6f(n)$.
Find all solutions of equation $f(k) + f(l) = 293$, where $k < l$.
($\mathbb{N}$ denotes the set of all natural numbers).

3 Find the minimun value of $\sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10} |k(x+y-10i)(3x-6y-36j)(19x+95y-95k)|$, where x, y are integers.

Day 2

1 Given four spheres with their radii equal to $2, 2, 3, 3$ respectively, each sphere externally touches the other spheres. Suppose that there is another sphere that is externally tangent to all those four spheres, determine the radius of this sphere.

2 Let $a_1, a_2, \cdots, a_{10}$ be pairwise distinct natural numbers with their sum equal to 1995. Find the minimal value of $a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1$.

3 Let $n (n > 1)$ be an odd. We define $x_k = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$ as follow: $x_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) = (1, 0, \cdots, 0, 1)$; $x_i^{(k)} = \begin{cases} 0, & x_i^{(k-1)} = x_{i+1}^{(k-1)}, \\ 1, & x_i^{(k-1)} \neq x_{i+1}^{(k-1)}, \end{cases}$ $\quad i = 1, 2, \cdots, n$, where $x_{n+1}^{(k-1)} = x_1^{(k-1)}$.
Let m be a positive integer satisfying $x_0 = x_m$. Prove that m is divisible by n.

China National Olympiad 1994

www.artofproblemsolving.com/community/c5217

by jred

Day 1

1 Let $ABCD$ be a trapezoid with $AB \parallel CD$. Points E,F lie on segments AB,CD respectively. Segments CE,BF meet at H, and segments ED,AF meet at G. Show that $S_{EHFG}\le \dfrac{1}{4}S_{ABCD}$. Determine, with proof, if the conclusion still holds when $ABCD$ is just any convex quadrilateral.

2 There are m pieces of candy held in n trays($n,m\ge 4$). An *operation* is defined as follow: take out one piece of candy from any two trays respectively, then put them in a third tray. Determine, with proof, if we can move all candies to a single tray by finite *operations*.

3 Find all functions $f:[1,\infty)\to [1,\infty)$ satisfying the following conditions:
(1) $f(x)\le 2(x+1)$;
(2) $f(x+1)=\dfrac{1}{x}[(f(x))^2-1]$.

Day 2

4 Let $f(z)=c_0z^n+c_1z^{n-1}+c_2z^{n-2}+\cdots+c_{n-1}z+c_n$ be a polynomial with complex coefficients. Prove that there exists a complex number z_0 such that $|f(z_0)|\ge |c_0|+|c_n|$, where $|z_0|\le 1$.

5 For arbitrary natural number n, prove that $\sum_{k=0}^{n}C_n^k 2^k C_{n-k}^{[(n-k)/2]}=C_{2n+1}^n$, where $C_0^0=1$ and $[\dfrac{n-k}{2}]$ denotes the integer part of $\dfrac{n-k}{2}$.

6 Let M be a point which has coordinates $(p\times 1994, 7p\times 1994)$ in the Cartesian plane (p is a prime). Find the number of right-triangles satisfying the following conditions:
(1) all vertexes of the triangle are lattice points, moreover M is on the right-angled corner of the triangle;
(2) the origin $(0,0)$ is the incenter of the triangle.

China National Olympiad 1993
www.artofproblemsolving.com/community/c5216
by jred

Day 1

1 Given an odd n, prove that there exist $2n$ integers $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$, such that for any integer k $(0 < k < n)$, the following $3n$ integers: $a_i + a_{i+1}, a_i + b_i, b_i + b_{i+k}$ $(i = 1, 2, \cdots, n; a_{n+1} = a_1, b_{n+j} = b_j, 0 < j < n)$ are of different remainders on division by $3n$.

2 Given a natural number k and a real number $a(a > 0)$, find the maximal value of $a^{k_1} + a^{k_2} + \cdots + a^{k_r}$, where $k_1 + k_2 + \cdots + k_r = k$ $(k_i \in \mathbb{N}, 1 \le r \le k)$.

