Andrei D. Polyanin
Valentin F. Zaitsev

HANDBOOK OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

SECOND EDITION

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HANDBOOK OF
NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

SECOND EDITION

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PREFACE TO THE NEW EDITION

The **Handbook of Nonlinear Partial Differential Equations**, a unique reference for scientists and engineers, contains over 3,000 nonlinear partial differential equations with solutions, as well as exact, symbolic, and numerical methods for solving nonlinear equations. First-, second-, third-, fourth-, and higher-order nonlinear equations and systems of equations are considered. Equations of parabolic, hyperbolic, elliptic, mixed, and general types are discussed. A large number of new exact solutions to nonlinear equations are described. In total, the handbook contains several times more nonlinear PDEs and exact solutions than any other book currently available.

In selecting the material, the authors gave the highest priority to the following five major types of equations:

- Equations that arise in various applications (heat and mass transfer theory, wave theory, nonlinear mechanics, hydrodynamics, gas dynamics, plasticity theory, nonlinear acoustics, combustion theory, nonlinear optics, theoretical physics, differential geometry, control theory, chemical engineering sciences, biology, and others).
- Equations of general form that depend on arbitrary functions; exact solutions of such equations are of principal value for testing numerical and approximate methods.
- Equations for which the general solution or solutions of quite general form, with arbitrary functions, could be obtained.
- Equations that involve many free parameters.
- Equations whose solutions are reduced to solving linear partial differential equations or linear integral equations.

The second edition has been substantially updated, revised, and expanded. More than 1,500 new equations with exact solutions, as well some methods and many examples, have been added. The new edition has been increased by a total of over 1,000 pages.

New to the second edition:

- First-order nonlinear partial differential equations with solutions.
- Some second-, third-, fourth-, and higher-order nonlinear equations with solutions.
- Parabolic, hyperbolic, elliptic, and other systems of equations with solutions.
- Some exact methods and transformations.
- Symbolic and numerical methods for solving nonlinear PDEs with Maple, Mathematica, and MATLAB.
- Many new examples and tables included for illustrative purposes.
- A large list of references consisting of over 1,300 sources.
- Extensive table of contents and detailed index.

Note that the first part of the book can be used as a database of test problems for numerical and approximate methods for solving nonlinear partial differential equations.

We would like to express our deep gratitude to Alexei Zhurov and Alexander Aksenov for fruitful discussions and valuable remarks. We are very thankful to Inna Shingareva and Carlos Lizárraga-Celaya who wrote three chapters (39–41) of the book at our request.

The authors hope that the handbook will prove helpful for a wide audience of researchers, university and college teachers, engineers, and students in various fields of applied mathematics, mechanics, physics, chemistry, biology, economics, and engineering sciences.

Andrei D. Polyanin
Valentin F. Zaitsev
September 2011

PREFACE TO THE FIRST EDITION

Nonlinear partial differential equations are encountered in various fields of mathematics, physics, chemistry, and biology, and numerous applications.

Exact (closed-form) solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions of nonlinear equations graphically demonstrate and allow unraveling the mechanisms of many complex nonlinear phenomena such as spatial localization of transfer processes, multiplicity or absence of steady states under various conditions, existence of peaking regimes, and many others. Furthermore, simple solutions are often used in teaching many courses as specific examples illustrating basic tenets of a theory that admit mathematical formulation.

Even those special exact solutions that do not have a clear physical meaning can be used as "test problems" to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving differential equations. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions.

This book contains more than 1,600 nonlinear mathematical physics equations and nonlinear partial differential equations and their solutions. A large number of new exact solutions to nonlinear equations are described. Equations of parabolic, hyperbolic, elliptic, mixed, and general types are discussed. Second-, third-, fourth-, and higher-order nonlinear equations are considered. The book presents exact solutions to equations of heat and mass transfer, wave theory, nonlinear mechanics, hydrodynamics, gas dynamics, plasticity theory, nonlinear acoustics, combustion theory, nonlinear optics, theoretical physics, differential geometry, control theory, chemical engineering sciences, biology, and other fields.

Special attention is paid to general-form equations that depend on arbitrary functions; exact solutions of such equations are of principal value for testing numerical and approximate methods. Almost all other equations contain one or more arbitrary parameters (in fact, this book deals with whole families of partial differential equations), which can be fixed by the reader at will. *In total, the handbook contains significantly more nonlinear PDEs and exact solutions than any other book currently available.*

The supplement of the book presents exact analytical methods for solving nonlinear mathematical physics equations. When selecting the material, the authors have given a pronounced preference to practical aspects of the matter; that is, to methods that allow effectively "constructing" exact solutions. Apart from the classical methods, the book also describes wide-range methods that have been greatly developed over the last decade (the nonclassical and direct methods for symmetry reductions, the differential constraints

method, the method of generalized separation of variables, and others). For the reader's better understanding of the methods, numerous examples of solving specific differential equations and systems of differential equations are given throughout the book.

For the convenience of a wide audience with different mathematical backgrounds, the authors tried to do their best, wherever possible, to avoid special terminology. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner, with necessary references made to books where these methods are considered in more detail. Many sections were written so that they could be read independently from each other. This allows the reader to quickly get to the heart of the matter.

The handbook consists of chapters, sections, and subsections. Equations and formulas are numbered separately in each subsection. The equations within subsections are arranged in increasing order of complexity. The extensive table of contents provides rapid access to the desired equations.

Separate parts of the book may be used by lecturers of universities and colleges for practical courses and lectures on nonlinear mathematical physics equations for graduate and postgraduate students. Furthermore, the books may be used as a database of test problems for numerical and approximate methods for solving nonlinear partial differential equations.

We would like to express our deep gratitude to Alexei Zhurov for fruitful discussions and valuable remarks.

The authors hope that this book will be helpful for a wide range of scientists, university teachers, engineers, and students engaged in the fields of mathematics, physics, mechanics, control, chemistry, and engineering sciences.

Andrei D. Polyanin
Valentin F. Zaitsev
September 2003

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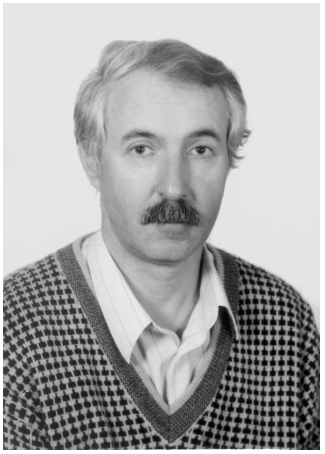
AUTHORS



Andrei D. Polyanin, D.Sc., Ph.D., is a well-known scientist of broad interests and is active in various areas of mathematics, mechanics, and chemical engineering sciences. He is one of the most prominent authors in the field of reference literature on mathematics.

Professor Polyanin graduated with honors from the Department of Mechanics and Mathematics at Moscow State University in 1974. He received his Ph.D. in 1981 and D.Sc. in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, Professor Polyanin has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences; he is also professor of mathematics at Bauman Moscow State Technical University. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and the Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation.

Professor Polyanin has made important contributions to the theory of differential and integral equations, mathematical physics, engineering mathematics, theory of heat and mass transfer, and chemical hydrodynamics. He has obtained exact solutions for several thousand ordinary differential, partial differential, and integral equations.

Professor Polyanin is the author of more than 30 books in English, Russian, German, and Bulgarian as well as over 150 research papers and three patents. He has written a number of fundamental handbooks, including: A. D. Polyanin and V. F. Zaitsev's *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, 1995 and 2003); A. D. Polyanin and A. V. Manzhirov's *Handbook of Integral Equations* (CRC Press, 1998 and 2008); A. D. Polyanin's *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, 2002); A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux's *Handbook of First Order Partial Differential Equations* (Taylor & Francis, 2002); A. D. Polyanin and V. F. Zaitsev's *Handbook of Nonlinear Partial Differential Equations* (Chapman & Hall/CRC Press, 2004); A. D. Polyanin and A. V. Manzhirov's *Handbook of Mathematics for Engineers and Scientists* (Chapman & Hall/CRC Press, 2007), and A. D. Polyanin and A. I. Chernoutsan's (Eds.) *A Concise Handbook of Mathematics, Physics, and Engineering Sciences* (Chapman & Hall/CRC Press, 2010).

Professor Polyanin is editor-in-chief of the international scientific educational website *EqWorld — The World of Mathematical Equations* (http://eqworld.ipmnet.ru), which is visited by several thousand users a day worldwide. He is also editor of the book series *Differential and Integral Equations and Their Applications* (Chapman & Hall/CRC Press, London/Boca Raton). Professor Polyanin is a member of the editorial board of the journal *Theoretical Foundations of Chemical Engineering.*

In 1991, Professor Polyanin was awarded a Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

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Valentin F. Zaitsev, Ph.D., D.Sc., is a noted scientist in the fields of ordinary differential equations, mathematical physics, and nonlinear mechanics.

Dr. Zaitsev graduated from the Radio Electronics Faculty of the Leningrad Polytechnical Institute (now St. Petersburg Technical University) in 1969 and received his Ph.D. degree in 1983 at the Leningrad State University. His Ph.D. thesis was devoted to the group approach of the study of some classes of ordinary differential equations. In 1992, Professor Zaitsev received his D.Sc. and his thesis was dedicated to the discrete group analysis of ordinary differential equations.

From 1971–1996, Dr. Zaitsev worked in the Research Institute for Computational Mathematics and Control Processes of the St. Petersburg State University. Since 1996, Professor Zaitsev has been a member of the staff of the Russian State Pedagogical University (St. Petersburg).

Professor Zaitsev has made important contributions to new methods in the theory of ordinary and partial differential equations. He is the author of more than 150 scientific publications, including 23 books, and 1 patent.

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SOME NOTATIONS AND REMARKS

Latin Characters

$C_1, C_2, \ldots$ are arbitrary constants;

$\operatorname{curl}\mathbf{u}$ curl of a vector $\mathbf{u}$, sometimes also denoted $\operatorname{rot}\mathbf{u}$;

$\operatorname{div}\mathbf{u}$ divergence of a vector $\mathbf{u}$; $\operatorname{div}\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$ in two-dimensional case $\mathbf{u} = (u_1, u_2)$;

$\operatorname{grad} a$ gradient of a scalar a, also denoted by ∇a, with ∇ being the nabla vector differential operator;

$r,\ \varphi,\ z$ cylindrical coordinates, $r = \sqrt{x^2 + y^2}$ and $x = r\cos\varphi$, $y = r\sin\varphi$;

$r,\ \theta,\ \varphi$ spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ and $x = r\sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = r\cos\theta$;

t time ($t \geq 0$);

w unknown function (dependent variable);

x, y, z space (Cartesian) coordinates;

$x_1, \ldots, x_n$ Cartesian coordinates in n-dimensional space.

Greek Characters

Δ Laplace operator:

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, in two-dimensional case;

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, in three-dimensional case;

$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$, in n-dimensional case;

$\Delta\Delta$ biharmonic operator; in two-dimensional case, $\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

Brief Notation for Derivatives

Partial derivatives:

$$w_x = \frac{\partial w}{\partial x}, \quad w_t = \frac{\partial w}{\partial t}, \quad w_{xx} = \frac{\partial^2 w}{\partial x^2}, \quad w_{xt} = \frac{\partial^2 w}{\partial x \partial t}, \quad w_{tt} = \frac{\partial^2 w}{\partial t^2}, \quad w_{xxx} = \frac{\partial^3 w}{\partial x^3}, \quad \ldots$$

Ordinary derivatives for $f = f(x)$:

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f''''_{xxxx} = \frac{d^4 f}{dx^4}, \quad f_x^{(n)} = \frac{d^n f}{dx^n} \quad \text{with} \quad n \geq 5.$$

Brief Notation for Differential Operators

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_t = \frac{\partial}{\partial t}, \quad \partial_w = \frac{\partial}{\partial w} \quad \text{(differential operators in } x, y, t, \text{ and } w\text{)};$$

$$\mathrm{D}_x = \frac{\partial}{\partial x} + w_x \frac{\partial}{\partial w} + w_{xx} \frac{\partial}{\partial w_x} + w_{xt} \frac{\partial}{\partial w_t} + \cdots \quad \text{(total differential operator in } x\text{)};$$

$$\mathrm{D}_t = \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_{xt} \frac{\partial}{\partial w_x} + w_{tt} \frac{\partial}{\partial w_t} + \cdots \quad \text{(total differential operator in } t\text{)}.$$

In the last two relations, w is assumed to be dependent on x and t, $w = w(x, t)$.

Remarks

1. The book presents solutions of the following types:

(a) expressible in terms of elementary functions explicitly, implicitly, or parametrically;
(b) expressible in terms of elementary functions and integrals of elementary functions;
(c) expressible in terms of elementary functions, functions involved in the equation (if the equation contains arbitrary functions), and integrals of the equation functions and/or other elementary functions;
(d) expressible in terms of ordinary differential equations or finite systems of ordinary differential equations;
(e) expressible in terms of solutions to linear partial differential equations or linear integral equations.

2. The book also deals with solutions described by equations with fewer new variables than those in the original equations. An expression that solves an equations in three independent variables and is determined by an equation in two independent variables will be called a *two-dimensional solution*.

3. As a rule, the book does not present simple solutions that depend on only one of the variables involved in the original equation.

4. Equations are numbered separately within each subsection. In Part I, when referencing a particular equation, we use a notation like 3.2.1.5, which implies equation 5 from Subsection 3.2.1. In Parts II and III, equation numbers are enclosed in parentheses; for example, (27.2.1.4) means equation 4 from Subsection 27.2.1.

5. If a formula or a solution contains an expression like $\dfrac{f(x)}{a-2}$, it is often not stated that the assumption $a \neq 2$ is implied.

6. If a formula or a solution contains derivatives of some functions, then the functions are assumed to be differentiable.

7. If a formula or a solution contains definite integrals, then the integrals are supposed to be convergent.

8. Though incomplete, very simple and graphical classification of solutions by their appearance is used in the book. For equations in two independent variables, x and t, and one unknown, w, the solution name and structure are as follows (x and t in the solutions below can be swapped):

No.	*Solution name*	*Solution structure*
1	Traveling-wave solution*	$w = F(z), \quad z = \alpha x + \beta t, \quad \alpha\beta \neq 0$
2	Additive separable solution	$w = \varphi(x) + \psi(t)$
3	Multiplicative separable solution	$w = \varphi(x)\psi(t)$
4	Self-similar solution**	$w = t^{\alpha} F(z), \quad z = xt^{\beta}$
5	Generalized self-similar solution	$w = \varphi(t)F(z), \quad z = x\psi(t)$
6	Generalized traveling-wave solution	$w = F(z), \quad z = \varphi(t)x + \psi(t)$
7	Generalized separable solution	$w = \varphi_1(x)\psi_1(t) + \cdots + \varphi_n(x)\psi_n(t)$
8	Functional separable solution	$w = F(z), \quad z = \varphi_1(x)\psi_1(t) + \cdots + \varphi_n(x)\psi_n(t)$

*For uniformity of presentation, we also use this term in the cases where the variable t plays the role of a spatial coordinate.

**Sometimes, a solution of the form $w = \bar{t}^{\alpha} F(z)$ with $z = \bar{x}\bar{t}^{\beta}$, where $\bar{x} = x + C_1$ and $\bar{t} = t + C_2$, will also be called a self-similar solution.

9. In this book, the term *integrable equation* applies to nonlinear partial differential equations that admit solution in terms of integrals or can be *linearized*, which means that their solutions can be expressed in terms of solutions to linear differential equations or linear integral equations.

10. The previous handbooks by Zaitsev and Polyanin (1996, 2003), Polyanin and Zaitsev (2002, 2004), and Polyanin, Zaitsev, and Moussiaux (2002) were extensively used in compiling this book; references to these sources are often omitted.

11. The ODE and PDE are conventional abbreviations for ordinary differential equation and partial differential equation, respectively.

⊙ This symbol indicates references to literature sources whenever:

(*a*) at least one of the solutions was obtained in the cited source (even though the solution contained "correctable" misprints in signs or coefficients);
(*b*) the cited source provides further information on the equation in question and their solutions.

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Part I

Exact Solutions of Nonlinear Partial Differential Equations

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Chapter 1
First-Order Quasilinear Equations

Preliminary remarks. The equations in each subsection are arranged in order of complexity of their form. The functions $\Phi(u_1, \ldots, u_n)$ appearing below in general solutions are arbitrary functions of their arguments u_k, and the u_k are some functions the independent and dependent variables.

1.1. Equations with Two Independent Variables Containing Arbitrary Parameters

1.1.1. Coefficients of Equations Contain Power-Law Functions

► **Equations of the form $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w)$.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw^n + sw^m.}$

General solution: $a\displaystyle\int \frac{dw}{cw^n + sw^m} = x + \Phi(bx - ay).$

2. $\boldsymbol{\dfrac{\partial w}{\partial x} + k\dfrac{\partial w}{\partial y} = (ax + by + cw)^n + s.}$

The substitution $v(x,y) = ax + by + cw(x,y)$ leads to an equation of the form 1.1.1.1 with $m = 0$:

$$\frac{\partial v}{\partial x} + k\frac{\partial v}{\partial y} = cv^n + a + bk + cs.$$

3. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = bx^n w + cx^m w^k.}$

General solution:

$$w = \left[F(x)\Phi(y - ax) + c(1-k)F(x)\int \frac{x^m\,dx}{F(x)}\right]^{\frac{1}{1-k}}, \qquad F(x) = \exp\left[\frac{b(1-k)}{n+1}x^{n+1}\right].$$

4. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + (\beta x^n + \lambda y^m)w^k.}$

This is a special case of equation 11.2.1.3 with $f(x) = \beta x^n$ and $g(y) = \lambda y^m$.

5. $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = bx^n y^m w^k.}$

This is a special case of equation 1.2.1.8 with $f(x) = bx^n$, $g(y) = y^m$, and $h(w) = w^k$.

6. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=cw^k}$.

General solution:

$$\frac{c}{a}\ln|x|+\Phi(bx-ay)=\begin{cases}\frac{1}{1-k}w^{1-k} & \text{if } k\neq 1,\\ \ln|w| & \text{if } k=1.\end{cases}$$

7. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cw^k}$.

General solution:

$$\frac{c}{a}\ln|x|+\Phi\big(|x|^b|y|^{-a}\big)=\begin{cases}\frac{1}{1-k}w^{1-k} & \text{if } k\neq 1,\\ \ln|w| & \text{if } k=1.\end{cases}$$

8. $\boldsymbol{ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=cw^k}$.

General solution for $ab>0$:

$$\frac{c}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x+ay\big|+\Phi\big(ay^2-bx^2\big)=\begin{cases}\frac{1}{1-k}w^{1-k} & \text{if } k\neq 1,\\ \ln|w| & \text{if } k=1.\end{cases}$$

9. $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=w-a\sqrt{w^2-x^2-y^2}}$.

General solution: $w+\sqrt{w^2-x^2-y^2}=\dfrac{1}{x^{a-1}}\Phi\Big(\dfrac{y}{x}\Big)$.

10. $\boldsymbol{ax^2\dfrac{\partial w}{\partial x}+by^2\dfrac{\partial w}{\partial y}=cw^2}$.

General solution: $w=\dfrac{1}{c}\left[\dfrac{1}{ax}+\Phi\left(\dfrac{1}{by}-\dfrac{1}{ax}\right)\right]^{-1}$.

11. $\boldsymbol{ax^2\dfrac{\partial w}{\partial x}-bx^2\dfrac{\partial w}{\partial y}=(w-bx-ay)^2}$.

General solution: $w=\dfrac{[(a+b)x+ay]\Phi(bx+ay)-ax(bx+ay)}{\Phi(bx+ay)-ax}$.

12. $\boldsymbol{(x^2\pm a^2)\dfrac{\partial w}{\partial x}+(y^2\pm a^2)\dfrac{\partial w}{\partial y}=bx^ny^m\left[\dfrac{n(x^2\pm a^2)}{x}+\dfrac{m(y^2\pm a^2)}{y}\right]w^2}$.

General solution:

$$w=\frac{1}{\Phi(v)-bx^ny^m},\quad \text{where}\quad v=\begin{cases}\dfrac{a(x-y)}{a^2+xy} & \text{for the upper sign,}\\ \dfrac{(x-a)(y+a)}{(x+a)(y-a)} & \text{for the lower sign.}\end{cases}$$

13. $(xy + a^2)\left(x\dfrac{\partial w}{\partial x} - y\dfrac{\partial w}{\partial y}\right) = a(x^2 + y^2)w^2.$

General solution: $w = \left[\dfrac{a(y^2 - x^2)}{2(xy + a^2)} + \Phi(xy)\right]^{-1}$.

14. $ax^n\dfrac{\partial w}{\partial x} + by^k\dfrac{\partial w}{\partial y} = cw^m + s.$

This is a special case of equation 1.2.1.11 with $f(w) = cw^m + s$.

15. $ay^n\dfrac{\partial w}{\partial x} + bx^k\dfrac{\partial w}{\partial y} = cw^m + s.$

This is a special case of equation 1.2.1.12 with $f(w) = cw^m + s$.

16. $y^{2a}\left(ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y}\right) = c(y^a w - x^b)^2.$

General solution: $\Phi\left(\dfrac{x^b}{y^a},\ y^{c/b}\exp\left(\dfrac{y^a}{y^a w - x^b}\right)\right) = 0.$

17. $b_1x^{n_1}\dfrac{\partial w}{\partial x} + b_2y^{n_2}\dfrac{\partial w}{\partial y} = aw + (c_1x^{m_1} + c_2y^{m_2})w^k.$

This is a special case of equation 1.2.1.17 with $f_1(x) = b_1x^{n_1}$, $f_2(y) = b_2y^{n_2}$, $g_1(x) = c_1x^{m_1}$, and $g_2(y) = c_2y^{m_2}$.

► **Equations of the form** $F(x, y, w)\frac{\partial w}{\partial x} + G(x, y, w)\frac{\partial w}{\partial y} = H(x, y, w).$

18. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = 0.$

Hopf's equation. It is used as a model equation of the nonlinear wave theory and gas dynamics, where the independent variables x and y play the role of time and the spatial coordinate, respectively.

1°. General solution:

$$\Phi(axw - y, w) = 0 \qquad \text{or} \qquad y = axw + \overline{\Phi}(w),$$

where Φ and $\overline{\Phi}$ are arbitrary functions.

2°. The solution of the Cauchy problem with the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0$$

can be represented in parametric form as

$$y = \xi + a\varphi(\xi)x, \qquad w = \varphi(\xi).$$

For $a > 0$ and $\varphi'(\xi) > 0$, these relations describe the classical single-valued solution.

3°. Consider the Cauchy problem with the discontinuous initial condition

$$w(0,y)=\begin{cases} w_1 & \text{for } y<0,\\ w_2 & \text{for } y>0.\end{cases}$$

It is assumed that $a>0$, $w_1>0$, and $w_2>0$.

For $w_1<w_2$, the generalized solution has the form

$$w(x,y)=\begin{cases} w_1 & \text{for } y/x<V_1,\\ y/(ax) & \text{for } V_1\le y/x\le V_2,\\ w_2 & \text{for } y/x>V_2,\end{cases}\qquad \text{where}\quad V_1=aw_1,\ \ V_2=aw_2.$$

This solution is continuous in the half-plane $x>0$ and describes a "rarefaction wave."

For $w_1>w_2$, the generalized solution has the form

$$w(x,y)=\begin{cases} w_1 & \text{for } y/x<V,\\ w_2 & \text{for } y/x>V,\end{cases}\qquad \text{where}\quad V=\tfrac{1}{2}a(w_1+w_2).$$

This solution experiences a discontinuity along the line $y=Vx$ and describes a "shock wave."

4°. In Subsection 23.3, examples are given which illustrate qualitative features of solutions to Hopf's equations, including the wave breaking effect and shock waves. These subsections also present general formulas that permit one to construct discontinuous solutions for arbitrary initial conditions. The references listed below contain a lot of solutions to Cauchy problems which describe the merging and decay of discontinuities, periodic waves, and other nonlinear physical effects.

⊙ *References for equation* 1.1.1.18: E. Hopf (1950), P. D. Lax (1954), O. A. Oleinik (1954, 1957, 1959), R. Courant (1962), G. B. Whitham (1974), J. Lighthill (1978), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. J. LeVeque (1992), A. G. Kulikovskii and E. I. Sveshnikova (1995).

19. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}=b.$

General solution: $\Phi\big(w-bx,\ aw^2-2by\big)=0.$

20. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}=bx.$

General solution: $y=axw-\frac{1}{3}abx^3+\Phi\big(w-\frac{1}{2}bx^2\big).$

21. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}=by.$

General solution: $x=\displaystyle\int_{y_0}^{y}\frac{dt}{\sqrt{ab(t^2-y^2)+a^2w^2}}+\Phi\big(aw^2-by^2\big)$, where y_0 may be taken arbitrary.

22. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}+bw=0.$

This is a *model equation of nonlinear waves with damping* ($a,\ b>0$).

1°. General solution: $\Phi\big(aw + by,\ we^{bx}\big) = 0.$

2°. The solution of the Cauchy problem with the initial condition $w(0, y) = f(y)$ can be represented in parametric form as

$$y = \xi + \frac{a}{b}\big(1 - e^{-bx}\big) f(\xi), \qquad w = e^{-bx} f(\xi).$$

⊙ *Reference*: G. B. Whitham (1974).

23. $\dfrac{\partial w}{\partial x} + (aw + bx)\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = axw + \frac{1}{2}bx^2 + \Phi(w).$

24. $\dfrac{\partial w}{\partial x} + (aw + by)\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \dfrac{1}{b}\ln|aw + by| + \Phi(w).$

25. $\dfrac{\partial w}{\partial x} + (w + ax + by + c)\dfrac{\partial w}{\partial y} = 0.$

General solution: $bx = \ln\big|a + b(w + ax + by + c)\big| + \Phi(w).$

26. $\dfrac{\partial w}{\partial x} + (aw + bx^k + c)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$y = \begin{cases} axw + \dfrac{b}{k+1}x^{k+1} + cx + \Phi(w) & \text{if } k \neq -1, \\ axw + b\ln|x| + cx + \Phi(w) & \text{if } k = -1. \end{cases}$$

27. $\dfrac{\partial w}{\partial x} + axw\dfrac{\partial w}{\partial y} = bx.$

General solution: $\Phi\big(aw^2 - 2by,\ 2w - bx^2\big) = 0.$

28. $(a - w)\dfrac{\partial w}{\partial x} + (a + w)\dfrac{\partial w}{\partial y} = 0.$

General solution: $\Phi\big(w,\ x(w + a) + y(w - a)\big) = 0.$

29. $(a_1 + a_2 w)\dfrac{\partial w}{\partial x} + (b_1 + b_2 w)\dfrac{\partial w}{\partial y} = c.$

1°. General solution for $c = 0$:

$$\Phi\big(w,\ x(b_1 + b_2 w) - y(a_1 + a_2 w)\big) = 0.$$

2°. General solution for $c \neq 0$:

$$\Phi\big(a_2 w^2 + 2a_1 w - 2cx,\ b_2 w^2 + 2b_1 w - 2cy\big) = 0.$$

30. $\boldsymbol{(w-ay)\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cw.}$

General solution: $c(bx+ay)-bw=\Phi\left(w^b y^{-c}\right)$.

31. $\boldsymbol{[m(w-c)-n(y-b)]\dfrac{\partial w}{\partial x}+[n(x-a)-l(w-c)]\dfrac{\partial w}{\partial y}=l(y-b)-m(x-a).}$

This is an equation of a helical surface and the surface of bodies of revolution.
General solution: $\Phi\left((x-a)^2+(y-b)^2+(w-c)^2,\ lx+my+nw\right)=0$.

32. $\boldsymbol{[b(x+y)-c(x+w)]\dfrac{\partial w}{\partial x}+[c(y+w)-a(x+y)]\dfrac{\partial w}{\partial y}=a(x+w)-b(y+w).}$

General solution: $\Phi(ax+by+cw,\ xy+yw+xw)=0$.

33. $\boldsymbol{(y^2-aw)\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cw.}$

General solution: $\Phi\left(w^b y^{-c},\ aw+cx-\dfrac{c}{2b}y^2\right)=0$.

34. $\boldsymbol{[x(y+a)-w]\dfrac{\partial w}{\partial x}+[1-(y+a)^2]\dfrac{\partial w}{\partial y}=(y+a)w-x.}$

General solution: $\Phi\left(x+(y+a)w,\ x(y+a)+w\right)=0$.

35. $\boldsymbol{\dfrac{\partial w}{\partial x}+(aw^2+bw+c)\dfrac{\partial w}{\partial y}=s.}$

General solution: $\Phi\left(w-sx,\ \frac{1}{3}aw^3+\frac{1}{2}bw^2+cw-sy\right)=0$.

36. $\boldsymbol{\dfrac{\partial w}{\partial x}+(aw^2+bw+cx+s)\dfrac{\partial w}{\partial y}=0.}$

General solution: $y=x(aw^2+bw+s)+\frac{1}{2}cx^2+\Phi(w)$.

37. $\boldsymbol{\dfrac{\partial w}{\partial x}+(aw^2+bw+cy+s)\dfrac{\partial w}{\partial y}=0.}$

General solution: $x=\dfrac{1}{c}\ln|aw^2+bw+cy+s|+\Phi(w)$.

38. $\boldsymbol{\dfrac{\partial w}{\partial x}+(a_2w^2+a_1w+b_2x^2+b_1x+b_0)\dfrac{\partial w}{\partial y}=0.}$

General solution: $y=x(a_2w^2+a_1w)+\frac{1}{3}b_2x^3+\frac{1}{2}b_1x^2+b_0x$.

39. $(a+bx^2)\dfrac{\partial w}{\partial x}+(\alpha+\beta w^2)\dfrac{\partial w}{\partial y}=\gamma.$

1°. General solution for $ab>0$:

$$\Phi\Big(\sqrt{ab}\,w-\gamma\arctan\big(\sqrt{b/a}\,x\big),\ \alpha w+\tfrac{1}{3}\beta w^3-\gamma y\Big)=0.$$

2°. General solution for $ab<0$:

$$\Phi\left(2\sqrt{-ab}\,w-\gamma\ln\frac{bx-\sqrt{-ab}}{bx+\sqrt{-ab}},\ \alpha w+\tfrac{1}{3}\beta w^3-\gamma y\right)=0.$$

40. $x(w+ax)\dfrac{\partial w}{\partial x}+y(w+ay)\dfrac{\partial w}{\partial y}=w^2-a^2xy.$

General solution: $\Phi\left(\dfrac{w}{x}+a\ln|y|,\ \dfrac{w}{y}+a\ln|x|\right)=0.$

41. $ax^2(by-cw)\dfrac{\partial w}{\partial x}+by^2(cw-ax)\dfrac{\partial w}{\partial y}=cw^2(ax-by).$

General solution: $\Phi(xyw,\ abxy+bcyw+acxw)=0.$

42. $b(by+cw)^2\dfrac{\partial w}{\partial x}-a^2x(by+2cw)\dfrac{\partial w}{\partial y}=a^2xw.$

General solution: $\Phi\big(w^b(by+y+2cw),\ a^2x^2+b^2y^2-bc^2w^2\big)=0.$

43. $(ax^2-by^2-cw^2)\dfrac{\partial w}{\partial x}+2axy\dfrac{\partial w}{\partial y}=2axw.$

General solution: $\Phi\left(\dfrac{ax^2+by^2+cw^2}{y},\ \dfrac{w}{y}\right)=0.$

44. $by(3ax^2+by^2+cw^2)\dfrac{\partial w}{\partial x}-2ax(ax^2+cw^2)\dfrac{\partial w}{\partial y}=2abxyw.$

General solution: $\Phi\left(\dfrac{ax^2+by^2+cw^2}{w},\ \dfrac{2ax^2+by^2}{w^2}\right)=0.$

45. $b(a^2xy-abyw-b^2w^2)\dfrac{\partial w}{\partial x}+ab(bxw-axy-ay^2)\dfrac{\partial w}{\partial y}$
$=a^2(axy+bxw+byw+ay^2-ax^2).$

General solution: $\Phi(ax^2+ay^2+2byw,\ a^2x^2+b^2w^2+2a^2xy)=0.$

46. $xy(a^2xy+2b^2w^2)\dfrac{\partial w}{\partial x}+byw(byw-ax^2)\dfrac{\partial w}{\partial y}=bw^2(byw-ax^2).$

General solution: $\Phi\left(\dfrac{w}{y},\ \dfrac{b^2w^2}{ax}+\dfrac{bxw}{y}+ay\right)=0.$

47. $\dfrac{\partial w}{\partial x} + (aw^k + b)\dfrac{\partial w}{\partial y} = c.$

General solution:

$$\Phi\big(w,\ y - aw^k x - bx\big) = 0 \quad \text{if} \quad c = 0,$$

$$\Phi\left(w - cx,\ \frac{aw^{k+1}}{k+1} + bw - cy\right) = 0 \quad \text{if} \quad c \neq 0.$$

48. $\dfrac{\partial w}{\partial x} + (aw^k + bx)\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = axw^k + \frac{1}{2}bx^2 + \Phi(w)$.

49. $\dfrac{\partial w}{\partial x} + (aw^k + by)\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \dfrac{1}{b}\ln|aw^k + by| + \Phi(w)$.

50. $\dfrac{\partial w}{\partial x} + (aw^k + bx^m)\dfrac{\partial w}{\partial y} = cx^n.$

This is a special case of equation 1.2.1.45 with $f(w) = aw^k$, $g(x) = bx^m$, and $h(x) = cx^n$.

51. $\dfrac{\partial w}{\partial x} + (aw^k + bx^m)\dfrac{\partial w}{\partial y} = cw^n.$

This is a special case of equation 1.2.1.46 with $f(w) = aw^k$, $g(x) = bx^m$, and $h(w) = cw^n$.

52. $\dfrac{\partial w}{\partial x} + ax^n y^m w^k \dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\frac{1}{1-m}y^{1-m} - \frac{a}{n+1}x^{n+1}w^k = \Phi(w) \quad \text{if} \quad n \neq -1,\ m \neq 1;$$

$$\ln|y| - \frac{a}{n+1}x^{n+1}w^k = \Phi(w) \quad \text{if} \quad n \neq -1,\ m = 1;$$

$$\frac{1}{1-m}y^{1-m} - aw^k\ln|x| = \Phi(w) \quad \text{if} \quad n = -1,\ m \neq 1;$$

$$\ln|y| - aw^k\ln|x| = \Phi(w) \quad \text{if} \quad n = -1,\ m = 1.$$

53. $(ax + by + cw)^n\left(\dfrac{\partial w}{\partial x} + k\dfrac{\partial w}{\partial y}\right) = 1.$

General solution: $\displaystyle\int \frac{du}{cu^{-n} + a + bk} = x + \Phi(y - kx), \quad \text{where} \quad u = ax + by + cw.$

54. $\Big(a + \sqrt{w - \alpha x - \beta y}\,\Big)\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \alpha a + \beta b.$

General solution:

$$\Phi\big(cy - bw,\ \alpha y + 2b\sqrt{w - \alpha x - \beta y}\,\big) = 0, \quad \text{where} \quad c = \alpha a + \beta b.$$

▷ *See also equations of the general form containing arbitrary functions in Section 1.2.*

1.1.2. Coefficients of Equations Contain Exponential Functions

► **Equations of the form** $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w)$.

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = e^{cw}$.

General solution: $w = -\dfrac{1}{c}\ln\left[\Phi(bx-ay) - \dfrac{c}{a}x\right]$.

2. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = b + ce^{\beta x+\lambda w}$.

General solution: $w = -\dfrac{1}{\lambda}\ln\left[e^{-b\lambda x}\Phi(y-ax) - \dfrac{c\lambda}{\beta+b\lambda}e^{\beta x}\right]$.

3. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = be^{\beta x} + ce^{\gamma x+\lambda w}$.

This is a special case of equation 1.2.1.2 with $f(x) = be^{\beta x}$ and $g(x) = ce^{\gamma x}$.

4. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c + (ke^{\beta x} + se^{\gamma y})e^{\lambda w}$.

This is a special case of equation 1.2.1.4 with $f(x) = ke^{\beta x}$ and $g(y) = se^{\gamma y}$.

5. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = e^{c(w+\lambda x+\beta y)}$.

General solution:

$$w = \begin{cases} -\dfrac{1}{c}\ln\left[\Phi(bx-ay) - \dfrac{1}{a\lambda+b\beta}e^{c(\lambda x+\beta y)}\right] & \text{if } \lambda \neq -b\beta/a, \\ \dfrac{\beta}{a}(bx-ay) - \dfrac{1}{c}\ln\left[\Phi(bx-ay) - \dfrac{c}{a}x\right] & \text{if } \lambda = -b\beta/a. \end{cases}$$

6. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = e^{cw}$.

General solution: $w = -\dfrac{1}{c}\ln\left[\Phi\left(|x|^b|y|^{-a}\right) - \dfrac{c}{a}\ln|x|\right]$.

7. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = e^{c(w+\lambda x+\beta y)}$.

General solution: $w = -\dfrac{1}{c}\ln\left[\Phi(u) - \dfrac{c}{a}\displaystyle\int \frac{\exp(c\lambda x + c\beta x^{b/a}u^{-1/a})}{x}dx\right]$, where $u = x^b y^{-a}$.
In the integration, u is treated as a constant.

8. $a y \dfrac{\partial w}{\partial x} + b x \dfrac{\partial w}{\partial y} = e^{c(w+\lambda x+\beta y)}.$

General solution:

$$w = -\frac{1}{c}\ln\left[\Phi\left(bx^2 - ay^2\right) - \frac{c}{a^{3/2}}\int \exp\left(c\lambda x + \frac{c\beta}{\sqrt{a}}\sqrt{bx^2 - u}\right)\frac{dx}{\sqrt{bx^2 - u}}\right],$$

where $u = bx^2 - ay^2$. In the integration, u is treated as a constant.

9. $a x^n \dfrac{\partial w}{\partial x} + b y^k \dfrac{\partial w}{\partial y} = c e^{\lambda w} + s.$

This is a special case of equation 1.2.1.11 with $f(w) = ce^{\lambda w} + s$.

10. $a y^n \dfrac{\partial w}{\partial x} + b x^k \dfrac{\partial w}{\partial y} = c e^{\lambda w} + s.$

This is a special case of equation 1.2.1.12 with $f(w) = ce^{\lambda w} + s$.

11. $a e^{\lambda x} \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = e^{\gamma(w+\beta y)}.$

General solution: $w = -\dfrac{1}{\gamma}\ln\left[\Phi\left(a\lambda y + be^{-\lambda x}\right) - \dfrac{1}{b\beta}e^{\gamma\beta y}\right].$

12. $a e^{\lambda x} \dfrac{\partial w}{\partial x} + b e^{\beta y} \dfrac{\partial w}{\partial y} = e^{\gamma w}.$

General solution: $w = -\dfrac{1}{\gamma}\ln\left[\Phi\left(b\beta e^{-\lambda x} - a\lambda e^{-\beta y}\right) + \dfrac{\gamma}{a\lambda}e^{-\lambda x}\right].$

13. $a e^{\lambda y} \dfrac{\partial w}{\partial x} + b e^{\beta y} \dfrac{\partial w}{\partial y} = c e^{\gamma w} + k.$

This is a special case of equation 1.2.1.14 with $f(w) = ce^{\gamma w} + k$.

14. $a e^{\alpha x} \dfrac{\partial w}{\partial x} + b e^{\beta y} \dfrac{\partial w}{\partial y} = e^{c(w+\lambda x+\mu y)}.$

General solution: $w = -\dfrac{1}{c}\ln\left[\Phi(u) - \dfrac{(a\alpha)^{c\mu/\beta}c}{a}\int e^{(c\lambda-\alpha)x}\left(b\beta e^{-\alpha x} - u\right)^{-c\mu/\beta}dx\right]$, where $u = b\beta e^{-\alpha x} - a\alpha e^{-\beta y}$. In the integration, u is treated as a constant.

15. $a e^{\beta y} \dfrac{\partial w}{\partial x} + b e^{\alpha x} \dfrac{\partial w}{\partial y} = e^{c(w+\lambda x+\mu y)}.$

General solution:

$$e^{-cw} = \begin{cases} \Phi(u) - \dfrac{c\alpha}{(a\alpha)^{c\mu/\beta}}\displaystyle\int e^{c\lambda x}\left(b\beta e^{\alpha x} - u\right)^{(c\mu-\beta)/\beta}dx & \text{if } \beta \neq c\mu, \\ \Phi(u) - \dfrac{1}{a\lambda}e^{c\lambda x} & \text{if } \beta = c\mu, \end{cases}$$

where $u = b\beta e^{\alpha x} - a\alpha e^{\beta y}$. In the integration, u is treated as a constant.

16. $ae^{\alpha x+\beta y}\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=e^{cw}.$

General solution: $w=-\dfrac{1}{c}\ln\Big[\Phi\big(b\beta e^{-\alpha x}+a\alpha e^{\beta y}\big)-\dfrac{c}{b}y\Big].$

17. $b_1e^{\beta_1 x}\dfrac{\partial w}{\partial x}+b_2e^{\beta_2 y}\dfrac{\partial w}{\partial y}=a+(c_1e^{\gamma_1 x}+c_2e^{\gamma_2 y})e^{\lambda w}.$

This is a special case of equation 1.2.1.18 with $f_1(x)=b_1e^{\beta_1 x}$, $f_2(y)=b_2e^{\beta_2 y}$, $g_1(x)=c_1e^{\gamma_1 x}$, and $g_2(y)=c_2e^{\gamma_2 y}$.

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x}+G(x,y,w)\frac{\partial w}{\partial y}=H(x,y,w).$

18. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+b)\dfrac{\partial w}{\partial y}=0.$

General solution: $y=x(ae^{\lambda w}+b)+\Phi(w).$

19. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+bx)\dfrac{\partial w}{\partial y}=0.$

General solution: $y=axe^{\lambda w}+\frac{1}{2}bx^2+\Phi(w).$

20. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+by)\dfrac{\partial w}{\partial y}=0.$

General solution: $x=\dfrac{1}{b}\ln|ae^{\lambda w}+by|+\Phi(w).$

21. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+be^{\beta x}+c)\dfrac{\partial w}{\partial y}=0.$

General solution: $y=axe^{\lambda w}+\dfrac{b}{\beta}e^{\beta x}+cx+\Phi(w).$

22. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+bx^m)\dfrac{\partial w}{\partial y}=cx^n.$

This is a special case of equation 1.2.1.45 with $f(w)=ae^{\lambda w}$, $g(x)=bx^m$, and $h(x)=cx^n$.

23. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+bx^m)\dfrac{\partial w}{\partial y}=cw^n.$

This is a special case of equation 1.2.1.46 with $f(w)=ae^{\lambda w}$, $g(x)=bx^m$, and $h(w)=cw^n$.

24. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+be^{\beta x})\dfrac{\partial w}{\partial y}=ce^{\gamma x}.$

This is a special case of equation 1.2.1.45 with $f(w)=ae^{\lambda w}$, $g(x)=be^{\beta x}$, and $h(x)=ce^{\gamma x}$.

25. $\dfrac{\partial w}{\partial x}+(ae^{\lambda w}+be^{\beta x})\dfrac{\partial w}{\partial y}=ce^{\gamma w}.$

This is a special case of equation 1.2.1.46 with $f(w)=ae^{\lambda w}$, $g(x)=be^{\beta x}$, and $h(w)=ce^{\gamma w}$.

▷ *See also equations of the general form containing arbitrary functions in Section 1.2.*

1.1.3. Coefficients of Equations Contain Hyperbolic Functions

► **Equations of the form $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w)$.**

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cosh^k(\lambda w) + s.$

This is a special case of equation 1.2.1.5 with $f(w) = c\cosh^k(\lambda w) + s$.

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\sinh^k(\lambda w) + s.$

This is a special case of equation 1.2.1.5 with $f(w) = c\sinh^k(\lambda w) + s$.

3. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cosh(w + \alpha x + \beta y).$

This is a special case of equation 1.2.1.6 with $f(u) = c\cosh u$.

4. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n\cosh^k(\lambda w).$

This is a special case of equation 1.2.1.7 with $f(x) = cx^n$, $g(w) = \cosh^k(\lambda w)$.

5. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n\sinh^k(\lambda w).$

This is a special case of equation 1.2.1.7 with $f(x) = cx^n$, $g(w) = \sinh^k(\lambda w)$.

6. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = c\tanh^k(\lambda w) + s.$

This is a special case of equation 1.2.1.9 with $f(w) = c\tanh^k(\lambda w) + s$.

7. $ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = c\tanh^k(\lambda w) + s.$

This is a special case of equation 1.2.1.10 with $f(w) = c\tanh^k(\lambda w) + s$.

8. $a\cosh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh^m(\beta y)\dfrac{\partial w}{\partial y} = \cosh^k(\gamma w).$

This is a special case of equation 1.2.1.15 with $f(x) = a\cosh^n(\lambda x)$, $g(y) = b\cosh^m(\beta y)$, and $h(w) = \cosh^k(\gamma w)$.

► **Equations of the form $F(x,y,w)\frac{\partial w}{\partial x} + G(x,y,w)\frac{\partial w}{\partial y} = H(x,y,w)$.**

9. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = b\cosh(\lambda x).$

General solution:

$$y = axu + \frac{ab}{\lambda^2}\cosh(\lambda x) + \Phi(u), \quad \text{where} \quad u = w - \frac{b}{\lambda}\sinh(\lambda x).$$

10. $\dfrac{\partial w}{\partial x} + \Big[a\cosh^k(\lambda w) + b\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\Big[a\cosh^k(\lambda w) + b\Big] + \Phi(w).$

11. $\dfrac{\partial w}{\partial x} + \Big[a\cosh^k(\lambda w) + b\cosh^n(\beta x) + c\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\cosh^k(\lambda w) + b\displaystyle\int \cosh^n(\beta x)\,dx + cx + \Phi(w).$

12. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = b\sinh(\lambda x).$

General solution:

$$y = axu + \frac{ab}{\lambda^2}\sinh(\lambda x) + \Phi(u), \quad \text{where} \quad u = w - \frac{b}{\lambda}\cosh(\lambda x).$$

13. $\dfrac{\partial w}{\partial x} + \Big[a\sinh^k(\lambda w) + b\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\Big[a\sinh^k(\lambda w) + b\Big] + \Phi(w).$

14. $\dfrac{\partial w}{\partial x} + \Big[a\sinh^k(\lambda w) + b\sinh^n(\beta x) + c\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\sinh^k(\lambda w) + b\displaystyle\int \sinh^n(\beta x)\,dx + cx + \Phi(w).$

15. $\dfrac{\partial w}{\partial x} + \Big[a\tanh^k(\lambda w) + b\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\Big[a\tanh^k(\lambda w) + b\Big] + \Phi(w).$

16. $\dfrac{\partial w}{\partial x} + \Big[a\tanh^k(\lambda w) + b\tanh^n(\beta x) + c\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\tanh^k(\lambda w) + b\displaystyle\int \tanh^n(\beta x)\,dx + cx + \Phi(w).$

17. $\dfrac{\partial w}{\partial x} + \Big[a\coth^k(\lambda w) + b\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\Big[a\coth^k(\lambda w) + b\Big] + \Phi(w).$

18. $\dfrac{\partial w}{\partial x} + \Big[a\coth^k(\lambda w) + b\coth^n(\beta x) + c\Big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\coth^k(\lambda w) + b\displaystyle\int \coth^n(\beta x)\,dx + cx + \Phi(w).$

1.1.4. Coefficients of Equations Contain Logarithmic Functions

► **Equations of the form** $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w)$.

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln^k(\lambda w) + s.$

This is a special case of equation 1.2.1.5 with $f(w) = c\ln^k(\lambda w) + s$.

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(w + \alpha x + \beta y).$

This is a special case of equation 1.2.1.6 with $f(u) = c\ln u$.

3. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n\ln^k(\lambda w).$

This is a special case of equation 1.2.1.7 with $f(x) = cx^n$ and $g(w) = \ln^k(\lambda w)$.

4. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = c\ln^k(\lambda w) + s.$

This is a special case of equation 1.2.1.9 with $f(w) = c\ln^k(\lambda w) + s$.

5. $ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = c\ln^k(\lambda w) + s.$

This is a special case of equation 1.2.1.10 with $f(w) = c\ln^k(\lambda w) + s$.

6. $a\ln^n(\lambda x)\dfrac{\partial w}{\partial x} + b\ln^m(\beta y)\dfrac{\partial w}{\partial y} = c\ln^k(\gamma w).$

This is a special case of equation 1.2.1.15 with $f(x) = a\ln^n(\lambda x)$, $g(y) = b\ln^m(\beta y)$, and $h(w) = c\ln^k(\gamma w)$.

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x} + G(x,y,w)\frac{\partial w}{\partial y} = H(x,y,w)$.

7. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = b\ln(\lambda x).$

General solution:

$$y = axu + \tfrac{1}{2}abx^2\ln(\lambda x) - \tfrac{3}{4}abx^2 + \Phi(u), \quad \text{where} \quad u = w + bx - bx\ln(\lambda x).$$

8. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda w) + b\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\big[a\ln^k(\lambda w) + b\big] + \Phi(w)$.

9. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda w) + bx\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\ln^k(\lambda w) + \frac{1}{2}bx^2 + \Phi(w)$.

10. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda w) + by\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \dfrac{1}{b}\ln|a\ln^k(\lambda w) + by| + \Phi(w).$

11. $\dfrac{\partial w}{\partial x} + \big[a\ln^k(\lambda w) + b\ln^n(\beta x) + c\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\ln^k(\lambda w) + \displaystyle\int \ln^n(\beta x)\,dx + cx + \Phi(w).$

1.1.5. Coefficients of Equations Contain Trigonometric Functions

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x} + G(x,y,w)\frac{\partial w}{\partial y} = H(x,y,w).$

1. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos^k(\lambda w) + s.$

This is a special case of equation 1.2.1.5 with $f(w) = c\cos^k(\lambda w) + s$.

2. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\sin^k(\lambda w) + s.$

This is a special case of equation 1.2.1.5 with $f(w) = c\sin^k(\lambda w) + s$.

3. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos(w + \alpha x + \beta y).$

This is a special case of equation 1.2.1.6 with $f(u) = c\cos u$.

4. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n\cos^k(\lambda w).$

This is a special case of equation 1.2.1.7 with $f(x) = cx^n$ and $g(w) = \cos^k(\lambda w)$.

5. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n\sin^k(\lambda w).$

This is a special case of equation 1.2.1.7 with $f(x) = cx^n$ and $g(w) = \sin^k(\lambda w)$.

6. $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = c\tan^k(\lambda w) + s.$

This is a special case of equation 1.2.1.9 with $f(w) = c\tan^k(\lambda w) + s$.

7. $ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = c\tan^k(\lambda w) + s.$

This is a special case of equation 1.2.1.10 with $f(w) = c\tan^k(\lambda w) + s$.

8. $a\cos^n(\lambda x)\dfrac{\partial w}{\partial x}+b\cos^m(\beta y)\dfrac{\partial w}{\partial y}=\cos^k(\gamma w)+c.$

This is a special case of equation 1.2.1.15 with $f(x)=a\cos^n(\lambda x)$, $g(y)=b\cos^m(\beta y)$, and $h(w)=\cos^k(\gamma w)+c$.

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x}+G(x,y,w)\frac{\partial w}{\partial y}=H(x,y,w).$

9. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}=b\cos(\lambda x).$

General solution:

$$y=axu-\frac{ab}{\lambda^2}\cos(\lambda x)+\Phi(u),\quad \text{where}\quad u=w-\frac{b}{\lambda}\sin(\lambda x).$$

10. $\dfrac{\partial w}{\partial x}+\Big[a\cos^k(\lambda w)+b\Big]\dfrac{\partial w}{\partial y}=0.$

General solution: $y=x\Big[a\cos^k(\lambda w)+b\Big]+\Phi(w).$

11. $\dfrac{\partial w}{\partial x}+\Big[a\cos^k(\lambda w)+b\cos^n(\beta x)+c\Big]\dfrac{\partial w}{\partial y}=0.$

General solution: $y=ax\cos^k(\lambda w)+b\displaystyle\int\cos^n(\beta x)\,dx+cx+\Phi(w).$

12. $\dfrac{\partial w}{\partial x}+aw\dfrac{\partial w}{\partial y}=b\sin(\lambda x).$

General solution:

$$y=axu-\frac{ab}{\lambda^2}\sin(\lambda x)+\Phi(u),\quad \text{where}\quad u=w+\frac{b}{\lambda}\cos(\lambda x).$$

13. $\dfrac{\partial w}{\partial x}+\Big[a\sin^k(\lambda w)+b\Big]\dfrac{\partial w}{\partial y}=0.$

General solution: $y=x\Big[a\sin^k(\lambda w)+b\Big]+\Phi(w).$

14. $\dfrac{\partial w}{\partial x}+\Big[a\sin^k(\lambda w)+b\sin^n(\beta x)+c\Big]\dfrac{\partial w}{\partial y}=0.$

General solution: $y=ax\sin^k(\lambda w)+b\displaystyle\int\sin^n(\beta x)\,dx+cx+\Phi(w).$

15. $\dfrac{\partial w}{\partial x}+\Big[a\tan^k(\lambda w)+b\Big]\dfrac{\partial w}{\partial y}=0.$

General solution: $y=x\Big[a\tan^k(\lambda w)+b\Big]+\Phi(w).$

16. $\dfrac{\partial w}{\partial x} + \left[a\tan^k(\lambda w) + b\tan^n(\beta x) + c\right]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\tan^k(\lambda w) + b\displaystyle\int \tan^n(\beta x)\,dx + cx + \Phi(w).$

17. $\dfrac{\partial w}{\partial x} + \left[a\cot^k(\lambda w) + b\right]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = x\left[a\cot^k(\lambda w) + b\right] + \Phi(w).$

18. $\dfrac{\partial w}{\partial x} + \left[a\cot^k(\lambda w) + b\cot^n(\beta x) + c\right]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = ax\cot^k(\lambda w) + b\displaystyle\int \cot^n(\beta x)\,dx + cx + \Phi(w).$

1.2. Equations with Two Independent Variables Containing Arbitrary Functions

1.2.1. Equations Contain Arbitrary Functions of One Variable

► **Equations of the form** $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w).$

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)w + g(x)w^k.$

General solution:

$$w^{1-k} = F(x)\Phi(y-ax) + (1-k)F(x)\int \frac{g(x)}{F(x)}\,dx, \qquad F(x) = \exp\left[(1-k)\int f(x)\,dx\right].$$

2. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x) + g(x)e^{\lambda w}.$

General solution:

$$e^{-\lambda w} = F(x)\Phi(y-ax) - \lambda F(x)\int \frac{g(x)}{F(x)}\,dx, \qquad F(x) = \exp\left[-\lambda\int f(x)\,dx\right].$$

3. $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \left[f(x) + g(y)\right]w^k.$

General solution:

$$\frac{ab}{1-k}w^{1-k} = E_1(x)\Phi(bx-ay) + bE_1(x)\int \frac{f(x)\,dx}{E_1(x)} + aE_2(y)\int \frac{g(y)\,dy}{E_2(y)},$$

where $E_1(x) = \exp\left[\dfrac{c}{a}(1-k)x\right]$ and $E_2(y) = \exp\left[\dfrac{c}{b}(1-k)y\right].$

4. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c+\big[f(x)+g(y)\big]e^{\lambda w}.}$

General solution:

$$\frac{ab}{\lambda}e^{-\lambda w}=E_1(x)\Phi(bx-ay)-bE_1(x)\int\frac{f(x)\,dx}{E_1(x)}-aE_2(y)\int\frac{g(y)\,dy}{E_2(y)},$$

where $E_1(x)=\exp\Big(-\dfrac{c\lambda}{a}x\Big)$ and $E_2(y)=\exp\Big(-\dfrac{c\lambda}{b}y\Big)$.

5. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(w).}$

General solution: $\displaystyle\int\frac{dw}{f(w)}=\frac{x}{a}+\Phi(bx-ay).$

6. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(w+\alpha x+\beta y).}$

The substitution $u=w+\alpha x+\beta y$ leads to an equation of the form 1.2.1.5:

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=f(u)+a\alpha+b\beta.$$

7. $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(x)g(w).}$

General solution: $\displaystyle\int\frac{dw}{g(w)}=\frac{1}{a}\int f(x)\,dx+\Phi(bx-ay).$

8. $\boldsymbol{\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=f(x)g(y)h(w).}$

General solution: $\displaystyle\int\frac{dw}{h(w)}=\int_{x_0}^{x}f(t)g(y-ax+at)\,dt+\Phi(y-ax)$, where x_0 may be taken arbitrary.

9. $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(w).}$

General solution: $\displaystyle\int\frac{dw}{f(w)}=\frac{1}{a}\ln|x|+\Phi\big(|x|^b|y|^{-a}\big).$

10. $\boldsymbol{ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=f(w).}$

General solution: $\displaystyle\int\frac{dw}{f(w)}=\frac{1}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x+ay\big|+\Phi\big(ay^2-bx^2\big),\qquad ab>0.$

11. $ax^n\dfrac{\partial w}{\partial x}+by^k\dfrac{\partial w}{\partial y}=f(w).$

General solution:

$$\int\frac{dw}{f(w)}=\frac{1}{a(1-n)}x^{1-n}+\Phi(u),\qquad u=\frac{1}{a(1-n)}x^{1-n}-\frac{1}{b(1-k)}y^{1-k}.$$

12. $ay^n\dfrac{\partial w}{\partial x}+bx^k\dfrac{\partial w}{\partial y}=f(w).$

General solution:

$$a\int\frac{dw}{f(w)}=\int\left(\frac{b}{a}\frac{n+1}{k+1}x^{k+1}-u\right)^{-\frac{n}{n+1}}dx,\qquad u=\frac{b}{a}\frac{n+1}{k+1}x^{k+1}-y^{n+1}.$$

In the integration, u is considered a parameter.

13. $ae^{\lambda x}\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}=f(w).$

General solution: $\displaystyle\int\frac{dw}{f(w)}=-\frac{1}{a\lambda}e^{-\lambda x}+\Phi(u),$ where $u=a\lambda e^{-\beta y}-b\beta e^{-\lambda x}.$

14. $ae^{\lambda y}\dfrac{\partial w}{\partial x}+be^{\beta x}\dfrac{\partial w}{\partial y}=f(w).$

General solution: $\displaystyle\int\frac{dw}{f(w)}=\frac{c(\beta x-\lambda y)}{u}+\Phi(u),$ where $u=a\beta e^{\lambda y}-b\lambda e^{\beta x}.$

15. $f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}=h(w).$

General solution: $\displaystyle\int\frac{dw}{h(w)}=\int\frac{dx}{f(x)}+\Phi(u),$ where $\displaystyle u=\int\frac{dx}{f(x)}-\int\frac{dy}{g(y)}.$

16. $f(y)\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}=h(w).$

The transformation $\displaystyle\xi=\int g(x)\,dx,\ \eta=\int f(y)\,dy$ leads to an equation of the form 1.2.1.8:

$$\frac{\partial w}{\partial\xi}+\frac{\partial w}{\partial\eta}=F(\xi)G(\eta)h(w),\qquad\text{where}\quad F(\xi)=\frac{1}{g(x)},\quad G(\eta)=\frac{1}{f(y)}.$$

17. $f_1(x)\dfrac{\partial w}{\partial x}+f_2(y)\dfrac{\partial w}{\partial y}=aw+\big[g_1(x)+g_2(y)\big]w^k.$

General solution:

$$w^{1-k}=E_1(x)\Phi(u)+(1-k)E_1(x)\int\frac{g_1(x)\,dx}{f_1(x)E_1(x)}+(1-k)E_2(y)\int\frac{g_2(y)\,dy}{f_2(y)E_2(y)},$$

where

$$E_1(x)=\exp\left[a(1-k)\int\frac{dx}{f_1(x)}\right],\quad E_2(y)=\exp\left[a(1-k)\int\frac{dy}{f_2(y)}\right],\quad u=\int\frac{dx}{f_1(x)}-\int\frac{dy}{f_2(y)}.$$

18. $f_1(x)\dfrac{\partial w}{\partial x} + f_2(y)\dfrac{\partial w}{\partial y} = a + \big[g_1(x) + g_2(y)\big]e^{\lambda w}.$

General solution:

$$e^{-\lambda w} = E_1(x)\Phi(u) - \lambda E_1(x)\int \frac{g_1(x)\,dx}{f_1(x)E_1(x)} - \lambda E_2(y)\int \frac{g_2(y)\,dy}{f_2(y)E_2(y)},$$

where

$$E_1(x) = \exp\left[-a\lambda\int \frac{dx}{f_1(x)}\right], \quad E_2(y) = \exp\left[-a\lambda\int \frac{dy}{f_2(y)}\right], \quad u = \int \frac{dx}{f_1(x)} - \int \frac{dy}{f_2(y)}.$$

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x} + G(x,y,w)\frac{\partial w}{\partial y} = H(x,y,w).$

19. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = f(x).$

General solution:

$$y = ax\big[w - F(x)\big] + a\int F(x)\,dx + \Phi\big(w - F(x)\big), \quad \text{where} \quad F(x) = \int f(x)\,dx.$$

20. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = f(y).$

General solution:

$$x = \pm\int_{y_0}^{y} \frac{dz}{\sqrt{2aF(z) - 2au}} + \Phi(u), \qquad u = F(y) - \frac{1}{2}aw^2,$$

where $F(y) = \int f(y)\,dy.$

21. $\dfrac{\partial w}{\partial x} + aw\dfrac{\partial w}{\partial y} = f(y - bx).$

This is a model equation describing nonlinear waves issuing from a moving source (the variables x and y play the role of time and the spatial coordinate, respectively, and b is the source velocity).

1°. General solution:

$$x = \pm\int_{z_0}^{y-bx} \frac{dz}{\sqrt{b^2 + 2aF(z) - 2au}} + \Phi(u), \qquad u = F(y - bx) - \frac{1}{2}aw^2 + bw,$$

where $F(z) = \int f(z)\,dz$; z_0 may be taken arbitrary.

2°. Solution with steady profile:

$$w = b - \left[(b - w_0)^2 - 2\int_{\xi}^{\infty} f(z)\,dz\right]^{1/2}, \qquad \xi = y - bx,$$

where w_0 is the constant of integration.

⊙ *References for equation* 1.2.1.21: A. L. Hoffman (1967) and G. B. Whitham (1974).

22. $\dfrac{\partial w}{\partial x} + \big[aw + f(x)\big]\dfrac{\partial w}{\partial y} = g(x).$

General solution:

$$y = ax\big[w - G(x)\big] + a\int G(x)\,dx + F(x) + \Phi\big(w - G(x)\big),$$

where $F(x) = \int f(x)\,dx$ and $G(x) = \int g(x)\,dx.$

23. $\dfrac{\partial w}{\partial x} + \big[aw + yf(x)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$yF(x) - aw\int F(x)\,dx = \Phi(w), \qquad \text{where} \quad F(x) = \exp\left[-\int f(x)\,dx\right].$$

24. $\dfrac{\partial w}{\partial x} + \big[aw + f(y)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \displaystyle\int_{y_0}^{y} \frac{dt}{f(t) + aw} + \Phi(w).$

25. $\dfrac{\partial w}{\partial x} + \big[ayw + f(x)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y\exp(-axw) - \displaystyle\int_{x_0}^{x} f(t)\exp(-atw)\,dt = \Phi(w).$

26. $\dfrac{\partial w}{\partial x} - f(y)w\dfrac{\partial w}{\partial y} = g(x)w + h(x).$

Complete integral:*

$$w = \varphi(x)\int \frac{dy}{f(y)} + \psi(x),$$

where

$$\varphi(x) = G(x)\left[C_1 - \int G(x)\,dx\right]^{-1}, \qquad G(x) = \exp\left[\int g(x)\,dx\right],$$
$$\psi(x) = S(x)\left[C_2 + \int \frac{h(x)}{S(x)}\,dx\right], \qquad S(x) = G(x)\exp\left[\int \varphi(x)\,dx\right].$$

* The general integral for equations 1.2.1.26–1.2.1.28 can be constructed according to formulas presented in Subsection 24.1.1.

27. $\dfrac{\partial w}{\partial x} + f(y)w\dfrac{\partial w}{\partial y} = aw^2 + g(x)w + h(x).$

Complete integral:

$$w(x,y) = \varphi(x) + C_1\psi(x)\exp\left[a\int \frac{dy}{f(y)}\right],$$

where C_1 is an arbitrary constant, and the functions $\varphi(x)$ and $\psi(x)$ are determined by the following system of first order ordinary differential equations:

$$\varphi'_x = a\varphi^2 + g(x)\varphi + h(x), \tag{1}$$
$$\psi'_x = \big[a\varphi + g(x)\big]\psi. \tag{2}$$

The Riccati equation (1) is integrable in quadrature for a lot of specific functions $g(x)$ and $h(x)$ [e.g., for $h(x) \equiv 0$ and any $g(x)$]. For details, see the books by Kamke (1976) and Polyanin and Zaitsev (2003).

If the solution of equation (1) is known, equation (2) is easy to integrate—it is linear in the unknown function ψ.

28. $\dfrac{\partial w}{\partial x} - f(y)w\dfrac{\partial w}{\partial y} = g(y)w^2 + h(x)w.$

Complete integral:

$$w = H(x)\left[C_1 + \int H(x)\,dx\right]^{-1}\frac{1}{\Psi(y)}\left[C_2 - \int \frac{\Psi(y)}{f(y)}\,dy\right],$$

where

$$H(x) = \exp\left[\int h(x)\,dx\right], \quad \Psi(y) = \exp\left[\int \frac{g(y)}{f(y)}\,dy\right].$$

29. $\dfrac{\partial w}{\partial x} - \big[aw + yf(x) + g(x)\big]\dfrac{\partial w}{\partial y} - h(x)w = 0.$

The transformation

$$w(x,y) = H(x)u(t,z), \quad z = yF(x) + \int g(x)F(x)\,dx, \quad t = \int F(x)H(x)\,dx,$$

where

$$F(x) = \exp\left[\int f(x)\,dx\right], \quad H(x) = \exp\left[\int h(x)\,dx\right],$$

leads to a simpler equation of the form 1.1.1.18: $\dfrac{\partial u}{\partial t} - au\dfrac{\partial u}{\partial z} = 0.$

30. $\dfrac{\partial w}{\partial x} + f(x)w^k\dfrac{\partial w}{\partial y} = g(x)w.$

The transformation

$$\xi = \int f(x)G^k(x)\,dx, \quad u(\xi,y) = \frac{1}{G(x)}w(x,y), \qquad \text{where} \quad G(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a special case of equation 1.1.1.47 with $b = c = 0$: $\dfrac{\partial u}{\partial \xi} + u^k\dfrac{\partial u}{\partial y} = 0.$

31. $\dfrac{\partial w}{\partial x} + f(y)w^k\dfrac{\partial w}{\partial y} = g(y)w^{k+1}.$

The transformation

$$\eta = \int \frac{dy}{f(y)\Psi^k(y)}, \quad u(x,\eta) = \frac{1}{\Psi(y)}w(x,y), \qquad \text{where} \quad \Psi(y) = \exp\left[\int \frac{g(y)}{f(y)}\,dy\right],$$

leads to a special case of equation 1.1.1.47 with $b = c = 0$: $\dfrac{\partial u}{\partial x} + u^k\dfrac{\partial u}{\partial \eta} = 0.$

32. $\dfrac{\partial w}{\partial x} + f(x)e^{\lambda w}\dfrac{\partial w}{\partial y} = g(x).$

The transformation

$$\xi = \int f(x)e^{\lambda G(x)}\,dx, \quad u(\xi,y) = w(x,y) - G(x), \qquad \text{where} \quad G(x) = \int g(x)\,dx,$$

leads to a special case of equation 1.1.2.18 with $b = 0$: $\dfrac{\partial u}{\partial \xi} + e^{\lambda u}\dfrac{\partial u}{\partial y} = 0.$

33. $\dfrac{\partial w}{\partial x} + f(y)e^{\lambda w}\dfrac{\partial w}{\partial y} = g(y)e^{\lambda w}.$

The transformation

$$\eta = \int \frac{dy}{f(y)e^{\lambda\Psi(y)}}, \quad u(x,\eta) = w(x,y) - \Psi(y), \qquad \text{where} \quad \Psi(y) = \int \frac{g(y)}{f(y)}\,dy,$$

leads to a special case of equation 1.1.2.18 with $b = 0$: $\dfrac{\partial u}{\partial x} + e^{\lambda u}\dfrac{\partial u}{\partial \eta} = 0.$

34. $\big[f(x) + aw\big]\dfrac{\partial w}{\partial x} + \big[g(y) + bw\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\int \frac{dx}{f(x) + aw} - \int \frac{dy}{g(y) + bw} = \Phi(w).$$

In the integration, w is considered a parameter.

35. $\big[f(y) + aw\big]\dfrac{\partial w}{\partial x} + \big[g(x) + bw\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $(bx - ay)w + \displaystyle\int g(x)\,dx - \int f(y)\,dy = \Phi(w).$

36. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = 0.$

Model equation of gas dynamics. This equation is also encountered in hydrodynamics, multiphase flows, wave theory, acoustics, chemical engineering, and other applications.

1°. General solution:

$$\Phi(y - xf(w), w) = 0, \qquad \text{or} \qquad y = xf(w) + \overline{\Phi}(w),$$

where Φ and $\overline{\Phi}$ are arbitrary functions.

2°. The solution of the Cauchy problem with the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0$$

can be represented in the parametric form

$$y = \xi + \mathcal{F}(\xi)x, \quad w = \varphi(\xi),$$

where $\mathcal{F}(\xi) = f\big(\varphi(\xi)\big)$.

3°. Consider the Cauchy problem with the discontinuous initial condition

$$w(0, y) = \begin{cases} w_1 & \text{for } y < 0, \\ w_2 & \text{for } y > 0. \end{cases}$$

It is assumed that $x \geq 0$, $f > 0$, and $f' > 0$ for $w > 0$, $w_1 > 0$, and $w_2 > 0$.

Generalized solution for $w_1 < w_2$:

$$w(x, y) = \begin{cases} w_1 & \text{for } y/x < V_1, \\ f^{-1}(y/x) & \text{for } V_1 \leq y/x \leq V_2, \\ w_2 & \text{for } y/x > V_2, \end{cases} \qquad \text{where} \quad V_1 = f(w_1), \ \ V_2 = f(w_2).$$

Here f^{-1} is the inverse of the function f, i.e., $f^{-1}\big(f(w)\big) \equiv w$. This solution is continuous in the half-plane $x > 0$ and describes a "rarefaction wave."

Generalized solution for $w_1 > w_2$:

$$w(x, y) = \begin{cases} w_1 & \text{for } y/x < V, \\ w_2 & \text{for } y/x > V, \end{cases} \qquad \text{where} \quad V = \frac{1}{w_2 - w_1} \int_{w_1}^{w_2} f(w)\, dw.$$

This solution undergoes a discontinuity along the line $y = Vx$ and describes a "shock wave."

4°. In Section 23.3, qualitative features of solutions to model equations of gas dynamics are considered, including the *wave breaking effect* and shock waves. This subsection also presents general formulas that permit one to construct generalized (discontinuous) solutions for arbitrary initial conditions. The references listed below contain a lot of solutions to Cauchy problems which describe the merging and decay of discontinuities, periodic waves, and other nonlinear physical effects.

⊙ *References for equation* 1.2.1.36: E. Hopf (1950), P. D. Lax (1954), O. A. Oleinik (1954, 1957, 1959), R. Courant (1962), G. B. Whitham (1974), J. Lighthill (1978), B. L. Rozhdestvenskii and N. N. Yanenko (1978), R. J. LeVeque (1992), A. G. Kulikovskii and E. I. Sveshnikova (1995).

37. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = g(x).$

General solution:

$$y = \int_{x_0}^{x} f\big(G(t) - G(x) + w\big)\, dt + \Phi\big(w - G(x)\big), \quad \text{where} \quad G(x) = \int g(x)\, dx.$$

38. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = g(y).$

General solution:

$$x = \int_{y_0}^{y} \psi\big(G(t) - G(y) + F(w)\big)\,dt + \Phi\big(F(w) - G(y)\big),$$

where $G(y) = \int g(y)\,dy$ and $F(w) = \int f(w)\,dw$. The function $\psi = \psi(z)$ is defined parametrically by $\psi = \dfrac{1}{f(w)}$, $z = F(w)$.

39. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = g(y - ax).$

This is a model equation describing nonlinear waves issuing from a moving source (the variables x and y play the role of time and the spatial coordinate, respectively, and a is the source velocity).

1°. The substitution $\xi = y - ax$ leads to an equation of the form 1.2.1.38:

$$\frac{\partial w}{\partial x} + \big[f(w) - a\big]\frac{\partial w}{\partial \xi} = g(\xi).$$

2°. Solution with steady profile:

$$\int f(w)\,dw - aw = \int g(\xi)\,d\xi + C, \quad \xi = y - ax,$$

where C is the constant of integration.

40. $\dfrac{\partial w}{\partial x} + f(w)\dfrac{\partial w}{\partial y} = g(w).$

General solution: $y = \displaystyle\int \frac{f(w)}{g(w)}dw + \Phi\left(x - \int \frac{dw}{g(w)}\right).$

41. $\dfrac{\partial w}{\partial x} + \big[f(w) + ax\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = xf(w) + \frac{1}{2}ax^2 + \Phi(w).$

42. $\dfrac{\partial w}{\partial x} + \big[f(w) + ay\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \dfrac{1}{a}\ln\big|ay + f(w)\big| + \Phi(w).$

43. $\dfrac{\partial w}{\partial x} + f(w + ax + by)\dfrac{\partial w}{\partial y} = 0.$

The substitution $u = w + ax + by$ leads to an equation of the form 1.2.1.38:

$$\frac{\partial u}{\partial x} + f(u)\frac{\partial u}{\partial y} = a + bf(u).$$

44. $\dfrac{\partial w}{\partial x} + \big[f(w) + g(x)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y = xf(w) + \int g(x)\,dx + \Phi(w).$

45. $\dfrac{\partial w}{\partial x} + \big[f(w) + g(x)\big]\dfrac{\partial w}{\partial y} = h(x).$

General solution:

$$y = \int_{x_0}^{x} f\big(H(t) - H(x) + w\big)\,dt + G(x) + \Phi\big(w - H(x)\big),$$

where $G(x) = \int g(x)\,dx$ and $H(x) = \int h(x)\,dx.$

46. $\dfrac{\partial w}{\partial x} + \big[f(w) + g(x)\big]\dfrac{\partial w}{\partial y} = h(w).$

General solution:

$$y = \int \frac{f(w)}{h(w)}\,dw + \int_{w_0}^{w} \frac{g\big(H(t) - H(w) + x\big)}{h(t)}\,dt + \Phi\big(x - H(w)\big),$$

where $H(x) = \int \dfrac{dw}{h(w)}.$

47. $\dfrac{\partial w}{\partial x} + \big[f(w) + g(y)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $x = \displaystyle\int_{y_0}^{y} \frac{dt}{g(t) + f(w)} + \Phi(w).$

48. $\dfrac{\partial w}{\partial x} + \big[f(w) + yg(x)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$yG(x) - f(w)\int G(x)\,dx = \Phi(w), \qquad \text{where} \quad G(x) = \exp\left[-\int g(x)\,dx\right].$$

49. $\dfrac{\partial w}{\partial x} + \big[f(w) + yg(x)\big]\dfrac{\partial w}{\partial y} = h(x).$

General solution:

$$yG(x) - \int G(x) f\big(H(t) - H(x) + w\big)\,dx = \Phi\big(w - H(x)\big),$$

where $G(x) = \exp\left[-\displaystyle\int g(x)\,dx\right]$ and $H(x) = \displaystyle\int h(x)\,dx.$

50. $\dfrac{\partial w}{\partial x} + \big[yf(w) + g(x)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y\exp\big[-xf(w)\big] - \displaystyle\int_{x_0}^{x} g(t)\exp\big[-tf(w)\big]\,dt = \Phi(w)$, where x_0 may be taken arbitrary.

51. $\dfrac{\partial w}{\partial x} + \big[xf(w) + yg(w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y + x\dfrac{f(w)}{g(w)} + \dfrac{f(w)}{g^2(w)} = \exp\big[g(w)x\big]\,\Phi(w).$

52. $\dfrac{\partial w}{\partial x} + \big[xf(w) + yg(w) + h(w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $y + \dfrac{xf(w) + h(w)}{g(w)} + \dfrac{f(w)}{g^2(w)} = \exp\big[g(w)x\big]\,\Phi(w).$

53. $\dfrac{\partial w}{\partial x} + \big[x^{2n+1}y^m f(w) + x^n y g(w)\big]\dfrac{\partial w}{\partial y} = 0.$

The transformation $\xi = x^{n+1}$, $\eta = y^{1-m}$ leads to an equation of the form 1.2.1.51:

$$\frac{\partial w}{\partial \xi} + \big[\xi F(w) + \eta G(w)\big]\frac{\partial w}{\partial \eta} = 0, \qquad \text{where} \quad F(w) = \frac{1-m}{1+n}f(w), \quad G(w) = \frac{1-m}{1+n}g(w).$$

54. $\dfrac{\partial w}{\partial x} + \big[e^{\lambda x}f(w) + e^{\beta y}g(w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution for $\lambda\beta \neq 0$:

$$e^{-\beta y}F(x,w) + \beta g(w)\int_{x_0}^{x} F(t,w)\,dt = \Phi(w), \qquad \text{where} \quad F(x,w) = \exp\left[\frac{\beta}{\lambda}e^{\lambda x}f(w)\right].$$

55. $\dfrac{\partial w}{\partial x} + f(x)g(y)h(w)\dfrac{\partial w}{\partial y} = 0.$

General solution: $\Phi(u,w) = 0$, where $u = \displaystyle\int \frac{dy}{g(y)} - h(w)\int f(x)\,dx.$

56. $\dfrac{\partial w}{\partial x} + f(x)g(y)h(w)\dfrac{\partial w}{\partial y} = p(x).$

General solution:

$$\int \frac{dy}{g(y)} = \int_{x_0}^{x} f(t)h\big(P(t) - P(x) + w\big)\,dt + \Phi\big(w - P(x)\big), \qquad \text{where} \quad P(x) = \int p(x)\,dx.$$

57. $\dfrac{\partial w}{\partial x} + f(x)g(y)h(w)\dfrac{\partial w}{\partial y} = p(w).$

General solution:

$$\int \frac{dy}{g(y)} = \int_{w_0}^{w} \frac{h(t)}{p(t)} f\big(P(t) - P(w) + x\big)\,dt + \Phi\big(x - P(w)\big), \qquad \text{where} \quad P(w) = \int \frac{dw}{p(w)}.$$

58. $\big[f(x) + g_1(w)\big]\dfrac{\partial w}{\partial x} + \big[h(y) + g_2(w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $\displaystyle\int \frac{dx}{f(x) + g_1(w)} - \int \frac{dy}{h(y) + g_2(w)} = \Phi(w).$ In the integration, w is considered a parameter.

59. $\big[f(y) + g_1(w)\big]\dfrac{\partial w}{\partial x} + \big[h(x) + g_2(w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution: $g_2(w)x - g_1(w)y + \displaystyle\int h(x)\,dx - \int f(y)\,dy = \Phi(w).$

1.2.2. Equations Contain Arbitrary Functions of Two Variables

► **Equations of the form** $F(x,y)\frac{\partial w}{\partial x} + G(x,y)\frac{\partial w}{\partial y} = H(x,y,w).$

1. $\dfrac{\partial w}{\partial x} + f(y)\dfrac{\partial w}{\partial y} = g(x,w).$

General solution:

$$x - \int \frac{dy}{f(y)} = I(x,w),$$

where $I(x,u) = C$ is a general integral of the ordinary differential equation $u'_x = g(x,u)$.

2. $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h_2(x,y)w^2 + h_1(x,y)w + h_0(x,y).$

Suppose a particular solution $w_0 = w_0(x,y)$ of the equation is known. Then the substitution $\zeta = w - w_0$ leads to an equation of the form 1.2.2.3 with $k = 2$:

$$f\frac{\partial \zeta}{\partial x} + g\frac{\partial \zeta}{\partial y} = h_2\zeta^2 + \big(2h_2w_0 + h_1\big)\zeta.$$

For $h_n = \text{const}$ ($n = 0, 1, 2$), a particular solution w_0 of the original equation can be determined by solving the quadratic equation $h_2w_0^2 + h_1w_0 + h_0 = 0$.

If the functions f and h_n ($n = 0, 1, 2$) are independent of y, then a particular solution should be sought in the form $w_0 = w_0(x)$.

3. $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h_1(x,y)w + h_2(x,y)w^k.$

The substitution $\xi = w^{1-k}$ leads to the nonhomogeneous linear equation

$$f(x,y)\frac{\partial \xi}{\partial x} + g(x,y)\frac{\partial \xi}{\partial y} = (1-k)h_1(x,y)\xi + (1-k)h_2(x,y).$$

Equations of this type are considered in Polyanin, Zaitsev, and Moussiaux (2002, Chapter 5).

4. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)+h_2(x,y)e^{\lambda w}.$

The substitution $\xi=e^{-\lambda w}$ leads to the nonhomogeneous linear equation

$$f(x,y)\frac{\partial \xi}{\partial x}+g(x,y)\frac{\partial \xi}{\partial y}=-\lambda h_1(x,y)\xi-\lambda h_2(x,y).$$

Equations of this type are considered in Polyanin, Zaitsev, and Moussiaux (2002, Chapter 5).

5. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_2(x,y)e^{\lambda w}+h_1(x,y)+h_0(x,y)e^{-\lambda w}.$

The substitution $\xi=e^{\lambda w}$ leads to an equation of the form 1.2.2.2:

$$f(x,y)\frac{\partial \xi}{\partial x}+g(x,y)\frac{\partial \xi}{\partial y}=\lambda h_2(x,y)\xi^2+\lambda h_1(x,y)\xi+\lambda h_0(x,y).$$

6. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)\cosh(\lambda w)+h_2(x,y).$

Applying the formula $\cosh(\lambda w)=\frac{1}{2}(e^{\lambda w}+e^{-\lambda w})$, we arrive at an equation of the form 1.2.2.5.

7. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)\sinh(\lambda w)+h_2(x,y).$

Applying the formula $\sinh(\lambda w)=\frac{1}{2}(e^{\lambda w}-e^{-\lambda w})$, we arrive at an equation of the form 1.2.2.5.

8. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)w\ln w+h_2(x,y)w.$

The substitution $w=e^{\xi}$ leads to the nonhomogeneous linear equation

$$f(x,y)\frac{\partial \xi}{\partial x}+g(x,y)\frac{\partial \xi}{\partial y}=h_1(x,y)\xi+h_2(x,y).$$

Equations of this type are considered in Polyanin, Zaitsev, and Moussiaux (2002, Chapter 5).

9. $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)w\ln w+h_2(x,y)w\ln^k w.$

The substitution $w=e^{\xi}$ leads to an equation of the form 1.2.2.3:

$$f(x,y)\frac{\partial \xi}{\partial x}+g(x,y)\frac{\partial \xi}{\partial y}=h_1(x,y)\xi+h_2(x,y)\xi^k.$$

10. $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h(x,y)\varphi(w).$

The substitution $\xi = \displaystyle\int \frac{dw}{\varphi(w)}$ leads to the nonhomogeneous linear equation

$$f(x,y)\frac{\partial \xi}{\partial x} + g(x,y)\frac{\partial \xi}{\partial y} = h(x,y).$$

Equations of this type are considered in Polyanin, Zaitsev, and Moussiaux (2002, Chapter 5); the equation can be solved using the *method of characteristics* described in changed to Subsection 24.3.5.

► **Equations of the form** $F(x,y,w)\frac{\partial w}{\partial x} + G(x,y,w)\frac{\partial w}{\partial y} = H(x,y,w).$

11. $\dfrac{\partial w}{\partial x} + f(x,w)\dfrac{\partial w}{\partial y} = g(x).$

General solution:

$$y = \int_{x_0}^{x} f\big(t,\, G(t) - G(x) + w\big)\, dt + \Phi\big(w - G(x)\big), \quad \text{where} \quad G(x) = \int g(x)\, dx.$$

12. $\dfrac{\partial w}{\partial x} + f(x,w)\dfrac{\partial w}{\partial y} = g(w).$

General solution:

$$y = \int_{w_0}^{w} \frac{f\big(G(t) - G(w) + x,\, t\big)}{g(t)}\, dt + \Phi\big(x - G(w)\big), \quad \text{where} \quad G(w) = \int \frac{dw}{g(w)}.$$

13. $\dfrac{\partial w}{\partial x} + f(x,w)g(y,w)\dfrac{\partial w}{\partial y} = 0.$

General solution: $\displaystyle\int_{x_0}^{x} f(\xi, w)\, d\xi = \int_{y_0}^{y} \frac{dt}{g(t,w)} + \Phi(w)$, where $\Phi(w)$ is an arbitrary function, and x_0 and y_0 are arbitrary numbers.

14. $\dfrac{\partial w}{\partial x} + \big[f(x,w) + yg(x,w)\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$yG(x,w) - \int_{x_0}^{x} f(t,w)G(t,w)\, dt = \Phi(w), \quad \text{where} \quad G(x,w) = \exp\left[-\int_{x_0}^{x} g(t,w)\, dt\right].$$

15. $\dfrac{\partial w}{\partial x} + \big[y^k f(x,w) + yg(x,w)\big]\dfrac{\partial w}{\partial y} = 0.$

The substitution $z = y^{1-k}$ leads to an equation of the form 1.2.2.14:

$$\frac{\partial w}{\partial x} + (1-k)\big[f(x,w) + zg(x,w)\big]\frac{\partial w}{\partial z} = 0.$$

16. $\dfrac{\partial w}{\partial x} + \big[e^{\lambda y} f(x,w) + g(x,w)\big]\dfrac{\partial w}{\partial y} = 0.$

The substitution $z = e^{-\lambda y}$ leads to an equation of the form 1.2.2.14:

$$\frac{\partial w}{\partial x} - \lambda\big[f(x,w) + zg(x,w)\big]\frac{\partial w}{\partial z} = 0.$$

17. $\dfrac{\partial w}{\partial x} + f(ax+by,w)\dfrac{\partial w}{\partial y} = 0.$

General solution for $b \neq 0$:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{a + bf(t,w)} - x, \quad z = ax + by, \quad c \text{ is any.}$$

18. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{y}{x}, w\right)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^{y/x} \frac{dt}{f(t,w) - t} - \ln|x|, \quad c \text{ is any.}$$

19. $\dfrac{\partial w}{\partial x} + f(x^n y^m, w)\dfrac{y}{x}\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{t\big[mf(t,w) + n\big]} - \ln|x|, \quad z = x^n y^m, \quad c \text{ is any.}$$

20. $\dfrac{\partial w}{\partial x} + e^{-ax} f(e^{ax}y, w)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{f(t,w) + at} - x, \quad z = e^{ax}y, \quad c \text{ is any.}$$

21. $\dfrac{\partial w}{\partial x} + e^{ay} f(xe^{ay}, w)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{t\big[atf(t,w) + 1\big]} - \ln|x|, \quad z = xe^{ay}, \quad c \text{ is any.}$$

22. $\dfrac{\partial w}{\partial x} + yf(e^{ax}y^m, w)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u,w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{t\big[a + mf(t,w)\big]} - x, \quad z = e^{ax}y^m, \quad c \text{ is any.}$$

23. $\dfrac{\partial w}{\partial x} + x^{n-1}e^{ay} f(x^n e^{ay}, w)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{t\big[atf(t, w) + n\big]} - \ln|x|, \quad z = x^n e^{ay}, \quad c \text{ is any.}$$

24. $\dfrac{\partial w}{\partial x} + \big[y^k f(x, w) g(e^{ax} y, w) - ay\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int \frac{dv}{v^k g(v, w)} - \int f(x, w)\exp[a(1-k)x]\,dx, \quad v = e^{ax} y.$$

In the integration, w is considered a parameter.

25. $\dfrac{\partial w}{\partial x} + e^{\alpha x - \beta y} f(ae^{\alpha x} + be^{\beta y},\ w)\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int_c^z \frac{dt}{a\alpha + b\beta f(t, w)} - \frac{1}{\alpha} e^{\alpha x}, \quad z = ae^{\alpha x} + be^{\beta y}, \quad c \text{ is any.}$$

26. $x\dfrac{\partial w}{\partial x} + \big[f(x, w) g(x^n e^y, w) - n\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int \frac{dv}{vg(v, w)} - \int \frac{f(x, w)}{x}\,dx, \quad v = x^n e^y.$$

In the integration, w is considered a parameter.

27. $\big[f(y, w) + amx^n y^{m-1}\big]\dfrac{\partial w}{\partial x} - \big[g(x, w) + anx^{n-1} y^m\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int f(y, w)\,dy + \int g(x, w)\,dx + ax^n y^m.$$

In the integration, w is considered a parameter.

28. $\big[e^{\alpha x} f(y, w) + c\beta\big]\dfrac{\partial w}{\partial x} - \big[e^{\beta y} g(x, w) + c\alpha\big]\dfrac{\partial w}{\partial y} = 0.$

General solution:

$$\Phi(u, w) = 0, \qquad \text{where} \quad u = \int e^{-\beta y} f(y, w)\,dy + \int e^{-\alpha x} g(x, w)\,dx - ce^{-\alpha x - \beta y}.$$

In the integration, w is considered a parameter.

29. $\dfrac{\partial w}{\partial x} + f(x,y,w)\dfrac{\partial w}{\partial y} = 0.$

General solution: $\Phi\big(u(x,y,w),w\big) = 0$, where $u(x,y,a) = C$ is the general integral of the ordinary differential equation $y'_x = f(x,y,a)$ with parameter a. Solutions for a large number of ordinary differential equations with various right-hand sides dependent on free parameters can be found in the handbook Polyanin and Zaitsev (2005); see also Kamke (1977) and Murphy (1960).

30. $f_y(x,y,w)\dfrac{\partial w}{\partial x} - f_x(x,y,w)\dfrac{\partial w}{\partial y} = 0.$

General solution: $\Phi\big(w,\ f(x,y,w)\big) = 0.$

31. $f(x,w)\dfrac{\partial w}{\partial x} + g(x,w)\dfrac{\partial w}{\partial y} = h(x,w).$

Solutions are sought in the implicit form (*hodograph transformations*)

$$y = y(x,w),$$

which means that x and w are taken to be the independent variables and y the dependent variable. Taking into account the fact that the old derivatives are expressed in terms of the new ones as $w_y = 1/y_w$ and $w_x = -y_x/y_w$, one arrives at the linear equation

$$f(x,w)\frac{\partial y}{\partial x} + h(x,w)\frac{\partial y}{\partial w} = g(x,w).$$

Equations of this type are considered in Polyanin, Zaitsev, and Moussiaux (2002, Chapter 5); this equation can be solved using the *method of characteristics* described in Subsection 24.3.5.

1.3. Other Quasilinear Equations

1.3.1. Equations with Three Independent Variables

1. $\dfrac{\partial w}{\partial x} + f(x,w)\dfrac{\partial w}{\partial y} + g(x,w)\dfrac{\partial w}{\partial z} = h(x).$

General solution: $\Phi(u_1,u_2,u_3) = 0$, where

$$u_1 = w - H(x), \quad H(x) = \int h(x)\,dx,$$

$$u_2 = y - \int_a^x f\big(t, H(t) - H(x) + w\big)\,dt, \quad u_3 = z - \int_a^x g\big(t, H(t) - H(x) + w\big)\,dt.$$

In the integration, w is treated as a parameter; a is an arbitrary number.