3 Let K, K_1 be two circles with the same center and their radii equal to R and $R_1 (R_1 > R)$ respectively. Quadrilateral $ABCD$ is inscribed in circle K. Quadrilateral $A_1B_1C_1D_1$ is inscribed in circle K_1 where A_1, B_1, C_1, D_1 lie on rays CD, DA, AB, BC respectively. Show that $\dfrac{S_{A_1B_1C_1D_1}}{S_{ABCD}} \ge \dfrac{R_1^2}{R^2}$.

Day 2

4 We are given a set $S = \{z_1, z_2, \cdots, z_{1993}\}$, where $z_1, z_2, \cdots, z_{1993}$ are nonzero complex numbers (also viewed as nonzero vectors in the plane). Prove that we can divide S into some groups such that the following conditions are satisfied:
(1) Each element in S belongs and only belongs to one group;
(2) For any group p, if we use $T(p)$ to denote the sum of all memebers in p, then for any memeber $z_i (1 \le i \le 1993)$ of p, the angle between z_i and $T(p)$ does not exceed 90°;
(3) For any two groups p and q, the angle between $T(p)$ and $T(q)$ exceeds 90° (use the notation introduced in (2)).

5 10 students bought some books in a bookstore. It is known that every student bought exactly three kinds of books, and any two of them shared at least one kind of book. Determine, with proof, how many students bought the most popular book at least? (Note: the most popular book means most students bought this kind of book)

6 Let $f : (0, +\infty) \to (0, +\infty)$ be a function satisfying the following condition: for arbitrary positive real numbers x and y, we have $f(xy) \le f(x)f(y)$. Show that for arbitrary positive real number x and natural number n, inequality $f(x^n) \le f(x)f(x^2)^{\frac{1}{2}} \dots f(x^n)^{\frac{1}{n}}$ holds.

China National Olympiad 1992

www.artofproblemsolving.com/community/c5215

by jred

Day 1

1 Let equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0$ with real coefficients satisfy $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1} \le 1$. Suppose that λ ($|\lambda| > 1$) is a complex root of the equation, prove that $\lambda^{n+1} = 1$.

2 Given nonnegative real numbers $x_1, x_2, \ldots, x_n$, let $a = min\{x_1, x_2, \ldots, x_n\}$. Prove that the following inequality holds:

$$\sum_{i=1}^{n} \frac{1+x_i}{1+x_{i+1}} \le n + \frac{1}{(1+a)^2} \sum_{i=1}^{n} (x_i - a)^2 \qquad (x_{n+1} = x_1),$$

and equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

3 Given a 9×9 grid, we assign either $+1$ or -1 to each square on the grid. We define an *adjustment* as follow: for each square on the grid, we make a product of all numbers of those squares which share a common side with the square (excluding itself).Then we have 81 products. Next we replace all the squares values with their corresponding products. Determine if we can make all values in the grid equal to 1 through finite *adjustments*.

Day 2

1 A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of $\triangle ABP$ and $\triangle CDP$ meet at P and Q (O, P, Q are pairwise distinct). Show that $\angle OQP = 90^\circ$.

2 Find the maximum possible number of edges of a simple graph with 8 vertices and without any quadrilateral. (a simple graph is an undirected graph that has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices.)

3 Let sequence $\{a_1, a_2, \ldots\}$ with integer terms satisfy the following conditions:
1) $a_{n+1} = 3a_n - 3a_{n-1} + a_{n-2}, n = 2, 3, \ldots$;
2) $2a_1 = a_0 + a_2 - 2$;
3) for arbitrary natural number m, there exist m consecutive terms $a_k, a_{k-1}, \ldots, a_{k+m-1}$ among the sequence such that all such m terms are perfect squares.
Prove that all terms of the sequence $\{a_1, a_2, \ldots\}$ are perfect squares.