2. $\dfrac{\partial w}{\partial x} + f(x,w)\dfrac{\partial w}{\partial y} + g(x,w)\dfrac{\partial w}{\partial z} = h(w).$

General solution: $\Phi(u_1,u_2,u_3) = 0$, where

$$u_1 = x - H(w), \quad H(w) = \int \frac{dw}{h(w)},$$

$$u_2 = y - \int_a^w \frac{f\big(H(t) - H(w) + x,\ t\big)}{h(t)}\,dt, \quad u_3 = z - \int_a^w \frac{g\big(H(t) - H(w) + x,\ t\big)}{h(t)}\,dt.$$

In the integration, w is treated as a parameter; a is an arbitrary number.

3. $\dfrac{\partial w}{\partial x} + f(x,w)\dfrac{\partial w}{\partial y} + g(x,w)\dfrac{\partial w}{\partial z} = h_1(x)h_2(w).$

The substitution $\overline{x} = \int h_1(x)\,dx$ leads to an equation of the form 1.3.1.2. The substitution $\overline{w} = \int \dfrac{dw}{h_2(w)}$ leads to an equation of the form 1.3.1.1 for $\overline{w}$.

4. $\dfrac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\dfrac{\partial w}{\partial y} + f_2(x,w)g_2(z,w)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int f_1(x,w)\,dx - \int \frac{dy}{g_1(y,w)}, \quad u_2 = \int f_2(x,w)\,dx - \int \frac{dz}{g_2(z,w)}.$$

In the integration, w is treated as a parameter.

5. $\dfrac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\dfrac{\partial w}{\partial y} + \big[f_2(x,w) + zg_2(x,w)\big]\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int f_1(x,w)\,dx - \int \frac{dy}{g_1(y,w)},$$

$$u_2 = zG_2(x,w) - \int f_2(x,w)G_2(x,w)\,dx, \quad G_2(x,w) = \exp\left[-\int g_2(x,w)\,dx\right].$$

In the integration, w is treated as a parameter.

6. $\dfrac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\dfrac{\partial w}{\partial y} + \big[z^n f_2(x,w) + zg_2(x,w)\big]\dfrac{\partial w}{\partial z} = 0.$

The substitution $\eta = z^{1-n}$ leads to an equation of the form 1.3.1.5:

$$\frac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\frac{\partial w}{\partial y} + (1-n)\big[f_2(x,w) + \eta g_2(x,w)\big]\frac{\partial w}{\partial \eta} = 0.$$

7. $\dfrac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\dfrac{\partial w}{\partial y} + \big[e^{\beta z} f_2(x,w) + g_2(x,w)\big]\dfrac{\partial w}{\partial z} = 0.$

The substitution $\eta = e^{-\beta z}$ leads to an equation of the form 1.3.1.5:

$$\frac{\partial w}{\partial x} + f_1(x,w)g_1(y,w)\frac{\partial w}{\partial y} - \beta\big[f_2(x,w) + \eta g_2(x,w)\big]\frac{\partial w}{\partial \eta} = 0.$$

8. $\dfrac{\partial w}{\partial x} + \big[f_1(x,w) + yg_1(x,w)\big]\dfrac{\partial w}{\partial y} + \big[f_2(x,w) + zg_2(x,w)\big]\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = yG_1(x,w) - \int f_1(x,w)G_1(x,w)\,dx, \qquad G_1(x,w) = \exp\left[-\int g_1(x,w)\,dx\right],$$

$$u_2 = zG_2(x,w) - \int f_2(x,w)G_2(x,w)\,dx, \qquad G_2(x,w) = \exp\left[-\int g_2(x,w)\,dx\right].$$

In the integration, w is treated as a parameter.

9. $\dfrac{\partial w}{\partial x}+\big[y^k f_1(x,w)+y g_1(x,w)\big]\dfrac{\partial w}{\partial y}+\big[z^n f_2(x,w)+z g_2(x,w)\big]\dfrac{\partial w}{\partial z}=0.$

The transformation $\xi = y^{1-k}$, $\eta = z^{1-n}$ leads to an equation of the form 1.3.1.8:

$$\frac{\partial w}{\partial x}+(1-k)\big[f_1(x,w)+\xi g_1(x,w)\big]\frac{\partial w}{\partial \xi}+(1-n)\big[f_2(x,w)+\eta g_2(x,w)\big]\frac{\partial w}{\partial \eta}=0.$$

10. $\dfrac{\partial w}{\partial x}+\big[e^{\alpha y} f_1(x,w)+g_1(x,w)\big]\dfrac{\partial w}{\partial y}+\big[e^{\beta z} f_2(x,w)+g_2(x,w)\big]\dfrac{\partial w}{\partial z}=0.$

The transformation $\xi = e^{-\alpha y}$, $\eta = e^{-\beta z}$ leads to an equation of the form 1.3.1.8:

$$\frac{\partial w}{\partial x}-\alpha\big[f_1(x,w)+\xi g_1(x,w)\big]\frac{\partial w}{\partial \xi}-\beta\big[f_2(x,w)+\eta g_2(x,w)\big]\frac{\partial w}{\partial \eta}=0.$$

11. $\dfrac{\partial w}{\partial x}+\big[y^k f_1(x,w)+y g_1(x,w)\big]\dfrac{\partial w}{\partial y}+\big[e^{\beta z} f_2(x,w)+g_2(x,w)\big]\dfrac{\partial w}{\partial z}=0.$

The transformation $\xi = y^{1-k}$, $\eta = e^{-\beta z}$ leads to an equation of the form 1.3.1.8:

$$\frac{\partial w}{\partial x}+(1-k)\big[f_1(x,w)+\xi g_1(x,w)\big]\frac{\partial w}{\partial \xi}-\beta\big[f_2(x,w)+\eta g_2(x,w)\big]\frac{\partial w}{\partial \eta}=0.$$

12. $\dfrac{\partial w}{\partial x}+f(a_1x+b_1y,w)\dfrac{\partial w}{\partial y}+g(a_2x+b_2z,w)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1 = x-\int_{c_1}^{\xi_1}\frac{dt}{a_1+b_1f(t,w)}, \qquad \xi_1 = a_1x+b_1y \quad (b_1\neq 0),$$
$$u_2 = x-\int_{c_2}^{\xi_2}\frac{dt}{a_2+b_2g(t,w)}, \qquad \xi_2 = a_2x+b_2z \quad (b_2\neq 0),$$

and c_1 and c_2 are arbitrary numbers. For $b_1 = 0$ or $b_2 = 0$, one should use the formulas

$$u_1 = y-\frac{1}{a_1}\int_{c_1}^{a_1x} f(t,w)\,dt, \quad u_2 = z-\frac{1}{a_2}\int_{c_2}^{a_2x} g(t,w)\,dt.$$

13. $\dfrac{\partial w}{\partial x}+f(y+ax,w)\dfrac{\partial w}{\partial y}+g\left(\dfrac{z}{x},w\right)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1 = x-\int_{c_1}^{y+ax}\frac{dt}{f(t,w)+a}, \quad u_2 = \int_{c_2}^{z/x}\frac{dt}{g(t,w)-t}-\ln|x|,$$

and c_1 and c_2 are arbitrary numbers.

14. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{y}{x}, w\right)\dfrac{\partial w}{\partial y} + g\left(\dfrac{z}{x}, w\right)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int_a^{y/x} \frac{dt}{f(t,w) - t} - \ln|x|, \quad u_2 = \int_b^{z/x} \frac{dt}{g(t,w) - t} - \ln|x|,$$

and a and b are arbitrary numbers.

15. $\dfrac{\partial w}{\partial x} + \dfrac{y}{x} f(x^n y^m, w)\dfrac{\partial w}{\partial y} + \dfrac{z}{x} g(x^k z^p, w)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int_a^{x^n y^m} \frac{dt}{t[mf(t,w) + n]} - \ln|x|, \quad u_2 = \int_b^{x^k z^p} \frac{dt}{t[pg(t,w) + k]} - \ln|x|,$$

and a and b are arbitrary numbers.

16. $\dfrac{\partial w}{\partial x} + e^{\alpha y} f(x e^{\alpha y}, w)\dfrac{\partial w}{\partial y} + e^{\beta z} g(x e^{\beta z}, w)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int_{c_1}^{x e^{\alpha y}} \frac{dt}{t[\alpha t f(t,w) + 1]} - \ln|x|, \quad u_2 = \int_{c_2}^{x e^{\beta z}} \frac{dt}{t[\beta t g(t,w) + 1]} - \ln|x|,$$

and c_1 and c_2 are arbitrary numbers.

17. $\dfrac{\partial w}{\partial x} + e^{\alpha y} f(x e^{\alpha y}, w)\dfrac{\partial w}{\partial y} + e^{-\beta x} g(z e^{\beta x}, w)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int_{c_1}^{x e^{\alpha y}} \frac{dt}{t[\alpha t f(t,w) + 1]} - \ln|x|, \quad u_2 = \int_{c_2}^{z e^{\beta x}} \frac{dt}{g(t,w) + \beta t} - x,$$

and c_1 and c_2 are arbitrary numbers.

18. $\dfrac{\partial w}{\partial x} + y f(e^{\alpha x} y^k, w)\dfrac{\partial w}{\partial y} + z g(e^{\beta x} z^n, w)\dfrac{\partial w}{\partial z} = 0.$

General solution: $\Phi(u_1, u_2, w) = 0$, where

$$u_1 = \int_{c_1}^{e^{\alpha x} y^k} \frac{dt}{t[\alpha + k f(t,w)]} - x, \quad u_2 = \int_{c_2}^{e^{\beta x} z^n} \frac{dt}{t[\beta + n g(t,w)]} - x,$$

and c_1 and c_2 are arbitrary numbers.

19. $\dfrac{\partial w}{\partial x}+e^{-\alpha x}f(ye^{\alpha x},w)\dfrac{\partial w}{\partial y}+e^{-\beta x}g(ze^{\beta x},w)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1=\int_{c_1}^{ye^{\alpha x}}\frac{dt}{f(t,w)+\alpha t}-x,\quad u_2=\int_{c_2}^{ze^{\beta x}}\frac{dt}{g(t,w)+\beta t}-x,$$

and c_1 and c_2 are arbitrary numbers.

20. $\dfrac{\partial w}{\partial x}+\dfrac{1}{x}f(x^ke^{\alpha y},w)\dfrac{\partial w}{\partial y}+\dfrac{1}{x}g(x^ne^{\beta z},w)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1=\int_{c_1}^{x^ke^{\alpha y}}\frac{dt}{t[k+\alpha f(t,w)]}-\ln|x|,\quad u_2=\int_{c_2}^{x^ne^{\beta z}}\frac{dt}{t[n+\beta g(t,w)]}-\ln|x|,$$

and c_1 and c_2 are arbitrary numbers.

21. $f(x,w)\dfrac{\partial w}{\partial x}+g(x,w)\dfrac{\partial w}{\partial y}+h(x,w)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1=y-\int\frac{g(x,w)}{f(x,w)}\,dx,\quad u_2=z-\int\frac{h(x,w)}{f(x,w)}\,dx.$$

In the integration, w is treated as a parameter.

22. $f(x,w)\dfrac{\partial w}{\partial x}+g(y,w)\dfrac{\partial w}{\partial y}+h(z,w)\dfrac{\partial w}{\partial z}=0.$

General solution: $\Phi(u_1,u_2,w)=0$, where

$$u_1=\int\frac{dy}{g(y,w)}-\int\frac{dx}{f(x,w)},\quad u_2=\int\frac{dz}{h(z,w)}-\int\frac{dx}{f(x,w)}.$$

In the integration, w is treated as a parameter.

1.3.2. Equations with Arbitrary Number of Independent Variables

1. $\dfrac{\partial w}{\partial x_1}+f_2(x_1,w)\dfrac{\partial w}{\partial x_2}+\cdots+f_n(x_1,w)\dfrac{\partial w}{\partial x_n}=g(x_1).$

General solution: $\Phi(u_1,u_2,\dots,u_n)=0$, where

$$u_1=w-G(x_1),\qquad G(x)=\int g(x)\,dx,$$

$$u_k=x_k-\int_a^{x_1}f_k\big(t,G(t)-G(x_1)+w\big)\,dt,\qquad k=2,\dots,n.$$

In the integration, w is treated as a parameter; a is an arbitrary number.

2. $\dfrac{\partial w}{\partial x_1} + f_2(x_1, w)\dfrac{\partial w}{\partial x_2} + \cdots + f_n(x_1, w)\dfrac{\partial w}{\partial x_n} = g(w).$

General solution: $\Phi(u_1, u_2, \ldots, u_n) = 0$, where

$$u_1 = x_1 - G(w), \qquad G(w) = \int \frac{dw}{g(w)},$$

$$u_k = x_k - \int_a^w \frac{f_k\big(G(t) - G(w) + x_1,\ t\big)}{g(t)}\,dt, \qquad k = 2, \ldots, n.$$

In the integration, w is treated as a parameter; a is an arbitrary number.

3. $\dfrac{\partial w}{\partial x_1} + f_2(x_1, w)\dfrac{\partial w}{\partial x_2} + \cdots + f_n(x_1, w)\dfrac{\partial w}{\partial x_n} = g(x_1)h(w).$

The substitution $x = \displaystyle\int g(x_1)\,dx_1$ leads to an equation of the form 1.3.2.2. The substitution $\overline{w} = \displaystyle\int \frac{dw}{h(w)}$ leads to an equation of the form 1.3.2.1 for $\overline{w}$.

4. $\dfrac{\partial w}{\partial x_1} + f_2(x_1, w)g_2(x_2, w)\dfrac{\partial w}{\partial x_2} + \cdots + f_n(x_1, w)g_n(x_n, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_k = \int f_k(x_1, w)\,dx_1 - \int \frac{dx_k}{g_k(x_k, w)}, \qquad k = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

5. $\dfrac{\partial w}{\partial x_1} + \big[f_2(x_1, w) + x_2 g_2(x_1, w)\big]\dfrac{\partial w}{\partial x_2} + \cdots + \big[f_n(x_1, w) + x_n g_n(x_1, w)\big]\dfrac{\partial w}{\partial x_n} = 0.$

General solution: $\Phi(w, u_2, \ldots, u_{n-1}) = 0$, where

$$u_k = x_k G_k(x_1, w) - \int f_k(x_1, w) G_k(x_1, w)\,dx_1, \quad G_k(x_1, w) = \exp\left[-\int g_k(x_1, w)\,dx_1\right].$$

In the integration, w is treated as a parameter; $k = 2, \ldots, n$.

6. $\dfrac{\partial w}{\partial x_1} + \big[x_2^{k_2} f_2(x_1, w) + x_2 g_2(x_1, w)\big]\dfrac{\partial w}{\partial x_2} + \cdots + \big[x_n^{k_n} f_n(x_1, w) + x_n g_n(x_1, w)\big]\dfrac{\partial w}{\partial x_n} = 0.$

The transformation $z_1 = x_1$, $z_2 = x_2^{1-k_2}$, $\ldots$, $z_n = x_n^{1-k_n}$ leads to an equation of the form 1.3.2.5:

$$\frac{\partial w}{\partial z_1} + (1-k_2)\big[f_2(z_1, w) + z_2 g_2(z_1, w)\big]\frac{\partial w}{\partial z_2} + \cdots + (1-k_n)\big[f_n(z_1, w) + z_n g_n(z_1, w)\big]\frac{\partial w}{\partial z_n} = 0.$$

7. $$\frac{\partial w}{\partial x_1} + \big[e^{a_2x_2}f_2(x_1, w) + g_2(x_1, w)\big]\frac{\partial w}{\partial x_2} + \cdots + \big[e^{a_nx_n}f_n(x_1, w) + g_n(x_1, w)\big]\frac{\partial w}{\partial x_n} = 0.$$

The transformation $z_1 = x_1$, $z_2 = e^{-a_2x_2}$, …, $z_n = e^{-a_nx_n}$ leads to an equation of the form 1.3.2.5:

$$\frac{\partial w}{\partial z_1} - a_2\big[f_2(z_1, w) + z_2g_2(z_1, w)\big]\frac{\partial w}{\partial z_2} - \cdots - a_n\big[f_n(z_1, w) + z_ng_n(z_1, w)\big]\frac{\partial w}{\partial z_n} = 0.$$

8. $$\frac{\partial w}{\partial x_1} + f_2(a_2x_1 + b_2x_2, w)\frac{\partial w}{\partial x_2} + \cdots + f_n(a_nx_1 + b_nx_n, w)\frac{\partial w}{\partial x_n} = 0.$$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_k = x_1 - \int_{c_k}^{z_k} \frac{dt}{a_k + b_kf_k(t, w)}, \qquad z_k = a_kx_1 + b_kx_k \quad \text{if } b_k \neq 0,$$
$$u_k = x_k - \frac{1}{a_k}\int_{c_k}^{z_k} f_k(t, w)\,dt, \qquad z_k = a_kx_1 \quad \text{if } b_k = 0,$$

and the c_k are arbitrary numbers. In the integration, w is treated as a parameter.

9. $$\frac{\partial w}{\partial x_1} + f_2\left(\frac{x_2}{x_1}, w\right)\frac{\partial w}{\partial x_2} + \cdots + f_n\left(\frac{x_n}{x_1}, w\right)\frac{\partial w}{\partial x_n} = 0.$$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_k = \int_{c_k}^{z_k} \frac{dt}{f_k(t, w) - t} - \ln|x_1|, \qquad z_k = \frac{x_k}{x_1}, \quad c_k \text{ are any}, \quad k = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

10. $$\frac{\partial w}{\partial x_1} + \frac{x_2}{x_1}f_2(x_1^{k_2}x_2^{m_2}, w)\frac{\partial w}{\partial x_2} + \cdots + \frac{x_n}{x_1}f_n(x_1^{k_n}x_n^{m_n}, w)\frac{\partial w}{\partial x_n} = 0.$$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_i = \int_{c_i}^{z_i} \frac{dt}{t[m_if_i(t, w) + k_i]} - \ln|x_1|, \qquad z_i = x_1^{k_i}x_i^{m_i}, \quad c_i \text{ are any}, \quad i = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

11. $$\frac{\partial w}{\partial x_1} + e^{a_2x_2}f_2(x_1e^{a_2x_2}, w)\frac{\partial w}{\partial x_2} + \cdots + e^{a_nx_n}f_n(x_1e^{a_nx_n}, w)\frac{\partial w}{\partial x_n} = 0.$$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_k = \int_{c_k}^{z_k} \frac{dt}{t[a_kt f_k(t, w) + 1]} - \ln|x_1|, \qquad z_k = x_1e^{a_kx_k}, \quad c_k \text{ are any}, \quad k = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

12. $\dfrac{\partial w}{\partial x_1} + x_2 f_2(e^{a_2x_1}x_2^{k_2}, w)\dfrac{\partial w}{\partial x_2} + \cdots + x_n f_n(e^{a_nx_1}x_n^{k_n}, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_i = \int_{c_i}^{z_i} \frac{dt}{t[a_i + k_i f_i(t, w)]} - x_1, \qquad z_i = e^{a_i x_1} x_i^{k_i}, \quad c_i \text{ are any}, \quad i = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

13. $\dfrac{\partial w}{\partial x_1} + e^{-a_2x_1} f_2(x_2 e^{a_2x_1}, w)\dfrac{\partial w}{\partial x_2} + \cdots + e^{-a_nx_1} f_n(x_n e^{a_nx_1}, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_k = \int_{c_k}^{z_k} \frac{dt}{f_k(t, w) + a_k t} - x_1, \qquad z_k = x_k e^{a_k x_1}, \quad c_k \text{ are any}, \quad k = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

14. $\dfrac{\partial w}{\partial x_1} + \dfrac{1}{x_1} f_2(x_1^{k_2} e^{a_2x_2}, w)\dfrac{\partial w}{\partial x_2} + \cdots + \dfrac{1}{x_1} f_n(x_1^{k_n} e^{a_nx_n}, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution: $\Phi(w, u_2, \ldots, u_n) = 0$, where

$$u_m = \int_{c_m}^{z_m} \frac{dt}{t[k_m + a_m f_m(t, w)]} - \ln|x_1|, \qquad z_m = x_1^{k_m} e^{a_m x_m}, \quad c_m \text{ are any}, \quad m = 2, \ldots, n.$$

In the integration, w is treated as a parameter.

15. $f_1(x_1, w)\dfrac{\partial w}{\partial x_1} + f_2(x_1, w)\dfrac{\partial w}{\partial x_2} + \cdots + f_n(x_1, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution:

$$\Phi(u_2, u_3, \ldots, u_n, w) = 0, \qquad \text{where} \quad u_k = x_k - \int \frac{f_k(x_1, w)}{f_1(x_1, w)}\, dx_1.$$

In the integration, w is treated as a parameter; $k = 2, \ldots, n$.

16. $f_1(x_1, w)\dfrac{\partial w}{\partial x_1} + f_2(x_2, w)\dfrac{\partial w}{\partial x_2} + \cdots + f_n(x_n, w)\dfrac{\partial w}{\partial x_n} = 0.$

General solution:

$$\Phi(u_1, u_2, \ldots, u_{n-1}, w) = 0, \qquad \text{where} \quad u_k = \int \frac{dx_k}{f_k(x_k, w)} - \int \frac{dx_n}{f_n(x_n, w)}.$$

In the integration, w is treated as a parameter; $k = 1, \ldots, n-1$.

Chapter 2
First-Order Equations with Two Independent Variables Quadratic in Derivatives

Preliminary remarks. For brevity, we only give complete integrals for most equations. Subsection 24.1.1 presents formulas that allow one to derive the general integral using a complete integral.

2.1. Equations Containing Arbitrary Parameters

2.1.1. Equations of the Form $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(x, y, w)$

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = a.$

Complete integral: $w = aC_1x + \dfrac{y}{C_1} + C_2.$

⊙ Reference: E. Kamke (1965).

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = axy + b.$

Complete integral: $w = y\sqrt{ax^2 + C_1} + b\displaystyle\int \frac{dx}{\sqrt{ax^2 + C_1}} + C_2.$

⊙ Reference: E. Kamke (1965).

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = axy + bx + cy + s.$

Complete integral: $w = y\sqrt{ax^2 + 2cx + C_1} + \displaystyle\int \frac{(bx + s)\,dx}{\sqrt{ax^2 + 2cx + C_1}} + C_2.$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ax^k y + bx^n.$

Complete integral:

$$w = \varphi(x)y + b\int \frac{x^n}{\varphi(x)}\,dx + C_1, \qquad \varphi(x) = \pm\left(\frac{2a}{k+1}x^{k+1} + C_2\right)^{1/2}.$$

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ax^n y^k + bx^m y^{2k+1}.$

This is a special case of equation 2.2.1.3 with $f(x) = ax^n$ and $g(x) = bx^m$.

6. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ax^n e^{\lambda y} + bx^m e^{2\lambda y}.$

This is a special case of equation 2.2.1.4 with $f(x) = ax^n$ and $g(x) = bx^m$.

7. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = aw + bx^n.$

Complete integral: $w = (ax + C_1)(y + C_2) + b(ax + C_1)\displaystyle\int \frac{x^n\,dx}{(ax + C_1)^2}.$

8. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = bw^a.$

Complete integral: $$w = \begin{cases} \left[\dfrac{2-a}{2C_1}\left(bC_1^2 x + y + C_2\right)\right]^{\frac{2}{2-a}} & \text{if } a \neq 2, \\ C_2 \exp\left(bC_1 x + \dfrac{y}{C_1}\right) & \text{if } a = 2. \end{cases}$$

⊙ Reference: E. Kamke (1965).

9. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = Ax^a y^b w^c.$

Complete integral: $$w = \begin{cases} \left[\dfrac{2-c}{2C_1}\left(AC_1^2 u + v + C_2\right)\right]^{\frac{2}{2-c}} & \text{if } c \neq 2, \\ C_2 \exp\left(AC_1 u + \dfrac{v}{C_1}\right) & \text{if } c = 2, \end{cases}$$

where $$u = \begin{cases} \dfrac{x^{a+1}}{a+1} & \text{if } a \neq -1, \\ \ln|x| & \text{if } a = -1, \end{cases} \qquad v = \begin{cases} \dfrac{y^{b+1}}{b+1} & \text{if } b \neq -1, \\ \ln|y| & \text{if } b = -1. \end{cases}$$

⊙ Reference: E. Kamke (1965).

10. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = \dfrac{bw^2}{w + a}.$

Complete integral in implicit form: $x + bC_1^2 y + C_2 = C_1\left(2\sqrt{w + a} + u\right)$, where

$$u = \begin{cases} \sqrt{a}\ln\dfrac{\sqrt{w+a} - \sqrt{a}}{\sqrt{w+a} + \sqrt{a}} & \text{if } a > 0, \\ 2\sqrt{-a}\arctan\sqrt{\dfrac{w+a}{-a}} & \text{if } a < 0. \end{cases}$$

11. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = (xy + a)w^k.$

The substitution $z = \dfrac{2}{2-k}w^{\frac{2-k}{2}}$ leads to an equation of the form 2.1.1.2: $\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} = xy + a.$

12. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = aw^{k+1} + bx^n w^{2k}, \qquad k \neq 1.$

The substitution $u = w^{1-k}$ leads to an equation of the form 2.1.1.7:

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = a(1-k)^2 u + b(1-k)^2 x^n.$$

13. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ae^{\lambda w} + bx^n e^{2\lambda w}.$

The substitution $u = e^{-\lambda w}$ leads to an equation of the form 2.1.1.7: $\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} = a\lambda^2 u + b\lambda^2 x^n.$

2.1.2. Equations of the Form $f(x,y,w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x,y,w)\dfrac{\partial w}{\partial x} = h(x,y,w)$

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x} = bw.$

Complete integral: $w = (by + C_1)\Big(x + C_2 - \dfrac{a}{b}\ln|by + C_1|\Big).$

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + (x + a)\dfrac{\partial w}{\partial x} + bx + c = 0.$

Complete integral: $w = C_1 y - bx + (ab + bC_1 - c)\ln|x + a + C_1| + C_2.$

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a(x + b)\dfrac{\partial w}{\partial x} + cy + s = 0.$

Complete integral for $a \neq 0$:

$$w = -(x+b)(ay + C_1) + \frac{1}{a^2}\big[acy + (as - cC_1)\ln|ay + C_1|\big] + C_2.$$

Complete integrals for $a = 0$ and $c \neq 0$:

$$w = -C_1(x+b) + \frac{y(cy + 2s)}{2C_1} + C_2, \qquad w = \pm\frac{cy + s}{c}\sqrt{C_1^2 - 2c(x+b)} + C_2.$$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + axy\dfrac{\partial w}{\partial x} + b = 0, \qquad a \neq 0.$

Complete integral:

$$w = -\frac{1}{2}axy^2 - \frac{1}{2}aC_1 x + \frac{b}{a}\varphi(y) + C_2, \quad \varphi(y) = \begin{cases} \dfrac{2}{\sqrt{C_1}}\arctan\dfrac{y}{\sqrt{C_1}} & \text{if } C_1 > 0, \\ \dfrac{1}{\sqrt{-C_1}}\ln\left|\dfrac{y - \sqrt{-C_1}}{y + \sqrt{-C_1}}\right| & \text{if } C_1 < 0. \end{cases}$$

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + axy\dfrac{\partial w}{\partial x} + by^k = 0.$

Complete integral: $w = -\dfrac{1}{2}axy^2 - \dfrac{1}{2}C_1x + 2b\displaystyle\int \frac{y^k\,dy}{ay^2 + C_1} + C_2.$

6. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + ax^k\dfrac{\partial w}{\partial x} + bx^n = 0.$

Complete integral: $w = C_1y - b\displaystyle\int \frac{x^n\,dx}{ax^k + C_1} + C_2.$

7. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + ay^k\dfrac{\partial w}{\partial x} + bx + c = 0.$

Complete integral: $w = \pm\Big(x + \dfrac{c}{b}\Big)\sqrt{C_1 - 2by} - \dfrac{a}{k+1}y^{k+1} + C_2.$

8. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + axy^k\dfrac{\partial w}{\partial x} + by^n = 0.$

Complete integral:

$$w = -x\varphi(y) + b\int \frac{y^n\,dy}{\varphi(y)} + C_2, \quad \varphi(y) = \frac{a}{k+1}y^{k+1} + C_1.$$

9. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + ae^x\dfrac{\partial w}{\partial x} + b = 0, \quad b \neq 0.$

Complete integral: $w = \dfrac{b}{C_1}\big(\ln|C_1 + ae^x| - x\big) + C_1y + C_2.$

10. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\sin x\dfrac{\partial w}{\partial x} + b = 0, \quad b \neq 0.$

Complete integral: $w = -\dfrac{2b}{\sqrt{C_1^2 - a^2}}\arctan\dfrac{a + C_1\tan(x/2)}{\sqrt{C_1^2 - a^2}} + C_1y + C_2.$

11. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\sin(\mu y)\dfrac{\partial w}{\partial x} = b\sin(\lambda x).$

Complete integral: $w = \dfrac{bC_1}{\lambda}\cos(\lambda x) + \dfrac{a}{\mu}\cos(\mu y) - \dfrac{y}{C_1} + C_2.$

12. $a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + (bx + cy + sw)\dfrac{\partial w}{\partial x} = 1, \quad s \neq 0.$

The substitution $z = bx + cy + sw - ac/s$ leads to an equation of the form 2.1.3.14: $(z_x - b)(az_y + sz) = s^2$.

⊙ Reference: E. Kamke (1965).

13. $a y \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial x} + b = 0.$

Complete integral: $w = \dfrac{y^{-1/a}}{2C_1}\left(2C_1^2 x - b y^{2/a} + C_2\right).$

Another complete integral for $a = -2$: $w^2 = 2(x - C_1)(C_2 y - b).$

14. $a y \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + (bx + cy + sw)\dfrac{\partial w}{\partial x} = 1, \quad a \neq 0.$

Complete integral:

$$w = \begin{cases} -\dfrac{c}{a+s}y + C_1 y^{-s/a} + \varphi(x) & \text{if } a + s \neq 0; \\ \dfrac{c}{s}y(\ln y - 1) + C_1 y + \varphi(x) & \text{if } a + s = 0. \end{cases}$$

Here $\varphi(x) = \dfrac{1}{s}\sqrt{2sx + C_2}$ for $b = 0$. If $b \neq 0$, the function φ is defined implicitly by $x = C_2 e^{b\varphi} - sb^{-2}(b\varphi + 1)$.

15. $w\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x} + bw = 0.$

Complete integral in implicit form:

$$\pm a \ln\left|\frac{a + \sqrt{a^2 - 4bC_1 w^2}}{w}\right| \mp \sqrt{a^2 - 4bC_1 w^2} - a \ln w = 2b(x + C_1 y + C_2).$$

2.1.3. Equations of the Form $f(x,y,w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x,y,w)\dfrac{\partial w}{\partial x} + h(x,y,w)\dfrac{\partial w}{\partial y} = s(x,y,w)$

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y}.$

Complete integrals:

$$w = C_1 + C_2 x + C_3 y, \qquad \text{where} \quad C_2 C_3 = aC_2 + bC_3,$$
$$w = bx + ay \pm \sqrt{C_1 x + 4abxy} + C_2,$$
$$w = bx + ay \pm \sqrt{(2ay + C_1)(2bx + C_1)} + C_2.$$

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = 0, \qquad ab \neq 0.$

Complete integral: $w = -\dfrac{bC_1 x^2}{2(a + C_1)} + C_1 y + C_2.$

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ax\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y}.$

Four different complete integrals:

$$w = \frac{a}{2C_1}(x + C_1 y)^2 + C_2, \quad w = ax\big(y + \sqrt{y^2 + C_1}\,\big) + C_2, \quad w = ay\big(x + \sqrt{x^2 + C_1}\,\big) + C_2,$$

$$w = a\left(xy + \frac{1}{2}\int \sqrt{u^2 + C_1}\,du + \frac{1}{2}\int \sqrt{v^2 + C_1}\,dv\right) + C_2, \qquad u = x + y, \quad v = x - y.$$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = 0, \qquad ab \neq 0.$

1°. Complete integral:

$$w = \frac{C_1 - 1}{2}\left(bx^2 - \frac{a}{C_1}y^2\right) + C_2.$$

2°. If $ab > 0$, then numbers α and β can be defined such that $a = \pm\alpha^2$ and $b = \pm\beta^2$, with both upper signs or both lower signs being taken at a time. If we set $u = \beta x + \alpha y$ and $v = \beta x - \alpha y$, then we arrive at an equation of the form 2.2.6.10: $w_u^2 \pm uw_u = w_v^2 \pm vw_v$. Thus, we obtain another complete integral.

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + ax + by + c.$

Complete integral:

$$w = xy \mp (x + b)\sqrt{y^2 + 2ay + C_1} \pm (ab - c)\ln\left|y + \sqrt{y^2 + 2ay + C_1} + a\right| + C_2.$$

6. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + axy + b.$

Complete integral for $a \neq -1$:

$$w = xy \pm x\sqrt{(a+1)y^2 + C_1} \pm \frac{b}{\sqrt{a+1}}\ln\left|\sqrt{a+1}\,y + \sqrt{(a+1)y^2 + C_1}\right| + C_2.$$

Complete integral for $a = -1$:

$$w = xy - C_1 x + \frac{b}{C_1}y + C_2.$$

7. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + axy + bx.$

Complete integrals for $a \neq -1$:

$$w = x\left[y \pm \sqrt{(a+1)y^2 + 2by + C_1}\,\right] + C_2,$$

$$w = xy \pm y\sqrt{(a+1)x^2 + C_1} \pm \frac{b}{a+1}\sqrt{(a+1)x^2 + C_1} + C_2.$$

Complete integrals for $a = -1$:

$$w = \frac{x}{C_1}(y - bx) - \frac{y}{2} + C_2, \qquad w = xy \pm x\sqrt{2by + C_1} + C_2.$$

8. $\left(\dfrac{\partial w}{\partial x}+ax\right)\left(\dfrac{\partial w}{\partial y}+by\right)=c.$

Complete integral: $w=-\frac{1}{2}x\left(ax+C_1\pm\sqrt{C_1^2-4c}\right)-\frac{1}{2}y\left(by+C_1\mp\sqrt{C_1^2-4c}\right)+C_2.$

9. $\left(\dfrac{\partial w}{\partial x}+ay\right)\left(\dfrac{\partial w}{\partial y}+ax\right)=b.$

Complete integral: $w=-axy+C_1x+\dfrac{b}{C_1}y+C_2.$

10. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}+cx^2+sy^2=0, \qquad 4cs=a^2b^2.$

The transformation

$$z=w+\frac{bx^2}{2}+\frac{ay^2}{2}, \quad u=\frac{\lambda x-\lambda^{-1}y}{\sqrt{2}}, \quad v=\frac{\lambda x+\lambda^{-1}y}{\sqrt{2}}, \qquad \lambda=\sqrt{\frac{2c}{ab}},$$

leads to an equation of the form 2.1.6.4: $z_u^2-z_v^2=2abu^2$.

11. $a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=w.$

Complete integral: $w=C_1x+C_2y+aC_1C_2.$

12. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+b(ax+by)\dfrac{\partial w}{\partial x}+a(ax+by)\dfrac{\partial w}{\partial y}+abc=0.$

Complete integral:

$$w=-\frac{(ax+by)^2}{2}-aC_1x+bC_1y\pm\frac{1}{2}(ax+by)\sqrt{(ax+by)^2-c+C_1^2}$$
$$\pm\frac{1}{2}(C_1^2-c)\ln\left|ax+by+\sqrt{(ax+by)^2-c+C_1^2}\,\right|+C_2.$$

13. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+b(ax+by)\dfrac{\partial w}{\partial x}-a(ax+by)\dfrac{\partial w}{\partial y}+abc=0.$

Complete integral: $w=C_1(by-ax)\mp\dfrac{1}{3C_1}(2aC_1x+2bC_1y-c+C_1^2)^{3/2}+C_2.$

14. $\left(\dfrac{\partial w}{\partial x}+a\right)\left(\dfrac{\partial w}{\partial y}+bw\right)=c.$

The transformation $x=u$, $y=\ln v$, $w=cz+v-au$ leads to an equation of the form 2.1.2.14:

$$cv\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}+[-abu+(b+1)v+bcz]\frac{\partial z}{\partial u}=1.$$

⊙ Reference: E. Kamke (1965).

15. $2x\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} - w\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x} = 0.$

Complete integral in implicit form: $(aC_1w + C_1^2x^2)^{3/2} - \frac{3}{2}aC_1^2(xw + ay) = C_1^3x^3 + C_2.$

16. $y\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} - w\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = 0, \qquad a \neq 0.$

Complete integrals:

$$w = C_1y \pm \sqrt{2aC_1x + C_2}, \quad w = \frac{x - C_2}{C_1}\left(y \pm \sqrt{y^2 + aC_1}\right), \quad w = \frac{C_2y}{ax + C_1} - \frac{ax + C_1}{2y}.$$

17. $(ax + by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = 0.$

Complete integral:

$$w = C_1(ax - by) - \ln|ax + by| \pm \sqrt{1 + C_1^2(ax + by)^2} \mp \operatorname{arctanh}\frac{1}{\sqrt{1 + C_1^2(ax + by)^2}} + C_2.$$

18. $(x^2 + a)\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y} - 1\right) + xy^2\dfrac{\partial w}{\partial y} = 0.$

Complete integral:

$$w = \pm\frac{C_1^2}{2}\ln(x^2 + a) + C_2 + \begin{cases} C_1 \arctan\dfrac{y}{C_1} & \text{for the upper sign,} \\ C_1 \operatorname{arctanh}\dfrac{y}{C_1} & \text{for the lower sign.} \end{cases}$$

19. $c(ax - by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b(ax - cw)\dfrac{\partial w}{\partial x} + a(cw - by)\dfrac{\partial w}{\partial y} = 0.$

General integral in parametric form:

$$x = \frac{1}{a}\left[\Phi(z) + \frac{z}{u}(2u - v + 1)\Phi'_z(z)\right], \quad y = \frac{1}{b}\left[\Phi(z) + \frac{z}{v}(2v - u + 1)\Phi'_z(z)\right],$$
$$w = \frac{1}{c}\left[xu + yv - (u + v - 1)\Phi(z)\right], \qquad z = \frac{1}{uv}(u + v - 1)^3.$$

20. $w\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = a\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right).$

Complete integral: $w^2 = \dfrac{a}{C_1C_2}(C_1x + C_2y)^2 + C_3.$

21. $aw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + x^2y\dfrac{\partial w}{\partial x} + xy^2\dfrac{\partial w}{\partial y} = xyw.$

Complete integral: $w^2 = C_1x^2 + C_2y^2 + aC_1C_2.$

22. $(a + b)w\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial x} + ax\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w^2 = C_3 - \dfrac{aC_2 + bC_1}{(a + b)C_1C_2}(C_1x^2 + C_2y^2).$

2.1.4. Equations of the Form $\dfrac{\partial w}{\partial x} + f(x,y,w)\left(\dfrac{\partial w}{\partial y}\right)^2 = g(x,y,w)$

▷ *Equations of this type are encountered in mechanics, where x plays the role of time and y plays the role of the spatial coordinate.*

1. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = by.$

This equation governs the free vertical drop of a point body near the Earth's surface (y is the vertical coordinate measured downward, x time, $m = \frac{1}{2a}$ the mass of the body, and $g = 2ab$ the gravitational acceleration).

Complete integral: $w = -C_1 x \pm \dfrac{2a}{3b}\left(\dfrac{by + C_1}{a}\right)^{3/2} + C_2.$

⊙ Reference: A. P. Markeev (1990).

2. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + by^2 = 0.$

This equation governs free oscillations of a point body of mass $m = 1/(2a)$ in an elastic field with elastic coefficient $k = 2b$ (x is time and y is the displacement from the equilibrium).

Complete integral: $w = -C_1 x + C_2 \pm \int \sqrt{\dfrac{C_1 - by^2}{a}}\, dx + C_2.$

⊙ Reference: F. R. Gantmakher (1966).

3. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^k + cy^n.$

This is a special case of equation 2.2.2.1 with $f(x) = bx^k$ and $g(y) = cy^n$. Complete integral:

$$w = -C_1 x + \frac{b}{k+1}x^{k+1} + \int \sqrt{\frac{cy^n + C_1}{a}}\, dy + C_2.$$

4. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^k y + cx^n.$

This is a special case of equation 2.2.2.2 with $f(x) = bx^k$ and $g(x) = cx^n$. Complete integral:

$$w = y\left(\frac{bx^{k+1}}{k+1} + C_1\right) + \frac{cx^{n+1}}{n+1} - \frac{ab^2x^{2k+3}}{(k+1)^2(2k+3)} - \frac{2abC_1x^{k+2}}{(k+1)(k+2)} - aC_1^2x + C_2.$$

5. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^k y^2 + cx^n y + sx^m.$

This is a special case of equation 2.2.2.3 with $f(x) = bx^k$, $g(x) = cx^n$, and $h(x) = sx^m$.

6. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=k(bx+cy)^n.$

This is a special case of equation 2.2.2.4 with $f(\xi)=k\xi^n$. Complete integral:

$$w=C_1x+C_2-\frac{b}{2ac^2}(bx+cy)\pm\frac{1}{2ac^2}\int\sqrt{4ac^2k\xi^n+b^2-4ac^2C_1}\,d\xi,\qquad \xi=bx+cy.$$

7. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bx^k+ce^{\beta y}.$

This is a special case of equation 2.2.2.1 with $f(x)=bx^k$ and $g(y)=ce^{\beta y}$. Complete integral:

$$w=-C_1x+\frac{b}{k+1}x^{k+1}+\int\sqrt{\frac{ce^{\beta y}+C_1}{a}}\,dy+C_2.$$

8. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}+ce^{\beta y}.$

This is a special case of equation 2.2.2.1 with $f(x)=be^{\lambda x}$ and $g(y)=ce^{\beta y}$. Complete integral:

$$w=-C_1x+\frac{b}{\lambda}e^{\lambda x}+\int\sqrt{\frac{ce^{\beta y}+C_1}{a}}\,dy+C_2.$$

9. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}+cy^k.$

This is a special case of equation 2.2.2.1 with $f(x)=be^{\lambda x}$ and $g(y)=cy^k$. Complete integral:

$$w=-C_1x+\frac{b}{\lambda}e^{\lambda x}+\int\sqrt{\frac{cy^k+C_1}{a}}\,dy+C_2.$$

10. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bx^ky+ce^{\beta x}.$

This is a special case of equation 2.2.2.2 with $f(x)=bx^k$ and $g(x)=ce^{\beta x}$.

11. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}y+ce^{\beta x}.$

This is a special case of equation 2.2.2.2 with $f(x)=be^{\lambda x}$ and $g(x)=ce^{\beta x}$. Complete integral:

$$w=y\left(\frac{b}{\lambda}e^{\lambda x}+C_1\right)+\frac{c}{\beta}e^{\beta x}-\frac{ab^2}{2\lambda^3}e^{2\lambda x}-\frac{2abC_1}{\lambda^2}e^{\lambda x}-aC_1^2x+C_2.$$

12. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}y+cx^k.$

This is a special case of equation 2.2.2.2 with $f(x)=be^{\lambda x}$ and $g(x)=cx^k$.

13. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=ke^{bx+cy}.$

This is a special case of equation 2.2.2.4 with $f(\xi)=ke^{\xi}$. Complete integral:

$$w=C_1x+C_2-\frac{b}{2ac^2}(bx+cy)\pm\frac{1}{2ac^2}\int\sqrt{4ac^2ke^{\xi}+b^2-4ac^2C_1}\,d\xi,\qquad \xi=bx+cy.$$

14. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bx^kw+cx^n.$

This is a special case of equation 2.2.2.7 with $f(x)=bx^k$ and $g(x)=cx^n$. Complete integral:

$$w=F(x)(C_1+C_2y)+F(x)\int\left[cx^n-aC_2^2F^2(x)\right]\frac{dx}{F(x)},\qquad F(x)=\exp\left(\frac{b}{k+1}x^{k+1}\right).$$

15. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bx^kw+ce^{\beta x}.$

This is a special case of equation 2.2.2.7 with $f(x)=bx^k$ and $g(x)=ce^{\beta x}$. Complete integral:

$$w=F(x)(C_1+C_2y)+F(x)\int\left[ce^{\beta x}-aC_2^2F^2(x)\right]\frac{dx}{F(x)},\qquad F(x)=\exp\left(\frac{b}{k+1}x^{k+1}\right).$$

16. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}w+cx^k.$

This is a special case of equation 2.2.2.7 with $f(x)=be^{\lambda x}$ and $g(x)=cx^k$. Complete integral:

$$w=F(x)(C_1+C_2y)+F(x)\int\left[cx^k-aC_2^2F^2(x)\right]\frac{dx}{F(x)},\qquad F(x)=\exp\left(\frac{b}{\lambda}e^{\lambda x}\right).$$

17. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=be^{\lambda x}w+ce^{\beta x}.$

This is a special case of equation 2.2.2.7 with $f(x)=be^{\lambda x}$ and $g(x)=ce^{\beta x}$. Complete integral:

$$w=F(x)(C_1+C_2y)+F(x)\int\left[ce^{\beta x}-aC_2^2F^2(x)\right]\frac{dx}{F(x)},\qquad F(x)=\exp\left(\frac{b}{\lambda}e^{\lambda x}\right).$$

18. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bw^2+cx^kw+sx^n.$

This is a special case of equation 2.2.2.8 with $f(x)=cx^k$ and $g(x)=sx^n$.

19. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2=bw^2+cx^kw+se^{\beta x}.$

This is a special case of equation 2.2.2.8 with $f(x)=cx^k$ and $g(x)=e^{\beta x}$.

20. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^2 + ce^{\lambda x}w + se^{\beta x}.$

This is a special case of equation 2.2.2.8 with $f(x) = ce^{\lambda x}$ and $g(x) = e^{\beta x}$.

21. $\dfrac{\partial w}{\partial x} + ax^k\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^n w + c_1 x^m y + c_2 x^l.$

This is a special case of equation 2.2.2.9 with $f(x) = ax^k$, $g(x) = bx^n$, $h(x) = c_1 x^m$, and $s(x) = c_2 x^l$.

22. $\dfrac{\partial w}{\partial x} + ay^k\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^2 + c_1 x^n w + c_2 x^m.$

This is a special case of equation 2.2.2.10 with $f(y) = ay^k$, $g(x) = c_1 x^n$, and $h(x) = c_2 x^m$.

23. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 = be^{\beta x}w + c_1 e^{\gamma x}y + c_2 e^{\mu x}.$

This is a special case of equation 2.2.2.9 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\beta x}$, $h(x) = c_1 e^{\gamma x}$, and $s(x) = c_2 e^{\mu x}$.

24. $\dfrac{\partial w}{\partial x} + ae^{\lambda y}\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^2 + c_1 e^{\beta x}w + c_2 e^{\gamma x}.$

This is a special case of equation 2.2.2.10 with $f(y) = ae^{\lambda y}$, $g(x) = c_1 e^{\beta x}$, and $h(x) = c_2 e^{\gamma x}$.

25. $\dfrac{\partial w}{\partial x} + \left(ax^k y + bx^n\right)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

This is a special case of equation 2.2.2.11 with $f(x) = ax^k$ and $g(x) = bx^n$. Complete integral:

$$w = \varphi(x)y - b\int x^n \varphi^2(x)\,dx + C_1, \qquad \varphi(x) = \left(\frac{a}{k+1}x^{k+1} + C_2\right)^{-1}.$$

26. $\dfrac{\partial w}{\partial x} + \left(ax^k y + bx^n\right)\left(\dfrac{\partial w}{\partial y}\right)^2 = cx^m w + s_0 x^l.$

This is a special case of equation 2.2.2.12 with $f(x) = ax^k$, $g(x) = bx^n$, $h(x) = cx^m$, $r(x) = 0$, and $s(x) = s_0 x^l$.

27. $\dfrac{\partial w}{\partial x} + \left(ax^k y + be^{\lambda x}\right)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

This is a special case of equation 2.2.2.11 with $f(x) = ax^k$ and $g(x) = be^{\lambda x}$. Complete integral:

$$w = \varphi(x)y - b\int e^{\lambda x}\varphi^2(x)\,dx + C_1, \qquad \varphi(x) = \left(\frac{a}{k+1}x^{k+1} + C_2\right)^{-1}.$$

28. $\dfrac{\partial w}{\partial x}+\left(ax^ny^k+bx^my^{2k-2}\right)\left(\dfrac{\partial w}{\partial y}\right)^2=0.$

This is a special case of equation 2.2.2.13 with $f(x)=ax^n$ and $g(x)=bx^m$.

29. $\dfrac{\partial w}{\partial x}+\left(ax^ny^k+bx^my^{2k-2}\right)\left(\dfrac{\partial w}{\partial y}\right)^2=cx^qw+s_0x^l.$

This is a special case of equation 2.2.2.14 with $f(x)=ax^n$, $g(x)=bx^m$, $h(x)=cx^q$, and $s(x)=s_0x^l$.

30. $\dfrac{\partial w}{\partial x}+A(ax+by)^k\left(\dfrac{\partial w}{\partial y}\right)^2=B(ax+by)^n.$

This is a special case of equation 2.2.2.17 with $f(\xi)=A\xi^k$ and $g(\xi)=B\xi^n$. Complete integral:

$$w=C_1x+C_2-\frac{a\xi^{1-k}}{2b^2A(1-k)}\pm\frac{a}{2b^2A}\int\xi^{-k}\sqrt{4b^2AB\xi^{k+n}-4b^2AC_1\xi^k+a^2}\,d\xi,\quad \xi=ax+by.$$

31. $\dfrac{\partial w}{\partial x}+A(ax+by)^k\left(\dfrac{\partial w}{\partial y}\right)^2=Be^{ax+by}.$

This is a special case of equation 2.2.2.17 with $f(\xi)=A\xi^k$ and $g(\xi)=Be^{\xi}$.

32. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda x}y+be^{\beta x}\right)\left(\dfrac{\partial w}{\partial y}\right)^2=0.$

This is a special case of equation 2.2.2.11 with $f(x)=ae^{\lambda x}$ and $g(x)=be^{\beta x}$. Complete integral:

$$w=\varphi(x)y-b\int e^{\beta x}\varphi^2(x)\,dx+C_1,\qquad \varphi(x)=\left(\frac{a}{\lambda}e^{\lambda x}+C_2\right)^{-1}.$$

33. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda x}y+bx^k\right)\left(\dfrac{\partial w}{\partial y}\right)^2=0.$

This is a special case of equation 2.2.2.11 with $f(x)=ae^{\lambda x}$ and $g(x)=bx^k$. Complete integral:

$$w=\varphi(x)y-b\int x^k\varphi^2(x)\,dx+C_1,\qquad \varphi(x)=\left(\frac{a}{\lambda}e^{\lambda x}+C_2\right)^{-1}.$$

34. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda x}y+be^{\beta x}\right)\left(\dfrac{\partial w}{\partial y}\right)^2=ce^{\gamma x}w+s_0e^{\mu x}.$

This is a special case of equation 2.2.2.12 with $f(x)=ae^{\lambda x}$, $g(x)=be^{\beta x}$, $h(x)=ce^{\gamma x}$, $r(x)=0$, and $s(x)=s_0e^{\mu x}$.

35. $\dfrac{\partial w}{\partial x}+\left(ae^{\beta x}+be^{\lambda y+\mu x}\right)\left(\dfrac{\partial w}{\partial y}\right)^2=0.$

This is a special case of equation 2.2.2.15 with $f(x)=ae^{\beta x}$ and $g(x)=be^{\mu x}$,

36. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda x}y^{k}+be^{\beta x}y^{2k-2}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=0.$

This is a special case of equation 2.2.2.13 with $f(x)=ae^{\lambda x}$ and $g(x)=be^{\beta x}$.

37. $\dfrac{\partial w}{\partial x}+\left(ae^{\lambda x}y^{k}+be^{\beta x}y^{2k-2}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=ce^{\gamma x}w+s_0e^{\mu x}.$

This is a special case of equation 2.2.2.14 with $f(x)=ae^{\lambda x}$, $g(x)=be^{\beta x}$, $h(x)=ce^{\gamma x}$, and $s(x)=s_0e^{\mu x}$.

38. $\dfrac{\partial w}{\partial x}+\left(ax^{k}+bx^{n}e^{\lambda y}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=0.$

This is a special case of equation 2.2.2.15 with $f(x)=ax^k$ and $g(x)=bx^n$.

39. $\dfrac{\partial w}{\partial x}+\left(ae^{\beta x}+be^{\lambda y+\mu x}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=cw+s_0e^{\gamma x}.$

This is a special case of equation 2.2.2.16 with $f(x)=ae^{\beta x}$, $g(x)=be^{\mu x}$, $h(x)=c$, and $s(x)=s_0e^{\gamma x}$.

40. $\dfrac{\partial w}{\partial x}+Ae^{ax+by}\left(\dfrac{\partial w}{\partial y}\right)^{2}=Be^{\lambda(ax+by)}.$

This is a special case of equation 2.2.2.17 with $f(\xi)=Ae^{\xi}$ and $g(\xi)=Be^{\lambda\xi}$. Complete integral:

$$w=C_1x+C_2+\frac{a}{2b^2A}e^{-\xi}\pm\frac{1}{2b^2A}\int e^{-\xi}\sqrt{4b^2ABe^{(\lambda+1)\xi}-4b^2AC_1e^{\xi}+a^2}\,d\xi,\qquad \xi=ax+by.$$

41. $\dfrac{\partial w}{\partial x}+Ae^{ax+by}\left(\dfrac{\partial w}{\partial y}\right)^{2}=B(ax+by)^{k}.$

This is a special case of equation 2.2.2.17 with $f(\xi)=Ae^{\xi}$ and $g(\xi)=B\xi^k$.

42. $\dfrac{\partial w}{\partial x}+\left(a_2x^{k_2}w+a_1x^{k_1}y+a_0x^{k_0}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=b_2x^{n_2}w+b_1x^{n_1}y+b_0x^{n_0}.$

This is a special case of equation 2.2.2.19 with $f_m(x)=a_mx^{k_m}$ and $g_m(x)=b_mx^{n_m}$ ($m=0,\,1,\,2$).

43. $\dfrac{\partial w}{\partial x}+\left(a_2e^{\lambda_2x}w+a_1e^{\lambda_1x}y+a_0e^{\lambda_0x}\right)\left(\dfrac{\partial w}{\partial y}\right)^{2}=b_2e^{\beta_2x}w+b_1e^{\beta_1x}y+b_0e^{\beta_0x}.$

This is a special case of equation 2.2.2.19 with $f_m(x)=a_me^{\lambda_mx}$ and $g_m(x)=b_me^{\beta_mx}$ ($m=0,\,1,\,2$).

44. $\dfrac{\partial w}{\partial x}-aw^{k}\left(\dfrac{\partial w}{\partial y}\right)^{2}=0.$

Complete integral: $w=\left[\dfrac{k+1}{a}\left(C_1^2x+C_1y+C_2\right)\right]^{\frac{1}{k+1}}.$

45. $\dfrac{\partial w}{\partial x} - aw^k\left(\dfrac{\partial w}{\partial y}\right)^2 - \left(bx^n y + cx^m\right)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 2.2.2.24 with $f(w) = aw^k$, $g(x) = bx^n$, and $h(x) = cx^m$.

46. $\dfrac{\partial w}{\partial x} + ax^n w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^m w.$

This is a special case of equation 2.2.2.20 with $f(x) = ax^n$ and $g(x) = bx^m$.

47. $\dfrac{\partial w}{\partial x} + ay^k w^n\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^m.$

This is a special case of equation 2.2.2.25 with $f(y) = ay^k$, $g(w) = w^n$, and $h(w) = bw^m$.

48. $\dfrac{\partial w}{\partial x} + by^n w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = aw^{k+2} + cx^m w.$

This is a special case of equation 2.2.2.21 with $f(y) = by^n$ and $g(x) = cx^m$.

49. $\dfrac{\partial w}{\partial x} + A(w + ax + by)^k\left(\dfrac{\partial w}{\partial y}\right)^2 = B(w + ax + by)^n.$

This is a special case of equation 2.2.2.27 with $f(u) = Au^k$ and $g(u) = Bu^n$.

50. $\dfrac{\partial w}{\partial x} + ae^{\lambda x} w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = be^{\beta x} w.$

This is a special case of equation 2.2.2.20 with $f(x) = ae^{\lambda x}$ and $g(x) = be^{\beta x}$.

51. $\dfrac{\partial w}{\partial x} + ae^{\lambda y} w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = be^{\beta w}.$

This is a special case of equation 2.2.2.25 with $f(y) = ae^{\lambda y}$, $g(w) = w^k$, and $h(w) = be^{\beta w}$.

52. $\dfrac{\partial w}{\partial x} + be^{\lambda y} w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = aw^{k+2} + ce^{\beta x} w.$

This is a special case of equation 2.2.2.21 with $f(y) = be^{\lambda y}$ and $g(x) = ce^{\beta x}$.

53. $\dfrac{\partial w}{\partial x} - ae^{\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \dfrac{1}{\beta}\ln\left[\beta(aC_1^2 x + C_1 y + C_2)\right].$

54. $\dfrac{\partial w}{\partial x} - ax^k e^{\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 - bx^n = 0.$

This is a special case of equation 2.2.2.22 with $f(x) = ax^k$ and $g(x) = bx^n$.

55. $\dfrac{\partial w}{\partial x} - ae^{\lambda x+\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 - be^{\gamma x} = 0.$

This is a special case of equation 2.2.2.22 with $f(x) = ae^{\lambda x}$ and $g(x) = be^{\gamma x}$.

56. $\dfrac{\partial w}{\partial x} - ae^{\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 - \left(be^{\lambda x}y + ce^{\mu x}\right)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 2.2.2.24 with $f(w) = ae^{\beta w}$, $g(x) = be^{\lambda x}$, and $h(x) = ce^{\mu x}$.

57. $\dfrac{\partial w}{\partial x} + ae^{\lambda y+\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 = be^{\gamma w}.$

This is a special case of equation 2.2.2.25 with $f(y) = ae^{\lambda y}$, $g(w) = e^{\beta w}$, and $h(w) = be^{\gamma w}$.

58. $\dfrac{\partial w}{\partial x} - ax^k y^n w^m\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \left[(m+1)\left(\dfrac{aC_1^2}{k+1}x^{k+1} + \dfrac{2C_1}{2-n}y^{\frac{2-n}{2}} + C_2\right)\right]^{\frac{1}{m+1}}.$

59. $\dfrac{\partial w}{\partial x} - ax^k e^{\lambda y} w^m\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \left[(m+1)\left(\dfrac{aC_1^2}{k+1}x^{k+1} - \dfrac{2C_1}{\lambda}e^{-\frac{1}{2}\lambda y} + C_2\right)\right]^{\frac{1}{m+1}}.$

60. $\dfrac{\partial w}{\partial x} - ae^{\lambda x} y^n w^m\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \left[(m+1)\left(\dfrac{aC_1^2}{\lambda}e^{\lambda x} + \dfrac{2C_1}{2-n}y^{\frac{2-n}{2}} + C_2\right)\right]^{\frac{1}{m+1}}.$

61. $\dfrac{\partial w}{\partial x} - ax^k y^n e^{\lambda w}\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \dfrac{1}{\lambda}\ln\left(\dfrac{aC_1^2\lambda}{k+1}x^{k+1} + \dfrac{2C_1\lambda}{2-n}y^{\frac{2-n}{2}} + C_2\right).$

62. $\dfrac{\partial w}{\partial x} - ae^{\lambda x+\beta y+\gamma w}\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral: $w = \dfrac{1}{\gamma}\ln\left(\dfrac{aC_1^2\gamma}{\lambda}e^{\lambda x} - \dfrac{2C_1\gamma}{\beta}e^{-\frac{1}{2}\beta y} + C_2\right).$

63. $\dfrac{\partial w}{\partial x} + Ae^{w+ax+by}\left(\dfrac{\partial w}{\partial y}\right)^2 = Be^{\lambda(w+ax+by)}.$

This is a special case of equation 2.2.2.27 with $f(u) = Ae^u$ and $g(u) = Be^{\lambda u}$.

2.1.5. Equations of the Form

$$\frac{\partial w}{\partial x} + f(x,y,w)\left(\frac{\partial w}{\partial x}\right)^2 + g(x,y,w)\frac{\partial w}{\partial y} = h(x,y,w)$$

▷ *Equations of this type are encountered in mechanics, where x plays the role of time and y plays the role of the spatial coordinate.*

1. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = cx^k + sy^n.$

Complete integral: $w = -C_1 x + C_2 + \dfrac{c}{k+1}x^{k+1} - \dfrac{b}{2a}y \pm \dfrac{1}{2a}\displaystyle\int \sqrt{4asy^n + b^2 + 4aC_1}\,dy.$

2. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = cx^k + se^{\beta y}.$

Complete integral: $w = -C_1 x + C_2 + \dfrac{c}{k+1}x^{k+1} - \dfrac{b}{2a}y \pm \dfrac{1}{2a}\displaystyle\int \sqrt{4ase^{\beta y} + b^2 + 4aC_1}\,dy.$

3. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = ce^{\lambda x} + sy^k.$

Complete integral: $w = -C_1 x + C_2 + \dfrac{c}{\lambda}e^{\lambda x} - \dfrac{b}{2a}y \pm \dfrac{1}{2a}\displaystyle\int \sqrt{4asy^k + b^2 + 4aC_1}\,dy.$

4. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = ce^{\lambda x} + se^{\beta y}.$

Complete integral: $w = -C_1 x + C_2 + \dfrac{c}{\lambda}e^{\lambda x} - \dfrac{b}{2a}y \pm \dfrac{1}{2a}\displaystyle\int \sqrt{4ase^{\beta y} + b^2 + 4aC_1}\,dy.$

5. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = cx^k y + sx^n.$

Complete integral:

$$w = y\left(\frac{cx^{k+1}}{k+1} + C_1\right) + \frac{sx^{n+1}}{n+1} - \frac{ac^2x^{2k+3}}{(k+1)^2(2k+3)} - \frac{c(2aC_1+b)x^{k+2}}{(k+1)(k+2)} - (aC_1^2 + bC_1)x + C_2.$$

6. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = cx^k y + se^{\beta x}.$

This is a special case of equation 2.2.3.2 with $f(x) = cx^k$ and $g(x) = se^{\beta x}$.

7. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = ce^{\lambda x} y + sx^k.$

This is a special case of equation 2.2.3.2 with $f(x) = ce^{\lambda x}$ and $g(x) = sx^k$.

8. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=ce^{\lambda x}y+se^{\beta x}.$

Complete integral:

$$w=y\left(\frac{c}{\lambda}e^{\lambda x}+C_1\right)+\frac{s}{\beta}e^{\beta x}-\frac{ac^2}{2\lambda^3}e^{2\lambda x}-\frac{c(2aC_1+b)}{\lambda^2}e^{\lambda x}-(aC_1^2+bC_1)x+C_2.$$

9. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=cx^kw+sx^n.$

Complete integral:

$$w=(C_1y+C_2)F(x)+F(x)\int\left[sx^n-aC_1^2F^2(x)-bC_1F(x)\right]\frac{dx}{F(x)},\quad F(x)=\exp\left(\frac{c}{k+1}x^{k+1}\right).$$

10. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=cx^kw+se^{\beta x}.$

Complete integral:

$$w=(C_1y+C_2)F(x)+F(x)\int\left[se^{\beta x}-aC_1^2F^2(x)-bC_1F(x)\right]\frac{dx}{F(x)},\quad F(x)=\exp\left(\frac{c}{k+1}x^{k+1}\right).$$

11. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=ce^{\lambda x}w+sx^k.$

Complete integral:

$$w=(C_1y+C_2)F(x)+F(x)\int\left[sx^k-aC_1^2F^2(x)-bC_1F(x)\right]\frac{dx}{F(x)},\quad F(x)=\exp\left(\frac{c}{\lambda}e^{\lambda x}\right).$$

12. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=ce^{\lambda x}w+se^{\beta x}.$

Complete integral:

$$w=(C_1y+C_2)F(x)+F(x)\int\left[se^{\beta x}-aC_1^2F^2(x)-bC_1F(x)\right]\frac{dx}{F(x)},\quad F(x)=\exp\left(\frac{c}{\lambda}e^{\lambda x}\right).$$

13. $\dfrac{\partial w}{\partial x}+ax^k\left(\dfrac{\partial w}{\partial y}\right)^2+bx^n\dfrac{\partial w}{\partial y}=cx^m.$

Complete integral: $w=C_1y-\dfrac{aC_1^2}{k+1}x^{k+1}-\dfrac{bC_1}{n+1}x^{n+1}+\dfrac{c}{m+1}x^{m+1}+C_2.$

14. $\dfrac{\partial w}{\partial x}+ax^k\left(\dfrac{\partial w}{\partial y}\right)^2+bx^n\dfrac{\partial w}{\partial y}=ce^{\beta x}.$

Complete integral: $w=C_1y-\dfrac{aC_1^2}{k+1}x^{k+1}-\dfrac{bC_1}{n+1}x^{n+1}+\dfrac{c}{\beta}e^{\beta x}+C_2.$

15. $\dfrac{\partial w}{\partial x} + ax^k\left(\dfrac{\partial w}{\partial y}\right)^2 + bx^n\dfrac{\partial w}{\partial y} = cx^m w + \beta x^\nu y + \gamma x^\mu.$

This is a special case of equation 2.2.3.5 with $f(x)=ax^k$, $g(x)=bx^n$, $h(x)=cx^m$, $p(x)=\beta x^\nu$, and $s(x)=\gamma x^\mu$.

16. $\dfrac{\partial w}{\partial x}+a_2x^{k_2}\left(\dfrac{\partial w}{\partial y}\right)^2+(a_1x^{k_1}y+a_0x^{k_0})\dfrac{\partial w}{\partial y}=bx^nw+c_2x^{m_2}y^2+c_1x^{m_1}y+c_0x^{m_0}.$

This is a special case of equation 2.2.3.6.

17. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + bx^k\dfrac{\partial w}{\partial y} = ce^{\beta x}.$

Complete integral: $w = C_1y - \dfrac{aC_1^2}{\lambda}e^{\lambda x} - \dfrac{bC_1}{k+1}x^{k+1} + \dfrac{c}{\beta}e^{\beta x} + C_2.$

18. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta x}\dfrac{\partial w}{\partial y} = ce^{\gamma x}.$

Complete integral: $w = C_1y - \dfrac{aC_1^2}{\lambda}e^{\lambda x} - \dfrac{bC_1}{\beta}e^{\beta x} + \dfrac{c}{\gamma}e^{\gamma x} + C_2.$

19. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta x}\dfrac{\partial w}{\partial y} = ce^{\gamma x}w + ke^{\nu x}y + me^{\mu x}.$

This is a special case of equation 2.2.3.5 with $f(x) = ae^{\lambda x}$, $g(x) = be^{\beta x}$, $h(x) = ce^{\gamma x}$, $p(x) = ke^{\nu x}$, and $s(x) = me^{\mu x}$.

20. $\dfrac{\partial w}{\partial x} + ay^k\left(\dfrac{\partial w}{\partial y}\right)^2 + by^n\dfrac{\partial w}{\partial y} = cy^m.$

Complete integral:

$$w = -C_1x + C_2 - \frac{by^{n-k+1}}{2a(n-k+1)} \pm \frac{1}{2a}\int y^{-k}\sqrt{b^2y^{2n} + 4acy^{k+m} + 4aC_1y^k}\,dy.$$

21. $\dfrac{\partial w}{\partial x} + ay^k\left(\dfrac{\partial w}{\partial y}\right)^2 + by^n\dfrac{\partial w}{\partial y} = ce^{\beta y}.$

This is a special case of equation 2.2.3.7 with $f(y) = ay^k$, $g(y) = by^n$, and $h(y) = ce^{\beta y}$.

22. $\dfrac{\partial w}{\partial x} + ay^k\left(\dfrac{\partial w}{\partial y}\right)^2 + by^n\dfrac{\partial w}{\partial y} = cx^m + sy^l.$

Complete integral:

$$w = -C_1x + C_2 + \frac{cx^{m+1}}{m+1} - \frac{by^{n-k+1}}{2a(n-k+1)} \pm \frac{1}{2a}\int y^{-k}\sqrt{b^2y^{2n} + 4asy^{k+l} + 4aC_1y^k}\,dy.$$

23. $\dfrac{\partial w}{\partial x} + ae^{\lambda y}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta y}\dfrac{\partial w}{\partial y} = ce^{\gamma y}.$

Complete integral:

$$w = -C_1 x + C_2 - \frac{be^{(\beta-\lambda)y}}{2a(\beta-\lambda)} \pm \frac{1}{2a}\int e^{-\lambda y}\sqrt{b^2e^{2\beta y} + 4ace^{(\lambda+\gamma)y} + 4C_1 ae^{\lambda y}}\,dy.$$

24. $\dfrac{\partial w}{\partial x} + ae^{\lambda y}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta y}\dfrac{\partial w}{\partial y} = ce^{\gamma x} + se^{\mu y}.$

Complete integral:

$$w = -C_1 x + C_2 + \frac{c}{\gamma}e^{\gamma x} - \frac{be^{(\beta-\lambda)y}}{2a(\beta-\lambda)} \pm \frac{1}{2a}\int e^{-\lambda y}\sqrt{b^2e^{2\beta y} + 4ase^{(\lambda+\mu)y} + 4C_1 ae^{\lambda y}}\,dy.$$

25. $\dfrac{\partial w}{\partial x} + A(ax+by)^k\left(\dfrac{\partial w}{\partial y}\right)^2 + B(ax+by)^n\dfrac{\partial w}{\partial y} = C(ax+by)^m.$

This is a special case of equation 2.2.3.9 with $f(z) = Az^k$, $g(z) = Bz^n$, and $h(z) = Cz^m$.

26. $\dfrac{\partial w}{\partial x} + Ae^{ax+by}\left(\dfrac{\partial w}{\partial y}\right)^2 + Be^{\lambda(ax+by)}\dfrac{\partial w}{\partial y} = Ce^{\beta(ax+by)}.$

This is a special case of equation 2.2.3.9 with $f(z) = Ae^z$, $g(z) = Be^{\lambda z}$, and $h(z) = Ce^{\beta z}$.

27. $\dfrac{\partial w}{\partial x} + aw^k\left(\dfrac{\partial w}{\partial y}\right)^2 + bw^n\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $C_1 x + C_2 y + \displaystyle\int \frac{aC_2^2 w^k\,dw}{C_1 + bC_2 w^n} = C_3.$

28. $\dfrac{\partial w}{\partial x} + aw^k\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\lambda w}\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $C_1 x + C_2 y + \displaystyle\int \frac{aC_2^2 w^k\,dw}{C_1 + bC_2 e^{\lambda w}} = C_3.$

29. $\dfrac{\partial w}{\partial x} + aw^k\left(\dfrac{\partial w}{\partial y}\right)^2 + bw^n\dfrac{\partial w}{\partial y} = cw^m.$

This is a special case of equation 2.2.3.12 with $f(w) = aw^k$, $g(w) = bw^n$, and $h(w) = cw^m$.

30. $\dfrac{\partial w}{\partial x} - aw^k\left(\dfrac{\partial w}{\partial y}\right)^2 - (bx^n y + cx^m)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 2.2.3.13 with $f(w) = aw^k$, $g(x) = bx^n$, and $h(x) = cx^m$.

31. $\dfrac{\partial w}{\partial x} + ae^{\lambda w}\left(\dfrac{\partial w}{\partial y}\right)^2 + bw^n\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $C_1x + C_2y + \displaystyle\int \frac{aC_2^2e^{\lambda w}\,dw}{C_1 + bC_2w^n} = C_3.$

32. $\dfrac{\partial w}{\partial x} + ae^{\lambda w}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta w}\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $C_1x + C_2y + \displaystyle\int \frac{aC_2^2e^{\lambda w}\,dw}{C_1 + bC_2e^{\beta w}} = C_3.$

33. $\dfrac{\partial w}{\partial x} + ae^{\lambda w}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta w}\dfrac{\partial w}{\partial y} = ce^{\gamma w}.$

This is a special case of equation 2.2.3.12 with $f(w) = ae^{\lambda w}$, $g(w) = be^{\beta w}$, and $h(w) = ce^{\gamma w}$.

34. $\dfrac{\partial w}{\partial x} - ae^{\lambda w}\left(\dfrac{\partial w}{\partial y}\right)^2 - (be^{\beta x}y + ce^{\gamma x})\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 2.2.3.13 with $f(w) = ae^{\lambda w}$, $g(x) = be^{\beta x}$, and $h(x) = ce^{\gamma x}$.

35. $\dfrac{\partial w}{\partial x} + A(w + ax + by)^k\left(\dfrac{\partial w}{\partial y}\right)^2 + B(w + ax + by)^n\dfrac{\partial w}{\partial y} = C(w + ax + by)^m.$

This is a special case of equation 2.2.3.14 with $f(z) = Az^k$, $g(z) = Bz^n$, and $h(z) = Cz^m$.

36. $\dfrac{\partial w}{\partial x} + Ae^{w+ax+by}\left(\dfrac{\partial w}{\partial y}\right)^2 + Be^{\lambda(w+ax+by)}\dfrac{\partial w}{\partial y} = Ce^{\beta(w+ax+by)}.$

This is a special case of equation 2.2.3.14 with $f(z) = Ae^z$, $g(z) = Be^{\lambda z}$, and $h(z) = Ce^{\beta z}$.

2.1.6. Equations of the Form

$$f(x, y, w)\left(\frac{\partial w}{\partial x}\right)^2 + g(x, y, w)\left(\frac{\partial w}{\partial y}\right)^2 = h(x, y, w)$$

▷ *Equations of this type are encountered in mechanics, geometric optics, and differential geometry. In particular, the equation* $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x, y)$ *describes the two-dimensional wave front in the process of propagation of light in an inhomogeneous medium with variable refractive index* $f(x, y)$.

1. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 = c.$

For $a = b$, this is a *differential equation of light rays*.
Complete integral: $w = C_1x + C_2y + C_3$, where $aC_1^2 + bC_2^2 = c$.
An alternative form of the complete integral: $\dfrac{w^2}{c} = \dfrac{(x - C_1)^2}{a} + \dfrac{(y - C_2)^2}{b}.$

2. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=a-2by.$

This equation governs parabolic motion of a point mass in vacuum (the coordinate x is measured along the Earth's surface, the coordinate y is measured vertically upward from the Earth's surface, and a is the gravitational acceleration).

Complete integral: $w=C_1x\pm\frac{1}{3b}(a-C_1^2-2by)^{3/2}+C_2$.

⊙ Reference: P. Appell (1953).

3. $\left(\dfrac{\partial w}{\partial x}\right)^2+a\left(\dfrac{\partial w}{\partial y}\right)^2=bx+cy.$

Complete integral for $b^2+ac^2\neq 0$:

$$w=\frac{bC_1}{b^2+ac^2}(by-acx)-\frac{2}{3}\frac{[(b^2+ac^2)(bx+cy)-ab^2C_1^2]^{3/2}}{(b^2+ac^2)^2}+C_2.$$

Complete integral for $b^2+ac^2=0$:

$$w=\frac{c}{4C_1b^2}(bx+cy)^2-\frac{C_1}{2}y+\frac{C_1b}{2c}x+C_2.$$

Another complete integral for $a=1$, $b=k^2$, and $c=\pm k^2$:

$$w=\tfrac{2}{3}k(x+C_1)^{3/2}\pm\tfrac{2}{3}k(y\mp C_1)^{3/2}+C_2.$$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2+a\left(\dfrac{\partial w}{\partial y}\right)^2=b_1x^2+b_2y^2+c_1x+c_2y+s.$

A separable equation. Complete integral:

$$w=\pm\int\sqrt{b_1x^2+c_1x+s-C_1}\,dx\pm\int\sqrt{\frac{b_2y^2+c_2y+C_1}{a}}\,dy+C_2.$$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=ax^2+bxy+cy^2+s.$

Let the numbers A, B, k_1, and k_2 satisfy the following algebraic system of equations (k_1 or k_2 can be set equal to unity):

$$Ak_1^2+Bk_2^2=a,\quad 2(A-B)k_1k_2=b,\quad Ak_2^2+Bk_1^2=c.$$

Then the transformation $\xi=k_1x+k_2y$, $\eta=k_2x-k_1y$ leads to an equation of the form 2.1.6.4:

$$\left(\frac{\partial w}{\partial\xi}\right)^2+\left(\frac{\partial w}{\partial\eta}\right)^2=\frac{A}{k_1^2+k_2^2}\xi^2+\frac{B}{k_1^2+k_2^2}\eta^2+\frac{s}{k_1^2+k_2^2}.$$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=ax^2+bxy+cy^2+\beta x+\gamma y+s.$

The transformation used for solving equation 2.1.6.5 brings the above equation to a separable equation of the form 2.1.6.4.

7. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bx^k.$

Complete integral: $w = C_1 y + C_2 \pm \int \sqrt{bx^k - aC_1^2}\, dx.$

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = ax^k + by^n + c.$

A separable equation. Complete integral:

$$w = \pm \int \sqrt{ax^k + C_1}\, dx \pm \int \sqrt{by^n + c - C_1}\, dy + C_2.$$

⊙ Reference: E. Kamke (1965).

9. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = \dfrac{a}{\sqrt{x^2 + y^2}} + b.$

This equation arises from the solution of the *two-body problem in celestial mechanics*.

By using the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, one can obtain a separable equation. This equation has the complete integral

$$w = \pm \int \sqrt{b + \frac{a}{r} - \frac{C_1^2}{r^2}}\, dr + C_1\theta + C_2.$$

⊙ References: P. Appell (1953), R. Courant (1962), and E. Kamke (1965).

10. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = a(x^2 + y^2)^k.$

This is a special case of equation 2.2.4.4 with $f(z) = az^k$.

11. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = a(x^2 + y^2)(xy)^k.$

This is a special case of equation 2.2.4.5 with $f(z) = az^k$.

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = a(x^2 + y^2)(x^2 - y^2)^k.$

This is a special case of equation 2.2.4.6 with $f(z) = az^k$.

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = A(ax + by)^k + B(bx - ay)^n + s.$

This is a special case of equation 2.2.4.3 with $f(z) = Az^k$ and $g(u) = Bu^n + s$.

14. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 = cw.$

Complete integral: $w = \dfrac{c}{4(aC_1^2 + bC_2^2)}(C_1 x + C_2 y + C_3)^2.$

15. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=aw^2+b.$

Complete integral in implicit form: $x\cos C_1+y\sin C_1+C_2=\pm\displaystyle\int\frac{dw}{\sqrt{aw^2+b}}$.
Whence, in particular, for $a=1$ and $b=0$ we have $w=\widetilde{C}_2\exp\bigl(x\cos C_1+y\sin C_1\bigr)$.

16. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+1=\dfrac{a^2}{w^2}.$

This equation describes a family of spherical surfaces of radius a with centers at points of the xy-plane. Complete integral in implicit form: $(x-C_1)^2+(y-C_2)^2+w^2=a^2$.

Another complete integral: $\dfrac{(y-C_1x-C_2)^2}{1+C_1^2}+w^2=a^2$.

17. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=aw^k.$

This is a special case of equation 2.2.4.7 with $f(w)=aw^k$.

18. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=\bigl(ax^2+bxy+cy^2+s\bigr)w^k.$

The substitution $u=\dfrac{2}{2-k}w^{\frac{2-k}{2}}$ leads to an equation of the form 2.1.6.5:

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=ax^2+bxy+cy^2+s.$$

19. $\left(\dfrac{\partial w}{\partial x}\right)^2+axy\left(\dfrac{\partial w}{\partial y}\right)^2=b.$

Complete integral: $w=-\dfrac{2\bigl(b-aC_1^2x\bigr)^{3/2}}{3aC_1^2}+2C_1\sqrt{y}+C_2$.

20. $\left(\dfrac{\partial w}{\partial x}\right)^2+A(ax+by)^k\left(\dfrac{\partial w}{\partial y}\right)^2=B(ax+by)^n.$

This is a special case of equation 2.2.4.15 with $f(z)=1$, $g(z)=Az^k$, and $h(z)=Bz^n$.

21. $a\left(\dfrac{\partial w}{\partial x}\right)^2+bw\left(\dfrac{\partial w}{\partial y}\right)^2=c^2.$

Complete integral: $bC_2^2w=-aC_1^2+\left[\dfrac{3bcC_2^2}{2}(C_1x+C_2y+C_3)\right]^{2/3}$.

22. $x\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\left(\dfrac{\partial w}{\partial y}\right)^2 = bx + cy.$

Complete integral:

$$w = \sqrt{x(bx + C_1)} + \frac{C_1}{2\sqrt{b}} \ln\left|2bx + 2\sqrt{bx(bx + C_1)} + C_1\right| + \sqrt{\frac{y(cy - C_1)}{a}} - \frac{C_1}{2\sqrt{ac}} \ln\left|2cy + 2\sqrt{cy(cy - C_1)} - C_1\right| + C_2.$$

23. $ax^2\left(\dfrac{\partial w}{\partial x}\right)^2 + by^2\left(\dfrac{\partial w}{\partial y}\right)^2 = w^c.$

The transformation

$$u = \ln x, \quad v = \ln y, \quad w = \begin{cases} \left(\dfrac{2-c}{2}z\right)^{\frac{2}{2-c}} & \text{if } c \neq 2, \\ e^z & \text{if } c = 2 \end{cases}$$

leads to an equation of the form 2.1.6.1: $az_u^2 + bz_v^2 = 1$.

24. $(x^2 - a^2)\left(\dfrac{\partial w}{\partial x}\right)^2 - (y^2 - a^2)\left(\dfrac{\partial w}{\partial y}\right)^2 = b(x^2 - y^2) + kx.$

This equation arises from the solution of the plane problem of attraction between two identical fixed point bodies (the variables x and y play the role of elliptic coordinates and a is the distance between the bodies).

Complete integral: $w = \pm \int \sqrt{\dfrac{bx^2 + kx + C_1}{x^2 - a^2}}\, dx \pm \int \sqrt{\dfrac{by^2 + C_1}{y^2 - a^2}}\, dy + C_2.$

⊙ Reference: V. I. Arnold (1980).

25. $(x + a_1)(x + a_2)\left(\dfrac{\partial w}{\partial x}\right)^2 - (y + a_1)(y + a_2)\left(\dfrac{\partial w}{\partial y}\right)^2 = k\sqrt{x + a_1} + b\sqrt{y + a_2} + c(x - y).$

Complete integral:

$$w = \int \sqrt{\frac{C_1 + cx + k\sqrt{x + a_1}}{(x + a_1)(x + a_2)}}\, dx + \int \sqrt{\frac{C_1 + cy - b\sqrt{y + a_1}}{(y + a_1)(y + a_2)}}\, dy + C_2.$$

For $2k = m_1 + m_2$, $2b = m_1 - m_2$, $x = \lambda_1$, and $y = \lambda_2$, this equation governs the motion of a point body of unit mass in the xy-plane under the action of the gravitational forces produced by two point masses m_1 and m_2 located at the points ($x = \pm 1$, $y = 0$) in elliptic coordinates (λ_1, λ_2).

⊙ Reference: E. Kamke (1965).

26. $$4y(a-x)(b-x)(c-x)\left(\frac{\partial w}{\partial x}\right)^2-4x(a-y)(b-y)(c-y)\left(\frac{\partial w}{\partial y}\right)^2=xy(x-y).$$

This equation is encountered in searching for geodesic lines on an ellipsoid with semiaxes a, b, and c. Complete integral:

$$w=\frac{1}{2}\int\sqrt{\frac{x(x+C_1)}{F(x)}}\,dx+\frac{1}{2}\int\sqrt{\frac{y(y+C_1)}{F(y)}}\,dy+C_2,\qquad F(t)=(a-t)(b-t)(c-t).$$

⊙ Reference: E. Kamke (1965).

27. $$ax^k\left(\frac{\partial w}{\partial x}\right)^2+by^n\left(\frac{\partial w}{\partial y}\right)^2=cx^m+sy^l+\beta.$$

A separable equation. Complete integral:

$$w=\pm\int\sqrt{\frac{cx^m+C_1}{ax^k}}\,dx\pm\int\sqrt{\frac{sy^l+\beta-C_1}{by^n}}\,dy+C_2.$$

The signs before each integral can chosen independently of each other.

28. $$(\sin x)^2\left[\left(\frac{\partial w}{\partial x}\right)^2-a\right]+b\left(\frac{\partial w}{\partial y}\right)^2=0.$$

This equation arises in introducing orthogonal geodesic parametric lines on a unit sphere. Complete integral:

$$w=C_1y+C_2\pm\int\sqrt{a-\frac{bC_1^2}{(\sin x)^2}}\,dx.$$

⊙ Reference: E. Kamke (1965).

29. $$A(ax+by)^k\left(\frac{\partial w}{\partial x}\right)^2+B(ax+by)^n\left(\frac{\partial w}{\partial y}\right)^2=C(ax+by)^m.$$

This is a special case of equation 2.2.4.15 with $f(z)=Az^k$, $g(z)=Bz^n$, and $h(z)=Cz^m$.

30. $$A(w+ax+by)^k\left(\frac{\partial w}{\partial x}\right)^2+B(w+ax+by)^n\left(\frac{\partial w}{\partial y}\right)^2=C(w+ax+by)^m.$$

This is a special case of equation 2.2.4.16 with $f(z)=Az^k$, $g(z)=Bz^n$, and $h(z)=Cz^m$.

31. $$ax^kw^{n_1}\left(\frac{\partial w}{\partial x}\right)^2+by^mw^{n_2}\left(\frac{\partial w}{\partial y}\right)^2=cw^{n_3}+s.$$

This is a special case of equation 2.2.4.17 with $f_1(x)=ax^k$, $g_1(w)=w^{n_1}$, $f_2(y)=by^m$, $g_2(w)=w^{n_2}$, and $h(w)=cw^{n_3}+s$.

32. $$Ae^{ax+by}\left(\frac{\partial w}{\partial x}\right)^2+Be^{\lambda(ax+by)}\left(\frac{\partial w}{\partial y}\right)^2=Ce^{(ax+by)}.$$

This is a special case of equation 2.2.4.15 with $f(z)=Ae^z$, $g(z)=Be^{\lambda z}$, and $h(z)=Ce^{\beta z}$.

33. $Ae^{w+ax+by}\left(\dfrac{\partial w}{\partial x}\right)^2 + Be^{\lambda(w+ax+by)}\left(\dfrac{\partial w}{\partial y}\right)^2 = Ce^{(w+ax+by)}.$

This is a special case of equation 2.2.4.16 with $f(z) = Ae^z$, $g(z) = Be^{\lambda z}$, and $h(z) = Ce^{\beta z}$.

34. $ae^{\lambda_1 x+\beta_1 w}\left(\dfrac{\partial w}{\partial x}\right)^2 + be^{\lambda_2 y+\beta_2 w}\left(\dfrac{\partial w}{\partial y}\right)^2 = ce^{\gamma w} + s.$

This is a special case of equation 2.2.4.17 with $f_1(x) = ae^{\lambda_1 x}$, $g_1(w) = e^{\beta_1 w}$, $f_2(y) = be^{\lambda_2 y}$, $g_2(w) = e^{\beta_2 w}$, and $h(w) = ce^{\gamma w} + s$.

2.1.7. Equations of the Form

$$f(x,y)\left(\frac{\partial w}{\partial x}\right)^2 + g(x,y)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = h(x,y,w)$$

1. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bx + cy = 0.$

1°. Complete integral for $b \neq 0$:

$$w = \frac{3ab^2x + b(ac-2b)y + aC_1(b+ac)}{3a^2b^2}\sqrt{a(aC_1-2by)} + C_2.$$

2°. Complete integral for $b \neq -ac$:

$$w = \frac{abC_1x - (ac+2b)C_1y}{2b(b+ac)} - \frac{\left[(aC_1)^2 - 4b(b+ac)(bx+cy)\right]^{-3/2}}{12b^2(b+ac)^2} + C_2.$$

3°. Complete integral for $b = 0$:

$$w = C_1x - \frac{y}{2aC_1}(cy + 2C_1^2) + C_2.$$

4°. Complete integral for $b = -ac$:

$$w = \frac{c^2(2y-ax)x}{2C_1} - \frac{(c^3y + 2C_1^2)y}{2acC_1} + C_2.$$

2. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bxy = 0.$

Complete integral: $w = \left(x - \dfrac{y}{2a}\right)\sqrt{C_1 - \dfrac{b}{a}y^2} - \dfrac{C_1}{2\sqrt{-ab}}\ln\left|\sqrt{-ab}\,y + \sqrt{a(aC_1 - by^2)}\right| + C_2.$

3. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw = 0.$

Complete integrals:

$$w = \frac{(by + C_1)(y - ax + C_2)}{a^2}, \qquad w = -\frac{b(C_1x + y + C_2)^2}{4C_1(a + C_1)}.$$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw + cy^k = 0.$

Complete integral: $w = \dfrac{by + C_1}{a^2}\left[y - ax + a^2 c\displaystyle\int \frac{y^k\,dy}{(by + C_1)^2} + C_2\right].$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by^k w = 0.$

Complete integral: $w = \dfrac{1}{a^2}(y - ax + C_1)\left(\dfrac{b}{k+1}y^{k+1} + C_2\right).$

6. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = cw^2.$

Complete integral: $w = C_3 \exp\bigl(C_1 x + C_2 y\bigr),$ where $aC_1^2 + bC_1C_2 = c.$

7. $\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + (bx + c)y = 0.$

Complete integral:

$$w = \begin{cases} \dfrac{abx + ac - 2b}{a^2 b}\sqrt{-a(2by + C_1)} + \dfrac{2\sqrt{aC_1}}{a^2}\arctan\sqrt{-1 - \dfrac{2by}{C_1}} + C_2 & \text{if } b \neq 0, \\ C_1 x - \dfrac{cy + C_1^2 \ln|y|}{aC_1} + C_2 & \text{if } b = 0. \end{cases}$$

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bxy + c = 0.$

Complete integral: $w = \dfrac{ax - 2}{a}\sqrt{C_1 - \dfrac{2by}{a}} - \dfrac{2(c + C_1)}{a(-C_1)^{1/2}}\arctan\sqrt{\dfrac{2by}{aC_1} - 1} + C_2.$

9. $\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw = 0.$

Complete integral: $w = \dfrac{(\ln|y| - ax + C_1)(b\ln|y| + C_2)}{a^2}.$

10. $\left(\dfrac{\partial w}{\partial x}\right)^2 + (ay + b)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + cyw = 0.$

Complete integral:

$$w = \begin{cases} \dfrac{\bigl(\ln|ay + b| - ax + C_1\bigr)\bigl(acy - bc\ln|ay + b| + C_2\bigr)}{a^3} & \text{if } a \neq 0, \\ \dfrac{(y - bx + C_1)(cy^2 + C_2)}{2b^2} & \text{if } a = 0. \end{cases}$$

11. $\left(\dfrac{\partial w}{\partial x}\right)^2 + axy\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.$

Complete integral:

$$w = -\frac{1}{4}x\left(aC_1x - \sqrt{a^2C_1^2x^2 - 4b}\right) + C_1\ln|y| - \frac{b}{aC_1}\ln\left|aC_1x + \sqrt{a^2C_1^2x^2 - 4b}\right| + C_2.$$

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{ax}{y}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.$

Complete integral:

$$w = -\frac{1}{4}x\left(aC_1x - \sqrt{a^2C_1^2x^2 - 4b}\right) + \frac{1}{2}C_1y^2 - \frac{b}{aC_1}\ln\left|aC_1x + \sqrt{a^2C_1^2x^2 - 4b}\right| + C_2.$$

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{a}{xy}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.$

Complete integral:

$$w = \frac{aC_1}{2}\ln|x| + \frac{1}{2}\sqrt{a^2{C_1}^2 - 4bx^2} - \frac{aC_1}{2}\operatorname{arctanh}\frac{aC_1}{\sqrt{a^2{C_1}^2 - 4bx^2}} - \frac{C_1}{2}y^2 + C_2.$$

14. $\left(\dfrac{\partial w}{\partial x}\right)^2 + y^a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + x^b = 0.$

Complete integral:

$$w = \begin{cases} -\dfrac{C_1}{2}x + \dfrac{1}{2}\displaystyle\int\sqrt{C_1^2 - 4x^b}\,dx - \dfrac{C_1y^{1-a}}{a-1} + C_2 & \text{if } a \neq 1, \\ -\dfrac{C_1}{2}x + \dfrac{1}{2}\displaystyle\int\sqrt{C_1^2 - 4x^b}\,dx + C_1\ln|y| + C_2 & \text{if } a = 1. \end{cases}$$

15. $\left(\dfrac{\partial w}{\partial x}\right)^2 + A(ax+by)^k\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + B(ax+by)^n = 0.$

This is a special case of equation 2.2.5.14 with $f(z) = az^k$, $g(z) = bz^n$, and $h(w) = 1$.