China National Olympiad 1991
www.artofproblemsolving.com/community/c5214
by jred

Day 1

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1 We are given a convex quadrilateral $ABCD$ in the plane.
(*i*) If there exists a point P in the plane such that the areas of $\triangle ABP, \triangle BCP, \triangle CDP, \triangle DAP$ are equal, what condition must be satisfied by the quadrilateral $ABCD$?
(*ii*) Find (with proof) the maximum possible number of such point P which satisfies the condition in (*i*).

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2 Given $I = [0, 1]$ and $G = \{(x, y)|x, y \in I\}$, find all functions $f : G \to I$, such that $\forall x, y, z \in I$ we have:
i. $f(f(x, y), z) = f(x, f(y, z))$;
ii. $f(x, 1) = x, f(1, y) = y$;
iii. $f(zx, zy) = z^k f(x, y)$.
(k is a positive real number irrelevant to x, y, z.)

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3 There are 10 birds on the ground. For any 5 of them, there are at least 4 birds on a circle. Determine the least possible number of birds on the circle with the most birds.

Day 2

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4 Find all positive integer solutions (x, y, z, n) of equation $x^{2n+1} - y^{2n+1} = xyz + 2^{2n+1}$, where $n \geq 2$ and $z \leq 5 \times 2^{2n}$.

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5 Find all natural numbers n, such that $\min_{k\in\mathbb{N}}(k^2 + [n/k^2]) = 1991$. ($[n/k^2]$ denotes the integer part of n/k^2.)

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6 A football is covered by some polygonal pieces of leather which are sewed up by three different colors threads. It features as follows:
i) any edge of a polygonal piece of leather is sewed up with an equal-length edge of another polygonal piece of leather by a certain color thread;
ii) each node on the ball is vertex to exactly three polygons, and the three threads joint at the node are of different colors.
Show that we can assign to each node on the ball a complex number (not equal to 1), such that the product of the numbers assigned to the vertices of any polygonal face is equal to 1.

China National Olympiad 1990

www.artofproblemsolving.com/community/c5213

by jred

Day 1

1 Given a convex quadrilateral $ABCD$, side AB is not parallel to side CD. The circle O_1 passing through A and B is tangent to side CD at P. The circle O_2 passing through C and D is tangent to side AB at Q. Circle O_1 and circle O_2 meet at E and F. Prove that EF bisects segment PQ if and only if $BC \parallel AD$.

2 Let x be a natural number. We call $\{x_0, x_1, \dots, x_l\}$ a *factor link* of x if the sequence $\{x_0, x_1, \dots, x_l\}$ satisfies the following conditions:
(1) $x_0 = 1, x_l = x$;
(2) $x_{i-1} < x_i, x_{i-1} | x_i, i = 1, 2, \dots, l$.
Meanwhile, we define l as the length of the *factor link* $\{x_0, x_1, \dots, x_l\}$. Denote by $L(x)$ and $R(x)$ the length and the number of the longest *factor link* of x respectively. For $x = 5^k \times 31^m \times 1990^n$, where k, m, n are natural numbers, find the value of $L(x)$ and $R(x)$.

3 A function $f(x)$ defined for $x \geq 0$ satisfies the following conditions:
i. for $x, y \geq 0$, $f(x)f(y) \leq x^2 f(y/2) + y^2 f(x/2)$;
ii. there exists a constant $M (M > 0)$, such that $|f(x)| \leq M$ when $0 \leq x \leq 1$.
Prove that $f(x) \leq x^2$.

Day 2

4 Given a positive integer number a and two real numbers A and B, find a necessary and sufficient condition on A and B for the following system of equations to have integer solution:

$$\begin{cases} x^2 + y^2 + z^2 = (Ba)^2 \\ x^2(Ax^2 + By^2) + y^2(Ay^2 + Bz^2) + z^2(Az^2 + Bx^2) = \dfrac{1}{4}(2A + B)(Ba)^4 \end{cases}$$

5 Given a finite set X, let f be a rule such that f maps every *even-element-subset* E of X (i.e. $E \subseteq X$, $|E|$ is even) into a real number $f(E)$. Suppose that f satisfies the following conditions:
(I) there exists an *even-element-subset* D of X such that $f(D) > 1990$;
(II) for any two disjoint *even-element-subsets* A, B of X, equation $f(A \cup B) = f(A) + f(B) - 1990$ holds.
Prove that there exist two subsets P, Q of X satisfying:
(1) $P \cap Q = \emptyset$, $P \cup Q = X$;

(2) for any *non-even-element-subset* S of P (i.e. $S \subseteq P$, $|S|$ is odd), we have $f(S) > 1990$;
(3) for any *even-element-subset* T of Q, we have $f(T) \leq 1990$.

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6 A convex n-gon and its $n-3$ diagonals which have no common point inside the polygon form a *subdivision graph*. Show that if and only if $3|n$, there exists a *subdivision graph* that can be drawn in one closed stroke. (i.e. start from a certain vertex, get through every edges and diagonals exactly one time, finally back to the starting vertex.)

China National Olympiad 1989
www.artofproblemsolving.com/community/c5212
by jred

Day 1

1 We are given two point sets A and B which are both composed of finite disjoint arcs on the unit circle. Moreover, the length of each arc in B is equal to $\dfrac{\pi}{m}$ ($m\in\mathbb{N}$). We denote by A^j the set obtained by a counterclockwise rotation of A about the center of the unit circle for $\dfrac{j\pi}{m}$ ($j=1,2,3,\dots$). Show that there exists a natural number k such that $l(A^k\cap B)\ge\dfrac{1}{2\pi}l(A)l(B)$.(Here $l(X)$ denotes the sum of lengths of all disjoint arcs in the point set X)

2 Let $x_1,x_2,\dots,x_n$ ($n\ge 2$) be positive real numbers satisfying $\sum_{i=1}^n x_i=1$. Prove that:

$$\sum_{i=1}^{n}\frac{x_i}{\sqrt{1-x_i}}\ge\frac{\sum_{i=1}^n\sqrt{x_i}}{\sqrt{n-1}}.$$

3 Let S be the unit circle in the complex plane (i.e. the set of all complex numbers with their moduli equal to 1).
We define function $f:S\rightarrow S$ as follow: $\forall z\in S$, $f^{(1)}(z)=f(z), f^{(2)}(z)=f(f(z)),\dots, f^{(k)}(z)=f(f^{(k-1)}(z))(k>1,k\in\mathbb{N}),\dots$
We call c an n-*period-point* of f if c ($c\in S$) and n ($n\in\mathbb{N}$) satisfy: $f^{(1)}(c)\neq c, f^{(2)}(c)\neq c, f^{(3)}(c)\neq c,\dots, f^{(n-1)}(c)\neq c, f^{(n)}(c)=c$.
Suppose that $f(z)=z^m$ ($z\in S; m>1, m\in\mathbb{N}$), find the number of 1989-*period-point* of f.

Day 2

4 Given a triangle ABC, points D,E,F lie on sides BC,CA,AB respectively. Moreover, the radii of incircles of $\triangle AEF,\triangle BFD,\triangle CDE$ are equal to r. Denote by r_0 and R the radii of incircles of $\triangle DEF$ and $\triangle ABC$ respectively. Prove that $r+r_0=R$.

5 Given 1989 points in the space, any three of them are not collinear. We divide these points into 30 groups such that the numbers of points in these groups are different from each other. Consider those triangles whose vertices are points belong to three different groups among the 30. Determine the numbers of points of each group such that the number of such triangles attains a maximum.

6 Find all functions $f:(1,+\infty)\to(1,+\infty)$ that satisfy the following condition: for arbitrary $x,y>1$ and $u,v>0$, inequality $f(x^uy^v)\le f(x)^{\frac{1}{4u}}f(y)^{\frac{1}{4v}}$ holds.