16. $\left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.$

Complete integral: $w = \dfrac{C_1}{a} + \left(\dfrac{9b}{4aC_1}\right)^{1/3}(y - C_1x + C_2)^{2/3}.$

17. $\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = b^2w^2.$

Complete integral in implicit form:

$$\pm b(C_1x + C_2y + C_3) = R + C_1\ln\frac{R - C_1}{w}, \qquad \text{where} \quad R^2 = aC_1C_2w^2 + C_1^2.$$

18. $\boldsymbol{x\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by^2 = 0.}$

Complete integral: $w = -C_1 x e^{-y/a} + \dfrac{b}{C_1}(y^2 - 2ay + 2a^2)e^{y/a} + C_2.$

19. $\boldsymbol{x\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = b.}$

Complete integral: $w = \dfrac{b}{C_1}y^{1/a} + C_1 x y^{-1/a} + C_2.$

Other complete integrals for $a = 1$:

$$w = \frac{2}{C_1}\sqrt{bC_1(C_1x + y)} + C_2, \qquad w = \sqrt{4bx + C_1^2} + C_1 \ln\left|\frac{\sqrt{4bx + C_1^2} - C_1}{y}\right| + C_2.$$

20. $\boldsymbol{ax\left(\dfrac{\partial w}{\partial x}\right)^2 - (ay + b)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + cy(ay + b)^2 = 0.}$

Complete integral: $w = C_1 x(ay + b) + \dfrac{cy^2}{2C_1} + C_2.$

21. $\boldsymbol{(x + y)\left(\dfrac{\partial w}{\partial x}\right)^2 + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.}$

Complete integral:

$$w = \begin{cases} C_1\left(x - \dfrac{y}{a-1}\right)y^{-1/a} - \dfrac{b}{C_1}y^{1/a} + C_2 & \text{if } a \neq 1, \\ \dfrac{C_1 x}{y} - \dfrac{by}{C_1} - C_1 \ln|y| + C_2 & \text{if } a = 1. \end{cases}$$

22. $\boldsymbol{w\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.}$

Complete integral: $w = \dfrac{a}{C_1} - \left(\dfrac{9b}{4C_1^2}\right)^{1/3}(C_1 x - y + C_2)^{2/3}.$

23. $\boldsymbol{w\left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b = 0.}$

Complete integral: $w = \left[\dfrac{9b}{4C_1(a - C_1)}\right]^{1/3}(y - C_1 x + C_2)^{2/3}.$

24. $\boldsymbol{y(w^2 + a)\left(\dfrac{\partial w}{\partial x}\right)^2 + bxw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = cx^2 y.}$

Complete integral in implicit form:

$$\int \sqrt{C_1(C_1 w^2 + bC_2 w + aC_1)}\, dw = \tfrac{1}{2}\sqrt{c}(C_1 x^2 + C_2 y^2) + C_3.$$

2.1.8. Other Equations

1. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial x} + cx + d = 0.$

Complete integral: $w = -\dfrac{1}{2}aC_1x - \dfrac{1}{12c}\left(a^2C_1^2 - 4d - 4cx\right)^{3/2} - \dfrac{b}{2a}y^2 + C_1y + C_2.$

2. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by^2\dfrac{\partial w}{\partial x} - cx^2 = 0.$

Complete integral:

$$w = -\frac{1}{2}aC_1x + \frac{x}{4}\sqrt{a^2C_1^2 + 4cx^2} - \frac{b}{3a}y^3 + C_1y + \frac{a^2C_1^2}{8\sqrt{c}}\ln\left|2\sqrt{c}\,x + \sqrt{a^2C_1^2 + 4cx^2}\,\right| + C_2.$$

3. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + be^{cw}\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \dfrac{1}{c}\ln\left|\dfrac{C_1(a - C_1)}{bc(C_1x - y + C_2)}\right|.$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial x} + c\dfrac{\partial w}{\partial y} + d = 0.$

Complete integral: $w = C_1x - \dfrac{C_1^2 + bC_1 + d}{aC_1 + c}y + C_2.$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = -\dfrac{b(x + C_1y)^2}{2(1 + aC_1)} + C_2.$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + (bw + c)\dfrac{\partial w}{\partial x} + (pw + q)\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \dfrac{1}{p - bC_1}\left\{\exp\left[\dfrac{p - bC_1}{C_1(a - C_1)}(y - C_1x + C_2)\right] + cC_1 - q\right\}.$

7. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw^2\dfrac{\partial w}{\partial x} + cw\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = c\left\{bC_1 - \exp\left[\dfrac{c}{C_1(a - C_1)}(C_1x - y + C_2)\right]\right\}^{-1}.$

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = \exp\left[\dfrac{b(C_1x - y)^2 + C_2}{2C_1(a - C_1)}\right].$

9. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{y}{x}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + \dfrac{a}{x}\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = -\dfrac{a}{C_1}\ln|C_1x + y| + C_2.$

10. $\left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw^2\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $\dfrac{aw\ln|w| + C_1}{bw} = \dfrac{y}{C_1} - x + C_2.$

11. $\left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = \dfrac{C_1^2 \pm \sqrt{abC_1(y - C_1x)^2 + C_2}}{aC_1}.$

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bw\dfrac{\partial w}{\partial x} + cw^2 + d.$

Complete integral in implicit form:

$$x + C_1y + C_2 = 2(aC_1^2 + 1)\int \frac{dw}{bw \pm \sqrt{(acC_1^2 + b^2)w^2 + adC_1^2}}.$$

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial x} + c\dfrac{\partial w}{\partial y} + d = 0.$

Complete integral:

$$w = Ax + C_1y + C_2, \quad A = -\tfrac{1}{2}b + \tfrac{1}{2}\sqrt{b^2 - 4d - 4cC_1 - 4aC_1^2}.$$

14. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 + cx\dfrac{\partial w}{\partial x} + sy\dfrac{\partial w}{\partial y} = k.$

This is a special case of equation 2.2.6.10.

15. $a\left(\dfrac{\partial w}{\partial x} + cy\right)^2 + b\left(\dfrac{\partial w}{\partial y} + cx\right)^2 = k.$

Complete integral: $w = -cxy + C_1x + C_2y + C_3,$ where $aC_1^2 + bC_2^2 = k.$

16. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 = c\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} - w\right).$

Complete integral: $w = C_1x + C_2y - \dfrac{aC_1^2 + bC_2^2}{c}.$

17. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+(x+a)\dfrac{\partial w}{\partial x}+(y+b)\dfrac{\partial w}{\partial y}=w+c.$

Complete integral: $w=C_1(x+a)+C_2(y+b)+C_1^2+C_2^2-c.$

18. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

Complete integral: $w=C_1x+C_2y+C_3,\quad$ where $\quad C_1^2+C_2^2=aC_1C_2.$

19. $x\left(\dfrac{\partial w}{\partial x}\right)^2+ay\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=0.$

Complete integral:

$$w=C_1(y-ax)+\frac{bC_1}{2}\ln\left|\frac{y}{x}\right|\pm C_1\sqrt{b^2-4axy}\mp|b|C_1\operatorname{arctanh}\frac{\sqrt{b^2-4axy}}{|b|}+C_2.$$

20. $\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}\right)^2-a^2\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+1\right]=0.$

Complete integral:

$$w=a\operatorname{arccosh}\frac{x\cos C_1+y\sin C_1}{a}+C_2.$$

By passing to the polar coordinates $x=\rho\cos\theta$ and $y=\rho\sin\theta$, one arrives at an alternative form of the complete integral:

$$w=a\ln\sqrt{\frac{\sigma+1}{\sigma-1}}-C_1\arctan\frac{a\sigma}{C_1}+C_1\theta+C_2,\qquad\text{where}\quad\sigma^2=\frac{\rho^2+C_1^2}{\rho^2-a^2}.$$

21. $\left(ay\dfrac{\partial w}{\partial x}-bx\dfrac{\partial w}{\partial y}\right)^2=a\left(\dfrac{\partial w}{\partial x}\right)^2+b\left(\dfrac{\partial w}{\partial y}\right)^2+ab.$

By passing to the polar coordinates $x\sqrt{b}=\rho\cos\theta$ and $y\sqrt{a}=\rho\sin\theta$, one arrives at a separable equation. A complete integral of this equation is

$$w=\sigma-C_1\arctan\frac{\sigma}{C_1}+C_1\theta+C_2,\quad\text{where}\quad\sigma^2=\rho^2(C_1^2-1)-C_1^2.$$

22. $\left(y\dfrac{\partial w}{\partial x}-x\dfrac{\partial w}{\partial y}\right)^2=a(x^2+y^2)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+1\right].$

By passing to the polar coordinates $x=\rho\cos\theta$ and $y=\rho\sin\theta$, one arrives at a separable equation. A complete integral of this equation (for $0\le a<1$) is

$$w=C_1\sqrt{\frac{a}{1-a}}\,\theta+\sigma+\frac{C_1}{2}\ln\left|\frac{\sigma-C_1}{\sigma+C_1}\right|+C_2,\quad\text{where}\quad\sigma^2=C_1^2-\rho^2$$

(hence, one must require that $\rho^2<C_1^2$).

23. $\boldsymbol{w\left(\dfrac{\partial w}{\partial x}-\dfrac{\partial w}{\partial y}\right)^2+a\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial w}{\partial y}\right)^2=b.}$

Complete integral:

$$w=-a\left(\frac{C_1+C_2}{C_1-C_2}\right)^2+b^{1/3}\left[\frac{3(C_1x+C_2y+C_3)}{2(C_1-C_2)}\right]^{2/3}\qquad\text{if}\quad C_1\neq C_2.$$

24. $\boldsymbol{\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}\right)^2=(1-w^2)\left[a^2\left(\dfrac{\partial w}{\partial x}\right)^2+b^2\left(\dfrac{\partial w}{\partial y}\right)^2\right].}$

The general integral can be found by solving the equation

$$\arctan\sqrt{\frac{b^2x^2+a^2y^2}{a^2b^2(1-w^2)}-1}-\arctan\frac{ay}{bx}=\sqrt{1-w^2}\,\Phi(w)$$

for w.

25. $\boldsymbol{\left(y\dfrac{\partial w}{\partial x}-x\dfrac{\partial w}{\partial y}\right)^2+a\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}\right)=b.}$

Complete integral: $w=\dfrac{b-C_1^2}{2a}\ln(x^2+y^2)-C_1\arctan\dfrac{y}{x}+C_2.$

26. $\boldsymbol{\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2=a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.}$

Complete integral: $w=C_1x+C_2y\pm\sqrt{aC_1C_2}.$

27. $\boldsymbol{\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2=a\left(\dfrac{\partial w}{\partial x}\right)^2+b\left(\dfrac{\partial w}{\partial y}\right)^2+c.}$

Complete integral: $w=C_1x+C_2y\pm\sqrt{aC_1^2+bC_2^2+c}.$

28. $\boldsymbol{\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2=ax\left(\dfrac{\partial w}{\partial x}\right)^2+by\left(\dfrac{\partial w}{\partial y}\right)^2.}$

Complete integral: $\sqrt{C_3w}=\sqrt{C_1x}+\sqrt{C_2y}-A,$ where $aC_1+bC_2=A^2.$

29. $\boldsymbol{\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2=a^2(x^2+y^2+w^2)^2\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+1\right].}$

Complete integral in implicit form:

$$(x-C_1)^2+(y-C_2)^2+(w-C_3)^2=\tfrac{1}{4}a^{-2},\qquad\text{where}\quad C_1^2+C_2^2+C_3^2=\tfrac{1}{4}a^{-2}.$$

30. $\boldsymbol{x^2\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2=ay^2\dfrac{\partial w}{\partial y}.}$

This equation describes a family of cones whose vertices lie on the w-axis. Complete integral: $w=-\dfrac{C_1^2x^2}{ay}+C_1+C_2x.$

31. $\left(w\dfrac{\partial w}{\partial x}+x\right)^2+\left(w\dfrac{\partial w}{\partial y}+y\right)^2-a^2w^2\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+1\right]=0.$

Complete integral: $(x-C_1)^2+(y-C_2)^2+w^2=\dfrac{C_1^2+C_2^2}{a^2}\quad(w\neq 0).$

2.2. Equations Containing Arbitrary Functions

2.2.1. Equations of the Form $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(x,y,w)$

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=ay+f(x).$

Complete integral: $w=\pm y\sqrt{2ax+C_1}\pm\displaystyle\int\frac{f(x)\,dx}{\sqrt{2ax+C_1}}+C_2.$

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(x)y+g(x).$

Complete integral:

$$w=\varphi(x)y+\int\frac{g(x)}{\varphi(x)}\,dx+C_1,\qquad \varphi(x)=\pm\left[2\int f(x)\,dx+C_2\right]^{1/2}.$$

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(x)y^k+g(x)y^{2k+1}.$

Complete integral:

$$w=\varphi(x)y^{k+1}+\frac{1}{k+1}\int\frac{f(x)}{\varphi(x)}\,dx+C_1,\qquad \varphi(x)=\pm\left[\frac{2}{k+1}\int g(x)\,dx+C_2\right]^{1/2}.$$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(x)e^{\lambda y}+g(x)e^{2\lambda y}.$

Complete integral:

$$w=\varphi(x)e^{\lambda y}+\frac{1}{\lambda}\int\frac{f(x)}{\varphi(x)}\,dx+C_1,\qquad \varphi(x)=\pm\left[\frac{2}{\lambda}\int g(x)\,dx+C_2\right]^{1/2}.$$

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(x)g(y).$

Complete integral: $w=C_1\displaystyle\int f(x)\,dx+\frac{1}{C_1}\int g(y)\,dy+C_2.$

6. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}=f(ax+by).$

Complete integral:

$$w=C_1(ax-by)\pm\int\left[C_1^2+\frac{1}{ab}f(z)\right]^{1/2}dz+C_2,\qquad z=ax+by.$$

7. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(xy).$

Complete integral:

$$w = C_1 \ln\frac{x}{y} \pm \int \frac{1}{z}\left[C_1^2 + zf(z)\right]^{1/2} dz + C_2, \qquad z = xy.$$

8. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = \dfrac{f(x^a y^b)}{xy}.$

Complete integral:

$$w = C_1 \ln\left(x^a y^{-b}\right) \pm \int \frac{1}{z}\left[C_1^2 + \frac{1}{ab}f(z)\right]^{1/2} dz + C_2, \qquad z = x^a y^b.$$

9. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = aw + f(x).$

Complete integral: $w = (ax + C_1)(y + C_2) + (ax + C_1)\displaystyle\int \frac{f(x)\,dx}{(ax + C_1)^2}.$

10. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(x)w + g(x)y + h(x).$

Complete integral:

$$w = \varphi(x)y + \psi(x),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the ordinary differential equations

$$\varphi\varphi'_x = f(x)\varphi + g(x), \tag{1}$$
$$\varphi\psi'_y = f(x)\psi + h(x). \tag{2}$$

The exact solutions of Abel's equation (1) for various $f(x)$ and $g(x)$ can be found in the book by Zaitsev and Polyanin (1993, 1994) and Polyanin and Zaitsev (2003). Equation (2) is linear in ψ and, hence, integrable for any $f(x)$ and $h(x)$, provided that the function φ is known.

For the simplest cases where f and/or g are identically zero, equation (1) has the solutions

$$\varphi(x) = \int f(x)\,dx + C_1 \qquad \text{for } g(x) \equiv 0 \text{ and any } f(x),$$

$$\varphi(x) = \pm\left[2\int g(x)\,dx + C_1\right]^{1/2} \qquad \text{for } f(x) \equiv 0 \text{ and any } g(x).$$

11. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = aw^{k+1} + f(x)w^{2k}.$

The substitution $u = w^{1-k}$ leads to an equation of the form 2.2.1.9:

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = a(1-k)^2 u + (1-k)^2 f(x).$$

12. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = ae^{\lambda w} + f(x)e^{2\lambda w}.$

The substitution $u = e^{-\lambda w}$ leads to an equation of the form 2.2.1.9: $\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} = a\lambda^2 u + \lambda^2 f(x)$.

13. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(ax + by)g(w).$

Complete integral in implicit form:

$$\int \frac{dw}{\sqrt{g(w)}} = C_1(ax - by) \pm \int \left[C_1^2 + \frac{1}{ab}f(z)\right]^{1/2} dz + C_2, \qquad z = ax + by.$$

14. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(xy)g(w).$

Complete integral in implicit form:

$$\int \frac{dw}{\sqrt{g(w)}} = C_1 \ln\frac{x}{y} \pm \int \frac{1}{z}\left[C_1^2 + zf(z)\right]^{1/2} dz + C_2, \qquad z = xy.$$

15. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = \dfrac{1}{xy}f(x^a y^b)g(w).$

Complete integral in implicit form:

$$\int \frac{dw}{\sqrt{g(w)}} = C_1 \ln\left(x^a y^{-b}\right) \pm \int \frac{1}{z}\left[C_1^2 + \frac{1}{ab}f(z)\right]^{1/2} dz + C_2, \qquad z = x^a y^b.$$

16. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(x)g(y)h(w).$

Complete integral in implicit form: $\displaystyle\int \frac{dw}{\sqrt{h(w)}} = C_1 \int f(x)\,dx + \frac{1}{C_1}\int g(y)\,dy + C_2.$

2.2.2. Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y,w)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(x,y,w)$

▷ *Equations of this type are encountered in mechanics, where x plays the role of time and y plays the role of the spatial coordinate.*

1. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x) + g(y).$

Complete integral: $\displaystyle w = -C_1 x + \int f(x)\,dx + \int \sqrt{\frac{g(y) + C_1}{a}}\,dy + C_2.$

2. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x)y + g(x).$

Complete integral:

$$w = \varphi(x)y + \int \left[g(x) - a\varphi^2(x)\right] dx + C_1, \quad \varphi(x) = \int f(x)\,dx + C_2.$$

3. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x)y^2 + g(x)y + h(x).$

Complete integral:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by solving the following system of ordinary differential equations:

$$\varphi'_x = -4a\varphi^2 + f(x), \tag{1}$$
$$\psi'_x = -4a\varphi\psi + g(x), \tag{2}$$
$$\chi'_x = -a\psi^2 + h(x). \tag{3}$$

Riccati's equation (1) can be integrated in quadrature for numerous $f(x)$. For details, see Kamke (1976) and Polyanin and Zaitsev (2003). For example, if $f(x) = \text{const}$, then the particular solution $\varphi = \frac{1}{2}\sqrt{f}$ can be used.

Given a solution of equation (1), equations (2) and (3) are easy to integrate, since they are linear in the unknowns ψ and χ.

4. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(bx + cy).$

Complete integral:

$$w = C_1 x + C_2 - \frac{b}{2ac^2}(bx + cy) \pm \frac{1}{2ac^2}\int \sqrt{4ac^2 f(\xi) + b^2 - 4ac^2 C_1}\, d\xi, \quad \xi = bx + cy.$$

5. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f\left(\dfrac{y}{x}\right).$

1°. Complete integral:

$$w = x\varphi\left(\frac{y}{x}, C_1\right) + C_2,$$

where the function $\varphi = \varphi(z, C_1)$ is the general solution of the first-order ordinary differential equation

$$\varphi - z\varphi'_z + a(\varphi'_z)^2 = f(z).$$

2°. Apply the Lagrange–Charpit method (see Subsection 24.1.3). The characteristic system has a first integral of the form

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} - w = C_1. \tag{1}$$

We use (1) to eliminate $\frac{\partial w}{\partial x}$ from the original equation, thus obtaining the equation

$$ax\left(\frac{\partial w}{\partial y}\right)^2 - y\frac{\partial w}{\partial y} + w + C_1 - xf\left(\frac{y}{x}\right) = 0,$$

which can be treated as an ordinary differential equation for w with parameter x. Making the change of variables

$$w(y,x) = -\frac{1}{4a}xu^2(\xi,x) + \frac{1}{4a}x\xi^2 + xf(\xi) - C_1, \qquad \xi = \frac{y}{x}, \tag{2}$$

we arrive at an Abel equation of the second kind for u:

$$uu'_\xi \pm u = \frac{2a}{x}f'_\xi(\xi), \tag{3}$$

where x is treated as a parameter. The constant of integration in the general solution of equation (3) will depend on x, i.e., $C_2 = C_2(x)$. This dependence can be found by substituting the general solution of equation (3) into (1) and by taking into account relation (2).

The books by Polyanin and Zaitsev (2003) and Zaitsev and Polyanin (1994) present cases where equation (3) is integrable in quadrature.

6. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x) + g\left(\dfrac{y}{x}\right).$

The substitution $w(x,y) = u(x,y) + \int f(x)\,dx$ leads to an equation of the form 2.2.2.5:

$$\frac{\partial u}{\partial x} + a\left(\frac{\partial u}{\partial y}\right)^2 = g\left(\frac{y}{x}\right).$$

7. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x)w + g(x).$

Complete integral:

$$w = F(x)(C_1 + C_2 y) + F(x)\int \left[g(x) - aC_2^2F^2(x)\right]\frac{dx}{F(x)}, \qquad F(x) = \exp\left[\int f(x)\,dx\right].$$

8. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^2 + f(x)w + g(x).$

For $b = 0$, see equation 2.2.2.7. Complete integral for $b \neq 0$:

$$w = \varphi(x) + C_1\psi(x)\exp\left(\pm y\sqrt{b/a}\,\right).$$

The function $\varphi = \varphi(x)$ is determined by the first order ordinary differential equation

$$\varphi' = b\varphi^2 + f(x)\varphi + g(x), \tag{1}$$

and the function $\psi(x)$ is related to $\varphi(x)$ by

$$\psi(x) = \exp\left\{\int \left[2b\varphi(x) + f(x)\right]dx\right\}. \tag{2}$$

Riccati's equation (1) can be integrated in quadrature for various f and g, in particular, for $g(x) \equiv 0$ and arbitrary $f(x)$ and for $f(x) = \text{const}$ and $g(x) = \text{const}$. For details, see the books by Kamke (1976) and Polyanin and Zaitsev (2003).

9. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 = g(x)w + h(x)y + s(x).$

Complete integral:

$$w = G(x)\big[y\varphi(x) + \psi(x)\big], \qquad G(x) = \exp\left[\int g(x)\,dx\right],$$

where

$$\varphi(x) = C_1 + \int \frac{h(x)}{G(x)}\,dx, \quad \psi(x) = C_2 + \int \big[s(x) - f(x)G^2(x)\varphi^2(x)\big]\,\frac{dx}{G(x)}.$$

10. $\dfrac{\partial w}{\partial x} + f(y)\left(\dfrac{\partial w}{\partial y}\right)^2 = bw^2 + g(x)w + h(x).$

The substitution $\xi = \displaystyle\int \frac{dy}{\sqrt{|f_1(y)|}}$ leads to an equation of the form 2.2.2.8.

11. $\dfrac{\partial w}{\partial x} + \big[f(x)y + g(x)\big]\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral:

$$w = \varphi(x)y - \int g(x)\varphi^2(x)\,dx + C_1, \qquad \varphi(x) = \left[C_2 + \int f(x)\,dx\right]^{-1}.$$

12. $\dfrac{\partial w}{\partial x} + \big[f(x)y + g(x)\big]\left(\dfrac{\partial w}{\partial y}\right)^2 = h(x)w + r(x)y + s(x).$

Complete integral:

$$w = \varphi(x)y + \psi(x).$$

The functions $\varphi(x)$ and $\psi(x)$ are determined by solving the ordinary differential equations

$$\varphi' + f(x)\varphi^2 = h(x)\varphi + r(x), \tag{1}$$
$$\psi' + g(x)\varphi^2 = h(x)\psi + s(x), \tag{2}$$

where the prime denotes the derivative with respect to x.

Riccati's equation (1) can be integrated in quadrature for various $f(x)$, $h(x)$, and $g(x)$. For details, see the books by Kamke (1976) and Polyanin and Zaitsev (2003). In particular, for $r(x) \equiv 0$ the general solution of equation (1) has the form

$$\varphi(x) = H(x)\left[C_1 + \int f(x)H(x)\,dx\right]^{-1}, \qquad H(x) = \exp\left[\int h(x)\,dx\right].$$

Equation (2) is linear in ψ and is easy to integrate (for known φ). We have

$$\psi(x) = C_2H(x) + H(x)\int \big[s(x) - g(x)\varphi^2(x)\big]\,\frac{dx}{H(x)}.$$

13. $\dfrac{\partial w}{\partial x} + \left[f(x)y^k + g(x)y^{2k-2}\right]\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

The substitution $\xi = y^{2-k}$ leads to an equation of the form 2.2.2.11:

$$\frac{\partial w}{\partial x} + (2-k)^2\left[f(x)\xi + g(x)\right]\left(\frac{\partial w}{\partial \xi}\right)^2 = 0.$$

14. $\dfrac{\partial w}{\partial x} + \left[f(x)y^k + g(x)y^{2k-2}\right]\left(\dfrac{\partial w}{\partial y}\right)^2 = h(x)w + s(x).$

1°. If $k \neq 2$, then the substitution $\xi = y^{2-k}$ leads to an equation of the form 2.2.2.12 with $r(x) = 0$:

$$\frac{\partial w}{\partial x} + (2-k)^2\left[f(x)\xi + g(x)\right]\left(\frac{\partial w}{\partial \xi}\right)^2 = h(x)w + s(x).$$

2°. If $k = 2$, then by introducing the new variable $\xi = \ln|y|$, we arrive at an equation of the form 2.2.2.12.

15. $\dfrac{\partial w}{\partial x} + \left[f(x) + g(x)e^{\lambda y}\right]\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

The substitution $\xi = e^{-\lambda y}$ leads to an equation of the form 2.2.2.11:

$$\frac{\partial w}{\partial x} - \lambda\left[f(x)\xi + g(x)\right]\left(\frac{\partial w}{\partial \xi}\right)^2 = 0.$$

16. $\dfrac{\partial w}{\partial x} + \left[f(x) + g(x)e^{\lambda y}\right]\left(\dfrac{\partial w}{\partial y}\right)^2 = h(x)w + s(x).$

The substitution $\xi = e^{-\lambda y}$ leads to an equation of the form 2.2.2.12 with $r(x) = 0$:

$$\frac{\partial w}{\partial x} - \lambda\left[f(x)\xi + g(x)\right]\left(\frac{\partial w}{\partial \xi}\right)^2 = h(x)w + s(x).$$

17. $\dfrac{\partial w}{\partial x} + f(ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2 = g(ax+by).$

Complete integral:

$$w = C_1 x + C_2 + \int \frac{-a \pm \sqrt{4b^2 f(\xi)g(\xi) - 4b^2 C_1 f(\xi) + a^2}}{2b^2 f(\xi)}\,d\xi, \qquad \xi = ax + by.$$

18. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{y}{x}\right)\left(\dfrac{\partial w}{\partial y}\right)^2 = g\left(\dfrac{y}{x}\right).$

Complete integral:

$$w = x\varphi\left(\frac{y}{x}, C_1\right) + C_2,$$

where the function $\varphi = \varphi(z, C_1)$ is the general solution of the first-order ordinary differential equation

$$\varphi - z\varphi'_z + f(z)(\varphi'_z)^2 = g(z).$$

19. $\dfrac{\partial w}{\partial x} + \big[f_2(x)w + f_1(x)y + f_0(x)\big]\left(\dfrac{\partial w}{\partial y}\right)^2 = g_2(x)w + g_1(x)y + g_0(x).$

Complete integral:

$$w = \varphi(x)y + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the following system of ordinary differential equations:

$$\varphi'_x + f_2(x)\varphi^3 + f_1(x)\varphi^2 - g_2(x)\varphi - g_1(x) = 0, \tag{1}$$

$$\psi'_x + \big[f_2(x)\varphi^2 - g_2(x)\big]\psi + f_0(x)\varphi^2 - g_0(x) = 0. \tag{2}$$

Equation (1) can be integrated in quadrature for a lot of functions $f_n(x)$ and $g_n(x)$, in particular, for $f_2(x) = g_1(x) = 0$ and $f_1(x) = g_1(x) = 0$. For details, see Kamke (1976) and Polyanin and Zaitsev (2003).

Given a solution of equation (1), equation (2) is easy to integrate, since it is linear in the unknown ψ.

20. $\dfrac{\partial w}{\partial x} + f(x)w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = g(x)w.$

1°. The substitution $u = \frac{1}{k+1}w^{k+1}$ leads to an equation of the form 2.2.2.9:

$$\frac{\partial u}{\partial x} + f(x)\left(\frac{\partial u}{\partial y}\right)^2 = (k+1)g(x)u.$$

2°. The transformation

$$w(x,y) = G(x)v(z,y), \quad z = \int f(x)G^{k+1}(x)\,dx, \qquad G(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a simpler equation of the form 2.1.4.44 with $a = -1$:

$$\frac{\partial v}{\partial z} + v^k\left(\frac{\partial v}{\partial y}\right)^2 = 0.$$

21. $\dfrac{\partial w}{\partial x} + f(y)w^k\left(\dfrac{\partial w}{\partial y}\right)^2 = aw^{k+2} + g(x)w.$

The substitution $u = \frac{1}{k+1}w^{k+1}$ leads to an equation of the form 2.2.2.10 with $h(x) \equiv 0$:

$$\frac{\partial u}{\partial x} + f(y)\left(\frac{\partial u}{\partial y}\right)^2 = a(k+1)^2u^2 + (k+1)g(x)u.$$

22. $\dfrac{\partial w}{\partial x} - f(x)e^{\beta w}\left(\dfrac{\partial w}{\partial y}\right)^2 - g(x) = 0.$

The transformation

$$w(x,y) = u(z,y) + G(x), \quad z = \int f(x)\exp\big[\beta G(x)\big]\,dx, \qquad G(x) = \int g(x)\,dx,$$

leads to a simpler equation, $\dfrac{\partial u}{\partial z} - e^{\beta u}\left(\dfrac{\partial u}{\partial y}\right)^2 = 0$, which has the complete integral

$$\beta u = 2\ln\big(C_1 - \tfrac{1}{2}\beta y\big) - \ln\big(C_2 - \beta z\big).$$

23. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral in implicit form: $\displaystyle\int f(w)\,dw = C_1^2 x + C_1 y + C_2.$

24. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^2 - \big[yg(x) + h(x)\big]\dfrac{\partial w}{\partial y} = 0.$

The transformation

$$t = \int \varphi^2(x)\,dx, \quad z = \varphi(x)y + \int h(x)\varphi(x)\,dx, \qquad \varphi(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a simpler equation of the form 2.2.2.23:

$$\frac{\partial w}{\partial t} - f(w)\left(\frac{\partial w}{\partial z}\right)^2 = 0.$$

25. $\dfrac{\partial w}{\partial x} + f(y)g(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(w).$

Complete integral: $\displaystyle x + C_1\int \frac{dy}{\sqrt{f(y)}} + \int \frac{2C_1^2 g(w)\,dw}{1 + \sqrt{1 + 4C_1^2 g(w)h(w)}} = C_2.$

26. $\dfrac{\partial w}{\partial x} - f(x)g(y)h(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

Complete integral in implicit form: $\displaystyle\int h(w)\,dw = C_1^2\int f(x)\,dx + C_1\int \frac{dy}{\sqrt{g(y)}} + C_2.$

27. $\dfrac{\partial w}{\partial x} + f(w + ax + by)\left(\dfrac{\partial w}{\partial y}\right)^2 = g(w + ax + by).$

Complete integral:

$$w = -ax - by + \varphi(\xi) + C_3, \qquad \xi = C_1 x + C_2 y,$$

where

$$\varphi(\xi) = \int \frac{2bC_2 f(\xi) - C_1 \pm \sqrt{4C_2^2 f(\xi)g(\xi) + 4C_2(aC_2 - bC_1)f(\xi) + C_1^2}}{2C_2^2 f(\xi)}\,d\xi.$$

One of the constants C_1 or C_2 can be set equal to unity.

28. $f_1(x)\dfrac{\partial w}{\partial x} + f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^2 = g_1(x) + g_2(y).$

Complete integral: $\displaystyle w = \int \frac{g_1(x) - C_1}{f_1(x)}\,dx + \int \sqrt{\frac{g_2(y) + C_1}{f_2(y)}}\,dy + C_2.$

29. $f(x)\dfrac{\partial w}{\partial x}+g(y)\left(\dfrac{\partial w}{\partial y}\right)^2=aw+h_1(x)+h_2(y).$

Complete integral:

$$w=\varphi(x)+\psi(y).$$

Here

$$\varphi(x)=E(x)\left[C_1+\int\frac{h_1(x)\,dx}{f(x)E(x)}\right],\quad E(x)=\exp\left[a\int\frac{dx}{f(x)}\right].$$

The function $\psi(y)$ is determined by the ordinary differential equation $g(y)(\psi'_y)^2=a\psi+h_2(y)$. By the change $z=\int\dfrac{dy}{\sqrt{|g(y)|}}$, this equation can be reduced to an equation which is discussed in detail in Zaitsev and Polyanin (1994) and Polyanin and Zaitsev (2003).

2.2.3. Equations of the Form

$$\frac{\partial w}{\partial x}+f(x,y,w)\left(\frac{\partial w}{\partial x}\right)^2+g(x,y,w)\frac{\partial w}{\partial y}=h(x,y,w)$$

▷ *Equations of this type are encountered in mechanics, where x plays the role of time and y plays the role of the spatial coordinate.*

1. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=f(x)+g(y).$

Complete integral:

$$w=-C_1x+C_2+\int f(x)\,dx-\frac{b}{2a}y\pm\frac{1}{2a}\int\sqrt{4ag(y)+b^2+4aC_1}\,dy.$$

2. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=f(x)y+g(x).$

Complete integral:

$$w=\varphi(x)y+\int\left[g(x)-a\varphi^2(x)-b\varphi(x)\right]dx+C_1,\qquad\varphi(x)=\int f(x)\,dx+C_2.$$

3. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\dfrac{\partial w}{\partial y}=f(x)w+g(x).$

Complete integral:

$$w=(C_1y+C_2)F(x)+F(x)\int\left[g(x)-aC_1^2F^2(x)-bC_1F(x)\right]\frac{dx}{F(x)},\quad F(x)=\exp\left[\int f(x)\,dx\right].$$

4. $\dfrac{\partial w}{\partial x}+f(x)\left(\dfrac{\partial w}{\partial y}\right)^2+g(x)\dfrac{\partial w}{\partial y}=h(x).$

Complete integral: $w=C_1y+C_2+\int\left[h(x)-C_1^2f(x)-C_1g(x)\right]dx.$

5. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(x)\dfrac{\partial w}{\partial y} = h(x)w + p(x)y + s(x).$

Complete integral:

$$w = y\varphi(x) + \psi(x),$$

where

$$\varphi(x) = C_1 H(x) + H(x)\int \frac{p(x)}{H(x)}\,dx, \quad H(x) = \exp\left[\int h(x)\,dx\right],$$
$$\psi(x) = C_2 H(x) + H(x)\int \left[s(x) - f(x)\varphi^2(x) - g(x)\varphi(x)\right]\frac{dx}{H(x)}.$$

6. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + \left[g_1(x)y + g_0(x)\right]\dfrac{\partial w}{\partial y} = s(x)w + h_2(x)y^2 + h_1(x)y + h_0(x).$

Complete integral:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by solving the following system of ordinary differential equations:

$$\varphi'_x + 4f(x)\varphi^2 + \left[2g_1(x) - s(x)\right]\varphi - h_2(x) = 0, \tag{1}$$
$$\psi'_x + \left[4f(x)\varphi + g_1(x) - s(x)\right]\psi + 2g_0(x)\varphi - h_1(x) = 0, \tag{2}$$
$$\chi'_x - s(x)\chi + f(x)\psi^2 + g_0(x)\psi - h_0(x) = 0. \tag{3}$$

Riccati's equation (1) can be integrated in quadrature for a lot of functions $f(x)$, $g_1(x)$, $s(x)$, and $h_2(x)$, in particular, for $h_2(x) \equiv 0$ (the other functions can be arbitrary). For details, see Kamke (1976) and Polyanin and Zaitsev (2003).

Given a solution of equation (1), equations (2) and (3) are easy to integrate, since they are linear in the unknowns ψ and χ.

7. $\dfrac{\partial w}{\partial x} + f(y)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(y)\dfrac{\partial w}{\partial y} = h(y).$

Complete integral: $w = -C_1 x + C_2 + \displaystyle\int \frac{-g(y) \pm \sqrt{g^2(y) + 4f(y)h(y) + 4C_1 f(y)}}{2f(y)}\,dy.$

8. $\dfrac{\partial w}{\partial x} + f(y)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(y)\dfrac{\partial w}{\partial y} = h(x) + r(y).$

Complete integral:

$$w = -C_1 x + C_2 + \int h(x)\,dx + \int \frac{-g(y) \pm \sqrt{g^2(y) + 4f(y)r(y) + 4C_1 f(y)}}{2f(y)}\,dy.$$

9. $\dfrac{\partial w}{\partial x} + f(ax + by)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(ax + by)\dfrac{\partial w}{\partial y} = h(ax + by).$

For $b = 0$, see equation 2.2.3.4. For $b \neq 0$, by passing from x and y to the new variables x and $\xi = ax + by$, we arrive at an equation of the form 2.2.3.7:

$$\frac{\partial w}{\partial x} + b^2 f(\xi)\left(\frac{\partial w}{\partial \xi}\right)^2 + \left[a + bg(\xi)\right]\frac{\partial w}{\partial \xi} = h(\xi).$$

10. $$\frac{\partial w}{\partial x} + \big[f_2(x)w + f_1(x)y + f_0(x)\big]\left(\frac{\partial w}{\partial y}\right)^2 + \big[g_2(x)w + g_1(x)y + g_0(x)\big]\frac{\partial w}{\partial y} = h_2(x)w + h_1(x)y + h_0(x).$$

Complete integral:

$$w = \varphi(x)y + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by solving the following system of ordinary differential equations:

$$\varphi'_x + f_2(x)\varphi^3 + \big[f_1(x) + g_2(x)\big]\varphi^2 + \big[g_1(x) - h_2(x)\big]\varphi - h_1(x) = 0, \tag{1}$$

$$\psi'_x + \big[f_2(x)\varphi^2 + g_2(x)\varphi - h_2(x)\big]\psi + f_0(x)\varphi^2 + g_0(x)\varphi - h_0(x) = 0. \tag{2}$$

Equation (1) can be integrated in quadrature for a lot of functions $f_n(x)$, $g_n(x)$, and $h_n(x)$, in particular, for $f_2(x) = 0$, $h_1(x) = 0$ and $f_1(x) = -g_2(x)$, $h_1(x) = 0$. For details, see Kamke (1976) and Polyanin and Zaitsev (2003).

Given a solution of equation (1), equation (2) is easy to integrate, since it is linear in the unknown ψ.

11. $$\frac{\partial w}{\partial x} + f(w)\left(\frac{\partial w}{\partial y}\right)^2 + g(w)\frac{\partial w}{\partial y} = 0.$$

Complete integral in implicit form: $C_1x + C_2y + \displaystyle\int \frac{C_2^2 f(w)\,dw}{C_1 + C_2 g(w)} = C_3.$

One of the constants C_1, C_2, or C_3 can be set equal to ± 1.

12. $$\frac{\partial w}{\partial x} + f(w)\left(\frac{\partial w}{\partial y}\right)^2 + g(w)\frac{\partial w}{\partial y} = h(w).$$

Complete integral in implicit form:

$$C_1x + C_2y + \int \frac{2C_2^2 f(w)\,dw}{C_1 + C_2 g(w) \pm \sqrt{[C_1 + C_2 g(w)]^2 + 4C_2^2 f(w)h(w)}} = C_3.$$

One of the constants C_1, C_2, or C_3 can be set equal to ± 1.

13. $$\frac{\partial w}{\partial x} - f(w)\left(\frac{\partial w}{\partial y}\right)^2 - \big[yg(x) + h(x)\big]\frac{\partial w}{\partial y} = 0.$$

The transformation

$$t = \int \varphi^2(x)\,dx, \quad z = \varphi(x)y + \int h(x)\varphi(x)\,dx, \qquad \varphi(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a simpler equation of the form 2.2.2.23:

$$\frac{\partial w}{\partial t} - f(w)\left(\frac{\partial w}{\partial z}\right)^2 = 0.$$

14. $\dfrac{\partial w}{\partial x} + f(w+ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(w+ax+by)\dfrac{\partial w}{\partial y} = h(w+ax+by).$

The substitution $u = w + ax + by$ leads to an equation of the form 2.2.3.12:

$$\frac{\partial u}{\partial x} + f(u)\left(\frac{\partial u}{\partial y}\right)^2 + \big[g(u) - 2bf(u)\big]\frac{\partial u}{\partial y} = h(u) - b^2 f(u) + bg(u) + a.$$

2.2.4. Equations of the Form $f(x,y,w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,y,w)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(x,y,w)$

▷ *Equations of this type are encountered in mechanics, geometric optics, and differential geometry.*

1. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = f(x).$

Complete integral: $w = C_1 y + C_2 \pm \displaystyle\int \sqrt{f(x) - C_1^2}\,dx.$

⊙ Reference: E. Kamke (1965).

2. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = f(x) + g(y).$

Complete integral: $w = \pm \displaystyle\int \sqrt{f(x) + C_1}\,dx \pm \int \sqrt{g_2(y) - C_1}\,dy + C_2$. The signs before each of the integrals can be chosen independently of each other.

3. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = f(ax+by) + g(bx-ay).$

The transformation $\xi = ax + by$, $\eta = bx - ay$ leads to an equation of the form 2.2.4.2:

$$\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2 = \frac{1}{a^2+b^2}f(\xi) + \frac{1}{a^2+b^2}g(\eta).$$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = f(x^2+y^2).$

Hamilton's equation for the plane motion of a point mass under the action of a central force. Complete integral:

$$w = C_1 \arctan\frac{x}{y} + C_2 \pm \frac{1}{2}\int \sqrt{zf(z) - C_1^2}\,\frac{dz}{z}, \qquad z = x^2 + y^2.$$

⊙ Reference: E. Kamke (1965).

5. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=(x^2+y^2)f(xy).$

Complete integral:

$$w=C_1(x^2-y^2)+C_2\pm\int\sqrt{f(z)-4C_1^2}\,dz,\qquad z=xy.$$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=(x^2+y^2)f(x^2-y^2).$

Complete integral:

$$w=C_1xy+C_2\pm\frac{1}{2}\int\sqrt{f(z)-C_1^2}\,dz,\qquad z=x^2-y^2.$$

7. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=f(w).$

Complete integral in implicit form: $\displaystyle\int\frac{dw}{\sqrt{f(w)}}=\pm\sqrt{(x+C_1)^2+(y+C_2)^2}.$

8. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=f(x,y)g(w).$

The substitution $u=\displaystyle\int\frac{dw}{\sqrt{g(w)}}$ leads to a simpler equation: $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=f(x,y)$.
For solutions of this equation for some types of the right-hand side, see 2.2.4.1–2.2.4.6.

9. $\left(\dfrac{\partial w}{\partial x}\right)^2+\dfrac{1}{x^2}\left(\dfrac{\partial w}{\partial y}\right)^2=f(x).$

This equation governs the plane motion of a point mass in a central force field, with x and y being the polar coordinates.

Complete integral: $\displaystyle w=C_1y\pm\int\sqrt{f(x)-\frac{C_1^2}{x^2}}\,dx+C_2.$

⊙ Reference: P. Appell (1953).

10. $\left(\dfrac{\partial w}{\partial x}\right)^2+f(x)\left(\dfrac{\partial w}{\partial y}\right)^2=g(x).$

Complete integral: $\displaystyle w=C_1y+C_2+\int\sqrt{g(x)-C_1^2f(x)}\,dx.$

11. $\left(\dfrac{\partial w}{\partial x}\right)^2+f(y)\left(\dfrac{\partial w}{\partial y}\right)^2=g(y).$

Complete integral: $\displaystyle w=C_1x+C_2+\int\sqrt{\frac{g(y)-C_1^2}{f(y)}}\,dy.$

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = g(w).$

Complete integral in implicit form: $\displaystyle\int \sqrt{\frac{C_1^2 + C_2^2 f(w)}{g(w)}}\, dw = C_1 x + C_2 y + C_3.$
One of the constants C_1 or C_2 can be set equal to ± 1.

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 - f(x,y,w)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

The equation can be factorized. Equating each of the factors to zero, we obtain two simpler equations: $\dfrac{\partial w}{\partial x} \pm \sqrt{f(x,y,w)}\,\dfrac{\partial w}{\partial y} = 0.$

14. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^2 = g_1(x) + g_2(y).$

A separable equation. This equation is encountered in differential geometry in studying geodesic lines of Liouville surfaces. Complete integral:

$$w = \pm \int \sqrt{\frac{g_1(x) + C_1}{f_1(x)}}\, dx \pm \int \sqrt{\frac{g_2(y) - C_1}{f_2(y)}}\, dy + C_2.$$

The signs before each of the integrals can be chosen independently of each other.

⊙ References: P. Appell (1953), E. Kamke (1965).

15. $f(ax+by)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(ax+by).$

Complete integral:

$$w = C_1 x + C_2 + \int \frac{-aC_1 f(z) \pm \sqrt{a^2 f(z)h(z) + b^2 g(z)h(z) - b^2 C_1^2 f(z)g(z)}}{a^2 f(z) + b^2 g(z)}\, dz, \quad z = ax + by.$$

16. $f(w+ax+by)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(w+ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(w+ax+by).$

Complete integral:

$$w = -ax - by + \varphi(\xi), \qquad \xi = C_1 x + C_2 y + C_3,$$

where the function $\varphi = \varphi(\xi)$ is defined implicitly by

$$\xi = \int \frac{\left[C_1^2 f(\varphi) + C_2^2 g(\varphi)\right] d\varphi}{aC_1 f(\varphi) + bC_2 g(\varphi) \pm \sqrt{C_1^2 f(\varphi)h(\varphi) + C_2^2 g(\varphi)h(\varphi) - (aC_2 - bC_1)^2 f(\varphi)g(\varphi)}}.$$

One of the constants C_1 or C_2 can be set equal to ± 1.

17. $f_1(x)g_1(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + f_2(y)g_2(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = h(w).$

Complete integral:

$$\int \sqrt{\frac{C_1^2 g_1(w) + C_2^2 g_2(w)}{h(w)}}\, dw = C_1 \int \frac{dx}{\sqrt{f_1(x)}} + C_2 \int \frac{dy}{\sqrt{f_2(y)}} + C_3.$$

One of the constants C_1 or C_2 can be set equal to ± 1.

2.2.5. Equations of the Form $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,y,w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = g(x,y,w)$

1. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(x).$

Complete integral: $w = C_1(ax - 2y) \pm \int \sqrt{a^2 C_1^2 + f(x)}\, dx + C_2.$

2. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(y).$

Complete integral: $w = C_1(y - ax) - \dfrac{1}{a^2 C_1}\int f(y)\, dy + C_2.$

3. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + bw + f(y) = 0.$

Complete integral: $w = \dfrac{by + C_1}{a^2}\left[y - ax + a^2 \int \dfrac{f(y)\, dy}{(by + C_1)^2} + C_2\right].$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(y)w = 0.$

Complete integral: $w = \dfrac{y - ax + C_1}{a^2}\left[\int f(y)\, dy + C_2\right].$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(w).$

Complete integral in implicit form: $\displaystyle\int \frac{dw}{\sqrt{f(w)}} = C_1 x + \frac{1 - C_1^2}{aC_1} y + C_2.$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(x)g(w).$

The substitution $z = \displaystyle\int \frac{dw}{\sqrt{|g(w)|}}$ leads to an equation of the form 2.2.5.1.

7. $\left(\dfrac{\partial w}{\partial x}\right)^2 + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = f(y)g(w).$

The substitution $z = \displaystyle\int \frac{dw}{\sqrt{|g(w)|}}$ leads to an equation of the form 2.2.5.2.

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x) = 0.$

Complete integral: $w = 2C_1 y + C_2 + \displaystyle\int \left[-C_1 f(x) \pm \sqrt{C_1^2 f^2(x) - g(x)}\right] dx.$

9. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x) = 0.$

Complete integral: $w = -\dfrac{C_1}{2}x \pm \dfrac{1}{2}\displaystyle\int \sqrt{C_1^2 - 4g(x)}\, + C_1 \int \frac{dy}{f(y)} + C_2.$

10. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(y) = 0.$

Complete integral: $w = C_1 x + C_2 - \displaystyle\int \frac{g(y) + C_1^2}{f(y)}\, dy.$

11. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = g(y)w + xh(y) + r(y).$

Complete integral:

$$w = x\varphi(y) + \psi(y),$$

where the function $\varphi(y)$ and $\psi(y)$ are determined by the following system of ordinary differential equations:

$$f(y)\varphi\varphi'_y = g(y)\varphi + h(y), \tag{1}$$

$$f(y)\varphi\psi'_y = g(y)\psi - \varphi^2 + r(y). \tag{2}$$

Abel's equation (1) can be considered independently. Given its solution, the solution of equation (2), which is linear in ψ, is easy to find.

For the simplest cases where g and/or h are identically zero, equation (1) has the solutions

$$\varphi(y) = \int \frac{g(y)}{f(y)}\, dy + C_1 \qquad \text{for } h(y) \equiv 0 \text{ and any } g(y),$$

$$\varphi(y) = \pm\left[2\int \frac{h(y)}{f(y)}\, dy + C_1\right]^{1/2} \qquad \text{for } g(y) \equiv 0 \text{ and any } h(y).$$

For other solutions of Abel's equation (1) see Polyanin and Zaitsev (2003).

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x)h(w) = 0.$

The substitution $z = \displaystyle\int \frac{dw}{\sqrt{|g(w)|}}$ leads to an equation of the form 2.2.5.9.

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(y)h(w) = 0.$

The substitution $z = \displaystyle\int \frac{dw}{\sqrt{|g(w)|}}$ leads to an equation of the form 2.2.5.10.

14. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(ax+by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(ax+by)h(w) = 0.$

Complete integral in implicit form:

$$\int \frac{dw}{\sqrt{h(w)}} = C_1 y + \int \frac{-aC_1 f(z) \pm \sqrt{a^2 C_1^2 f^2(z) - 4ag(z)\big[a + bf(z)\big]}}{2a\big[a + bf(z)\big]}\,dz + C_2,$$

where $z = ax + by$.

15. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(w+ax+by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(w+ax+by) = 0.$

Complete integral:

$$w = -ax - by + \varphi(z), \qquad z = C_1 x + C_2 y + C_3,$$

where the function $\varphi = \varphi(z)$ is defined implicitly by

$$z = \int \frac{2C_1[C_1 + C_2 f(\varphi)]\,d\varphi}{\Delta(\varphi) \pm \sqrt{\Delta^2(\varphi) - 4C_1[C_1 + C_2 f(\varphi)][a^2 + abf(\varphi) + g(\varphi)]}},$$
$$\Delta(\varphi) = 2aC_1 + (aC_2 + bC_1)f(\varphi).$$

One of the constants C_1 or C_2 can be set equal to ± 1.

16. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)g(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + h(x)\varphi(w) = 0.$

Complete integral in implicit form:

$$\int \frac{dw}{\sqrt{\varphi(w)}} = \int \Big[C_1 f(x) \pm \sqrt{C_1^2 f^2(x) - h(x)}\Big]\,dx - 2C_1 \int \frac{dy}{g(y)} + C_2.$$

17. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)g(w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} - h(w) = 0.$

Complete integral in implicit form: $\displaystyle\int \sqrt{\frac{1 + C_1 g(w)}{h(w)}}\,dw = x + C_1 \int \frac{dy}{f(y)} + C_2.$

2.2.6. Other Equations

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(x)\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y} + h(x) = 0.$

Complete integral: $w = -\displaystyle\int \frac{C_1 g(x) + h(x)}{C_1 + f(x)}\,dx + C_1 y + C_2.$

2. $\left(\dfrac{\partial w}{\partial x} + ay\right)\left(\dfrac{\partial w}{\partial y} + ax\right) = f(x)g(y).$

Complete integral: $w = -axy + C_1\displaystyle\int f(x)\,dx + \frac{1}{C_1}\int g(y)\,dy + C_2.$

3. $\left[\dfrac{\partial w}{\partial x} + f_1(x)\right]\left[\dfrac{\partial w}{\partial y} + f_2(y)\right] = g_1(x)g_2(y).$

Complete integral: $w = \displaystyle\int \left[C_1 g_1(x) - f_1(x)\right]dx + \int \left[\frac{1}{C_1}g_2(y) - f_2(y)\right]dy + C_2.$

4. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x)\dfrac{\partial w}{\partial x} = h(x).$

Complete integral: $w = -\dfrac{C_1}{2}x - \dfrac{1}{2}\displaystyle\int \left\{g(x) - \sqrt{\left[g(x) + C_1\right]^2 + 4h(x)}\right\}dx + C_1\int \frac{dy}{f(y)} + C_2.$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial x} = h(x).$

Complete integral: $w = C_1 x \pm \displaystyle\int \sqrt{C_1^2 + h(x)}\,dx - \int \frac{2C_1 + g(y)}{f(y)}\,dy + C_2.$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(x)g(y)\dfrac{\partial w}{\partial x} = h(x).$

Complete integral: $w = 2C_1 y - \displaystyle\int g(y)\,dy + \int \left[-C_1 f(x) \pm \sqrt{C_1^2 f^2(x) + h(x)}\right]dx + C_2.$

7. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(x)\dfrac{\partial w}{\partial x} + h(x)\dfrac{\partial w}{\partial y} + s(x) = 0.$

Complete integral:

$$w = \frac{1}{2}\int \left\{-C_1 f(x) - g(x) + \sqrt{\left[C_1 f(x) + g(x)\right]^2 - 4C_1 h(x) - 4s(x)}\right\}dx + C_1 y + C_2.$$

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial x} + h(y)\dfrac{\partial w}{\partial y} + s(y) = 0.$

Complete integral: $w = C_1 x - \displaystyle\int \frac{C_1^2 + C_1 g(y) + s(y)}{C_1 f(y) + h(y)}\,dy + C_2.$

9. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2=f(x^2+y^2)\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}\right)^2+g(x^2+y^2)w^2.$

Complete integral:

$$w=C_1\exp\left[C_2\arcsin\frac{x}{\sqrt{z}}\pm\frac{1}{2}\int\sqrt{\frac{zg(z)-C_2^2}{1-zf(z)}}\frac{dz}{z}\right],\qquad z=x^2+y^2.$$

10. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^2+g_1(x)\dfrac{\partial w}{\partial x}+g_2(y)\dfrac{\partial w}{\partial y}=h_1(x)+h_2(y).$

A separable equation. Complete integral:

$$w=\int\frac{-g_1(x)\pm\sqrt{g_1^2(x)+4f_1(x)h_1(x)+4C_1f_1(x)}}{2f_1(x)}\,dx$$
$$+\int\frac{-g_2(y)\pm\sqrt{g_2^2(y)+4f_2(y)h_2(y)-4C_1f_2(y)}}{2f_2(y)}\,dy+C_2.$$

The signs before the roots can be chosen independently of each other.

11. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+f_3(x)\left(\dfrac{\partial w}{\partial y}\right)^2=g(w).$

Complete integral in implicit form:

$$\int\frac{dw}{\sqrt{g(w)}}=C_1y+C_2-\int\frac{C_1f_2(x)\pm\sqrt{C_1^2f_2^2(x)+4f_1(x)-4C_1^2f_1(x)f_3(x)}}{2f_1(x)}\,dx.$$

12. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+f_3(x)\left(\dfrac{\partial w}{\partial y}\right)^2$
$$+g_1(x)\frac{\partial w}{\partial x}+g_2(x)\frac{\partial w}{\partial y}+h(x)=0.$$

Complete integral:

$$w=C_1y+C_2-\int\frac{g_1(x)+C_1f_2(x)\pm\sqrt{[g_1(x)+C_1f_2(x)]^2-4f_1(x)\Delta(x)}}{2f_1(x)}\,dx,$$
$$\Delta(x)=C_1^2f_3(x)+C_1g_2(x)+h(x).$$

13. $f_1(ax+by)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(ax+by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+f_3(ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2$
$$+g_1(ax+by)\frac{\partial w}{\partial x}+g_2(ax+by)\frac{\partial w}{\partial y}+h(ax+by)=0.$$

For $b=0$, see equation 2.2.6.12. For $b\neq0$, the transformation $w(x,y)=u(\xi,x)$, $\xi=ax+by$ leads to an equation of the form 2.2.6.12:

$$\left[a^2f_1(\xi)+abf_2(\xi)+b^2f_3(\xi)\right]\left(\frac{\partial u}{\partial\xi}\right)^2+\left[2af_1(\xi)+bf_2(\xi)\right]\frac{\partial u}{\partial\xi}\frac{\partial u}{\partial x}+f_1(\xi)\left(\frac{\partial u}{\partial x}\right)^2$$
$$+\left[ag_1(\xi)+bg_2(\xi)\right]\frac{\partial u}{\partial\xi}+g_1(\xi)\frac{\partial u}{\partial x}+h(\xi)=0.$$

14. $f(x^2+y^2)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right]=\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w\right)^2.$

Complete integral:

$$w=C_1\exp\left(C_2\arcsin\frac{x}{\sqrt{x^2+y^2}}\right)\Psi(x^2+y^2),$$

where $\Psi=\Psi(z)$ is a solution of the nonlinear ordinary differential equation

$$z(2z\Psi'_z-\Psi)^2=f(z)[4z^2(\Psi'_z)^2+C_2^2\Psi^2],\qquad z=x^2+y^2.$$

In particular, if $f(z)=z$, then $\Psi(z)=z^{(1-C_2^2)/4}$.

15. $f_1(w)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+f_3(w)\left(\dfrac{\partial w}{\partial y}\right)^2$
$$+g_1(w)\frac{\partial w}{\partial x}+g_2(w)\frac{\partial w}{\partial y}+h(w)=0.$$

Complete integral in implicit form:

$$C_1x+C_2y+\int\frac{2F(w)\,dw}{G(w)\pm\sqrt{G^2(w)-4F(w)h(w)}}=C_3,$$

where $F(w)=C_1^2f_1(w)+C_1C_2f_2(w)+C_2^2f_3(w)$ and $G(w)=C_1g_1(w)+C_2g_2(w)$.

16. $f_1(w+ax+by)\left(\dfrac{\partial w}{\partial x}\right)^2+f_2(w+ax+by)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+f_3(w+ax+by)\left(\dfrac{\partial w}{\partial y}\right)^2$
$$+g_1(w+ax+by)\frac{\partial w}{\partial x}+g_2(w+ax+by)\frac{\partial w}{\partial y}+h(w+ax+by)=0.$$

The substitution $u(x,y)=w(x,y)+ax+by$ leads to an equation of the form 2.2.6.15:

$$f_1(u)\left(\frac{\partial u}{\partial x}\right)^2+f_2(u)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+f_3(u)\left(\frac{\partial u}{\partial y}\right)^2+\left[g_1(u)-2af_1(u)-bf_2(u)\right]\frac{\partial u}{\partial x}$$
$$+\left[g_2(u)-2bf_3(u)-af_2(u)\right]\frac{\partial u}{\partial y}+h(u)+a^2f_1(u)+abf_2(u)+b^2f_3(u)-ag_1(u)-bg_2(u)=0.$$

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Chapter 3
First-Order Nonlinear Equations with Two Independent Variables of General Form

3.1. Nonlinear Equations Containing Arbitrary Parameters

3.1.1. Equations Contain the Fourth Powers of Derivatives

1. $\left(\dfrac{\partial w}{\partial x}\right)^4 = a\dfrac{\partial w}{\partial y} + bx + c.$

Complete integrals:

$$w = \frac{4}{5b}(bx + aC_1 + c)^{5/4} + C_1 y + C_2, \qquad w = \frac{aC_1 x - bxy - cy}{a} - \frac{(aC_1 - by)^5}{5a^5 b} + C_2.$$

2. $\left(\dfrac{\partial w}{\partial x}\right)^4 = a\dfrac{\partial w}{\partial y} + by + c.$

Complete integral: $w = C_1 x - \dfrac{y(by + 2c - 2C_1^4)}{2a} + C_2.$

3. $\left(\dfrac{\partial w}{\partial x}\right)^4 + a\dfrac{\partial w}{\partial y} = bx^k.$

Complete integral: $w = \displaystyle\int \left(bx^k + aC_1\right)^{1/4} dy - C_1 y + C_2.$

4. $\left(\dfrac{\partial w}{\partial x}\right)^4 + a\dfrac{\partial w}{\partial y} = by^k.$

Complete integral: $w = aC_1 x - a^3 C_1^4 y + \dfrac{b}{a(k+1)} y^{k+1} + C_2.$

5. $\left(\dfrac{\partial w}{\partial x}\right)^4 + a\dfrac{\partial w}{\partial y} = bxy^k + cy^n.$

Complete integral:

$$w = x\varphi(y) + \frac{c}{a(n+1)} y^{n+1} - \frac{1}{a}\int \varphi^4(y)\, dy + C_1, \qquad \varphi(y) = \frac{b}{a(k+1)} y^{k+1} + C_2.$$

6. $\left(\dfrac{\partial w}{\partial x}\right)^4 + a\dfrac{\partial w}{\partial y} = bx^k + cy^n.$

Complete integral: $w = \displaystyle\int \left(bx^k + aC_1\right)^{1/4} dx - C_1 y + \frac{c}{a(n+1)} y^{n+1} + C_2.$

7. $\left(\dfrac{\partial w}{\partial x}\right)^4 - aw^k \dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \left[\dfrac{3-k}{3}\left(aC_1 x + a^3 C_1^4 y + C_2\right)\right]^{\frac{3}{3-k}}.$

8. $\left(\dfrac{\partial w}{\partial x}\right)^4 - ax^n y^m w^k \dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \left[\dfrac{3-k}{3}\left(\dfrac{4aC_1}{n+4} x^{\frac{n+4}{4}} + \dfrac{a^3 C_1^4}{1-m} y^{1-m} + C_2\right)\right]^{\frac{3}{3-k}}.$

9. $\left(\dfrac{\partial w}{\partial x}\right)^4 + a\left(\dfrac{\partial w}{\partial y}\right)^3 + b\left(\dfrac{\partial w}{\partial x}\right)^2 + c\dfrac{\partial w}{\partial y} + d = 0.$

Complete integral: $w = \pm\dfrac{x}{2}\sqrt{-2b + 2\sqrt{b^2 - 4d - 4aC_1^3 - 4cC_1}} + C_1 y + C_2.$

10. $\left[a^2\left(\dfrac{\partial w}{\partial x}\right)^2 - 2w\right]^2 = b\left(\dfrac{\partial w}{\partial y}\right)^3.$

Complete integral: $w = \dfrac{1}{2}\left(\dfrac{x}{a} + C_1\right)^2 + \dfrac{4}{27b}(y + C_2)^3.$

11. $ay^k\left(\dfrac{\partial w}{\partial x}\right)^4 = \left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y}.$

Complete integral: $w = C_1 x - \dfrac{1}{2}\displaystyle\int \left(b \pm \sqrt{b^2 + 4aC_1^4 y^k}\right) dy + C_2.$

12. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^3 = ay + bx^k.$

Complete integral: $w = (4ax + C_1)^{1/4} y + b\displaystyle\int x^k (4ax + C_1)^{-3/4}\, dx + C_2.$

13. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^3 = ax^k y + bx^n.$

Complete integral:

$$w = \varphi(x) y + b\int \frac{x^n\, dx}{[\varphi(x)]^3} + C_2, \qquad \varphi(x) = \left(\frac{4a}{k+1} x^{k+1} + C_1\right)^{1/4}.$$

14. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^3 = ax^k y^n.$

Complete integral: $w = \dfrac{aC_1^3}{k+1}x^{k+1} + \dfrac{3}{C_1(n+3)}y^{\frac{n+3}{3}} + C_2.$

15. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^3 = aw + bx^k.$

Complete integral: $w = (3ax + C_1)^{1/3}(y + C_2) + b(3ax + C_1)^{1/3}\displaystyle\int x^k(3ax + C_1)^{-4/3}\,dx.$

16. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^3 = ax^n w + bx^m y + cx^l.$

This is a special case of equation 3.2.1.22 with $k = 3$, $f(x) = ax^n$, $g(x) = bx^m$, and $h(x) = cx^l$.

17. $w^2\left(\dfrac{\partial w}{\partial x}\right)^2\left(\dfrac{\partial w}{\partial y}\right)^2 = a^2y^2\left(\dfrac{\partial w}{\partial x}\right)^2 + b^2x^2\left(\dfrac{\partial w}{\partial y}\right)^2.$

Complete integral: $w^2 = \dfrac{\sqrt{C_1^2+1}}{C_1}(bx^2 + aC_1y^2) + C_2.$

3.1.2. Equations Contain Derivatives in Radicands

1. $\dfrac{\partial w}{\partial x} + \sqrt{a + b\left(\dfrac{\partial w}{\partial y}\right)^2} = 0.$

This equation is encountered in optimal control and differential games; see Subbotin (1991, 1995).

Complete integral: $w = C_1y - \sqrt{a + bC_1^2}\,x + C_2.$

See also Example 24.7 in Subsection 24.3.6, where the Cauchy problem for this equation with $a = b = 1$ is considered.

2. $\dfrac{\partial w}{\partial x} + \sqrt{a + b\left(\dfrac{\partial w}{\partial y}\right)^2} - x\sqrt{c + k\left(\dfrac{\partial w}{\partial y}\right)^2} = 0.$

Complete integral: $w = C_1y + \frac{1}{2}x^2\sqrt{c + kC_1^2} - x\sqrt{a + bC_1^2} + C_2.$

The Cauchy problem for this equation with $b = k = 1$, $a > 0$, and $c > 0$ was considered in the book by Melikyan (1998).

3. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} - w + \sqrt{a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 + c} = 0.$

Complete integral: $w = C_1x + C_2y + \sqrt{aC_1^2 + bC_2^2 + c}.$

3.1.3. Equations Contain Arbitrary Powers of Derivatives

1. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\dfrac{\partial w}{\partial y} = bx^n.$

Complete integral: $w = \displaystyle\int (bx^n + aC_1)^{1/k}\,dy - C_1 y + C_2.$

2. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\dfrac{\partial w}{\partial y} = by^n.$

Complete integral: $w = aC_1 x - a^{k-1}C_1^k y + \dfrac{b}{a(n+1)}y^{n+1} + C_2.$

3. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\dfrac{\partial w}{\partial y} = bx^n + cy^m.$

Complete integral: $w = \displaystyle\int (bx^n + aC_1)^{1/k}\,dx - C_1 y + \frac{c}{a(m+1)}y^{m+1} + C_2.$

4. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\dfrac{\partial w}{\partial y} = bxy^n + cy^m.$

Complete integral:

$$w = x\varphi(y) + \frac{c}{a(m+1)}y^{m+1} - \frac{1}{a}\int \varphi^k(y)\,dy + C_1, \qquad \varphi(y) = \frac{b}{a(n+1)}y^{n+1} + C_2.$$

5. $\left(\dfrac{\partial w}{\partial x}\right)^k - aw^n\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \left[\dfrac{n-k+1}{1-k}\left(aC_1 x + a^{k-1}C_1^k y + C_2\right)\right]^{\frac{1-k}{n-k+1}}.$

6. $\left(\dfrac{\partial w}{\partial x}\right)^k - ax^n y^m w^l\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \left[\dfrac{l-k+1}{1-k}\left(\dfrac{akC_1}{n+k}x^{\frac{n+k}{k}} + \dfrac{a^{k-1}C_1^k}{1-m}y^{1-m} + C_2\right)\right]^{\frac{1-k}{l-k+1}}.$

7. $\left(\dfrac{\partial w}{\partial x}\right)^k + ay^n\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = by^m w + cxy^l + dy^s.$

This is a special case of equation 3.2.1.17 with $f(y) = ay^n$, $g(y) = by^m$, $h(y) = cy^l$, and $r(y) = dy^s$.

8. $y^n\left(\dfrac{\partial w}{\partial x}\right)^k = a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y}.$

Complete integral: $w = C_1 x + C_2 - \dfrac{b}{2a}y \pm \dfrac{1}{2a}\displaystyle\int \sqrt{b^2 + 4aC_1^k y^n}\,dy.$

9. $(axy^n + by^m)\left(\dfrac{\partial w}{\partial x}\right)^k - \dfrac{\partial w}{\partial y} = 0.$

Complete integral:

$$w = x\varphi(y) + b\int y^m\varphi^k(y)\,dy + C_1, \qquad \varphi(y) = \left[\frac{a(1-k)}{n+1}y^{n+1} + C_2\right]^{\frac{1}{1-k}}.$$

10. $ay^m w^n\left(\dfrac{\partial w}{\partial x}\right)^k - \dfrac{\partial w}{\partial y} + by^s w = 0.$

The transformation

$$w(x,y) = G(y)u(x,z), \quad z = a\int y^m G^{n+k-1}(y)\,dy, \qquad G(y) = \exp\left(\frac{b}{s+1}y^{s+1}\right),$$

leads to a simpler equation of the form 3.1.3.5: $\left(\dfrac{\partial u}{\partial x}\right)^k - u^{-n}\dfrac{\partial u}{\partial z} = 0.$

11. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = ay + bx^n.$

Complete integral: $w = \big[a(k+1)x + C_1\big]^{\frac{1}{k+1}}y + b\displaystyle\int x^n\big[a(k+1)x + C_1\big]^{-\frac{k}{k+1}}dx + C_2.$

12. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = ax^n y + bx^m.$

This is a special case of equation 3.2.1.19 with $f(x) = ax^n$ and $g(x) = bx^m$.

13. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = ax^n y^m.$

Complete integral: $w = \dfrac{aC_1^k}{n+1}x^{n+1} + \dfrac{k}{C_1(m+k)}y^{\frac{m+k}{k}} + C_2.$

14. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = aw + bx^n.$

Complete integral: $w = (akx + C_1)^{\frac{1}{k}}(y + C_2) + b(akx + C_1)^{\frac{1}{k}}\displaystyle\int x^n(akx + C_1)^{-\frac{k+1}{k}}dx.$

15. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = ax^n w + bx^m y + cx^l.$

This is a special case of equation 3.2.1.22 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x) = cx^l$.

16. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k + a\dfrac{\partial w}{\partial y} + bx + c = 0.$

Complete integral: $w = -\dfrac{bx + 2aC_1 + 2c}{2C_1^k}x + C_1 y + C_2.$

17. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k + a\dfrac{\partial w}{\partial x} + bx + c = 0.$

Complete integral: $w = -\dfrac{bx + 2c}{2(a + C_1^k)}x + C_1 y + C_2.$

18. $\left(\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\right)^k = x\dfrac{\partial w}{\partial x} - y\dfrac{\partial w}{\partial y}.$

Complete integral:

$$C_1^{-k} w = \frac{1}{2}\ln\left|\frac{x}{y}\right| \pm R \pm \frac{1}{2}\ln\left|\frac{R-1}{R+1}\right| + C_2, \qquad \text{where} \quad R = \sqrt{4xyC_1^{1-2k} + 1}.$$

19. $\left(x\dfrac{\partial w}{\partial y} - y\dfrac{\partial w}{\partial x}\right)^k = \left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2.$

Complete integral:

$$w = C_1 \arctan\frac{y}{x} + C_2 + \int \sqrt{C_1^k R^2 - C_1^2}\,\frac{dR}{R}, \qquad \text{where} \quad R = \sqrt{x^2 + y^2}.$$

20. $\left(y\dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial y}\right)^k = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2.$

Complete integral:

$$C_1 w = \frac{1}{2}\ln\left|\frac{x+y}{x-y}\right| \pm R \pm \frac{1}{2}\ln\left|\frac{R-1}{R+1}\right| + C_2, \qquad \text{where} \quad R = \sqrt{(x+y)(x-y)C_1^{2-k} + 1}.$$

21. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\left(\dfrac{\partial w}{\partial y}\right)^n = b.$

Complete integral: $w = (b - aC_1)^{1/k}x + C_1^{1/n}y + C_2.$

22. $\left(\dfrac{\partial w}{\partial x}\right)^k + a\left(\dfrac{\partial w}{\partial y}\right)^k = w.$

Complete integral: $w = \left(1 + aC_1^k\right)^{\frac{1}{1-k}}\left[\dfrac{k-1}{k}(x + C_1 y + C_2)\right]^{\frac{k}{k-1}}.$

23. $ax^m\left(\dfrac{\partial w}{\partial x}\right)^k + by^l\left(\dfrac{\partial w}{\partial y}\right)^n = cx^p + sy^q.$

Complete integral: $w = \displaystyle\int\left(\frac{cx^p - C_1}{ax^m}\right)^{1/k} dx + \int\left(\frac{sy^q + C_1}{by^l}\right)^{1/n} dy + C_2.$

24. $axy^m\left(\dfrac{\partial w}{\partial x}\right)^k + by^l\left(\dfrac{\partial w}{\partial x}\right)^n - \dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 3.2.2.5 with $f(y) = ay^m$ and $g(y) = by^l$.

25. $e^{ax}\left(\dfrac{\partial w}{\partial x}\right)^k + e^{by}\left(\dfrac{\partial w}{\partial y}\right)^k = c.$

Complete integral:

$$w = \begin{cases} -\dfrac{kC_1}{a}e^{-ax/k} - \dfrac{k(c-C_1^k)^{1/k}}{b}e^{-by/k} + C_2 & \text{if } a \neq 0, \\ C_1 x - \dfrac{k(c-C_1^k)^{1/k}}{b}e^{-by/k} + C_2 & \text{if } a = 0. \end{cases}$$

26. $a\left(\dfrac{\partial w}{\partial x} + cy\right)^k + b\left(\dfrac{\partial w}{\partial y} + cx\right)^n = cx^m + sy^l.$

This is a special case of equation 3.2.2.10 with $f(x) = cx^m$ and $g(y) = sy^l$.

27. $\left(\dfrac{\partial w}{\partial x}\right)^k \left(\dfrac{\partial w}{\partial y}\right)^n = ax^m y^l.$

Complete integral: $w = \dfrac{ka^{1/k}}{(m+k)C_1^n} x^{\frac{m+k}{k}} + \dfrac{nC_1^k}{n+l} y^{\frac{n+l}{n}} + C_2.$

28. $\left(\dfrac{\partial w}{\partial x}\right)^k \left(\dfrac{\partial w}{\partial y}\right)^n = ax^m y^l w^s.$

This is a special case of equation 3.2.2.8 with $f(x) = ax^m$, $g(y) = y^l$, and $h(w) = w^s$.

29. $\left(\dfrac{\partial w}{\partial x}\right)^k \left(\dfrac{\partial w}{\partial y}\right)^n = ax^m \dfrac{\partial w}{\partial y} + bx^l.$

Complete integral: $w = C_1 y + C_1^{-n/k} \displaystyle\int (aC_1 x^m + bx^l)^{1/k}\, dx + C_2.$

30. $\left(\dfrac{\partial w}{\partial x} + ay\right)^k \left(\dfrac{\partial w}{\partial y} + ax\right)^n = bx^m y^l.$

This is a special case of equation 3.2.2.11 with $f(x) = bx^m$ and $g(y) = y^l$.

3.1.4. More Complicated Equations

1. $\dfrac{\partial w}{\partial x} - a\exp\left(\lambda \dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = ae^{\lambda C_1} x + C_1 y + C_2.$

2. $\dfrac{\partial w}{\partial x} - a\exp\left(\lambda \dfrac{\partial w}{\partial y}\right) = bx^k.$

Complete integral: $w = C_1 y + ae^{\lambda C_1} x + \dfrac{b}{k+1} x^{k+1} + C_2.$

3. $\dfrac{\partial w}{\partial x} - (ay + bx^k)\exp\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

This is a special case of equation 3.3.1.6 with $f(u) = \exp(\lambda u)$ and $g(x) = bx^k$.

4. $ay\dfrac{\partial w}{\partial x} - \exp\left(\dfrac{\partial w}{\partial y}\right) + bxy = 0.$

Complete integral: $w = \dfrac{2C_1x - bx^2}{2a} + y\ln\bigl(C_1y\bigr) - y + C_2.$

5. $\dfrac{\partial w}{\partial x} - a\sinh\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = a\sinh(\lambda C_1)x + C_1y + C_2.$

6. $\dfrac{\partial w}{\partial x} - ax^k\sinh\left(\lambda\dfrac{\partial w}{\partial y}\right) = bx^n.$

Complete integral: $w = C_1y + \dfrac{a\sinh(\lambda C_1)}{k+1}x^{k+1} + \dfrac{b}{n+1}x^{n+1} + C_2.$

7. $\dfrac{\partial w}{\partial x} - aw\sinh\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

This is a special case of equation 3.3.1.10 with $f(u) = a\sinh(\lambda u)$ and $\beta = 0$.

8. $\dfrac{\partial w}{\partial x} - a\ln\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = a\ln(\lambda C_1)x + C_1y + C_2.$

9. $\dfrac{\partial w}{\partial x} - a\ln\left(\lambda\dfrac{\partial w}{\partial y}\right) = bx^k.$

Complete integral: $w = C_1y + a\ln(\lambda C_1)x + \dfrac{b}{k+1}x^{k+1} + C_2.$

10. $\dfrac{\partial w}{\partial x} - ax^k\ln\left(\lambda\dfrac{\partial w}{\partial y}\right) = bx^n.$

Complete integral: $w = C_1y + \dfrac{a\ln(\lambda C_1)}{k+1}x^{k+1} + \dfrac{b}{n+1}x^{n+1} + C_2.$

11. $\dfrac{\partial w}{\partial x} - aw\ln\left(\dfrac{\lambda}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = C_1\exp\bigl[C_2y + a\ln(\lambda C_2)x\bigr].$

12. $\dfrac{\partial w}{\partial x} - a\cos\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = a\cos(\lambda C_1)x + C_1y + C_2.$

13. $\dfrac{\partial w}{\partial x} + a\cos\left(\lambda\dfrac{\partial w}{\partial y}\right) = bx^k y + cx^n.$

This is a special case of equation 3.3.1.3 with $f(u) = a\cos(\lambda u)$, $g(x) = bx^k$, $h(x) = cx^n$.

14. $\dfrac{\partial w}{\partial x} - ax^k \sin\left(\lambda\dfrac{\partial w}{\partial y}\right) - bx^n\dfrac{\partial w}{\partial y} = 0.$

Complete integral: $w = \dfrac{a\sin(\lambda C_1)}{k+1}x^{k+1} + \dfrac{bC_1}{n+1}x^{n+1} + C_1 y + C_2.$

15. $\dfrac{\partial w}{\partial x} - (ax^k y + bx^n)\sin\left(\lambda\dfrac{\partial w}{\partial y}\right) = 0.$

This is a special case of equation 3.3.1.7 with $f(u) = \sin(\lambda u)$, $g(x) = ax^k$, and $h(x) = bx^n$.

16. $\dfrac{\partial w}{\partial x} - a\sin\left(\lambda x\dfrac{\partial w}{\partial y}\right) = b.$

Complete integral: $w = \dfrac{a}{\lambda C_1}\cos(\lambda C_1 x) + bx - C_1 y + C_2.$

17. $\dfrac{\partial w}{\partial x} - aw\sin\left(\dfrac{\lambda}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = C_1\exp\bigl[C_2 y + a\sin(\lambda C_2)x\bigr].$

▷ *For solutions of other nonlinear equations, see also Sections 3.2 and 3.3, where equations of a general form with arbitrary functions are considered.*

3.2. Equations Containing Arbitrary Functions of Independent Variables

3.2.1. Equations Contain One Arbitrary Power of Derivative

1. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^k = f(x).$

Complete integral: $w = -aC_1^k x + \displaystyle\int f(x)\,dx + C_1 y + C_2.$

2. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^k = f(y).$

Complete integral: $w = -C_1 x + \displaystyle\int\left[\frac{f(y)+C_1}{a}\right]^{1/k} dy + C_2.$

3. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^k = yf(x) + g(x).$

Complete integral:

$$w = y\varphi(x) + \int\bigl[g(x) - a\varphi^k(x)\bigr]\,dx + C_1, \quad \varphi(x) = \int f(x)\,dx + C_2.$$

4. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^k=f(x)+g(y).$

Complete integral: $w=-C_1x+\displaystyle\int f(x)\,dx+\int\left[\frac{g(y)+C_1}{a}\right]^{1/k}dy+C_2.$

5. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^k=f(x)w+g(x).$

Complete integral:

$$w=C_1F(x)+C_2yF(x)+F(x)\int\left[g(x)-aC_2^kF^k(x)\right]\frac{dx}{F(x)},\qquad F(x)=\exp\left[\int f(x)\,dx\right].$$

6. $\dfrac{\partial w}{\partial x}+f(x)\left(\dfrac{\partial w}{\partial y}\right)^k=g(x)w+h(x)y+s(x).$

Complete integral:

$$w=G(x)\left[y\varphi(x)+\psi(x)\right],\qquad G(x)=\exp\left[\int g(x)\,dx\right],$$

where

$$\varphi(x)=C_1+\int\frac{h(x)}{G(x)}\,dx,\quad \psi(x)=C_2+\int\left[s(x)-f(x)G^k(x)\varphi^k(x)\right]\frac{dx}{G(x)}.$$

7. $\dfrac{\partial w}{\partial x}+f(x)\left(\dfrac{\partial w}{\partial y}\right)^k+g(x)\dfrac{\partial w}{\partial y}=0.$

Complete integral: $w=C_1y+C_2-\displaystyle\int\left[C_1^kf(x)+C_1g(x)\right]dx.$

8. $\dfrac{\partial w}{\partial x}-\left[yf(x)+g(x)\right]\left(\dfrac{\partial w}{\partial y}\right)^k=0.$

Complete integral:

$$w=y\varphi(x)+\int\varphi^k(x)g(x)\,dx+C_1,\qquad \varphi(x)=\left[(1-k)\int f(x)\,dx+C_2\right]^{\frac{1}{1-k}}.$$

9. $\dfrac{\partial w}{\partial x}-f(x)w^n\left(\dfrac{\partial w}{\partial y}\right)^k-g(x)w=0.$

The transformation

$$w(x,y)=G(x)u(z,y),\quad z=\int f(x)G^{n+k-1}(x)\,dx,\qquad G(x)=\exp\left[\int g(x)\,dx\right],$$

leads to a simpler equation, $\dfrac{\partial u}{\partial z}-u^n\left(\dfrac{\partial u}{\partial y}\right)^k=0$. For $n+k\neq1$, the last equation has the complete integral

$$u=\left(C_1y+C_1^k\lambda^{k-1}z+C_2\right)^{\lambda},\qquad \lambda=\frac{k-1}{n+k-1}.$$

10. $\dfrac{\partial w}{\partial x} - f(x)e^{\beta w}\left(\dfrac{\partial w}{\partial y}\right)^k - g(x) = 0.$

The transformation

$$w(x,y) = u(z,y) + G(x), \quad z = \int f(x)\exp\bigl[\beta G(x)\bigr]\,dx, \qquad G(x) = \int g(x)\,dx,$$

leads to a simpler equation, $\dfrac{\partial u}{\partial z} - e^{\beta u}\left(\dfrac{\partial u}{\partial y}\right)^k = 0$, which has the complete integral

$$u = \frac{k}{\beta}\ln\left(C_1 - \frac{\beta}{k}y\right) - \frac{1}{\beta}\ln(C_2 - \beta z).$$

11. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^k = 0.$

Complete integral in implicit form: $\displaystyle\int \bigl[f(w)\bigr]^{\frac{1}{k-1}}\,dw = C_1^k x + C_1 y + C_2.$

12. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^k - \bigl[yg(x) + h(x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

The transformation

$$t = \int \varphi^k(x)\,dx, \quad z = \varphi(x)y + \int h(x)\varphi(x)\,dx, \qquad \varphi(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a simpler equation of the form 3.2.1.11: $\dfrac{\partial w}{\partial t} - f(w)\left(\dfrac{\partial w}{\partial z}\right)^k = 0.$

13. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^k + g(w)\dfrac{\partial w}{\partial y} = 0.$

Complete integral in implicit form: $\displaystyle\int \left[\frac{f(w)}{C_1 + g(w)}\right]^{\frac{1}{k-1}} dw = C_1 x + y + C_2.$

14. $\dfrac{\partial w}{\partial x} - f(x)g(y)h(w)\left(\dfrac{\partial w}{\partial y}\right)^k = 0.$

Complete integral in implicit form:

$$\int \bigl[h(w)\bigr]^{\frac{1}{k-1}}\,dw = C_1^k \int f(x)\,dx + C_1 \int [g(y)]^{-\frac{1}{k}}\,dy + C_2.$$

15. $f(x)\dfrac{\partial w}{\partial x} + g(y)\left(\dfrac{\partial w}{\partial y}\right)^k + h(x) = 0.$

Complete integral: $\displaystyle w = C_1 \int \frac{dy}{\bigl[g(y)\bigr]^{1/k}} - \int \frac{C_1^k + h(x)}{f(x)}dx + C_2.$

16. $\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x} = g(x)\left(\dfrac{\partial w}{\partial y}\right)^k + h(x).$

Complete integral: $w = C_1 y - \dfrac{1}{2}\displaystyle\int\left[f(x) \pm \sqrt{f^2(x) + 4C_1^k g(x) + 4h(x)}\right] dx + C_2.$

17. $\left(\dfrac{\partial w}{\partial x}\right)^k + f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} = g(y)w + xh(y) + r(y).$

Complete integral:

$$w = x\varphi(y) + \psi(y),$$

where the function $\varphi(y)$ and $\psi(y)$ are determined by solving the following system of ordinary differential equations:

$$f(y)\varphi\varphi'_y = g(y)\varphi + h(y), \tag{1}$$

$$f(y)\varphi\psi'_y = g(y)\psi - \varphi^k + r(y). \tag{2}$$

Abel's equation (1) can be considered independently. Given its solution, the solution of equation (2), which is linear in ψ, is easy to find.

For the simplest cases where g and/or h are identically zero, equation (1) has the solutions

$$\varphi(y) = \int \frac{g(y)}{f(y)}\,dy + C_1 \qquad \text{for} \quad h(y) \equiv 0 \quad \text{and any} \quad g(y),$$

$$\varphi(y) = \pm\left[2\int \frac{h(y)}{f(y)}\,dy + C_1\right]^{1/2} \qquad \text{for} \quad g(y) \equiv 0 \quad \text{and any} \quad h(y).$$

18. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = ay + f(x).$

Complete integral: $w = \big[a(k+1)x + C_1\big]^{\frac{1}{k+1}} y + \displaystyle\int f(x)\big[a(k+1)x + C_1\big]^{-\frac{k}{k+1}}\,dx + C_2.$

19. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = f(x)y + g(x).$

Complete integral:

$$w = \varphi(x)y + \int \frac{g(x)}{[\varphi(x)]^k}\,dx + C_1, \qquad \varphi(x) = \left[(k+1)\int f(x)\,dx + C_2\right]^{\frac{1}{k+1}}.$$

20. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = f(x)g(y).$

Complete integral: $w = C_1^k \displaystyle\int f(x)\,dx + \frac{1}{C_1}\int \big[g(y)\big]^{1/k}\,dy + C_2.$

21. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = aw + f(x).$

Complete integral: $w = (akx + C_1)^{\frac{1}{k}}(y + C_2) + (akx + C_1)^{\frac{1}{k}} \int f(x)(akx + C_1)^{-\frac{k+1}{k}}\,dx.$

22. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k = f(x)w + g(x)y + h(x).$

Complete integral:

$$w = y\varphi(x) + \psi(x),$$

where the function $\varphi(x)$ and $\psi(x)$ are determined by solving the following system of ordinary differential equations:

$$\varphi^k \varphi'_x = f(x)\varphi + g(x), \tag{1}$$
$$\varphi^k \psi'_x = f(x)\psi + h(x). \tag{2}$$

Equation (1) can be considered independently. Given its solution, the solution of equation (2), which is linear in ψ, is easy to find.

3.2.2. Equations Contain Two or Three Arbitrary Powers of Derivatives

1. $f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(y)\left(\dfrac{\partial w}{\partial y}\right)^n = h_1(x) + h_2(y).$

Complete integral: $w = \displaystyle\int \left[\frac{h_1(x) + C_1}{f(x)}\right]^{1/k} dx + \int \left[\frac{h_2(y) - C_1}{g(y)}\right]^{1/n} dy + C_2.$

2. $f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(y)\left(\dfrac{\partial w}{\partial y}\right)^n = aw + h_1(x) + h_2(y).$

Complete integral:

$$w = \varphi(x) + \psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined from the ordinary differential equations

$$f(x)(\varphi'_x)^k - a\varphi - h_1(x) = C_1,$$
$$g(y)(\psi'_y)^n - a\psi - h_2(y) = -C_1.$$

By the substitutions $z = \int |f(x)|^{-1/k}\,dx$ and $t = \int |g(y)|^{-1/n}\,dy$, respectively, these equations can be reduced to equations some solvable cases of which can be found in Zaitsev and Polyanin (1994) and Polyanin and Zaitsev (1995).

3. $\left(\dfrac{\partial w}{\partial x}\right)^k - f(w)\left(\dfrac{\partial w}{\partial y}\right)^n = 0, \qquad n \neq k.$

Complete integral in implicit form: $\displaystyle\int [f(w)]^{\frac{1}{n-k}}\,dw = C_1^n x + C_1^k y + C_2.$

4. $\left(\dfrac{\partial w}{\partial x}\right)^k - f(x)g(y)h(w)\left(\dfrac{\partial w}{\partial y}\right)^n = 0, \qquad n \neq k.$

Complete integral in implicit form:

$$\int \big[h(w)\big]^{\frac{1}{n-k}}\,dw = C_1^n \int \big[f(x)\big]^{\frac{1}{k}}\,dx + C_1^k \int \big[g(y)\big]^{-\frac{1}{n}}\,dy + C_2.$$

5. $xf(y)\left(\dfrac{\partial w}{\partial x}\right)^k + g(y)\left(\dfrac{\partial w}{\partial x}\right)^n - \dfrac{\partial w}{\partial y} = 0.$

Complete integral:

$$w = x\varphi(y) + \int g(y)\varphi^n(y)\,dy + C_1.$$

$$\varphi(y) = \left[(1-k)\int f(y)\,dy + C_2\right]^{\frac{1}{1-k}} \quad \text{if} \quad k \neq 1,$$

$$\varphi(y) = C_2 \exp\left[\int f(y)\,dy\right] \quad \text{if} \quad k = 1.$$

6. $f(x)w^n\left(\dfrac{\partial w}{\partial x}\right)^k + g(y)w^k\left(\dfrac{\partial w}{\partial y}\right)^n = w^{n+k}\ln w.$

Complete integral:

$$w = \exp\left\{\left[\frac{k-1}{k}\int \big[f(x)\big]^{-\frac{1}{k}}\,dx + C_1\right]^{\frac{k}{k-1}} + \left[\frac{n-1}{n}\int \big[g(y)\big]^{-\frac{1}{n}}\,dy + C_2\right]^{\frac{n}{n-1}}\right\}.$$

7. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y}\right)^k + \big[f_1(x)y + f_2(x)\big]\left(\dfrac{\partial w}{\partial y}\right)^n + f_3(x)\dfrac{\partial w}{\partial x} = g_1(x)w + g_2(x)y + g_3(x).$

A complete integral of this equation can be sought in the form $w = y\varphi(x) + \psi(x)$.