China National Olympiad 1988
www.artofproblemsolving.com/community/c5211
by jred

Day 1

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1 Let $r_1, r_2, \dots, r_n$ be real numbers. Given n reals $a_1, a_2, \dots, a_n$ that are not all equal to 0, suppose that inequality

$$r_1(x_1 - a_1) + r_2(x_2 - a_2) + \cdots + r_n(x_n - a_n) \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} - \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

holds for arbitrary reals $x_1, x_2, \dots, x_n$. Find the values of $r_1, r_2, \dots, r_n$.

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2 Given two circles C_1, C_2 with common center, the radius of C_2 is twice the radius of C_1. Quadrilateral $A_1A_2A_3A_4$ is inscribed in C_1. The extension of A_4A_1 meets C_2 at B_1; the extension of A_1A_2 meets C_2 at B_2; the extension of A_2A_3 meets C_2 at B_3; the extension of A_3A_4 meets C_2 at B_4. Prove that $P(B_1B_2B_3B_4) \geq 2P(A_1A_2A_3A_4)$, and in what case the equality holds? ($P(X)$ denotes the perimeter of quadrilateral X)

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3 Given a finite sequence of real numbers $a_1, a_2, \dots, a_n$ ($\ast$), we call a segment $a_k, \dots, a_{k+l-1}$ of the sequence ($\ast$) a *long*(Chinese dragon) and a_k *head* of the *long* if the arithmetic mean of $a_k, \dots, a_{k+l-1}$ is greater than 1988. (especially if a single item $a_m > 1988$, we still regard a_m as a *long*). Suppose that there is at least one *long* among the sequence ($\ast$), show that the arithmetic mean of all those items of sequence ($\ast$) that could be *head* of a certain *long* individually is greater than 1988.

Day 2

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4 (1) Let a, b, c be positive real numbers satisfying $(a^2 + b^2 + c^2)^2 > 2(a^4 + b^4 + c^4)$. Prove that a, b, c can be the lengths of three sides of a triangle respectively.
(2) Let $a_1, a_2, \dots, a_n$ be n ($n > 3$) positive real numbers satisfying $(a_1^2 + a_2^2 + \cdots + a_n^2)^2 > (n-1)(a_1^4 + a_2^4 + \cdots + a_n^4)$. Prove that any three of $a_1, a_2, \dots, a_n$ can be the lengths of three sides of a triangle respectively.

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5 Given three tetrahedrons $A_iB_iC_iD_i$ $(i = 1, 2, 3)$, planes $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$ are drawn through B_i, C_i, D_i respectively, and they are perpendicular to edges A_iB_i, A_iC_i, A_iD_i $(i = 1, 2, 3)$ respectively. Suppose that all nine planes $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$ meet at a point E, and points A_1, A_2, A_3 lie on line l. Determine the intersection (shape and position) of the circumscribed spheres of the three tetrahedrons.

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6 Let n ($n \geq 3$) be a natural number. Denote by $f(n)$ the least natural number by which n is not divisible (e.g. $f(12) = 5$). If $f(n) \geq 3$, we may have $f(f(n))$ in the same way. Similarly, if

$f(f(n)) \geq 3$, we may have $f(f(f(n)))$, and so on. If $\underbrace{f(f(\dots f}_{k \text{ times}}(n)\dots)) = 2$, we call k the *length* of n (also we denote by l_n the *length* of n). For arbitrary natural number n ($n \geq 3$), find l_n with proof.

China National Olympiad 1987

www.artofproblemsolving.com/community/c5210

by jred

Day 1

1 Let n be a natural number. Prove that a necessary and sufficient condition for the equation $z^{n+1}-z^n-1=0$ to have a complex root whose modulus is equal to 1 is that $n+2$ is divisible by 6.

2 We are given an equilateral triangle ABC with the length of its side equal to 1. There are $n-1$ points on each side of the triangle ABC that equally divide the side into n segments. We draw all possible lines that pass through any two of all those $3(n-1)$ points such that they are parallel to one of three sides of triangle ABC. All such lines divide triangle ABC into some lesser triangles whose vertices are called *nodes*. We assign a real number for each *node* such that the following conditions are satisfied:
(I) real numbers a, b, c are assigned to A, B, C respectively;
(II) for any rhombus that is consisted of two lesser triangles that share a common side, the sum of the numbers of vertices on its one diagonal is equal to that of vertices on the other diagonal.
1) Find the minimum distance between the *node* with the maximal number to the *node* with the minimal number;
2) Denote by S the sum of the numbers of all *nodes*, find S.

3 Some players participate in a competition. Suppose that each player plays one game against every other player and there is no draw game in the competition. Player A is regarded as an excellent player if the following condition is satisfied: for any other player B, either A beats B or there exists another player C such that C beats B and A beats C. It is known that there is only one excellent player in the end, prove that this player beats all other players.

Day 2

4 Five points are arbitrarily put inside a given equilateral triangle ABC whose area is equal to 1. Show that we can draw three equilateral triangles within triangle ABC such that the following conditions are all satisfied:
i) the five points are covered by the three equilateral triangles;
ii) any side of the three equilateral triangles is parallel to a certain side of the triangle ABC;
iii) the sum of the areas of the three equilateral triangles is not larger than 0.64.

5 Let $A_1A_2A_3A_4$ be a tetrahedron. We construct four mutually tangent spheres S_1, S_2, S_3, S_4 with centers A_1, A_2, A_3, A_4 respectively. Suppose that there exists a point Q such that we can con-

struct two spheres centered at Q satisfying the following conditions:
i) One sphere with radius r is tangent to S_1, S_2, S_3, S_4;
ii) One sphere with radius R is tangent to every edges of tetrahedron $A_1A_2A_3A_4$.
Prove that $A_1A_2A_3A_4$ is a regular tetrahedron.

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6 Sum of m pairwise different positive even numbers and n pairwise different positive odd numbers is equal to 1987. Find, with proof, the maximum value of $3m + 4n$.

China National Olympiad 1986

www.artofproblemsolving.com/community/c5209

by jred

Day 1

1 We are given n reals $a_1, a_2, \cdots, a_n$ such that the sum of any two of them is non-negative. Prove that the following statement and its converse are both true: if n non-negative reals $x_1, x_2, \cdots, x_n$ satisfy $x_1 + x_2 + \cdots + x_n = 1$, then the inequality $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$ holds.

2 In $\triangle ABC$, the length of altitude AD is 12, and the bisector AE of $\angle A$ is 13. Denote by m the length of median AF. Find the range of m when $\angle A$ is acute, orthogonal and obtuse respectively.

3 Let $Z_1, Z_2, \cdots, Z_n$ be complex numbers satisfying $|Z_1| + |Z_2| + \cdots + |Z_n| = 1$. Show that there exist some among the n complex numbers such that the modulus of the sum of these complex numbers is not less than $1/6$.

Day 2

4 Given a $\triangle ABC$ with its area equal to 1, suppose that the vertices of quadrilateral $P_1P_2P_3P_4$ all lie on the sides of $\triangle ABC$. Show that among the four triangles $\triangle P_1P_2P_3, \triangle P_1P_2P_4, \triangle P_1P_3P_4, \triangle P_2P_3P_4$ there is at least one whose area is not larger than $1/4$.

5 Given a sequence $1, 1, 2, 2, 3, 3, \ldots, 1986, 1986$, determine, with proof, if we can rearrange the sequence
so that for any integer $1 \leq k \leq 1986$ there are exactly k numbers between the two ks.

6 Suppose that each point on the plane is colored either white or black. Show that there exists an equilateral triangle with the side length equal to 1 or $\sqrt{3}$ whose three vertices are in the same color.