8. $\left(\dfrac{\partial w}{\partial x}\right)^k\left(\dfrac{\partial w}{\partial y}\right)^n = f(x)g(y)h(w).$

Complete integral in implicit form:

$$\int \big[h(w)\big]^{-\frac{1}{k+n}}\,dw = C_1^n \int \big[f(x)\big]^{\frac{1}{k}}\,dx + \frac{1}{C_1^k}\int \big[g(y)\big]^{\frac{1}{n}}\,dy + C_2.$$

9. $\left(\dfrac{\partial w}{\partial x}\right)^k\left(\dfrac{\partial w}{\partial y}\right)^n = f(x)\dfrac{\partial w}{\partial y} + g(x).$

Complete integral: $w = C_1 y + C_1^{-n/k}\int \big[C_1 f(x) + g(x)\big]^{1/k}\,dx + C_2.$

10. $a\left(\dfrac{\partial w}{\partial x}+cy\right)^k+b\left(\dfrac{\partial w}{\partial y}+cx\right)^n=f(x)+g(y).$

Complete integral: $w=-cxy+\displaystyle\int\left[\frac{f(x)+C_1}{a}\right]^{1/k}dx+\int\left[\frac{g(y)-C_1}{b}\right]^{1/n}dy+C_2.$

11. $\left(\dfrac{\partial w}{\partial x}+ay\right)^k\left(\dfrac{\partial w}{\partial y}+ax\right)^n=f(x)g(y).$

Complete integral: $w=-axy+C_1^n\displaystyle\int\big[f(x)\big]^{1/k}dx+\frac{1}{C_1^k}\int\big[g(y)\big]^{1/n}dy+C_2.$

12. $\left[\dfrac{\partial w}{\partial x}+f_1(x)\right]^k\left[\dfrac{\partial w}{\partial y}+f_2(y)\right]^n=g_1(x)g_2(y).$

Complete integral: $w=\displaystyle\int\big[C_1^n g_1^{1/k}(x)-f_1(x)\big]\,dx+\int\left[\frac{1}{C_1^k}g_2^{1/n}(y)-f_2(y)\right]dy+C_2.$

13. $\dfrac{\partial w}{\partial x}+f_1(x)\sqrt{g_1(x)+\left(\dfrac{\partial w}{\partial y}\right)^k}+f_2(x)\sqrt{g_2(x)+\left(\dfrac{\partial w}{\partial y}\right)^n}=0.$

Such equations are encountered in optimal control and differential games ($k=n=2$); see Melikyan (1998).

Complete integral: $w=C_1y+C_2-\displaystyle\int\left[f_1(x)\sqrt{g_1(x)+C_1^k}+f_2(x)\sqrt{g_2(x)+C_1^n}\right]dx.$

14. $\left(\dfrac{\partial w}{\partial x}\right)^k+f(y)\left(\dfrac{\partial w}{\partial x}\right)^{k-n}\left(\dfrac{\partial w}{\partial y}\right)^n=g(w).$

Complete integral in implicit form: $\displaystyle\int\frac{dw}{[g(w)]^{1/k}}=C_1x+\left(\frac{1-C_1^k}{C_1^{k-n}}\right)^{1/n}\int\frac{dy}{[f(y)]^{1/n}}+C_2.$

15. $\left(\dfrac{\partial w}{\partial x}\right)^k\left(\dfrac{\partial w}{\partial y}\right)^n=f(x)\left(\dfrac{\partial w}{\partial y}\right)^m+g(x).$

Complete integral: $w=C_1y+C_1^{-n/k}\displaystyle\int\big[C_1^m f(x)+g(x)\big]^{1/k}dx+C_2.$

3.3. Equations Containing Arbitrary Functions of Derivatives

3.3.1. Equations Contain Arbitrary Functions of One Variable

1. $\dfrac{\partial w}{\partial x}+f\left(\dfrac{\partial w}{\partial y}\right)=0.$

This equation is encountered in optimal control and differential games; see Subbotin (1991, 1995).

1°. Complete integral: $w = C_1 y - f(C_1)x + C_2$.

2°. On differentiating the equation with respect to y, we arrive at the quasilinear equation

$$\frac{\partial u}{\partial x} + f'(u)\frac{\partial u}{\partial y} = 0, \qquad u = \frac{\partial w}{\partial y},$$

which is discussed in detail in Section 23.3 (see equation (23.3.1.1)).

3°. The solution of the Cauchy problem with the initial condition $w(0, y) = \varphi(y)$ can be written in parametric form as

$$y = f'(\zeta)x + \xi, \quad w = \big[\zeta f'(\zeta) - f(\zeta)\big]x + \varphi(\xi), \qquad \text{where} \quad \zeta = \varphi'(\xi).$$

See also Examples 24.7 and 24.8 in Subsection 24.3.6.

2. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = g(x).$

Complete integral: $w = C_1 y - f(C_1)x + \displaystyle\int g(x)\,dx + C_2.$

3. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = g(x)y + h(x).$

Complete integral:

$$w = \varphi(x)y + \int \big[h(x) - f\big(\varphi(x)\big)\big]\,dx + C_1, \qquad \varphi(x) = \int g(x)\,dx + C_2.$$

4. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = g(x)w + h(x).$

Complete integral:

$$w = (C_1 y + C_2)\varphi(x) + \varphi(x)\int \big[h(x) - f(C_1\varphi(x))\big]\,\frac{dx}{\varphi(x)}, \qquad \varphi(x) = \exp\left[\int g(x)\,dx\right].$$

5. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = g(x)w + h(x)y + s(x).$

This is a special case of equation 3.3.2.4 with $F(x, u) = f(u) - s(x)$.

6. $\dfrac{\partial w}{\partial x} - \big[ay + g(x)\big]f\left(\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = \varphi(x)y + \int f\big(\varphi(x)\big)g(x)\,dx + C_1,$$

where the function $\varphi(x)$ is defined implicitly by the relation $\displaystyle\int \frac{d\varphi}{f(\varphi)} = ax + C_2.$

7. $\dfrac{\partial w}{\partial x} - \big[g(x)y + h(x)\big] f\left(\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = \varphi(x)y + \int f\big(\varphi(x)\big)h(x)\,dx + C_1,$$

where the function $\varphi(x)$ is defined implicitly by the relation $\displaystyle\int \frac{d\varphi}{f(\varphi)} = \int g(x)\,dx + C_2.$

8. $\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_0(x)\big] f\left(\dfrac{\partial w}{\partial y}\right) + h_2(x)w + h_1(x)y + h_0(x) = 0.$

1°. Complete integral:

$$w = \varphi(x)y + \psi(x),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by solving the ordinary differential equations

$$\varphi'_x + g_1(x)f(\varphi) + h_2(x)\varphi + h_1(x) = 0, \tag{1}$$
$$\psi'_x + g_0(x)f(\varphi) + h_2(x)\psi + h_0(x) = 0. \tag{2}$$

2°. For $g_1(x) \equiv 0$, the general solutions of equations (1) and (2) are

$$\varphi(x) = C_1 H(x) - H(x)\int \frac{h_1(x)}{H(x)}\,dx, \qquad H(x) = \exp\left[-\int h_2(x)\,dx\right],$$
$$\psi(x) = C_2 H(x) - H(x)\int \frac{h_0(x) + g_0(x)f(\varphi(x))}{H(x)}\,dx.$$

9. $\dfrac{\partial w}{\partial x} - wf\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = C_1 \exp\big[C_2 y + f(C_2)x\big].$

10. $\dfrac{\partial w}{\partial x} - wf\left(w^{\beta}\dfrac{\partial w}{\partial y}\right) = 0.$

For $\beta = -1$, see equation 3.3.1.9. Complete integral for $\beta \neq -1$:

$$w = \big[(1+\beta)y + C_1\big]^{\frac{1}{1+\beta}}\varphi(x + C_2),$$

where the function $\varphi(x)$ is defined implicitly by the relation $\displaystyle x = \int \frac{d\varphi}{\varphi f\left(\varphi^{\beta+1}\right)}.$

11. $\dfrac{\partial w}{\partial x} - f\left(e^{\beta w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = \frac{1}{\beta}\ln(\beta y + C_1) + \varphi(x + C_2),$$

where the function $\varphi(x)$ is defined implicitly by the relation $\displaystyle x = \int \frac{d\varphi}{f\left(e^{\beta\varphi}\right)}.$

12. $\dfrac{\partial w}{\partial x} - g(x)w^{\beta} f\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = h(x)w.$

The transformation

$$w(x,y) = H(x)u(z,y), \quad z = \int g(x)H^{\beta-1}(x)\,dx, \qquad H(x) = \exp\left[\int h(x)\,dx\right],$$

leads to a simpler equation, $\dfrac{\partial u}{\partial z} - u^{\beta} f\left(\dfrac{1}{u}\dfrac{\partial u}{\partial y}\right) = 0,$ which, by solving for $\dfrac{\partial u}{\partial y}$, may be rewritten in the form of equation 3.3.1.10.

13. $\dfrac{\partial w}{\partial x} - g(x)e^{\beta w} f\left(\dfrac{\partial w}{\partial y}\right) = h(x).$

The transformation

$$w(x,y) = u(z,y) + H(x), \quad z = \int g(x)\exp\big[\beta H(x)\big]\,dx, \qquad H(x) = \int h(x)\,dx,$$

leads to a simpler equation, $\dfrac{\partial u}{\partial z} - e^{\beta u} f\left(\dfrac{\partial u}{\partial y}\right) = 0,$ which, by solving for $\dfrac{\partial u}{\partial y}$, may be rewritten in the form of equation 3.3.1.11.

14. $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = \left(\dfrac{\partial w}{\partial x}\right)^{k} f\left(\dfrac{\partial w}{\partial y} \Big/ \dfrac{\partial w}{\partial x}\right).$

Complete integral: $w = \dfrac{k-1}{k}\big[f(C_1)\big]^{\frac{1}{1-k}}(x + C_1 y)^{\frac{k}{k-1}} + C_2.$

15. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = f\left(x\dfrac{\partial w}{\partial y} - y\dfrac{\partial w}{\partial x}\right).$

Complete integral: $w = C_1 \arctan\dfrac{y}{x} + C_2 + \displaystyle\int \sqrt{f(C_1)R^2 - C_1^2}\,\frac{dR}{R},$ where $R = \sqrt{x^2 + y^2}.$

16. $\Phi\left(\dfrac{\partial w}{\partial x} + f_1(x)\right)\Psi\left(\dfrac{\partial w}{\partial y} + f_2(y)\right) = g_1(x)g_2(y).$

Complete integral:

$$w = \int \big[\varphi\big(C_1 g_1(x)\big) - f_1(x)\big]\,dx + \int \big[\psi\big(C_1^{-1} g_2(y)\big) - f_2(y)\big]\,dy + C_2,$$

where φ is the inverse of Φ and ψ is the inverse of Ψ.

3.3.2. Equations Contain Arbitrary Functions of Two Variables

1. $\dfrac{\partial w}{\partial x} - F\left(x, \dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = \displaystyle\int F(x, C_1)\,dx + C_1 y + C_2.$

⊙ Reference: E. Kamke (1965).

2. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}\right) = aw.$

Complete integral: $w = e^{ax}(C_1 y + C_2) - e^{ax} \int e^{-ax} F(x, C_1 e^{ax})\, dx.$

3. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}\right) = g(x)w.$

Complete integral:

$$w = \varphi(x)(C_1 y + C_2) - \varphi(x) \int F\big(x, C_1\varphi(x)\big) \frac{dx}{\varphi(x)}, \qquad \varphi(x) = \exp\left[\int g(x)\, dx\right].$$

4. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}\right) = g(x)w + h(x)y.$

Complete integral:

$$w = y\varphi(x) + \psi(x),$$

where

$$\varphi(x) = C_1 G(x) + G(x) \int \frac{h(x)}{G(x)}\, dx, \qquad G(x) = \exp\left[\int g(x)\, dx\right],$$
$$\psi(x) = C_2 G(x) - G(x) \int F\big(x, \varphi(x)\big) \frac{dx}{G(x)}.$$

5. $\dfrac{\partial w}{\partial x} - F\left(\dfrac{y}{x}, \dfrac{\partial w}{\partial y}\right) = 0.$

This is a special case of equation 3.3.4.6.

6. $\dfrac{\partial w}{\partial x} - wF\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = C_1 \exp\left[C_2 y + \int F(x, C_2)\, dx\right].$

7. $\dfrac{\partial w}{\partial x} - g(x)F\left(w, \dfrac{\partial w}{\partial y}\right) - h(x)\dfrac{\partial w}{\partial y} = 0.$

By passing to the new variables $t = \int g(x)\, dx$, $z = y + \int h(x)\, dx$, we obtain a simpler equation, $\dfrac{\partial w}{\partial t} - F\left(w, \dfrac{\partial w}{\partial z}\right) = 0$, which is a special case of equation 3.3.3.4.

8. $\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x)\dfrac{\partial w}{\partial x} = F\left(x, \dfrac{\partial w}{\partial y}\right).$

Complete integral: $w = -\dfrac{1}{2} \int \left[g(x) \pm \sqrt{g^2(x) + 4F(x, C_1)}\right] dx + C_1 y + C_2.$

9. $F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = C_1 x + C_2 y + C_3,$$

where C_1 and C_3 are arbitrary constants and the constant C_2 is related to C_1 by $F(C_1, C_2) = 0$.

⊙ Reference: E. Kamke (1965).

10. $w = x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right).$

Clairaut's equation. Complete integral: $w = C_1 x + C_2 y + F(C_1, C_2)$.

⊙ References: R. Courant (1962), E. Kamke (1965).

11. $F_1\left(x, \dfrac{\partial w}{\partial x}\right) = F_2\left(y, \dfrac{\partial w}{\partial y}\right).$

A separable equation. Complete integral:

$$w = \varphi(x) + \psi(y) + C_1,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined from the ordinary differential equations

$$F_1\left(x, \varphi'_x\right) = C_2, \qquad F_2\left(y, \psi'_y\right) = C_2.$$

⊙ Reference: E. Kamke (1965).

12. $F_1\left(x, \dfrac{\partial w}{\partial x}\right) + F_2\left(y, \dfrac{\partial w}{\partial y}\right) + aw = 0.$

A separable equation. Complete integral:

$$w = \varphi(x) + \psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined from the ordinary differential equations

$$F_1\left(x, \varphi'_x\right) + a\varphi = C_1,$$
$$F_2\left(y, \psi'_y\right) + a\psi = -C_1,$$

where C_1 is an arbitrary constant. If $a \neq 0$, one can set $C_1 = 0$ in these equations.

⊙ Reference: E. Kamke (1965).

13. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + w^k F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w(x, y) = \varphi(x)\psi(y).$$

The functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by solving the ordinary differential equations

$$\varphi^{-k} F_1\left(x, \frac{\varphi'_x}{\varphi}\right) = C, \qquad \psi^k F_2\left(y, \frac{\psi'_y}{\psi}\right) = -C,$$

where C is an arbitrary constant.

14. $\boldsymbol{F_1\left(x, \dfrac{\partial w}{\partial x}\right) + e^{\lambda w} F_2\left(y, \dfrac{\partial w}{\partial y}\right) = 0.}$

Complete integral:

$$w(x, y) = \varphi(x) + \psi(y).$$

The functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by solving the ordinary differential equations

$$e^{-\lambda\varphi} F_1\big(x, \varphi'_x\big) = C, \qquad e^{\lambda\psi} F_2\big(y, \psi'_y\big) = -C,$$

where C is an arbitrary constant.

15. $\boldsymbol{F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = k \ln w.}$

Complete integral:

$$w(x, y) = \varphi(x)\psi(y).$$

The functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by solving the ordinary differential equations

$$F_1\left(x, \frac{\varphi'_x}{\varphi}\right) - k \ln \varphi = C, \qquad F_2\left(y, \frac{\psi'_y}{\psi}\right) - k \ln \psi = -C,$$

where C is an arbitrary constant.

16. $\boldsymbol{\dfrac{\partial w}{\partial x} + y F_1\left(x, \dfrac{\partial w}{\partial y}\right) + F_2\left(x, \dfrac{\partial w}{\partial y}\right) = 0.}$

Complete integral:

$$w = \varphi(x)y - \int F_2\big(x, \varphi(x)\big)\, dx + C_1,$$

where the function $\varphi(x)$ is determined by solving the ordinary differential equation

$$\varphi'_x + F_1(x, \varphi) = 0.$$

17. $\boldsymbol{\dfrac{\partial w}{\partial x} + w F_1\left(x, \dfrac{\partial w}{\partial y}\right) + y F_2\left(x, \dfrac{\partial w}{\partial y}\right) + F_3\left(x, \dfrac{\partial w}{\partial y}\right) = 0.}$

Complete integral:

$$w = \varphi(x)y + \psi(x),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by solving the following system of ordinary differential equations:

$$\varphi'_x + F_1(x, \varphi)\varphi + F_2(x, \varphi) = 0, \tag{1}$$
$$\psi'_x + F_1(x, \varphi)\psi + F_3(x, \varphi) = 0. \tag{2}$$

Equation (1) can be considered independently. Given its solution, the integration of equation (2), which is linear in ψ, is easy.

18. $\boldsymbol{F_1\left(y, \dfrac{\partial w}{\partial x}\right)\dfrac{\partial w}{\partial y} + wF_2\left(y, \dfrac{\partial w}{\partial x}\right) + xF_3\left(y, \dfrac{\partial w}{\partial x}\right) + F_4\left(y, \dfrac{\partial w}{\partial x}\right) = 0.}$

Complete integral:

$$w = x\varphi(y) + \psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by solving the following system of ordinary differential equations:

$$F_1(y,\varphi)\varphi'_y + F_2(y,\varphi)\varphi + F_3(y,\varphi) = 0, \tag{1}$$

$$F_1(y,\varphi)\psi'_y + F_2(y,\varphi)\psi + F_4(y,\varphi) = 0. \tag{2}$$

Equation (1) can be considered independently. Given its solution, the solution of equation (2), which is linear in ψ, is easy to find.

19. $\boldsymbol{F\left(\dfrac{\partial w}{\partial x} + ay,\ \dfrac{\partial w}{\partial y} + ax\right) = 0.}$

Complete integral: $w = -axy + C_1x + C_2y + C_3,$ where $F(C_1, C_2) = 0.$

20. $\boldsymbol{F\left(\dfrac{\partial w}{\partial x} + akx^{k-1}y^n,\ \dfrac{\partial w}{\partial y} + anx^k y^{n-1}\right) = 0.}$

Complete integral: $w = -ax^k y^n + C_1x + C_2y + C_3,$ where $F(C_1, C_2) = 0.$

21. $\boldsymbol{\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = F\left(x^2 + y^2,\ y\dfrac{\partial w}{\partial x} - x\dfrac{\partial w}{\partial y}\right).}$

Complete integral:

$$w = -C_1 \arctan\frac{y}{x} + \frac{1}{2}\int \sqrt{\xi F(\xi, C_1) - C_1^2}\,\frac{d\xi}{\xi} + C_2, \qquad \xi = x^2 + y^2.$$

⊙ Reference: E. Kamke (1965).

3.3.3. Equations Contain Arbitrary Functions of Three Variables

1. $\boldsymbol{\dfrac{\partial w}{\partial x} + F\left(x, y, \dfrac{\partial w}{\partial y}\right) = g(x)w.}$

The substitution

$$w(x,y) = G(x)u(x,y), \qquad G(x) = \exp\left[\int g(x)\,dx\right],$$

leads to a similar equation which does not depend on u explicitly:

$$\frac{\partial u}{\partial x} + \mathcal{F}\left(x, y, \frac{\partial u}{\partial y}\right) = 0, \qquad \text{where} \quad \mathcal{F}\big(x, y, p\big) = \frac{1}{G(x)}F\big(x, y, G(x)p\big).$$

It follows that the original equation has a complete integral of the following structure: $w = C_1G(x) + \varphi(x, y, C_2).$

2. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral: $w = C_1 y + \varphi(x, C_1) + C_2$, where the function $\varphi = \varphi(x, C_1)$ is determined from the ordinary differential equation $F\left(x, \varphi'_x, C_1\right) = 0$.

3. $F\left(ax + by, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

For $b = 0$, see equation 3.3.3.2. Complete integral for $b \neq 0$:

$$w = C_1 x + \varphi(z, C_1) + C_2, \qquad z = ax + by,$$

where the function $\varphi = \varphi(z)$ is determined from the nonlinear ordinary differential equation $F\left(z, a\varphi'_z + C_1, b\varphi'_z\right) = 0$.

4. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

We seek a particular solution of the equation in the form

$$w = w(z), \qquad z = C_1 x + C_2 y,$$

where C_1 and C_2 are arbitrary constants. We arrive at the following ordinary differential equation for $w = w(z)$:

$$F\left(w, C_1 w'_z, C_2 w'_z\right) = 0.$$

Solving this equation for the derivative yields $w'_z = f(w)$. By integrating the last equation, we obtain a complete integral of the original partial differential equation in implicit form:

$$\int \frac{dw}{f(w)} = C_1 x + C_2 y + C_3.$$

This equation defines cylindrical surfaces whose elements are parallel to the xy-plane.

⊙ Reference: E. Kamke (1965).

5. $F\left(ax + by + cw, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

For $c = 0$, see equation 3.3.3.3. If $c \neq 0$, then the substitution $cu = ax + by + cw$ leads to an equation of the form 3.3.3.4: $F\left(cu, \dfrac{\partial u}{\partial x} - \dfrac{a}{c}, \dfrac{\partial u}{\partial y} - \dfrac{b}{c}\right) = 0$.

6. $F\left(x, \dfrac{\partial w}{\partial x} + ay, \dfrac{\partial w}{\partial y} + ax\right) = 0.$

Complete integral:

$$w = -axy + C_1 y + \varphi(x) + C_2,$$

where the function $\varphi(x)$ is determined by solving the first-order ordinary differential equation $F\left(x, \varphi'_x, C_1\right) = 0$.

7. $F\left(w,\ f(x)\dfrac{\partial w}{\partial x},\ g(y)\dfrac{\partial w}{\partial y}\right)=0.$

The transformation $\xi=\int\dfrac{dx}{f(x)}$, $\eta=\int\dfrac{dy}{g(y)}$ leads to the equation of the form 3.3.3.4: $F\left(w,w_\xi,w_\eta\right)=0$.

⊙ Reference: E. Kamke (1965).

8. $F\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}\right)=0.$

This is a special case of equation 3.3.4.3.

9. $F\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ w-x\dfrac{\partial w}{\partial x}-y\dfrac{\partial w}{\partial y}\right)=0.$

Clairaut's equation (*in implicit form*). Complete integral: $w=C_1x+C_2y+C_3$, where the constants C_1, C_2, and C_3 are related by $F(C_1,C_2,C_3)=0$.

⊙ References: R. Courant (1962), E. Kamke (1965).

10. $F\left(y,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}+f(y)w+xg(y)+h(y)\right)=0.$

Complete integral:

$$w=x\varphi(y)+\psi(y),\quad \text{where } \varphi(y)=C_1F(y)-F(y)\int\frac{g(y)}{F(y)}\,dy,\ \ F(y)=\exp\left[-\int f(y)\,dy\right].$$

The function $\psi(y)$ is determined by solving the nonlinear ordinary differential equation $F\left(y,\ \varphi(y),\ \psi'_y+f(y)\psi+h(y)\right)=0$.

11. $xF\left(y,\dfrac{\partial w}{\partial x},x\dfrac{\partial w}{\partial x}-w\right)+\dfrac{\partial w}{\partial y}G\left(y,\dfrac{\partial w}{\partial x},x\dfrac{\partial w}{\partial x}-w\right)=H\left(y,\dfrac{\partial w}{\partial x},x\dfrac{\partial w}{\partial x}-w\right).$

The Euler transformation $x=W_X$, $y=Y$, $w=XW_X-W$, where $W=W(X,Y)$, brings the nonlinear equation in question to the quasilinear equation

$$F(Y,X,W)\frac{\partial W}{\partial X}-G(Y,X,W)\frac{\partial W}{\partial Y}=H(Y,X,W).$$

Equations of this type are considered in Chapter 23.

⊙ Reference: E. Kamke (1965).

12. $xF\left(u,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\right)+yG\left(u,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\right)=H\left(u,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\right),\quad u=x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}-w.$

The Legendre transformation $x=W_X$, $y=W_Y$, $w=XW_X+YW_Y-W$, where $W=W(X,Y)$, brings the nonlinear equation in question to the quasilinear equation

$$F(W,X,Y)\frac{\partial W}{\partial X}+G(W,X,Y)\frac{\partial W}{\partial Y}=H(W,X,Y).$$

Equations of this type are considered in Chapter 23.

⊙ Reference: E. Kamke (1965).

3.3.4. Equations Contain Arbitrary Functions of Four Variables

1. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, w - y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = C_1 y + \varphi(x),$$

where the function $\varphi(x)$ is determined from the nonlinear ordinary differential equation $F\big(x, \varphi'_x, C_1, \varphi\big) = 0$.

⊙ Reference: E. Kamke (1965).

2. $F\left(y, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, w - x\dfrac{\partial w}{\partial x}\right) = 0.$

Complete integral:

$$w = C_1 x + \varphi(y),$$

where the function $\varphi(y)$ is determined from the nonlinear ordinary differential equation $F\big(y, C_1, \varphi'_y, \varphi\big) = 0$.

⊙ Reference: E. Kamke (1965).

3. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = \varphi(\xi), \qquad \xi = C_1 x + C_2 y,$$

where the function $\varphi(\xi)$ is determined by solving the nonlinear ordinary differential equation $F\big(\varphi, C_1\varphi'_\xi, C_2\varphi'_\xi, \xi\varphi'_\xi\big) = 0$.

4. $F\left(ax + by, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, w - x\dfrac{\partial w}{\partial x} - y\dfrac{\partial w}{\partial y}\right) = 0.$

Complete integral:

$$w = C_1 x + C_2 y + \varphi(\xi), \qquad \xi = ax + by,$$

where the function $\varphi(\xi)$ is determined by solving the nonlinear ordinary differential equation $F\big(\xi, a\varphi'_\xi + C_1, b\varphi'_\xi + C_2, \varphi - \xi\varphi'_\xi\big) = 0$.

5. $F\left(x, \dfrac{\partial w}{\partial x}, G\left(y, \dfrac{\partial w}{\partial y}\right)\right) = 0.$

Complete integral:

$$w = \varphi(x, C_1) + \psi(y, C_1) + C_2,$$

where the functions φ and ψ are determined by the ordinary differential equations

$$F(x, \varphi'_x, C_1) = 0, \qquad G(y, \psi'_y) = C_1.$$

On solving these equations for the derivatives, we obtain linear separable equations, which are easy to integrate.

6. $\boldsymbol{F\left(\dfrac{y}{x}, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} - w\right) = 0.}$

1°. Complete integral:

$$w = x\varphi\left(\frac{y}{x}, C_1, C_2\right) + C_2,$$

where the function $\varphi = \varphi(z, C_1, C_2)$ is the general solution of the first-order ordinary differential equation

$$F\big(z, \varphi - z\varphi'_z, \varphi'_z, C_2\big) = 0.$$

2°. A complete integral, $w = w(x, y, C_1, C_2)$, may be obtained by solving the equations

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} - w = C_1, \qquad F\left(\frac{y}{x}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, C_1\right) = 0,$$

for the derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ followed by determining w with the aid of the Lagrange–Charpit method (see Subsection 24.1.3).

⊙ Reference: E. Kamke (1965).

Chapter 4
First-Order Nonlinear Equations with Three or More Independent Variables

4.1. Nonlinear Equations with Three Variables Quadratic in Derivatives

4.1.1. Equations Contain Squares of One or Two Derivatives

1. $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial x}\right)^2 = k.$

Complete integral: $w = C_1x + C_2y + C_3z + C_4$, where $C_1 + aC_2 + bC_3 + cC_1^2 = k.$

2. $\dfrac{\partial w}{\partial x} + ax\dfrac{\partial w}{\partial y} + bx\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial y}\right)^2 = k.$

Complete integral: $w = C_1y + C_2z - \frac{1}{2}(aC_1 + bC_2)x^2 + (k - cC_1^2)x + C_3.$

3. $\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial y}\right)^2 = k.$

Complete integral:

$$w = aC_1z + (C_2e^{-ax} - bC_1)y + (k - b^2cC_1^2)x - \frac{2bc}{a}C_1C_2e^{-ax} + \frac{cC_2^2}{2a}e^{-2ax} + C_3.$$

4. $\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y} + bz\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial x}\right)^2 = k.$

Complete integral: $w = C_1x + C_2\ln|y| + C_3|z| + C_4$, where $C_1 + aC_2 + bC_3 + cC_1^2 = k.$

5. $\dfrac{\partial w}{\partial x} + az\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial y}\right)^2 = kx + s.$

Complete integral:

$$w = \left(C_1e^{\lambda x} + C_2e^{-\lambda x}\right)y - \frac{\lambda}{b}\left(C_1e^{\lambda x} - C_2e^{-\lambda x}\right)z + \tfrac{1}{2}kx^2 + (s - 2cC_1C_2)x - \frac{cC_1^2}{2\lambda}e^{2\lambda x} + \frac{cC_2^2}{2\lambda}e^{-2\lambda x} + C_3, \qquad \lambda = \sqrt{ab}.$$

6. $\dfrac{\partial w}{\partial x} + axz\dfrac{\partial w}{\partial y} + bxy\dfrac{\partial w}{\partial z} + c\left(\dfrac{\partial w}{\partial y}\right)^2 = kx + s.$

Complete integral:

$$w = y\left[C_1\exp(\lambda x^2) + C_2\exp(-\lambda x^2)\right] - \frac{2\lambda}{b}z\left[C_1\exp(\lambda x^2) - C_2\exp(-\lambda x^2)\right]$$
$$+ \tfrac{1}{2}kx^2 + sx - c\int\left[C_1\exp(\lambda x^2) + C_2\exp(-\lambda x^2)\right]^2 dx + C_3, \qquad \lambda = \tfrac{1}{2}\sqrt{ab}.$$

7. $\dfrac{\partial w}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial z}\right)^2 + az = 0.$

A problem of the motion of a material point under gravity is reduced to this equation; x is time, y and z are the longitudinal and transverse spatial coordinates, and a is the acceleration due to gravity.

Complete integral: $w = -C_1x + C_2y + C_3 \pm \dfrac{1}{3a}(2C_1 - C_2^2 - 2az)^{3/2}.$

⊙ Reference: G. K. Suslov (1946).

8. $\dfrac{\partial w}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial z}\right)^2 = \dfrac{a}{\sqrt{y^2+z^2}}.$

A problem on the motion of two point bodies attracting to each other according to Newton's law of universal gravitation is reduced to this equation. By passing to the polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we arrive at the separable equation

$$\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{a}{r}.$$

Complete integral:

$$w = -C_1x - C_2\theta \mp \int\left(2C_1 + \frac{2a}{r} - \frac{C_2^2}{r^2}\right)^{1/2} dr + C_3.$$

⊙ Reference: R. Courant (1962).

9. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\left(\dfrac{\partial w}{\partial z}\right)^2 = cw + kx^n.$

This is a special case of equation 4.3.1.2 with $f(x) = c$ and $g(x) = kx^n$. Complete integral:

$$w = (C_1 + C_2y + C_3z)e^{cx} - \frac{1}{c}(aC_2^2 + bC_3^2)e^{2cx} + ke^{cx}\int x^n e^{-cx}\,dx.$$

10. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\left(\dfrac{\partial w}{\partial z}\right)^2 = cw + ke^{\beta x}.$

This is a special case of equation 4.3.1.2 with $f(x) = c$ and $g(x) = ke^{\beta x}$. Complete integral:

$$w = (C_1 + C_2y + C_3z)e^{cx} - \frac{1}{c}(aC_2^2 + bC_3^2)e^{2cx} + \frac{k}{\beta - c}e^{\beta x}.$$

11. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2=bw^2+cx^nw+kx^m.$

This is a special case of equation 4.3.1.3 with $f(x)=cx^n$ and $g(x)=kx^m$.

12. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+b_1\dfrac{\partial w}{\partial y}+b_2\dfrac{\partial w}{\partial z}=0.$

Complete integral: $w=-(a_1C_1^2+a_2C_2^2+b_1C_1+b_2C_2)x+C_1y+C_2z+C_3.$

13. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+b_1z\dfrac{\partial w}{\partial y}+b_2y\dfrac{\partial w}{\partial z}=0.$

Complete integral:

$$w=y\varphi_1(x)+z\varphi_2(x)-\int\left[a_1\varphi_1^2(x)+a_2\varphi_2^2(x)\right]dx+C_3,$$

where

$$\varphi_1(x)=C_1e^{\lambda x}+C_2e^{-\lambda x},\quad \varphi_2(x)=-\frac{\lambda}{b_2}\left(C_1e^{\lambda x}-C_2e^{-\lambda x}\right),\quad \lambda=\sqrt{b_1b_2}.$$

14. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+\left(b_{11}y+b_{12}z\right)\dfrac{\partial w}{\partial y}+\left(b_{21}y+b_{22}z\right)\dfrac{\partial w}{\partial z}=sw+c_1y+c_2z+c_0.$

This is a special case of equation 4.3.1.17.

15. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+b_1x^kz\dfrac{\partial w}{\partial y}+b_2x^ky\dfrac{\partial w}{\partial z}=cx^n.$

Complete integral:

$$w=y\varphi_1(x)+z\varphi_2(x)+\frac{c}{n+1}x^{n+1}-\int\left[a_1\varphi_1^2(x)+a_2\varphi_2^2(x)\right]dx+C_3,$$

where

$$\varphi_1(x)=C_1\exp(\lambda x^{k+1})+C_2\exp(-\lambda x^{k+1}),\quad \lambda=\frac{\sqrt{b_1b_2}}{k+1},$$
$$\varphi_2(x)=-\frac{\lambda(k+1)}{b_2}\left[C_1\exp(\lambda x^{k+1})-C_2\exp(-\lambda x^{k+1})\right].$$

16. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+b_1e^{\beta x}z\dfrac{\partial w}{\partial y}+b_2e^{\beta x}y\dfrac{\partial w}{\partial z}=0.$

Complete integral:

$$w=y\varphi_1(x)+z\varphi_2(x)-\int\left[a_1\varphi_1^2(x)+a_2\varphi_2^2(x)\right]dx+C_3,$$

where

$$\varphi_1(x)=b_2\left[C_1\exp\left(\lambda e^{\beta x}\right)+C_2\exp\left(-\lambda e^{\beta x}\right)\right],\quad \lambda=\frac{\sqrt{b_1b_2}}{\beta},$$
$$\varphi_2(x)=\beta\lambda\left[-C_1\exp\left(\lambda e^{\beta x}\right)+C_2\exp\left(-\lambda e^{\beta x}\right)\right].$$

17. $\dfrac{\partial w}{\partial x}+ax^k\left(\dfrac{\partial w}{\partial y}\right)^2+bx^n\left(\dfrac{\partial w}{\partial z}\right)^2=cx^mw.$

Complete integral:

$$w=\varphi(x)(C_1+C_2y+C_3z)-\varphi(x)\int\varphi(x)\left(aC_2^2x^k+bC_3^2x^n\right)dx,\qquad \varphi(x)=\exp\left(\frac{c}{m+1}x^{m+1}\right).$$

18. $\dfrac{\partial w}{\partial x}+a_1x^k\left(\dfrac{\partial w}{\partial y}\right)^2+a_2x^n\left(\dfrac{\partial w}{\partial z}\right)^2=bw+c_1x^my+c_2x^sz.$

Complete integral:

$$w=e^{bx}\left[y\varphi_1(x)+z\varphi_2(x)+\psi(x)\right],$$

where

$$\varphi_1(x)=C_1+c_1\int x^me^{-bx}\,dx,\quad \varphi_2(x)=C_2+c_2\int x^se^{-bx}\,dx,$$
$$\psi(x)=C_3-\int e^{bx}\left[a_1x^k\varphi_1^2(x)+a_2x^n\varphi_2^2(x)\right]dx.$$

19. $\dfrac{\partial w}{\partial x}+ax^k\left(\dfrac{\partial w}{\partial y}+cz\right)^2+bx^n\left(\dfrac{\partial w}{\partial z}+cy\right)^2=sx^m.$

Complete integral: $w=-cyz+C_1y+C_2z-\dfrac{aC_1^2}{k+1}x^{k+1}-\dfrac{bC_2^2}{n+1}x^{n+1}+\dfrac{s}{m+1}x^{m+1}+C_3.$

20. $\dfrac{\partial w}{\partial x}+a_1y^k\left(\dfrac{\partial w}{\partial y}\right)^2+a_2z^n\left(\dfrac{\partial w}{\partial z}\right)^2=bx^m+c_1y^p+c_2z^q.$

Complete integral:

$$w=C_1-(C_2+C_3)x+\frac{b}{m+1}x^{m+1}+\int\sqrt{\frac{C_2+c_1y^p}{a_1y^k}}\,dy+\int\sqrt{\frac{C_3+c_2z^q}{a_2z^n}}\,dz.$$

21. $\dfrac{\partial w}{\partial x}+a_1y^k\left(\dfrac{\partial w}{\partial y}\right)^2+a_2z^n\left(\dfrac{\partial w}{\partial z}\right)^2=bw^2+cx^mw+sx^q.$

This is a special case of equation 4.3.1.10 with $f_1(y)=a_1y^k$, $f_2(z)=a_2z^n$, $g(x)=cx^m$, and $h(x)=sx^q$.

22. $\dfrac{\partial w}{\partial x}+\left(a_1x^{k_1}y+b_1x^{n_1}\right)\left(\dfrac{\partial w}{\partial y}\right)^2+\left(a_2x^{k_2}z+b_2x^{n_2}\right)\left(\dfrac{\partial w}{\partial z}\right)^2=c_1x^mw+c_2x^s.$

This is a special case of equation 4.3.1.12.

23. $\dfrac{\partial w}{\partial x}+A(ax+by+cz)^k\left(\dfrac{\partial w}{\partial y}\right)^2+B(ax+by+cz)^n\left(\dfrac{\partial w}{\partial z}\right)^2=D(ax+by+cz)^m.$

This is a special case of equation 4.3.1.14 with $f(u)=Au^k$, $g(u)=Bu^n$, and $h(u)=Du^m$.

24. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta x}\left(\dfrac{\partial w}{\partial z}\right)^2 = cw.$

Complete integral: $w = C_1e^{cx} + e^{cx}(C_2y + C_3z) - \dfrac{aC_2^2}{c+\lambda}e^{(2c+\lambda)x} - \dfrac{bC_3^2}{c+\beta}e^{(2c+\beta)x}.$

25. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + be^{\beta x}\left(\dfrac{\partial w}{\partial z}\right)^2 = ce^{\gamma x}w.$

Complete integral:

$$w = \varphi(x)(C_1 + C_2y + C_3z) - \varphi(x)\int \varphi(x)\big(aC_2^2e^{\lambda x} + bC_3^2e^{\beta x}\big)\,dx, \qquad \varphi(x) = \exp\left(\frac{c}{\gamma}e^{\gamma x}\right).$$

26. $\dfrac{\partial w}{\partial x} + a_1e^{\lambda x}\left(\dfrac{\partial w}{\partial y}\right)^2 + a_2e^{\beta x}\left(\dfrac{\partial w}{\partial z}\right)^2 = bw + c_1e^{\gamma x}y + c_2e^{\mu x}z.$

Complete integral:

$$w = e^{bx}\big[y\varphi_1(x) + z\varphi_2(x) + \psi(x)\big],$$

where

$$\varphi_1(x) = C_1 + \frac{c_1}{\gamma - b}e^{(\gamma-b)x}, \quad \varphi_2(x) = C_2 + \frac{c_2}{\mu - b}e^{(\mu-b)x},$$
$$\psi(x) = C_3 - \int e^{2bx}\big[a_1e^{\lambda x}\varphi_1^2(x) + a_2e^{\beta x}\varphi_2^2(x)\big]\,dx.$$

27. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y} + cz\right)^2 + be^{\beta x}\left(\dfrac{\partial w}{\partial z} + cy\right)^2 = ke^{\gamma x} + s.$

Complete integral: $w = -cyz + C_1y + C_2z - \dfrac{a}{\lambda}C_1^2e^{\lambda x} - \dfrac{b}{\beta}C_2^2e^{\beta x} + \dfrac{k}{\gamma}e^{\gamma x} + sx + C_3.$

28. $\dfrac{\partial w}{\partial x} + \dfrac{a}{\sin^2 z}\left(\dfrac{\partial w}{\partial y}\right)^2 + a\left(\dfrac{\partial w}{\partial z}\right)^2 + b\cos z = 0.$

A problem of the motion of a rod leaning on a horizontal plane and a vertical axis is reduced to this equation (x is time, and y and z are angular coordinates).

Complete integral: $w = -aC_1x + C_2y \pm \displaystyle\int\left(C_1 - \frac{C_2^2}{\sin^2 z} - \frac{b}{a}\cos z\right)^{1/2} dz + C_3.$

⊙ Reference: A. P. Markeev (1990).

29. $\dfrac{\partial w}{\partial x} + ay^kw^m\left(\dfrac{\partial w}{\partial y}\right)^2 + bz^nw^s\left(\dfrac{\partial w}{\partial z}\right)^2 = 0.$

Complete integral: $x + \dfrac{aC_1^2}{m+1}w^{m+1} + \dfrac{bC_2^2}{s+1}w^{s+1} = \dfrac{2C_1}{2-k}y^{\frac{2-k}{2}} + \dfrac{2C_2}{2-n}z^{\frac{2-n}{2}} + C_3.$

30. $\dfrac{\partial w}{\partial x} + ay^k w^m \left(\dfrac{\partial w}{\partial y}\right)^2 + bz^n w^s \left(\dfrac{\partial w}{\partial z}\right)^2 = cw^q.$

Complete integral:

$$x + \int \frac{2\varphi(w)\,dw}{1 + \sqrt{1 + 4cw^q\varphi(w)}} = \frac{2C_1}{2-k} y^{\frac{2-k}{2}} + \frac{2C_2}{2-n} z^{\frac{2-n}{2}} + C_3,$$

where $\varphi(w) = aC_1^2 w^m + bC_2^2 w^s$.

31. $x^k \dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\left(\dfrac{\partial w}{\partial z}\right)^2 = cx^n.$

Complete integral: $w = C_1 y + C_2 z - \dfrac{aC_1^2 + bC_2^2}{1-k} x^{1-k} + \dfrac{c}{n-k+1} x^{n-k+1} + C_3.$

32. $w \dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\left(\dfrac{\partial w}{\partial z}\right)^2 = cw^2.$

Complete integral: $\ln|w| = (c - aC_1^2 - bC_2^2)x + C_1 y + C_2 z + C_3.$

4.1.2. Equations Contain Squares of Three Derivatives

1. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 + c\left(\dfrac{\partial w}{\partial z}\right)^2 = 1.$

Complete integral: $w = C_1 x + C_2 y + C_3 z + C_4,$ where $aC_1^2 + bC_2^2 + cC_3^2 = 1.$

⊙ Reference: E. Kamke (1965).

2. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2 = ax^k + by^n + cz^m.$

This is a special case of equation 4.3.1.23. Complete integral:

$$w = \pm \int \sqrt{ax^k + C_1}\,dx \pm \int \sqrt{by^n + C_2}\,dy \pm \int \sqrt{cz^m - C_1 - C_2}\,dz + C_3.$$

3. $a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\left(\dfrac{\partial w}{\partial y}\right)^2 + c\left(\dfrac{\partial w}{\partial z}\right)^2 = w^k.$

Complete integral:

$$\frac{2}{2-k} w^{\frac{2-k}{2}} = C_1 x + C_2 y + C_3 z + C_4, \qquad \text{where} \quad aC_1^2 + bC_2^2 + cC_3^2 = 1.$$

4. $a_1 x^{k_1}\left(\dfrac{\partial w}{\partial x}\right)^2 + a_2 y^{k_2}\left(\dfrac{\partial w}{\partial y}\right)^2 + a_3 z^{k_3}\left(\dfrac{\partial w}{\partial z}\right)^2 = b_1 x^{n_1} + b_2 y^{n_2} + b_3 z^{n_3}.$

This is a special case of equation 4.3.1.23 with $f_1(x) = a_1 x^{k_1}$, $f_2(y) = a_2 y^{k_2}$, $f_3(z) = a_3 z^{k_3}$, $g_1(x) = b_1 x^{n_1}$, $g_2(y) = b_2 y^{n_2}$, and $g_3(z) = b_3 z^{n_3}$.

5. $a_1x^{k_1}\left(\dfrac{\partial w}{\partial x}\right)^2 + a_2y^{k_2}\left(\dfrac{\partial w}{\partial y}\right)^2 + \left(b_1x^{n_1} + b_2y^{n_2}\right)\left(\dfrac{\partial w}{\partial z}\right)^2 = c_1x^{m_1} + c_2y^{m_2}.$

This is a special case of equation 4.3.1.24 with $f_1(x) = a_1x^{k_1}$, $f_2(y) = a_2y^{k_2}$, $g_1(x) = b_1x^{n_1}$, $g_2(y) = b_2y^{n_2}$, $h_1(x) = c_1x^{m_1}$, and $h_2(y) = c_2y^{m_2}$.

6. $a_1x^{k_1}\left(\dfrac{\partial w}{\partial x}\right)^2 + a_2y^{k_2}\left(\dfrac{\partial w}{\partial y}\right)^2 + a_3z^{k_3}\left(\dfrac{\partial w}{\partial z}\right)^2 = \left(b_1x^{n_1} + b_2y^{n_2} + b_3z^{n_3}\right)w^m.$

This is a special case of equation 4.3.1.26 with $f_1(x) = a_1x^{k_1}$, $f_2(y) = a_2y^{k_2}$, $f_3(z) = a_3z^{k_3}$, $g_1(x) = b_1x^{n_1}$, $g_2(y) = b_2y^{n_2}$, $g_3(z) = b_3z^{n_3}$, and $h(w) = w^m$.

7. $A(ax+by+c)^k\left(\dfrac{\partial w}{\partial x}\right)^2 + B(ax+by+c)^n\left(\dfrac{\partial w}{\partial y}\right)^2 + C(ax+by+c)^m\left(\dfrac{\partial w}{\partial y}\right)^2 + s = 0.$

This is a special case of equation 4.3.1.27.

8. $a_1x^{k_1}w^{n_1}\left(\dfrac{\partial w}{\partial x}\right)^2 + a_2y^{k_2}w^{n_2}\left(\dfrac{\partial w}{\partial y}\right)^2 + a_3z^{k_3}w^{n_3}\left(\dfrac{\partial w}{\partial z}\right)^2 = bw^m.$

This is a special case of equation 4.3.1.28.

4.1.3. Equations Contain Products of Derivatives with Respect to Different Variables

1. $\dfrac{\partial w}{\partial x} + ax^k\left(\dfrac{\partial w}{\partial y} + bz\right)\left(\dfrac{\partial w}{\partial z} + by\right) = cx^n.$

Complete integral: $w = -byz + C_1y + C_2z + \dfrac{c}{n+1}x^{n+1} - \dfrac{aC_1C_2}{k+1}x^{k+1} + C_3.$

2. $\dfrac{\partial w}{\partial x} + ae^{\lambda x}\left(\dfrac{\partial w}{\partial y} + bz\right)\left(\dfrac{\partial w}{\partial z} + by\right) = ce^{\beta x} + k.$

Complete integral: $w = -byz + C_1y + C_2z + \dfrac{c}{\beta}e^{\beta x} + kx - \dfrac{a}{\lambda}C_1C_2e^{\lambda x} + C_3.$

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} + by\dfrac{\partial w}{\partial y} + cz\dfrac{\partial w}{\partial z} = kx^n + s.$

Complete integral:

$$w = y\varphi(x) + z\psi(x) + \int \frac{kx^n + s}{\varphi(x) + a\psi(x)}\,dx + C_3,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the following autonomous system of ordinary differential equations:

$$(\varphi + a\psi)\varphi' + b\varphi = 0, \quad (\varphi + a\psi)\psi' + c\psi = 0.$$

If $bc \neq 0$, the general solution of this system is given by

$$\varphi + abC_1\varphi^{c/b} + bx = C_2, \quad \psi = cC_1\varphi^{c/b}.$$

In two degenerate cases, we have

$$\varphi = C_1, \quad C_1\ln|\psi| + a\psi + cx = C_2 \qquad \text{if } b = 0;$$
$$\psi = C_1, \quad aC_1\ln|\varphi| + \varphi + bx = C_2 \qquad \text{if } c = 0.$$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} + bz\dfrac{\partial w}{\partial y} + cy\dfrac{\partial w}{\partial z} = kx^n + s.$

Complete integral:

$$w = y\varphi(x) + z\psi(x) + \int \frac{kx^n + s}{\varphi(x) + a\psi(x)}\,dx + C_3,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the following autonomous system of ordinary differential equations:

$$(\varphi + a\psi)\varphi' + c\psi = 0, \quad (\varphi + a\psi)\psi' + b\varphi = 0.$$

If $bc \neq 0$, the general solution of this system is given by

$$bc\varphi^2 + cC_1 = (C_2 + bcx + a\varphi)^2, \quad c\psi^2 = b\varphi^2 + C_1.$$

In the degenerate cases, we have

$$\psi = C_1, \quad \varphi^2 + 2aC_1\varphi + 2cC_1x = C_2 \qquad \text{if } b = 0;$$
$$\varphi = C_1, \quad a\psi^2 + 2C_1\psi + 2bC_1x = C_2 \qquad \text{if } c = 0.$$

5. $az\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + by\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} + x + cy^2 - s = 0.$

The Legendre transformation (both direct and inverse transformations are given)

$$x = \frac{\partial W}{\partial X}, \quad y = \frac{\partial W}{\partial Y}, \quad z = \frac{\partial W}{\partial Z}, \quad w = X\frac{\partial W}{\partial X} + Y\frac{\partial W}{\partial Y} + Z\frac{\partial W}{\partial Z} - W;$$
$$X = \frac{\partial w}{\partial x}, \quad Y = \frac{\partial w}{\partial y}, \quad Z = \frac{\partial w}{\partial z}, \quad W = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} - w,$$

brings this equation to an equation of the form 4.1.1.6 (with $k = 0$):

$$\frac{\partial W}{\partial X} + bXZ\frac{\partial W}{\partial Y} + aXY\frac{\partial W}{\partial Z} + c\left(\frac{\partial W}{\partial Y}\right)^2 = s.$$

6. $z\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + ay\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} = bx^n w^k\dfrac{\partial w}{\partial x} + cx^m w^{k+1}.$

There is a solution in the form of the product of a function of one variable and a function of two variables:

$$w = f(x)g(y, \xi), \qquad \xi = ay^2 - z^2.$$

The dependence $f = f(x)$ can be found by solving the first-order ordinary differential equation

$$(bx^n f^{k-1} - C_1)f'_x + cx^m f^k = 0,$$

which is integrable by quadrature, e.g., for $n = 0$ or $k = 1$. The function $g = g(y, \xi)$ is determined by

$$g(y, \xi) = \begin{cases} \left[\Phi(\xi) \pm C_1(1-k)\displaystyle\int_{y_0}^{y} \frac{dt}{\sqrt{at^2 - \xi}}\right]^{\frac{1}{1-k}} & \text{if } k \neq 1, \\ \Phi(\xi)\exp\left(\pm C_1 \displaystyle\int_{y_0}^{y} \frac{dt}{\sqrt{at^2 - \xi}}\right) & \text{if } k = 1, \end{cases}$$

where $\Phi = \Phi(\xi)$ is an arbitrary function. To obtain a complete integral, one should set $\Phi = C_3$ (the constant C_2 occurs in the solution of the equation for f).

7. $a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+b\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+c\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=k.$

Complete integral: $w=C_1x+C_2y+C_3z+C_4$, where $aC_1C_2+bC_1C_3+cC_2C_3=k$.

⊙ Reference: E. Kamke (1965).

8. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+b\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=cw.$

Complete integral: $w=\dfrac{cC_1C_2}{4(C_1+aC_2+b)}\left(x+\dfrac{y}{C_2}+\dfrac{z}{C_1}+C_3\right)^2.$

9. $az^k\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+by^n\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+cx^m\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=0.$

$$w=\frac{aC_1}{k+1}z^{k+1}+\frac{bC_2}{n+1}y^{n+1}+\frac{cC_3}{m+1}x^{m+1}+C_4,$$

where $C_1C_2+C_1C_3+C_2C_3=0$.

10. $aw^k\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+bw^n\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+cw^m\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=1.$

Complete integral:

$$\int\sqrt{aC_1C_2w^k+bC_1C_3w^n+cC_2C_3w^m}\,dw=C_1x+C_2y+C_3z+C_4.$$

One of the constants C_1, C_2, or C_3 can be set equal to unity.

4.1.4. Equations Contain Squares and Products of Derivatives

1. $\dfrac{\partial w}{\partial x}+a_1w^{k_1}\left(\dfrac{\partial w}{\partial y}\right)^2+a_2w^{k_2}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}+a_3w^{k_3}\left(\dfrac{\partial w}{\partial z}\right)^2=bw^n+c.$

Complete integral:

$$C_1x+C_2y+C_3z+2\int\frac{f(w)\,dw}{C_1+\sqrt{C_1^2+4(bw^n+c)f(w)}}=C_4,$$

where $f(w)=a_1C_2^2w^{k_1}+a_2C_2C_3w^{k_2}+a_3C_3^2w^{k_3}$. One of the constants C_1, C_2, or C_3 can be set equal to ±1.

2. $\left(\dfrac{\partial w}{\partial x}\right)^2+ax^k\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+bxz^n\left(\dfrac{\partial w}{\partial y}\right)^2+cxz^m\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=sx^p.$

Complete integral:

$$w=C_1y-\frac{aC_1}{2(k+1)}x^{k+1}\pm\int\sqrt{\tfrac14a^2C_1^2x^{2k}+sx^p-C_2x}\,dx+\frac{C_2}{cC_1(1-m)}z^{1-m}-\frac{bC_1}{c(n-m+1)}z^{n-m+1}.$$

3. $\left(\dfrac{\partial w}{\partial x}\right)^2+a_1\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+a_2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+a_3\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=w^k\left(b_1\dfrac{\partial w}{\partial x}+b_2\dfrac{\partial w}{\partial y}+b_3\dfrac{\partial w}{\partial z}\right).$

Complete integral:

$$\frac{b_1C_1+b_2C_2+b_3C_3}{C_1^2+a_1C_1C_2+a_2C_1C_3+a_3C_2C_3}\big(C_1x+C_2y+C_3z\big)=\begin{cases}\frac{1}{1-k}w^{1-k} & \text{if } k\neq 1,\\ |\ln w| & \text{if } k=1.\end{cases}$$

4. $a_1\left(\dfrac{\partial w}{\partial x}\right)^2+a_1\left(\dfrac{\partial w}{\partial y}\right)^2+a_2\left(\dfrac{\partial w}{\partial z}\right)^2+b_1\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}+b_2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}+b_3\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=cw+s.$

Complete integral:

$$w=\begin{cases}\frac{1}{4}(C_1x+C_2y+C_3z+C_4)^2-s/c & \text{if } c\neq 0,\\ C_1x+C_2y+C_3z+C_4 & \text{if } c=0,\end{cases}$$

where the constants C_1, C_2, and C_3 are related by the one constraint

$$a_1C_1^2+a_2C_2^2+a_3C_3^2+b_1C_1C_2+b_2C_1C_3+b_3C_2C_3=c.$$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial z}\right)^2=\big[a(x^2+y^2+z^2)+b\big]^c\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+z\dfrac{\partial w}{\partial z}\right)^2.$

This is a special case of equation 4.3.2.3; see item 2° for $k=2$ and $f(r)=(ar+b)^c$ [for $c=-1$, see also E. Kamke (1965)].

4.2. Other Nonlinear Equations with Three Variables Containing Parameters

4.2.1. Equations Cubic in Derivatives

1. $\left(\dfrac{\partial w}{\partial x}\right)^3-aw^k\dfrac{\partial w}{\partial x}-bw^n\dfrac{\partial w}{\partial y}-cw^m\dfrac{\partial w}{\partial z}=0.$

Complete integral: $\displaystyle\int\frac{dw}{\sqrt{aw^k+bC_1w^n+cC_2w^m}}=x+C_1y+C_2z+C_3.$

2. $\left(\dfrac{\partial w}{\partial x}\right)^3-aw^k\left(\dfrac{\partial w}{\partial x}\right)^2-bw^n\left(\dfrac{\partial w}{\partial y}\right)^2-cw^m\left(\dfrac{\partial w}{\partial z}\right)^2=0.$

Complete integral: $\displaystyle\int\frac{dw}{aw^k+bC_1^2w^n+cC_2^2w^m}=x+C_1y+C_2z+C_3.$

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}=ax^ky^nz^m.$

This is a special case of equation 4.3.2.1 with $f(x)=ax^k$, $g(y)=y^n$, and $h(z)=z^m$.

Complete integral: $\displaystyle w=\frac{aC_1}{k+1}x^{k+1}+\frac{C_2}{n+1}y^{n+1}+\frac{1}{C_1C_2(m+1)}z^{m+1}+C_3.$

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = ax^k y^n z^m w^s.$

This is a special case of equation 4.3.2.2 with $f(x) = ax^k$, $g(y) = y^n$, $h(z) = z^m$, and $\varphi(w) = w^s$.

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = ax\left(\dfrac{\partial w}{\partial x}\right)^2 + by\left(\dfrac{\partial w}{\partial y}\right)^2 + cz\left(\dfrac{\partial w}{\partial z}\right)^2.$

The Legendre transformation (both direct and inverse transformations are specified)

$$x = \frac{\partial W}{\partial X}, \quad y = \frac{\partial W}{\partial Y}, \quad z = \frac{\partial W}{\partial Z}, \quad w = X\frac{\partial W}{\partial X} + Y\frac{\partial W}{\partial Y} + Z\frac{\partial W}{\partial Z} - W;$$
$$X = \frac{\partial w}{\partial x}, \quad Y = \frac{\partial w}{\partial y}, \quad Z = \frac{\partial w}{\partial z}, \quad W = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} - w,$$

leads to the linear equation

$$aX^2\frac{\partial W}{\partial X} + bY^2\frac{\partial W}{\partial Y} + cZ^2\frac{\partial W}{\partial Z} = XYZ.$$

The general solution of this equation is (Polyanin, Zaitsev, and Moussiaux (2002, p. 178))

$$W = W_0(X, Y, Z) + \Phi\left(\frac{1}{aX} - \frac{1}{bY}, \frac{1}{aX} - \frac{1}{cZ}\right),$$

where $W_0 = W_0(X, Y, Z)$ is a particular solution,

$$W_0 = XYZ\left[\frac{aX\ln(aX)}{(aX - bY)(aX - cZ)} + \frac{bY\ln(bY)}{(bY - aX)(bY - cZ)} + \frac{cZ\ln(cZ)}{(cZ - aX)(cZ - bY)}\right].$$

6. $\dfrac{\partial w}{\partial x}\left(\dfrac{\partial w}{\partial y} + az\right)\left(\dfrac{\partial w}{\partial z} + ay\right) = b.$

Complete integral: $w = -ayz + C_1x + C_2y + \dfrac{b}{C_1C_2}z + C_3.$

7. $\left(\dfrac{\partial w}{\partial x} + ayz\right)\left(\dfrac{\partial w}{\partial y} + axz\right)\left(\dfrac{\partial w}{\partial z} + axy\right) = b.$

Complete integral: $w = -axyz + C_1x + C_2y + \dfrac{b}{C_1C_2}z + C_3.$

8. $\left(\dfrac{\partial w}{\partial x} - aw\right)\left(\dfrac{\partial w}{\partial y} - bw\right)\left(\dfrac{\partial w}{\partial z} - cw\right) = k\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}.$

Complete integral: $\ln|w| = C_1x + C_2y + C_3z + C_4$, where $(C_1 - a)(C_2 - b)(C_3 - c) = kC_1C_2C_3$.

4.2.2. Equations Contain Roots and Moduli of Derivatives

1. $\dfrac{\partial w}{\partial x}+\sqrt{a+b\left(\dfrac{\partial w}{\partial y}\right)^2+c\left(\dfrac{\partial w}{\partial z}\right)^2}=0.$

Equations of this type are encountered in differential games [e.g., see Subbotin (1991, 1995)]. Complete integral: $w=-Ax+C_1y+C_2z+C_3,$ where $A=\sqrt{a+bC_1^2+cC_2^2}.$

2. $\dfrac{\partial w}{\partial x}+a_1\sqrt{1+b_1\left(\dfrac{\partial w}{\partial y}\right)^2}+a_2\sqrt{1+b_2\left(\dfrac{\partial w}{\partial z}\right)^2}=0.$

Complete integral: $w=-Ax+C_1y+C_2z+C_3,$ where $A=a_1\sqrt{1+b_1C_1^2}+a_2\sqrt{1+b_2C_2^2}.$

3. $\dfrac{\partial w}{\partial x}+\sqrt{a+b\left(\dfrac{\partial w}{\partial y}+kz\right)^2+c\left(\dfrac{\partial w}{\partial z}+ky\right)^2}=0.$

Complete integral: $w=-kyz-Ax+C_1y+C_2z+C_3,$ where $A=\sqrt{a+bC_1^2+cC_2^2}.$

4. $\dfrac{\partial w}{\partial x}+z\dfrac{\partial w}{\partial y}+\left|\dfrac{\partial w}{\partial y}\right|-\left|\dfrac{\partial w}{\partial z}\right|=0.$

This equation is encountered in differential games.

An exact closed-form solution of this equation subject to the initial condition

$$w=\max\{|y|,|z|\}\qquad\text{at}\qquad x=2$$

was obtained in Taras'ev (1985). In the domain $1\le x\le 2$, this solution is given by

$$w(x,y,z)=\begin{cases}\max\{\varphi_1,\varphi_3,\varphi_6\} & \text{if } \varphi_1+\varphi_3\ge 0,\\ \max\{\varphi_2,\varphi_4,\varphi_5\} & \text{if } \varphi_2+\varphi_4\ge 0,\\ \max\{\varphi_5,\varphi_6\} & \text{if } \varphi_1+\varphi_3\le 0 \text{ and } \varphi_2+\varphi_4\le 0,\end{cases}$$

where

$$\begin{aligned}&\varphi_1=\varphi_1(x,y,z)=x+y+2z-\tfrac{1}{2}x^2-xz,\\ &\varphi_3=\varphi_3(x,y,z)=2-x+z,\\ &\varphi_5=\varphi_5(x,y,z)=2-\varphi_3-2(1-\varphi_1-\varphi_3)^{1/2},\\ &\varphi_k=\varphi_k(x,y,z)=\varphi_{k-1}(x,-y,-z),\qquad k=2,\ 4,\ 6.\end{aligned}$$

At the points (x,y,z) of matching various φ_n and φ_m, the function w is continuous but can be nondifferentiable.

In the domain $0\le x\le 1$, the solution has a more sophisticated structure and is defined by 12 different functions, 6 of which are defined implicitly.

⊙ References: A. M. Taras'ev (1985) and A. I. Subbotin (1991).

4.2.3. Equations Contain Arbitrary Powers of Derivatives

1. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^k+b\left(\dfrac{\partial w}{\partial z}\right)^n=cw+sx^m.$

This is a special case of equation 4.3.2.4 with $f(x)=c$ and $g(x)=sx^m$.

2. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^k+a_2\left(\dfrac{\partial w}{\partial z}\right)^n+b_1x^m z\dfrac{\partial w}{\partial y}+b_2x^m y\dfrac{\partial w}{\partial z}=0.$

This is a special case of equation 4.3.2.5 with $f(x)=x^m$.

3. $\dfrac{\partial w}{\partial x}+a_1\left(\dfrac{\partial w}{\partial y}\right)^k+a_2\left(\dfrac{\partial w}{\partial z}\right)^n+b_1e^{\lambda x} z\dfrac{\partial w}{\partial y}+b_2e^{\lambda x} y\dfrac{\partial w}{\partial z}=0.$

This is a special case of equation 4.3.2.5 with $f(x)=e^{\lambda x}$.

4. $\dfrac{\partial w}{\partial x}+a_1x^{n_1}\left(\dfrac{\partial w}{\partial y}\right)^{k_1}+a_2x^{n_2}\left(\dfrac{\partial w}{\partial z}\right)^{k_2}=bx^s w+c_1x^{m_1}y+c_2x^{m_2}z.$

This is a special case of equation 4.3.2.6 with $f_1(x)=a_1x^{n_1}$, $f_2(x)=a_2x^{n_2}$, $g(x)=bx^s$, $h_1(x)=c_1x^{m_1}$, and $h_2(x)=c_2x^{m_2}$.

5. $\dfrac{\partial w}{\partial x}+a_1e^{\lambda_1 x}\left(\dfrac{\partial w}{\partial y}\right)^{k_1}+a_2e^{\lambda_2 x}\left(\dfrac{\partial w}{\partial z}\right)^{k_2}=be^{\beta x}w+c_1e^{\gamma_1 x}y+c_2e^{\gamma_2 x}z.$

This is a special case of equation 4.3.2.6 with $f_1(x)=a_1e^{\lambda_1 x}$, $f_2(x)=a_2e^{\lambda_2 x}$, $g(x)=be^{\beta x}$, $h_1(x)=c_1e^{\gamma_1 x}$, and $h_2(x)=c_2e^{\gamma_2 x}$.

6. $\dfrac{\partial w}{\partial x}+a_1x^{k_1}\left(\dfrac{\partial w}{\partial y}+bz\right)^{n_1}+a_2x^{k_2}\left(\dfrac{\partial w}{\partial z}+by\right)^{n_2}=cx^m.$

Complete integral: $w=-byz+C_1y+C_2z+\dfrac{c}{m+1}x^{m+1}-\dfrac{a_1C_1^{n_1}}{k_1+1}x^{k_1+1}-\dfrac{a_2C_2^{n_2}}{k_2+1}x^{k_2+1}+C_3.$

7. $\dfrac{\partial w}{\partial x}+a_1e^{\lambda_1 x}\left(\dfrac{\partial w}{\partial y}+bz\right)^{n_1}+a_2e^{\lambda_2 x}\left(\dfrac{\partial w}{\partial z}+by\right)^{n_2}=ce^{\beta x}.$

Complete integral: $w=-byz+C_1y+C_2z+\dfrac{c}{\beta}e^{\beta x}-\dfrac{a_1}{\lambda_1}C_1^{n_1}e^{\lambda_1 x}-\dfrac{a_2}{\lambda_2}C_2^{n_2}e^{\lambda_2 x}+C_3.$

8. $\dfrac{\partial w}{\partial x}+ax^k\left(\dfrac{\partial w}{\partial y}+bz\right)^n\left(\dfrac{\partial w}{\partial z}+by\right)^m=cx^s.$

Complete integral: $w=-byz+C_1y+C_2z+\dfrac{c}{s+1}x^{s+1}-\dfrac{aC_1^nC_2^m}{k+1}x^{k+1}+C_3.$

9. $\dfrac{\partial w}{\partial x}+ae^{\lambda x}\left(\dfrac{\partial w}{\partial y}+bz\right)^n\left(\dfrac{\partial w}{\partial z}+by\right)^m=ce^{\beta x}.$

Complete integral: $w=-byz+C_1y+C_2z+\dfrac{c}{\beta}e^{\beta x}-\dfrac{a}{\lambda}C_1^nC_2^me^{\lambda x}+C_3.$

10. $a\left(\dfrac{\partial w}{\partial x}\right)^k+b\left(\dfrac{\partial w}{\partial y}\right)^k+c\left(\dfrac{\partial w}{\partial z}\right)^k=\left(x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+z\dfrac{\partial w}{\partial z}\right)^n.$

Complete integral: $w=\dfrac{k-n}{k}\left(aC_1^k+bC_2^k+cC_3^k\right)^{\frac{1}{n-k}}\left(C_1x+C_2y+C_3z\right)^{\frac{k}{k-n}}+C_4.$
One of the constants C_1, C_2, or C_3 can be set equal to ±1.

11. $$a_1 w^{k_1}\left(\frac{\partial w}{\partial x}\right)^n + a_2 w^{k_2}\left(\frac{\partial w}{\partial y}\right)^n + a_3 w^{k_3}\left(\frac{\partial w}{\partial z}\right)^n - b_1 w^{s_1}\left(\frac{\partial w}{\partial x}\right)^m - b_2 w^{s_2}\left(\frac{\partial w}{\partial y}\right)^m - b_3 w^{s_3}\left(\frac{\partial w}{\partial z}\right)^m = 0.$$

Complete integral:

$$\int\left(\frac{a_1 C_1^n w^{k_1} + a_2 C_2^n w^{k_2} + a_3 C_3^n w^{k_3}}{b_1 C_1^m w^{s_1} + b_2 C_2^m w^{s_2} + b_3 C_3^m w^{s_3}}\right)^{\frac{1}{n-m}} dw = C_1 x + C_2 y + C_3 z + C_4.$$

One of the constants C_1, C_2, or C_3 can be set equal to unity.

12. $$a\left(\frac{\partial w}{\partial x}\right)^k + b\left(\frac{\partial w}{\partial y}\right)^n + c\left(\frac{\partial w}{\partial z}\right)^m = 1.$$

Complete integral: $w = C_1 x + C_2 y + C_3 z + C_4$, where $aC_1^k + bC_2^n + cC_3^m = 1$.

13. $$a\left(\frac{\partial w}{\partial x}\right)^k + b\left(\frac{\partial w}{\partial y}\right)^n + c\left(\frac{\partial w}{\partial z}\right)^m + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = w.$$

Complete integral: $w = C_1 x + C_2 y + C_3 z + aC_1^k + bC_2^n + cC_3^m$.

14. $$a_1\left(\frac{\partial w}{\partial x} + byz\right)^{k_1} + a_2\left(\frac{\partial w}{\partial y} + bxz\right)^{k_2} + a_3\left(\frac{\partial w}{\partial z} + bxy\right)^{k_3} = c.$$

Complete integral: $w = -bxyz + C_1 x + C_2 y + C_3 z + C_4$, where $a_1 C_1^{k_1} + a_2 C_2^{k_2} + a_3 C_3^{k_3} = c$.

15. $$a_1 x^{k_1}\left(\frac{\partial w}{\partial x}\right)^{n_1} + a_2 y^{k_2}\left(\frac{\partial w}{\partial y}\right)^{n_2} + a_3 z^{k_3}\left(\frac{\partial w}{\partial z}\right)^{n_3} = b_1 x^{m_1} + b_2 y^{m_2} + b_3 z^{m_3}.$$

This is a special case of equation 4.3.2.9.

16. $$\left(\frac{\partial w}{\partial x}\right)^{k_1}\left(\frac{\partial w}{\partial y}\right)^{k_2}\left(\frac{\partial w}{\partial z}\right)^{k_3} = a x^{n_1} y^{n_2} z^{n_3} w^m.$$

This is a special case of equation 4.3.2.10.

17. $$a\left(\frac{\partial w}{\partial x}\right)^k\left(\frac{\partial w}{\partial y}\right)^n\left(\frac{\partial w}{\partial z}\right)^m = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}.$$

Complete integral:

$$w = \frac{k+n+m-1}{k+n+m}\left(C_1 x + C_2 y + \beta z\right)^{\frac{k+n+m}{k+n+m-1}} + C_3, \qquad \beta = \left(aC_1^k C_2^n\right)^{-\frac{1}{m}}.$$

18. $$a\left(\frac{\partial w}{\partial x}\right)^k\left(\frac{\partial w}{\partial y}\right)^n\left(\frac{\partial w}{\partial z}\right)^m + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = w.$$

Complete integral: $w = C_1 x + C_2 y + C_3 z + aC_1^k C_2^n C_3^m$.

19. $\left(\dfrac{\partial w}{\partial x}+ayz\right)^k\left(\dfrac{\partial w}{\partial y}+axz\right)^n\left(\dfrac{\partial w}{\partial z}+axy\right)^m=b.$

Complete integral: $w=-axyz+C_1x+C_2y+C_3z+C_4,$ where $C_1^kC_2^nC_3^m=b.$

20. $\dfrac{\partial w}{\partial x}+a_1x^{n_1}\left(\dfrac{\partial w}{\partial y}\right)^{k_1}\left(\dfrac{\partial w}{\partial z}\right)^{m_1}+a_2x^{n_2}\left(\dfrac{\partial w}{\partial y}\right)^{k_2}\left(\dfrac{\partial w}{\partial z}\right)^{m_2}=bx^sw.$

This is a special case of equation 4.3.4.3 with $F(x,u,v)=a_1x^{n_1}u^{k_1}v^{m_1}+a_2x^{n_2}u^{k_2}v^{m_2}$, and $g(x)=bx^s$.

21. $\dfrac{\partial w}{\partial x}+a_1x^{n_1}\left(\dfrac{\partial w}{\partial y}\right)^{k_1}\left(\dfrac{\partial w}{\partial z}\right)^{m_1}+a_2x^{n_2}\left(\dfrac{\partial w}{\partial y}\right)^{k_2}\left(\dfrac{\partial w}{\partial z}\right)^{m_2}$
$=bx^sw+c_1x^{q_1}y+c_2x^{q_2}z.$

This is a special case of equation 4.3.4.4.

22. $\dfrac{\partial w}{\partial x}+a_1e^{\lambda_1x}\left(\dfrac{\partial w}{\partial y}\right)^{k_1}\left(\dfrac{\partial w}{\partial z}\right)^{m_1}+a_2e^{\lambda_2x}\left(\dfrac{\partial w}{\partial y}\right)^{k_2}\left(\dfrac{\partial w}{\partial z}\right)^{m_2}=be^{\beta x}w.$

This is a special case of equation 4.3.4.3.

4.3. Nonlinear Equations with Three Variables Containing Arbitrary Functions

4.3.1. Equations Quadratic in Derivatives

1. $\left(\dfrac{\partial w}{\partial x}\right)^2+f(x)\dfrac{\partial w}{\partial x}+g_1(y)\dfrac{\partial w}{\partial y}+g_2(z)\dfrac{\partial w}{\partial z}+h(x)=0.$

Complete integral:

$$w=\frac{1}{2}\int\left[-f(x)+\sqrt{f^2(x)-4h(x)-4C_1-4C_2}\right]dx+C_1\int\frac{dy}{g_1(y)}+C_2\int\frac{dz}{g_2(z)}+C_3.$$

2. $\dfrac{\partial w}{\partial x}+a\left(\dfrac{\partial w}{\partial y}\right)^2+b\left(\dfrac{\partial w}{\partial z}\right)^2=f(x)w+g(x).$

Complete integral:

$$w=C_1F(x)+(C_2y+C_3z)F(x)+F(x)\int\frac{g(x)-\lambda F^2(x)}{F(x)}\,dx,$$

where

$$F(x)=\exp\left[\int f(x)\,dx\right],\qquad \lambda=aC_2^2+bC_3^2.$$

3. $\dfrac{\partial w}{\partial x} + a_1\left(\dfrac{\partial w}{\partial y}\right)^2 + a_2\left(\dfrac{\partial w}{\partial z}\right)^2 = bw^2 + f(x)w + g(x).$

For $b = 0$, see equation 4.3.1.2. Complete integral for $b \neq 0$:

$$w = \varphi(x) + C_3\psi(x)\exp(C_1 y + C_2 z).$$

The constants C_1 and C_2 are related by the constraint $a_1C_1^2 + a_2C_2^2 = b$. The function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x = b\varphi^2 + f(x)\varphi + g(x), \tag{1}$$

and the function $\psi = \psi(x)$ is expressed via $\varphi(x)$ as follows:

$$\psi(x) = \exp\left\{\int \left[2b\varphi(x) + f(x)\right] dx\right\}. \tag{2}$$

The Riccati equation (1) can be integrated by quadrature for many f and g, in particular, for $g(x) \equiv 0$ and any $f(x)$, as well as for $f(x) = \text{const}$ and $g(x) = \text{const}$. For details, see the books Kamke (1976) and Polyanin and Zaitsev (2003).

4. $\dfrac{\partial w}{\partial x} + a_1\left(\dfrac{\partial w}{\partial y}\right)^2 + a_2\left(\dfrac{\partial w}{\partial z}\right)^2 + b_1 f(x) z\dfrac{\partial w}{\partial y} + b_2 f(x) y\dfrac{\partial w}{\partial z} = 0.$

This is a special case of equation 4.3.1.17, item 2°. Complete integral:

$$w = y\varphi_1(x) + z\varphi_2(x) - \int \left[a_1\varphi_1^2(x) + a_2\varphi_2^2(x)\right] dx + C_3,$$

where the functions $\varphi_1(x)$ and $\varphi_2(x)$ are given by

$$\varphi_1(x) = C_1 b_2 \exp\left[\lambda \int f(x)\,dx\right] + C_2 b_2 \exp\left[-\lambda \int f(x)\,dx\right], \quad \lambda = \sqrt{b_1 b_2},$$
$$\varphi_2(x) = -C_1\lambda \exp\left[\lambda \int f(x)\,dx\right] + C_2\lambda \exp\left[-\lambda \int f(x)\,dx\right].$$

5. $\dfrac{\partial w}{\partial x} + a_1\left(\dfrac{\partial w}{\partial y}\right)^2 + a_2\left(\dfrac{\partial w}{\partial z}\right)^2 + f(x) z\dfrac{\partial w}{\partial y} + g(x) y\dfrac{\partial w}{\partial z} = 0.$

Complete integral:

$$w = y\varphi_1(x) + z\varphi_2(x) - \int \left[a_1\varphi_1^2(x) + a_2\varphi_2^2(x)\right] dx + C_3, \qquad \varphi_2(x) = -\frac{1}{g(x)}\frac{d\varphi_1}{dx},$$

where the function $\varphi_1 = \varphi_1(x)$ is determined by the second-order linear ordinary differential equation

$$\frac{d}{dx}\left[\frac{1}{g(x)}\frac{d\varphi_1}{dx}\right] - f(x)\varphi_1 = 0.$$

6. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(x)\left(\dfrac{\partial w}{\partial z}\right)^2 = aw.$

Complete integral: $w = C_1 e^{ax} + e^{ax}(C_2 y + C_3 z) - e^{ax}\displaystyle\int e^{ax}\left[C_2^2 f(x) + C_3^2 g(x)\right] dx.$

7. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(x)\left(\dfrac{\partial w}{\partial z}\right)^2 = h(x)w.$

Complete integral:

$$w = \varphi(x)(C_1 + C_2 y + C_3 z) - \varphi(x)\int \varphi(x)\left[C_2^2 f(x) + C_3^2 g(x)\right] dx,$$

where $\varphi(x) = \exp\left[\displaystyle\int h(x)\,dx\right]$.

8. $\dfrac{\partial w}{\partial x} + f_1(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_2(x)\left(\dfrac{\partial w}{\partial z}\right)^2 = g(x)w + h_1(x)y + h_2(x)z.$

Complete integral:

$$w = G(x)\left[y\varphi_1(x) + z\varphi_2(x) + \psi(x)\right],$$

where

$$\varphi_1(x) = C_1 + \int \frac{h_1(x)}{G(x)}\,dx, \quad \varphi_2(x) = C_2 + \int \frac{h_2(x)}{G(x)}\,dx,$$
$$\psi(x) = C_3 - \int G(x)\left[f_1(x)\varphi_1^2(x) + f_2(x)\varphi_2^2(x)\right] dx, \quad G(x) = \exp\left[\int g(x)\,dx\right].$$

9. $\dfrac{\partial w}{\partial x} + f_1(y)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_2(z)\left(\dfrac{\partial w}{\partial z}\right)^2 = g(x) + h_1(y) + h_2(z).$

Equations of this type are often encountered in mechanics; the variable x plays the role of time, and y and z are generalized coordinates.

Complete integral:

$$w = C_1 - (C_2 + C_3)x + \int g(x)\,dx + \int \sqrt{\frac{C_2 + h_1(y)}{f_1(y)}}\,dy + \int \sqrt{\frac{C_2 + h_2(z)}{f_2(z)}}\,dz.$$

10. $\dfrac{\partial w}{\partial x} + f_1(y)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_2(z)\left(\dfrac{\partial w}{\partial z}\right)^2 = bw^2 + g(x)w + h(x).$

The transformation $\xi = \displaystyle\int \frac{dy}{\sqrt{|f_1(y)|}}$, $\eta = \displaystyle\int \frac{dz}{\sqrt{|f_2(z)|}}$ leads to an equation of the form 4.3.1.3.

11. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y} + az\right)^2 + g(x)\left(\dfrac{\partial w}{\partial z} + ay\right)^2 = h(x).$

Complete integral: $w = -ayz + C_1 y + C_2 z + \displaystyle\int \left[h(x) - C_1^2 f(x) - C_2^2 g(x)\right] dx + C_3.$

12. $$\frac{\partial w}{\partial x} + \big[f_1(x)y + g_1(x)\big]\left(\frac{\partial w}{\partial y}\right)^2 + \big[f_2(x)z + g_2(x)\big]\left(\frac{\partial w}{\partial z}\right)^2 = a(x)w + b(x) + h_1(x)y + h_2(x)z.$$

Complete integral:

$$w = \varphi_1(x)y + \varphi_2(x)z + \psi(x),$$

where the functions $\varphi_1(x)$, $\varphi_2(x)$, and $\psi(x)$ are determined by solving the ordinary differential equations

$$\varphi_k' + f_k(x)\varphi_k^2 = a(x)\varphi_k + h_k(x), \qquad k = 1,\ 2; \tag{1}$$

$$\psi' + g_1(x)\varphi_1^2 + g_2(x)\varphi_2^2 = a(x)\psi + b(x). \tag{2}$$

The prime denotes the derivative with respect to x.

The Riccati equations (1) can be integrated by quadrature for many $f_k(x)$, $h_k(x)$, and $a(x)$; for details, see the books Kamke (1976) and Polyanin and Zaitsev (1995). In particular, for $h_k \equiv 0$ the solutions of equations (1) are given by

$$\varphi_k(x) = A(x)\left[C_k + \int A(x)f_k(x)\,dx\right]^{-1}, \qquad A(x) = \exp\left[\int a(x)\,dx\right].$$

Equation (2) is linear with respect to ψ and can be readily integrated (for known φ_k):

$$\psi(x) = C_3A(x) + A(x)\int \big[b(x) - g_1(x)\varphi_1^2(x) - g_2(x)\varphi_2^2(x)\big]\frac{dx}{A(x)}.$$

13. $$\frac{\partial w}{\partial x} + \big[f_1(x)y^k + g_1(x)y^{2k-2}\big]\left(\frac{\partial w}{\partial y}\right)^2 + \big[f_2(x)z^n + g_2(x)z^{2n-2}\big]\left(\frac{\partial w}{\partial z}\right)^2 = a(x)w + b(x).$$

1°. If $k \neq 2$ and $n \neq 2$, then the transformation $\xi = y^{2-k}$, $\eta = z^{2-n}$ leads to an equation of the form 4.3.1.12:

$$\frac{\partial u}{\partial x} + (2-k)^2\big[f_1(x)\xi + g_1(x)\big]\left(\frac{\partial u}{\partial \xi}\right)^2 + (2-n)^2\big[f_2(x)\eta + g_2(x)\big]\left(\frac{\partial u}{\partial \eta}\right)^2 = a(x)u + b(x).$$

2°. If $k = 2$ and $n \neq 2$, then we introduce the new variables $\xi = \ln|y|$ and $\eta = z^{2-n}$ to arrive at an equation of the form 4.3.1.12. The case $k \neq 2$ and $n = 2$ is treated analogously.

14. $$\frac{\partial w}{\partial x} + f(ax+by+cz)\left(\frac{\partial w}{\partial y}\right)^2 + g(ax+by+cz)\left(\frac{\partial w}{\partial z}\right)^2 = h(ax+by+cz).$$

Complete integral:

$$w = C_1x + C_2y + C_3z + \varphi(\xi) + C_4, \qquad \xi = ax + by + cz,$$

where

$$\varphi(\xi) = \int \frac{-G(\xi) \pm \sqrt{G^2(\xi) - 4F(\xi)H(\xi)}}{2F(\xi)}\,d\xi, \quad F(\xi) = b^2f(\xi) + c^2g(\xi),$$

$$G(\xi) = a + 2bC_2f(\xi) + 2cC_3g(\xi), \quad H(\xi) = C_1 + C_2^2f(\xi) + C_3^2g(\xi) - h(\xi).$$

One of the constants C_1, C_2, or C_3 can be set equal to unity.

15. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y} + az\right)\left(\dfrac{\partial w}{\partial z} + ay\right) = g(x).$

Complete integral: $w = -ayz + C_1 y + C_2 z + \displaystyle\int \big[g(x) - C_1 C_2 f(x)\big]\,dx + C_3.$

16. $\dfrac{\partial w}{\partial x} + f_{11}(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_{12}(x)\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} + f_{22}(x)\left(\dfrac{\partial w}{\partial z}\right)^2$
$= g(x)w + h_{11}(x)y^2 + h_{12}(x)yz + h_{22}(x)z^2 + h_1(x)y + h_2(x)z + h_0(x).$

This is a special case of equation 4.6.1.10. A complete integral can be sought in the form

$$w = \varphi_{11}(x)y^2 + \varphi_{12}(x)yz + \varphi_{22}(x)z^2 + \psi_1(x)y + \psi_2(x)z + \psi_0(x).$$

Substituting this expression into the original partial differential equation yields a system of ordinary differential equations for $\varphi_{km}(x)$ and $\psi_k(x)$. If all $h_k(x) \equiv 0$, then we may set $\psi_k(x) \equiv 0$.

17. $\dfrac{\partial w}{\partial x} + f_{11}(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_{12}(x)\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} + f_{22}(x)\left(\dfrac{\partial w}{\partial z}\right)^2 + \big[g_{11}(x)y + g_{12}(x)z\big]\dfrac{\partial w}{\partial y}$
$+ \big[g_{21}(x)y + g_{22}(x)z\big]\dfrac{\partial w}{\partial z} = s(x)w + h_1(x)y + h_2(x)z + h_0(x).$

This is a special case of equation 4.3.4.5.

1°. Complete integral:

$$w = \varphi_1(x)y + \varphi_2(x)z + \psi(x).$$

The functions $\varphi_1(x)$, $\varphi_2(x)$, and $\psi(x)$ are determined by solving the system of ordinary differential equations

$$\varphi_1' + (g_{11} - s)\varphi_1 + g_{21}\varphi_2 = h_1, \tag{1}$$
$$\varphi_2' + g_{12}\varphi_1 + (g_{22} - s)\varphi_2 = h_2, \tag{2}$$
$$\psi' - s\psi = h_0 - f_{11}\varphi_1^2 - f_{12}\varphi_1\varphi_2 - f_{22}\varphi_2^2, \tag{3}$$

where $f_{ij} = f_{ij}(x)$, $g_{ij} = g_{ij}(x)$, $h_k = h_k(x)$, and $s = s(x)$; the prime denotes the derivative with respect to x.

Having found $\varphi_1 = \varphi_1(x)$ and $\varphi_2 = \varphi_2(x)$ from the linear system of equations (1) and (2), one can determine the solution of equation (3) according to the formula

$$\psi(x) = C_3 S + S\int (h_0 - f_{11}\varphi_1^2 - f_{12}\varphi_1\varphi_2 - f_{22}\varphi_2^2)\,\frac{dx}{S}, \qquad S = \exp\left(\int s\,dx\right).$$

2°. Consider the special case $h_1 \equiv 0$ and $h_2 \equiv 0$. In addition, assume that

$$g_{11} = a_1 g(x) + s(x), \quad g_{12} = a_2 g(x), \quad g_{21} = a_3 g(x), \quad g_{22} = a_4 g(x) + s(x), \tag{4}$$

where $g = g(x)$ and $s = s(x)$ are arbitrary functions and a_1, a_2, a_3, and a_4 are arbitrary numbers. With these assumptions, the general solution of the linear homogeneous system (1), (2) has the form

$$\begin{aligned}
\varphi_1 &= C_1 \exp\left(\lambda_1 \int g\,dx\right) + C_2 \exp\left(\lambda_2 \int g\,dx\right),\\
\varphi_2 &= -\frac{a_1 + \lambda_1}{a_3} C_1 \exp\left(\lambda_1 \int g\,dx\right) - \frac{a_1 + \lambda_2}{a_3} C_2 \exp\left(\lambda_2 \int g\,dx\right),
\end{aligned} \tag{5}$$

where λ_1 and λ_2 are the roots of the quadratic equation $(\lambda + a_1)(\lambda + a_4) - a_2 a_3 = 0$.

3°. Let $h_1 = h_1(x)$ and $h_2 = h_2(x)$ be arbitrary functions and let the $g_{ij} = g_{ij}(x)$ be defined by (4). In this case, the general solution of the linear nonhomogeneous system of equations (1) and (2) can be obtained by variation of constants or with the aid of the Wronskian determinant [see, for example, Murphy (1960), Kamke (1977), and Zwillinger (1998)].

18. $$\frac{\partial w}{\partial x} + w^k f(x)\left(\frac{\partial w}{\partial y}\right)^2 + w^k g(x)\left(\frac{\partial w}{\partial z}\right)^2 = h(x)w.$$

The substitution $u = \dfrac{1}{k+1}w^{k+1}$ leads to an equation of the form 4.3.1.7:

$$\frac{\partial u}{\partial x} + f(x)\left(\frac{\partial u}{\partial y}\right)^2 + g(x)\left(\frac{\partial u}{\partial z}\right)^2 = (k+1)h(x)u.$$

19. $$\frac{\partial w}{\partial x} + w^k f_1(y)\left(\frac{\partial w}{\partial y}\right)^2 + w^k f_2(z)\left(\frac{\partial w}{\partial z}\right)^2 = aw^{k+2} + g(x)w.$$

The substitution $u = \dfrac{1}{k+1}w^{k+1}$ leads to an equation of the form 4.3.1.10 with $h(x) \equiv 0$:

$$\frac{\partial u}{\partial x} + f_1(y)\left(\frac{\partial u}{\partial y}\right)^2 + f_2(z)\left(\frac{\partial u}{\partial z}\right)^2 = a(k+1)^2u^2 + (k+1)g(x)u.$$

20. $$\frac{\partial w}{\partial x} + f_1(y)g_1(w)\left(\frac{\partial w}{\partial y}\right)^2 + f_2(z)g_2(w)\left(\frac{\partial w}{\partial z}\right)^2 = 0.$$

Complete integral:

$$x + \int \left[C_1^2 g_1(w) + C_2^2 g_2(w)\right] dw = C_1 \int \frac{dy}{\sqrt{f_1(y)}} + C_2 \int \frac{dz}{\sqrt{f_2(z)}} + C_3.$$

It has been assumed in the products $f_1 g_1$ and $f_2 g_2$ that both f_1 and f_2 are nonnegative, and g_1 and g_2 can have any sign.

21. $$\frac{\partial w}{\partial x} + f_1(y)g_1(w)\left(\frac{\partial w}{\partial y}\right)^2 + f_2(z)g_2(w)\left(\frac{\partial w}{\partial z}\right)^2 = h(w).$$

Complete integral:

$$x + C_1 \int \frac{dy}{\sqrt{f_1(y)}} + C_2 \int \frac{dz}{\sqrt{f_2(z)}} + \int \frac{2\varphi(w)\,dw}{1 + \sqrt{1 + 4h(w)\varphi(w)}} = C_3,$$

where $\varphi(w) = C_1^2 g_1(w) + C_2^2 g_2(w)$.

22. $$\frac{\partial w}{\partial x} + \frac{1}{f_1(y) + f_2(z)}\left[g_1(y)\left(\frac{\partial w}{\partial y}\right)^2 + g_2(z)\left(\frac{\partial w}{\partial z}\right)^2 - h_1(y) - h_2(z)\right] = 0.$$

Liouville's equation. It is encountered in problems of mechanics (x is time, and y and z are generalized coordinates). Complete integral:

$$w = -C_1 x + \int \sqrt{\frac{C_1 f_1(y) + h_1(y) + C_2}{g_1(y)}}\, dy + \int \sqrt{\frac{C_1 f_2(z) + h_2(z) - C_2}{g_2(z)}}\, dz + C_3.$$

⊙ Reference: E. N. Berezkin (1968).

23. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 + f_3(z)\left(\frac{\partial w}{\partial z}\right)^2 = g_1(x) + g_2(y) + g_3(z).$

Complete integral:

$$w = \pm\int\left[\frac{g_1(x)+C_1}{f_1(x)}\right]^{1/2}dx \pm \int\left[\frac{g_2(y)+C_2}{f_2(y)}\right]^{1/2}dy \pm \int\left[\frac{g_3(z)-C_1-C_2}{f_3(z)}\right]^{1/2}dz + C_3.$$

24. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 + \big[g_1(x)+g_2(y)\big]\left(\frac{\partial w}{\partial z}\right)^2 = h_1(x) + h_2(y).$

A problem of the motion of a point mass being attracted by two fixed centers according to Newton's law of universal gravitation can be reduced to a special case of this equation [see Appell (1953)]. Complete integral:

$$w = C_1 z + C_2 \pm \int\sqrt{\frac{h_1(x) - C_1^2 g_1(x) + C_3}{f_1(x)}}\,dx \pm \int\sqrt{\frac{h_2(y) - C_1^2 g_2(y) - C_3}{f_2(y)}}\,dy.$$

The signs before integrals may be chosen independently.

25. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 + f_3(z)\left(\frac{\partial w}{\partial z}\right)^2 = aw + g_1(x) + g_2(y) + g_3(z).$

This is a special case of equation 4.6.1.17 with $n = 3$.

26. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 + f_3(z)\left(\frac{\partial w}{\partial z}\right)^2 = \big[g_1(x)+g_2(y)+g_3(z)\big]h(w).$

The substitution $u = \int\frac{dw}{\sqrt{h(w)}}$ leads to an equation of the form 4.3.1.23:

$$f_1(x)\left(\frac{\partial u}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial u}{\partial y}\right)^2 + f_3(z)\left(\frac{\partial u}{\partial z}\right)^2 = g_1(x) + g_2(y) + g_3(z).$$

27. $f(ax+by+cz)\left(\frac{\partial w}{\partial x}\right)^2 + g(ax+by+cz)\left(\frac{\partial w}{\partial y}\right)^2 + h(ax+by+cz)\left(\frac{\partial w}{\partial z}\right)^2 = k.$

Complete integral:

$$w = C_1 x + C_2 y + C_3 z + \varphi(\xi) + C_4, \qquad \xi = ax + by + cz,$$

where

$$\varphi(\xi) = \int\frac{-G(\xi) \pm \sqrt{G^2(\xi) - 4F(\xi)H(\xi)}}{2F(\xi)}\,d\xi, \quad F(\xi) = a^2 f(\xi) + b^2 g(\xi) + c^2 h(\xi),$$
$$G(\xi) = 2aC_1 f(\xi) + 2bC_2 g(\xi) + 2cC_3 h(\xi), \quad H(\xi) = C_1^2 f(\xi) + C_2^2 g(\xi) + C_3^2 h(\xi) - k.$$

One of the constants C_1, C_2, or C_3 can be set equal to unity.

28. $f_1(x)g_1(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + f_2(y)g_2(w)\left(\dfrac{\partial w}{\partial y}\right)^2 + f_3(z)g_3(w)\left(\dfrac{\partial w}{\partial z}\right)^2 = h(w).$

Complete integral:

$$\int\sqrt{\frac{C_1^2g_1(w)+C_2^2g_2(w)+C_3^2g_3(w)}{h(w)}}\,dw = C_1\int\frac{dx}{\sqrt{f_1(x)}}+C_2\int\frac{dy}{\sqrt{f_2(y)}}+C_3\int\frac{dz}{\sqrt{f_3(z)}}+C_4.$$

One of the constants C_1, C_2, or C_3 can be set equal to ±1.

29. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(x)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} + g(x)\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = h(x).$

Complete integral: $w = C_1y + C_2z + \displaystyle\int\frac{h(x)-C_1C_2g(x)}{C_1+C_2f(x)}\,dx + C_3.$

30. $f(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(z)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} = h(x).$

Complete integral: $w = \dfrac{1}{C_1+C_2}\displaystyle\int h(x)\,dx + C_1\int\frac{dy}{f(y)} + C_2\int\frac{dz}{g(z)} + C_3.$

31. $f(z)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z} + h(x)\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = 0.$

Complete integral: $w = C_1\displaystyle\int h(x)\,dx - \frac{C_1}{C_2+1}\int g(y)\,dy + \frac{C_1}{C_2}\int f(z)\,dz + C_3.$

4.3.2. Equations with Power Nonlinearity in Derivatives

1. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = f(x)g(y)h(z).$

Complete integral: $w = C_1\displaystyle\int f(x)\,dx + C_2\int g(y)\,dy + \frac{1}{C_1C_2}\int h(z)\,dz + C_3.$

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} = f(x)g(y)h(z)\varphi(w).$

Complete integral:

$$\int\frac{dw}{\varphi^{1/3}(w)} = C_1\int f(x)\,dx + C_2\int g(y)\,dy + \frac{1}{C_1C_2}\int h(z)\,dz + C_3.$$

3. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2 = f\left(x^2+y^2+z^2\right)\left(x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z}\right)^k.$

1°. By passing from x, y, z to the spherical coordinates r, θ, φ,

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

we arrive at the equation

$$r^2\left(\frac{\partial w}{\partial r}\right)^2 - r^2f(r^2)\left(r\frac{\partial w}{\partial r}\right)^k + \left(\frac{\partial w}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial w}{\partial\varphi}\right)^2 = 0.$$

A complete integral of this equation can be found by the method of separation of variables as the sum of functions depending on different variables (linear in φ).

2°. Complete integral for $k = 2$:

$$w = C_1 \int \frac{dr}{r\sqrt{r^2 f(r^2) - 1}} + \int \sqrt{C_1^2 - \frac{C_2^2}{\sin^2\theta}}\, d\theta + C_2\varphi + C_3.$$

4. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^k + b\left(\dfrac{\partial w}{\partial z}\right)^n = f(x)w + g(x).$

Complete integral:

$$w = C_1 F(x) + (C_2 y + C_3 z)F(x) + F(x)\int \left[g(x) - aC_2^k F^k(x) - bC_3^n F^n(x)\right]\frac{dx}{F(x)},$$

where $F(x) = \exp\left[\int f(x)\,dx\right]$.

5. $\dfrac{\partial w}{\partial x} + a_1\left(\dfrac{\partial w}{\partial y}\right)^k + a_2\left(\dfrac{\partial w}{\partial z}\right)^n + b_1 f(x) z\dfrac{\partial w}{\partial y} + b_2 f(x) y\dfrac{\partial w}{\partial z} = 0.$

Complete integral:

$$w = y\varphi_1(x) + z\varphi_2(x) - \int \left[a_1\varphi_1^k(x) + a_2\varphi_2^n(x)\right]dx + C_3,$$

where

$$\varphi_1(x) = C_1 b_2 \exp\left[\lambda\int f(x)\,dx\right] + C_2 b_2 \exp\left[-\lambda\int f(x)\,dx\right], \quad \lambda = \sqrt{b_1 b_2},$$
$$\varphi_2(x) = -C_1\lambda \exp\left[\lambda\int f(x)\,dx\right] + C_2\lambda \exp\left[-\lambda\int f(x)\,dx\right].$$

6. $\dfrac{\partial w}{\partial x} + f_1(x)\left(\dfrac{\partial w}{\partial y}\right)^k + f_2(x)\left(\dfrac{\partial w}{\partial z}\right)^n = g(x)w + h_1(x)y + h_2(x)z.$

Complete integral:

$$w = G(x)\left[y\varphi_1(x) + z\varphi_2(x) + \psi(x)\right],$$

where

$$G(x) = \exp\left[\int g(x)\,dx\right], \quad \varphi_1(x) = C_1 + \int \frac{h_1(x)}{G(x)}\,dx, \quad \varphi_2(x) = C_2 + \int \frac{h_2(x)}{G(x)}\,dx,$$
$$\psi(x) = C_3 - \int \left[f_1(x)G^{k-1}(x)\varphi_1^k(x) + f_2(x)G^{n-1}(x)\varphi_2^n(x)\right]dx.$$

7. $\dfrac{\partial w}{\partial x} + f(x)\left(\dfrac{\partial w}{\partial y} + az\right)^k + g(x)\left(\dfrac{\partial w}{\partial z} + ay\right)^n = h(x).$

Complete integral: $w = -ayz + C_1 y + C_2 z + \int \left[h(x) - C_1^k f(x) - C_2^n g(x)\right]dx + C_3.$

8. $\dfrac{\partial w}{\partial x}+f(x)\left(\dfrac{\partial w}{\partial y}+az\right)^k\left(\dfrac{\partial w}{\partial z}+ay\right)^n=g(x).$

Complete integral: $w=-ayz+C_1y+C_2z+\displaystyle\int\left[g(x)-C_1^kC_2^nf(x)\right]dx+C_3.$

9. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^k+f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^n+f_3(z)\left(\dfrac{\partial w}{\partial z}\right)^m=g_1(x)+g_2(y)+g_3(z).$

Complete integral:

$$w=\int\left[\frac{g_1(x)+C_1}{f_1(x)}\right]^{1/k}dx+\int\left[\frac{g_2(y)+C_2}{f_2(y)}\right]^{1/n}dy+\int\left[\frac{g_3(z)-C_1-C_2}{f_3(z)}\right]^{1/m}dz+C_3.$$

10. $\left(\dfrac{\partial w}{\partial x}\right)^k\left(\dfrac{\partial w}{\partial y}\right)^n\left(\dfrac{\partial w}{\partial z}\right)^m=f(x)g(y)h(z)\varphi(w).$

This is a special case of equation 4.6.2.4.

4.3.3. Equations with Arbitrary Dependence on Derivatives

1. $f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^k+f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^n+\left[g_1(x)+g_2(y)\right]F\left(\dfrac{\partial w}{\partial z}\right)=h_1(x)+h_2(y).$

Complete integral:

$$w=C_1z+C_2+\int\left[\frac{h_1(x)-F(C_1)g_1(x)+C_3}{f_1(x)}\right]^{1/k}dx+\int\left[\frac{h_2(y)-F(C_1)g_2(y)-C_3}{f_2(y)}\right]^{1/n}dy.$$

2. $\dfrac{\partial w}{\partial x}+f(x)F\left(ay+\dfrac{\partial w}{\partial y}\right)+g(x)G\left(bz+\dfrac{\partial w}{\partial z}\right)=h(x).$

Complete integral:

$$w=-\int\left[F(C_1)f(x)+G(C_2)g(x)-h(x)\right]dx-\tfrac{1}{2}ay^2-\tfrac{1}{2}bz^2+C_1y+C_2z+C_3.$$

3. $\dfrac{\partial w}{\partial x}+f(x)F\left(y^b\dfrac{\partial w}{\partial y}\right)+g(x)G\left(z^a\dfrac{\partial w}{\partial z}\right)=h(x).$

1°. Complete integral for $a\neq1$ and $b\neq1$:

$$w=-\int[F(C_2)f(x)+G(C_1)g(x)-h(x)]\,dx+\frac{C_1}{1-a}z^{1-a}+\frac{C_2}{1-b}y^{1-b}+C_3.$$

2°. Complete integral for $a=1$ and $b\neq1$:

$$w=-\int[F(C_2)f(x)+G(C_1)g(x)-h(x)]\,dx+C_1\ln|z|+\frac{C_2}{1-b}y^{1-b}+C_3.$$

3°. Complete integral for $a=1$ and $b=1$:

$$w=-\int[F(C_2)f(x)+G(C_1)g(x)-h(x)]\,dx+C_1\ln|z|+C_2\ln|y|+C_3.$$

4. $\dfrac{\partial w}{\partial x}+f(x)F\left(by+\dfrac{\partial w}{\partial y}\right)+g(x)G\left(z^a\dfrac{\partial w}{\partial z}\right)=h(x).$

1°. Complete integral for $a\neq 1$:

$$w=-\int[F(C_2)f(x)+G(C_1)g(x)-h(x)]\,dx+\frac{C_1}{1-a}z^{1-a}-\frac{b}{2}y^2+C_2y+C_3.$$

2°. Complete integral for $a=1$:

$$w=-\int[F(C_2)f(x)+G(C_1)g(x)-h(x)]\,dx+C_1\ln|z|-\frac{b}{2}y^2+C_2y+C_3.$$

5. $\left(\dfrac{\partial w}{\partial x}\right)^2+p(x)\dfrac{\partial w}{\partial x}+f(x)F\left(ay+\dfrac{\partial w}{\partial y}\right)+g(x)G\left(bz+\dfrac{\partial w}{\partial z}\right)=h(x).$

Complete integral:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx$$
$$-\frac{a}{2}y^2-\frac{b}{2}z^2+C_1z+C_2y+C_3.$$

6. $\left(\dfrac{\partial w}{\partial x}\right)^2+p(x)\dfrac{\partial w}{\partial x}+f(x)F\left(y^a\dfrac{\partial w}{\partial y}\right)+g(x)G\left(z^b\dfrac{\partial w}{\partial z}\right)=h(x).$

1°. Complete integral for $a\neq 1$ and $b\neq 1$:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx+\frac{C_1z^{1-b}}{1-b}+\frac{C_2y^{1-a}}{1-a}+C_3.$$

2°. Complete integral for $a\neq 1$ and $b=1$:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx+C_1\ln|z|+\frac{C_2}{1-a}y^{1-a}+C_3.$$

3°. Complete integral for $a=1$ and $b=1$:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx+C_1\ln|z|+C_2\ln|y|+C_3.$$

7. $\left(\dfrac{\partial w}{\partial x}\right)^2+p(x)\dfrac{\partial w}{\partial x}+f(x)F\left(ay+\dfrac{\partial w}{\partial y}\right)+g(x)G\left(z^b\dfrac{\partial w}{\partial z}\right)=h(x).$

1°. Complete integral for $b\neq 1$:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx+\frac{C_1z^{1-b}}{1-b}-\frac{a}{2}y^2+C_2y+C_3.$$

2°. Complete integral for $b=1$:

$$w=\frac{1}{2}\int\left[-p(x)+\sqrt{p^2(x)-4F(C_2)f(x)-4G(C_1)g(x)+4h(x)}\right]dx+C_1\ln|z|-\frac{a}{2}y^2+C_2y+C_3.$$

4.3.4. Nonlinear Equations of General Form

1. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral: $w = C_1 y + C_2 z + C_3 - \displaystyle\int F(x, C_1, C_2)\, dx.$

2. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = aw.$

Complete integral: $w = e^{ax}(C_1 y + C_2 z + C_3) - e^{ax} \displaystyle\int e^{-ax} F(x, C_1 e^{ax}, C_2 e^{ax})\, dx.$

3. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = g(x)w.$

Complete integral:

$$w = \varphi(x)(C_1 y + C_2 z + C_3) - \varphi(x) \int F\big(x, C_1\varphi(x), C_2\varphi(x)\big) \frac{dx}{\varphi(x)},$$

where $\varphi(x) = \exp\left[\displaystyle\int g(x)\, dx\right].$

4. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = g(x)w + h_1(x)y + h_2(x)z.$

Complete integral:

$$w = y\varphi(x) + z\psi(x) + \chi(x),$$

where

$$\varphi(x) = C_1 G(x) + G(x) \int \frac{h_1(x)}{G(x)}\, dx, \quad \psi(x) = C_2 G(x) + G(x) \int \frac{h_2(x)}{G(x)}\, dx,$$
$$\chi(x) = C_3 G(x) - G(x) \int F(x, \varphi, \psi) \frac{dx}{G(x)}, \quad G(x) = \exp\left[\int g(x)\, dx\right].$$

5. $\dfrac{\partial w}{\partial x} + \big[f_{12}(x)y + f_{11}(x)z + f_{10}(x)\big]\dfrac{\partial w}{\partial y} + \big[f_{22}(x)y + f_{21}(x)z + f_{20}(x)\big]\dfrac{\partial w}{\partial z}$
$+ F\left(x, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = g(x)w + h_1(x)y + h_2(x)z.$

Complete integral:

$$w = y\varphi_1(x) + z\varphi_2(x) + \psi(x).$$

The functions $\varphi_1(x)$, $\varphi_2(x)$, and $\psi(x)$ are determined by solving the system of ordinary differential equations

$$\varphi_1' + (f_{12} - g)\varphi_1 + f_{22}\varphi_2 = h_1, \tag{1}$$
$$\varphi_2' + f_{11}\varphi_1 + (f_{21} - g)\varphi_2 = h_2, \tag{2}$$
$$\psi' - g\psi = -f_{10}\varphi_1 - f_{20}\varphi_2 - F(x, \varphi_1, \varphi_2), \tag{3}$$

where $f_{ij} = f_{ij}(x)$, $g = g(x)$, and $h_k = h_k(x)$; the prime denotes the derivative with respect to x.

Having found $\varphi_1 = \varphi_1(x)$ and $\varphi_2 = \varphi_2(x)$ from the linear system of equation (1) and (2), one can determine the solution of equation (3) according to the formula

$$\psi(x) = C_3 G - G\int \big[f_{10}\varphi_1 + f_{20}\varphi_2 + F(x,\varphi_1,\varphi_2)\big]\,\frac{dx}{G}, \qquad G = \exp\Big(\int g\,dx\Big).$$

2°. Consider the special case $h_1 \equiv 0$ and $h_2 \equiv 0$. In addition, assume that

$$f_{11} = a_1 f(x), \quad f_{12} = a_2 f(x) + g(x), \quad f_{21} = a_3 f(x) + g(x), \quad f_{22} = a_4 f(x), \tag{4}$$

where $f = f(x)$ and $g = g(x)$ are arbitrary functions and a_1, a_2, a_3, and a_4 are arbitrary numbers. With these assumptions, the general solution of the linear homogeneous system (1), (2) has the form

$$\begin{aligned}\varphi_1 &= C_1\exp\Big(\lambda_1\int f\,dx\Big) + C_2\exp\Big(\lambda_2\int f\,dx\Big),\\ \varphi_2 &= -\frac{a_2+\lambda_1}{a_4}C_1\exp\Big(\lambda_1\int f\,dx\Big) - \frac{a_2+\lambda_2}{a_4}C_2\exp\Big(\lambda_2\int f\,dx\Big),\end{aligned} \tag{5}$$

where λ_1 and λ_2 are the roots of the quadratic equation $(\lambda + a_2)(\lambda + a_3) - a_1 a_4 = 0$.

3°. Let $h_1 = h_1(x)$ and $h_2 = h_2(x)$ be arbitrary functions and let the $f_{ij} = f_{ij}(x)$ be defined by relations (4). In this case, the general solution of the linear nonhomogeneous system of equations (1), (2) can be obtained by variation of constants or with the aid of the Wronskian determinant [see, for example, Murphy (1960), Kamke (1977), and Zwillinger (1998)].

6. $$\frac{\partial w}{\partial x} + wF\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) + yG\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) + zH\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) + R\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) = 0.$$

A complete integral can be sought in the form $w = \varphi(x)y + \psi(x)z + \chi(x)$.

7. $$\frac{\partial w}{\partial x} + F\Big(x, \frac{\partial w}{\partial y} + az, \frac{\partial w}{\partial z} + ay\Big) = 0.$$

Complete integral: $w = -ayz + C_1 y + C_2 z - \int F\big(x, C_1, C_2\big)\,dx + C_3.$

8. $$\frac{\partial w}{\partial x} + F\Big(x, g_1(y)\frac{\partial w}{\partial y} + h_1(y), g_2(z)\frac{\partial w}{\partial z} + h_2(z)\Big) = 0.$$

Complete integral:

$$w = \int \frac{C_1 - h_1(y)}{g_1(y)}\,dy + \int \frac{C_2 - h_2(z)}{g_2(z)}\,dz - \int F(x, C_1, C_2)\,dx + C_3.$$

9. $$\Big(\frac{\partial w}{\partial x}\Big)^2 + \frac{\partial w}{\partial x}F\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) + G\Big(x, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\Big) = 0.$$

Complete integral:

$$w = C_1 y + C_2 z + C_3 + \frac{1}{2}\int\Big[-F(x, C_1, C_2) + \sqrt{F^2(x, C_1, C_2) - 4G(x, C_1, C_2)}\Big]\,dx.$$

10. $F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral: $w = C_1x + C_2y + C_3z + C_4$, where the first three constants are related by the one constraint $F(C_1, C_2, C_3) = 0$.

⊙ Reference: E. Kamke (1965).

11. $F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) + x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 0.$

Complete integral:

$$w = \varphi(\xi) + C_4, \quad \xi = C_1x + C_2y + C_3z,$$

where the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$F\left(C_1\varphi'_\xi, C_2\varphi'_\xi, C_3\varphi'_\xi\right) + \xi\varphi'_\xi = 0.$$

One of the constants C_1, C_2, or C_3 can be set equal to unity.

12. $F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) + x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = w.$

Clairaut's equation. Complete integral: $w = C_1x + C_2y + C_3z + F(C_1, C_2, C_3)$.

⊙ Reference: E. Kamke (1965).

13. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = w.$

Complete integral: $w = C_2y + C_3z + \varphi(x)$, where the function $\varphi(x) = \varphi(x, C_1, C_2, C_3)$ is determined by the ordinary differential equation $F\left(x, \varphi'_x, C_2, C_3\right) = \varphi$.

14. $F\left(ax + by + cz, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral:

$$w = C_1x + C_2y + C_3 + \varphi(\xi, C_1, C_2), \qquad \xi = ax + by + cz,$$

where the function $\varphi = \varphi(\xi, C_1, C_2)$ is determined by the ordinary differential equation

$$F\left(\xi, a\varphi'_\xi + C_1, b\varphi'_\xi + C_2, c\varphi'_\xi\right) = 0.$$

15. $F\left(\dfrac{\partial w}{\partial x} + ayz, \dfrac{\partial w}{\partial y} + axz, \dfrac{\partial w}{\partial z} + axy\right) = 0.$

Complete integral: $w = -axyz + C_1y + C_2z + C_3z + C_4$, where $F(C_1, C_2, C_3) = 0$.

16. $F\left(ax + by + cz, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) + x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = w.$

Complete integral:

$$w = C_1x + C_2y + C_3z + \varphi(\xi), \quad \xi = ax + by + cz,$$

where the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi, a\varphi'_\xi + C_1, b\varphi'_\xi + C_2, c\varphi'_\xi + C_3\right) + \xi\varphi'_\xi = \varphi.$$

17. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral:

$$w = \varphi(\xi), \quad \xi = C_1 x + C_2 y + C_3 z,$$

where the function $\varphi(\xi)$ is determined by solving the first-order autonomous ordinary differential equation $F\big(\varphi, C_1\varphi'_\xi, C_2\varphi'_\xi, C_3\varphi'_\xi\big) = 0$.

⊙ Reference: E. Kamke (1965).

18. $F\left(ax + by + cz + kw, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}\right) = 0.$

For $k = 0$, see equation 4.3.4.14. For $k \neq 0$, the substitution $ku = ax + by + cz + kw$ leads to an equation of the form 4.3.4.17: $F\Big(ku, \dfrac{\partial u}{\partial x} - \dfrac{a}{k}, \dfrac{\partial u}{\partial y} - \dfrac{b}{k}, \dfrac{\partial u}{\partial z} - \dfrac{c}{k}\Big) = 0.$

19. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y} + az, \dfrac{\partial w}{\partial z} + ay\right) = 0.$

Complete integral: $w = -ayz + C_1 y + C_2 z + \varphi(x) + C_3$, where the function $\varphi(\xi)$ is determined by the first-order ordinary differential equation $F\big(x, \varphi'_x, C_1, C_2\big) = 0$.

20. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}, x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral:

$$w = \varphi(\xi), \quad \xi = C_1 x + C_2 y + C_3 z,$$

where the function $\varphi(\xi)$ is determined by the first-order ordinary differential equation

$$F\big(\varphi, C_1\varphi'_\xi, C_2\varphi'_\xi, C_3\varphi'_\xi, \xi\varphi'_\xi\big) = 0.$$

21. $F\left(x, \dfrac{\partial w}{\partial x}, G\left(y, \dfrac{\partial w}{\partial y}\right), H\left(z, \dfrac{\partial w}{\partial z}\right)\right) = 0.$

Complete integral:

$$w = \varphi(x, C_1, C_2) + \psi(y, C_1) + \chi(z, C_2) + C_3.$$

The functions φ, ψ, and χ are determined by the ordinary differential equations

$$F(x, \varphi'_x, C_1, C_2) = 0, \quad G(y, \psi'_y) = C_1, \quad H(z, \chi'_z) = C_2.$$

Solving these equations for the derivatives, we obtain linear separable equations, which are easy to integrate.

22. $F\left(G\left(H\left(x, \dfrac{\partial w}{\partial x}\right), y, \dfrac{\partial w}{\partial y}\right), z, \dfrac{\partial w}{\partial z}\right) = 0.$

Complete integral:

$$w = \varphi(x, C_1) + \psi(y, C_2, C_1) + \chi(z, C_2) + C_3.$$

The functions φ, ψ, and χ are determined by the ordinary differential equations

$$H\big(x, \varphi'_x\big) = C_1, \quad G\big(C_1, y, \psi'_y\big) = C_2, \quad F\big(C_2, z, \chi'_z\big) = 0.$$

Solving these equations for the derivatives, we obtain linear separable equations, which are easy to integrate.

⊙ Reference: A. P. Markeev (1990).

4.4. Nonlinear Equations with Four Independent Variables

▷ *This section deals with specific nonlinear equations with four independent variables which contain arbitrary parameters. For nonlinear equations containing arbitrary functions, see Section 4.5 with* $n = 4$.

4.4.1. Equations Quadratic in Derivatives

1. $\dfrac{\partial w}{\partial x_1} + a\left(\dfrac{\partial w}{\partial x_2}\right)^2 + b\left(\dfrac{\partial w}{\partial x_3}\right)^2 + c\left(\dfrac{\partial w}{\partial x_4}\right)^2 = kx_1^n w + sx_1^m.$

Complete integral:

$$w = C_1 F(x_1) + (C_2 x_2 + C_3 x_3 + C_4 x_4) F(x_1) + F(x_1) \int \frac{s x_1^m - \lambda F^2(x_1)}{F(x_1)}\, dx_1,$$

where $F(x) = \exp\left(\dfrac{k}{n+1} x_1^{n+1}\right)$, $\lambda = aC_2^2 + bC_3^2 + cC_4^2$.

2. $\dfrac{\partial w}{\partial x_1} + a\left(\dfrac{\partial w}{\partial x_2}\right)^2 + b\left(\dfrac{\partial w}{\partial x_3}\right)^2 + c\left(\dfrac{\partial w}{\partial x_4}\right)^2 = kw^2 + sx_1^m w.$

This is a special case of equation 4.5.1.2.

3. $\dfrac{\partial w}{\partial x_1} + \dfrac{1}{2c}\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \dfrac{1}{2a}\left(\dfrac{\partial w}{\partial x_3}\right)^2 + \dfrac{1}{2a \sin^2 x_3}\left(\dfrac{\partial w}{\partial x_4} - \cos x_3 \dfrac{\partial w}{\partial x_2}\right)^2 + k \cos x_3 = 0.$

This equation governs the motion of a rigid body with one fixed point in the Lagrange–Poisson case (x_1 is time; x_2, x_3, and x_4 are Eulerian angles; w is the Hamiltonian function; $a = b \neq c$ are mass moments of inertia; and k is the body weight times the distance from the center of mass to the fixed point). Complete integral:

$$w = -C_1 x_1 + C_2 x_2 + C_3 x_4 + C_4 \pm \int \sqrt{2aC_1 - 2ak \cos x_3 - \frac{a}{c} C_2^2 - \frac{(C_3 - C_2 \cos x_3)^2}{\sin^2 x_3}}\, dx_3.$$

⊙ Reference: E. N. Berezkin (1962).

4. $\dfrac{\partial w}{\partial x_1} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \dfrac{1}{2x_2^2}\left(\dfrac{\partial w}{\partial x_3}\right)^2 + \dfrac{1}{2x_2^2 \sin^2 x_3}\left(\dfrac{\partial w}{\partial x_4}\right)^2 - \dfrac{a}{x_2} = 0.$

This equation governs the motion of a planet in a central Newtonian gravitational field (x_1 is time, x_2 is the radial coordinate, x_3 and x_4 are the angular coordinates, and w is the Hamiltonian function). Complete integral:

$$w = -C_1 x_1 + C_2 x_4 + C_3 \pm \int \sqrt{2C_1 + \frac{2a}{x_2} - \frac{C_4^2}{x_2^2}}\, dx_2 \pm \int \sqrt{C_4^2 - \frac{C_2^2}{\sin^2 x_3}}\, dx_3.$$

⊙ Reference: P. Appell (1953) and E. N. Berezkin (1962).

5. $\dfrac{\partial w}{\partial x_1}+ax_1^k\left(\dfrac{\partial w}{\partial x_2}\right)^2+bx_1^n\left(\dfrac{\partial w}{\partial x_3}\right)^2+cx_1^m\left(\dfrac{\partial w}{\partial x_4}\right)^2=sx_1^qw.$

Complete integral:

$$w=\varphi(x_1)(C_1+C_2x_2+C_3x_3+C_4x_4)-\varphi(x_1)\int\varphi(x_1)\bigl(aC_2^2x_1^k+bC_3^2x_1^n+cC_4^2x_1^m\bigr)\,dx_1,$$

where $\varphi(x_1)=\exp\left(\dfrac{s}{q+1}x_1^{q+1}\right)$.

6. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+a_3x_1^{k_3}\left(\dfrac{\partial w}{\partial x_3}\right)^2+a_4x_1^{k_4}\left(\dfrac{\partial w}{\partial x_4}\right)^2$
$=bx_1^nw+c_2x_1^{m_2}x_2+c_3x_1^{m_3}x_3+c_4x_1^{m_4}x_4.$

This is a special case of equation 4.6.1.5.

7. $\dfrac{\partial w}{\partial x_1}+a_2x_2^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+a_3x_3^{k_3}\left(\dfrac{\partial w}{\partial x_3}\right)^2+a_4x_4^{k_4}\left(\dfrac{\partial w}{\partial x_4}\right)^2$
$=c_1x_1^{n_1}+c_2x_2^{n_2}+c_3x_3^{n_3}+c_4x_4^{n_4}.$

This is a special case of equation 4.2.2.6.

8. $\dfrac{\partial w}{\partial x_1}+a_1e^{\lambda_1x_1}\left(\dfrac{\partial w}{\partial x_2}\right)^2+a_2e^{\lambda_2x_1}\left(\dfrac{\partial w}{\partial x_3}\right)^2+a_3e^{\lambda_3x_1}\left(\dfrac{\partial w}{\partial x_4}\right)^2=be^{\beta x_1}w.$

Complete integral:

$$w=\varphi(x_1)(C_1+C_2x_2+C_3x_3+C_4x_4)-\varphi(x_1)\int\varphi(x_1)\bigl(a_1C_2^2e^{\lambda_1x_1}+a_2C_3^2e^{\lambda_2x_1}+a_3C_4^2e^{\lambda_3x_1}\bigr)dx_1,$$

where $\varphi(x_1)=\exp\left(\dfrac{b}{\beta}e^{\beta x_1}\right)$.

9. $\dfrac{\partial w}{\partial x_1}+a_2x_2^{k_2}w^{n_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+a_3x_3^{k_3}w^{n_3}\left(\dfrac{\partial w}{\partial x_3}\right)^2+a_4x_4^{k_4}w^{n_4}\left(\dfrac{\partial w}{\partial x_4}\right)^2=0.$

Complete integral:

$$C_1-x_1+\frac{2C_2}{2-k_2}x_2^{\frac{2-k_2}{2}}+\frac{2C_3}{2-k_3}x_3^{\frac{2-k_3}{2}}+\frac{2C_4}{2-k_4}x_4^{\frac{2-k_4}{2}}$$
$$=\frac{a_2C_2^2}{n_2+1}w^{n_2+1}+\frac{a_3C_3^2}{n_3+1}w^{n_3+1}+\frac{a_4C_4^2}{n_4+1}w^{n_4+1}.$$

10. $\dfrac{\partial w}{\partial x_1}+a_2x_2^{k_2}w^{n_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+a_3x_3^{k_3}w^{n_3}\left(\dfrac{\partial w}{\partial x_3}\right)^2+a_4x_4^{k_4}w^{n_4}\left(\dfrac{\partial w}{\partial x_4}\right)^2=bw^m.$

This is a special case of equation 4.6.1.15.

4.4.2. Equations Contain Power-Law Functions of Derivatives

1. $\dfrac{\partial w}{\partial x_1}+\sqrt{k+a\left(\dfrac{\partial w}{\partial x_2}\right)^2+b\left(\dfrac{\partial w}{\partial x_3}\right)^2+c\left(\dfrac{\partial w}{\partial x_4}\right)^2}=0.$

Complete integral:

$$w=-Ax_1+C_1x_2+C_2x_3+C_3x_4+C_4,\qquad A=\sqrt{k+aC_1^2+bC_2^2+cC_3^2}.$$

2. $\dfrac{\partial w}{\partial x_1}+a_1\sqrt{1+b_1\left(\dfrac{\partial w}{\partial x_2}\right)^2}+a_2\sqrt{1+b_2\left(\dfrac{\partial w}{\partial x_3}\right)^2}+a_3\sqrt{1+b_3\left(\dfrac{\partial w}{\partial x_4}\right)^2}=0.$

Complete integral:

$$w=-Ax_1+C_1x_2+C_2x_3+C_3x_4+C_4,$$

where $A=a_1\sqrt{1+b_1C_1^2}+a_2\sqrt{1+b_2C_2^2}+a_3\sqrt{1+b_3C_3^2}$.

3. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\dfrac{\partial w}{\partial x_3}\dfrac{\partial w}{\partial x_4}=ax_1^kx_2^nx_3^mx_4^s.$

Complete integral: $w=\dfrac{C_1}{k+1}x_1^{k+1}+\dfrac{C_2}{n+1}x_2^{n+1}+\dfrac{C_3}{m+1}x_3^{m+1}+\dfrac{a}{C_1C_2C_3(s+1)}x_4^{s+1}.$

4. $\dfrac{\partial w}{\partial x_1}+a_1\left(\dfrac{\partial w}{\partial x_2}+bx_3x_4\right)^{k_1}+a_2\left(\dfrac{\partial w}{\partial x_3}+bx_2x_4\right)^{k_2}+a_3\left(\dfrac{\partial w}{\partial x_4}+bx_2x_3\right)^{k_3}=cx_1^n.$

Complete integral:

$$w=-bx_2x_3x_4+C_1x_2+C_2x_3+C_3x_4-\left(a_1C_1^{k_1}+a_2C_2^{k_2}+a_3C_3^{k_3}\right)x_1+\frac{c}{n+1}x_1^{n+1}+C_4.$$

5. $a_1x_1^{k_1}\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}+a_2x_2^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}+a_3x_3^{k_3}\left(\dfrac{\partial w}{\partial x_3}\right)^{k_3}+a_4x_4^{k_4}\left(\dfrac{\partial w}{\partial x_4}\right)^{k_4}$
$=b_1x_1^{n_1}+b_2x_2^{n_2}+b_3x_3^{n_3}+b_4x_4^{n_4}.$

This is a special case of equation 4.6.2.6.

6. $a_1x_1^{k_1}\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}+a_2x_2^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}+a_3x_3^{k_3}\left(\dfrac{\partial w}{\partial x_3}\right)^{k_3}+a_4x_4^{k_4}\left(\dfrac{\partial w}{\partial x_4}\right)^{k_4}=bw^n.$

This is a special case of equation 4.6.2.7.

7. $a_1\left(\dfrac{\partial w}{\partial x_1}+bx_2\right)^{k_1}+a_2\left(\dfrac{\partial w}{\partial x_2}+bx_1\right)^{k_2}+a_3\left(\dfrac{\partial w}{\partial x_3}+cx_4\right)^{k_3}+a_4\left(\dfrac{\partial w}{\partial x_4}+cx_3\right)^{k_4}=s.$

Complete integral:

$$w=-bx_1x_2-cx_3x_4+C_1x_1+C_2x_2+C_3x_3+C_4x_4+C_5,$$

where $a_1C_1^{k_1}+a_2C_2^{k_2}+a_3C_3^{k_3}+a_4C_4^{k_4}=s$.

8. $a\left(\dfrac{\partial w}{\partial x_1}\right)^k\left(\dfrac{\partial w}{\partial x_2}\right)^n+b\left(\dfrac{\partial w}{\partial x_3}\right)^m\left(\dfrac{\partial w}{\partial x_4}\right)^s+x_1\dfrac{\partial w}{\partial x_1}+x_2\dfrac{\partial w}{\partial x_2}+x_3\dfrac{\partial w}{\partial x_3}+x_4\dfrac{\partial w}{\partial x_4}=w.$

Complete integral: $w = C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + aC_1^kC_2^n + bC_3^mC_4^s.$

9. $a_1\left(\dfrac{\partial w}{\partial x_1}+bx_2\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}+bx_1\right)^{k_2}+a_2\left(\dfrac{\partial w}{\partial x_3}+cx_4\right)^{n_1}\left(\dfrac{\partial w}{\partial x_4}+cx_3\right)^{n_2}=s.$

Complete integral:

$$w = -bx_1x_2 - cx_3x_4 + C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + C_5,$$

where $a_1C_1^{k_1}C_2^{k_2} + a_2C_3^{n_1}C_4^{n_2} = s.$

10. $a_1\left(\dfrac{\partial w}{\partial x_1}+bx_2\right)^{k_1}\left(\dfrac{\partial w}{\partial x_3}+cx_4\right)^{k_2}+a_2\left(\dfrac{\partial w}{\partial x_2}+bx_1\right)^{n_1}\left(\dfrac{\partial w}{\partial x_4}+cx_3\right)^{n_2}=s.$

Complete integral:

$$w = -bx_1x_2 - cx_3x_4 + C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + C_5,$$

where $a_1C_1^{k_1}C_3^{k_2} + a_2C_2^{n_1}C_4^{n_2} = s.$

11. $\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}\left(\dfrac{\partial w}{\partial x_3}\right)^{k_3}\left(\dfrac{\partial w}{\partial x_4}\right)^{k_4}=ax_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}w^m.$

This is a special case of equation 4.6.2.4.

12. $a\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}\left(\dfrac{\partial w}{\partial x_3}\right)^{k_3}\left(\dfrac{\partial w}{\partial x_4}\right)^{k_4}=x_1\dfrac{\partial w}{\partial x_1}+x_2\dfrac{\partial w}{\partial x_2}+x_3\dfrac{\partial w}{\partial x_3}+x_4\dfrac{\partial w}{\partial x_4}.$

Complete integral:

$$w = \frac{k_1 + k_2 + k_3 + k_4 - 1}{k_1 + k_2 + k_3 + k_4}\left(C_1x_1 + C_2x_2 + C_3x_3 + Ax_4\right)^{\frac{k_1+k_2+k_3+k_4}{k_1+k_2+k_3+k_4-1}} + C_4,$$

where $A = \left(aC_1^{k_1}C_2^{k_2}C_3^{k_3}\right)^{-1/k_4}.$

13. $a\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}\left(\dfrac{\partial w}{\partial x_3}\right)^{k_3}\left(\dfrac{\partial w}{\partial x_4}\right)^{k_4}+x_1\dfrac{\partial w}{\partial x_1}+x_2\dfrac{\partial w}{\partial x_2}+x_3\dfrac{\partial w}{\partial x_3}+x_4\dfrac{\partial w}{\partial x_4}=w.$

Complete integral: $w = C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + aC_1^{k_1}C_2^{k_2}C_3^{k_3}C_4^{k_4}.$

14. $\left(\dfrac{\partial w}{\partial x_1}+ax_2\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}+ax_1\right)^{k_2}\left(\dfrac{\partial w}{\partial x_3}+bx_4\right)^{k_3}\left(\dfrac{\partial w}{\partial x_4}+bx_3\right)^{k_4}=c.$

Complete integral:

$$w = -ax_1x_2 - bx_3x_4 + C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + C_5,$$

where $C_1^{k_1}C_2^{k_2}C_3^{k_3}C_4^{k_4} = c.$

4.5. Nonlinear Equations with Arbitrary Number of Variables Containing Arbitrary Parameters

4.5.1. Equations Quadratic in Derivatives

1. $\dfrac{\partial w}{\partial x_1}+a_2\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_n\left(\dfrac{\partial w}{\partial x_n}\right)^2=bw+cx_1^k.$

This is a special case of equation 4.6.1.1 with $f(x_1)=b$ and $g(x_1)=cx_1^k$. Complete integral:

$$w=(C_1+C_2x_2+\cdots+C_nx_n)e^{bx_1}-\frac{1}{b}(a_2C_2^2+\cdots+a_nC_n^2)e^{2bx_1}+ce^{bx_1}\int e^{-bx_1}x_1^k\,dx_1.$$

2. $\dfrac{\partial w}{\partial x_1}+a_2\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_n\left(\dfrac{\partial w}{\partial x_n}\right)^2=bw^2+cx_1^kw.$

This is a special case of equation 4.6.1.2 with $f(x_1)=cx_1^k$ and $g(x_1)=0$. Complete integral:

$$w=\varphi(x_1)+C_1\psi(x_1)\exp(C_2x_2+\cdots+C_nx_n).$$

The arbitrary constants $C_2,\ldots,C_n$ are related by the one constraint $a_2C_2^2+\cdots+a_nC_n^2=b$. The function $\varphi(x)$ and $\psi(x)$ are given by

$$\varphi(x)=F(x)\left[C_{n+1}-b\int F(x)\,dx\right]^{-1},\quad F(x)=\exp\left(\frac{c}{k+1}x^{k+1}\right),$$
$$\psi(x)=\exp\left\{\int\left[2b\varphi(x)+cx^k\right]dx\right\}.$$

3. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_1^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=bw.$

Complete integral:

$$w=C_1e^{bx_1}+e^{bx_1}(C_2x_2+\cdots+C_nx_n)-e^{bx_1}\int e^{bx_1}\left(a_2C_2^2x_1^{k_2}+\cdots+a_nC_n^2x_1^{k_n}\right)dx_1.$$

4. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_1^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=bx_1^mw.$

This is a special case of equation 4.6.1.4 with $f_i(x_1)=a_ix_1^{k_i}$ and $g(x_1)=bx_1^m$. Complete integral:

$$w=\varphi(x_1)(C_1+C_2x_2+\cdots+C_nx_n)-\varphi(x_1)\int\varphi(x_1)\left(a_2C_2^2x_1^{k_2}+\cdots+a_nC_n^2x_1^{k_n}\right)dx_1,$$

where $\varphi(x_1)=\exp\left(\dfrac{b}{m+1}x_1^{m+1}\right)$.

5. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_1^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=\beta w+b_2x_1^{m_2}x_2+\cdots+b_nx_1^{m_n}x_n.$

This is a special case of equation 4.6.1.5 with $f_i(x_1)=a_ix_1^{k_i}$, $g(x_1)=\beta$, and $h_i(x_1)=b_ix_1^{m_i}$. Complete integral:

$$w=x_2\varphi_2(x_1)+\cdots+x_n\varphi_n(x_1)+\psi(x_1),$$

where the functions $\varphi_k(x_1)$ and $\psi(x_1)$ are given by

$$\varphi_k(x_1)=C_ke^{\beta x_1}+b_ke^{\beta x_1}\int e^{-\beta x_1}x_1^{m_k}\,dx_1,\qquad k=2,\ldots,n;$$

$$\psi(x_1)=C_1e^{\beta x_1}-e^{\beta x_1}\int e^{-\beta x_1}\left[a_2x_1^{k_2}\varphi_2^2(x_1)+\cdots+a_nx_1^{k_n}\varphi_n^2(x_1)\right]dx_1.$$

6. $\dfrac{\partial w}{\partial x_1}+a_2x_2^{k_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_n^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=cx_1^{\beta}+b_2x_2^{m_2}+\cdots+b_nx_n^{m_n}.$

Complete integral:

$$w=C_1-(C_2+\cdots+C_n)x_1+\frac{c}{\beta+1}x_1^{\beta+1}+\int\sqrt{\frac{C_2+b_2x_2^{m_2}}{a_2x_2^{k_2}}}\,dx_2+\cdots+\int\sqrt{\frac{C_n+b_nx_n^{m_n}}{a_nx_n^{k_n}}}\,dx_n.$$

7. $\dfrac{\partial w}{\partial x_1}+a_2e^{\lambda_2x_1}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_ne^{\lambda_nx_1}\left(\dfrac{\partial w}{\partial x_n}\right)^2=be^{\beta x_1}w.$

This is a special case of equation 4.6.1.4 with $f_i(x_1)=a_ie^{\lambda_ix_1}$ and $g(x_1)=be^{\beta x_1}$. Complete integral:

$$w=\varphi(x_1)(C_1+C_2x_2+\cdots+C_nx_n)-\varphi(x_1)\int\varphi(x_1)\left(a_2C_2^2e^{\lambda_2x_1}+\cdots+a_nC_n^2e^{\lambda_nx_1}\right)dx_1,$$

where $\varphi(x_1)=\exp\left(\dfrac{b}{\beta}e^{\beta x_1}\right)$.

8. $\dfrac{\partial w}{\partial x_1}+a_2e^{\lambda_2x_1}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_ne^{\lambda_nx_1}\left(\dfrac{\partial w}{\partial x_n}\right)^2$
$=cw+b_2e^{\beta_2x_1}x_2+\cdots+b_ne^{\beta_nx_1}x_n.$

This is a special case of equation 4.6.1.5 with $f_i(x_1)=a_ie^{\lambda_ix_1}$, $g(x_1)=c$, and $h_i(x_1)=b_ie^{\beta_ix_1}$.

9. $\dfrac{\partial w}{\partial x_1}+\sum\limits_{k=2}^{n}\left(a_kx_1^{\beta_k}x_k^{m_k}+b_kx_1^{\gamma_k}x_k^{2m_k-2}\right)\left(\dfrac{\partial w}{\partial x_k}\right)^2=c_1x_1^{\mu_1}w+c_0x_1^{\mu_0}.$

This is a special case of equation 4.6.1.9 with $f_k(x_1)=a_kx_1^{\beta_k}$, $g_k(x_1)=b_kx_1^{\gamma_k}$, $a(x_1)=c_1x_1^{\mu_1}$, and $b(x_1)=c_0x_1^{\mu_0}$.

10. $\dfrac{\partial w}{\partial x_1}+\sum\limits_{k,m=2}^{n}a_{km}x_1^{\beta_{km}}\dfrac{\partial w}{\partial x_k}\dfrac{\partial w}{\partial x_m}=bw+\sum\limits_{k,m=2}^{n}c_{km}x_1^{\gamma_{km}}x_kx_m.$

This is a special case of equation 4.6.1.10.

11. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{k_2}w^{\beta}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_1^{k_n}w^{\beta}\left(\dfrac{\partial w}{\partial x_n}\right)^2=bx_1^mw.$

The substitution $u=\dfrac{1}{\beta+1}w^{\beta+1}$ leads to an equation of the form 4.5.1.4:

$$\frac{\partial u}{\partial x_1}+a_2x_1^{k_2}\left(\frac{\partial u}{\partial x_2}\right)^2+\cdots+a_nx_1^{k_n}\left(\frac{\partial u}{\partial x_n}\right)^2=b(\beta+1)x_1^mu.$$

12. $\dfrac{\partial w}{\partial x_1}+a_2x_1^{m_2}w^{k}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_1^{m_n}w^{k}\left(\dfrac{\partial w}{\partial x_n}\right)^2=bw^{k+2}+cx_1^sw.$

This is a special case of equation 4.6.1.13 with $f_i(x_1)=a_ix_1^{m_i}$ and $h(x_1)=cx_1^s$.

13. $\dfrac{\partial w}{\partial x_1}+a_2x_2^{k_2}w^{m_2}\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_nx_n^{k_n}w^{m_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=0.$

This is a special case of equation 4.6.1.14 with $f_i(x_i)=a_ix_i^{k_i}$, $g_i(w)=w^{m_i}$. Complete integral for $k_i\neq 2$ and $m_i\neq -1$:

$$x_1+\frac{a_2(2-k_2)^2C_2^2}{m_2+1}w^{m_2+1}+\cdots+\frac{a_n(2-k_n)^2C_n^2}{m_n+1}w^{m_n+1}=C_1+2C_2x_2^{\frac{2-k_2}{2}}+\cdots+2C_nx_n^{\frac{2-k_n}{2}}.$$

14. $a_1\left(\dfrac{\partial w}{\partial x_1}\right)^2+a_2\left(\dfrac{\partial w}{\partial x_2}\right)^2+\cdots+a_n\left(\dfrac{\partial w}{\partial x_n}\right)^2=w.$

Complete integral: $w=\displaystyle\sum_{k=1}^{n}\frac{1}{4a_k}\left(x_k+C_k\right)^2.$

⊙ Reference: A. M. Vinogradov and I. S. Krasilshchik (1997).

15. $a_1x_1^{k_1}\left(\dfrac{\partial w}{\partial x_1}\right)^2+\cdots+a_nx_n^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=b_1x_1^{m_1}+\cdots+b_nx_n^{m_n}.$

This is a special case of equation 4.6.1.16 with $f_i(x_i)=a_ix_i^{k_i}$ and $g_i(x_i)=b_ix_i^{m_i}$.

16. $a_1x_1^{k_1}\left(\dfrac{\partial w}{\partial x_1}\right)^2+\cdots+a_nx_n^{k_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=\left(b_1x_1^{m_1}+\cdots+b_nx_n^{m_n}\right)w^{\beta}.$

This is a special case of equation 4.6.1.18 with $f_i(x_i)=a_ix_i^{k_i}$, $g_i(x_i)=b_ix_i^{m_i}$, and $h(w)=w^{\beta}$.

17. $a_1e^{\lambda_1x_1}\left(\dfrac{\partial w}{\partial x_1}\right)^2+\cdots+a_ne^{\lambda_nx_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=b_1e^{\beta_1x_1}+\cdots+b_ne^{\beta_nx_n}.$

This is a special case of equation 4.6.1.16 with $f_k(x_k)=a_ke^{\lambda_kx_k}$ and $g_k(x_k)=b_ke^{\beta_kx_k}$.

18. $a_1e^{\lambda_1x_1}\left(\dfrac{\partial w}{\partial x_1}\right)^2+\cdots+a_ne^{\lambda_nx_n}\left(\dfrac{\partial w}{\partial x_n}\right)^2=\left(b_1e^{\beta_1x_1}+\cdots+b_ne^{\beta_nx_n}\right)e^{\gamma w}.$

This is a special case of equation 4.6.1.18 with $f_k(x_k)=a_ke^{\lambda_kx_k}$, $g_k(x_k)=b_ke^{\beta_kx_k}$, and $h(w)=e^{\gamma w}$.

4.5.2. Equations with Power-Law Nonlinearity in Derivatives

1. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\ldots\dfrac{\partial w}{\partial x_n} = x_1x_2\ldots x_n.$

Complete integrals:

$$\text{(a)}\qquad w = \tfrac{1}{2}\sum_{k=1}^{n-1} C_k x_k^2 + \tfrac{1}{2}Ax_n^2 + C_n, \qquad A = \frac{1}{C_1C_2\ldots C_{n-1}},$$

$$\text{(b)}\qquad (w - C_n)^n = \left(\frac{n}{2}\right)^n (x_n^2 - A)\prod_{k=1}^{n-1}(x_k^2 - C_k).$$

2. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\ldots\dfrac{\partial w}{\partial x_n} = ax_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}w^m.$

Complete integral:

$$\frac{n}{n-m}w^{\frac{n-m}{n}} = \frac{C_1}{k_1+1}x^{k_1+1} + \cdots + \frac{C_{n-1}}{k_{n-1}+1}x^{k_{n-1}+1} + \frac{a}{(k_n+1)C_1C_2\ldots C_{n-1}}x^{k_n+1} + C_n.$$

3. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\ldots\dfrac{\partial w}{\partial x_n} = x_1\dfrac{\partial w}{\partial x_1} + x_2\dfrac{\partial w}{\partial x_2} + \cdots + x_n\dfrac{\partial w}{\partial x_n}.$

Complete integral:

$$w = \frac{n-1}{n}(C_1\ldots C_n)^{\frac{1}{1-n}}(C_1x_1 + \cdots + C_nx_n)^{\frac{n}{n-1}} + C_{n+1}.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to unity.

4. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\ldots\dfrac{\partial w}{\partial x_n} = \left(a_1\dfrac{\partial w}{\partial x_1} - w\right)\left(a_2\dfrac{\partial w}{\partial x_2} - w\right)\ldots\left(a_n\dfrac{\partial w}{\partial x_n} - w\right).$

Complete integral: $\ln|w| = \sum\limits_{k=1}^{n} C_kx_k + C_{n+1}$, where the arbitrary constants $C_1, \ldots, C_n$ are related by the one constraint $C_1C_2\ldots C_n = (a_1C_1 - 1)(a_2C_2 - 1)\ldots(a_nC_n - 1)$.

5. $a_1x_1^{k_1}w^{m_1}\left(\dfrac{\partial w}{\partial x_1}\right)^{\beta} + \cdots + a_nx_n^{k_n}w^{m_n}\left(\dfrac{\partial w}{\partial x_n}\right)^{\beta} = bw^{\gamma}.$

This is a special case of equation 4.6.2.7.

6. $a_1\left(\dfrac{\partial w}{\partial x_1}\right)^{m_1} + a_2\left(\dfrac{\partial w}{\partial x_2}\right)^{m_2} + \cdots + a_n\left(\dfrac{\partial w}{\partial x_n}\right)^{m_n} = w.$

Complete integral:

$$w = \sum_{k=1}^{n} b_k\big(x_k + C_k\big)^{\frac{m_k}{m_k-1}}, \qquad \text{where} \quad b_k = a_k^{\frac{1}{1-m_k}}\left(\frac{m_k}{m_k-1}\right)^{\frac{m_k}{1-m_k}}.$$

7. $w = \sum_{k=1}^{n} x_k \dfrac{\partial w}{\partial x_k} + a\left(\dfrac{\partial w}{\partial x_1}\right)^{m_1}\left(\dfrac{\partial w}{\partial x_2}\right)^{m_2}\cdots\left(\dfrac{\partial w}{\partial x_n}\right)^{m_n}.$

Complete integral:

$$w = \sum_{k=1}^{n} C_k x_k + aC_1^{m_1}C_2^{m_2}\ldots C_n^{m_n}.$$

▷ *Complete integrals of some other equations containing arbitrary parameters can be obtained by utilizing the results of Section 4.6, where equations containing arbitrary functions are discussed.*

4.6. Nonlinear Equations with Arbitrary Number of Variables Containing Arbitrary Functions

4.6.1. Equations Quadratic in Derivatives

1. $\dfrac{\partial w}{\partial x_1} + a_2\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + a_n\left(\dfrac{\partial w}{\partial x_n}\right)^2 = f(x_1)w + g(x_1).$

Complete integral:

$$w = C_1F(x_1) + (C_2x_2 + \cdots + C_nx_n)F(x_1) + F(x_1)\int \frac{g(x_1) - bF^2(x_1)}{F(x_1)}\,dx_1,$$

where $F(x) = \exp\left[\int f(x)\,dx\right]$, $b = a_2C_2^2 + \cdots + a_nC_n^2$.

2. $\dfrac{\partial w}{\partial x_1} + a_2\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + a_n\left(\dfrac{\partial w}{\partial x_n}\right)^2 = bw^2 + f(x_1)w + g(x_1).$

1°. For $b = 0$, see equation 4.6.1.1. Complete integral for $b \neq 0$:

$$w = \varphi(x_1) + C_1\psi(x_1)\exp(C_2x_2 + \cdots + C_nx_n).$$

The arbitrary constants $C_2, \ldots, C_n$ are related by the one constraint $a_2C_2^2 + \cdots + a_nC_n^2 = b$. The function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x = b\varphi^2 + f(x)\varphi + g(x), \tag{1}$$

and the function $\psi = \psi(x)$ is expressed via $\varphi(x)$ as

$$\psi(x) = \exp\left\{\int \left[2b\varphi(x) + f(x)\right]dx\right\}. \tag{2}$$

Riccati's equation (1) can be integrated by quadrature for many f and g; for details, see the books Kamke (1976) and Polyanin and Zaitsev (2003). Consider two special cases.

2°. For $g(x) \equiv 0$ and arbitrary $f(x)$, equation (1) has the general solution

$$\varphi(x) = F(x)\left[C_{n+1} - b\int F(x)\,dx\right]^{-1}, \qquad F(x) = \exp\left[\int f(x)\,dx\right].$$

3°. For $f(x) = f = \text{const}$ and $g(x) = g = \text{const}$, equation (1) is separable. Integrating yields

$$\int \frac{d\varphi}{b\varphi^2 + f\varphi + g} = x + C_{n+1}.$$

3. $\dfrac{\partial w}{\partial x_1} + f_2(x_1)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_1)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = aw.$

Complete integral:

$$w = C_1 e^{ax_1} + e^{ax_1}(C_2x_2 + \cdots + C_nx_n) - e^{ax_1}\int e^{ax_1}\left[C_2^2 f_2(x_1) + \cdots + C_n^2 f_n(x_1)\right] dx_1.$$

4. $\dfrac{\partial w}{\partial x_1} + f_2(x_1)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_1)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = g(x_1)w.$

Complete integral:

$$w = \varphi(x_1)(C_1 + C_2x_2 + \cdots + C_nx_n) - \varphi(x_1)\int \varphi(x_1)\left[C_2^2 f_2(x_1) + \cdots + C_n^2 f_n(x_1)\right] dx_1,$$

where $\varphi(x_1) = \exp\left[\int g(x_1)\,dx_1\right]$.

5. $\dfrac{\partial w}{\partial x_1} + f_2(x_1)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_1)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = g(x_1)w + h_2(x_1)x_2 + \cdots + h_n(x_1)x_n.$

Complete integral:

$$w = x_2\varphi_2(x_1) + \cdots + x_n\varphi_n(x_1) + \psi(x_1),$$

where the functions $\varphi_k(x_1)$ and $\psi(x_1)$ are given by

$$\varphi_k(x_1) = C_kG(x_1) + G(x_1)\int \frac{h_k(x_1)}{G(x_1)}\,dx_1, \quad G(x_1) = \exp\left[\int g(x_1)\,dx_1\right], \quad k = 2, \ldots, n;$$

$$\psi(x_1) = C_1G(x_1) - G(x_1)\int \left[f_2(x_1)\varphi_2^2(x_1) + \cdots + f_n(x_1)\varphi_n^2(x_1)\right]\frac{dx_1}{G(x_1)}.$$

6. $\dfrac{\partial w}{\partial x_1} + f_2(x_2)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = g(x_1) + h_2(x_2) + \cdots + h_n(x_n).$

Equations of this form are often encountered in mechanics; the variable x_1 plays the role of time, and $x_2, \ldots, x_n$ are generalized coordinates.

Complete integral:

$$w = C_1 - (C_2 + \cdots + C_n)x_1 + \int g(x_1)\,dx_1 + \int \sqrt{\frac{C_2 + h_2(x_2)}{f_2(x_2)}}\,dx_2 + \cdots + \int \sqrt{\frac{C_n + h_n(x_n)}{f_n(x_n)}}\,dx_n.$$

7. $\dfrac{\partial w}{\partial x_1} + f_2(x_2)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = bw^2 + g(x_1)w + h(x_1).$

The transformation $\xi_k = \displaystyle\int \frac{dx_k}{\sqrt{|f_k(x_k)|}}$ $(k = 2, \ldots, n)$ leads to an equation of the form 4.6.1.2.

8. $\dfrac{\partial w}{\partial x_1}+\sum\limits_{k=2}^{n}\Big[f_k(x_1)x_k+g_k(x_1)\Big]\left(\dfrac{\partial w}{\partial x_k}\right)^2=a(x_1)w+b(x_1)+\sum\limits_{k=2}^{n}h_k(x_1)x_k.$

Complete integral:

$$w=\sum_{k=2}^{n}\varphi_k(x_1)x_k+\psi(x_1),$$

where the functions $\varphi_k(x_1)$ and $\psi(x_1)$ are determined by the ordinary differential equations

$$\varphi_k'+f_k(x_1)\varphi_k^2=a(x_1)\varphi_k+h_k(x_1),\qquad k=2,\ldots,n; \tag{1}$$

$$\psi'+\sum_{k=2}^{n}g_k(x_1)\varphi_k^2=a(x_1)\psi+b(x_1), \tag{2}$$

where the prime denotes the derivative with respect to x_1. Riccati's equation (1) can be integrated by quadrature for many $f_k(x_1)$, $h_k(x_1)$, and $a(x_1)$, in particular, for $h_k(x_1)=0$; see the books Kamke (1976), Polyanin and Zaitsev (2003). Equation (2) is linear with respect to ψ and can be readily integrated (for known φ_k).

9. $\dfrac{\partial w}{\partial x_1}+\sum\limits_{k=2}^{n}\Big[f_k(x_1)x_k^{m_k}+g_k(x_1)x_k^{2m_k-2}\Big]\left(\dfrac{\partial w}{\partial x_k}\right)^2=a(x_1)w+b(x_1).$

1°. If all $m_k\neq 2$, then the transformation $\xi_k=x_k^{2-m_k}$ $(k=2,\ldots,n)$ leads to an equation of the form 4.6.1.8:

$$\frac{\partial w}{\partial x_1}+\sum_{k=2}^{n}(2-m_k)^2\big[f_k(x_1)\xi_k+g_k(x_1)\big]\left(\frac{\partial w}{\partial \xi_k}\right)^2=a(x_1)w+b(x_1).$$

2°. If there is an $m_l=2$, then we introduce the new variable $\xi_l=\ln|x_l|$ instead of x_l and transform the other variables just as in item 1°. As a result, we obtain an equation of the form 4.6.1.8.

10. $\dfrac{\partial w}{\partial x_1}+\sum\limits_{k,m=2}^{n}f_{km}(x_1)\dfrac{\partial w}{\partial x_k}\dfrac{\partial w}{\partial x_m}+\sum\limits_{k=2}^{n}\left[\sum\limits_{m=2}^{n}g_{km}(x_1)x_m+g_k(x_1)\right]\dfrac{\partial w}{\partial x_k}$

$$=\psi(x_1)w+\sum_{k,m=2}^{n}h_{km}(x_1)x_kx_m+\sum_{k=2}^{n}h_k(x_1)x_k+h_0(x_1).$$

A complete integral can be sought in the form

$$w=\sum_{k,m=2}^{n}\varphi_{km}(x_1)x_kx_m+\sum_{k=2}^{n}\varphi_k(x_1)x_k+\varphi_0(x_1).$$

Substituting this expression into the original partial differential equation yields a system of ordinary differential equations for $\varphi_{km}(x_1)$, $\varphi_k(x_1)$, and $\varphi_0(x_1)$. If all $g_k(x_1)\equiv 0$, $h_k(x_1)\equiv 0$, and $h_0(x_1)\equiv 0$, then one may set $\varphi_k(x_1)\equiv 0$ and $\varphi_0(x_1)\equiv 0$.

11. $$\frac{\partial w}{\partial x_1}+\frac{1}{\sum\limits_{k=2}^{n} f_k(x_k)}\sum_{k=2}^{n}\left[g_k(x_k)\left(\frac{\partial w}{\partial x_k}\right)^2-h_k(x_k)\right]=0.$$

Liouville's equation. Complete integral:

$$w=-C_1x_1+\sum_{k=2}^{n}\int\sqrt{\frac{C_1f_k(x_k)+h_k(x_k)+C_k}{g_k(x_k)}}\,dx_k+C_{n+1},$$

where the arbitrary constants $C_2,\dots,C_n$ are related by the one constraint $C_2+\cdots+C_n=0$.

Note that the equation in question governs the motion of a mechanical system with ideal holonomic constraints in the case where the kinetic energy T and the force function U have the form

$$T=\frac{1}{4}\left[\sum_{k=2}^{n}f_k(x_k)\right]\left[\sum_{k=2}^{n}\frac{1}{g_k(x_k)}(x_k')^2\right],\quad U=\sum_{k=2}^{n}h_k(x_k)\Big/\sum_{k=2}^{n}f_k(x_k).$$

Here x_1 is time and $x_2,\dots,x_n$ are generalized coordinates.

⊙ References: E. N. Berezkin (1968), V. V. Kozlov (1995).

12. $$\frac{\partial w}{\partial x_1}+w^kf_2(x_1)\left(\frac{\partial w}{\partial x_2}\right)^2+\cdots+w^kf_n(x_1)\left(\frac{\partial w}{\partial x_n}\right)^2=g(x_1)w.$$

The substitution $u=\dfrac{1}{k+1}w^{k+1}$ leads to an equation of the form 4.6.1.4:

$$\frac{\partial u}{\partial x_1}+f_2(x_1)\left(\frac{\partial u}{\partial x_2}\right)^2+\cdots+f_n(x_1)\left(\frac{\partial u}{\partial x_n}\right)^2=(k+1)g(x_1)u.$$

13. $$\frac{\partial w}{\partial x_1}+w^kf_2(x_2)\left(\frac{\partial w}{\partial x_2}\right)^2+\cdots+w^kf_n(x_n)\left(\frac{\partial w}{\partial x_n}\right)^2=aw^{k+2}+g(x_1)w.$$

The substitution $u=\dfrac{1}{k+1}w^{k+1}$ leads to an equation of the form 4.6.1.7 with $h(x_1)\equiv 0$:

$$\frac{\partial u}{\partial x_1}+f_2(x_2)\left(\frac{\partial u}{\partial x_2}\right)^2+\cdots+f_n(x_n)\left(\frac{\partial u}{\partial x_n}\right)^2=a(k+1)^2u^2+(k+1)g(x_1)u.$$

14. $$\frac{\partial w}{\partial x_1}+f_2(x_2)g_2(w)\left(\frac{\partial w}{\partial x_2}\right)^2+\cdots+f_n(x_n)g_n(w)\left(\frac{\partial w}{\partial x_n}\right)^2=0.$$

Complete integral:

$$x_1+\int\left[C_2^2g_2(w)+\cdots+C_n^2g_n(w)\right]dw=C_1+C_2\int\frac{dx_2}{\sqrt{f_2(x_2)}}+\cdots+C_n\int\frac{dx_n}{\sqrt{f_n(x_n)}}.$$

It has been assumed in the products f_kg_k that the f_k are all nonnegative and the g_k can have any sign.

15. $\dfrac{\partial w}{\partial x_1} + f_2(x_2)g_2(w)\left(\dfrac{\partial w}{\partial x_2}\right)^2 + \cdots + f_n(x_n)g_n(w)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = h(w).$

Complete integral:

$$x_1 + C_2\int \frac{dx_2}{\sqrt{f_2(x_2)}} + \cdots + C_n\int \frac{dx_n}{\sqrt{f_n(x_n)}} + \int \frac{2\varphi(w)\,dw}{1+\sqrt{1+4h(w)\varphi(w)}} = C_1,$$

where $\varphi(w) = C_2^2 g_2(w) + \cdots + C_n^2 g_n(w)$.

16. $f_1(x_1)\left(\dfrac{\partial w}{\partial x_1}\right)^2 + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = g_1(x_1) + \cdots + g_n(x_n).$

Complete integral:

$$w = \pm\int \left[\frac{g_1(x_1)+C_1}{f_1(x_1)}\right]^{1/2} dx_1 \pm \cdots \pm \int \left[\frac{g_n(x_n)+C_n}{f_n(x_n)}\right]^{1/2} dx_n + C_{n+1},$$

where the arbitrary constants $C_1, \ldots, C_n$ are related by the one constraint $C_1 + \cdots + C_n = 0$.

17. $f_1(x_1)\left(\dfrac{\partial w}{\partial x_1}\right)^2 + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = aw + g_1(x_1) + \cdots + g_n(x_n).$

Complete integral:

$$w = \varphi_1(x_1, C_1) + \cdots + \varphi_n(x_n, C_n).$$

The functions $\varphi_k = \varphi_k(x_k, C_k)$ are determined by the ordinary differential equations $f_k(x_k)(\varphi_k')^2 - a\varphi_k - g_k(x_k) = 0$. By the substitutions $\psi_k = \sqrt{a\varphi_k - g_k(x_k)}$, these equations can be reduced to Abel's equations of the second kind for ψ_k [integrable cases of these equations can be found in Polyanin and Zaitsev (2003) and Zaitsev and Polyanin (1994)].

18. $f_1(x_1)\left(\dfrac{\partial w}{\partial x_1}\right)^2 + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = \big[g_1(x_1) + \cdots + g_n(x_n)\big]h(w).$

The substitution $u = \displaystyle\int \frac{dw}{\sqrt{h(w)}}$ leads to an equation of the form 4.6.1.16:

$$f_1(x_1)\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + f_n(x_n)\left(\frac{\partial u}{\partial x_n}\right)^2 = g_1(x_1) + \cdots + g_n(x_n).$$

19. $f_1(x_1)g_1(w)\left(\dfrac{\partial w}{\partial x_1}\right)^2 + \cdots + f_n(x_n)g_n(w)\left(\dfrac{\partial w}{\partial x_n}\right)^2 = h(w).$

Complete integral:

$$\int \left[\frac{C_1^2 g_1(w) + \cdots + C_n^2 g_n(w)}{h(w)}\right]^{1/2} dw = C_1\int \frac{dx_1}{[f_1(x_1)]^{1/2}} + \cdots + C_n\int \frac{dx_n}{[f_n(x_n)]^{1/2}} + C_{n+1}.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to unity.

4.6.2. Equations with Power-Law Nonlinearity in Derivatives

1. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\dots\dfrac{\partial w}{\partial x_n} = f_1(x_1)f_2(x_2)\dots f_n(x_n).$

Complete integral:

$$w = C_1\int f_1(x_1)\,dx_1 + \dots + C_{n-1}\int f_{n-1}(x_{n-1})\,dx_{n-1} + \frac{1}{C_1C_2\dots C_{n-1}}\int f_n(x_n)\,dx_n + C_n.$$

2. $\dfrac{\partial w}{\partial x_1}\dfrac{\partial w}{\partial x_2}\dots\dfrac{\partial w}{\partial x_n} = f_1(x_1)f_2(x_2)\dots f_n(x_n)\varphi(w).$

Complete integral:

$$\int\frac{dw}{\varphi^{1/n}(w)} = C_1\int f_1(x_1)\,dx_1 + \dots + C_{n-1}\int f_{n-1}(x_{n-1})\,dx_{n-1} + \frac{1}{C_1\dots C_{n-1}}\int f_n(x_n)\,dx_n + C_n.$$

3. $f_n(x_n)\left(\dfrac{\partial w}{\partial x_1}\dots\dfrac{\partial w}{\partial x_{n-1}}\right) + \dots + f_k(x_k)\left(\dfrac{\partial w}{\partial x_1}\dots\dfrac{\partial w}{\partial x_{k-1}}\dfrac{\partial w}{\partial x_{k+1}}\dots\dfrac{\partial w}{\partial x_n}\right) + \dots + f_1(x_1)\left(\dfrac{\partial w}{\partial x_2}\dots\dfrac{\partial w}{\partial x_n}\right) = 0.$

The large parentheses contain products of $n-1$ different partial derivatives. Complete integral:

$$w = \sum_{k=1}^{n} C_k\int f_k(x_k)\,dx_k + C_{n+1},$$

where the arbitrary constants $C_1,\dots,C_n$ are related by the one constraint

$$C_1\dots C_{n-1} + \dots + C_1\dots C_{k-1}C_{k+1}\dots C_n + \dots + C_2\dots C_n = 0.$$

4. $\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1}\left(\dfrac{\partial w}{\partial x_2}\right)^{k_2}\dots\left(\dfrac{\partial w}{\partial x_n}\right)^{k_n} = f_1(x_1)f_2(x_2)\dots f_n(x_n)\varphi(w).$

Complete integral:

$$\int\big[\varphi(w)\big]^{-\frac{1}{k_1+\dots+k_n}}\,dw = C_1\int\big[f_1(x_1)\big]^{\frac{1}{k_1}}\,dx_1 + \dots + C_n\int\big[f_n(x_n)\big]^{\frac{1}{k_n}}\,dx_n + C_{n+1},$$

where $C_1^{k_1}\dots C_n^{k_n} = 1$.

5. $f_1(x_1)\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1} + \dots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^{k_n} = w.$

Complete integral:

$$w = \varphi_1(x_1) + \dots + \varphi_n(x_n), \quad \text{where} \quad \varphi_m(x_m) = \left[\frac{k_m-1}{k_m}\int f_m^{-\frac{1}{k_m}}(x_m)\,dx_m + C_m\right]^{\frac{k_m}{k_m-1}}.$$

6. $f_1(x_1)\left(\dfrac{\partial w}{\partial x_1}\right)^{k_1} + \cdots + f_n(x_n)\left(\dfrac{\partial w}{\partial x_n}\right)^{k_n} = g_1(x_1) + \cdots + g_n(x_n).$

Complete integral:

$$w = \int \left[\frac{g_1(x_1) + C_1}{f(x_1)}\right]^{1/k_1} dx_1 + \cdots + \int \left[\frac{g_n(x_n) + C_n}{f(x_n)}\right]^{1/k_n} dx_n + C_{n+1},$$

where the arbitrary constants $C_1, \ldots, C_n$ are related by the one constraint $C_1 + \cdots + C_n = 0$.

7. $f_1(x_1)g_1(w)\left(\dfrac{\partial w}{\partial x_1}\right)^{k} + \cdots + f_n(x_n)g_n(w)\left(\dfrac{\partial w}{\partial x_n}\right)^{k} = h(w).$

Complete integral:

$$\int \left[\frac{C_1^k g_1(w) + \cdots + C_n^k g_n(w)}{h(w)}\right]^{1/k} dw = C_1 \int \frac{dx_1}{[f_1(x_1)]^{1/k}} + \cdots + C_n \int \frac{dx_n}{[f_n(x_n)]^{1/k}} + C_{n+1}.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to unity.

8. $\dfrac{\partial w}{\partial x_1} + f_2(x_2)g_2(w)\left(\dfrac{\partial w}{\partial x_2}\right)^{k} + \cdots + f_n(x_n)g_n(w)\left(\dfrac{\partial w}{\partial x_n}\right)^{k} = 0.$

Complete integral:

$$x_1 + C_1 + \int \left[C_2^k g_2(w) + \cdots + C_n^k g_n(w)\right]^{\frac{1}{k-1}} dw = C_2 \int \frac{dx_2}{[f_2(x_2)]^{1/k}} + \cdots + C_n \int \frac{dx_n}{[f_n(x_n)]^{1/k}}.$$

9. $\dfrac{\partial w}{\partial x_1} + \dfrac{1}{\sum\limits_{k=2}^{n} f_k(x_k)} \sum_{k=2}^{n} \left[g_k(x_k)\left(\dfrac{\partial w}{\partial x_k}\right)^{m_k} - h_k(x_k)\right] = 0.$

Complete integral:

$$w = -C_1 x_1 + \sum_{k=2}^{n} \int \left[\frac{C_1 f_k(x_k) + h_k(x_k) + C_k}{g_k(x_k)}\right]^{1/m_k} dx_k + C_{n+1},$$

where the arbitrary constants $C_2, \ldots, C_n$ are related by the one constraint $C_2 + \cdots + C_n = 0$.

4.6.3. Equations Contain Arbitrary Functions of Two Variables

1. $f_1\left(x_1, \dfrac{\partial w}{\partial x_1}\right) + f_2\left(x_2, \dfrac{\partial w}{\partial x_2}\right) + \ldots + f_n\left(x_n, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral:

$w = \varphi_1(x_1, C_1) + \varphi_2(x_2, C_2) + \cdots + \varphi_n(x_n, C_n) + C_{n+1}$, where $C_n = -C_1 - \cdots - C_{n-1}$.

The functions $\varphi_k = \varphi_k(x_k, C_k)$ are determined by the ordinary differential equations

$$f_k\left(x_k, \frac{d\varphi_k}{dx_k}\right) = C_k \qquad (k = 1, \ldots, n).$$

Solving these equations for the derivatives, one obtains linear separable equations, which are easy to integrate.

⊙ Reference: E. Kamke (1965).

2. $f_1\left(x_1, \dfrac{\partial w}{\partial x_1}\right) f_2\left(x_2, \dfrac{\partial w}{\partial x_2}\right) \ldots f_n\left(x_n, \dfrac{\partial w}{\partial x_n}\right) = a, \qquad a \neq 0.$

Complete integral:

$$w = \varphi_1(x_1, C_1) + \varphi_2(x_2, C_2) + \cdots + \varphi_n(x_n, C_n) + C_{n+1}, \quad \text{where} \quad C_n = \frac{a}{C_1 C_2 \ldots C_{n-1}}.$$

The functions $\varphi_k = \varphi_k(x_k, C_k)$ are determined by the ordinary differential equations

$$f_k\left(x_k, \frac{d\varphi_k}{dx_k}\right) = C_k \qquad (k = 1, \ldots, n).$$

Solving these equations for the derivatives, one obtains linear separable equations, which are easy to integrate.

⊙ Reference: E. Kamke (1965).

3. $f_1\left(\dfrac{\partial w}{\partial x_1} + a_1 y_1, \dfrac{\partial w}{\partial y_1} + a_1 x_1\right) + \ldots + f_n\left(\dfrac{\partial w}{\partial x_n} + a_n y_n, \dfrac{\partial w}{\partial y_n} + a_n x_n\right) = 0.$

Complete integral:

$$w = -a_1 x_1 y_1 - \cdots - a_n x_n y_n + A_1 x_1 + \cdots + A_n x_n + B_1 y_1 + \cdots + B_n y_n,$$

where the arbitrary constants A_k and B_m are related by the one constraint

$$f_1(A_1, B_1) + \cdots + f_n(A_n, B_n) = 0.$$

4. $f_1\left(\dfrac{\partial w}{\partial x_1} + a_1 y_1, \dfrac{\partial w}{\partial y_1} + a_1 x_1\right) \ldots f_n\left(\dfrac{\partial w}{\partial x_n} + a_n y_n, \dfrac{\partial w}{\partial y_n} + a_n x_n\right) = b.$

The left-hand side of the equation consists of the product of n functions with complicated arguments. Complete integral:

$$w = -a_1 x_1 y_1 - \cdots - a_n x_n y_n + A_1 x_1 + \cdots + A_n x_n + B_1 y_1 + \cdots + B_n y_n,$$

where the arbitrary constants A_k and B_m are related by the one constraint

$$f_1(A_1, B_1) \ldots f_n(A_n, B_n) = b.$$

4.6.4. Nonlinear Equations of General Form

1. $\dfrac{\partial w}{\partial x_1} + F\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral: $w = C_1 + C_2 x_2 + \cdots + C_n x_n - \displaystyle\int F(x_1, C_2, \ldots, C_n)\, dx_1.$

2. $\dfrac{\partial w}{\partial x_1} + F\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = aw.$

Complete integral:

$$w = C_1 e^{ax_1} + e^{ax_1}(C_2 x_2 + \cdots + C_n x_n) - e^{ax_1} \int e^{-ax_1} F(x_1, C_2 e^{ax_1}, \ldots, C_n e^{ax_1})\, dx_1.$$

3. $\dfrac{\partial w}{\partial x_1} + F\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = g(x_1)w.$

Complete integral:

$$w = \varphi(x_1)(C_1 + C_2x_2 + \cdots + C_nx_n) - \varphi(x_1)\int F\big(x_1, C_2\varphi(x_1), \ldots, C_n\varphi(x_1)\big)\,\frac{dx_1}{\varphi(x_1)},$$

where $\varphi(x_1) = \exp\left[\int g(x_1)\,dx_1\right]$.

4. $\dfrac{\partial w}{\partial x_1} + F\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = g(x_1)w + h_2(x_1)x_2 + \cdots + h_n(x_1)x_n.$

Complete integral:

$$w = x_2\varphi_2(x_1) + \cdots + x_n\varphi_n(x_1) + \psi(x_1),$$

where

$$\varphi_k(x_1) = C_kG(x_1) + G(x_1)\int \frac{h_k(x_1)}{G(x_1)}\,dx_1, \qquad k = 2, \ldots, n;$$
$$\psi(x_1) = C_1G(x_1) - G(x_1)\int F(x_1, \varphi_2, \ldots, \varphi_n)\frac{dx_1}{G(x_1)}, \qquad G(x_1) = \exp\left[\int g(x_1)\,dx_1\right].$$

5. $\dfrac{\partial w}{\partial x_1} + wF\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + \sum\limits_{k=2}^{n} x_kG_k\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + H\left(x_1, \dfrac{\partial w}{\partial x_2}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral:

$$w = \varphi(x_1, C_1, \ldots, C_n) + \sum_{k=2}^{n} x_k\psi_k(x_1, C_1, \ldots, C_n),$$

where the functions φ, ψ_2, …, ψ_n are determined by solving the nonlinear system of ordinary differential equations

$$\varphi' + F(x_1, \psi_2, \ldots, \psi_n)\varphi + H(x_1, \psi_2, \ldots, \psi_n) = 0,$$
$$\psi_k' + F(x_1, \psi_2, \ldots, \psi_n)\psi_k + G_k(x_1, \psi_2, \ldots, \psi_n) = 0, \quad k = 2, \ldots, n.$$

The prime denotes the derivative with respect to x_1.

6. $\dfrac{\partial w}{\partial x_1} + F\left(x_1, \varphi_2(x_2)\dfrac{\partial w}{\partial x_2} + \psi_2(x_2), \ldots, \varphi_n(x_n)\dfrac{\partial w}{\partial x_n} + \psi_n(x_n)\right) = 0.$

Complete integral:

$$w = \int \frac{C_2 - \psi_2(x_2)}{\varphi_2(x_2)}\,dx_2 + \cdots + \int \frac{C_n - \psi_n(x_n)}{\varphi_n(x_n)}\,dx_n - \int F(x_1, C_2, \ldots, C_n)\,dx_1 + C_1.$$

7. $F\left(\dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral:

$$w = C_1x_1 + \ldots + C_nx_n + C_{n+1}.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by the one constraint $F(C_1, \ldots, C_n) = 0$.

⊙ Reference: E. Kamke (1965).

8. $F\left(x_1, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral:

$$w = C_1 + C_2x_2 + \cdots + C_nx_n + \varphi(x_1),$$

where the function $\varphi = \varphi(x_1)$ is determined by the ordinary differential equation

$$F\big(x_1, \varphi', C_2, \ldots, C_n\big) = 0.$$

9. $F\left(a_1x_1 + \cdots + a_nx_n, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.$

Complete integral:

$$w = C_1x_1 + \cdots + C_nx_n + C_{n+1} + \varphi(z), \qquad z = a_1x_1 + \cdots + a_nx_n,$$

where the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$F\big(z, a_1\varphi'_z + C_1, \ldots, a_n\varphi'_z + C_n\big) = 0.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to unity.

10. $F\left(\dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + x_1\dfrac{\partial w}{\partial x_1} + \cdots + x_n\dfrac{\partial w}{\partial x_n} = 0.$

Complete integral:

$$w = \varphi(z) + C_{n+1}, \qquad z = C_1x_1 + \cdots + C_nx_n,$$

where the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$F(C_1\varphi'_z, \ldots, C_n\varphi'_z) + z\varphi'_z = 0.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to ± 1.

11. $F\left(\dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + x_{k+1}\dfrac{\partial w}{\partial x_{k+1}} + \cdots + x_n\dfrac{\partial w}{\partial x_n} = 0.$

Complete integral:

$$w = C_1x_1 + \ldots + C_kx_k + \varphi(z) + C_{n+1}, \qquad z = C_{k+1}x_{k+1} + \cdots + C_nx_n,$$

where the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$F(C_1, \ldots, C_k, C_{k+1}\varphi'_z, \ldots, C_n\varphi'_z) + z\varphi'_z = 0.$$

One of the constants $C_{k+1}, \ldots, C_n$ can be set equal to ± 1.

12. $\boldsymbol{F\left(\dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + x_1\dfrac{\partial w}{\partial x_1} + \cdots + x_n\dfrac{\partial w}{\partial x_n} = w.}$

Clairaut's equation. Complete integral: $w = C_1x_1 + \cdots + C_nx_n + F\left(C_1, \ldots, C_n\right)$.

⊙ References: E. Kamke (1965), A. M. Vinogradov and I. S. Krasilshchik (1997).

13. $\boldsymbol{F\left(x_1, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + x_2\dfrac{\partial w}{\partial x_2} + \cdots + x_n\dfrac{\partial w}{\partial x_n} = w.}$

Complete integral: $w = C_2x_2 + \cdots + C_nx_n + \varphi(x_1)$. The function $\varphi(x_1) = \varphi(x_1, C_1, \ldots, C_n)$ is determined by the ordinary differential equation $F\left(x_1, \varphi', C_2, \ldots, C_n\right) = \varphi$.

14. $\boldsymbol{F\left(a_1x_1 + \cdots + a_nx_n, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) + x_1\dfrac{\partial w}{\partial x_1} + \cdots + x_n\dfrac{\partial w}{\partial x_n} = w.}$

Complete integral:

$$w = C_1x_1 + \cdots + C_nx_n + \varphi(z), \qquad z = a_1x_1 + \cdots + a_nx_n,$$

where the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$F\left(z, a_1\varphi'_z + C_1, \ldots, a_n\varphi'_z + C_n\right) + z\varphi'_z = \varphi.$$

15. $\boldsymbol{F\left(w, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = 0.}$

Complete integral:

$$w = \varphi(z), \quad z = C_1x_1 + \cdots + C_nx_n,$$

where the function $\varphi(z)$ is determined by the first-order autonomous ordinary differential equation

$$F\left(\varphi, C_1\varphi'_z, \ldots, C_n\varphi'_z\right) = 0.$$

⊙ Reference: E. Kamke (1965).

16. $\boldsymbol{F\left(w, \dfrac{\partial w}{\partial x_1}, \ldots, \dfrac{\partial w}{\partial x_n}\right) = G\left(x_1\dfrac{\partial w}{\partial x_1} + x_2\dfrac{\partial w}{\partial x_2} + \cdots + x_n\dfrac{\partial w}{\partial x_n}\right).}$

1°. Complete integral:

$$w = \varphi(z), \qquad z = C_1x_1 + \cdots + C_nx_n,$$

where the function $\varphi(z)$ is determined by the first-order ordinary differential equation

$$F\left(\varphi, C_1\varphi'_z, \ldots, C_n\varphi'_z\right) = G\left(z\varphi'_z\right).$$

2°. If the left-hand side of the equation does not explicitly depend on w and is a homogeneous function of degree k in the derivatives, i.e., $F(\lambda p_1, \ldots, \lambda p_n) = \lambda^k F(p_1, \ldots, p_n)$, and the function G is linear, $G(\xi) = a\xi$, then a complete integral of the original equation is given by [Kamke (1965)]:

$$w = \frac{k-1}{k}\left[\frac{a}{F(C_1, \ldots, C_n)}\left(\sum_{m=1}^{n} C_mx_m\right)^k\right]^{\frac{1}{k-1}} + C_{n+1}.$$

One of the constants $C_1, \ldots, C_n$ can be set equal to unity.

17. $\boldsymbol{F\left(a_1x_1+\cdots+a_nx_n+bw,\ \dfrac{\partial w}{\partial x_1},\ \ldots,\ \dfrac{\partial w}{\partial x_n}\right)=0.}$

For $b = 0$, see equation 4.6.4.14. For $b \neq 0$, the substitution $bu = a_1x_1 + \cdots + a_nx_n + bw$ leads to an equation of the form 4.6.4.15: $F\left(bu,\ \dfrac{\partial u}{\partial x_1}-\dfrac{a_1}{b},\ \ldots,\ \dfrac{\partial u}{\partial x_n}-\dfrac{a_n}{b}\right)=0.$

18. $\boldsymbol{F\big(\varphi_1(x_1,p_1),\ \varphi_2(x_2,p_2),\ \ldots,\ \varphi_n(x_n,p_n)\big)=0,\qquad p_k=\dfrac{\partial w}{\partial x_k}.}$

Here F is a composite function of the n functions $\varphi_1, \ldots, \varphi_n$ each of which depends on only one pair of Hamiltonian variables x_k, p_k ($k = 1, \ldots, n$).

Complete integral:

$$w = w_1(x_1, C_1) + w_2(x_2, C_2) + \cdots + w_n(x_n, C_n) + C_{n+1}.$$

The arbitrary constants $C_1, \ldots, C_n$ are linked by the one constraint $F(C_1, \ldots, C_n) = 0$ and the functions $w_k = w_k(x_k, C_k)$ are determined by the ordinary differential equations

$$\varphi_k\left(x_k,\ \frac{dw_k}{dx_k}\right)=C_k;\qquad k=1,\ 2,\ \ldots,\ n.$$

Solving these equations for the derivatives, one obtains linear separable equations, which are easy to integrate.

⊙ References: E. Kamke (1965), A. P. Markeev (1990).

19. $\boldsymbol{F\big(\varphi_1(x_1,\psi p_1),\ \varphi_2(x_2,\psi p_2),\ \ldots,\ \varphi_n(x_n,\psi p_n)\big)=0,}$

$$\boldsymbol{\psi=\psi(w),\quad p_1=\frac{\partial w}{\partial x_1},\ \ \ldots,\ \ p_n=\frac{\partial w}{\partial x_n}.}$$

The substitution $u = \int \psi(w)\,dw$ leads to an equation of the form 4.6.4.18:

$$F\big(\varphi_1(x_1,q_1),\varphi_2(x_2,q_2),\ldots,\varphi_n(x_n,q_n)\big)=0,\qquad \text{where}\quad q_k=\frac{\partial u}{\partial x_k}.$$

⊙ Reference: E. Kamke (1965).

20. $\boldsymbol{F_1(x_1,\ldots,x_k,p_1,\ldots,p_k)+F_2(x_{k+1},\ldots,x_n,p_{k+1},\ldots,p_n)=aw.}$

A complete integral can be represented as the sum of two functions,

$$w = w_1(x_1, \ldots, x_k) + w_2(x_{k+1}, \ldots, x_n),$$

which are determined by solving the two simpler equations

$$\begin{aligned} F_1(x_1,\ldots,x_k,q_1,\ldots,q_k)-aw_1&=C_1,\\ F_2(x_{k+1},\ldots,x_n,q_{k+1},\ldots,q_n)-aw_2&=-C_1,\end{aligned}\qquad q_m=\begin{cases}\dfrac{\partial w_1}{\partial x_m} & \text{for } m=1,\ldots,k,\\[2mm] \dfrac{\partial w_2}{\partial x_m} & \text{for } m=k+1,\ldots,n,\end{cases}$$

where C_1 is an arbitrary constant.

21. $\boldsymbol{F_1(x_1, \ldots, x_k, p_1, \ldots, p_k) + e^{\lambda w} F_2(x_{k+1}, \ldots, x_n, p_{k+1}, \ldots, p_n) = 0.}$

A complete integral can be represented as the sum of two functions,

$$w = w_1(x_1, \ldots, x_k) + w_2(x_{k+1}, \ldots, x_n),$$

which are determined by solving the two simpler equations

$$\begin{aligned} e^{-\lambda w_1} F_1(x_1, \ldots, x_k, q_1, \ldots, q_k) &= C_1, \\ e^{\lambda w_2} F_2(x_{k+1}, \ldots, x_n, q_{k+1}, \ldots, q_n) &= -C_1, \end{aligned} \qquad q_m = \begin{cases} \dfrac{\partial w_1}{\partial x_m} & \text{for } m = 1, \ldots, k, \\ \dfrac{\partial w_2}{\partial x_m} & \text{for } m = k+1, \ldots, n, \end{cases}$$

where C_1 is an arbitrary constant.

22. $\boldsymbol{F_n\big(\ldots F_3(F_2(F_1(x_1, p_1), x_2, p_2), x_3, p_3), \ldots, x_n, p_n\big) = 0,} \qquad \boldsymbol{p_k = \dfrac{\partial w}{\partial x_k}.}$

Complete integral:

$$w = w_1(x_1, C_1) + w_2(x_2, C_2, C_1) + \cdots + w_{n-1}(x_{n-1}, C_{n-1}, C_{n-2}) + w_n(x_n, C_{n-1}) + C_n.$$

The functions w_k are determined by the ordinary differential equations

$$\begin{aligned} F_1(x_1, w_1') &= C_1, \\ F_k(C_{k-1}, x_k, w_k') &= C_k, \quad k = 2, \ldots, n-1, \\ F_n(C_{n-1}, x_n, w_n') &= 0, \end{aligned}$$

where w_k' stands for the derivative of w_k with respect to x_k. Solving these equations for the derivatives, one obtains linear separable equations, which are easy to integrate.

⊙ References: A. P. Markeev (1990), V. V. Kozlov (1995).

Chapter 5
Second-Order Parabolic Equations with One Space Variable

5.1. Equations with Power-Law Nonlinearities

5.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw + cw^2$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - bw^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w = \frac{a}{b}\frac{12(4-\sqrt{6})x^2 + 12(4-\sqrt{6})C_1 x + 120(12-5\sqrt{6})at + 12(2-\sqrt{6})C_2 + 6C_1^2}{\left[x^2 + C_1 x + 10(3-\sqrt{6})at + C_2\right]^2},$$

$$w = \frac{a}{b}\frac{12(4+\sqrt{6})x^2 + 12(4+\sqrt{6})C_1 x + 120(12+5\sqrt{6})at + 12(2+\sqrt{6})C_2 + 6C_1^2}{\left[x^2 + C_1 x + 10(3+\sqrt{6})at + C_2\right]^2},$$

where C_1 and C_2 are arbitrary constants.

3°. Traveling-wave solution (λ is an arbitrary constant):

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z - bw^2 = 0.$$

4°. Self-similar solution:

$$w = t^{-1}u(\xi), \qquad \xi = xt^{-1/2},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \tfrac{1}{2}\xi u'_\xi + u - bu^2 = 0$$

⊙ *References for equation* 5.1.1.1: T. Barannyk (2002), A. D. Polyanin and V. F. Zaitsev (2004, p. 1).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw(1-w).$

Fisher equation. This equation arises in heat and mass transfer, combustion theory, biology, and ecology. For example, it describes the mass transfer in a two-component medium at rest with a volume chemical reaction of quasi-first order. The kinetic function $f(w) = aw(1-w)$ models also an autocatalytic chain reaction in combustion theory.

This is a special case of equation 5.1.3.2 with $m = 2$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions (C is an arbitrary constant):

$$w(x,t) = \big[1 + C\exp\big(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{6a}\,x\big)\big]^{-2},$$
$$w(x,t) = \big[-1 + C\exp\big(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{6a}\,x\big)\big]^{-2},$$
$$w(x,t) = \frac{1 + 2C\exp\big(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{-6a}\,x\big)}{\big[1 + C\exp\big(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{-6a}\,x\big)\big]^2}.$$

3°. Traveling-wave solutions:

$$w(x,t) = \pm\xi^2\varphi(\xi), \qquad \xi = C_1\exp\big(\tfrac{1}{6}\sqrt{6a}\,x + \tfrac{5}{6}at\big),$$

where the function $\varphi(\xi)$ is defined implicitly by

$$\xi = \int \frac{d\varphi}{\sqrt{\pm(4\varphi^3 - 1)}} - C_2,$$

and C_1 and C_2 are arbitrary constants. For the upper sign, the inversion of this relation corresponds to the classical Weierstrass elliptic function, $\varphi(\xi) = \wp(\xi + C_3,\, 0,\, 1)$.

4°. The substitution $U = 1 - w$ leads to an equation of the similar form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - aU(1-U).$$

⊙ *References for equation* 5.1.1.2: R. A. Fisher (1937), M. J. Ablowitz and A. Zeppetella (1979), V. G. Danilov, V. P. Maslov, and K. A. Volosov (1995).

5.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - bw^3.$

This is a special case of equation 5.1.2.5 with $b_0 = b_1 = b_2 = 0$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = \pm\sqrt{\frac{2a}{b}}\,\frac{2C_1x + C_2}{C_1x^2 + C_2x + 6aC_1t + C_3}.$$

3°. Traveling-wave solution (λ is an arbitrary constant):

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z - bw^3 = 0.$$

4°. Self-similar solution:

$$w = t^{-1/2}u(\xi), \qquad \xi = xt^{-1/2},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \tfrac{1}{2}\xi u'_\xi + \tfrac{1}{2}u - bu^3 = 0.$$

5°. Solution:

$$w = xU(\zeta), \qquad \zeta = t + \frac{1}{6a}x^2,$$

where the function $U(\zeta)$ is determined by the ordinary differential equation $U''_{\zeta\zeta} - 9abU^3 = 0$.

⊙ *References for equation* 5.1.2.1: P. A. Clarkson and E. L. Mansfield (1994), T. A. Barannyk and A. G. Nikitin (2004), G. Cicogna (2004).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw - bw^3$.

This is a special case of equation 5.1.2.5 with $b_0 = b_2 = 0$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm w(\pm x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation (the signs are chosen arbitrarily).

2°. Solutions with $a > 0$ and $b > 0$:

$$w = \sqrt{\frac{a}{b}}\,\frac{C_1\exp\left(\frac{1}{2}\sqrt{2a}\,x\right) - C_2\exp\left(-\frac{1}{2}\sqrt{2a}\,x\right)}{C_1\exp\left(\frac{1}{2}\sqrt{2a}\,x\right) + C_2\exp\left(-\frac{1}{2}\sqrt{2a}\,x\right) + C_3\exp\left(-\frac{3}{2}at\right)},$$

$$w = \sqrt{\frac{a}{b}}\left[\frac{2C_1\exp\left(\sqrt{2a}\,x\right) + C_2\exp\left(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\right)}{C_1\exp\left(\sqrt{2a}\,x\right) + C_2\exp\left(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\right) + C_3} - 1\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solution with $a < 0$ and $b > 0$:

$$w = \sqrt{\frac{|a|}{b}}\,\frac{\sin\left(\frac{1}{2}\sqrt{2|a|}\,x + C_1\right)}{\cos\left(\frac{1}{2}\sqrt{2|a|}\,x + C_1\right) + C_2\exp\left(-\frac{3}{2}at\right)},$$

where C_1 and C_2 are arbitrary constants.

4°. Solution with $a > 0$ (generalizes the first solution of Item 2°):

$$w = \big[C_1 \exp\big(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\big) - C_2 \exp\big(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\big)\big]U(z),$$
$$z = C_1 \exp\big(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\big) + C_2 \exp\big(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\big) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation $aU''_{zz} = 2bU^3$ (whose solution can be written out in implicit form).

5°. Solution with $a < 0$ (generalizes the solution of Item 3°):

$$w = \exp\big(\tfrac{3}{2}at\big)\sin\big(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\big)V(\xi),$$
$$\xi = \exp\big(\tfrac{3}{2}at\big)\cos\big(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\big) + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $V = V(\xi)$ is determined by the autonomous ordinary differential equation $aV''_{\xi\xi} = -2bV^3$ (whose solution can be written out in implicit form).

6°. See also equation 5.1.3.2 with $m = 3$.

⊙ *References for equation* 5.1.2.2: F. Cariello and M. Tabor (1989), M. C. Nucci and P. A. Clarkson (1992), A. D. Polyanin and V. F. Zaitsev (2004, p. 3).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - bw^3 - cw^2.$

This is a special case of equation 5.1.2.5 with $b_1 = b_0 = 0$.

1°. Traveling-wave solutions:

$$w(x,t) = \left(ct \pm \sqrt{\frac{b}{2a}}\,x + C\right)^{-1},$$

where C is an arbitrary constant.

2°. Solutions:

$$w(x,t) = k\sqrt{\frac{2a}{b}}\,\frac{1}{F}\,\frac{\partial F}{\partial x},$$

where

$$F = C_1\left(x + kc\sqrt{\frac{2a}{b}}\,t\right) + C_2\exp\left(-\frac{kc}{\sqrt{2ab}}x + \frac{c^2}{2b}t\right) + C_3, \qquad k = \pm 1,$$

and C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x,t) = k\sqrt{\frac{2a}{b}}\,\frac{1}{F}\,\frac{\partial F}{\partial x} - \frac{c}{b},$$

where

$$F = C_1\exp\left(\frac{kc}{\sqrt{2ab}}x - \frac{c^2}{2b}t\right) + C_2\left(\frac{kc}{\sqrt{2ab}}x + \frac{c^2}{b}t\right)\exp\left(\frac{kc}{\sqrt{2ab}}x - \frac{c^2}{2b}t\right) + C_3, \qquad k = \pm 1.$$

4°. The transformation

$$t = \frac{b}{c^2}\tau, \quad x = \frac{\sqrt{2ab}}{c}z, \quad w = -\frac{c}{b}u$$

leads to the equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + u^2(1-u). \tag{1}$$

The *Volosov's solution* of Eq. (1) in implicit form:

$$z = F(u, \tau),$$

where the function $F = F(u, \tau)$ is determined by the system of first-order partial differential equations

$$\frac{\partial F}{\partial \tau} = \frac{1}{2} + \frac{3uR}{2(u^2 - u - R)}, \qquad \frac{\partial F}{\partial u} = \frac{1}{u^2 - u - R}, \tag{2}$$

where the function $R = R(u, \tau)$ is determined by the implicit formula

$$\frac{1-u}{R} + \ln\frac{u+R}{R} = \frac{3}{2}\tau. \tag{3}$$

Relation (3) is the consistency condition for system (2).

Remark. For overdetermined systems of two equations of the form (2) and their consistency condition, see Section 38.1.

⊙ *References for equation* 5.1.2.3: A. D. Polyanin and V. F. Zaitsev (2004, pp. 3–4), K. A. Volosov (2007, http://eqworld.ipmnet.ru/eqarchive/view.php?id=56).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} - w(1-w)(a-w)$.

FitzHugh–Nagumo equation. This equation arises in population genetics and models the transmission of nerve impulses.

1°. There are three stationary solutions: $w = w_k$, where $w_1 = 0$, $w_2 = 1$, and $w_3 = a$. The linear stability analysis shows that

if $-1 \le a < 0$: the solutions $w = a$, $w = 1$ are stable, $w = 0$ is unstable;
if $0 < a < 1$: the solutions $w = 0$, $w = 1$ are stable, $w = a$ is unstable.

There is a stationary nonhomogeneous solution that can be represented in implicit form (A and B are arbitrary constants):

$$\int \frac{dw}{\sqrt{\frac{1}{4}w^4 - \frac{1}{3}(a+1)w^3 + \frac{1}{2}aw^2 + A}} = \pm x + B.$$

2°. Traveling-wave solutions (A is an arbitrary constant):

$$w(x,t) = \frac{1}{1 + A\exp\left[\pm\frac{1}{2}\sqrt{2}\,x + \frac{1}{2}(2a-1)t\right]},$$

$$w(x,t)=\frac{a}{1+A\exp\left[\pm\frac{1}{2}\sqrt{2}\,a\,x+\frac{1}{2}a(2-a)t\right]},$$

$$w(x,t)=\frac{A\exp\left[\pm\frac{1}{2}\sqrt{2}\,(1-a)x+\frac{1}{2}(1-a^2)t\right]+a}{A\exp\left[\pm\frac{1}{2}\sqrt{2}\,(1-a)x+\frac{1}{2}(1-a^2)t\right]+1},$$

$$w(x,t)=\tfrac{1}{2}+\tfrac{1}{2}\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,x+\tfrac{1}{4}(1-2a)t+A\right],$$

$$w(x,t)=\tfrac{1}{2}a+\tfrac{1}{2}a\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,ax+\tfrac{1}{4}a(a-2)t+A\right],$$

$$w(x,t)=\tfrac{1}{2}(1+a)+\tfrac{1}{2}(1-a)\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,(1-a)x+\tfrac{1}{4}(1-a^2)t+A\right],$$

$$w(x,t)=\frac{2a}{(1+a)-(1-a)\tanh\left[\pm\frac{1}{4}\sqrt{2}\,(1-a)x+\frac{1}{4}(1-a^2)t+A\right]},$$

$$w(x,t)=\tfrac{1}{2}+\tfrac{1}{2}\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,x+\tfrac{1}{4}(1-2a)t+A\right],$$

$$w(x,t)=\tfrac{1}{2}a+\tfrac{1}{2}a\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,ax+\tfrac{1}{4}a(a-2)t+A\right],$$

$$w(x,t)=\tfrac{1}{2}(1+a)+\tfrac{1}{2}(1-a)\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,(1-a)x+\tfrac{1}{4}(1-a^2)t+A\right],$$

$$w(x,t)=\frac{2a}{(1+a)-(1-a)\coth\left[\pm\frac{1}{4}\sqrt{2}\,(1-a)x+\frac{1}{4}(1-a^2)t+A\right]}.$$

3°. "Two-phase" solution:

$$w(x,t)=\frac{A\exp(z_1)+aB\exp(z_2)}{A\exp(z_1)+B\exp(z_2)+C},$$

$$z_1=\pm\tfrac{\sqrt{2}}{2}\,x+\left(\tfrac{1}{2}-a\right)t,\quad z_2=\pm\tfrac{\sqrt{2}}{2}\,ax+a\left(\tfrac{1}{2}a-1\right)t,$$

where A, B, and C are arbitrary constants.

4°. The solutions of Item 2° are special cases of the traveling-wave solution

$$w(x,t)=w(\xi),\qquad \xi=x+\lambda t,$$

where λ is an arbitrary constant, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$w''_{\xi\xi}-\lambda w'_\xi=w(1-w)(a-w).$$

The substitution $w'_\xi=\lambda y(w)$ leads to an Abel equation of the second kind:

$$yy'_w-y=\lambda^{-2}\left[aw-(a+1)w^2+w^3\right].$$

The general solution of this equation with $a=-1$ and $\lambda=\pm\frac{3}{\sqrt{2}}$ can be found in Polyanin and Zaitsev (2003).

5°. Let us give two transformations that preserve the form of the original equation.

The substitution $u=1-w$ leads to an equation of the similar form with parameter $a_1=1-a$:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-u(1-u)(1-a-u).$$

The transformation

$$v(z,\tau) = 1 - \frac{1}{a}w(x,t), \quad \tau = a^2 t, \quad z = ax$$

leads to an equation of the similar form with parameter $a_2 = 1 - a^{-1}$:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial z^2} - v(1-v)\left(1 - \frac{1}{a} - v\right).$$

Therefore, if $w = w(x,t;\, a)$ is a solution of the equation in question, then the functions

$$w_1 = 1 - w\big(x,t;\, 1-a\big),$$
$$w_2 = a - aw\big(ax, a^2 t;\, 1 - a^{-1}\big)$$

are also solutions of the equation. The aforesaid allows us to "multiply" exact solutions.

⊙ *References for equation* 5.1.2.4: T. Kawahara and M. Tanaka (1983), M. C. Nucci and P. A. Clarkson (1992), N. H. Ibragimov (1994, p. 142), V. F. Zaitsev and A. D. Polyanin (1996, p. 392).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3.$

1°. Solutions are given by

$$w(x,t) = \frac{\beta}{F}\frac{\partial F}{\partial x} + \lambda, \quad \beta = \pm\sqrt{-\frac{2a}{b_3}}, \tag{1}$$

where λ is any of the roots of the cubic equation

$$b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0 \tag{2}$$

and the specific form of $F = F(x,t)$ depends on the equation coefficients.

Introduce the notation

$$p_1 = -3a, \quad p_2 = \beta(b_2 + 3b_3\lambda), \quad q_1 = -\frac{\beta}{2a}(b_2 + 3b_3\lambda), \quad q_2 = -\frac{1}{2a}(3b_3\lambda^2 + 2b_2\lambda + b_1). \tag{3}$$

Four cases are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$F(x,t) = C_1\exp(k_1 x + s_1 t) + C_2\exp(k_2 x + s_2 t) + C_3,$$
$$k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2 p_1 - k_n p_2, \tag{4}$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1,\ 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$, we have

$$F(x,t) = C_1\exp(kx + s_1 t) + C_2(kx + s_2 t)\exp(kx + s_1 t) + C_3,$$
$$k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1, \quad s_2 = -\tfrac{1}{2}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1.$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$F(x,t) = C_1(x - p_2 t) + C_2 \exp[-q_1 x + q_1(p_2 - p_1 q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$F(x,t) = C_1(x - p_2 t)^2 + C_2(x - p_2 t) - 2C_1 p_1 t + C_3.$$

Special case. Let

$$a = 1, \quad b_0 = 0, \quad b_1 w + b_2 w^2 + b_3 w^3 = -bw(w - \lambda_1)(w - \lambda_2).$$

By formulas (1)–(4) with $\lambda = 0$, one can obtain the solution

$$w(x,t) = \frac{C_1\lambda_1 \exp(z_1) + C_2\lambda_2 \exp(z_2)}{C_1 \exp(z_1) + C_2 \exp(z_2) + C_3},$$

where

$$z_1 = \pm\tfrac{1}{2}\sqrt{2b}\,\lambda_1 x + \tfrac{1}{2}b\lambda_1(\lambda_1 - 2\lambda_2)t,$$
$$z_2 = \pm\tfrac{1}{2}\sqrt{2b}\,\lambda_2 x + \tfrac{1}{2}b\lambda_2(\lambda_2 - 2\lambda_1)t.$$

2°. There is a traveling-wave solution, $w = w(x + \gamma t)$.

⊙ *References for equation* 5.1.2.5: V. G. Danilov and P. Yu. Sybochev (1991), N. A. Kudryashov (1993), P. A. Clarkson and E. L. Mansfield (1994).

5.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k + cw^m + sw^n$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-1} x + C_2, C_1^{2k-2} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z + bw^k = 0.$$

3°. Self-similar solution:

$$w = t^{\frac{1}{1-k}} u(\xi), \qquad \xi = \frac{x}{\sqrt{t}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{1}{k-1}u + bu^k = 0.$$

4°. For $k = 2$ and $k = 3$, see equations 5.1.1.1 and 5.1.2.1, respectively, where other solutions can be found.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^m.$

Kolmogorov–Petrovskii–Piskunov equation (a special case). This equation arises in heat and mass transfer, combustion theory, biology, and ecology.

1°. Traveling-wave solutions:

$$w(x,t) = \big[\beta + C\exp(\lambda t \pm \mu x)\big]^{\frac{2}{1-m}}, \tag{1}$$

$$w(x,t) = \big[-\beta + C\exp(\lambda t \pm \mu x)\big]^{\frac{2}{1-m}}, \tag{2}$$

where C is an arbitrary constant and the parameters λ, μ, and β are given by

$$\lambda = \frac{a(1-m)(m+3)}{2(m+1)}, \quad \mu = \sqrt{\frac{a(1-m)^2}{2(m+1)}}, \quad \beta = \sqrt{-\frac{b}{a}}.$$

2°. Solutions (1) and (2) are special cases of a wider class of solutions, the class of traveling-wave solutions:

$$w = w(z), \quad z = \pm\mu x + \lambda t.$$

These are determined by the autonomous equation

$$\mu^2 w''_{zz} - \lambda w'_z + aw + bw^m = 0. \tag{3}$$

For

$$\mu = \sqrt{\frac{a(m+3)^2}{2(m+1)}}, \quad \lambda = \mu^2 \qquad (m \neq \pm 1,\ m \neq -3)$$

the solution of equation (3) can be represented in parametric form as

$$z = \frac{m+3}{m-1}\ln f(\zeta), \quad w = \zeta\big[f(\zeta)\big]^{\frac{2}{m-1}},$$

where the function $f(\zeta)$ is given by

$$f(\zeta) = \pm\int\left[C_1 - \frac{4b}{a(m-1)^2}\zeta^{m+1}\right]^{-1/2} d\zeta + C_2,$$

and C_1 and C_2 are arbitrary constants.

3°. By the change of variable $U(w) = \mu^2\lambda^{-1}w'_z$, equation (3) can be reduced to an Abel equation of the second kind:

$$UU'_w - U = a_1 w + b_1 w^m, \qquad a_1 = -a\mu^2\lambda^{-2}, \quad b_1 = -b\mu^2\lambda^{-2}.$$

The books by Polyanin and Zaitsev (2003) and Zaitsev and Polyanin (1994) present exact solutions of this equation for some values of m and a_1 (b_1 is any).

⊙ *References for equation* 5.1.3.2: P. Kaliappan (1984), V. G. Danilov, V. P. Maslov, and K. A. Volosov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 393).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^m + cw^{2m-1}.$

This equation arises in heat and mass transfer, combustion theory, biology, and ecology. For $m = 2$, see Subsection 5.1.2.

1°. Traveling-wave solutions:

$$w(x,t) = \big[\beta + C\exp(\lambda t + \mu x)\big]^{\frac{1}{1-m}}, \tag{1}$$

where C is an arbitrary constant and the parameters β, λ, and μ are determined by the system of algebraic equations

$$a\beta^2 + b\beta + c = 0, \tag{2}$$

$$\mu^2 - (1-m)\lambda + a(1-m)^2 = 0, \tag{3}$$

$$\mu^2 - \lambda + (1-m)[2a + (b/\beta)\big] = 0. \tag{4}$$

The quadratic equation (2) for β can be solved independently. In the general case, system (2)–(4) gives four sets of the parameters, which generate four exact solutions of the original equation.

2°. Solution (1) is a special case of a wider class of traveling-wave solutions,

$$w = w(z), \quad z = x + \sigma t,$$

that are determined by the autonomous equation

$$w''_{zz} - \sigma w'_z + aw + bw^m + cw^{2m-1} = 0. \tag{5}$$

The substitution $U(w) = w'_z$ brings (5) to the Abel equation

$$UU'_w - \sigma U + aw + bw^m + cw^{2m-1} = 0,$$

whose general solutions for some m (no constraints are imposed on a, b, and c) can be found in the books by Polyanin and Zaitsev (1995, 2003).

3°. The substitution

$$u = w^{1-m}$$

leads to an equation with quadratic nonlinearity:

$$u\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} + \frac{m}{1-m}\left(\frac{\partial u}{\partial x}\right)^2 + a(1-m)u^2 + b(1-m)u + c(1-m). \tag{6}$$

Solution (1) corresponds to a particular solution of (6) that has the form $u = \beta + C\exp(\omega t + \mu x)$.

For $a = 0$, equation (6) has also other traveling-wave solutions:

$$u(x,t) = (1-m)\left(bt \pm \sqrt{-\frac{c}{m}}\,x\right) + C.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 393–394).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw^{m-1} + bmw^m - mb^2w^{2m-1}.$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} - \lambda w'_z + aw^{m-1} + bmw^m - mb^2w^{2m-1} = 0. \tag{1}$$

For $\lambda = 1$, it can be shown that a one-parameter family of solutions to equation (1) satisfies the first-order equation

$$w'_z = w - bw^m + \frac{a}{mb}. \tag{2}$$

Integrating (2) yields a solution in implicit form (A is any):

$$\int \frac{dw}{a + mbw - mb^2w^m} = \frac{1}{mb}z + A. \tag{3}$$

In the special case $a = 0$, it follows from (3) that

$$w(z) = \left\{C\exp\left[(1-m)z\right] + b\right\}^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

5.1.4. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x, t, w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s_1(bx + ct)^k + s_2w^n.$

This is a special case of equation 5.6.1.2 with $f(z, w) = s_1z^k + s_2w^n$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s(w + bx + ct)^k.$

This is a special case of equation 5.6.1.2 with $f(z, w) = s(w + z)^k$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s(bx + ct)^k w^n.$

This is a special case of equation 5.6.1.2 with $f(z, w) = sz^kw^n$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n x^m w^k.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C^{2n+m+2}w(C^{k-1}x, C^{2k-2}t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Self-similar solution:

$$w = t^{\frac{2n+m+2}{2(1-k)}}u(\xi), \quad \xi = \frac{x}{\sqrt{t}},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{2n + m + 2}{2(k-1)}u + b\xi^m u^k = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + se^{bx+ct}w^n.$

This is a special case of equation 5.6.1.2 with $f(z, w) = se^z w^n$.

5.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k\dfrac{\partial w}{\partial x} + f(w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + cw + k_1 w^{n_1} + k_2 w^{n_2}.$

This is a special case of equation 5.6.2.3 with $f(t) = b$. On passing from t, x to the new variables t, $z = x + bt$, one arrives at the simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + cw + k_1 w^{n_1} + k_2 w^{n_2},$$

special cases of which are discussed in Subsections 5.1.1 to 5.1.3.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + w\dfrac{\partial w}{\partial x}.$

Burgers equation. It is used for describing wave processes in gas dynamics, hydrodynamics, and acoustics.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1 x + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \frac{A - x}{B + t},$$

$$w(x,t) = \lambda + \frac{2}{x + \lambda t + A},$$

$$w(x,t) = \frac{4x + 2A}{x^2 + Ax + 2t + B},$$

$$w(x,t) = \frac{6(x^2 + 2t + A)}{x^3 + 6xt + 3Ax + B},$$

$$w(x,t) = \frac{2\lambda}{1 + A\exp(-\lambda^2 t - \lambda x)},$$

$$w(x,t) = -\lambda + A\frac{\exp\big[A(x - \lambda t)\big] - B}{\exp\big[A(x - \lambda t)\big] + B},$$

$$w(x,t) = -\lambda + 2A\tanh\big[A(x - \lambda t) + B\big],$$

$$w(x,t) = \frac{\lambda}{\lambda^2 t + A}\left[2\tanh\left(\frac{\lambda x + B}{\lambda^2 t + A}\right) - \lambda x - B\right],$$

$$w(x,t) = -\lambda + 2A\tan\big[A(\lambda t - x) + B\big],$$

$$w(x,t) = \frac{2\lambda\cos(\lambda x + A)}{B\exp(\lambda^2 t) + \sin(\lambda x + A)},$$

$$w(x,t) = \frac{2A}{\sqrt{\pi(t + \lambda)}}\exp\left[-\frac{(x + B)^2}{4(t + \lambda)}\right]\left[A\operatorname{erf}\left(\frac{x + B}{2\sqrt{t + \lambda}}\right) + C\right]^{-1},$$

where A, B, C, and λ are arbitrary constants, and $\operatorname{erf} z \equiv \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z \exp(-\xi^2)\, d\xi$ is the error function (also called the probability integral).

3°. Other solutions can be obtained using the following formula (*Hopf–Cole transformation*):

$$w(x,t) = \frac{2}{u}\frac{\partial u}{\partial x}, \tag{1}$$

where $u = u(x,t)$ is a solution of the linear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

For details about this equation, see the books Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. *Cauchy problem*. Initial condition:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad -\infty < x < \infty.$$

Solution:

$$w(x,t) = 2\frac{\partial}{\partial x}\ln F(x,t),$$

where

$$F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4t} - \frac{1}{2}\int_0^{\xi} f(\xi')\, d\xi'\right] d\xi.$$

5°. The Burgers equation is connected with the linear heat equation (2) by the Bäcklund transformation

$$\frac{\partial u}{\partial x} - \frac{1}{2}uw = 0,$$
$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial (uw)}{\partial x} = 0.$$

Remark 1. The equation $w_t + ww_x = w_{xx}$ is often call the Burgers equation; it is obtained from the original equation by substituting $-w$ for $-w$.

Remark 2. The Hopf–Cole transformation (1) and equation 5.6.3.2, which is a generalized Burgers equation, were encountered much earlier in Fortsyth (1906).

⊙ *References for equation* 5.1.5.2: E. Hopf (1950), J. Cole (1951), J. M. Burgers (1948), E. R. Benton and G. W. Platzman (1972), O. V. Rudenko and C. I. Soluyan (1975), N. H. Ibragimov (1994, pp. 180–182), V. F. Zaitsev and A. D. Polyanin (1996, pp. 395–396), M. Qin, F. Mei, and G. Fan (2007).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x}$.

Unnormalized Burgers equation. The scaling of the independent variables $x = \dfrac{a}{b}z$, $t = \dfrac{a}{b^2}\tau$ leads to an equation of the form 5.1.5.2:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} + w\frac{\partial w}{\partial z}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + c.$

The transformation

$$w = u(z,t) + ct, \qquad z = x + \tfrac{1}{2}bct^2,$$

leads to the Burgers equation 5.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + bu\frac{\partial u}{\partial z}.$$

5. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + bw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm w\big(\pm x - C_1\sigma e^{bt} + C_2,\ t + C_3\big) + Cbe^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (one takes either the upper or the lower signs).

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \sigma w w'_z - \lambda w'_z + bw = 0.$$

3°. Degenerate solution:

$$w(x,t) = \frac{b(x + C_1)}{\sigma(1 + C_2 e^{-bt})}.$$

6. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + b_1 w + b_0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x - C_1\sigma e^{b_1 t} + C_2,\ t + C_3\big) + Cb_1 e^{b_1 t},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The transformation

$$w = u(z,t) - \frac{b_0}{b_1}, \qquad z = x + \sigma\frac{b_0}{b_1}t,$$

leads to a simpler equation of the form 5.1.5.5:

$$\frac{\partial u}{\partial t} + \sigma u\frac{\partial u}{\partial z} = a\frac{\partial^2 u}{\partial z^2} + b_1 u.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + \dfrac{b^2}{9a}w(w-k)(w+k).$

Solution:

$$w = \frac{k(-1 + C_1 e^{4\lambda x})}{1 + C_1 e^{4\lambda x} + C_2 e^{2\lambda x + bk\lambda t}}, \qquad \lambda = \frac{bk}{12a},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Private communication*: K. A. Volosov (2000).

8. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3.$

For $b_0 = 0$, it is the *Burgers–Huxley equation*.

Exact solutions of the equation are given by

$$w(x,t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda. \tag{1}$$

Here, β and λ are any of the roots of the respective quadratic and cubic equations

$$b_3\beta^2 + \sigma\beta + 2a = 0,$$
$$b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0,$$

and the specific form of $z = z(x,t)$ depends on the equation coefficients.

1°. *Case* $b_3 \neq 0$. Introduce the notation:

$$p_1 = -\beta\sigma - 3a, \qquad p_2 = \lambda\sigma + \beta b_2 + 3\beta\lambda b_3,$$
$$q_1 = -\frac{\beta b_2 + 3\beta\lambda b_3}{\beta\sigma + 2a}, \quad q_2 = -\frac{3b_3\lambda^2 + 2b_2\lambda + b_1}{\beta\sigma + 2a}.$$

Four cases are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$z(x,t) = C_1\exp(k_1 x + s_1 t) + C_2\exp(k_2 x + s_2 t) + C_3,$$
$$k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2 p_1 - k_n p_2,$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1, 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$,

$$z(x,t) = C_1\exp(kx + s_1 t) + C_2(kx + s_2 t)\exp(kx + s_1 t) + C_3,$$
$$k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1, \quad s_2 = -\tfrac{1}{2}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1.$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$z(x,t) = C_1(x - p_2 t) + C_2\exp[-q_1 x + q_1(p_2 - p_1 q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$z(x,t) = C_1(x - p_2 t)^2 + C_2(x - p_2 t) - 2C_1 p_1 t + C_3.$$

2°. *Case* $b_3 = 0$ *and* $b_2 \neq 0$. Solutions are given by (1) with

$$\beta = -\frac{2a}{\sigma}, \qquad z(x,t) = C_1 + C_2\exp\left[Ax + A\left(\frac{b_1\sigma}{2b_2} + \frac{2ab_2}{\sigma}\right)t\right], \qquad A = \frac{\sigma(b_1 + 2b_2\lambda)}{2ab_2},$$

where λ is a root of the quadratic equation $b_2\lambda^2 + b_1\lambda + b_0 = 0$.

3°. *Case* $b_3 = b_2 = 0$. See equations 5.1.5.4–5.1.5.6.

⊙ *References for equation* 5.1.5.8: P. G. Estevez and P. R. Gordoa (1990), N. A. Kudryashov (1993), P. G. Esteves (1994).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^m\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\big(C_1^m x + C_2,\ C_1^{2m} t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w(x,t) = \left[C\exp\left(-\frac{\lambda m}{a}z\right) + \frac{b}{\lambda(m+1)}\right]^{-1/m}, \qquad z = x + \lambda t;$$

$$w(x,t) = \left\{\frac{(m+1)\lambda}{2b}\left[1 + \tanh\left(\frac{m\lambda}{2a}z + C\right)\right]\right\}^{1/m}, \quad z = x + \lambda t,$$

where C and λ are arbitrary constants. A wider family of traveling-wave solutions is presented in 5.6.3.8 for $f(w) = bw^m$.

3°. There is a self-similar solution of the form

$$w(\xi,t) = |t|^{-\frac{1}{2m}}\varphi(\xi), \qquad \xi = x|t|^{-\frac{1}{2}}.$$

⊙ *References for equation* 5.1.5.9: A. D. Polyanin and V. F. Zaitsev (2004, p. 13), L. Zhang (2011).

5.1.6. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + sw^k.$

This is a special case of equation 5.6.2.1 with $f(w) = sw^k$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + bzw'_z + sw^k = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n\dfrac{\partial w}{\partial x} + cw + k_1 w^{m_1} + k_2 w^{m_2}.$

This is a special case of equation 5.6.2.3 with $f(t) = bt^n$. On passing from t, x to the new variables t, $z = x + \dfrac{b}{n+1}t^{n+1}$, one arrives at the simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + cw + k_1 w^{m_1} + k_2 w^{m_2},$$

special cases of which are discussed in Subsections 5.1.1 to 5.1.3.

3. $\dfrac{\partial w}{\partial t} + \dfrac{k}{t}w + bw\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2}.$

Modified Burgers equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= C_1 w(C_1 x + C_2, C_1^2 t), \\ w_2 &= w(x - bC_3 t^{1-k}, t) + C_3(1-k)t^{-k} \quad \text{if } k \neq 1, \\ w_3 &= w(x - bC_3 \ln|t|, t) + C_3 t^{-1} \quad \text{if } k = 1, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Degenerate solution linear in x:

$$\begin{aligned} w(x,t) &= \frac{(1-k)x + C_1}{C_2 t^k + bt} \quad \text{if } k \neq 1, \\ w(x,t) &= \frac{x + C_1}{t(C_2 + b\ln|t|)} \quad \text{if } k = 1, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution:

$$w(x,t) = u(z)t^{-1/2}, \quad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$au''_{zz} + \left(\tfrac{1}{2}z - bu\right)u'_z + \left(\tfrac{1}{2} - k\right)u = 0.$$

⊙ *Reference for equation* 5.1.6.3: A. D. Polyanin and V. F. Zaitsev (2002, p. 17).

4. $\dfrac{\partial w}{\partial t} + bw\dfrac{\partial w}{\partial x} = a\left[\dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial w}{\partial x}\right) - \dfrac{w}{x^2}\right].$

Cylindrical Burgers equation. The variable x plays the role of the radial coordinate.

Solution:

$$w(x,t) = -\frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation with axial symmetry

$$\frac{\partial \theta}{\partial t} = \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x}\right).$$

⊙ *Reference*: S. Nerney, E. J. Schmahl, and Z. E. Musielak (1996).

5. $\dfrac{\partial w}{\partial t} + bw\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + cx^k\dfrac{\partial w}{\partial x} + ckx^{k-1}w.$

Solution:

$$w(x,t) = -\frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + cx^k\frac{\partial \theta}{\partial x}.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + c(x+st)^k.$

This is a special case of equation 5.6.3.2 with $f(x,t) = c(x+st)^k$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + cx^k + st^n.$

This is a special case of equation 5.6.3.2 with $f(x,t) = cx^k + st^n$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + cw^k)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + (bz + cw^k)w'_z = 0.$$

⊙ *Reference for equation* 5.1.6.8: A. D. Polyanin and V. F. Zaitsev (2004, p. 14).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw^m + ct + s)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.3.12 with $f(w) = bw^m$, $g(t) = ct + s$, and $h(w) = 0$.

On passing from t, x to the new variables t, $z = x + \frac{1}{2}ct^2 + st$, we obtain an equation of the form 5.1.5.9:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + bw^m\frac{\partial w}{\partial z}.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw^m + ct^k)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.3.12 with $f(w) = bw^m$, $g(t) = ct^k$, and $h(w) = 0$.

On passing from t, x to the new variables t, $z = x + \dfrac{c}{k+1}t^{k+1}$, we obtain an equation of the form 5.1.5.9:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + bw^m\frac{\partial w}{\partial z}.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s_1(bx + ct)^k w^n\dfrac{\partial w}{\partial x} + s_2(bx + ct)^p w^q.$

This is a special case of equation 5.6.3.14 with $f(z,w) = s_1 z^k w^n$ and $g(z,w) = s_2 z^p w^q$.

5.1.7. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Solutions:

$$w(x) = \frac{a}{b}\ln|Ax+B| + C,$$
$$w(x,t) = A^2bt \pm Ax + B,$$
$$w(x,t) = -\frac{(x+A)^2}{4bt} - \frac{a}{2b}\ln t + B,$$
$$w(x,t) = \frac{a}{b}\ln|x^2 + 2at + Ax + B| + C,$$
$$w(x,t) = \frac{a}{b}\ln|x^3 + 6axt + Ax + B| + C,$$
$$w(x,t) = \frac{a}{b}\ln|x^4 + 12ax^2t + 12a^2t^2 + A| + B,$$
$$w(x,t) = -\frac{a^2\lambda^2}{b}t + \frac{a}{b}\ln|\cos(\lambda x + A)| + B,$$

where A, B, C, and λ are arbitrary constants.

2°. The substitution

$$w(x,t) = \frac{a}{b}\ln|u(x,t)|$$

leads to the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + c.$

The substitution

$$w(x,t) = \frac{a}{b}\ln|u(x,t)| + ct$$

leads to the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + s.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t + C_2) + C_3e^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{c(x+C_2)^2}{C_1e^{-ct} - 4b} - \frac{2a}{C_1}e^{ct}\ln|C_1e^{-ct} - 4b| + C_3e^{ct} - \frac{s}{c}.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2.$

1°. Solutions for $a < 0$:

$$w(x,t) = C_1 \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$
$$w(x,t) = \frac{1}{C_1 - at} + \frac{C_2}{(C_1 - at)^2} \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$

where C_1 and C_2 are arbitrary constants. The first formula represents a multiplicative separable solution and the second one, a generalized separable solution.

2°. Generalized separable solution for $a < 0$:

$$w(x,t) = \varphi(t) + \psi(t)\bigl[A \exp\bigl(x\sqrt{-a}\,\bigr) + B \exp\bigl(-x\sqrt{-a}\,\bigr)\bigr],$$

where A and B are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = a\bigl(\varphi^2 + 4AB\psi^2\bigr), \tag{1}$$
$$\psi'_t = a(2\varphi - 1)\psi. \tag{2}$$

Dividing equation (1) by (2) termwise yields the first-order equation $(2\varphi - 1)\psi\varphi'_\psi = \varphi^2 + 4AB\psi^2$.

3°. Generalized separable solution for $a > 0$:

$$w(x,t) = \varphi(t) + \psi(t)\cos\bigl(x\sqrt{a} + C\bigr),$$

where C is an arbitrary constant, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = a\bigl(\varphi^2 + \psi^2\bigr), \tag{3}$$
$$\psi'_t = a(2\varphi - 1)\psi. \tag{4}$$

Dividing equation (3) by (4) termwise yields a first-order equation.

⊙ *References for equation* 5.1.7.4: V. A. Galaktionov and S. A. Posashkov (1989), V. F. Zaitsev and A. D. Polyanin (1996, p. 399).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + bcw^2 + sw + k.$

1°. Generalized separable solution for $c < 0$:

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-c}\,\bigr), \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = bc\varphi^2 + s\varphi + k, \tag{2}$$
$$\psi'_t = (2bc\varphi + s - ac)\psi. \tag{3}$$

The solution of system (2), (3) is given by

$$\varphi(t) = \lambda + \frac{2bc\lambda + s}{C_1 \exp\big[-(2bc\lambda + s)t\big] - bc},$$
$$\psi(t) = \frac{C_1 C_2 \exp\big[-(2bc\lambda + s + ac)t\big]}{\big\{C_1 \exp\big[-(2bc\lambda + s)t\big] - bc\big\}^2},$$

where C_1 and C_2 are arbitrary constants and $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are roots of the quadratic equation

$$bc\lambda^2 + s\lambda + k = 0.$$

2°. For more complicated generalized separable solutions that involve hyperbolic and trigonometric functions of x, see equation 5.6.6.2 with f, g, h = const.

⊙ *References for equation* 5.1.7.5: V. A. Galaktionov and S. A. Posashkov (1989), V. F. Zaitsev and A. D. Polyanin (1996, p. 399).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + st^n w + kt^m.$

This is a special case of equation 5.6.6.2 with f = const, $g = st^n$, and $h = kt^m$.

5.1.8. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\dfrac{\partial w}{\partial x} + c.$

The substitution $u = e^w$ leads to the constant coefficient linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu,$$

which can be reduced to the linear heat equation [see Polyanin (2002, p. 93)].

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bt^n\dfrac{\partial w}{\partial x} + ct^m.$

This is a special case of equation 5.6.5.4 with $f(x,t) = bt^n$ and $g(x,t) = ct^m$.

The substitution $u = e^w$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bt^n\frac{\partial u}{\partial x} + ct^m u.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n\left(\dfrac{\partial w}{\partial x}\right)^2 + ct^m w + st^k.$

This is a special case of equation 5.6.6.1 with $f(t) = bt^n$, $g(t) = ct^m$, and $h(t) = st^k$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda t}\left(\dfrac{\partial w}{\partial x}\right)^2 + ce^{\mu t}w + se^{\nu t}.$

This is a special case of equation 5.6.6.1 with $f(t) = be^{\lambda t}$, $g(t) = ce^{\mu t}$, and $h(t) = se^{\nu t}$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Traveling-wave solutions with $bc < 0$:

$$w = C\exp\left[\pm\sqrt{-\frac{c}{b}}\,x + \left(s - \frac{ac}{b}\right)t\right],$$

where C is an arbitrary constant.

Remark. The more general equation

$$w_t = aw_{xx} + f(w)\big(bw_x^2 + cw^2\big) + sw,$$

where $f(w)$ is an arbitrary function, has the same solution.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\big(x\sqrt{|c/b|}\,\big) + \psi(t)\exp\big(-x\sqrt{|c/b|}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_t' = \left(s - \frac{ac}{b}\right)\varphi + 4c\varphi^2\psi,$$
$$\psi_t' = \left(s - \frac{ac}{b}\right)\psi + 4c\varphi\psi^2.$$

Integrating this system, we get the multiplicative separable solution

$$w = \left[C_1\exp\big(x\sqrt{|c/b|}\,\big) + C_2\exp\big(-x\sqrt{|c/b|}\,\big)\right]\left[C_3\exp\left(\frac{2(ac-bs)}{b}t\right) + \frac{4bcC_1C_2}{ac-bs}\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants. By setting $C_1 = 0$ or $C_2 = 0$, we get the solutions of Item 1°.

3°. Generalized separable solution with $bc > 0$:

$$w = \varphi(t)\cos\big(x\sqrt{c/b}\,\big) + \psi(t)\sin\big(x\sqrt{c/b}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_t' = \left(s - \frac{ac}{b}\right)\varphi + c(\varphi^2 + \psi^2)\varphi,$$
$$\psi_t' = \left(s - \frac{ac}{b}\right)\psi + c(\varphi^2 + \psi^2)\psi.$$

Integrating this system, we get the multiplicative separable solution

$$w = \left[C_1\cos\big(x\sqrt{c/b}\,\big) + C_2\sin\big(x\sqrt{c/b}\,\big)\right]\left[C_3\exp\left(\frac{2(ac-bs)}{b}t\right) + \frac{bc(C_1^2 + C_2^2)}{ac-bs}\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References for equation* 5.1.8.5: A. D. Polyanin and V. F. Zaitsev (2002, p. 381; 2004, p. 699).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{a}{w}\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = a/w$.

The substitution

$$u = \begin{cases} \dfrac{1}{a+1} w^{a+1} & \text{if } a \neq -1, \\ \ln|w| & \text{if } a = -1 \end{cases}$$

leads to the constant coefficient linear equation $\partial_t u = \partial_{xx} u$.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a w^k \left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = aw^k$. For $k = 0$, see equation 5.1.7.1, and for $k = -1$, see equation 5.1.8.6.

The substitution

$$u = \int \exp\left(\frac{a}{k+1} w^{k+1}\right) dw$$

leads to the constant coefficient linear equation $\partial_t u = \partial_{xx} u$.

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a w^m \left(\dfrac{\partial w}{\partial x}\right)^2 + (bx + ct + s)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = aw^m$, $g(t) = b$, and $h(t) = ct + s$.

5.1.9. Equations of the Form $\dfrac{\partial w}{\partial t} = a w^k \dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} C_2 w(C_1 x + C_3, C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x, t) = \frac{x^2 + Ax + B}{C - 2at},$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$ak^2 \int \frac{dw}{\lambda \ln|w| + C_1} = kx + \lambda t + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

4°. For other exact solutions, see equation 5.1.9.26 with $m = 1$, Items 5° to 8°.

2. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = Ax + B + bt,$$
$$w(x,t) = \frac{x^2 + Ax + B}{C - 2at} - \frac{b}{4a}(C - 2at),$$

where A, B, and C are arbitrary constants. The first solution is degenerate and the second one is a generalized separable solution.

3°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 w w''_{zz} - \lambda w'_z + b = 0.$$

4°. Self-similar solution:

$$w = tU(\xi), \qquad \xi = x/t,$$

where the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$aUU''_{\xi\xi} + \xi U'_\xi - U + b = 0.$$

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + bw + c.$

1°. Generalized separable solutions:

$$w(x,t) = Ae^{bt}x + Be^{bt} - \frac{c}{b},$$
$$w(x,t) = \frac{b(x + A)^2 - Bce^{-bt} - 2act + C}{Bbe^{-bt} - 2a},$$

where A, B, and C are arbitrary constants (the first solution is degenerate).

2°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 w w''_{zz} - \lambda w'_z + bw + c = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + cw^2 + kw + s.$

This is a special case of equation 5.1.9.10 with $b = 0$.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + bw^2 + (ct + d)w + st + k.$

This is a special case of equation 5.6.9.3 with $f(t) = ct + d$ and $g(t) = st + k$.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ct + d)w + pt + k.$

This is a special case of equation 5.6.10.2 with $f(t)\equiv 0$, $g(t)=b$, $h(t)=ct+d$, and $s(t)=pt+k$.

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - \dfrac{2}{3}a\left(\dfrac{\partial w}{\partial x}\right)^2 + b.$

Generalized separable solution:

$$w(x,t) = \frac{1}{a}\big[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\big],$$

where A is an arbitrary constant and the functions $f_2(t)$, $f_1(t)$, and $f_0(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f_2' &= 6Af_1 - \tfrac{2}{3}f_2^2, \\ f_1' &= 18Af_0 - \tfrac{2}{3}f_1f_2, \\ f_0' &= 2f_0f_2 - \tfrac{2}{3}f_1^2 + ab. \end{aligned}$$

The general solution of this system with $A \neq 0$ has the form

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi_t'(t),$$

where the function $\varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi_{tt}'' + 12\varphi^2 - 36A^2ab = 0.$$

The solution of the last equation is defined implicitly by

$$\int (C_1 + 72A^2ab\varphi - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and B, C_1, and C_2 are arbitrary constants.

⊙ *Reference*: J. R. King (1993), V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2004, p. 20).

8. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - a\left(\dfrac{\partial w}{\partial x}\right)^2 - b.$

1°. Generalized separable solution quadratic in x:

$$w = \frac{(x + C_1)^2}{2at} + C_2 t - bt \ln|t|,$$

where C_1 and C_2 are arbitrary constants (one more arbitrary constant can be added by a translation in t).

2°. Generalized separable solution:

$$w = \varphi(t) + \psi(t)(C_1 e^{\lambda x} + C_2 e^{-\lambda x}),$$

where C_1, C_2, and λ are arbitrary constants and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = 4aC_1C_2\lambda^2\psi^2 - b,$$
$$\psi'_t = a\lambda^2\varphi\psi.$$

3°. Generalized separable solution:

$$w = \varphi(t) + \psi(t)\sin(\lambda x + C),$$

where C and λ are arbitrary constants and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = -a\lambda^2\psi^2 - b,$$
$$\psi'_t = -a\lambda^2\varphi\psi.$$

4°. Generalized separable solution:

$$w = C_1 x + C_2 - (aC_1^2 + b)t + C_3 t^\beta \exp\left(-\frac{x}{aC_1 t} - \frac{C_2}{aC_1^2 t}\right), \qquad \beta = 1 - \frac{b}{aC_1^2},$$

where C_1, C_2, and C_3 are arbitrary constants ($C_1 \neq 0$).

⊙ *Reference for equation* 5.1.11.8: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 37–38).

9. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + c\dfrac{\partial w}{\partial x} + pw + q.$

This is a special case of equation 5.6.10.2 with $f(t) = b$, $g(t) = c$, $h(t) = p$, and $s(t) = q$.

10. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + kw + s.$

1°. Generalized separable solutions involving an exponential of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\pm\lambda x), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = c\varphi^2 + k\varphi + s, \tag{2}$$

$$\psi'_t = (a\lambda^2\varphi + 2c\varphi + k)\psi. \tag{3}$$

Integrating (2) yields

$$\int \frac{d\varphi}{c\varphi^2 + k\varphi + s} = t + C_1.$$

On computing the integral, one can find $\varphi = \varphi(t)$ in explicit form. The solution of equation (3) is expressed in terms of $\varphi(t)$ as

$$\psi(t) = C_2 \exp\left[\int (a\lambda^2\varphi + 2c\varphi + k)\, dt\right],$$

where C_1 and C_2 are arbitrary constants.

2°. There are also generalized separable solutions that involve hyperbolic and trigonometric functions (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cosh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2};$$

$$w(x,t) = \varphi(t) + \psi(t)\sinh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2};$$

$$w(x,t) = \varphi(t) + \psi(t)\cos(\lambda x + A), \quad \lambda = \left(\frac{c}{a+b}\right)^{1/2}.$$

The functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by autonomous systems of first-order ordinary differential equations (these systems can be reduced to a single first-order equation each).

For details about these solutions, see Items 2° to 4° of equation 5.6.10.1 with $f(t) = k$ and $g(t) = s$.

⊙ *References for equation* 5.1.9.10: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin and V. F. Zaitsev (2002, p. 21).

11. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - \dfrac{3}{4}a\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + kw + s.$

This is a special case of equation 5.1.9.10 with $b = -\frac{3}{4}a$.

1°. Solution with $ac > 0$:

$$w = \varphi_1(t) + \varphi_2(t)\cos(\lambda x) + \varphi_3(t)\sin(\lambda x) + \varphi_4(t)\cos(2\lambda x) + \varphi_5(t)\sin(2\lambda x), \quad \lambda = \sqrt{c/a},$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\begin{aligned}
\varphi'_1 &= c[\varphi_1^2 - \tfrac{3}{8}(\varphi_2^2 + \varphi_3^2) - 3(\varphi_4^2 + \varphi_5^2)] + k\varphi_1 + s,\\
\varphi'_2 &= c[\varphi_1\varphi_2 - 3(\varphi_2\varphi_4 + \varphi_3\varphi_5)] + k\varphi_2,\\
\varphi'_3 &= c[\varphi_1\varphi_3 + 3(\varphi_3\varphi_4 - \varphi_2\varphi_5)] + k\varphi_3,\\
\varphi'_4 &= c[-2\varphi_1\varphi_4 + \tfrac{3}{8}(\varphi_2^2 - \varphi_3^2)] + k\varphi_4,\\
\varphi'_5 &= c(-2\varphi_1\varphi_5 + \tfrac{3}{4}\varphi_2\varphi_3) + k\varphi_5.
\end{aligned}$$

2°. Solution with $ac < 0$:

$$w = \varphi_1(t) + \varphi_2(t)\cosh(\lambda x) + \varphi_3(t)\sinh(\lambda x) + \varphi_4(t)\cosh(2\lambda x) + \varphi_5(t)\sinh(2\lambda x), \quad \lambda = \sqrt{-c/a},$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\begin{aligned}
\varphi_1' &= c[\varphi_1^2 + \tfrac{3}{8}(\varphi_3^2 - \varphi_2^2) + 3(\varphi_5^2 - \varphi_4^2)] + k\varphi_1 + s,\\
\varphi_2' &= c[\varphi_1\varphi_2 + 3(\varphi_3\varphi_5 - \varphi_2\varphi_4)] + k\varphi_2,\\
\varphi_3' &= c[\varphi_1\varphi_3 + 3(\varphi_3\varphi_4 - \varphi_2\varphi_5)] + k\varphi_3,\\
\varphi_4' &= c[-2\varphi_1\varphi_4 + \tfrac{3}{8}(\varphi_2^2 + \varphi_3^2)] + k\varphi_4,\\
\varphi_5' &= c(-2\varphi_1\varphi_5 + \tfrac{3}{4}\varphi_2\varphi_3) + k\varphi_5.
\end{aligned}$$

3°. Solution with $c = 0$:

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 + \varphi_4(t)x^3 + \varphi_5(t)x^4,$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\begin{aligned}
\varphi_1' &= a(2\varphi_1\varphi_3 - \tfrac{3}{4}\varphi_2^2) + k\varphi_1 + s,\\
\varphi_2' &= a(6\varphi_1\varphi_4 - \varphi_2\varphi_3) + k\varphi_2,\\
\varphi_3' &= a(5\varphi_3^2 + \tfrac{3}{2}\varphi_2\varphi_4) + k\varphi_3,\\
\varphi_4' &= a(6\varphi_2\varphi_5 - \varphi_3\varphi_4) + k\varphi_4,\\
\varphi_5' &= a(2\varphi_3\varphi_5 - \tfrac{3}{4}\varphi_4^2) + k\varphi_5.
\end{aligned}$$

Remark. In the original equation, the parameters k and s can be replaced by arbitrary functions $k = k(t)$ and $s = s(t)$.

12. $$\frac{\partial w}{\partial t} = w\frac{\partial^2 w}{\partial x^2} - 2\left(\frac{\partial w}{\partial x}\right)^2 + bw\frac{\partial w}{\partial x} + cw^2 + sw^3.$$

This is a special case of equation 5.1.9.30 with $m = 1$. The substitution

$$w = 1/f$$

leads to the equation

$$f\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + b\frac{\partial f}{\partial x} - cf - s.$$

This equation has a generalized separable solution of the form

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
f_{xx}'' + bf_x' - cf &= f^2,\\
g_{xx}'' + bg_x' - cg &= fg + s.
\end{aligned}$$

Remark. In the original equation, the parameters b, c, and s can be replaced by arbitrary functions $b = b(x)$, $c = c(x)$, and $s = s(x)$.

13. $\dfrac{\partial w}{\partial t} = aw^2\dfrac{\partial^2 w}{\partial x^2}.$

The substitution $w = 1/v$ leads to an equation of the form 5.1.10.3:

$$\frac{\partial v}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{v^2}\frac{\partial v}{\partial x}\right).$$

Therefore the solutions of the original equation are expressed via solutions of the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial y^2}$$

by the relations

$$w = \frac{\partial u}{\partial y}, \quad x = u.$$

The variable y should be eliminated to obtain $w = w(x, t)$ in explicit form.

⊙ *Reference*: N. H. Ibragimov (1985).

14. $\dfrac{\partial w}{\partial t} = aw^2\dfrac{\partial^2 w}{\partial x^2} + bw^2.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1^{-1}x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The transformation $w = 1/u$, $\tau = at$ leads to an equation of the form 5.1.11.2:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right) - \frac{b}{a}.$$

15. $\dfrac{\partial w}{\partial t} = aw^2\dfrac{\partial^2 w}{\partial x^2} + bw^{-1}.$

This is a special case of equation 5.1.9.27 with $m = 2$ and $b = -1$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-1} w(\pm C_1^2 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solutions:

$$w(x, t) = \pm\left[C_1(x + C_2)^2 + C_3\exp(2aC_1 t) - \frac{b}{aC_1}\right]^{1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

16. $\dfrac{\partial w}{\partial t} = aw^2\dfrac{\partial^2 w}{\partial x^2} + bw + cw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\left[bC_1e^{2bt}(x+C_2)^2 + C_3F(t) + 2cF(t)\int\frac{dt}{F(t)}\right]^{1/2}, \qquad F(t) = \exp\bigl(aC_1e^{2bt} + 2bt\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants.

17. $\dfrac{\partial w}{\partial t} = w^2\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}w\left(\dfrac{\partial w}{\partial x}\right)^2.$

Multiplicative separable solution:

$$w = \frac{C_2x^2 + C_1x + C_0}{\sqrt{(C_1^2 - 4C_0C_2)t + C_3}},$$

where C_0, C_1, C_2, and C_3 are arbitrary constants.

18. $\dfrac{\partial w}{\partial t} = w^2\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}w\left(\dfrac{\partial w}{\partial x}\right)^2 + aw + b.$

Generalized separable solution

$$w = \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_2' &= \varphi_2\bigl(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\bigr) + a\varphi_2,\\
\varphi_1' &= \varphi_1\bigl(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\bigr) + a\varphi_1,\\
\varphi_0' &= \varphi_0\bigl(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\bigr) + a\varphi_0 + b.
\end{aligned}$$

Remark. In the original equation, the parameters a and b can be replaced by arbitrary functions $a = a(t)$ and $b = b(t)$.

⊙ *Reference*: V. A. Galaktionov, S. A. Posashkov (1995).

19. $\dfrac{\partial w}{\partial t} = w^2\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}w\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{2}w^3 + aw + b.$

Functional separable solution:

$$w = \varphi_1(t) + \varphi_2(t)\cos x + \varphi_3(t)\sin x,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= \tfrac{1}{2}\varphi_1(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + a\varphi_1 + b,\\
\varphi_2' &= \tfrac{1}{2}\varphi_2(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + a\varphi_2,\\
\varphi_3' &= \tfrac{1}{2}\varphi_3(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + a\varphi_3.
\end{aligned}$$

It follows from the last two equations that $\varphi_3 = C\varphi_2$, where C is an arbitrary constant.

Remark. The parameters of the equation can be arbitrary functions of time, $a = a(t)$ and $b = b(t)$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 32).

20. $\dfrac{\partial w}{\partial t} = w^2\dfrac{\partial^2 w}{\partial x^2} - 3w\left(\dfrac{\partial w}{\partial x}\right)^2 + bw^2\dfrac{\partial w}{\partial x} + cw^3 + sw^5.$

This is a special case of equation 5.1.9.30 with $m = 2$. The substitution

$$w = f^{-1/2}$$

leads to the equation

$$f\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + b\frac{\partial f}{\partial x} - 2cf - 2s.$$

This equation has a generalized separable solution of the form

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + bf'_x - 2cf = f^2,$$
$$g''_{xx} + bg'_x - 2cg = fg + 2s.$$

Remark. In the original equation, the parameters b, c, and s can be replaced by arbitrary functions $b = b(x)$, $c = c(x)$, and $s = s(x)$.

21. $\dfrac{\partial w}{\partial t} = aw^3\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 5.1.9.20 with $m = 3$.

Functional separable solution:

$$w(x,t) = a^{-1/3}\left[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\right]^{1/3}.$$

Here,

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi'_t(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int (C_1 - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants. Setting $C_1 = 0$ in the last relation, one obtains the function φ in explicit form: $\varphi = -\frac{1}{2}(t + C_2)^{-2}$.

⊙ *Reference*: G. A. Rudykh and E. I. Semenov (1998).

22. $\dfrac{\partial w}{\partial t} = aw^3\dfrac{\partial^2 w}{\partial x^2} + bw^{-2}.$

This is a special case of equation 5.1.9.21 with $m = 3$ and $b = -2$.

The substitution $w = u^{1/3}$ leads to an equation of the form 5.1.9.7:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}a\left(\frac{\partial u}{\partial x}\right)^2 + 3b.$$

Therefore the equation in question has a generalized separable solution of the form

$$w(x,t) = a^{-1/3}\big[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\big]^{1/3}.$$

23. $\dfrac{\partial w}{\partial t} = w^3\dfrac{\partial^2 w}{\partial x^2} - 4w^2\left(\dfrac{\partial w}{\partial x}\right)^2 + bw^3\dfrac{\partial w}{\partial x} + cw^4 + sw^7.$

This is a special case of equation 5.1.9.30 with $m = 3$. The substitution

$$w = f^{-1/3}$$

leads to the equation

$$f\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + b\frac{\partial f}{\partial x} - 3cf - 3s.$$

This equation has a generalized separable solution of the form

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + bf'_x - 3cf = f^2,$$
$$g''_{xx} + bg'_x - 3cg = fg + 3s.$$

Remark. In the original equation, the parameters b, c, and s can be replaced by arbitrary functions $b = b(x)$, $c = c(x)$, and $s = s(x)$.

24. $\dfrac{\partial w}{\partial t} = aw^4\dfrac{\partial^2 w}{\partial x^2} + bw + cw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{\varphi(t)x^2 + \psi(t)x + \chi(t)},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \tfrac{1}{2}a\varphi(4\varphi\chi - \psi^2) + 2b\varphi,$$
$$\psi'_t = \tfrac{1}{2}a\psi(4\varphi\chi - \psi^2) + 2b\psi,$$
$$\chi'_t = \tfrac{1}{2}a\chi(4\varphi\chi - \psi^2) + 2b\chi + 2c.$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

Remark. In the original equation, the parameters a, b, and c can be replaced by arbitrary function s $a = a(t)$, $b = b(t)$, and $c = c(t)$.

⊙ *Reference*: V. A. Galaktionov, S. A. Posashkov, and S. R. Svirshchevskii (1995).

25. $\dfrac{\partial w}{\partial t} = aw^4\dfrac{\partial^2 w}{\partial x^2} + bx^k w^5.$

This is a special case of equation 5.6.11.4 with $m = 4$ and $f(x) = bx^k$.

26. $\dfrac{\partial w}{\partial t} = aw^m\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2/m}C_2^{1/m}w(\pm C_1 x + C_3, C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x) = Ax + B,$$

$$w(x,t) = (\pm\beta x + \beta\lambda t + A)^{1/m}, \quad \beta = \frac{m\lambda}{a(1-m)},$$

$$w(x,t) = \left[\frac{m(x-A)^2}{2a(2-m)(B-t)}\right]^{1/m},$$

$$w(x,t) = \left[A|t+B|^{\frac{m}{m-2}} + \frac{m}{2a(m-2)}\frac{(x+C)^2}{t+B}\right]^{1/m},$$

$$w(x,t) = \left[\frac{(x+A)^2}{\varphi(t)} + B(x+A)^m|\varphi(t)|^{\frac{m(m-3)}{2}}\right]^{1/m}, \quad \varphi(t) = C + \frac{2a(m-2)}{m}t,$$

where A, B, C, and λ are arbitrary constants (the first solution is degenerate).

3°. Traveling-wave solution in implicit form:

$$\int \frac{dw}{\lambda w^{1-m} + C_1} = \frac{\beta x + \lambda t + C_2}{a\beta^2(1-m)},$$

where C_1, C_2, β, and λ are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution, and to $C_1 = 0$ there corresponds the second solution in Item 2°.

4°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x),$$

where λ is an arbitrary constant, and the function $f = f(x)$ is determined by the autonomous ordinary differential equation $amf''_{xx} + \lambda f^{1-m} = 0$ (its solution can be written out in implicit form).

5°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}},$$

where the function $w(z)$ is determined by the ordinary differential equation $2aw^m w''_{zz} + zw'_z = 0$.

6°. Self-similar solution of a more general form:

$$w = t^{\beta}U(\zeta), \quad \zeta = xt^{-\frac{m\beta+1}{2}},$$

where β is an arbitrary constant, and the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$aU^m U''_{\zeta\zeta} = \beta U - \tfrac{1}{2}(m\beta + 1)\zeta U'_{\zeta}.$$

This equation is generalized homogeneous, and, hence, its order can be reduced.

7°. Generalized self-similar solution:

$$w = e^{-2\lambda t}\varphi(\xi), \quad \xi = xe^{\lambda m t},$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$a\varphi^m \varphi''_{\xi\xi} = \lambda m \xi \varphi'_{\xi} - 2\lambda\varphi.$$

This equation is generalized homogeneous, and, hence, its order can be reduced.

8°. Solution:

$$w = (At + B)^{-1/m}\psi(u), \quad u = x + k\ln(At + B),$$

where A, B, and k are arbitrary constants, and the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$a\psi^m \psi''_{uu} = Ak\psi'_u - \frac{A}{m}\psi.$$

9°. The substitution $u = w^{1-m}$ leads to the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{\frac{m}{1-m}}\frac{\partial u}{\partial x}\right),$$

which is considered in Subsection 5.1.10.

27. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + bw^k.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-m-1}x + C_2, C_1^{2k-2}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions:

$$w = w(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw^m w''_{zz} - \lambda w'_z + bw^k = 0.$$

3°. Self-similar solution for $k \neq 1$:

$$w = t^{\frac{1}{1-k}}u(\xi), \quad \xi = xt^{\frac{k-m-1}{2(1-k)}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au^m u''_{\xi\xi} + \frac{m-k+1}{2(1-k)}\xi u'_{\xi} + bu^k - \frac{1}{1-k}u = 0.$$

4°. For $m \neq 1$, the substitution $u = w^{1-m}$ leads to the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{\frac{m}{1-m}}\frac{\partial u}{\partial x}\right) + b(1-m)u^{\frac{k-m}{1-m}},$$

which is considered in Subsection 5.1.11.

5°. For $k = 1$, the transformation

$$w(x,t) = e^{bt}U(x,\tau), \quad \tau = \frac{1}{bm}e^{bmt} + \text{const},$$

leads to an equation of the form 5.1.9.26:

$$\frac{\partial U}{\partial \tau} = aU^m \frac{\partial^2 U}{\partial x^2}.$$

28. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + bxt^n \dfrac{\partial w}{\partial x} + ct^k w.$

This is a special case of equation 5.6.11.5 with $f(t) = bt^n$ and $g(t) = ct^k$.

29. $\dfrac{\partial w}{\partial t} = w^{2m} \dfrac{\partial^2 w}{\partial x^2} + (m-1)w^{2m-1}\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^{m+1}.$

The substitution $w = u^{-1/m}$ leads to an equation of the form 5.1.11.3:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right) - am,$$

that can be linearized.

30. $\dfrac{\partial w}{\partial t} = w^m \dfrac{\partial^2 w}{\partial x^2} - (m+1)w^{m-1}\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^m \dfrac{\partial w}{\partial x} + bw^{m+1} + cw^{2m+1}.$

For $m \neq 0$, the substitution

$$w = f^{-1/m}$$

leads to the equation

$$f\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} - bmf - cm.$$

This equation has a generalized separable solution of the form

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + af'_x - bmf = f^2,$$
$$g''_{xx} + ag'_x - bmg = fg + cm.$$

Remark. In the original equation, the parameters a, b, and c can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

5.1.10. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right)$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 5.1.10.8 with $m = 1$. It is a quadratic *porous medium equation*, which describes unsteady flow of groundwater with the presence of a free surface (J. Boussinesq, 1904), where w is the groundwater pressure.

1°. Solutions:

$$w(x,t) = C_1 x + aC_1^2 t + C_2,$$
$$w(x,t) = -\frac{(x+C_1)^2}{6a(t+C_2)} + \frac{C_3}{|t+C_2|^{1/3}},$$
$$w(x,t) = \frac{(x+C_1)^2}{C_2 - 6at} + C_3|x+C_1|^{1/2}|C_2 - 6at|^{-5/8},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Traveling-wave solution in implicit form:

$$w - C_2 \ln|w + C_2| = C_1 x + aC_1^2 t + C_3.$$

3°. Solution in parametric form:

$$x = (6at + C_1)\xi + C_2\xi^2 + C_3,$$
$$w = -(6at + C_1)\xi^2 - 2C_2\xi^3.$$

4°. Solution in parametric form:

$$x = tf(\xi) + g(\xi),$$
$$w = tf'_\xi(\xi) + g'_\xi(\xi),$$

where the functions $f = f(\xi)$ and $g(\xi)$ are determined by the system of ordinary differential equations

$$(f'_\xi)^2 - ff''_{\xi\xi} = af'''_{\xi\xi\xi}, \tag{1}$$
$$f'_\xi g'_\xi - fg''_{\xi\xi} = ag'''_{\xi\xi\xi}. \tag{2}$$

The order of equation (1) can be reduced by two. Suppose a solution of equation (1) is known. Equation (2) is linear in g and has two linearly independent particular solutions

$$g_1 = 1, \quad g_2 = f(\xi).$$

The second particular solution follows from the comparison of (1) and (2). The general solution of equation (1) can be represented in the form (see Polyanin and Zaitsev, 2003):

$$g(\xi) = C_1 + C_2 f + C_3\left(f\int \psi\, d\xi - \int f\psi\, d\xi\right),$$
$$f = f(\xi), \quad \psi = \frac{1}{(f'_\xi)^2}\exp\left(-\frac{1}{a}\int f\, d\xi\right). \tag{3}$$

It is not difficult to verify that equation (1) has the following particular solutions:

$$f(\xi) = 6a(\xi + C)^{-1},$$
$$f(\xi) = Ce^{\lambda\xi} - a\lambda, \tag{4}$$

where C and λ are arbitrary constants. One can see, taking into account (1) and (3), that the first solution in (4) leads to the solution of Item 3°. Substituting the second relation of (4) into (1), we obtain another solution.

Remark. The above solution was obtained, with the help of the Mises transformation, from a solution of the hydrodynamic boundary layer equation (see 13.3.1.1, Items 5° and 7°).

5°. For other solutions, see Items 4° to 9° of equation 5.1.10.8 with $m = 1$.

⊙ *References for equation* 5.1.10.1: G. I. Barenblatt and Ya. B. Zel'dovich (1957), D. Zwillinger (1989, p. 311), A. D. Polyanin and V. F. Zaitsev (2002, pp. 22–23; 2004, p. 25).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right)$.

Fast diffusion equation. This is a special case of equation 5.1.10.8 with $m = -1$.

Solutions:

$$w(x,t) = (C_1x - aC_1^2t + C_2)^{-1},$$
$$w(x,t) = (2at + C_1)(x + C_2)^{-2},$$
$$w(x,t) = \frac{2a(t + C_1)}{(x + C_2)^2 + C_3(t + C_1)^2},$$
$$w(x,t) = \frac{C_1^2}{C_2 + C_3\exp(aC_2t - C_1x)},$$
$$w(x,t) = \frac{C_1^2}{at + C_2}\left[C_3\exp\left(-\frac{C_1x}{at + C_2}\right) - 1 + \frac{C_1x}{at + C_2}\right]^{-1},$$
$$w(x,t) = \frac{2aC_1^2t + C_2}{\sinh^2(C_1x + C_3)},$$
$$w(x,t) = \frac{C_2 - 2aC_1^2t}{\cosh^2(C_1x + C_3)},$$
$$w(x,t) = \frac{2aC_1^2t + C_2}{\cos^2(C_1x + C_3)},$$
$$w(x,t) = \frac{2\sin(2at + C_1)}{\cos(2at + C_1) - \cos(2x + C_2)},$$
$$w(x,t) = \frac{2\sinh(2at + C_1)}{\cosh(2x + C_2) - \cosh(2at + C_1)},$$
$$w(x,t) = \frac{2\cosh(2at + C_1)}{\sinh(2x + C_2) - \sinh(2at + C_1)},$$
$$w(x,t) = -\frac{2\sinh(2at + C_1)}{\cosh(2x + C_2) + \cosh(2at + C_1)},$$
$$w(x,t) = \frac{2\sin(2at + C_1)}{\cosh(2x + C_2) - \cos(2at + C_1)},$$
$$w(x,t) = \frac{2\sinh(2at + C_1)}{\cosh(2at + C_1) - \cos(2x + C_2)},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References*: O. V. Voinov (1994), V. V. Pukhnachov (1987), S. N. Aristov (1999), A. D. Polyanin and V. F. Zaitsev (2002, p. 23), R. O. Popovich, O. O. Vaneeva, and N. M. Ivanova (2007), N. M. Ivanova (2008).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right)$.

Fujita–Storm equation. This is a special case of equation 5.1.10.8 with $m = -2$.

1°. Solutions:

$$w(x,t) = \pm(2C_1x - 2aC_1^2t + C_2)^{-1/2},$$

$$w(x,t) = \pm\frac{\sqrt{2at}}{x}\left[\ln\left(\frac{C_1}{x^2t}\right)\right]^{-1/2},$$

$$w(x,t) = \pm\left[\frac{C_1(x+C_2)^2}{2a} + C_3\exp(C_1t)\right]^{-1/2},$$

$$w(x,t) = \frac{\pm1}{f(\tau)\sqrt{4 - e^{8\tau}\left[ce^{-\tau} \pm f(\tau)\right]^2}}, \qquad \tau = at, \quad f(\tau) = \sqrt{c^2e^{-2\tau} + 2e^{-8\tau} + 2e^{-4\tau}x},$$

$$w(x,t) = \frac{\pm1}{f(-\tau)\sqrt{e^{-8\tau}\left[ce^{-\tau} \pm f(\tau)\right]^2 - 4}}, \quad \tau = at, \quad f(\tau) = \sqrt{c^2e^{-2\tau} + 2e^{-8\tau} + 2e^{-4\tau}x},$$

where C_1, C_2, C_3, and c are arbitrary constants. The first solution is of the traveling-wave type, the second is self-similar, and the third is a functional separable solution. In the last two equations, the plus or minus signs can be selected arbitrarily.

2°. Introduce a new unknown $z = z(x,t)$ by $w = \dfrac{\partial z}{\partial x}$ and then integrate the resulting equation with respect to x to obtain

$$\frac{\partial z}{\partial t} = a\left(\frac{\partial z}{\partial x}\right)^{-2}\frac{\partial^2 z}{\partial x^2}. \tag{1}$$

By the hodograph transformation

$$x = u, \quad z = y, \tag{2}$$

equation (1) can be reduced to a linear heat equation for $u = u(y,t)$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial y^2}. \tag{3}$$

Transformation (2) means that the dependent variable z is taken to be the independent variable, and the independent variable x, the dependent one.

Solutions $w = w(x,t)$ of the original equation are expressed via solutions $u = u(y,t)$ of the linear equation (3) according to

$$w = \left(\frac{\partial u}{\partial y}\right)^{-1}, \quad x = u(y,t). \tag{4}$$

The variable y should be eliminated from (4) to obtain $w = w(x,t)$ in explicit form.

3°. The transformation

$$\bar{x}=\int_{x_0}^{x} w(y,t)\,dy+a\int_{t_0}^{t}\left[w^{-2}(x,\tau)\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0} d\tau,\quad \bar{t}=t-t_0,\quad \bar{w}(\bar{x},\bar{t})=\frac{1}{w(x,t)},\quad (5)$$

where x_0 and t_0 are any numbers, leads to the linear equation

$$\frac{\partial \bar{w}}{\partial \bar{t}}=a\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}.$$

The inversion of transformation (5) is given by

$$x=\int_{\bar{x}_0}^{\bar{x}} \bar{w}(x',\bar{t})\,dx'+\int_{\bar{t}_0}^{\bar{t}}\left(\frac{\partial \bar{w}(\bar{x},t')}{\partial \bar{x}}\right)_{\bar{x}=\bar{x}_0} dt',\quad t=\bar{t}-\bar{t}_0,\quad w(x,t)=\frac{1}{\bar{w}(\bar{x},\bar{t})}.$$

4°. Suppose $w=w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=w-\frac{2tw_t+xw_x+w}{at(w_x-\frac{1}{2}w^2)+\frac{1}{2}xw}$$

is also a solution of the equation.

⊙ *References for equation* 5.1.10.3: M. L. Storm (1951), G. W. Bluman and S. Kumei (1980), A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981), N. H. Ibragimov (1985), V. Tychynin, O. Petrova, and O. Tertyshnyk (2007), N. M. Ivanova (2008).

4. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 5.1.10.8 with $m=-4/3$ (the equation admits more invariant solutions than for $m\neq -4/3$).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=\frac{|A_1B_2-A_2B_1|^{3/2}C_1^{-3/4}}{(A_2x+B_2)^3}\,w\left(\frac{A_1x+B_1}{A_2x+B_2},\,C_1t+C_2\right),$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants ($A_1B_2-A_2B_1\neq 0$), is also a solution of the equation.

2°. Solutions:

$$w(x,t)=(\pm 2C_1x-3aC_1^2t+C_2)^{-3/4},$$
$$w(x,t)=(at+C_1)^{3/4}[(x+C_2)(C_3x+C_2C_3+1)]^{-3/2},$$
$$w(x,t)=(\pm 2C_1x^3+C_2x^4-3aC_1^2x^4t)^{-3/4},$$
$$w(x,t)=\left[\frac{(x+C_1)^2}{a(t+C_2)}+C_3(t+C_2)^2\right]^{-3/4},$$
$$w(x,t)=\left[\frac{(x+C_1)^2}{a(t+C_2)}+C_3(t+C_2)^2(x+C_1)^4\right]^{-3/4},$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is of the traveling-wave type, the second is a solution in multiplicative separable form, and the others are functional separable solutions.

3°. Functional separable solution:

$$w(x,t) = \big[\varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\big]^{-3/4},$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -a\tfrac{3}{4}\varphi_1^2 + 2a\varphi_0\varphi_2,\\
\varphi_1' &= -a\varphi_1\varphi_2 + 6a\varphi_0\varphi_3,\\
\varphi_2' &= -a\varphi_2^2 + \tfrac{3}{2}a\varphi_1\varphi_3 + 12a\varphi_0\varphi_4,\\
\varphi_3' &= -a\varphi_2\varphi_3 + 6a\varphi_1\varphi_4,\\
\varphi_4' &= -\tfrac{3}{4}a\varphi_3^2 + 2a\varphi_2\varphi_4.
\end{aligned}$$

The prime denotes a derivative with respect to t.

4°. There are exact solutions of the following forms:

$$w(x,t) = x^{-3}F(y), \quad y = t - \frac{1}{x};$$

$$w(x,t) = x^{-3}G(z), \quad z = \frac{tx^2}{(x+1)^2}.$$

5°. For other solutions, see equation 5.1.10.8 with $m = -4/3$.

⊙ *References for equation* 5.1.10.4: L. V. Ovsiannikov (1959, 1982), J. R. King (1991), N. H. Ibragimov (1994, pp. 120, 128), V. A. Galaktionov (1995), G. A. Rudykh and E. I. Semenov (1998).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2/3}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.1.10.8 with $m = -2/3$.

1°. Solution:

$$w = (C - 4at)^{3/2}\big[(C - 4at)^{3/2} - x^2\big]^{-3/2}.$$

2°. The transformation

$$t = \tau, \quad x = v, \quad w = 1/u,$$

where $\frac{\partial v}{\partial \xi} = u$, leads to an equation of the form 5.1.10.4:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial \xi}\left(u^{-4/3}\frac{\partial u}{\partial \xi}\right).$$

⊙ *References for equation* 5.1.10.5: A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), J. R. King (1991), N. H. Ibragimov (1994, pp. 127–128).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-3/2}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.1.10.8 with $m = -3/2$.

1°. Functional separable solution:

$$w(x,t) = a^{2/3}\left[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\right]^{-2/3},$$

where A is an arbitrary constant and the functions $f_2(t)$, $f_1(t)$, and $f_0(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f_2' &= 6Af_1 - \tfrac{2}{3}f_2^2, \\ f_1' &= 18Af_0 - \tfrac{2}{3}f_1f_2, \\ f_0' &= 2f_0f_2 - \tfrac{2}{3}f_1^2. \end{aligned}$$

The general solution of this system with $A \neq 0$ has the form

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi_t'(t),$$

where the function $\varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi_{tt}'' + 12\varphi^2 = 0.$$

The solution of the last equation is given implicitly by

$$\int (C_1 - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants. Setting $C_1 = 0$ in this relation, we find φ in explicit form: $\varphi = -\frac{1}{2}(t + C_2)^{-2}$.

2°. For other solutions, see equation 5.1.10.8 with $m = -3/2$.

⊙ *References for equation* 5.1.10.6: J. R. King (1993), A. D. Polyanin and V. F. Zaitsev (2004, p. 28).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-1/2}\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 5.1.10.8 with $m = -1/2$.

Functional separable solution:

$$w = [\varphi(x)t + \psi(x)]^2,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$a\varphi_{xx}'' = \varphi^2, \quad a\psi_{xx}'' = \varphi\psi.$$

A particular solution of this system is

$$\varphi = \frac{6a}{(x + C_1)^2}, \quad \psi = C_2(x + C_1)^3 + \frac{C_3}{(x + C_1)^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References*: T. K. Amerov (1990), J. R. King (1992).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right)$.

This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory, and flows in porous media. For example, it describes unsteady heat transfer in a quiescent medium with the heat diffusivity being a power-law function of temperature. For $m = 1, -1, -2, -4/3, -2/3, -3/2, -1/2$, see also equations 5.1.10.1 to 5.1.10.7.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^m C_2^2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:*

$$w(x) = (Ax + B)^{\frac{1}{m+1}},$$

$$w(x,t) = (\pm kx + k\lambda t + A)^{1/m}, \quad k = \lambda m/a,$$

$$w(x,t) = \left[\frac{m(x-A)^2}{2a(m+2)(B-t)}\right]^{\frac{1}{m}},$$

$$w(x,t) = \left[A|t+B|^{-\frac{m}{m+2}} - \frac{m}{2a(m+2)}\frac{(x+C)^2}{t+B}\right]^{\frac{1}{m}},$$

$$w(x,t) = \left[\frac{m(x+A)^2}{\varphi(t)} + B|x+A|^{\frac{m}{m+1}}|\varphi(t)|^{-\frac{m(2m+3)}{2(m+1)^2}}\right]^{\frac{1}{m}}, \quad \varphi(t) = C - 2a(m+2)t,$$

where A, B, C, and λ are arbitrary constants. The third solution for $B > 0$ and the fourth solution for $B < 0$ correspond to *blow-up regimes* (the solution increases without bound on a finite time interval).

Special case. A solution satisfying the initial and boundary conditions

$$w = 0 \quad \text{at} \quad t = 0 \quad (x > 0),$$
$$w = kt^{1/m} \quad \text{at} \quad x = 0 \quad (t > 0),$$

is given by

$$w(x,t) = \begin{cases} k(t - x/\lambda)^{1/m} & \text{for } 0 \le x \le \lambda t, \\ 0 & \text{for } x > \lambda t, \end{cases}$$

where $\lambda = \sqrt{ak^m/m}$.

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is defined implicitly by

$$a\int \frac{w^m\,dw}{\lambda w + C_1} = C_2 + z,$$

and λ, C_1, and C_2 are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution, and to $C_1 = 0$ there corresponds the second solution in Item 2°.

* For the sake of brevity, here and henceforth, exact solutions of nonlinear equations are given only for the domain of their spatial localization, where $w \not\equiv 0$.

4°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x), \tag{1}$$

where the function $f = f(x)$ is defined implicitly by

$$\int \frac{f^m\,df}{\sqrt{C_1 - bf^{m+2}}} = \pm x + C_2, \qquad b = \frac{2\lambda}{am(m+2)},$$

and λ, C_1, and C_2 are arbitrary constants.

5°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$2a\big(w^m w_z'\big)_z' + zw_z' = 0. \tag{2}$$

Solutions of this sort usually describe situations where the unknown function assumes constant values at the initial and boundary conditions.

To the particular solution of equation (2) with $w(z) = k_2 z^{2/m}$ there corresponds the third solution in Item 2°.

Fujita (1952) obtained the general solution of equation (2) for $m = -1$ and $m = -2$; see also the book by Lykov (1967).

With the boundary conditions

$$w = 1 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = \infty$$

the solution of equation (2) is localized and has the structure

$$w = \begin{cases} (1-Z)^{1/m}\dfrac{P(1-Z,\,m)}{P(1,\,m)} & \text{for } 0 \le Z \le 1, \\ 0 & \text{for } 1 \le Z < \infty, \end{cases}$$

where

$$Z = \frac{z}{z_0}, \quad z_0^2 = \frac{2a}{mP(1,\,m)}, \quad P(\xi,\,m) = \sum_{k=0}^{\infty} b_k \xi^k,$$

$b_0 = 1$, $b_1 = -\frac{1}{2}[m(m+1)]^{-1}$, . . . ; see Samarskii and Sobol' (1963).

6°. Self-similar solution:

$$w = t^{-\frac{1}{m+2}} F(\xi), \quad \xi = xt^{-\frac{1}{m+2}} \qquad (0 \le x < \infty).$$

Here, the function $F = F(\xi)$ is determined by the first-order ordinary differential equation

$$a(m+2)F^m F_\xi' + \xi F = C, \tag{3}$$

where C is an arbitrary constant.

To $C = 0$ in (3) there corresponds the fourth solution in Item 2°, which describes the propagation of a thermal wave coming from a plane source. For details, see the book by Zel'dovich and Raiser (1966).

Performing the change of variable $\varphi = F^m$ in equation (3), one obtains

$$\varphi_\xi' = \alpha\varphi^{-1/m} - \beta\xi, \tag{4}$$

where $\alpha = \frac{mC}{a(m+2)}$ and $\beta = \frac{m}{a(m+2)}$. The books by Polyanin and Zaitsev (1995, 2003) present general solutions of equation (4) for $m = -1$ and $m = 1$.

7°. Self-similar solution of a more general form:

$$w = t^{\beta} g(\zeta), \quad \zeta = x t^{-\frac{m\beta+1}{2}}, \qquad \beta \text{ is any.}$$

Here, the function $g = g(\zeta)$ is determined by the ordinary differential equation

$$G''_{\zeta\zeta} = A_1 \zeta G^{-\frac{m}{m+1}} G'_{\zeta} + A_2 G^{\frac{1}{m+1}}, \qquad G = g^{m+1}, \tag{5}$$

where $A_1 = -(m\beta + 1)/(2a)$ and $A_2 = \beta(m + 1)/a$. This equation is homogeneous, and, therefore, its order can be reduced (and then it can be transformed to an Abel equation of the second kind). Exact analytical solutions of equation (5) for various values of m can be found in Polyanin and Zaitsev (2003).

8°. Generalized self-similar solution:

$$w = e^{-2\lambda t}\varphi(u), \quad u = x e^{\lambda m t}, \qquad \lambda \text{ is any,}$$

where the function $\varphi = \varphi(u)$ is determined by the ordinary differential equation

$$a(\varphi^m \varphi'_u)'_u = \lambda m u \varphi'_u - 2\lambda\varphi. \tag{6}$$

This equation is homogeneous, and, hence, its order can be reduced (and then it can be transformed to an Abel equation of the second kind). The substitution $\Phi = \varphi^{m+1}$ brings (6) to an equation that coincides, up to notation, with (5).

9°. Solution:

$$w = (t + A)^{-1/m}\psi(u), \quad u = x + b\ln(t + A), \qquad A,\ b \text{ are any,}$$

where the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$a(\psi^m \psi'_u)'_u = b\psi'_u - \psi/m. \tag{7}$$

Introduce the new dependent variable $p(\psi) = \dfrac{a}{b}\psi^m \psi'_u$. Taking into account the identity $\dfrac{d}{du} = \dfrac{b}{a}\psi^{-m} p \dfrac{d}{d\psi}$, we arrive at an Abel equation of the second kind:

$$pp'_{\psi} = p - s\psi^{m+1}, \quad s = a/(mb^2).$$

The general solutions of this equation with $m = -3, -2, -\frac{3}{2}, -1$ can be found in Polyanin and Zaitsev (2003).

10°. Unsteady point source solution with $a = 1$:

$$w(x,t) = \begin{cases} A t^{-1/(m+2)} \left(\eta_0^2 - \dfrac{x^2}{t^{2/(m+2)}} \right)^{1/m} & \text{for } |x| \le \eta_0 t^{1/(m+2)}, \\ 0 & \text{for } |x| > \eta_0 t^{1/(m+2)}, \end{cases}$$

where

$$A = \left[\frac{m}{2(m+2)} \right]^{1/m}, \quad \eta_0 = \left[\frac{\Gamma(1/m + 3/2)}{A\sqrt{\pi}\,\Gamma(1/m + 1)} E_0 \right]^{m/(m+2)}, \quad \Gamma(z) = \int_0^{\infty} e^{-\xi} \xi^{z-1}\, d\xi,$$

with $\Gamma(z)$ being the gamma function.

The above solution satisfies the initial condition

$$w(x,0) = E_0\delta(x),$$

where $\delta(x)$ is the Dirac delta function, and the condition of conservation of energy

$$\int_{-\infty}^{\infty} w(x,t)\,dx = E_0 > 0.$$

11°. The transformation

$$\widetilde{t} = t - t_0, \quad \widetilde{x} = \int_{x_0}^{x} w(y,t)\,dy + a\int_{t_0}^{t}\left[w^m(x,\tau)\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0} d\tau, \quad \widetilde{w}(\widetilde{x},\widetilde{t}) = \frac{1}{w(x,t)}$$

takes a nonzero solution $w(x,t)$ of the original equation to a solution $\widetilde{w}(\widetilde{x},\widetilde{t})$ of a similar equation

$$\frac{\partial \widetilde{w}}{\partial \widetilde{t}} = a\frac{\partial}{\partial \widetilde{x}}\left(\widetilde{w}^{-m-2}\frac{\partial \widetilde{w}}{\partial \widetilde{x}}\right).$$

⊙ *References for equation* 5.1.10.8: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952), L. V. Ovsiannikov (1959, 1962, 1982), N. H. Ibragimov (1994, pp. 120–124), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), G. A. Rudykh and E. I. Semenov (1998), R. Z. Zhdanov and V. I. Lahno (1998).

5.1.11. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^k$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1}w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1 x + (aC_1^2 + b)t + C_2,$$
$$w(x,t) = -\frac{(x+C_2)^2}{6a(t+C_1)} + C_3|t+C_1|^{-1/3} + \frac{3}{4}b(t+C_1),$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

3°. Traveling-wave solution in implicit form:

$$a\int \frac{u\,du}{au^2 - C_1 u + b} = -\ln|\pm x + C_1 t + C_2| + C_3, \qquad u = \frac{w}{\pm x + C_1 t + C_2}.$$

4°. For other solutions, see equation 5.1.11.14 with $m = 1$ and $k = 0$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw.$

This is a special case of equation 5.1.11.10 with $m = 1$.

1°. Additive separable solution:

$$w = -\frac{b}{6a}(x + C_1)^2 + C_2 \exp\left(\tfrac{2}{3}bt\right),$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution:

$$w = -be^{bt}(6ae^{bt} + C_1)^{-1}(x + C_3)^2 + C_2e^{bt}(6ae^{bt} + C_1)^{-1/3},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Generalized separable solution:

$$w = -be^{bt}(6ae^{bt} + C_1)^{-1}(x + C_3)^2 + C_2e^{bt}(6ae^{bt} + C_1)^{-5/8}(x + C_3)^{1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1w(\pm C_1x + C_2, C_1^{-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$x = -\frac{2}{bu}\frac{\partial u}{\partial y}, \quad w(x, t) = -\frac{b}{2}\left[\frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right]^{-1} \tag{1}$$

leads to the equation

$$\frac{\partial}{\partial y}\left(\Phi\frac{\partial \Psi}{\partial y}\right) = 0, \quad \text{where} \quad \Phi = \left[\frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right]^{-1}, \quad \Psi = \frac{1}{u}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2}\right).$$

It follows that any solution $u = u(x, t)$ of the linear heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

generates a solution (1) of the original nonlinear equation.

⊙ *References for equation* 5.1.11.3: V. A. Dorodnitsyn and S. R. Svirshchevskii (1983), S. I. Svinolupov (1985).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) - bw^3.$

Functional separable solutions:

$$w(x,t) = \pm\left[C_1(x+C_2)^2 + C_3\exp(2aC_1t) - \frac{b}{aC_1}\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-1}\dfrac{\partial w}{\partial x}\right) + bw^2.$

1°. Generalized separable solution:

$$w = \left[\frac{(x+C_1)^2}{2at} + C_2t - bt\ln|t|\right]^{-1},$$

where C_1 and C_2 are arbitrary constants (one more arbitrary constant can be added using a translation in t).

2°. Functional separable solution:

$$w = \frac{1}{\varphi(t) + \psi(t)(C_1e^{\lambda x} + C_2e^{-\lambda x})},$$

where C_1, C_2, and λ are arbitrary constants and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = 4aC_1C_2\lambda^2\psi^2 - b,$$
$$\psi'_t = a\lambda^2\varphi\psi.$$

3°. Functional separable solution:

$$w = \frac{1}{\varphi(t) + \psi(t)\sin(\lambda x + C)},$$

where C and λ are arbitrary constants and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = -a\lambda^2\psi^2 - b,$$
$$\psi'_t = -a\lambda^2\varphi\psi.$$

4°. Functional separable solution:

$$w = \left[C_1x + C_2 - (aC_1^2 + b)t + C_3t^\beta\exp\left(-\frac{x}{aC_1t} - \frac{C_2}{aC_1^2t}\right)\right]^{-1}, \qquad \beta = 1 - \frac{b}{aC_1^2},$$

where C_1, C_2, and C_3 are arbitrary constants ($C_1 \neq 0$).

⊙ *Reference for equation* 5.1.11.5: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 37–38).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw^{-1/3}.$

1°. For $ab > 0$, the transformation

$$w(x,t) = \exp(\pm 3\lambda x)z(\xi,t), \quad \xi = \frac{1}{2\lambda}\exp(\pm 2\lambda x), \qquad \lambda = \left(\frac{b}{3a}\right)^{1/2},$$

leads to a simpler equation of the form 5.1.10.4:

$$\frac{\partial z}{\partial t} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right). \tag{1}$$

2°. For $ab < 0$, the transformation

$$w(x,t) = \frac{z(\xi,t)}{\cos^3(\lambda x)}, \quad \xi = \frac{1}{\lambda}\tan(\lambda x), \qquad \lambda = \left(-\frac{b}{3a}\right)^{1/2},$$

also leads to simpler equation (1) (for its solutions, see equation 5.1.10.4).

3°. Multiplicative separable solution:

$$w(x,t) = (t+C)^{3/4}u(x),$$

where C is an arbitrary constant, and the function $u = u(x)$ is determined by the autonomous ordinary differential equation

$$a(u^{-4/3}u'_x)'_x + bu^{-1/3} - \tfrac{3}{4}u = 0.$$

4°. The substitution $u = w^{-4/3}$ leads to an equation with quadratic nonlinearity of the form 5.1.9.11 with $c = -\frac{4}{3}b$ and $k = s = 0$:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - \frac{3}{4}a\left(\frac{\partial u}{\partial x}\right)^2 - \frac{4}{3}bu^2. \tag{2}$$

For $a = -b$, equation (2) has a solution of the form

$$u = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx) + \varphi_4(t)\cos(2kx) + \varphi_5(t)\sin(2kx), \quad k = \tfrac{2}{\sqrt{3}},$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of first-order ordinary differential equations (not written out here).

For $a = b$, equation (2) has a solution of the form

$$u = \varphi_1(t) + \varphi_2(t)\cosh(kx) + \varphi_3(t)\sinh(kx) + \varphi_4(t)\cosh(2kx) + \varphi_5(t)\sinh(2kx), \quad k = \tfrac{2}{\sqrt{3}}.$$

⊙ *References for equation* 5.1.11.6: N. H. Ibragimov (1994, pp. 134, 136), V. A. Galaktionov (1995), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), A. D. Polyanin and V. F. Zaitsev (2002, pp. 27–28).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-3/2}\dfrac{\partial w}{\partial x}\right) + bw^{5/2}.$

1°. Functional separable solution:

$$w(x,t) = a^{2/3}\left[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\right]^{-2/3}.$$

Here,

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi_t'(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int (C_1 - 108A^2ab\varphi - 8\varphi^3)^{-1/2}\,d\varphi = t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants.

2°. For other solutions, see equations 5.1.11.12 with $m = -3/2$ and 5.1.11.14 with $m = -3/2$ and $k = 5/2$.

3°. The substitution $w = u^{-2/3}$ leads to an equation of the form 5.1.9.7:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}a\left(\frac{\partial u}{\partial x}\right)^2 - \frac{3}{2}b.$$

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + cw^{7/3}.$

This is a special case of equation 5.1.12.5 with $b = 0$. See also equation 5.1.11.14 with $m = -4/3$ and $k = 7/3$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b\sqrt{w}.$

This is a special case of equation 5.1.11.8 with $m = -\frac{1}{2}$. This equation with $a = \frac{1}{2}$ and $b = -1$ arises in plasma physics; it describes energy diffusion in a strong magnetic field in the presence of energy sinks due to plasma radiation.

Generalized separable solution:

$$w = \varphi^2(x)\left(C\int \frac{dx}{\varphi^2(x)} - t\right)^2,$$

where C is an arbitrary constant and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$2a\varphi_{xx}'' + b\varphi + 2\varphi^2 = 0.$$

⊙ *Reference*: A. Oron and P. Rosenau (1986).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw.$

1°. Solutions:

$$w(x,t) = e^{bt}(Ax+B)^{\frac{1}{m+1}},$$

$$w(x,t) = e^{bt}\left(\pm\frac{\lambda m}{a}x + \frac{\lambda^2}{ab}e^{bmt} + A\right)^{\frac{1}{m}},$$

$$w(x,t) = e^{bt}\left[\frac{bm^2(x-A)^2}{2a(m+2)(B-e^{bmt})}\right]^{\frac{1}{m}},$$

$$w(x,t) = \left[A\exp\left(\frac{2bmt}{m+2}\right) - \frac{bm^2(x+B)^2}{2a(m+2)}\right]^{\frac{1}{m}},$$

$$w(x,t) = e^{bt}\left[A|e^{bmt}+B|^{-\frac{m}{m+2}} - \frac{bm^2}{2a(m+2)}\frac{(x+C)^2}{e^{bmt}+B}\right]^{\frac{1}{m}},$$

where A, B, C, and λ are arbitrary constants.

2°. By the transformation

$$w(x,t) = e^{bt}v(x,\tau), \quad \tau = \frac{1}{bm}e^{bmt} + \text{const}$$

the original equation can be reduced to an equation of the form 5.1.10.8:

$$\frac{\partial v}{\partial \tau} = a\frac{\partial}{\partial x}\left(v^m\frac{\partial v}{\partial x}\right).$$

3°. See also equation 5.1.11.14 with $k = 1$.

⊙ *References for equation* 5.1.11.10: L. K. Martinson and K. B. Pavlov (1972), A. D. Polyanin and V. F. Zaitsev (2002, pp. 28–31).

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^{m+1}.$

1°. Multiplicative separable solution ($a = b = 1$, $m > 0$):

$$w(x,t) = \begin{cases} \left[\dfrac{2(m+1)}{m(m+2)}\dfrac{\cos^2(\pi x/L)}{(t_0-t)}\right]^{1/m} & \text{for } |x| \le \dfrac{L}{2}, \\ 0 & \text{for } |x| > \dfrac{L}{2}, \end{cases} \tag{1}$$

where $L = 2\pi(m+1)^{1/2}/m$. Solution (1) describes a blow-up regime that exists on a limited time interval $t \in [0,\, t_0)$. The solution is localized in the interval $|x| < L/2$.

2°. Multiplicative separable solution:

$$w(x,t) = \left(\frac{Ae^{\mu x} + Be^{-\mu x} + D}{m\lambda t + C}\right)^{1/m},$$

$$B = \frac{\lambda^2(m+1)^2}{4b^2A(m+2)^2}, \quad D = -\frac{\lambda(m+1)}{b(m+2)}, \quad \mu = m\sqrt{-\frac{b}{a(m+1)}},$$

where A, C, and λ are arbitrary constants, $ab(m+1) < 0$.

3°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$a(\varphi^m \varphi'_x)'_x + b\varphi^{m+1} + \lambda\varphi = 0. \tag{2}$$

Equation (2) has the following solution in implicit form:

$$\int \varphi^m \left[A - \frac{2\lambda}{a(m+2)}\varphi^{m+2} - \frac{b}{a(m+1)}\varphi^{2m+2}\right]^{-1/2} d\varphi = \pm x + B,$$

where A and B are arbitrary constants.

4°. Functional separable solutions [it is assumed that $ab(m+1) < 0$]:

$$w(x,t) = \left[f(t) + g(t)e^{\lambda x}\right]^{1/m}, \qquad \lambda = \pm m\sqrt{\frac{-b}{a(m+1)}},$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2, \qquad g'_t = \frac{bm(m+2)}{m+1}fg.$$

Integrating yields

$$f(t) = (C_1 - bmt)^{-1}, \quad g(t) = C_2(C_1 - bmt)^{-\frac{m+2}{m+1}},$$

where C_1 and C_2 are arbitrary constants.

5°. Functional separable solution (A and B are arbitrary constants):

$$w(x,t) = \left[f(t) + g(t)(Ae^{\lambda x} + Be^{-\lambda x})\right]^{1/m}, \qquad \lambda = m\sqrt{\frac{-b}{a(m+1)}}, \tag{3}$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2 + \frac{4bmAB}{m+1}g^2, \quad g'_t = \frac{bm(m+2)}{m+1}fg. \tag{4}$$

On eliminating t from this system, one obtains a homogeneous first-order equation:

$$f'_g = \frac{m+1}{m+2}\frac{f}{g} + \frac{4AB}{m+2}\frac{g}{f}. \tag{5}$$

The substitution $\zeta = f/g$ leads to a separable equation. Integrating yields the solution of equation (5) in the form

$$f = \pm g\left(4AB + C_1 g^{-\frac{2}{m+2}}\right)^{\frac{1}{2}}, \qquad C_1 \text{ is any.}$$

Substituting this expression into the second equation of system (4), one obtains a separable equation for $g = g(t)$.

6°. The functional separable solutions

$$w(x,t) = \big[f(t) + g(t)\cosh(\lambda x)\big]^{1/m},$$
$$w(x,t) = \big[f(t) + g(t)\sinh(\lambda x)\big]^{1/m}$$

are special cases of formula (3) with $A = \frac{1}{2}$, $B = \frac{1}{2}$ and $A = \frac{1}{2}$, $B = -\frac{1}{2}$, respectively.

7°. Functional separable solution [it is assumed that $ab(m+1) > 0$]:

$$w(x,t) = \big[f(t) + g(t)\cos(\lambda x + C)\big]^{1/m}, \qquad \lambda = m\sqrt{\frac{b}{a(m+1)}},$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2 + \frac{bm}{m+1}g^2, \quad g'_t = \frac{bm(m+2)}{m+1}fg,$$

which coincides with system (4) for $AB = \frac{1}{4}$.

⊙ *References for equation* 5.1.11.11: N. V. Zmitrenko, S. P. Kurdyumov, A. P. Mikhailov, and A. A. Samarskii (1976), M. Bertsch, R. Kersner, and L. A. Peletier (1985), V. A. Galaktionov and S. A. Posashkov (1989), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), V. F. Zaitsev and A. D. Polyanin (1996, pp. 407–408).

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^{1-m}.$

This is a special case of equation 5.1.11.14 with $k = 1 - m$, where some solutions are also given.

Functional separable solution for $m \neq -1$ and $m \neq -2$:

$$w(x,t) = \left[\frac{1}{F}(x+A)^2 + B|F|^{-\frac{m}{m+2}} - \frac{bm^2}{4a(m+1)}F\right]^{1/m},$$
$$F = F(t) = C - \frac{2a(m+2)}{m}t,$$

where A, B, and C are arbitrary constants.

For $m = -2$ see also equation 5.1.11.4 and for $m = -1$ see equation 5.1.11.5.

⊙ *References*: R. Kersner (1978 *a, b*).

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{2n}\dfrac{\partial w}{\partial x}\right) + bw^{1-n}.$

This is a special case of equation 5.1.11.14 with $m = 2n$ and $k = 1 - n$.

Generalized traveling-wave solution:

$$w(x,t) = \left[\pm\frac{x + C_1}{\sqrt{C_2 - kt}} - \frac{bn^2}{3a(n+1)}(C_2 - kt)\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 36).

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^k.$

This is a special case of equation 5.6.15.2 with $f(w) = aw^m$ and $g(w) = bw^k$. For $b = 0$, see Subsection 5.1.10.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-m-1}x + C_2, C_1^{2k-2}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. A space-homogeneous solution and a stationary solution are given by (the latter is written out in implicit form):

$$w(t) = \begin{cases} \big[(1-k)bt + C\big]^{\frac{1}{1-k}} & \text{if } k \neq 1, \\ Ce^{bt} & \text{if } k = 1, \end{cases}$$

$$\int w^m\left[A - \frac{2b}{a(m+k+1)}w^{m+k+1}\right]^{-1/2} dw = \pm x + B,$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$a(w^m w_z')_z' - \lambda w_z' + bw^k = 0. \tag{1}$$

The substitution

$$u(w) = \frac{a}{\lambda}w^m w_z'$$

brings (1) to the Abel equation

$$uu_w' - u = -ab\lambda^{-2}w^{m+k}. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents exact solutions of equation (2) with $m + k = -2, -1, -\frac{1}{2}, 0, 1$.

4°. Self-similar solution for $k \neq 1$:

$$w = t^{\frac{1}{1-k}}u(\xi), \quad \xi = xt^{\frac{k-m-1}{2(1-k)}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$a(u^m u_\xi')_\xi' + \frac{m-k+1}{2(1-k)}\xi u_\xi' + bu^k - \frac{1}{1-k}u = 0.$$

⊙ *Reference for equation* 5.1.11.14: V. A. Dorodnitsyn (1982).

5.1.12. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw + c_1w^{k_1} + c_2w^{k_2} + c_3w^{k_3}$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw + c.$

For $b = 0$, see equation 5.1.11.1 and for $c = 0$, see equation 5.1.11.2.

Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1xe^{bt} + C_2e^{bt} + \frac{aC_1^2}{b}e^{2bt} - \frac{c}{b},$$

$$w(x,t) = \frac{be^{bt}(x+C_2)^2}{\varphi(t)} + C_3\frac{e^{bt}}{\varphi^{1/3}(t)} + \frac{ce^{bt}}{\varphi^{1/3}(t)}\int e^{-bt}\varphi^{1/3}(t)dt, \qquad \varphi(t) = C_1 - 6ae^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 37).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw + cw^2.$

This is a special case of equation 5.6.10.1.

There are generalized separable solutions of the forms:

$$w = f(t) + g(t)[a\cosh(kx) + b\sinh(kx)],$$
$$w = f(t) + g(t)[a\cos(kx) + b\sin(kx)],$$

where a and b are arbitrary constants, and the functions $f(t)$ and $g(t)$ and the constant k are determined by substituting these solutions into the original equation.

⊙ *References*: W. I. Newman (1980), V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, pp. 425–427).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b + c\sqrt{w}.$

This is a special case of equation 5.1.12.12 with $m = -1/2$.

Functional separable solution:

$$w = [\varphi(x) + \psi(x)t]^2,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$a\varphi''_{xx} - \varphi\psi + \tfrac{1}{2}c\varphi + \tfrac{1}{2}b = 0,$$
$$a\psi''_{xx} - \psi^2 + \tfrac{1}{2}c\psi = 0.$$

This system can be easily integrated by starting from the latter equation. In particular, by setting $\psi = \text{const}$, one arrives at the special solution

$$w = \left(\frac{1}{2}ct - \frac{b}{4a}x^2 + C_1x + C_2\right)^2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 280–281).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b + cw + d\sqrt{w}.$

Generalized separable solution:

$$w = \left[f(x)\exp\left(\tfrac{1}{2}ct\right) + g(x)\right]^2,$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the system of ordinary differential equations

$$2af''_{xx} + cfg + df = 0,$$
$$2ag''_{xx} + cg^2 + dg + b = 0.$$

The latter equation has two constant special solutions: $g = k_{1,2} = \text{const}$. In both these cases, the former equation is a linear equation with constant coefficients, which is easily integrable and has solutions that are expressed in terms of exponentials or sines and cosines.

Remark. The constants a, b, and d in the above equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $d = d(x)$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) - bw - cw^3.$

Functional separable solutions for $b \neq 0$:

$$w(x,t) = \pm\left[bC_1e^{2bt}(x+C_2)^2 + C_3F(t) + 2cF(t)\int\frac{dt}{F(t)}\right]^{-1/2}, \qquad F(t) = \exp\left(aC_1e^{2bt} + 2bt\right),$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 37).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw^{-1/3} + cw.$

1°. Multiplicative separable solutions:

$$w(x,t) = e^{ct}\left(Ae^{kx} + Be^{-kx}\right)^{-3} \qquad \text{if} \quad b/(3a) = k^2 > 0,$$
$$w(x,t) = e^{ct}\left[A\cos(kx) + B\sin(kx)\right]^{-3} \qquad \text{if} \quad b/(3a) = -k^2 < 0,$$

where A and B are arbitrary constants.

2°. The transformation

$$w = e^{ct}u(x,\tau), \qquad \tau = -\frac{3}{4c}e^{-\frac{4}{3}ct} + \text{const}$$

leads to a simpler equation of the form 5.1.11.6:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right) + bu^{-1/3}.$$

3°. See also equation 5.1.12.10.

⊙ *Reference for equation* 5.1.12.6: A. D. Polyanin and V. F. Zaitsev (2004, p. 38).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw + cw^{5/3}.$

Functional separable solution:

$$w(x,t) = \left[\varphi(t)x^2 + \psi(t)x + \chi(t)\right]^{-3/2},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi_t' &= \tfrac{1}{2}a\varphi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\varphi,\\
\psi_t' &= \tfrac{1}{2}a\psi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\psi,\\
\chi_t' &= \tfrac{1}{2}a\chi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\chi - \tfrac{2}{3}c.
\end{aligned}$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

Remark. In the original equation, the constants a, b, and c can be replaced by arbitrary functions $a = a(t)$, $b = b(t)$, and $c = c(t)$.

⊙ *Reference*: V. A. Galaktionov, S. A. Posashkov, and S. R. Svirshchevskii (1995).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw + cw^{7/3}.$

Functional separable solution:

$$w(x,t) = \left[\varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\right]^{-3/4},$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -\tfrac{3}{4}a\varphi_1^2 + 2a\varphi_0\varphi_2 - \tfrac{4}{3}b\varphi_0 - \tfrac{4}{3}c,\\
\varphi_1' &= -a\varphi_1\varphi_2 + 6a\varphi_0\varphi_3 - \tfrac{4}{3}b\varphi_1,\\
\varphi_2' &= -a\varphi_2^2 + \tfrac{3}{2}a\varphi_1\varphi_3 + 12a\varphi_0\varphi_4 - \tfrac{4}{3}b\varphi_2,\\
\varphi_3' &= -a\varphi_2\varphi_3 + 6a\varphi_1\varphi_4 - \tfrac{4}{3}b\varphi_3,\\
\varphi_4' &= -\tfrac{3}{4}a\varphi_3^2 + 2a\varphi_2\varphi_4 - \tfrac{4}{3}b\varphi_4.
\end{aligned}$$

The prime denotes a derivative with respect to t.

Remark. In the original equation, the constants a, b, and c can be replaced by arbitrary functions $a = a(t)$, $b = b(t)$, and $c = c(t)$.

⊙ *Reference*: V. A. Galaktionov (1995).

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) - \dfrac{3}{4}w^{-1/3} + bw^{5/3} + cw.$

Functional separable solution:

$$w = [\varphi_1(t) + \varphi_2(t)\cos x + \varphi_3(t)\sin x]^{-3/2},$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= \tfrac{1}{2}\varphi_1(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) - \tfrac{2}{3}c\varphi_1 - \tfrac{2}{3}b,\\
\varphi_2' &= \tfrac{1}{2}\varphi_2(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) - \tfrac{2}{3}c\varphi_2,\\
\varphi_3' &= \tfrac{1}{2}\varphi_3(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) - \tfrac{2}{3}c\varphi_3.
\end{aligned}$$

It follows from the last two equations that $\varphi_3 = C\varphi_2$, where C is an arbitrary constant.

Remark. In the original equation, the constants b and c can be replaced by arbitrary functions $b = b(t)$ and $c = c(t)$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 32).

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) - aw^{-1/3} + bw^{7/3} + cw.$

The substitution $u = w^{-4/3}$ leads to an equation with quadratic nonlinearity:

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} - \frac{3}{4}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{4}{3}\left(au^2 - cu - b\right).$$

1°. For $a = 1$, there is a solution of the form

$$u = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx) + \varphi_4(t)\cos(2kx) + \varphi_5(t)\sin(2kx), \quad k = 2\times 3^{-1/2},$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of first-order ordinary differential equations (not written out here).

2°. For $a = -1$, there is a solution of the form

$$u=\varphi_1(t)+\varphi_2(t)\cosh(kx)+\varphi_3(t)\sinh(kx)+\varphi_4(t)\cosh(2kx)+\varphi_5(t)\sinh(2kx), \quad k=2\times 3^{-1/2}.$$

⊙ *Reference for equation* 5.1.12.10: V. A. Galaktionov (1995).

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^{m+1} + cw.$

1°. Multiplicative separable solutions:

$$w(x,t) = e^{ct}\left[A\cos(kx) + B\sin(kx)\right]^{\frac{1}{m+1}} \quad \text{if} \quad b(m+1)/a = k^2 > 0,$$
$$w(x,t) = e^{ct}\left[A\exp(kx) + B\exp(-kx)\right]^{\frac{1}{m+1}} \quad \text{if} \quad b(m+1)/a = -k^2 < 0,$$

where A and B are arbitrary constants.

2°. The transformation

$$w = e^{ct}u(x,\tau), \quad \tau = \frac{1}{cm}e^{cmt} + \text{const}$$

leads to a simpler equation of the form 5.1.11.11:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^m\frac{\partial u}{\partial x}\right) + bu^{m+1}.$$

Special case. Multiplicative separable solution for $m = -1$:

$$w = A\exp\left(ct - \frac{b}{2a}x^2 + Bx\right),$$

where A and B are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + b + cw^{-m}.$

Functional separable solution:

$$w = \left[c(m+1)t - \frac{b(m+1)}{2a}x^2 + C_1x + C_2\right]^{\frac{1}{m+1}},$$

where C_1 and C_2 are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw + cw^{1-m}.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \left(C_1e^{bmt}x + \frac{a}{bm^2}e^{2bmt} + C_2e^{bmt} - \frac{c}{b}\right)^{1/m},$$

where C_1 and C_2 are arbitrary constants.

2°. For a more complicated solution, see 5.6.13.6 with $f(t) = b$ and $g(t) = c$.

⊙ *Reference for equation* 5.1.12.13: V. A. Galaktionov and S. A. Posashkov (1989).

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^{1+m} + cw + sw^{1-m}.$

This is a special case of equation 5.6.13.7 with $f(t) = c$ and $g(t) = s$.

The substitution $u = w^m$ leads to an equation of the form 5.1.9.10:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + bmu^2 + cmu + sm.$$

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

15. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{2n}\dfrac{\partial w}{\partial x}\right) + bw + cw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left[\varphi(t)(\pm x + C_1) + cn\varphi(t)\int\frac{dt}{\varphi(t)}\right]^{1/n}, \qquad \varphi(t) = \left[C_2e^{-2bnt} - \frac{a(n+1)}{bn^2}\right]^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 40).

16. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + bw + ck + bkw^{n+1} + cw^{-n}.$

Functional separable solutions:

$$w(x,t) = \left\{\exp\big[b(n+1)t\big]\big[C_1\cos\big(x\sqrt{\lambda}\,\big) + C_2\sin\big(x\sqrt{\lambda}\,\big)\big] - \frac{c}{b}\right\}^{\frac{1}{n+1}} \qquad \text{if } \lambda > 0,$$

$$w(x,t) = \left\{\exp\big[b(n+1)t\big]\big[C_1\cosh\big(x\sqrt{|\lambda|}\,\big) + C_2\sinh\big(x\sqrt{|\lambda|}\,\big)\big] - \frac{c}{b}\right\}^{\frac{1}{n+1}} \qquad \text{if } \lambda < 0,$$

where C_1 and C_2 are arbitrary constants and $\lambda = \dfrac{bk}{a}(n+1)$.

⊙ *Reference*: V. A. Galaktionov (1994).

5.1.13. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^2 + bw)\dfrac{\partial w}{\partial x}\right]$.

This is a special case of equation 5.6.15.1 with $f(w) = aw^2 + bw$ and a special case of equation 5.1.13.4 with n=1.

Solutions:

$$w(x,t) = \pm\sqrt{2C_1x + 2aC_1^2t + C_2} - \frac{b}{a},$$
$$w(x,t) = \pm\frac{x + C_1}{\sqrt{C_2 - 4at}} - \frac{b}{2a},$$

where C_1 and C_2 are arbitrary constants. The first solution is of the traveling-wave type and the second one is self-similar.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{a}{w^2 + b^2}\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 5.6.15.1 with $f(w) = a(w^2 + b^2)^{-1}$.

1°. Solutions (A and B are arbitrary constants):

$$w(x) = b\tan(Ax + B),$$
$$w(x,t) = \pm bx\left(A - 2ab^{-2}t - x^2\right)^{-1/2},$$
$$w(x,t) = Ab\exp(ab^{-2}t - x)\left\{1 - A^2\exp\left[2(ab^{-2}t - x)\right]\right\}^{-1/2}.$$

2°. Traveling-wave solution in implicit form:

$$\lambda(kx + \lambda t) + B = \frac{ak^2}{A^2 + b^2}\left[\ln|w + A| - \frac{1}{2}\ln(w^2 + b^2) + \frac{A}{b}\arctan\frac{w}{b}\right],$$

where A, B, k, and λ are arbitrary constants.

3°. The substitution

$$w = \frac{bu}{\sqrt{1 - u^2}} \tag{1}$$

leads to the equation

$$\frac{\partial u}{\partial t} = \frac{a}{b^2}\left[(1 - u^2)\frac{\partial^2 u}{\partial x^2} + u\left(\frac{\partial u}{\partial x}\right)^2\right], \tag{2}$$

which is a special case of 12.1.2.12 with $F(t,\xi,\eta) = ab^{-2}(\xi - \eta)$. Equation (2) has multiplicative separable solutions

$$u = \frac{Ae^{\lambda x} + Be^{-\lambda x}}{\sqrt{4AB + Ce^{-kt}}}, \qquad k = \frac{2a\lambda^2}{b^2};$$
$$u = \frac{A\sin(\lambda x) + B\cos(\lambda x)}{\sqrt{A^2 + B^2 + Ce^{kt}}}, \quad k = \frac{2a\lambda^2}{b^2}, \tag{3}$$

where A, B, C, and λ are arbitrary constants. Formulas (1) and (3) provide two solutions of the original equation.

4°. Solution:

$$w = b\tan\left(\pm\frac{1}{2}z \pm \frac{a}{b^2}t + C\right),$$
$$z = x^2\cos^{-2}\left(\pm\frac{1}{2}z - \arctan\big(\psi(z)\big) \pm \frac{a}{b^2}t + C\right),$$

where C is an arbitrary constant and the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi'_z = \frac{1}{2}(1+\psi^2)\left(\pm 1 - \frac{\psi}{z}\right).$$

Here the function $z = z(x,t)$ is defined implicitly.

5°. Solution:

$$w = b\tan\left(\varphi(z) + \arctan\big(\psi(z)\big) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$
$$z = \frac{b^2x^2}{at}\cos^{-2}\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$

where C is an arbitrary constant and the functions $\varphi(z)$ and $\psi(z)$ are determined by the system of ordinary differential equations

$$\varphi'_z = \frac{\psi}{2z}, \quad \psi'_z = \frac{1}{2}(1+\psi^2)\left(\frac{C}{2} - \frac{\psi}{2} - \frac{\psi}{z}\right).$$

Here the function $z = z(x,t)$ is defined implicitly.

⊙ *References for equation* 5.1.13.2: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994, pp. 125–126), P. W. Doyle and P. J. Vassiliou (1998).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{a}{1-w^2}\dfrac{\partial w}{\partial x}\right)$.

Solutions:

$$w = \pm\frac{x + C_1}{\sqrt{(x+C_1)^2 + 2at + C_2}},$$
$$w = \pm\frac{e^{at-x}}{C\sqrt{1 + C^2e^{2(x-at)}}},$$
$$w = \pm\frac{\sinh(x+C_1)}{\sqrt{\cosh^2(x+C_1) - C_2e^{-2at}}},$$
$$w = \pm\frac{\cosh(x+C_1)}{\sqrt{\sinh^2(x+C_1) + C_2e^{-2at}}},$$
$$w = \pm\frac{\sin(x+C_1)}{\sqrt{C_2e^{2at} - \cos^2(x+C_1)}},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: P. W. Doyle and P. J. Vassiliou (1998).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = aw^{2n} + bw^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \left(\pm\sqrt{2C_1 nx + 2aC_1^2 nt + C_2} - \frac{b}{a}\right)^{1/n},$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution:

$$w(x,t) = \left[\pm\frac{x + C_1}{\sqrt{C_2 - kt}} - \frac{b}{a(n+1)}\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n}.$$

⊙ *Reference for equation* 5.1.13.4: A. D. Polyanin and V. F. Zaitsev (2004, pp. 41–42).

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\dfrac{\partial w}{\partial x}\right] + cw^{1-n}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\pm\frac{x + C_1}{\sqrt{C_2 - kt}} - \frac{cn^2}{3a(n+1)}(C_2 - kt) - \frac{b}{a(n+1)}\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 42).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\dfrac{\partial w}{\partial x}\right] + cw + sw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left[\varphi(t)(\pm x + C_1) + \frac{b}{n}\varphi(t)\int \varphi(t)\,dt + sn\varphi(t)\int \frac{dt}{\varphi(t)}\right]^{1/n},$$
$$\varphi(t) = \left[C_2 e^{-2cnt} - \frac{a(n+1)}{cn^2}\right]^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

5.1.14. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bx^m w^{-1/3}.$

For $m = 0$, see equation 5.1.11.6. For $m \neq 0$, the original equation can be reduced to a simpler equation 5.1.10.4 that corresponds to the case $b = 0$ [see equation 5.6.13.1 with $f(x) = bx^m$].

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w.$

This is a special case of equation 5.6.13.4 with $f(t) = bt^n$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w.$

This is a special case of equation 5.6.13.4 with $f(t) = be^{\lambda t}$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w^{1-m}.$

This is a special case of equation 5.6.13.5 with $f(t) = bt^n$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w^{1-m}.$

This is a special case of equation 5.6.13.5 with $f(t) = be^{\lambda t}$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w + ct^k w^{1-m}.$

This is a special case of equation 5.6.13.6 with $f(t) = bt^n$ and $g(t) = ct^k$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w + ce^{\mu t} w^{1-m}.$

This is a special case of equation 5.6.13.6 with $f(t) = be^{\lambda t}$ and $g(t) = ce^{\mu t}$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bw^{1+m} + ct^n w + st^k w^{1-m}.$

This is a special case of equation 5.6.13.7 with $f(t) = ct^n$ and $g(t) = st^k$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bx^n w^{1+m}.$

This is a special case of equation 5.6.13.8 with $f(x) = bx^n$.

5.1.15. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x}$

1. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.1.15.8 with $n = 1$.

Degenerate solution linear in x:

$$w(x,t) = \frac{ax + b\ln|t + C_1| + C_2}{a^2(t + C_1)},$$

where C_1 and C_2 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(w^2\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.1.15.8 with $n = 2$.

1°. Traveling-wave solution in implicit form:

$$2b\int \frac{w^2\,dw}{aw^2 + 2\lambda w + C_1} = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

2°. Degenerate solution linear in x:

$$w(x,t) = (x + C_1)f(t).$$

Here, C_1 is an arbitrary constant, and the function $f = f(t)$ is determined by the ordinary differential equation

$$f'_t + af^2 = 2bf^3,$$

whose solution can be represented in implicit form:

$$\frac{1}{af} + \frac{2b}{a^2}\ln\left|\frac{2bf - a}{f}\right| = t + C_2.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + aw^{-2}\dfrac{\partial w}{\partial x}.$

Fokas–Yortsos equation.

1°. This equation can be linearized; see equation 5.6.15.5 (the special case in Item 2°) and equation 5.6.13.3 with $f(x) = a$.

2°. Traveling-wave solution in implicit form:

$$\int \frac{dw}{w(\lambda w^2 + C_1 w + a)} = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants (the integral can be calculated explicitly).

3°. Multiplicative separable solution:

$$w(x,t) = f(x)\sqrt{C_1 t + C_2},$$

where C_1 and C_2 are arbitrary constants, and the function $f = f(x)$ is determined by the autonomous ordinary differential equation

$$(f^{-2}f'_x)'_x + af^{-2}f'_x = \tfrac{1}{2}C_1 f.$$

4°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = \frac{aw^2}{aw - w_x}$$

is also a solution of the equation.

⊙ *References for equation* 5.1.15.3: A. S. Fokas and Y. C. Yortsos (1982), V. Tychynin and O. Rasin (2004), N. M. Ivanova (2008).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\dfrac{a}{(w+b)^2}\dfrac{\partial w}{\partial x}\right] + c\dfrac{\partial w}{\partial x}.$

The transformation

$$u(z,t) = w(x,t) + b, \qquad z = x + ct$$

leads to an equation of the form 5.1.10.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

5. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = \dfrac{\partial}{\partial x}\left[(bw^2 + cw)\dfrac{\partial w}{\partial x}\right].$

1°. Degenerate solution linear in x:

$$w(x,t) = f(t)x + g(t),$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the system of ordinary differential equations

$$f'_t + af^2 = 2bf^3,$$
$$g'_t + afg = 2bf^2 g + cf^2.$$

The solution of the first equation can be found in 5.1.15.2, Item 2°. The second equation is easy to integrate, since it is linear in g.

2°. Traveling-wave solution in implicit form:

$$2\int \frac{bw^2 + cw}{aw^2 + 2\lambda w + C_1}\,dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + b\,\operatorname{sign} x\dfrac{\partial w}{\partial x}.$

This equation arises in biology.

By changing from t and x to the new variables t and $z = |x| + bt$, we arrive at a simpler equation of the form 5.1.10.8:

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial z}\left(w^m\frac{\partial w}{\partial z}\right).$$

⊙ *References*: J. D. Murray (1989), N. Shigesada, K. Kawasaki, and E. Teramoto (1979).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = Aw\big(Bx + b(B - A^nB^2)t + C_1, A^nB^2t + C_2\big),$$

where A, B, C_1, and C_2 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$w = U(z,t), \qquad z = x + bt,$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial U}{\partial t} = a\frac{\partial}{\partial z}\left(U^n\frac{\partial U}{\partial z}\right).$$

8. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1w(C_1^{1-n}x + C_2, C_1^{2-n}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$2b\int\frac{w^n\,dw}{aw^2 + 2\lambda w + C_1} = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

3°. Self-similar solution for $n \neq 2$:

$$w(x,t) = u(z)t^{1/(n-2)}, \qquad z = xt^{-(n-1)/(n-2)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$bu^nu''_{zz} + 2bnu^{n-1}(u'_z)^2 - \left(au - \frac{n-1}{n-2}z\right)u'_z - \frac{1}{n-2}u = 0.$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + (bw^n + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\frac{C_2 - x}{b(t+C_1)} + \frac{a\ln|t+C_1|}{b^2 n(t+C_1)} - \frac{c}{b}\right]^{1/n},$$

where C_1 and C_2 are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{2n}\dfrac{\partial w}{\partial x}\right) + bw^n\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.1.15.11 with $k = 2n$ and is a special case of equation 5.1.15.12 with $b = s = 0$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1^{-n}x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$a\int \frac{w^{2n}\,dw}{C_1 w + C_2 - \frac{b}{n+1}w^{n+1}} = x + C_1 t + C_3 \quad \text{if} \quad n \neq -1,$$
$$a\int \frac{dw}{w^2(C_1 w + C_2 - b\ln|x|)} = x + C_1 t + C_3 \quad \text{if} \quad n = -1,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Multiplicative separable solution:

$$w = (x + C_1)^{1/n}\varphi(t),$$

where C_1 is an arbitrary constant and the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi_t' = a\frac{n+1}{n^2}\varphi^{2n+1} + \frac{b}{n}\varphi^{n+1}.$$

Integrating yields the general solution in implicit form

$$\varphi^{-n} - \frac{a(n+1)}{bn}\ln\left[\varphi^{-n} + \frac{a(n+1)}{bn}\right] = -bt + C_2,$$

where C_2 is an arbitrary constant.

⊙ *References for equation* 5.1.15.10: C. M. Yung, K. Verburg, and P. Baveye (1994), A. D. Polyanin and V. F. Zaitsev (2004, p. 46), N. M. Ivanova (2008).

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^k\dfrac{\partial w}{\partial x}\right) + bw^n\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1^{n-k}x + C_2, C_1^{2n-k}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$a\int \frac{w^k\,dw}{C_1 w + C_2 - \frac{b}{n+1}w^{n+1}} = x + C_1 t + C_3 \quad \text{if} \quad n \neq -1,$$

$$a\int \frac{w^k\,dw}{C_1 w + C_2 - b\ln|x|} = x + C_1 t + C_3 \quad \text{if} \quad n = -1,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Self-similar solution:

$$w = t^{\frac{1}{k-2n}}U(z), \qquad z = xt^{\frac{n-k}{k-2n}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$\frac{1}{k-2n}\big[U + (n-k)zU_z'\big] = a(U^kU_z')_z' + bU^nU_z'.$$

Special case. For $k = n$, there is the generalized traveling-wave solution

$$w = \left\{\frac{C_2 - x}{b(t+C_1)} + \frac{a\ln|t+C_1|}{b^2n(t+C_1)}\right\}^{1/n},$$

where C_1 and C_2 are arbitrary constants (this solution is a special case of the one given in equation 5.1.15.12).

⊙ *References for equation* 5.1.15.11: C. M. Yung, K. Verburg, and P. Baveye (1994), N. M. Ivanova (2008).

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(aw^{2n} + bw^n\right)\dfrac{\partial w}{\partial x}\right] + \left(cw^n + s\right)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\varphi(t)x + (st + C_1)\varphi(t) + \frac{b}{n}\varphi(t)\int \varphi(t)\,dt\right]^{1/n},$$

where C_1 is an arbitrary constant and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi_t' = \frac{a(n+1)}{n}\varphi^3 + c\varphi^2.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 46).

5.1.16. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-1}\dfrac{\partial w}{\partial x}\right) + bx\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.1.16.4 with $n = -1$.

Solutions:

$$w(x,t) = C_1 \exp\left(C_2 x e^{bt}\right),$$

$$w(x,t) = \frac{2b}{2bxe^{bt} - e^{2bt} + C_1},$$

$$w(x,t) = \frac{4be^{2bt}}{4b^2(xe^{bt} + C_1)^2 + C_2 e^{4bt}},$$

$$w(x,t) = \frac{C_1 C_2^2 \exp(2bt + C_2 x e^{bt})}{b[1 - C_1 \exp(C_2 x e^{bt})]^2},$$

$$w(x,t) = \frac{C_1}{2bC_2 + C_3 \exp(C_1 x e^{bt} + C_1 C_2 e^{2bt})},$$

$$w(x,t) = \frac{C_1^2 e^{2bt}}{b\cos^2(C_1 x e^{bt} + C_2)},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: N. M. Ivanova (2008).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial w}{\partial x} + cw.$

This is a special case of equation 5.6.13.10 with $m = -2$, $f(t) = b$, and $g(t) = c$.

The transformation (A and B are arbitrary constants)

$$w(x,t) = e^{ct}u(z,\tau), \quad z = x + bt + A, \quad \tau = B - \frac{1}{2c}e^{-2ct}$$

leads to an equation of the form 5.1.10.3:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. A. Dorodnitsyn and S. R. Svirshchevskii (1983); the case $b = 0$ was treated.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + (bx + c)\dfrac{\partial w}{\partial x} + sw.$

This is a special case of equation 5.6.13.12 with $m = -2$, $f(t) = b$, $g(t) = c$, and $h(t) = s$.

1°. For $b = 0$, see equation 5.1.16.2.

2°. Let $b \neq 0$ and $b \neq s$. The transformation

$$w(x,t) = e^{st}u(z,\tau), \quad z = e^{bt}\left(x + \frac{c}{b}\right), \quad \tau = \frac{1}{2(b-s)}e^{2(b-s)t}$$

leads to a simpler equation of the form 5.1.10.3:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

3°. Let $b \neq 0$ and $b = s$. The transformation

$$w(x,t) = e^{bt}u(z,t), \quad z = e^{bt}\left(x + \frac{c}{b}\right)$$

leads to a simpler equation of the form 5.1.10.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

⊙ *References for equation* 5.1.16.3: S. I. Svinolupov (1985), V. F. Zaitsev and A. D. Polyanin (1996, pp. 434–435).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) - bx\dfrac{\partial w}{\partial x}$.

This is a special case of equation 5.1.16.5 with $c = s = 0$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^n x + C_2 e^{bt}, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized traveling-wave solutions:

$$w(x,t) = \left(\pm x + C_1 e^{bt}\right)^{2/n}\left[C_2 e^{2bt} + \frac{a(n+2)}{bn}\right]^{-1/n},$$

where C_1 and C_2 are arbitrary constants.

3°. Generalized traveling-wave solutions:

$$w = w(z), \quad z = \pm x + Ce^{bt},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$a(w^n w_z')_z' - bzw_z' = 0.$$

4°. The transformation

$$w(x,t) = u(z,\tau), \quad z = e^{-bt}x, \quad \tau = C - \frac{1}{2b}e^{-2bt},$$

where C is an arbitrary constant, leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^n\frac{\partial u}{\partial z}\right).$$

If $n = -1$ or $n = -2$, see 5.1.10.2 or 5.1.10.3 for solutions of this equation.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + (bx + c)\dfrac{\partial w}{\partial x} + sw.$

This is a special case of equation 5.6.13.12.

The transformation

$$w(x,t) = e^{st}u(z,\tau), \quad z = e^{bt}\left(x + \frac{c}{b}\right), \quad \tau = \frac{1}{2b + sn}e^{(2b+sn)t},$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^n\frac{\partial u}{\partial z}\right).$$

If $n = -1$ or $n = -2$, see 5.1.10.2 or 5.1.10.3 for solutions of this equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 434–435).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + aw^n\dfrac{\partial^2 w}{\partial x^2} + bw^{n-1}\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^n C_2^2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Functional separable solution for $b = \frac{1}{3}n(a-2) - a - 1$:

$$w(x,t) = \left[n\sum_{k=0}^{3}\varphi_k(t)x^k\right]^{1/n}.$$

Here,

$$\varphi_3(t) = A, \quad \varphi_2(t) = \int \psi(t)\,dt + B, \quad \varphi_1(t) = \frac{1}{3A}\left[\int \psi(t)\,dt + B\right]^2 + \frac{1}{2A\beta n}\psi(t),$$

$$\varphi_0(t) = \frac{1}{27A^2}\left[\int \psi(t)\,dt + B\right]^3 + \frac{1}{6A^2\beta n}\psi(t)\left[\int \psi(t)\,dt + B\right] + \frac{1}{12A^2\beta^2 n^2}\psi_t'(t),$$

where the function $\psi = \psi(t)$ is defined implicitly by

$$\int \left(C_1 - \tfrac{8}{3}\beta n\psi^3\right)^{-1/2} d\psi = C_2 + t,$$

A, B, C_1, and C_2 are arbitrary constants, $\beta = a + 1$; $A \neq 0$, $n \neq 0$, $a > -1$.

3°. Functional separable solution for $b = \frac{1}{4}n(a-3) - a - 1$:

$$w(x,t) = \left[n\sum_{k=0}^{4}\varphi_k(t)x^k\right]^{1/n}.$$

Here, the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -\tfrac{3}{4}\beta\varphi_1^2 + 2\beta\varphi_2\varphi_0,\\
\varphi_1' &= -\beta\varphi_1\varphi_2 + 6\beta\varphi_3\varphi_0,\\
\varphi_2' &= -\beta\varphi_2^2 + \tfrac{3}{2}\beta\varphi_1\varphi_3 + 12\beta\varphi_4\varphi_0,\\
\varphi_3' &= -\beta\varphi_2\varphi_3 + 6\beta\varphi_1\varphi_4,\\
\varphi_4' &= -\tfrac{3}{4}\beta\varphi_3^2 + 2\beta\varphi_2\varphi_4,
\end{aligned}$$

where $\beta = n(a+1)$; the prime denotes a derivative with respect to t.

4°. There are exact solutions of the following forms:

$$w(x,t) = F(z), \quad z = Ax + Bt;$$
$$w(x,t) = (At + B)^{-1/n} G(x);$$
$$w(x,t) = t^{\beta} H(\xi), \quad \xi = xt^{-\frac{\beta n+1}{2}};$$
$$w(x,t) = e^{-2t} U(\eta), \quad \eta = xe^{nt};$$
$$w(x,t) = (At + B)^{-1/n} V(\zeta), \quad \zeta = x + C\ln(At + B),$$

where A, B, C, and β are arbitrary constants. The first solution is of the traveling-wave type, the second is a solution in multiplicative separable form, and the third is self-similar.

⊙ *Reference for equation* 5.1.16.6: G. A. Rudykh and E. I. Semenov (1998).

5.1.17. Other Equations

1. $$\boldsymbol{w\frac{\partial w}{\partial t} = w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2 + aw\frac{\partial w}{\partial x} + bw^2.}$$

The substitution $w = e^u$ leads to the linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + b.$$

Remark. In the original equation, the parameters a and b can be replaced by arbitrary functions $a = a(x,t)$ and $b = b(x,t)$.

2. $$\boldsymbol{w\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw\frac{\partial w}{\partial x} + sw^2.}$$

1°. For $a \neq -b$, the substitution

$$w = f^{\frac{a}{a+b}}$$

leads to the linear equation

$$\frac{\partial f}{\partial t} = a\frac{\partial^2 f}{\partial x^2} + c\frac{\partial f}{\partial x} + \frac{s(a+b)}{a}f.$$

2°. For $a = -b$, the substitution $\tau = at$ leads to an equation of the form 5.1.17.1.

Remark. In the original equation, the parameters c and s can be replaced by arbitrary functions $c = c(x,t)$ and $s = s(x,t)$.

3. $$\boldsymbol{w\frac{\partial w}{\partial t} = w\frac{\partial^2 w}{\partial x^2} + a\left(\frac{\partial w}{\partial x}\right)^2 + bw\frac{\partial w}{\partial x} + cw^2 + sw^{1-a}.}$$

1°. For $a \neq -1$, the substitution

$$w = f^{\frac{1}{1+a}}$$

leads to the linear equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + b\frac{\partial f}{\partial x} + c(a+1)f + s(a+1).$$

Remark. In the original equation, the parameters b, c, and s can be replaced by arbitrary functions $b = b(x)$, $c = c(x)$, and $s = s(x)$.

4. $\dfrac{\partial w}{\partial t} = w\dfrac{\partial^2 w}{\partial x^2} + (k-1)\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^{k+1}\dfrac{\partial w}{\partial x}.$

The substitution $w = u^{1/k}$ leads to an unnormalized Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au\frac{\partial u}{\partial x},$$

that can be linearized (see equation 5.1.5.3).

5. $\dfrac{\partial w}{\partial t} = (aw^2 + bw^4)\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 5.6.16.3 with $f(w) = aw^2 + bw^4$.

1°. Self-similar solutions:

$$w(x,t) = \pm\left[\frac{(x+C_1)^2}{2a(t+C_2)} - \frac{a}{b}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

2°. Functional separable solutions:

$$w(x,t) = \pm\left[\varphi(t)(x^2 + C_1x + C_2) - \frac{a}{b}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order separable equation

$$\varphi'_t = -2a\varphi^2 + \tfrac{1}{2}b(4C_2 - C_1^2)\varphi^3,$$

whose solution can be written out in implicit form.

6. $\dfrac{\partial w}{\partial t} = (aw^2 + bw^4)\dfrac{\partial^2 w}{\partial x^2} + cw + kw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{\varphi(t)x^2 + \psi(t)x + \chi(t)},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi'_t &= \tfrac{1}{2}b\varphi(4\varphi\chi - \psi^2) + 2c\varphi,\\
\psi'_t &= \tfrac{1}{2}b\psi(4\varphi\chi - \psi^2) + 2c\psi,\\
\chi'_t &= \tfrac{1}{2}(b\chi + a)(4\varphi\chi - \psi^2) + 2c\chi + 2k.
\end{aligned}$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

Remark. The above remains true if the equation coefficients are arbitrary functions of time: $a = a(t)$, $b = b(t)$, $c = c(t)$, and $k = k(t)$.

7. $\dfrac{\partial w}{\partial t} = ax^{4-k}w^k\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 5.6.16.4 with $f(u) = au^k$.

The transformation

$$w(x,t) = xu(z,t), \quad z = 1/x$$

leads to a simpler equation of the form 5.1.9.26:

$$\frac{\partial u}{\partial t} = au^k\frac{\partial^2 u}{\partial z^2}.$$

8. $\dfrac{\partial w}{\partial t} = ax^n w^k\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x, C_1^k C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The substitution $u = w^{1-k}$ leads to an equation of the form 5.1.17.10:

$$\frac{\partial u}{\partial t} = ax^n\frac{\partial}{\partial x}\left(u^{\frac{k}{1-k}}\frac{\partial u}{\partial x}\right).$$

3°. The transformation

$$w(x,t) = xu(z,t), \quad z = 1/x$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = az^{4-n-k}u^k\frac{\partial^2 u}{\partial z^2}.$$

9. $\dfrac{\partial w}{\partial t} = ax^{\frac{3m+4}{m+1}}\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.1.17.10.

The transformation

$$w(x,t) = x^{\frac{1}{m+1}}u(z,t), \quad z = \frac{1}{x}$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

10. $\dfrac{\partial w}{\partial t} = ax^n\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right).$

This equation occurs in nonlinear problems of heat and mass transfer and is a special case of equation 5.6.17.16 with $f(w) = aw^m$. For $n = 0$, see equation 5.1.10.8.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x, C_1^m C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x) = (Ax + B)^{\frac{1}{m+1}},$$

$$w(x,t) = k(\lambda t + A)^{-\frac{1}{m}} x^{\frac{2-n}{m}}, \quad k = \left[\frac{m\lambda}{a(n-2)(2+m-n-nm)}\right]^{\frac{1}{m}},$$

$$w(x,t) = t^{(1-n)\beta}\left[\frac{m\beta}{a(2-n)}\left(xt^{\beta}\right)^{2-n} + A\right]^{\frac{1}{m}}, \quad \beta = \frac{1}{nm+n-m-2},$$

$$w(x,t) = \exp(-\lambda t)\left[\frac{\lambda}{a}(m+1)^2 x^{\frac{m}{m+1}} \exp(\lambda m t) + A\right]^{\frac{1}{m}}, \quad n = \frac{m+2}{m+1},$$

where A, B, and λ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x),$$

where the function $f = f(x)$ is expressed via solutions to the Emden–Fowler equation

$$F''_{xx} + \frac{\lambda(m+1)}{am} x^{-n} F^{\frac{1}{m+1}} = 0, \qquad F = f^{m+1}. \tag{1}$$

To the power-law particular solution of this equation there corresponds the second solution of the original equation in Item 2°.

The order of equation (1) can be reduced; the equation is analyzed in detail in Polyanin and Zaitsev (2003), where its exact solutions for 26 different pairs of values of the parameters n and m are presented.

4°. Self-similar solution for $n \neq -2$:

$$w = w(z), \quad z = xt^{\frac{1}{n-2}},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$a(2-n)\left(w^m w'_z\right)'_z + z^{1-n} w'_z = 0. \tag{2}$$

The book by Zaitsev and Polyanin (1993) presents the general solution of equation (2) for $m = -1$ and any n.

5°. Self-similar solution:

$$w = t^{\alpha} g(\zeta), \quad \zeta = xt^{\beta}, \qquad \beta = \frac{m\alpha + 1}{n-2}, \quad \alpha \text{ is any},$$

where the function $g(\zeta)$ is determined by the ordinary differential equation

$$a\zeta^n \left(g^m g'_\zeta\right)'_\zeta = \beta\zeta g'_\zeta + \alpha g. \tag{3}$$

This equation is homogeneous, and, hence, its order can be reduced (thereafter it can be transformed to an Abel equation of the second kind).

In the special case

$$\alpha = \frac{1-n}{nm+n-m-2}, \quad \beta = \frac{1}{nm+n-m-2},$$

a first integral of equation (3) is given by

$$ag^m g'_\zeta = \beta\zeta^{1-n} g + C. \tag{4}$$

To $C = 0$ in (4) there corresponds the third solution in Item 2°.

In the general case, the change of variable $G = g^{m+1}$ brings (3) to the equation

$$G''_{\zeta\zeta} = A_1\zeta^{1-n} G^{-\frac{m}{m+1}} G'_\zeta + A_2\zeta^{-n} G^{\frac{1}{m+1}}, \tag{5}$$

where $A_1 = \beta/a$ and $A_2 = \alpha(m+1)/a$. Exact analytical solutions of equation (5) for various values of the parameters n and m can be found in the books by Polyanin and Zaitsev (1995, 2003).

6°. Generalized self-similar solution:

$$w = e^{\lambda(n-2)t}\varphi(u), \quad u = xe^{\lambda mt}, \qquad \lambda \text{ is any},$$

where the function $\varphi(u)$ is determined by the ordinary differential equation

$$au^n(\varphi^m\varphi'_u)'_u = \lambda mu\varphi'_u + \lambda(n-2)\varphi. \tag{6}$$

This equation is homogeneous, so its order can be reduced (thereafter it can be transformed to an Abel equation of the second kind).

In the special case $n = \dfrac{m+2}{m+1}$, equation (6) has the first integral

$$a\varphi^m\varphi'_u = \lambda mu^{-\frac{1}{m+1}}\varphi + C.$$

To $C = 0$ there corresponds the last solution in Item 2°.

In the general case, the change of variable $\Phi = \varphi^{m+1}$ brings (6) to an equation that coincides, up to notation, with (5).

7°. For $n = 2$, there are solutions of the form

$$w = w(\xi), \quad \xi = \ln|x| - \lambda t,$$

which are defined implicitly by

$$a(m+1)\int \frac{w^m\,dw}{aw^{m+1} - \lambda(m+1)w + C_1} = \xi + C_2,$$

where λ, C_1, and C_2 are arbitrary constants. To the special case $C_1 = 0$ there corresponds the solution

$$w(x,t) = \left[\frac{\lambda(m+1)}{a} + C|x|^{\frac{m}{m+1}}\exp\left(-\frac{m\lambda}{m+1}t\right)\right]^{\frac{1}{m}},$$

where C is an arbitrary constant.

8°. The transformation

$$w(x,t) = x^{\frac{1}{m+1}}u(z,t), \quad z = \frac{1}{x}$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = az^{\frac{4+3m-n-nm}{m+1}}\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

⊙ *Reference for equation* 5.1.17.10: V. F. Zaitsev and A. D. Polyanin (1996, pp. 410–412).

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(\dfrac{ax+b}{cw+k}\right)^2 \dfrac{\partial w}{\partial x}\right].$

The substitution $u = \dfrac{cw+k}{ax+b}$ $(c \neq 0)$ leads to an equation of the form 5.1.10.3:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right).$$

⊙ *Reference*: A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981).

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(x^n w^m \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.6.17.5 with $f(x) = ax^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x, C_1^m C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Let $m \neq -1$ and $2m - 2n - nm + 3 \neq 0$. The transformation

$$w(x,t) = x^{\frac{1-n}{m+1}} u(\xi, t), \quad \xi = x^{\frac{2m-2n-nm+3}{m+1}}$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(\xi^{\frac{3m-3n-2nm+4}{2m-2n-nm+3}} u^m \frac{\partial u}{\partial \xi}\right), \tag{1}$$

where $A = a\left(\dfrac{2m-2n-nm+3}{m+1}\right)^2$.

3°. In the special case $n = \dfrac{3m+4}{2m+3}$, the transformed equation is very simple and coincides, up to notation, with equation 5.1.10.8:

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(u^m \frac{\partial u}{\partial \xi}\right).$$

4°. In the special case of $n = 2$ and $m = -2$, the transformed equation becomes

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(u^{-2}\frac{\partial u}{\partial \xi}\right),$$

so it coincides with equation 5.1.10.3 (which can further be reduced to the linear heat equation).

⊙ *Reference for equation* 5.1.17.12: V. F. Zaitsev and A. D. Polyanin (1996, pp. 412–413).

13. $\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n w^m \dfrac{\partial w}{\partial x}\right).$

This equation occurs in nonlinear problems of heat and mass transfer and other applications. For $n = 0$, see equation 5.1.10.8. To $n = 1$ there correspond two-dimensional problems with axial symmetry, and to $n = 2$ there correspond three-dimensional spherically symmetric problems. Equation with $n = 5$ are encountered in the theory of static turbulence. To $m > 0$ there corresponds *slow diffusion* and to $m < 0$, *fast diffusion*.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x, C_1^m C_2^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x) = \left(Ax^{1-n} + B\right)^{\frac{1}{m+1}},$$

$$w(x,t) = \left(\frac{mx^2}{A - kt}\right)^{\frac{1}{m}}, \quad k = 2a(nm + m + 2),$$

$$w(x,t) = \left(A|kt + B|^{-\frac{m(n+1)}{nm+m+2}} - \frac{mx^2}{kt + B}\right)^{\frac{1}{m}}, \quad k = 2a(nm + m + 2),$$

$$w(x,t) = \left[A\exp\left(-\frac{4a\lambda}{m}t\right) + \lambda x^2\right]^{\frac{1}{m}}, \quad n = -\frac{m+2}{m},$$

where A, B, and λ are arbitrary constants.

3°. Solutions with $m = -\dfrac{2}{n+1}$:

$$w(x,t) = e^{-a\lambda(n+1)t}\left[\frac{\lambda}{n+1}\left(C + x^2 e^{-2a\lambda t}\right)\right]^{-\frac{n+1}{2}},$$

$$w(x,t) = t^{\frac{1+n}{1-n}}\left(\frac{\xi^2}{n-1}\ln\frac{\xi}{C}\right)^{-\frac{n+1}{2}}, \quad \xi = xt^{\frac{1+n}{2(1-n)}},$$

where C and λ are arbitrary constants.

4°. Some other solutions with $m = -1$, $m = -\dfrac{1}{2}$, $m = -\dfrac{2}{n+1}$, and $m = \dfrac{n-3}{2}$ can be found in King (1990 *c*) and Ivanova (2008).

⊙ *References for equation* 5.1.17.13: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952, 1989), R. E. Pattle (1959), Ya. B. Zel'dovich and Yu. P. Raiser (1966), J. R. King (1990 *c*), L. I. Sedov (1993), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 413), N. M. Ivanova (2008).

14. $\dfrac{\partial w}{\partial t} = k(ax^2 + bx + c)^m w^{4-2m}\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 5.6.16.5 with $f(u) = ku^{-2m}$.

1°. The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c} \tag{1}$$

leads to the equation

$$\frac{\partial u}{\partial t} = ku^{4-2m}\frac{\partial^2 u}{\partial z^2} + k(ac-\tfrac{1}{4}b^2)u^{5-2m}, \tag{2}$$

which has a traveling-wave solution $u = u(z+\lambda t)$ and a multiplicative separable solution $u = f(t)g(z)$.

Using the change of variable $\varphi = u^{2m-3}$, one obtains from (2) an equation of the form 5.1.11.11,

$$\frac{\partial \varphi}{\partial t} = k\frac{\partial}{\partial z}\left(\varphi^n \frac{\partial \varphi}{\partial z}\right) + p\varphi^{n+1},$$
$$n = \frac{4-2m}{2m-3}, \qquad p = k(2m-3)(ac-\tfrac{1}{4}b^2),$$

which admits a wide class of exact solutions.

2°. By the transformation

$$w(x,t) = \big[v(\xi,t)\big]^{\frac{1}{2m+3}}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}, \tag{3}$$

the original equation can be reduced to equation 5.6.17.5:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)v^{\frac{4-2m}{2m-3}}\frac{\partial v}{\partial \xi}\right], \tag{4}$$

where the function $F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{k}{(ax^2+bx+c)^m}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}. \tag{5}$$

Note some special cases of equation (4) where the function $F = F(\xi)$ can be written out in explicit form:

$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{\cos^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1, \quad a=1, \quad b=0, \quad c=1;$$
$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{\cosh^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1, \quad a=-1, \quad b=0, \quad c=1;$$
$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{v^{-3/2}}{\cos\xi}\frac{\partial v}{\partial \xi}\right), \qquad m=\frac{1}{2}, \quad a=-1, \quad b=0, \quad c=1.$$

⊙ *Reference for equation* 5.1.17.14: V. F. Zaitsev and A. D. Polyanin (1996, pp. 413–414).

15. $\dfrac{\partial w}{\partial t} = ae^{\lambda x}\dfrac{\partial}{\partial x}\left(w^{-1}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial w}{\partial x}.$

For exact solutions to this equations, see Ivanova (2008).

16. $\dfrac{\partial f(t)w}{\partial t} = \dfrac{k}{x^4}\dfrac{\partial}{\partial x}x^5\left[\sqrt{f(t)(1-w)}\,\dfrac{\partial f(t)w}{\partial x}\right].$

Closed von *Kármán–Howarth equation*. It is used for the statistical description of decaying isotropic turbulence in the limit of large Reynolds numbers (see von Kármán and Howarth (1938) and Oberlack and Peters (1993)). Here w is the normalized double correlation function and $f(t) = \overline{u'^2}(t)$ is the turbulence intensity, and $B_{LL} = f(t)w$ is the longitudinal correlation function.

1°. Self-similar solution for $f(t) = t^{-2(\sigma+1)/(\sigma+3)}$ (σ is an arbitrary constant):

$$w = w(\xi), \qquad \xi = xt^{-2/(\sigma+3)}$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\frac{k}{\xi^4}\frac{d}{d\xi}\left(\xi^5\sqrt{1-w}\,\frac{dw}{d\xi}\right) + \frac{2}{\sigma+3}\xi\frac{dw}{d\xi} + 2\frac{\sigma+1}{\sigma+3}w = 0$$

subject to the boundary conditions

$$w(\xi) = 1 \quad \text{at} \quad \xi = 0, \qquad w(\xi) \to 0 \quad \text{as} \quad \xi \to \infty.$$

Special case. For *Loitsianskii's decay law*, which corresponds to $\sigma = 4$, we have a solution in implicit form

$$\xi = \frac{7}{2}k\left[\ln\left(\frac{1+\sqrt{1-w}}{1-\sqrt{1-w}}\right) - 2\sqrt{1-w}\right].$$

Geometric interpretation: in the new variables $x = \xi/(7k)$, $y = \sqrt{w}$ this solution is transformed to the *tractrix equation*

$$x = -\sqrt{1-y^2} + \frac{1}{2}\ln\left(\frac{1+\sqrt{1-y^2}}{1-\sqrt{1-y^2}}\right),$$

arising in the differential geometry. The curve $x = x(y)$ is the element of Beltarmi surface.

⊙ *Reference*: V. N. Grebenev and M. Oberlack (2009).

2°. The substitution $u = f(t)(1-w)$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = \frac{k}{x^4}\frac{\partial}{\partial x}\left(x^5\sqrt{u}\,\frac{\partial u}{\partial x}\right) + f_t'(t). \tag{1}$$

3°. For a linear function $f(t) = at + b$, equation (1) admits a stationary solution

$$u = \left(C_1x^{-4} + C_2 - \frac{3a}{10k}x\right)^{2/3},$$

where C_1 and C_2 are arbitrary constants. The corresponding nonstationary solution of the original equation is expressed as

$$w = 1 - \frac{1}{at+b}\left(C_1x^{-4} + C_2 - \frac{3a}{10k}x\right)^{2/3}. \tag{2}$$

4°. For a suitable $f(t)$, equation (1) admits a generalized separable solution of the form

$$u = [\varphi(t)x + \psi(t)]^2.$$

5.2. Equations with Exponential Nonlinearities

5.2.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 e^{\lambda w} + b_2 e^{2\lambda w}$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda} x + C_2, C_1^{2\lambda} t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution (k and β are arbitrary constants):

$$w = w(z), \qquad z = kx + \beta t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 w''_{zz} - \beta w'_z + be^{\lambda w} = 0.$$

3°. Solution:

$$w = u(\xi) - \frac{1}{\lambda}\ln t, \qquad \xi = \frac{x}{\sqrt{t}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{1}{\lambda} + be^{\lambda u} = 0.$$

⊙ *Reference for equation* 5.2.1.1: N. H. Ibragimov (1994, p. 135), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a + be^{\lambda w}.$

This equation occurs in heat and mass transfer and combustion theory.

1°. Traveling-wave solutions:

$$w(x,t) = -\frac{2}{\lambda}\ln\big[\beta + C\exp\big(\pm\mu x - \tfrac{1}{2}a\lambda t\big)\big],$$
$$w(x,t) = -\frac{2}{\lambda}\ln\big[-\beta + C\exp\big(\pm\mu x - \tfrac{1}{2}a\lambda t\big)\big], \qquad \beta = \sqrt{-\frac{b}{a}},\ \mu = \sqrt{\frac{a\lambda}{2}},$$

where C is an arbitrary constant.

2°. The solutions of Item 1° are special cases of the traveling-wave solutions

$$w = w(z), \qquad z = \pm\mu x + \sigma t,$$

that satisfy the autonomous equation

$$\mu^2 w''_{zz} - \sigma w'_z + a + be^{\lambda w} = 0. \tag{1}$$

For

$$\mu = \sqrt{\tfrac{1}{2}a\lambda}, \qquad \sigma = \mu^2,$$

the general solution of equation (1) can be written out in parametric form as

$$z = 2\int \frac{f'_\tau(\tau)\,d\tau}{f(\tau)\big[\lambda\tau f(\tau) + 2\big]} + C_1, \qquad w = \frac{2}{\lambda}\ln|f(\tau)|,$$

where the function $f(\tau)$ is defined by

$$f(\tau) = \frac{C_2 - 2\ln\big|\tau + \sqrt{\tau^2 + k}\big|}{\lambda\sqrt{\tau^2 + k}}, \qquad k = \frac{4b}{a\lambda^2},$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference for equation* 5.2.1.2: V. F. Zaitsev and A. D. Polyanin (1996, p. 414).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a + be^{\lambda w} + ce^{2\lambda w}$.

Equations of this form are encountered in problems of heat and mass transfer and combustion theory.

1°. Traveling-wave solutions for $a \neq 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\big[\beta + C\exp(\pm\mu x - a\lambda t)\big], \qquad \mu = \frac{1}{\beta}\sqrt{-c\lambda}, \tag{1}$$

where C is an arbitrary constant and the parameter β is determined by solving the quadratic equation

$$a\beta^2 + b\beta + c = 0.$$

2°. Traveling-wave solutions for $a = 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\big(\pm\sqrt{-c\lambda}\,x - b\lambda t + C\big). \tag{2}$$

3°. The substitution $u = e^{-\lambda w}$ leads to an equation with quadratic nonlinearity:

$$u\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 - a\lambda u^2 - b\lambda u - c\lambda.$$

The particular solution $u = \beta + C\exp(\lambda t + \mu x)$ of this equation generates a solution (1).

4°. Solutions (1) and (2) are special cases of a wider class of traveling-wave solutions $w = w(x + \sigma t)$.

⊙ *Reference for equation* 5.2.1.3: V. F. Zaitsev and A. D. Polyanin (1996, p. 415).

5.2.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(w)$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)$.

This equation governs unsteady heat transfer in a quiescent medium in the case where the thermal diffusivity is exponentially dependent on temperature.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_3 t + C_4) + \frac{1}{\lambda}\ln\frac{C_3}{C_1^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \frac{2}{\lambda}\ln\left(\frac{\pm x + A}{\sqrt{B - 2at}}\right),$$
$$w(x,t) = -\frac{1}{\lambda}\ln\big(C - 2a\lambda\mu t\big) + \frac{1}{\lambda}\ln\big(\lambda\mu x^2 + Ax + B\big),$$

where A, B, C, and μ are arbitrary constants. The first solution is self-similar and the second one is an additive separable solution.

3°. Traveling-wave solution in implicit form:

$$x + \beta t + C_1 = a\int \frac{e^{\lambda w}\,dw}{\beta w + C_2}.$$

4°. Self-similar solution:

$$w = w(y), \qquad y = x/\sqrt{t},$$

where the function $w(y)$ is determined by the ordinary differential equation

$$a(e^{\lambda w}w_y')_y' + \tfrac{1}{2}yw_y' = 0.$$

5°. Solution:

$$w(x,t) = U(\xi) + 2kt, \quad \xi = xe^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U = U(\xi)$ is determined by the ordinary differential equation

$$2k - k\lambda\xi U_\xi' = a(e^{\lambda U}U_\xi')_\xi'.$$

6°. Solution:

$$w(x,t) = F(\zeta) - \frac{1}{\lambda}\ln t, \quad \zeta = x + \beta\ln t,$$

where β is an arbitrary constant, and the function $F = F(\zeta)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$-\zeta + \beta\lambda F = a\lambda e^{\lambda F}F_\zeta' + C.$$

7°. Solution:

$$w(x,t) = G(\theta) - \frac{2b+1}{\lambda}\ln t, \quad \theta = xt^b,$$

where b is an arbitrary constant, and the function $G = G(\theta)$ is determined by the ordinary differential equation

$$-\frac{2b+1}{\lambda} + b\theta G'_\theta = (ae^{\lambda G}G'_\theta)'_\theta.$$

8°. The substitution $\varphi = e^{\lambda w}$ leads to an equation of the form 5.1.9.1:

$$\frac{\partial \varphi}{\partial t} = a\varphi\frac{\partial^2 \varphi}{\partial x^2}.$$

⊙ *References for equation* 5.2.2.1: L. V. Ovsiannikov (1959, 1982), N. H. Ibragimov (1994, pp. 110, 119), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^w\dfrac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, t + C_3) - 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solutions:

$$w(x,t) = \ln|C_1x + C_2| + bt + C_3,$$
$$w(x,t) = 2\ln|\pm x + C_1| - \ln\left(C_2e^{-bt} - \frac{2a}{b}\right).$$

The first solution is degenerate.

3°. The transformation

$$w = bt + u(x,\tau), \quad \tau = \frac{1}{b}e^{bt} + \text{const}$$

leads to an equation of the form 5.2.2.1:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(e^u\frac{\partial u}{\partial x}\right).$$

⊙ *Reference for equation* 5.2.2.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 41).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(e^w\dfrac{\partial w}{\partial x}\right) - a^2e^w.$

Solutions:

$$w(x,t) = \ln\left|\frac{\pm A\exp[2(\pm ax + B)] + 2\exp(\pm ax + B) + D}{2a^2(t + C)}\right| \mp ax - B,$$

where A, B, C, and D are arbitrary constants (either the upper or the lower signs must be taken).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{w}\dfrac{\partial w}{\partial x}\right) - be^{w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, C_2 t + C_3) + \ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = u(x) - \ln(aC_1 t + C_2),$$

where C_1 and C_2 are arbitrary constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$u''_{xx} + (u'_x)^2 + C_1 e^{-u} - \frac{b}{a} = 0.$$

Integrating yields the general solution in implicit form:

$$\int \left(C_3 e^{-2u} - 2C_1 e^{-u} + \frac{b}{a}\right)^{-1/2} du = \pm x + C_4.$$

The integral is computable, so the solution can be rewritten in explicit form (if $a = 1$ and $b > 0$, see 5.2.2.3 for a solution).

3°. The substitution $u = e^{w}$ leads to an equation of the form 5.1.9.10:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - bu^2.$$

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(e^{w}\dfrac{\partial w}{\partial x}\right) + ae^{w} + b, \qquad a \neq 0, \quad b \neq 0.$

1°. Additive separable solution for $a = k^2 > 0$:

$$w(x,t) = \ln\left[C_1\cos(kx) + C_2\sin(kx)\right] + bt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Additive separable solution for $a = -k^2 < 0$:

$$w(x,t) = \ln\left[C_1\cosh(kx) + C_2\sinh(kx)\right] + bt + C_3.$$

3°. The transformation

$$w = bt + u(x,\tau), \quad \tau = \frac{1}{b}e^{bt} + \text{const}$$

leads to an equation of the form 5.2.2.4:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x}\left(e^{u}\frac{\partial u}{\partial x}\right) + ae^{u}.$$

⊙ *Reference for equation* 5.2.2.5: A. D. Polyanin and V. F. Zaitsev (2002, pp. 41–42).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-1}x + C_2, C_1^{2\lambda}t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution for $\lambda \neq 0$:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \qquad z = 2\ln|x| + \frac{1-\lambda}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2a\lambda e^{-z}\left[2(e^{u}u_z')_z' - e^{u}u_z'\right] + b\lambda e^{\lambda u} = (1-\lambda)u_z' - 1.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}.$

Functional separable solution:

$$w = \frac{1}{\lambda}\ln\left(c\lambda t - \frac{b\lambda}{2a}x^2 + C_1 x + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w} + c + se^{-\lambda w}.$

This is a special case of equation 5.6.14.4 with $f(t) = c$ and $g(t) = s$.

Functional separable solutions:

$$w = \frac{1}{\lambda}\ln\left\{e^{\alpha t}\left[C_1\cos(x\sqrt{\beta}) + C_2\sin(x\sqrt{\beta})\right] + \gamma\right\} \qquad \text{if } ab\lambda > 0,$$

$$w = \frac{1}{\lambda}\ln\left\{e^{\alpha t}\left[C_1\cosh(x\sqrt{-\beta}) + C_2\sinh(x\sqrt{-\beta})\right] + \gamma\right\} \qquad \text{if } ab\lambda < 0.$$

Here, C_1 and C_2 are arbitrary constants and

$$\alpha = \lambda(b\gamma + c), \qquad \beta = b\lambda/a,$$

where $\gamma = \gamma_{1,2}$ are roots of the quadratic equation $b\gamma^2 + c\gamma + s = 0$.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

5.2.3. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)$

▷ *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

Traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1 x + \frac{a}{\lambda}C_1^2 t + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1 e^{b\lambda t}x + \frac{aC_1^2}{b\lambda^2}e^{2b\lambda t} + C_2 e^{b\lambda t}\right),$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 56).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{w}\dfrac{\partial w}{\partial x}\right) + b(w+2).$

Functional separable solution:

$$w(x,t) = \ln\left[C_1 e^{2bt} - \frac{b}{2a}(x+C_2)^2\right],$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{-\lambda w}.$

Traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left[C_1 x + \left(\frac{aC_1^2}{\lambda} + b\lambda\right)t + C_2\right],$$

where C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1 e^{b\lambda t}x + \frac{aC_1^2}{b\lambda^2}e^{2b\lambda t} + C_2 e^{b\lambda t} - \frac{c}{b}\right),$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 57).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(awe^{\lambda w} + be^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right] + b + ce^{-\lambda w}.$

The substitution $w = u - b/a$ leads to an equation of the form 5.2.3.5:

$$\frac{\partial u}{\partial t} = ae^{-b\lambda/a}\frac{\partial}{\partial x}\left(ue^{\lambda u}\frac{\partial u}{\partial x}\right) + b + ce^{b\lambda/a}e^{-\lambda u}.$$

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(ae^{2\lambda w} + bwe^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right].$

Self-similar solutions:

$$w(x,t) = \frac{1}{\lambda}\ln\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}} - \frac{b}{a\lambda}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(ae^{2\lambda w} + bwe^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right] + c.$

Generalized traveling-wave solutions:

$$w(x,t) = \frac{1}{\lambda}\ln\left[\pm\varphi(t)x + C_1\varphi(t) + \frac{b}{\lambda}\varphi(t)\int\varphi(t)\,dt\right], \qquad \varphi(t) = \left(C_2e^{-2c\lambda t} - \frac{a}{c\lambda}\right)^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[w^n\exp\left(\lambda w^n\right)\dfrac{\partial w}{\partial x}\right] + bw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left(\frac{1}{\lambda}\ln z\right)^{1/n}, \qquad z = C_1e^{bn\lambda t}x + \frac{aC_1^2}{bn^2\lambda^2}e^{2bn\lambda t} + C_2e^{bn\lambda t},$$

where C_1 and C_2 are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}e^{1/w}\dfrac{\partial w}{\partial x}\right).$

Solution:

$$w = \frac{z^2 + c}{2at},$$

where

$$z = \frac{x}{\ln\frac{x^2+c}{2e^2at} + \frac{2\sqrt{c}}{x}\arctan\frac{x}{\sqrt{c}} + \frac{b}{x}} \quad \text{if} \quad c > 0,$$

$$z = \frac{x}{\ln\frac{x^2+c}{2e^2at} + \frac{2\sqrt{-c}}{x}\ln\frac{x-\sqrt{-c}}{x+\sqrt{-c}} + \frac{b}{x}} \quad \text{if} \quad c < 0,$$

where b and c are arbitrary constants.

⊙ *References*: I. Tsyfra, A. Messina, A. Napoli, and V. Tretynyk (2004), N. M. Ivanova (2008).

5.2.4. Other Equations Explicitly Independent of x and t

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \beta\dfrac{\partial w}{\partial x} + a + be^{\lambda w} + ce^{2\lambda w}.$

On passing from t, x to the new variables t, $z = x + \beta t$, one arrives at a simpler equation of the form 5.2.1.3:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + a + be^{\lambda w} + ce^{2\lambda w}.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.3.8 with $f(w) = be^{\lambda w}$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3) + \frac{1}{\lambda}\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Apart from the traveling wave $w = w(x + \lambda t)$, there is also an exact solution of the form

$$w = \varphi(\xi) - \frac{1}{2\lambda}\ln t, \qquad \xi = \frac{x}{\sqrt{t}}.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\lambda w}\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = ae^{\lambda w}$.

The substitution

$$u = \int \exp\left(\frac{a}{\lambda}e^{\lambda w}\right) dw$$

leads to the linear heat equation $\partial_t u = \partial_{xx} u$.

4. $\dfrac{\partial w}{\partial t} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x + C_2, C_3 t + C_4) + \frac{1}{\lambda}\ln\frac{C_3}{C_1^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x, t) = \frac{1}{\lambda}\ln\left[C_2\exp\left(\frac{C_1}{ak^2}z\right) + \frac{\beta}{C_1}\right], \qquad z = kx + \beta t,$$

where C_1, C_2, k, and β are arbitrary constants.

3°. Additive separable solutions:

$$w(x,t)=\frac{1}{\lambda}\ln\left[\frac{\cos^2(C_2x+C_3)}{2C_2^2(at+C_1)}\right],$$
$$w(x,t)=\frac{1}{\lambda}\ln\left[\frac{\sinh^2(C_2x+C_3)}{2C_2^2(at+C_1)}\right],$$
$$w(x,t)=\frac{1}{\lambda}\ln\left[\frac{\cosh^2(C_2x+C_3)}{2C_2^2(C_1-at)}\right],$$

where C_1, C_2, and C_3 are arbitrary constants; note that $\ln(A/B)=\ln|A|-\ln|B|$ for $AB>0$.

4°. Self-similar solution:

$$w=w(y),\qquad y=x/\sqrt{t},$$

where the function $w(y)$ is determined by the ordinary differential equation

$$ae^{\lambda w}w''_{yy}+\tfrac{1}{2}yw'_y=0.$$

5°. Solution:

$$w(x,t)=U(\xi)+2kt,\quad \xi=xe^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U=U(\xi)$ is determined by the ordinary differential equation

$$2k-k\lambda\xi U'_\xi=ae^{\lambda U}U''_{\xi\xi}.$$

6°. Solution:

$$w(x,t)=F(\zeta)-\frac{1}{\lambda}\ln t,\quad \zeta=x+\beta\ln t,$$

where β is an arbitrary constant, and the function $F=F(\zeta)$ is determined by the autonomous ordinary differential equation

$$\beta\lambda F'_\zeta-1=a\lambda e^{\lambda F}F''_{\zeta\zeta}.$$

7°. Solution:

$$w(x,t)=G(\theta)-\frac{2b+1}{\lambda}\ln t,\quad \theta=xt^b,$$

where b is an arbitrary constant, and the function $G=G(\theta)$ is determined by the ordinary differential equation

$$-\frac{2b+1}{\lambda}+b\theta G'_\theta=ae^{\lambda G}G''_{\theta\theta}.$$

5. $\dfrac{\partial w}{\partial t}=ae^{w}\dfrac{\partial^2 w}{\partial x^2}+be^{\lambda w}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1^{\lambda-1}x+C_2,C_1^{2\lambda}t+C_3)+2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = kx + \beta t,$$

where k and β are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^2 e^w w''_{\xi\xi} - \beta w'_\xi + be^{\lambda w} = 0.$$

3°. Solution for $\lambda \neq 0$:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \qquad z = 2\ln x + \frac{1-\lambda}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2a\lambda e^{u-z}\left(2u''_{zz} - u'_z\right) + b\lambda e^{\lambda u} = (1-\lambda)u'_z - 1.$$

4°. Additive separable solution for $\lambda = 1$:

$$w(x,t) = -\ln(kt + C) + \varphi(x),$$

where the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$a\varphi''_{xx} + b + ke^{-\varphi} = 0.$$

5°. Additive separable solutions for $\lambda = 0$:

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\frac{\cos^2(C_2x + C_3)}{a - C_1e^{-bt}}\right],$$

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\frac{\sinh^2(C_2x + C_3)}{a - C_1e^{-bt}}\right],$$

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\frac{\cosh^2(C_2x + C_3)}{C_1e^{-bt} - a}\right],$$

where C_1, C_2, and C_3 are arbitrary constants; note that $\ln(A/B) = \ln|A| - \ln|B|$ for $AB > 0$.

6. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\left(e^{C_1}x + \frac{a}{\lambda}C_1e^{C_1}t + C_2, e^{C_1}t + C_3\right) - \frac{1}{\lambda}C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$2b\int \frac{e^{\lambda w}\,dw}{aw^2 + 2\beta w + C_1} = x + \beta t + C_2,$$

where C_1, C_2, and β are arbitrary constants.

3°. Solution:

$$w(x,t) = u(z) + \frac{1}{\lambda}\ln t, \qquad z = \frac{x}{t} - \frac{a}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\left(au - z - \frac{a}{\lambda}\right)u'_z + \frac{1}{\lambda} = b(e^\lambda uu'_z)'_z.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\beta w}\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(e^{(\beta-\lambda)k}x + A, e^{(2\beta-\lambda)k}t + B\big) + k,$$

where A, B, and k are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$a\int \frac{e^{\lambda w}dw}{C_1 w + C_2 - (b/\beta)e^{\beta w}} = x + C_1 t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solution with $\lambda \neq 2\beta$:

$$w = U(\zeta) + \frac{1}{\lambda - 2\beta}\ln t, \qquad \zeta = xt^{\frac{\beta-\lambda}{\lambda-2\beta}},$$

where the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$\frac{1}{\lambda-2\beta}\big[(\beta-\lambda)\zeta U'_\zeta + 1\big] = a(e^{\lambda U}U'_\zeta)'_\zeta + be^{\beta U}U'_\zeta.$$

4°. Additive separable solution with $\lambda = 2\beta$:

$$w = \varphi(t) + \frac{1}{\beta}\ln x,$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\beta\varphi'_t = ae^{2\beta\varphi} + be^{\beta\varphi}.$$

⊙ *References for equation* 5.2.4.7: M. P. Edwards (1994), N. M. Ivanova (2008).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\big(ae^{2w} + ce^{w}\big)\dfrac{\partial w}{\partial x}\right] + \big(ke^{w} + s\big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.2.4.10 with $b = 0$ and $\lambda = 1$.

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\big(ae^{2w} + bwe^{w}\big)\dfrac{\partial w}{\partial x}\right] + \big(ke^{w} + s\big)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.2.4.10 with $c = 0$ and $\lambda = 1$.

Generalized traveling-wave solution:

$$w(x,t) = \ln\left[\varphi(t)x + (st + C_1)\varphi(t) + b\varphi(t)\int \varphi(t)\,dt\right],$$

where C_1 is an arbitrary constant, and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi'_t = a\varphi^3 + k\varphi^2,$$

whose general solution can be written out in implicit form

$$-\frac{1}{k\varphi}+\frac{a}{k^2}\ln\left|\frac{a\varphi+k}{\varphi}\right|+C_2=t.$$

Here C_2 is an arbitrary constant and $k\neq 0$.

In special cases, we have

$$\varphi(t)=(C_2-2at)^{-1/2} \quad \text{if } k=0,$$
$$\varphi(t)=(C_2-kt)^{-1} \quad \text{if } a=0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 60).

10. $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[\left(ae^{2\lambda w}+bwe^{\lambda w}+ce^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right]+\left(ke^{\lambda w}+s\right)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w=\frac{1}{\lambda}\ln\left[\varphi(t)x+(st+C_1)\varphi(t)+\frac{b}{\lambda}\varphi(t)\int\varphi(t)\,dt\right],$$

where C_1 is an arbitrary constant, and the function $\varphi=\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi_t'=a\varphi^3+k\varphi^2.$$

For solutions to this equation, see the previous equation 5.2.4.9.

5.2.5. Equations Explicitly Dependent on x and/or t

1. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+ce^{\lambda w+bx+ct}.$

This is a special case of equation 5.6.1.2 with $f(z,w)=ce^{z+\lambda w}$.

2. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2+be^{\beta x+\mu t-\lambda w}.$

This is a special case of equation 5.6.4.9 with $f(x,t)\equiv 0$ and $g(x,t)=be^{\beta x+\mu t}$.

3. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)+bt^n.$

This is a special case of equation 5.6.14.1 with $f(t)=bt^n$.

4. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)+be^{\mu t}.$

This is a special case of equation 5.6.14.1 with $f(t)=be^{\mu t}$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w} + ce^{\mu t}.$

This is a special case of equation 5.6.14.4 with $f(t) = ce^{\mu t}$ and $g(t) = 0$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + bt^n e^{-\lambda w}.$

This is a special case of equation 5.6.14.2 with $f(t) = 0$ and $g(t) = bt^n$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{-\lambda w+\mu t}.$

This is a special case of equation 5.6.14.2 with $f(t) = 0$ and $g(t) = be^{\mu t}$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\mu t} + ce^{-\lambda w+\nu t}.$

This is a special case of equation 5.6.14.2 with $f(t) = be^{\mu t}$ and $g(t) = ce^{\nu t}$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + (bx + c)e^{\lambda w}.$

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where C is an arbitrary constant and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda(bx + c)\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w+\mu x}.$

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where C is an arbitrary constant and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda be^{\mu x}\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(x^n e^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.6.17.12 with $f(x) = ax^n$.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w+\mu x}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 5.6.17.12 with $f(x) = ae^{\mu x}$.

5.3. Equations with Hyperbolic Nonlinearities

5.3.1. Equations Involving Hyperbolic Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\cosh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\cosh^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\cosh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\cosh^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\cosh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\cosh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cosh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\cosh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cosh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + 2b\cosh(\lambda w) + c\cosh^k(\beta t).$

This is a special case of equation 5.6.14.4 with $f(t) = c\cosh^k(\beta t)$ and $g(t) = b$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cosh^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\cosh^2(\beta w)$.

Self-similar solutions:

$$w(x, t) = \frac{1}{\beta}\operatorname{arcsinh}\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cosh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\cosh^k(\beta w)$.

5.3.2. Equations Involving Hyperbolic Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\sinh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\sinh^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\sinh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\sinh^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\sinh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\sinh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\sinh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\sinh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\sinh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + 2b\sinh(\lambda w) + c\sinh^k(\beta t).$

This is a special case of equation 5.6.14.4 with $f(t) = c\sinh^k(\beta t)$ and $g(t) = -b$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sinh^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\sinh^2(\beta w)$.

Self-similar solutions:

$$w(x, t) = \frac{1}{\beta}\operatorname{arccosh}\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sinh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\sinh^k(\beta w)$.

5.3.3. Equations Involving Hyperbolic Tangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\tanh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\tanh^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\tanh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\tanh^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\tanh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\tanh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\tanh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\tanh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\tanh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\tanh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\tanh^k(\beta w)$.

5.3.4. Equations Involving Hyperbolic Cotangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\coth^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\coth^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\coth^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\coth^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\coth^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\coth^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\coth^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\coth^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\coth^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\coth^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\coth^k(\beta w)$.

5.4. Equations with Logarithmic Nonlinearities

5.4.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x, t, w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\ln w.$

This is a special case of equation 5.6.1.1 with $f(w) = b\ln w$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw\ln w.$

This is a special case of equation 5.6.1.1 with $f(w) = aw\ln w$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{at})w(\pm x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solutions:

$$w(x, t) = \exp\left(Ae^{at}x + \frac{A^2}{a}e^{2at} + Be^{at}\right),$$
$$w(x, t) = \exp\left[\tfrac{1}{2} - \tfrac{1}{4}a(x + A)^2 + Be^{at}\right],$$
$$w(x, t) = \exp\left[-\frac{a(x + A)^2}{4(1 + Be^{-at})} + \frac{1}{2B}e^{at}\ln(1 + Be^{-at}) + Ce^{at}\right],$$

where A, B, and C are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x, t) = \exp\left[Ae^{at} + f(x)\right], \tag{1}$$

where the function $f(x)$ is defined implicitly by

$$\int \left(Be^{-2f} - af + \tfrac{1}{2}a\right)^{-1/2} df = \pm x + C. \tag{2}$$

Relations (1) and (2) involve three arbitrary constants, A, B, and C.

4°. There are more complicated solutions of the form

$$w(x, t) = \exp\left[Ae^{at} + f(x + bt)\right],$$

where the function $f(\xi)$ is determined by the autonomous ordinary differential equation

$$f''_{\xi\xi} + (f'_\xi)^2 - bf'_\xi + af = 0.$$

⊙ *References for equation* 5.4.1.2: V. A. Dorodnitsyn (1979, 1982), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), A. D. Polyanin and V. F. Zaitsev (2002, p. 46).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw\ln w + bw.$

The substitution $w = e^{-b/a}u$ leads to an equation of the form 5.4.1.2:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au\ln u.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw\ln w + (bx + c)w.$

This is a special case of equations 5.6.1.5 and 5.6.1.7.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw\ln w + (bx + ct + k)w.$

This is a special case of equation 5.6.1.7.

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw\ln w + (bx^2 + cx + k)w.$

This is a special case of equation 5.6.1.9.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a(w + b)\ln^2(w + b).$

1°. The substitution $w = e^u - b$ leads to an equation of the form 5.1.7.4:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + au^2. \tag{1}$$

2°. Solutions of equation (1) for $a < 0$:

$$u(x,t) = C_1\exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$
$$u(x,t) = \frac{1}{C_1 - at} + \frac{C_2}{(C_1 - at)^2}\exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$

where C_1 and C_2 are arbitrary constants. The first solution is a traveling-wave solution and the second one is a generalized separable solution.

3°. Equation (1) has also generalized separable solutions of the following forms:

$$u(x,t) = \varphi(t) + \psi(t)\bigl[A\exp(x\sqrt{-a}\,) + B\exp(x\sqrt{-a}\,)\bigr] \quad \text{if} \quad a < 0,$$
$$u(x,t) = \varphi(t) + \psi(t)\bigl[A\sin(x\sqrt{a}\,) + B\cos(x\sqrt{a}\,)\bigr] \quad \text{if} \quad a > 0.$$

For details, see 5.1.7.4.

⊙ *References for equation* 5.4.1.7: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin and V. F. Zaitsev (2002, p. 47).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + (1 + kw)\bigl[a\ln^2(1 + kw) + b\ln(1 + kw) + c\bigr].$

This is a special case of equation 5.6.1.10.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

5.4.2. Other Equations

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x} + bw\ln w.$

On passing from t, x to the new variables t, $z = x + at$, one obtains a simpler solution of the form 5.4.1.2:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + bw\ln w.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt\dfrac{\partial w}{\partial x} + cw\ln w.$

This is a special case of equation 5.6.2.6 with $f(t) = 0$, $g(t) = bt$, $h(t) = c$, and $p(t) = s(t) = 0$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + cw\ln w.$

This is a special case of equation 5.6.2.6 with $f(t) = b$, $g(t) = 0$, $h(t) = c$, and $p(t) = s(t) = 0$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{a}{x^k}\dfrac{\partial}{\partial x}\left(x^k\dfrac{\partial w}{\partial x}\right) + bw\ln w.$

The values $k = 1$ and $k = 2$ correspond to problems with axial and central symmetry, respectively.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{bt})\, w(\pm x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution:

$$w(x,t) = \exp\left[-\frac{bx^2}{4a(1 + Ae^{-bt})} + Be^{bt} + \frac{1}{2A}(k+1)e^{bt}\ln(1 + Ae^{-bt})\right],$$

where A and B are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,t) = \exp\left(Ae^{bt}\right)\theta(x),$$

where A is an arbitrary constant and the function $\theta(x)$ is determined by the second-order ordinary differential equation

$$\frac{a}{x^k}\frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) + b\theta\ln\theta = 0.$$

⊙ *References for equation* 5.4.2.4: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), A. D. Polyanin and V. F. Zaitsev (2002, pp. 47–48).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = e^{C_1} w(x + bC_1 t + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \exp\left[C_1 \exp\left(-\frac{b}{a}x + b^2 C_2 t\right) + 1 - aC_2 - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

3°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{C_1 - x}{b(t + C_2)} + \frac{a}{b^2}\frac{\ln|t + C_2|}{t + C_2} - \frac{c}{b}\right].$$

4°. Solution:

$$w(x,t) = e^{\lambda t} u(z), \qquad z = x + \tfrac{1}{2}b\lambda t^2 + kt,$$

where k and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$a u''_{zz} + (b\ln u + c - k)u'_z - \lambda u = 0.$$

To $\lambda = 0$, there corresponds a traveling-wave solution.

⊙ *Reference for equation* 5.4.2.5: A. D. Polyanin and V. F. Zaitsev (2004, p. 67).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a\ln^k(bw)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = a\ln^k(bw)$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = Aw(A^{-n}x + bA^{-n}\ln A\, t + B, A^{-n}t + C),$$

where A, B, and C are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$a\int \frac{w^n\, dw}{bw\ln w + (c - b + C_1)w + C_2} = -x + C_1 t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solution:

$$w = t^{1/n}U(z), \qquad z = \frac{x}{t} + \frac{b}{n}\ln t,$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$a(U^n U'_z)'_z + \left(b\ln U + z + c - \frac{b}{n}\right)U'_z - \frac{1}{n}U = 0.$$

⊙ *References for equation* 5.4.2.7: C. M. Yung, K. Verburg, and P. Baveye (1994), N. M. Ivanova (2008).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a\ln w + b)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\ln w + b$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \exp\left(\pm\sqrt{2C_1 x + 2aC_1^2 t + C_2} - \frac{b}{a}\right),$$

$$w(x,t) = \exp\left(\frac{C_2 \pm x}{\sqrt{C_1 - 2at}} - \frac{a+b}{a}\right),$$

where C_1 and C_2 are arbitrary constants. The first solution represents a traveling wave and the second one is self-similar.

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a\ln w + b)\dfrac{\partial w}{\partial x}\right] + cw.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{C_2 \pm x}{\sqrt{C_1 - 2at}} - \frac{c}{3a}(C_1 - 2at) - \frac{a+b}{a}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 68).

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a\ln w + b)\dfrac{\partial w}{\partial x}\right] + cw\ln w + sw.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\varphi(t)(C_1 \pm x) + (a+b)\varphi(t)\int \varphi(t)\,dt + s\varphi(t)\int \frac{dt}{\varphi(t)}\right],$$

$$\varphi(t) = \left(C_2 e^{-2ct} - \frac{a}{c}\right)^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

11. $w\dfrac{\partial w}{\partial t} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2 + cw^2\ln w.$

The substitution $w = e^u$ leads to the linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu + b.$$

Remark. In the original equation, the parameters a, b, and c can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

5.5. Equations with Trigonometric Nonlinearities

5.5.1. Equations Involving Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\cos^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\cos^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\cos^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\cos^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\cos^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\cos^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cos^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\cos^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cos^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\cos^2(\lambda w + \beta)\dfrac{\partial^2 w}{\partial x^2}.$

The substitution $u = \tan(\lambda w + \beta)$ leads to an equation of the form 5.1.13.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{u^2 + 1}\frac{\partial u}{\partial x}\right).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cos^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\cos^2(\beta w)$.

Self-similar solutions:

$$w(x, t) = \frac{1}{\beta}\arcsin\frac{\pm x + C_1}{\sqrt{2at + C_2}},$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cos^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\cos^k(\beta w)$.

5.5.2. Equations Involving Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w).$

This is a special case of equation 5.6.1.1 with $f(w) = b\sin^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\sin^k(\lambda w + bx + ct).$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\sin^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\sin^k(\lambda w).$

This is a special case of equation 5.6.2.1 with $f(w) = s\sin^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 5.6.6.8 with $f(w) = b\sin^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\sin^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\sin^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\sin^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\sin^2(\lambda w)\dfrac{\partial^2 w}{\partial x^2}.$

The substitution $u = \cot(\lambda w)$ leads to an equation of the form 5.1.13.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{u^2+1}\frac{\partial u}{\partial x}\right).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sin^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\sin^2(\beta w)$.

Self-similar solutions:

$$w(x,t) = \frac{1}{\beta}\arccos\frac{\pm x + C_1}{\sqrt{2at + C_2}},$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sin^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\sin^k(\beta w)$.

5.5.3. Equations Involving Tangent

1. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w).$$

This is a special case of equation 5.6.1.1 with $f(w) = b\tan^k(\lambda w)$.

2. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \beta\tan^k(\lambda w + bx + ct).$$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\tan^k(z + \lambda w)$.

3. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx + c)\frac{\partial w}{\partial x} + s\tan^k(\lambda w).$$

This is a special case of equation 5.6.2.1 with $f(w) = s\tan^k(\lambda w)$.

4. $$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w)\left(\frac{\partial w}{\partial x}\right)^2.$$

This is a special case of equation 5.6.6.8 with $f(w) = b\tan^k(\lambda w)$.

5. $$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w)\left(\frac{\partial w}{\partial x}\right)^2 + c\tan^k(\beta t)\frac{\partial w}{\partial x}.$$

This is a special case of equation 5.6.6.10 with $f(w) = b\tan^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\tan^k(\beta t)$.

6. $$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left[\tan^k(\beta w)\frac{\partial w}{\partial x}\right].$$

This is a special case of equation 5.6.15.1 with $f(w) = a\tan^k(\beta w)$.

5.5.4. Equations Involving Cotangent

1. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w).$$

This is a special case of equation 5.6.1.1 with $f(w) = b\cot^k(\lambda w)$.

2. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \beta\cot^k(\lambda w + bx + ct).$$

This is a special case of equation 5.6.1.2 with $f(z, w) = \beta\cot^k(z + \lambda w)$.

3. $$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx + c)\frac{\partial w}{\partial x} + s\cot^k(\lambda w).$$

This is a special case of equation 5.6.2.1 with $f(w) = s\cot^k(\lambda w)$.

4. $$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w)\left(\frac{\partial w}{\partial x}\right)^2.$$

This is a special case of equation 5.6.6.8 with $f(w) = b\cot^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cot^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 5.6.6.10 with $f(w) = b\cot^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cot^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cot^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 5.6.15.1 with $f(w) = a\cot^k(\beta w)$.

5.5.5. Equations Involving Inverse Trigonometric Functions

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{e^{\lambda \arctan w}}{1+w^2}\dfrac{\partial w}{\partial x}\right).$

Solution in parametric form:

$$w = \tan\left(\varphi(z) + \arctan\left(2z\varphi'_z\right) - \frac{1}{\lambda}\ln t\right),$$
$$x^2 = z\cos^2\left(\varphi(z) + \frac{1}{\lambda}\ln t\right),$$

where z is the parameter and the function $\varphi = \varphi(z)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$\varphi'_z = \frac{1}{2z}\tan\left(\frac{1}{\lambda}\ln\frac{C-z}{2} - \varphi\right) - \frac{1}{\lambda(C-z)}.$$

⊙ *Reference*: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989, pp. 29–30).

2. $\dfrac{\partial w}{\partial t} = \left[\left(\dfrac{\partial w}{\partial x}\right)^2 + 1\right]^{-1}\exp\left[k\arctan\left(\dfrac{\partial w}{\partial x}\right)\right]\dfrac{\partial^2 w}{\partial x^2}, \qquad k \neq 0.$

Solution:

$$w^2 = u(z) - x^2, \qquad z = t\exp\left[-k\arctan(x/w)\right],$$

where the function $u(z)$ is determined by the ordinary differential equation

$$2k^2z^2uu''_{zz} - k^2z(3zu'_z - 2u)u'_z - 4u^2 - \tfrac{1}{2}(k^2z^2u'^2_z + 4u^2)u'_z\exp\left[k\arctan\left(\tfrac{1}{2}kzu^{-1}u'_z\right)\right] = 0.$$

⊙ *Reference*: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989, pp. 29–33).

5.6. Equations Involving Arbitrary Functions

5.6.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w).$

Kolmogorov–Petrovskii–Piskunov equation. Equations of this form are often encountered in various problems of heat and mass transfer (with f being the rate of a volume chemical reaction), combustion theory, biology, and ecology. For $f = f(w)$ having power-law, exponential, or logarithmic form, see Subsections 5.1.1 to 5.1.3, equations 5.2.1.1 to 5.2.1.3, or equations 5.4.1.2, 5.4.1.3, 5.4.1.7, and 5.4.1.8, respectively.

1°. The equation has an implicit nonstationary solution independent of the space variable:

$$\int \frac{dw}{f(w)} = t + C, \qquad C \text{ is an arbitrary constant.}$$

2°. Stationary solution in implicit form:

$$\int \left[C_1 - \frac{2}{a} \int f(w)\, dw \right]^{-1/2} dw = C_2 \pm x.$$

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where λ is an arbitrary constant. The function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$a w''_{zz} - \lambda w'_z + f(w) = 0. \tag{1}$$

The transformation

$$\xi = (\lambda/a)z, \quad U(w) = w'_\xi$$

brings (1) to the Abel equation

$$U U'_w - U + a\lambda^{-2} f(w) = 0. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents a considerable number of solutions to equation (2) for various $f = f(w)$.

⊙ *References for equation* 5.6.1.1: A. N. Kolmogorov, I. G. Petrovskii, and I. S. Piskunov (1937), Ya. B. Zel'dovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze (1985).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(bx + ct, w).$

Solution:

$$w = w(\xi), \quad \xi = bx + ct,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$ab^2 w''_{\xi\xi} - c w'_\xi + f(\xi, w) = 0.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{t} f\left(\dfrac{x}{\sqrt{t}}, w\right).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(Cx, C^2 t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. The transformation

$$\tau = \ln t, \quad \xi = \frac{x}{\sqrt{t}}$$

leads to the equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial \xi^2} + \frac{1}{2}\xi\frac{\partial w}{\partial \xi} + f(\xi, w),$$

which admits exact solutions of the form $w = w(\xi)$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + f(t)w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \exp\bigl(C_1 e^{bt}\bigr)\,w\bigl(\pm x + C_2, t\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{a}{b}A^2e^{2bt} + e^{bt}\int e^{-bt}f(t)\,dt\right],$$

where A and B are arbitrary constants.

3°. Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = \frac{be^{bt}}{A - 4ae^{bt}}, \quad \psi(t) = Be^{bt} + e^{bt}\int e^{-bt}\bigl[2a\varphi(t) + f(t)\bigr]\,dt,$$

where A and B are arbitrary constants.

4°. There are also functional separable solutions of the more general form

$$w(x,t) = \exp\bigl[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\bigr],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 5.6.1.9) that can be integrated.

5°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt}f(t)\,dt + \Phi(x+\lambda t)\right],$$

where A and λ are arbitrary constants, and the function $\Phi = \Phi(z)$ is determined by the autonomous ordinary differential equation

$$a\Phi''_{zz} + a(\Phi'_z)^2 - \lambda\Phi'_z + b\Phi = 0,$$

the order of which can be reduced by one.

6°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}f(t)\,dt\right]u(x,t)$$

leads to a simpler equation of the form 5.4.1.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\ln u.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + \big[f(x) + g(t)\big]w.$

1°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}g(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b\varphi\ln\varphi + f(x)\varphi = 0.$$

2°. With the substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}g(t)\,dt\right]u(x,t)$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\ln u + f(x)u.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + g(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\big[\Phi(t)x + \Psi(t)\big],$$

where the functions $\Phi(t)$ and $\Psi(t)$ are given by

$$\Phi(t) = Ae^F, \quad \Psi(t) = Be^F + e^F\int e^{-F}(aA^2e^{2F} + g)\,dt, \qquad F = \int f\,dt,$$

and A and B are arbitrary constants.

2°. Functional separable solution:

$$w(x,t) = \exp\big[\varphi(t)x^2 + \psi(t)\big],$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = e^F\left(A - 4a\int e^F dt\right)^{-1}, \quad F = \int f\,dt,$$
$$\psi(t) = Be^F + e^F\int e^{-F}(2a\varphi + g)\,dt,$$

and A and B are arbitrary constants.

3°. There are also functional separable solutions of the more general form

$$w(x,t) = \exp\big[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\big],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 5.6.1.9), which can be integrated.

⊙ *Reference for equation* 5.6.1.6: V. F. Zaitsev and A. D. Polyanin (1996, pp. 419–420).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + \big[g(t)x + h(t)\big]w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\big[\varphi(t)x + \psi(t)\big],$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F + e^F\int e^{-F}g\,dt, \quad F = \int f\,dt,$$

$$\psi(t) = Be^F + e^F\int e^{-F}(a\varphi^2 + h)\,dt,$$

and A and B are arbitrary constants.

2°. There are also functional separable solutions of the form

$$w(x,t) = \exp\big[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\big],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 5.6.1.9), which can be integrated.

⊙ *Reference for equation* 5.6.1.7: V. F. Zaitsev and A. D. Polyanin (1996, p. 420).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\exp\big[\varphi(x)\big],$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + a(\varphi'_x)^2 + f(x)\varphi + g(x) + b = 0.$$

For f, g = const, this equation can be reduced by the substitution $u(\varphi) = (\varphi'_x)^2$ to a first-order linear equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 420–421).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + \big[g(t)x^2 + h(t)x + s(t)\big]w.$

Functional separable solution:

$$w(x,t) = \exp\big[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\big],$$

where the functions $\varphi_n(t)$ $(n = 1,\ 2,\ 3)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_2 = 4a\varphi_2^2 + f\varphi_2 + g, \tag{1}$$

$$\varphi'_1 = 4a\varphi_2\varphi_1 + f\varphi_1 + h, \tag{2}$$

$$\varphi'_0 = f\varphi_0 + a\varphi_1^2 + 2a\varphi_2 + s. \tag{3}$$

Here, the arguments of the functions f, g, h, and s are not specified, and the prime denotes a derivative with respect to t.

Equation (1) for $\varphi_2 = \varphi_2(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977), Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f and g.

Given a solution of equation (1), the solutions of equations (2) and (3) can be constructed successively, due to the linearity of each of them in the unknown.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 421).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw + c)\left[k\ln^2(bw + c) + f(t)\ln(bw + c) + g(t)\right].$

The substitution

$$bw + c = \exp u, \quad u = u(x, t)$$

leads to an equation of the form 5.6.6.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + a\left(\frac{\partial u}{\partial x}\right)^2 + bku^2 + bf(t)u + bg(t),$$

which has exponential and sinusoidal solutions with respect to x.

5.6.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x, t)\dfrac{\partial w}{\partial x} + g(x, t, w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + f(w).$

This equation governs unsteady mass transfer with a volume chemical reaction in an inhomogeneous fluid flow.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + C_1e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + (bz + c)w'_z + f(w) = 0.$$

⊙ *Reference for equation* 5.6.2.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 56).

2. $\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + f(t)w\ln w.$

This equation can be rewritten as

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{an}{x}\frac{\partial w}{\partial x} + f(t)w\ln w.$$

Functional separable solution:

$$w(x, t) = \exp\left[\varphi(t)x^2 + \psi(t)\right],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4a\varphi^2 + f\varphi,$$
$$\psi'_t = 2a(n + 1)\varphi + f\psi;$$

the arguments of the functions f and g are not specified. Integrating the first equation and then the second, we obtain

$$\varphi(t) = e^F\left(A - 4a\int e^F dt\right)^{-1}, \quad F = \int f\,dt,$$

$$\psi(t) = Be^F + 2a(n + 1)e^F\int \varphi e^{-F} dt,$$

where A and B are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(w).$

On passing from t, x to the new variables $t, z = x + \int f(t)\,dt$, one obtains a simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + g(w),$$

which has a traveling-wave solution $w = w(kz + \lambda t)$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t, w).$

On passing from t, x to the new variables $t, z = x + \int f(t)\,dt$, one obtains a simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + g(t, w).$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x, t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}h(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\varphi'_x + b\varphi\ln\varphi + g(x)\varphi = 0.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \big[xp(t) + s(t)\big]w.$

1°. Generalized traveling-wave solution:

$$w(x, t) = \exp\big[x\varphi(t) + \psi(t)\big],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = \big[f(t) + h(t)\big]\varphi + p(t), \tag{1}$$

$$\psi'_t = h(t)\psi + a\varphi^2 + g(t)\varphi + s(t). \tag{2}$$

Integrating (1) and then (2), we obtain (C_1 and C_2 are arbitrary constants)

$$\varphi(t) = C_1E(t) + E(t)\int\frac{p(t)}{E(t)}\,dt, \qquad E(t) = \exp\left[\int f(t)\,dt + \int h(t)\,dt\right],$$

$$\psi(t) = C_2H(t) + H(t)\int\frac{a\varphi^2(t) + g(t)\varphi(t) + s(t)}{H(t)}\,dt, \qquad H(t) = \exp\left[\int h(t)\,dt\right].$$

2°. See equation 5.6.2.7 with $r(t) = 0$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \big[x^2 r(t) + xp(t) + s(t)\big]w.$

Functional separable solution:

$$w(x,t) = \exp\big[x^2\varphi(t) + x\psi(t) + \chi(t)\big],$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = 4a\varphi^2 + (2f + h)\varphi + r, \tag{1}$$
$$\psi_t' = (4a\varphi + f + h)\psi + 2g\varphi + p, \tag{2}$$
$$\chi_t' = h\chi + 2a\varphi + a\psi^2 + g\psi + s. \tag{3}$$

For $r \equiv 0$, equation (1) is a Bernoulli equation, so it is easy to integrate. In the general case, equation (1) for $\varphi = \varphi(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to the Riccati equation for various f, h, and r. With equation (1) solved, the solutions of equations (2) and (3) can be obtained with ease, since these are linear in their unknowns $\psi = \psi(t)$ and $\chi = \chi(t)$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left[xf(t) + \dfrac{g(t)}{x}\right]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \big[x^2 p(t) + s(t)\big]w.$

Functional separable solution:

$$w(x,t) = \exp\big[\varphi(t)x^2 + \psi(t)\big],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = 4a\varphi^2 + (2f + h)\varphi + p, \tag{1}$$
$$\psi_t' = h\psi + 2(a + g)\varphi + s. \tag{2}$$

For $p \equiv 0$, equation (1) is a Bernoulli equation, so it is easy to integrate. In the general case, equation (1) for $\varphi = \varphi(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to the Riccati equation for various f, h, and r. With equation (1) solved, the solution of the linear equation (2) can be obtained with ease.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 55).

5.6.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x + b\int_{t_0}^{t}(t-\tau)f(\tau)\,d\tau,$$

where t_0 is any number, leads to the Burgers equation 5.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\frac{\partial u}{\partial x}.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + f(x,t).$

This is a special case of equation 5.6.3.4.

Let us substitute

$$w = \frac{\partial u}{\partial x}$$

and then integrate the resulting equation with respect to x to arrive at an equation of the form 5.6.4.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{b}{2}\left(\frac{\partial u}{\partial x}\right)^2 + F(x,t),$$

where $F(x,t) = \int f(x,t)\,dx + g(t)$ with $g(t)$ being an arbitrary function.

⊙ *Reference*: A. R. Fortsyth (1906).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + \dfrac{\partial}{\partial x}\Big[f(x,t)w\Big].$

This is a special case of equation 5.6.3.4.

Solution:

$$w(x,t) = \frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + f(x,t)\frac{\partial \theta}{\partial x}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + \dfrac{\partial}{\partial x}\Big[f(x,t)w\Big] + g(x,t).$

The Bäcklund transformation

$$\frac{\partial Z}{\partial t} = \frac{b}{2a}Z\left[a\frac{\partial w}{\partial x} + \frac{1}{2}bw^2 + f(x,t)w + \int g(x,t)\,dx\right],$$
$$\frac{\partial Z}{\partial x} = \frac{b}{2a}Zw,$$

links the equation in question with the linear equation

$$\frac{\partial Z}{\partial t} = a\frac{\partial^2 Z}{\partial x^2} + f(x,t)\frac{\partial Z}{\partial x} + \frac{b}{2a}Z\int g(x,t)\,dx.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[bw + f(t)\big]\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + \int_{t_0}^{t} f(\tau)\,d\tau + b\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

where t_0 is any number, leads to the Burgers equation 5.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\frac{\partial u}{\partial x}.$$

6. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + g(t)w + h(t).$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + \psi(t),\, t\big) + \varphi(t), \quad \varphi(t) = C\exp\left[\int g(t)\,dt\right], \quad \psi(t) = -\int f(t)\varphi(t)\,dt,$$

where C is an arbitrary constant, is also a solution of the equation.

Remark. This remains true if the equation coefficient a is an arbitrary function of time, $a = a(t)$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(t)\ln w + g(t)\big]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left\{\varphi(t)(x + C_1) + \varphi(t)\int\big[a\varphi(t) + g(t)\big]\,dt\right\}, \quad \varphi(t) = -\left[\int f(t)\,dt + C_2\right]^{-1},$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x}.$

Generalized Burgers equation.

1°. Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t$$

are defined implicitly by

$$a\int\frac{dw}{\lambda w - F(w) + A} = z + B, \qquad F(w) = \int f(w)\,dw,$$

where A and B are arbitrary constants.

2°. For the special forms of the function $f(w) = bw$, $f(w) = bw^m$, $f(w) = be^{\lambda w}$, and $f(w) = b\ln w + c$, see equations 5.1.5.3, 5.1.5.9, 5.2.4.2, and 5.4.2.5, respectively, where other solutions are also given.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x} + g(w).$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w = w(z)$ is determined by the second-order autonomous ordinary differential equation

$$aw''_{zz} + \big[f(w) - \lambda\big]w'_z + g(w) = 0,$$

which can be reduced with the change of variable $w'_z = u(w)$ to a first-order equation. For exact solutions of the above ordinary differential equation with various $f(w)$ and $g(w)$, see Polyanin and Zaitsev (2003).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + bx\big]\dfrac{\partial w}{\partial x} + g(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + \big[f(w) + bz\big]w'_z + g(w) = 0.$$

⊙ *Reference for equation* 5.6.3.10: A. D. Polyanin and V. F. Zaitsev (2002, p. 56).

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t)\big]\dfrac{\partial w}{\partial x}.$

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one obtains an equation of the form 5.6.3.8:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(w)\frac{\partial w}{\partial z}.$$

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t)\big]\dfrac{\partial w}{\partial x} + h(w).$

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one obtains a simpler equation of the form 5.6.3.9:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(w)\frac{\partial w}{\partial z} + h(w).$$

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t) + bx\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + Ce^{-bt}, t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-bt} + e^{-bt}\int e^{bt} g(t)\,dt,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + \big[f(w) + bz\big]w'_z + h(w) = 0.$$

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(bx + ct, w)\dfrac{\partial w}{\partial x} + g(bx + ct, w).$

Solution:

$$w = w(\xi), \qquad \xi = bx + ct,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$ab^2 w''_{\xi\xi} + \big[bf(\xi, w) - c\big]w'_\xi + g(\xi, w) = 0.$$

5.6.4. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x) + g(t).$

This is a special case of equation 5.6.4.3.

Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + f(x) - A = 0,$$

which can be reduced, with the change of variable $\varphi'_x = \dfrac{a}{b}\dfrac{\psi'_x}{\psi}$, to a second-order linear equation:

$$\psi''_{xx} + ba^{-2}\big[f(x) - A\big]\psi = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t)x^2 + g(t)x + h(t).$

This is a special case of equation 5.6.4.3.

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4b\varphi^2 + f, \tag{1}$$
$$\psi'_t = 4b\varphi\psi + g, \tag{2}$$
$$\chi'_t = 2a\varphi + b\psi^2 + h. \tag{3}$$

Equation (1) for φ is a Riccati equation. In the special case $f = \text{const}$, it can be easily integrated by separation of variables. Having determined φ, one finds the solutions of equation (2) and then (3), which are linear in the unknowns ψ and χ, respectively.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t).$

The substitution $u = \exp\left(\dfrac{b}{a}w\right)$ leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{b}{a}f(x,t)u.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 e^{ct},$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{c(x+C_2)^2}{C_1 e^{-ct} - 4b} - \frac{2a}{C_1} e^{ct} \ln|C_1 e^{-ct} - 4b| + C_3 e^{ct} + e^{ct} \int e^{-ct} f(t)\,dt.$$

3°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct} \int e^{-ct} f(t)\,dt + \Theta(\xi), \quad \xi = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$a\Theta''_{\xi\xi} + b\left(\Theta'_\xi\right)^2 - \lambda\Theta'_\xi + c\Theta = 0.$$

4°. The substitution

$$w = U(x,t) + e^{ct} \int e^{-ct} f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

⊙ *Reference for equation* 5.6.4.4: A. D. Polyanin and V. F. Zaitsev (2002, p. 57).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(x).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, t + C_1) + C_2 e^{ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = Ae^{ct} + \varphi(x),$$

where A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + c\varphi + f(x) = 0.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(x) + g(t).$

Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{ct} + e^{ct}\int e^{-ct}g(t)\,dt,$$

where A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + c\varphi + f(x) = 0.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

This is a special case of equation 5.6.6.1 with $f(t) = b$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + f(t)w + g(t).$

This is a special case of equation 5.6.6.2.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t) + g(x,t)e^{-\lambda w}.$

The substitution $u = \exp(\lambda w)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \lambda f(x,t)u + \lambda g(x,t).$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{t}f\left(\dfrac{x}{\sqrt{t}},\ w\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(Cx, C^2t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. The transformation $\tau = \ln t$, $\xi = xt^{-1/2}$ leads to the equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial \xi^2} + b\left(\frac{\partial w}{\partial \xi}\right)^2 + \frac{1}{2}\xi\frac{\partial w}{\partial \xi} + f(\xi, w),$$

which admits an exact solution of the form $w = w(\xi)$.

3°. In the special case $f = f(\xi)$, there is also a exact solution of the form $w = C\tau + \varphi(\xi)$, where C is an arbitrary constant, and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$a\varphi''_{\xi\xi} + b(\varphi'_\xi)^2 + \tfrac{1}{2}\xi\varphi'_\xi + f(\xi) - C = 0.$$

5.6.5. Equations of the Form

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x} + kw + g(x) + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{kt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{kt} + e^{kt}\int e^{-kt}h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + f(x)\varphi'_x + k\varphi + g(x) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t)\dfrac{\partial w}{\partial x} + cw^2 + g(t)w + h(t).$

This is a special case of equation 5.6.6.5.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

There are generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $b\lambda^2 + c\lambda + k = 0$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t).$

The substitution $u = \exp\left(\dfrac{b}{a}w\right)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + \frac{b}{a}g(x,t)u.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t) + h(x,t)e^{-\lambda w}.$

The substitution $u = \exp(\lambda w)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + \lambda g(x,t)u + \lambda h(x,t).$$

5.6.6. Equations of the Form

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x,t,w)\left(\frac{\partial w}{\partial x}\right)^2 + g(x,t,w)\frac{\partial w}{\partial x} + h(x,t,w)$$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 \exp\left[\int g(t)\,dt\right],$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = 4f\varphi^2 + g\varphi, \tag{1}$$
$$\psi_t' = (4f\varphi + g)\psi, \tag{2}$$
$$\chi_t' = g\chi + 2a\varphi + f\psi^2 + h. \tag{3}$$

Equation (1) is a Bernoulli equation for φ, so it can be readily integrated. Having determined φ, we can obtain the solutions of equation (2) and then (3), which are linear in the unknowns ψ and χ, respectively. Finally, we have

$$\begin{aligned}
&\varphi = e^G\left(A_1 - 4\int e^G f\,dt\right)^{-1}, \quad G = \int g\,dt,\\
&\psi = A_2 \exp\left[\int (4f\varphi + g)\,dt\right],\\
&\chi = A_3 e^G + e^G \int e^{-G}(2a\varphi + f\psi^2 + h)\,dt,
\end{aligned} \tag{4}$$

where A_1, A_2, and A_3 are arbitrary constants.

To the limit passage $A_1 \to \infty$ in (4) there corresponds a degenerate solution with $\varphi \equiv 0$.

⊙ *Reference for equation* 5.6.6.1: V. F. Zaitsev and A. D. Polyanin (1996, p. 422).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions:

$$w(x,t) = \varphi(t) + \psi(t)\exp\left(\pm x\sqrt{-b}\right), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f, g, and h are not indicated)

$$\varphi_t' = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi_t' = (2bf\varphi + g - ab)\psi. \tag{3}$$

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f, g, and h.

Given a solution of (2), one can solve the linear equation (3) for $\psi = \psi(t)$ to obtain

$$\psi(t) = C\exp\left[-abt + \int (2bf\varphi + g)\,dt\right], \tag{4}$$

where C is an arbitrary constant.

Note two special integrable cases of equation (2).

Solution of equation (2) with $h \equiv 0$:

$$\varphi(t) = e^G\left(C_1 - b\int fe^G\,dt\right)^{-1}, \quad G = \int g\,dt,$$

where C_1 is an arbitrary constant.

If the functions f, g, and h are proportional,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

then the solution of (2) is given by

$$\int \frac{d\varphi}{b\varphi^2 + \alpha\varphi + \beta} = \int f\,dt + C_2, \tag{5}$$

where C_2 is an arbitrary constant. On integrating the left-hand side of equation (5) and solving for φ, one can find $\varphi = \varphi(t)$ in explicit form.

2°. Generalized separable solution of a more general form:

$$w(x,t) = \varphi(t) + \psi(t)\left[A\exp\left(x\sqrt{-b}\right) + B\exp\left(-x\sqrt{-b}\right)\right], \qquad b < 0, \tag{6}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\left(\varphi^2 + 4AB\psi^2\right) + g\varphi + h, \tag{7}$$

$$\psi'_t = 2bf\varphi\psi + g\psi - ab\psi. \tag{8}$$

Solving (8) for φ to express it in terms of ψ and then substituting the resulting expression into (7), one arrives at a second-order nonlinear equation for ψ (if f, g, h = const, this equation is autonomous and, hence, its order can be reduced).

Note two special cases of solution (6) where the exponentials combine to form hyperbolic functions:

$$w(x,t) = \varphi(t) + \psi(t)\cosh\left(x\sqrt{-b}\right), \qquad A = \tfrac{1}{2}, \quad B = \tfrac{1}{2},$$

$$w(x,t) = \varphi(t) + \psi(t)\sinh\left(x\sqrt{-b}\right), \qquad A = \tfrac{1}{2}, \quad B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos\left(x\sqrt{b} + c\right), \qquad b > 0, \tag{9}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\left(\varphi^2 + \psi^2\right) + g\varphi + h, \tag{10}$$

$$\psi'_t = 2bf\varphi\psi + g\psi - ab\psi. \tag{11}$$

Solving (11) for φ to express it in terms of ψ and then substituting the resulting expression into (10), one arrives at a second-order nonlinear equation for ψ (if f, g, h = const, this equation is autonomous and, hence, its order can be reduced).

⊙ *References for equation* 5.6.6.2: V. A. Galaktionov and S. A. Posashkov (1989, the case f, g, h = const was considered), V. F. Zaitsev and A. D. Polyanin (1996, pp. 422–423).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w\dfrac{\partial w}{\partial x} + cf(t)w^2 + g(t)w + h(t).$

There are generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x)\dfrac{\partial w}{\partial x} + bw + h(x) + p(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}p(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)(\varphi'_x)^2 + g(x)\varphi'_x + b\varphi + h(x) = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)\dfrac{\partial w}{\partial x} + bf(t)w^2 + h(t)w + p(t).$

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one arrives at an equation of the form 5.6.6.2:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(t)\left(\frac{\partial w}{\partial z}\right)^2 + bf(t)w^2 + h(t)w + p(t).$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[g_1(t)x + g_0(t)\big]\dfrac{\partial w}{\partial x} + h(t)w + p(t)x^2 + q(t)x + s(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4f\varphi^2 + (2g_1 + h)\varphi + p, \tag{1}$$

$$\psi'_t = (4f\varphi + g_1 + h)\psi + 2g_0\varphi + q, \tag{2}$$

$$\chi'_t = h\chi + 2a\varphi + f\psi^2 + g_0\psi + s. \tag{3}$$

Equation (1) is a Riccati equation for $\varphi = \varphi(t)$ and, hence, can be reduced to a second-order linear equation. For solution of such equations, see Kamke (1977) and Polyanin and Zaitsev (2003). In the special case $p \equiv 0$, (1) is a Bernoulli equation, so it can be readily integrated.

Given a solution of (1), equations (2) and (3) can be easily solved, since these are linear in their respective unknowns ψ and χ.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - ak\dfrac{1}{w}\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t)w + h(x,t)w^k.$

The substitution $u = w^{1-k}$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + (1-k)g(x,t)u + (1-k)h(x,t).$$

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x)\dfrac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x)\frac{\partial u}{\partial x}.$$

Some exact solutions of this equation, for arbitrary g, can be found in Polyanin (2002).

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \big[xg(t) + h(t)\big]\frac{\partial u}{\partial x}.$$

This equation can be reduced to the linear heat equation; see Polyanin (2002, p. 133).

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,t)\dfrac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t)\frac{\partial u}{\partial x}.$$

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} + f(w)(w'_z)^2 + [g(w) - \lambda]w'_z + h(w) = 0. \tag{1}$$

The substitution $w'_z = u(w)$ leads to the first-order equation

$$auu'_w + f(w)u^2 + [g(w) - \lambda]u + h(w) = 0. \tag{2}$$

For exact solutions of the ordinary differential equations (1) and (2) for various $f(w)$, $g(w)$, and $h(w)$, see the book by Polyanin and Zaitsev (2003).

Note that in the special case $h \equiv 0$, equation (2) becomes linear and, hence, can be readily integrated.

5.6.7. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(x)\dfrac{\partial w}{\partial x} + bw + h(x) + p(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}p(t)\,dt,$$

where the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$a\varphi''_{xx} + f(x)(\varphi'_x)^k + g(x)\varphi'_x + b\varphi + h(x) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}g(t)\,dt,$$

where the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$a\varphi''_{xx} + f(x, \varphi'_x) + b\varphi = 0.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^2 t + \int f(t,\lambda)\,dt\right],$$

where A and λ are arbitrary constants.

5.6.8. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = (ax^2 + b)\dfrac{\partial^2 w}{\partial x^2} + ax\dfrac{\partial w}{\partial x} + f(w).$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2 + b}}$ leads to an equation of the form 5.6.1.1:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{f(t)}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + g(t)w\ln w.$

This equation can be rewritten in the form

$$\frac{\partial w}{\partial t} = f(t)\frac{\partial^2 w}{\partial x^2} + \frac{nf(t)}{x}\frac{\partial w}{\partial x} + g(t)w\ln w.$$

Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = 4f\varphi^2 + g\varphi,$$
$$\psi_t' = 2(n+1)f\varphi + g\psi;$$

the arguments of f and g are omitted. Successively integrating, one obtains

$$\varphi(t) = e^G\left(A - 4\int f e^G dt\right)^{-1}, \quad G = \int g\,dt,$$
$$\psi(t) = Be^G + 2(n+1)e^G\int f\varphi e^{-G}dt,$$

where A and B are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + \left[xg(t) + \dfrac{h(t)}{x}\right]\dfrac{\partial w}{\partial x} + s(t)w\ln w + \bigl[x^2 p(t) + q(t)\bigr]w.$

Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = 4f\varphi^2 + (2g+s)\varphi + p, \tag{1}$$
$$\psi_t' = s\psi + 2(f+h)\varphi + q. \tag{2}$$

For $p \equiv 0$, equation (1) is a Bernoulli equation and, hence, can be easily integrated. In the general case, (1) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a considerable number of solutions to this equation for various f, g, s, and p. Having solved equation (1), one can find $\psi = \psi(t)$ from the linear equation (2).

4. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(e^{\lambda x}\dfrac{\partial w}{\partial x}\right) + g(t)w\ln w + h(t)w.$

Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)e^{-\lambda x} + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the ordinary differential equations

$$\varphi_t' = \lambda^2 f(t)\varphi^2 + g(t)\varphi,$$
$$\psi_t' = g(t)\psi + h(t).$$

Integrating yields

$$\varphi(t) = G(t)\left[A - \lambda^2\int f(t)G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = BG(t) + G(t)\int\frac{h(t)}{G(t)}\,dt,$$

where A and B are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + aw\ln w.$

This equation can be rewritten in the form

$$\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + f'_x(x)\frac{\partial w}{\partial x} + aw\ln w.$$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = \exp(C_1 e^{at})\, w(x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp(Ce^{at})\,\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$(f\varphi'_x)'_x + a\varphi\ln\varphi = 0.$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + aw\ln w + \big[g(x) + h(t)\big]w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = \exp(Ce^{at})w(x,t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}h(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$(f\varphi'_x)'_x + a\varphi\ln\varphi + g(x)\varphi = 0.$$

7. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + aw\ln w + \big[h(x) + s(t)\big]w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = \exp(Ce^{at})w(x,t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}s(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + a\varphi\ln\varphi + h(x)\varphi = 0.$$

8. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + h(x)\dfrac{\partial w}{\partial x} + aw + p(x) + q(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}q(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)(\varphi'_x)^2 + h(x)\varphi'_x + a\varphi + p(x) = 0.$$

9. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\left(\dfrac{\partial w}{\partial x}\right)^k + h(x)\dfrac{\partial w}{\partial x} + aw + p(x) + q(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}q(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)(\varphi'_x)^k + h(x)\varphi'_x + a\varphi + p(x) = 0.$$

10. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, \dfrac{\partial w}{\partial x}\right) + aw + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x,\,\varphi'_x) + a\varphi = 0.$$

5.6.9. Equations of the Form $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w + bx + c.$

Generalized separable solution:

$$w(x,t) = (bx + c)t + Ax + B - \frac{1}{a}\int_{x_0}^{x}(x-\xi)f(\xi)\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)w + g(t).$

1°. Degenerate solution linear in x:

$$w(x,t) = F(t)(Ax + B) + F(t)\int \frac{g(t)}{F(t)}dt, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)(x^2 + Ax + B) + \varphi(t)\int \frac{g(t)}{\varphi(t)}dt,$$

$$\varphi(t) = F(t)\left[C - 2a\int F(t)\,dt\right]^{-1}, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where A, B, and C are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + cw^2 + f(t)w + g(t).$

This is a special case of equation 5.6.10.1 with $b = 0$.

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - ak^2w^2 + f(x)w + b_1\sinh(kx) + b_2\cosh(kx).$

Generalized separable solution:

$$w(x,t) = t\big[b_1\sinh(kx) + b_2\cosh(kx)\big] + \varphi(x).$$

Here, the function $\varphi(x)$ is determined by the linear nonhomogeneous ordinary differential equation with constant coefficients

$$a\varphi''_{xx} - ak^2\varphi + f(x) = 0,$$

whose general solution is given by

$$\varphi(x) = C_1\sinh(kx) + C_2\cosh(kx) - \frac{1}{ak}\int_{x_0}^{x} f(\xi)\sinh\big[k(x-\xi)\big]\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + ak^2w^2 + f(x)w + b_1\sin(kx) + b_2\cos(kx).$

Generalized separable solution:

$$w(x,t) = t\big[b_1\sin(kx) + b_2\cos(kx)\big] + \varphi(x).$$

Here, the function $\varphi(x)$ is determined by the linear nonhomogeneous ordinary differential equation with constant coefficients

$$a\varphi''_{xx} + ak^2\varphi + f(x) = 0,$$

whose general solution is given by

$$\varphi(x) = C_1\sin(kx) + C_2\cos(kx) - \frac{1}{ak}\int_{x_0}^{x} f(\xi)\sin\big[k(x-\xi)\big]\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = G(t)u(z,\tau), \quad z = x + \int f(t)\,dt, \quad \tau = \int G(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right]$$

leads to a simpler equation of the form 5.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

This is a special case of equation 5.6.10.5.

8. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = G(t)u(z,\tau), \quad z = xF(t), \quad \tau = \int F^2(t)G(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

9. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = H(t)u(z,\tau), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

10. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\varphi'_t = C\varphi^2 + g(t)\varphi,\\
&\psi'_t = \big[C\varphi + g(t)\big]\psi + h(t),\\
&a\Theta''_{xx} + f(x)\Theta'_x = C,
\end{aligned}$$

where C is an arbitrary constant. Integrating successively, one obtains

$$\begin{aligned}
&\varphi(t) = G(t)\left[A_1 - C\int G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right],\\
&\psi(t) = A_2\varphi(t) + \varphi(t)\int \frac{h(t)}{\varphi(t)}\,dt,\\
&\Theta(x) = B_1\int \frac{dx}{F(x)} + B_2 + \frac{C}{a}\int\left[\int F(x)\,dx\right]\frac{dx}{F(x)}, \quad F(x) = \exp\left[\frac{1}{a}\int f(x)\,dx\right],
\end{aligned}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)H(t)\left[A - B\int H(t)\,dt\right]^{-1}, \quad H(t) = \exp\left[\int h(t)\,dt\right].$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\varphi''_{xx} + f(x)\varphi'_x + g(x)\varphi = B.$$

For exact solutions of this equation with various $f(x)$ and $g(x)$, see Kamke (1977) and Polyanin and Zaitsev (2003).

5.6.10. Equations of the Form $\dfrac{\partial w}{\partial t} = (aw+b)\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,t,w)\dfrac{\partial w}{\partial x} + h(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + f(t)w + g(t).$

1°. Generalized separable solutions involving an exponential of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\pm\lambda x), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f and g are not indicated)

$$\varphi'_t = c\varphi^2 + f\varphi + g, \tag{2}$$
$$\psi'_t = (a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{3}$$

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f and g.

In particular, for $g \equiv 0$, equation (2) is a Bernoulli equation, which is easy to integrate. In another special case, f, g = const, a particular solution of (2) is a number, $\varphi = \varphi_0$, which is a root of the quadratic equation $c\varphi_0^2 + f\varphi_0 + g = 0$. The substitution $u = \varphi - \varphi_0$ leads to a Bernoulli equation.

Given a solution of (2), the solution of equation (3) can be obtained in the form

$$\psi(t) = C\exp\left[\int (a\lambda^2\varphi + 2c\varphi + f)\,dt\right], \tag{4}$$

where C is an arbitrary constant.

2°. Generalized separable solution involving hyperbolic cosine (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cosh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{5}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f and g are not specified)

$$\varphi'_t = c\varphi^2 - b\lambda^2\psi^2 + f\varphi + g, \tag{6}$$
$$\psi'_t = (a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{7}$$

Solving equation (7) for φ to express it in terms of ψ and then substituting the resulting expression into (6), one arrives at a second-order nonlinear equation for ψ, which is autonomous if f, g = const and, hence, its order can be reduced.

3°. Generalized separable solution involving hyperbolic sine (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\sinh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = c\varphi^2 + b\lambda^2\psi^2 + f\varphi + g,$$
$$\psi'_t = (a\lambda^2\varphi + 2c\varphi + f)\psi.$$

4°. Generalized separable solution involving a trigonometric function (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos(\lambda x + A), \quad \lambda = \left(\frac{c}{a+b}\right)^{1/2}, \tag{8}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = c\varphi^2 + b\lambda^2\psi^2 + f\varphi + g, \tag{9}$$
$$\psi'_t = (-a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{10}$$

Solving equation (10) for φ to express it in terms of ψ and substituting the resulting expression into (9), one arrives at a second-order nonlinear equation for ψ, which is autonomous if f, g = const and, hence, its order can be reduced.

⊙ *References for equation* 5.6.10.1: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, pp. 425–427).

2. $$\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + f(t)\left(\frac{\partial w}{\partial x}\right)^2 + g(t)\frac{\partial w}{\partial x} + h(t)w + s(t).$$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f, g, h, and s are not specified)

$$\varphi'_t = 2(2f + a)\varphi^2 + h\varphi, \tag{1}$$
$$\psi'_t = (4f\varphi + 2a\varphi + h)\psi + 2g\varphi, \tag{2}$$
$$\chi'_t = (2a\varphi + h)\chi + f\psi^2 + g\psi + s. \tag{3}$$

Equation (1) is a Bernoulli equation for $\varphi = \varphi(t)$, so it is easy to integrate. After that, equation (2) and then (3) can be solved with ease, since both are linear in their respective unknowns ψ and χ.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 427).

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x)w\dfrac{\partial w}{\partial x} + h(x)w^2 + p(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where $\varphi(x)$ and $\psi(t)$ are determined by the following system of ordinary differential equations (C is an arbitrary constant):

$$a\varphi\varphi''_{xx} + f(x)(\varphi'_x)^2 + g(x)\varphi\varphi'_x + h(x)\varphi^2 = C\varphi, \tag{1}$$
$$\psi'_t = C\psi^2 + p(t)\psi. \tag{2}$$

The general solution of equation (2) is given by

$$\psi(t) = P(t)\left[A - C\int P(t)\,dt\right]^{-1}, \quad P(t) = \exp\left[\int p(t)\,dt\right],$$

where A is an arbitrary constant. In the special case $f \equiv 0$, equation (1) can be reduced, on dividing it by φ, to a second-order linear equation; for exact solutions of this equation with various $g(x)$ and $h(x)$, see Kamke (1997) and Polyanin and Zaitsev (2003).

4. $\dfrac{\partial w}{\partial t} = (aw + b)\dfrac{\partial^2 w}{\partial x^2} + c\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t)\dfrac{\partial w}{\partial x} + kw^2 + g(t)w + h(t).$

The transformation

$$u(z,t) = w(x,t) + \frac{b}{a}, \quad z = x + \int f(t)\,dt$$

leads to an equation of the form 5.6.10.1 for $u = u(z,t)$.

5. $\dfrac{\partial w}{\partial t} = (aw + b)\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[g_1(t)x + g_0(x)\big]\dfrac{\partial w}{\partial x}$
$$+ h(t)w + p_2(t)x^2 + p_1(t)x + p_0(t).$$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of first-order ordinary differential equations with variable coefficients, which is not written out here.

5.6.11. Equations of the Form $\dfrac{\partial w}{\partial t} = aw^m\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = w^2\dfrac{\partial^2 w}{\partial x^2} + w^2\dfrac{\partial}{\partial x}[f(x)w].$

1°. Consider the nonlinear equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + f(x)\left(\frac{\partial x}{\partial z}\right)^2, \tag{1}$$

which is reduced to the linear heat equation (see Equation 5.6.6.8). Dividing the equation by x_z, differentiating with respect to z, changing to the von Mises variables

$$t,\ z,\ x = x(t,z) \quad \Longrightarrow \quad t,\ x,\ w = w(t,x), \quad \text{where} \quad w = \frac{\partial x}{\partial z},$$

and taking into account the formula

$$\frac{\partial}{\partial z}\left(\frac{x_t}{x_z}\right) \equiv \frac{x_z x_{tz} - x_t x_{zz}}{x_z^2} = \frac{w_t}{w},$$

we arrive at the original equation.

2°. The original equation is related to equation (1) by the Bäcklund transformation

$$x_t = w_z + f(x)w^2,$$
$$x_z = w.$$

2. $\dfrac{\partial w}{\partial t} = aw^4\dfrac{\partial^2 w}{\partial x^2} + f(x)w^5.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^4 t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Suppose $u = u(x)$ is a nontrivial solution of the second-order linear ordinary differential equation

$$au''_{xx} + f(x)u = 0. \tag{1}$$

Then the transformation

$$\xi = \int \frac{dx}{u^2}, \quad z = \frac{w}{u}$$

simplifies the original equation bringing it to the form

$$\frac{\partial z}{\partial t} = az^4\frac{\partial^2 z}{\partial \xi^2}.$$

Using the change of variable $v = z^{-3}$, we obtain an equation of the form 5.1.10.4:

$$\frac{\partial v}{\partial t} = a\frac{\partial}{\partial \xi}\left(v^{-4/3}\frac{\partial v}{\partial \xi}\right).$$

3°. Multiplicative separable solution:

$$w(x,t) = (4\lambda t + C)^{-1/4} g(x),$$

where C and λ are arbitrary constants, and the function $g = g(x)$ is determined by *Yermakov's equation*

$$ag''_{xx} + f(x)g + \lambda g^{-3} = 0. \tag{2}$$

Given a particular solution, $u = u(x)$, of the linear equation (1), the general solution of the nonlinear equation (2) can be expressed as (e.g., see Polyanin and Zaitsev, 2003)

$$Ag^2 = -\frac{\lambda}{a}u^2 + u^2\left(B + A\int \frac{dx}{u^2}\right)^2,$$

where A and B are arbitrary constants ($A \neq 0$).

⊙ *Reference for equation* 5.6.11.2: V. F. Zaitsev and A. D. Polyanin (1996, pp. 427–428).

3. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + f(t)w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^m x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$w(x,t) = F(t)u(x,\tau), \quad \tau = \int F^m(t)\,dt, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.9.20:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial x^2}.$$

4. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + f(x)w^{m+1}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$a\varphi^m \varphi''_{xx} + f(x)\varphi^{m+1} + \lambda\varphi = 0.$$

5. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = u(z,\tau)G(t), \quad z = xF(t), \quad \tau = \int F^2(t)G^m(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.9.26:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial z^2}.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 428).

6. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + \big[f(t)x + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.9.26:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial z^2}.$$

5.6.12. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(t).$

Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1 x + aC_1^2 t + C_2 + \int f(t)\,dt,$$

$$w(x,t) = -\frac{(x+C_2)^2}{6a(t+C_1)} + C_3(t+C_1)^{-1/3} + (t+C_1)^{-1/3}\int (t+C_1)^{1/3} f(t)\,dt,$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(t)w + g(t).$

This is a special case of equation 5.6.13.6 with $m = 1$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw^2 + f(t)w + g(t).$

This is a special case of equation 5.6.13.7 with $m = 1$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(x)w^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1 t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w = (\lambda t + C)^{-1}\varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a(\varphi\varphi'_x)'_x + f(x)\varphi^2 + \lambda\varphi = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

This is a special case of equation 5.6.13.8 with $m = 1$.

The transformation

$$w(x,t) = G(t)u(z,\tau), \quad z = x + \int f(t)\,dt, \quad \tau = \int G(t)\,dt, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial z}\right).$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(t,x) = u(z,\tau)G(t), \quad z = xF(t), \quad \tau = \int F^2(t)G(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.1:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial z}\right).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

This is a special case of equation 5.6.13.10 with $m = 1$.

5.6.13. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + f(x)w^{-1/3}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^{-4/3}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. The substitution $w = v^{-3}$ leads to an equation of the form 5.6.11.2:

$$\frac{\partial v}{\partial t} = av^4\frac{\partial^2 v}{\partial x^2} - \tfrac{1}{3}f(x)v^5.$$

3°. Suppose $u = u(x)$ is any nontrivial particular solution of the second-order linear ordinary differential equation

$$au''_{xx} - \tfrac{1}{3}f(x)u = 0.$$

The transformation

$$\xi = \pm\int \frac{dx}{u^2}, \quad z = wu^3$$

simplifies the original equation, bringing it to equation 5.1.10.4:

$$\frac{\partial z}{\partial t} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right).$$

⊙ *Reference for equation* 5.6.13.1: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{w^2}\dfrac{\partial w}{\partial x}\right) + f(x).$

This equation can be reduced to the generalized Burgers equation (see Example 26.36).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{w^2}\dfrac{\partial w}{\partial x}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{f(x)}{w}\right).$

The substitution $w = 1/u$ leads to an equation of the form 5.6.11.1:

$$\frac{\partial u}{\partial t} = u^2\frac{\partial^2 u}{\partial x^2} + u^2\frac{\partial}{\partial x}[f(x)u].$$

Therefor the original equation can be reduced to a second-order linear equation.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(t)w.$

The transformation

$$w(x,t) = u(x,\tau)F(t), \quad \tau = \int F^m(t)\,dt, \qquad F(t) = \exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^m\frac{\partial u}{\partial x}\right).$$

If $m = -1$ or $m = -2$, see 5.1.10.2 or 5.1.10.3 for solutions of this equation.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(t)w^{1-m}.$

The substitution $u = w^m$ leads to an equation of the form 5.6.10.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + mf(t),$$

which admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(t)w + g(t)w^{1-m}.$

The substitution $u = w^m$ leads to an equation of the form 5.6.10.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + mf(t)u + mg(t),$$

which admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 429).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^{1+m} + f(t)w + g(t)w^{1-m}.$

For $b = 0$, see equation 5.6.13.6.

The substitution $u = w^m$ leads to an equation of the form 5.6.10.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + bmu^2 + mf(t)u + mg(t),$$

which admits generalized separable solutions of the following forms:

$$u(x,t) = \varphi(t) + \psi(t)\exp(\pm\lambda x),$$
$$u(x,t) = \varphi(t) + \psi(t)\cosh(\lambda x + C),$$
$$u(x,t) = \varphi(t) + \psi(t)\sinh(\lambda x + C),$$
$$u(x,t) = \varphi(t) + \psi(t)\cos(\lambda x + C),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by systems of appropriate first-order ordinary differential equations, the parameter λ is a root of a quadratic equation, and C is an arbitrary constant.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 429).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(x)w^{1+m}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w = (\lambda mt + C)^{-1/m}\varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the equation

$$a\psi''_{xx} + (m+1)f(x)\psi + \lambda(m+1)\psi^{\frac{1}{m+1}} = 0, \qquad \psi = \varphi^{m+1}.$$

The book by Polyanin and Zaitsev (2003) presents exact solutions of this equation for various $f(x)$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + g(x)w^{m+1} + f(t)w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations

$$a(\varphi^m\varphi_x')_x' + g(x)\varphi^{m+1} + C_1\varphi = 0,$$
$$\psi_t' - f(t)\psi + C_1\psi^{m+1} = 0,$$

and C_1 is an arbitrary constant. The general solution of the second equation for ψ is given by

$$\psi(t) = e^F\left(C_2 + mC_1\int e^{mF}\,dt\right)^{-1/m}, \qquad F = \int f(t)\,dt,$$

where C_2 is an arbitrary constant.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = u(z,\tau)G(t), \quad z = x + \int f(t)\,dt, \quad \tau = \int G^m(t)\,dt, \quad G(t) = \exp\left[\int g(t)dt\right],$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(t,x) = u(z,\tau)G(t), \quad z = xF(t), \quad \tau = \int F^2(t)G^m(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

In the special case $m = -2$, this equation can be transformed to the linear heat equation (see 5.1.10.3).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 430).

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

If $m = -1$ or $m = -2$, see 5.1.10.2 or 5.1.10.3 for solutions of this equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 434–435).

5.6.14. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, t) - \frac{2}{\lambda}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. The transformation

$$w(x,t) = u(x,\tau) + F(t), \quad \tau = \int \exp\big[\lambda F(t)\big]\,dt, \qquad F(t) = \int f(t)\,dt,$$

leads to a simpler equation of the form 5.2.2.1:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(e^{\lambda u}\frac{\partial u}{\partial x}\right).$$

⊙ *Reference for equation* 5.6.14.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 74).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(t) + g(t)e^{-\lambda w}.$

The substitution $u = e^{\lambda w}$ leads to an equation of the form 5.6.9.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \lambda f(t)u + \lambda g(t).$$

This equation admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(x) + (bx + c)e^{-\lambda w}.$

The substitution $u = e^{\lambda w}$ leads to an equation of the form 5.6.9.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \lambda f(x)u + \lambda(bx + c).$$

This equation admits a generalized separable solution of the form $u = \lambda(bx + c)t + \varphi(x)$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w} + f(t) + g(t)e^{-\lambda w}.$

For $b = 0$, see equation 5.6.14.2.

The substitution $u = e^{\lambda w}$ leads to an equation of the form 5.6.10.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + bu^2 + \lambda f(t)u + \lambda g(t).$$

This equation admits generalized separable solutions of the following forms:

$$u(x, t) = \varphi(t) + \psi(t)\exp(\pm\mu x),$$
$$u(x, t) = \varphi(t) + \psi(t)\cosh(\mu x + C),$$
$$u(x, t) = \varphi(t) + \psi(t)\sinh(\mu x + C),$$
$$u(x, t) = \varphi(t) + \psi(t)\cos(\mu x + C),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by systems of appropriate first-order ordinary differential equations, μ is a root of a quadratic equation, and C is an arbitrary constant.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 430–431).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(x)e^{\lambda w}.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda f(x)\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

⊙ *Reference for equation* 5.6.14.5: V. F. Zaitsev and A. D. Polyanin (1996, p. 431).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + g(x)e^{\lambda w} + f(t).$

Additive separable solution:

$$w = \varphi(x) + \psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations

$$a(e^{\lambda\varphi}\varphi'_x)'_x + g(x)e^{\lambda\varphi} + C_1 = 0, \tag{1}$$

$$\psi'_t - f(t) + C_1 e^{\lambda\psi} = 0, \tag{2}$$

and C_1 is an arbitrary constant. Equation (1) can be reduced, with the change of variable $U = e^{\lambda\varphi}$, to the linear equation $aU''_{xx} + \lambda g(x)U + \lambda C_1 = 0$. The general solution of equation (2) is given by

$$\psi(t) = F - \frac{1}{\lambda}\ln\left(C_2 + \lambda C_1\int e^{\lambda F}\,dt\right), \qquad F = \int f(t)\,dt,$$

where C_2 is an arbitrary constant.

5.6.15. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

This equation is frequently encountered in nonlinear problems of heat and mass transfer (with f being the thermal diffusivity or diffusion coefficient) and the theory of flows through porous media. For $f(w) = aw^m$, see Subsection 5.1.10; for $f(w) = e^{\lambda w}$, see equation 5.2.2.1; and for $f(w) = a\ln w + b$, see equation 5.4.2.8.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$k^2\int\frac{f(w)\,dw}{\lambda w + C_1} = kx + \lambda t + C_2, \tag{1}$$

where C_1, C_2, k, and λ are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution.

3°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$\left[f(w)w'_z\right]'_z + \tfrac{1}{2}zw'_z = 0. \tag{2}$$

Solutions of this form usually correspond to constant w in the initial and boundary conditions for the original partial differential equation (w_0, w_1 = const):

$$w = w_0 \quad \text{at} \quad t = 0 \qquad \text{(initial condition)},$$
$$w = w_1 \quad \text{at} \quad x = 0 \qquad \text{(boundary condition)},$$
$$w \to w_0 \quad \text{as} \quad x \to \infty \qquad \text{(boundary condition)}.$$

Then the boundary conditions for the ordinary differential equation (2) are as follows:

$$w = w_1 \quad \text{at} \quad z = 0, \qquad w \to w_0 \quad \text{as} \quad z \to \infty. \tag{3}$$

For $f(w) = aw^{-1}$, $f(w) = aw^{-2}$, and $f(w) = (\alpha w^2 + \beta w + \gamma)^{-1}$, the general solutions of (2) were obtained by Fujita (1952); see also the book by Lykov (1967).

4°. We now describe a simple method for finding an $f(w)$ such that equation (2) admits an exact solution. To this end, we integrate equation (2) with respect to z and then apply the hodograph transformation (where w is regarded as the independent variable and z as the dependent one) to obtain

$$f(w) = -\frac{1}{2}z'_w\left(\int z\,dw + A\right), \qquad A \text{ is an arbitrary constant.} \tag{4}$$

Substituting a specific expression $z = z(w)$ for z on the right-hand side of relation (4), one obtains a one-parameter family of functions $f(w)$ for which $z = z(w)$ solves equation (2). The explicit form of $w = w(z)$ is obtained by the inversion of $z = z(w)$.

The method just outlined was devised by Philip (1960); he obtained a large number of exact solutions to the original equation for various $f = f(w)$. Some of his results, those corresponding to a problem with the initial and boundary conditions of (3) with $w_0 = 0$ and $w_1 = 1$, are listed below in Table 5.1. All solutions are written out in implicit form, $z = z(w)$, and are valid within the range of their spatial localization $0 \le w \le 1$.

TABLE 5.1
Solutions of equation 5.6.15.1 for various $f = f(w)$, where $z = xt^{-1/2}$

No	Function $f = f(w)$	Solution $z = z(w)$	Conditions
1	$\frac{n}{2}w^n - \frac{n}{2(n+1)}w^{2n}$	$1 - w^n$	$n > 0$
2	$\frac{n}{2(n+1)}\left[(1-w)^{n-1} - (1-w)^{2n}\right]$	$(1-w)^n$	$n > 0$
3	$\frac{n}{2(1-n)}w^{-2n} - \frac{n}{2}w^{-n}$	$w^{-n} - 1$	$0 < n < 1$
4	$\frac{1}{2}\sin^2\left(\frac{1}{2}\pi w\right)$	$\cos\left(\frac{1}{2}\pi w\right)$	
5	$\frac{1}{8}\sin(\pi w)\left[\pi w + \sin(\pi w)\right]$	$\cos^2\left(\frac{1}{2}\pi w\right)$	
6	$\frac{1}{16}\sin^2(\pi w)\left[5 + \cos(\pi w)\right]$	$\cos^3\left(\frac{1}{2}\pi w\right)$	
7	$\frac{1}{2}\cos\left(\frac{1}{2}\pi w\right)\left[\cos\left(\frac{1}{2}\pi w\right) + \frac{1}{2}\pi w - 1\right]$	$1 - \sin(\frac{1}{2}\pi w)$	
8	$\frac{w\arccos w + 1}{2\sqrt{1-w^2}} - \frac{1}{2}$	$\arccos w$	
9	$\frac{\pi - 2(1-w)\arcsin(1-w)}{4\sqrt{2w-w^2}} - \frac{1}{2}$	$\arcsin(1-w)$	
10	$\frac{w\arcsin w}{4\sqrt{1-w^2}} + \frac{1}{4}w^2$	$\sqrt{1-w^2}$	
11	$\frac{1}{2}(1 - \ln w)$	$-\ln w$	

5°. There is another way to find an $f(w)$ for which equation (2) admits exact solutions. By direct substitution, one can verify that equation (2) is satisfied by

$$w = \phi'_z, \quad f(w) = \frac{s + \phi - z\phi'_z}{2\phi''_{zz}}, \tag{5}$$

where $\phi = \phi(z)$ is an arbitrary function, and s is an arbitrary constant. Expressions (5) define a parametric representation of $f = f(w)$; the explicit representation is obtained by eliminating z.

For example, assuming in (5) that

$$\phi(z) = w_0 z + \frac{1}{\lambda}(w_0 - w_1)e^{-\lambda z} \qquad (\lambda > 0,\ w_1 > w_0),$$

and eliminating z, one obtains

$$f(w) = \frac{A}{w - w_0} + B + C\ln(w - w_0), \qquad w = w_0 + (w_1 - w_0)e^{-\lambda z},$$

where $A = -\frac{1}{2}s\lambda^{-1}$, $B = \frac{1}{2}\lambda^{-2}\left[1 + \ln(w_1 - w_0)\right]$, and $C = -\frac{1}{2}\lambda^{-2}$. Note that this solution satisfies the boundary conditions of (3). Likewise, one can construct other $f(w)$.

6°. Here is one more method for constructing an $f(w)$ for which equation (2) admits exact solutions. Suppose $\bar{w} = \bar{w}(z)$ is a solution of equation (2) with an $f(w)$. Then $\bar{w} = \bar{w}(z)$ is also a solution of the more complicated equation $\left[F(w)w'_z\right]'_z + \frac{1}{2}zw'_z = 0$ with

$$F(w) = f(w) + Ag(w) \qquad (A \text{ is an arbitrary constant}), \tag{6}$$

where the function $g = g(w)$ is defined parametrically by

$$g(w) = \frac{1}{\bar{w}'_z}, \quad w = \bar{w}(z). \tag{7}$$

For example, the function $\bar{w} = bz^{2/m}$, where b is some constant, is a particular solution of equation (2) if $f(w)$ is a power-law function, $f(w) = aw^m$. It follows from (6) and (7) that $\bar{w}$ is also a solution of equation (2) with $f(w) = aw^m + Aw^{\frac{m-2}{2}}$.

For the first solution presented in Table 5.1, the method outlined gives the following one-parameter family of functions:

$$f(w) = \frac{n}{2}w^n - \frac{n}{2(n+1)}w^{2n} + Aw^{n-1},$$

for which $z = 1 - w^n$ is a solution of the equation.

7°. The transformation

$$\begin{aligned} &\bar{t} = t - t_0, \qquad \bar{w}(\bar{x}, \bar{t}) = \frac{1}{w(x,t)}, \\ &\bar{x} = \int_{x_0}^{x} w(y,t)\,dy + \int_{t_0}^{t} f\big(w(x_0,\tau)\big)\left[\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0} d\tau \end{aligned} \tag{8}$$

takes a nonzero solution $w(x,t)$ of the original equation to a solution $\bar{w}(\bar{x},\bar{t})$ of a similar equation,

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left[\bar{f}(\bar{w})\frac{\partial \bar{w}}{\partial \bar{x}}\right], \qquad \bar{f}(w) = \frac{1}{w^2} f\left(\frac{1}{w}\right). \tag{9}$$

In the special case of power-law dependence, $f(w) = aw^m$, transformation (8) leads to equation (9) where $\bar{f}(w) = aw^{-m-2}$.

8°. The equation in question is represented in conservative form, i.e., in the form of a conservation law.

Another conservation law:

$$\frac{\partial}{\partial t}(xw) + \frac{\partial}{\partial x}\left[F(w) - xf(w)\frac{\partial w}{\partial x}\right] = 0,$$

where $F(w) = \int f(w)\,dw$.

9°. For $f(w) = a(w^2 + b)^{-1}$, see equation 5.1.13.2 and Subsection 30.3.2 (Example 30.9).

10°. For $f(w) = aw^{2n} + bw^n$, see equation 5.1.13.3.

⊙ *References for equation* 5.6.15.1: L. V. Ovsiannikov (1959, 1962, 1982), V. A. Dorodnitsyn and S. R. Svirshchevskii (1983), W. Strampp (1982), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), J. R. King (1990 *b*), N. H. Ibragimov (1994, pp. 110, 118, 119), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), V. F. Zaitsev and A. D. Polyanin (1996, pp. 435–438), P. W. Doyle and P. J. Vassiliou (1998).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w).$

This equation governs unsteady heat conduction in a quiescent medium in the case where the thermal diffusivity and the rate of reaction are arbitrary functions of temperature.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w)w_z']_z' - \lambda w_z' + g(w) = 0. \tag{1}$$

The substitution

$$y(w) = \frac{1}{\lambda} f(w)w_z'$$

brings (1) to an Abel equation of the second kind:

$$yy_w' - y = \varphi(w), \qquad \text{where} \quad \varphi(w) = -\lambda^{-2} f(w)g(w). \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents a considerable number of solutions to equation (2) for various $\varphi = \varphi(w)$.

3°. Let the function $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \frac{A}{f(w)} + B,$$

where A and B are some numbers. In this case, there is a functional separable solution, which is defined implicitly by

$$\int f(w)\,dw = At - \frac{1}{2}Bx^2 + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants.

4°. Let now $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{A_1A_2w + B}{g(w)} + \frac{A_2A_3}{g(w)}\int Z\,dw, \tag{3}$$

$$Z = -A_2\int \frac{dw}{g(w)}, \tag{4}$$

where A_1, A_2, and A_3 are some numbers. Then there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \frac{\pm x + C_2}{\sqrt{2A_3t + C_1}} - \frac{A_1}{A_3} - \frac{A_2}{3A_3}(2A_3t + C_1),$$

where the function $w(Z)$ is determined by the inversion of (4), and C_1 and C_2 are arbitrary constants.

5°. Let $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{1}{g(w)}\left(A_1w + A_3\int Z\,dw\right)\exp\left[-A_4\int \frac{dw}{g(w)}\right], \tag{5}$$

$$Z = \frac{1}{A_4}\exp\left[-A_4\int \frac{dw}{g(w)}\right] - \frac{A_2}{A_4}, \tag{6}$$

where A_1, A_2, A_3, and A_4 are some numbers ($A_4 \neq 0$). In this case, there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \varphi(t)x + \psi(t),$$

where the function $w(Z)$ is determined by the inversion of (6),

$$\varphi(t) = \pm\left(C_1e^{2A_4t} - \frac{A_3}{A_4}\right)^{-1/2}, \quad \psi(t) = -\varphi(t)\left[A_1\int \varphi(t)\,dt + A_2\int \frac{dt}{\varphi(t)} + C_2\right],$$

and C_1 and C_2 are arbitrary constants.

6°. Let $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \left(\frac{a}{f(w)} + b\right)\left(\int f(w)\,dw + c\right),$$

where a, b, and c are arbitrary constants. In this case, there are generalized separable solutions defined implicitly by

$$\int f(w)\,dw + c = e^{at}z(x),$$

where

$$\begin{aligned} z(x) &= C_1\exp\bigl(x\sqrt{|b|}\,\bigr) + C_2\exp\bigl(-x\sqrt{|b|}\,\bigr) && \text{if } b < 0,\\ z(x) &= C_1\cos\bigl(x\sqrt{b}\,\bigr) + C_2\sin\bigl(x\sqrt{b}\,\bigr) && \text{if } b > 0,\\ z(x) &= C_1x + C_2 && \text{if } b = 0, \end{aligned}$$

and C_1 and C_2 are arbitrary constants.

7°. Let $f(w)$ and $g(w)$ be as follows:

$$f(w) = w\varphi'_w(w), \qquad g(w) = a\left[w + 2\frac{\varphi(w)}{\varphi'_w(w)}\right],$$

where $\varphi(w)$ is an arbitrary function and a is any number. Then there are functional separable solutions defined implicitly by

$$\varphi(w) = C_1e^{2at} - \tfrac{1}{2}a(x + C_2)^2,$$

where C_1 and C_2 are arbitrary constants.

8°. Let $f(w)$ and $g(w)$ be as follows:

$$f(w) = A\frac{V(z)}{V'_z(z)}, \qquad g(w) = B\bigl[2z^{-1/2}V'_z(z) + z^{-3/2}V(z)\bigr],$$

where A and B are arbitrary constants ($AB \neq 0$), $V(z)$ is an arbitrary function, and the function $z = z(w)$ is determined implicitly by the formula

$$w = \int z^{-1/2}V'_z(z)\,dz + C_1; \tag{7}$$

C_1 is an arbitrary constant. Then there exists a functional separable solution of the form (7) where

$$z = -\frac{(x + C_3)^2}{4At + C_2} + 2Bt + \frac{BC_2}{2A},$$

and C_2 and C_3 are arbitrary constants.

9°. Group classification of solutions to the equation in question was carried out by Dorodnitsyn (1979, 1982); see also Dorodnitsyn and Svirshchevskii (1983), Galaktionov, Dorodnitsyn, Elenin, Kurdyumov, and Samarskii (1986), and Ibragimov (1994). As a result, only a limited number of equations were extracted that possess symmetries other than translations. This means that symmetry analysis does not allow one to obtain the solutions given in Items 3°–8° above (for equations with functional arbitrariness).

10°. If $f = dF(w)/dw$ and $g = aF(w) + bw + c$, where $F(w)$ is an arbitrary function, and a, b, and c are arbitrary constants, then there is a conservation law

$$\big[e^{-bt}p(x)w\big]_t + \big\{e^{-bt}\big[p(x)_xF(w) - p(x)(F(w))_x + \varphi(x)\big]\big\}_x = 0.$$

Here,

$$p(x) = \begin{cases} C_1\sin(\sqrt{a}x) + C_2\cos(\sqrt{a}x) & \text{if } a > 0, \\ C_1e^{\sqrt{-a}x} + C_2e^{-\sqrt{-a}x} & \text{if } a < 0, \\ C_1x + C_2 & \text{if } a = 0, \end{cases}$$

where $\varphi'_x = cp(x)$; C_1 and C_2 are arbitrary constants.

11°. For specific equations of this form, see Subsections 5.1.1 to 5.1.3, 5.1.11 to 5.1.13, 5.2.1 to 5.2.3, and 5.4.1.

⊙ *References for equation* 5.6.15.2: V. A. Dorodnitsyn (1979), V. A. Galaktionov (1994), A. D. Polyanin and V. F. Zaitsev (2004, pp. 107–108), A. D. Polyanin and E. A. Vyazmina (2005).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{g(t)}{f(w)} + h(x).$

Functional separable solution in implicit form:

$$\int f(w)\,dw = \int g(t)\,dt - \int_{x_0}^{x}(x-\xi)h(\xi)\,d\xi + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants, and x_0 is any number.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] - (ax+b)\dfrac{\partial w}{\partial x}.$

This equation governs unsteady heat and mass transfer in an inhomogeneous fluid flow in the cases where the thermal diffusivity is arbitrarily dependent on temperature.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1e^{at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1e^{at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[f(w)w'_z]'_z - (az+b)w'_z = 0.$$

3°. Functional separable solution in implicit form:

$$\int f(w)\,dw = C_1e^{-at}(ax+b) + C_2.$$

4°. On passing from t, x to the new variables

$$\tau = \frac{1}{2a}\big(1 - e^{-2at}\big), \qquad \zeta = e^{-at}\left(x + \frac{b}{a}\right),$$

one obtains a simpler equation of the form 5.6.15.1 for $w(\zeta,\tau)$:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial \zeta}\left[f(w)\frac{\partial w}{\partial \zeta}\right].$$

⊙ *Reference for equation* 5.6.15.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 109).

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x}.$

1°. Traveling-wave solution in implicit form:

$$k^2 \int \frac{f(w)\,dw}{\lambda w - kG(w) + C_1} = kx + \lambda t + C_2, \qquad G(w) = \int g(w)\,dw,$$

where C_1, C_2, k, and λ are arbitrary constants.

2°. The transformation

$$dz = w\,dx + \big[f(w)w_x + G(w)\big]\,dt, \quad d\tau = dt, \quad u = 1/w \qquad \big(dz = z_x\,dx + z_t\,dt\big)$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left[\Phi(u)\frac{\partial u}{\partial z}\right] + \Psi(u)\frac{\partial u}{\partial z},$$

where

$$\Phi(u) = \frac{1}{u^2} f\left(\frac{1}{u}\right), \quad \Psi(u) = \frac{1}{u} g\left(\frac{1}{u}\right) - G\left(\frac{1}{u}\right), \quad G(w) = \int g(w)\,dw.$$

Special case. For $f(w) = a$ and $g(w) = bw$, the original equation is an unnormalized Burgers equation 5.1.5.3. The above transformation brings it to the solvable equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(\frac{a}{u^2}\frac{\partial u}{\partial z}\right) + \frac{b}{2u^2}\frac{\partial u}{\partial z},$$

which was studied by Fokas and Yortsos (1982).

3°. Let $f(w)$ and $g(w)$ be defined as

$$f(w) = Z'_w\left(A_1 w + A_3 \int Z\,dw\right), \qquad g(w) = A_2 + A_4 Z,$$

where

$$Z = Z(w) \tag{1}$$

is a prescribed function (chosen arbitrarily). Then the original equation has the following generalized traveling-wave solution:

$$w = w(Z), \qquad Z = \varphi(t)x + (A_2 t + C_1)\varphi(t) + A_1\varphi(t)\int \varphi(t)\,dt,$$

where C_1 is an arbitrary constant, the function $w(Z)$ is determined by the inversion of (1), and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi'_t = A_3\varphi^3 + A_4\varphi^2, \tag{2}$$

whose general solution can be written out in implicit form.

In special cases, solutions of equation (2) are given by

$$\varphi(t) = (C_2 - 2A_3 t)^{-1/2} \quad \text{if } A_4 = 0,$$
$$\varphi(t) = (C_2 - A_4 t)^{-1} \quad \text{if } A_3 = 0.$$

4°. Conservation law:

$$D_t(w) + D_x\big[-f(w)w_x - G(w)\big] = 0, \qquad G(w) = \int g(w)\,dw.$$

⊙ *Reference for equation* 5.6.15.5: A. D. Polyanin and V. F. Zaitsev (2004, pp. 109–110).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(t)\dfrac{\partial w}{\partial x}.$

This equation governs unsteady heat conduction in a moving medium in the case where the thermal diffusivity is arbitrarily dependent on temperature.

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one obtains a simpler equation of the form 5.6.15.1:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right].$$

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + xg(t)\dfrac{\partial w}{\partial x}.$

On passing from t, x to the new variables (A and B are arbitrary constants)

$$\tau = \int G^2(t)\,dt + A, \quad z = xG(t), \qquad \text{where} \quad G(t) = B\exp\left[\int g(t)\,dt\right],$$

one obtains a simpler equation of the form 5.6.15.1 for $w(\tau, z)$:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right].$$

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation

$$w = U(z,\tau), \quad z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^2(t)\,dt, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.6.15.1:

$$\frac{\partial U}{\partial \tau} = \frac{\partial}{\partial z}\left[f(U)\frac{\partial U}{\partial z}\right].$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 441).

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x} + h(w).$

For $g \equiv \text{const}$, this equation governs unsteady heat conduction in a medium moving at a constant velocity in the case where the thermal diffusivity and the reaction rate are arbitrary functions of temperature.

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w)w_z']_z' + [g(w) - \lambda]w_z' + h(w) = 0. \tag{1}$$

The substitution $y(w) = f(w)w_z'$ brings (1) to the Abel equation

$$yy_w' + [g(w) - \lambda]y + f(w)h(w) = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a large number of exact solutions to equation (2) for various $f(w)$, $g(w)$, and $h(w)$.

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \big[ax + g(w)\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[f(w)w_z']_z' + [az + g(w)]w_z' + h(w) = 0.$$

⊙ *Reference for equation* 5.6.15.10: A. D. Polyanin and V. F. Zaitsev (2004, p. 111).

5.6.16. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,w)\dfrac{\partial^2 w}{\partial x^2}$

1. $\dfrac{\partial w}{\partial t} = f(x)w^m\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the generalized Emden–Fowler equation

$$\varphi_{xx}'' + \lambda\big[f(x)\big]^{-1}\varphi^{1-m} = 0. \tag{1}$$

For $m = 1$, a solution of equation (1) is given by

$$\varphi(x) = -\lambda\int_{x_0}^{x}\frac{(x-\xi)}{f(\xi)}d\xi + Ax + B,$$

where A, B, and x_0 are arbitrary constants.

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (1) for various $f(x)$.

3°. The transformation $u = w/x$, $\xi = 1/x$ leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = F(\xi)u^m\frac{\partial^2 u}{\partial \xi^2}, \qquad F(\xi) = \xi^{4-m}f(1/\xi).$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{f(x)}{aw+b}\dfrac{\partial^2 w}{\partial x^2}.$

Generalized separable solution linear in t:

$$w(x,t) = \frac{1}{a}\big[\varphi(x)t + \psi(x) - b\big],$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$f(x)\varphi''_{xx} - \varphi^2 = 0,$$
$$f(x)\psi''_{xx} - \varphi\psi = 0.$$

The first equation can be treated independently from the second. The second equation has a particular solution $\psi(x) = \varphi(x)$, so its general solution is given by

$$\psi(x) = C_1\varphi(x) + C_2\varphi(x)\int \frac{dx}{\varphi^2(x)},$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$k^2\int \frac{dw}{\lambda F(w) + C_1} = kx + \lambda t + C_2, \qquad F(w) = \int \frac{dw}{f(w)},$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. Self-similar solution:

$$w = U(z), \qquad z = \frac{x + C_1}{\sqrt{C_2 t + C_3}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U(z)$ is determined by the ordinary differential equation

$$f(U)U''_{zz} + \tfrac{1}{2}C_2 z U'_z = 0.$$

4°. The substitution $u = \displaystyle\int \frac{dw}{f(w)}$ leads to an equation of the form 5.6.15.1:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right],$$

where the function F is defined parametrically as

$$F(u) = f(w), \qquad u = \int \frac{dw}{f(w)}.$$

To obtain $F = F(u)$ in explicit form, one should eliminate w from the two relations.

5°. Conservation laws:

$$D_t(u) + D_x(-w_x) = 0,$$
$$D_t(xu) + D_x(w - xw_x) = 0,$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$; w_x is the partial derivative of w with respect to x; and u is defined in Item 4°.

4. $\dfrac{\partial w}{\partial t} = x^4 f\left(\dfrac{w}{x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

The transformation $u = w/x$, $\xi = 1/x$ leads to a simpler equation of the form 5.6.16.3:

$$\frac{\partial u}{\partial t} = f(u)\frac{\partial^2 u}{\partial \xi^2}.$$

5. $\dfrac{\partial w}{\partial t} = w^4 f\left(\dfrac{w}{\sqrt{ax^2+bx+c}}\right)\dfrac{\partial^2 w}{\partial x^2}.$

With the transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t} = u^4 f(u)\frac{\partial^2 u}{\partial z^2} + (ac - \tfrac{1}{4}b^2)u^5 f(u),$$

which has a traveling-wave solution $u = u(z + \lambda t)$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 432).

5.6.17. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t,w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + w\left(\dfrac{\partial w}{\partial x}\right)^2 - aw^3.$

1°. Multiplicative separable solutions for $a > 0$:

$$w(x,t) = C\exp\left[\pm x\sqrt{a} + a\int f(t)\,dt\right],$$

where C is an arbitrary constant.

2°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \left(C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}}\right)e^F\left(C_3 + 8aC_1C_2\int e^{2F}\,dt\right)^{-1/2}, \qquad F = a\int f(t)\,dt,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \left[C_1\sin\left(x\sqrt{|a|}\right) + C_2\cos\left(x\sqrt{|a|}\right)\right]e^F\left[C_3 + 2a(C_1^2 + C_2^2)\int e^{2F}\,dt\right]^{-1/2},$$

where $F = a\int f(t)\,dt$; C_1, C_2, and C_3 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_t' &= 6f(t)\varphi^2 + g(t)\varphi + h_2(t),\\
\psi_t' &= 6f(t)\varphi\psi + g(t)\psi + h_1(t),\\
\chi_t' &= 2f(t)\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).
\end{aligned}$$

3. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = g(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right).$

Degenerate solution linear in x:

$$w(x,t) = \frac{1}{F(t)}\left[x + \int \frac{g(t)}{F(t)}\,dt + C_1\right], \qquad F(t) = \int f(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = g(t)\dfrac{\partial}{\partial x}\left(w^2\dfrac{\partial w}{\partial x}\right).$

Degenerate solution linear in x:

$$w(x,t) = (x + C_1)\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi_t' = 2g(t)\varphi^3 - f(t)\varphi^2.$$

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)w^m\dfrac{\partial w}{\partial x}\right].$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$[f(x)\varphi^m\varphi_x']_x' + \lambda\varphi = 0. \tag{1}$$

The transformation

$$z = \int \frac{dx}{f(x)}, \quad \Phi = \varphi^{m+1}$$

brings (1) to the generalized Emden–Fowler equation

$$\Phi''_{zz} + F(z)\Phi^{\frac{1}{m+1}} = 0, \tag{2}$$

where the function $F = F(z)$ is defined parametrically by

$$F = \lambda(m+1)f(x), \quad z = \int \frac{dx}{f(x)}.$$

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of solutions to equation (2) for various $F = F(z)$.

3°. The transformation

$$w(x,t) = \big[\psi(x)\big]^{\frac{1}{m+1}} u(\xi,t), \quad \xi = -\int \big[\psi(x)\big]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)},$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)u^m \frac{\partial u}{\partial \xi}\right],$$

where the function $F = F(\xi)$ is defined parametrically by

$$F = f(x)[\psi(x)]^{\frac{3m+4}{m+1}}, \quad \xi = -\int \big[\psi(x)\big]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)}.$$

6. $\dfrac{\partial w}{\partial t} = f(x)w^m \dfrac{\partial^2 w}{\partial x^2} + g(x)w^{m+1}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi^m\varphi''_{xx} + g(x)\varphi^{m+1} + \lambda\varphi = 0. \tag{1}$$

In the special case of $f(x) = ax^n$ and $g(x) = bx^k$, equation (1) becomes

$$\varphi''_{xx} + (b/a)x^{k-n}\varphi + (\lambda/a)x^{-n}\varphi^{1-m} = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (2) for various values of n, m, and k.

7. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + g(t)w^{1-m}.$

Functional separable solution:

$$w(x,t) = \left[\varphi(t)x^2 + \psi(t)\right]^{1/m},$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \frac{2(m+2)}{m}f\varphi^2, \qquad \psi'_t = 2f\varphi\psi + mg.$$

Integrating yields

$$\varphi = \frac{1}{F}, \quad \psi = F^{-\frac{m}{m+2}}\left(A + m\int gF^{\frac{m}{m+2}}\,dt\right),$$
$$F = B - \frac{2(m+2)}{m}\int f\,dt,$$

where A and B are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 433).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)w^m\dfrac{\partial w}{\partial x}\right] + g(x)w^{m+1}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$[f(x)\varphi^m\varphi'_x]'_x + g(x)\varphi^{m+1} + \lambda\varphi = 0. \tag{1}$$

The transformation

$$z = \int \frac{dx}{f(x)}, \quad \Phi = \varphi^{m+1}$$

brings (1) to the equation

$$\Phi''_{zz} + F(z)\Phi^{\frac{1}{m+1}} + G(z)\Phi = 0, \tag{2}$$

where the functions $F = F(z)$ and $G = G(z)$ are defined parametrically by

$$\begin{cases} F = \lambda(m+1)f(x), \\ z = \displaystyle\int \frac{dx}{f(x)}, \end{cases} \qquad \begin{cases} G = (m+1)f(x)g(x), \\ z = \displaystyle\int \frac{dx}{f(x)}. \end{cases}$$

In the special case of $f(x) = ax^n$ and $g(x) = bx^k$, equation (2) becomes

$$\Phi''_{zz} + Az^{\frac{n}{1-n}}\Phi^{\frac{1}{m+1}} + Bz^{\frac{n+k}{1-n}}\Phi = 0, \qquad n \neq 1, \tag{3}$$

where $A = \lambda a(m+1)\left[a(1-n)\right]^{\frac{n}{1-n}}$ and $B = ab(m+1)\left[a(1-n)\right]^{\frac{n+k}{1-n}}$.

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (3) for various values of n, m, and k.

⊙ *Reference for equation* 5.6.17.8: V. F. Zaitsev and A. D. Polyanin (1996, p. 434).

9. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + g(t)\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)\exp\left[\int h(t)\,dt\right], \quad z = x + \int g(t)\,dt, \quad \tau = \int f(t)\exp\left[m\int h(t)dt\right]dt$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

10. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x} + s(t)w.$

The transformation

$$w(x,t) = u(z,\tau)S(t), \quad z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)S^m(t)\,dt,$$

where the functions $S(t)$ and $G(t)$ are given by

$$S(t) = \exp\left[\int s(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.10.8:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, pp. 434–435).

11. $\dfrac{\partial w}{\partial t} = x^k f(t)\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + xg(t)\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(t,x) = u(z,\tau)H(t), \quad z = xG(t), \quad \tau = \int f(t)G^{2-k}(t)H^m(t)\,dt,$$

where the functions $G(t)$ and $H(t)$ are given by

$$G(t) = \exp\left[\int g(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 5.1.17.10:

$$\frac{\partial u}{\partial \tau} = z^k\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 440).

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)e^{\beta w}\dfrac{\partial w}{\partial x}\right].$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(\beta t + C) + \frac{1}{\beta}\ln\left[\int \frac{A-\beta x}{f(x)}\,dx + B\right],$$

where A, B, and C are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)e^{\beta w}\dfrac{\partial w}{\partial x}\right] + g(x)e^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(\beta t + C) + \varphi(x),$$

where β and C are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$[f(x)\psi_x']_x' + \beta g(x)\psi + \beta = 0, \qquad \psi = e^{\beta\varphi}.$$

⊙ *Reference for equation* 5.6.17.13: V. F. Zaitsev and A. D. Polyanin (1996, p. 435).

14. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left\{\left[f_2(t)w^{2n} + f_1(t)w^n\right]\dfrac{\partial w}{\partial x}\right\} + g_1(t)w + g_2(t)w^{1-n}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\varphi(t)x + \psi(t)\right]^{1/n},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = \frac{(n+1)}{n}f_2(t)\varphi^3 + ng_1(t)\varphi,$$
$$\psi_t' = \frac{(n+1)}{n}f_2(t)\varphi^2\psi + ng_1(t)\psi + \frac{1}{n}f_1(t)\varphi^2 + ng_2(t),$$

which is easy to integrate (the first equation is a Bernoulli equation and the second one is linear in ψ).

15. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^2 w}{\partial x^2} + \big[ax + g(w)\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$f(w)w''_{zz} + [az + g(w)]w'_z + h(w) = 0.$$

16. $\dfrac{\partial w}{\partial t} = x^{1-n}\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

Nonlinear problems of the diffusion boundary layer, defined by equation 5.6.20.2, are reducible to equations of this form. For $n = 1$, see equation 5.6.15.1, and for $f(w) = aw^m$, see equation 5.1.17.10.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x, C_1^{n+1} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution for $n \neq -1$:

$$w = w(z), \qquad z = xt^{-\frac{1}{n+1}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(n+1)[f(w)w'_z]'_z + z^n w'_z = 0, \tag{1}$$

which is often accompanied by the boundary conditions of (3) in 5.6.15.1.

The general solution of equation (1) with $f(w) = a(w+b)^{-1}$ and arbitrary n can be found in Zaitsev and Polyanin (1993).

3°. We now describe a simple way to find functions $f(w)$ for which equation (1) admits exact solutions. Let us integrate (1) with respect to z and then apply the hodograph transformation (with w regarded as the independent variable and z as the dependent one) to obtain

$$f(w) = -\frac{1}{n+1} z'_w \left(\int z^n \, dw + A \right), \qquad A \text{ is any.} \tag{2}$$

Substituting a specific $z = z(w)$ for z on the right-hand side of (2), one obtains a one-parameter family of functions $f(w)$ for which $z = z(w)$ solves equation (1). An explicit form of the solution, $w = w(z)$, is determined by the inversion of $z = z(w)$.

For example, setting $z = (1-w)^k$, one obtains from (2) the corresponding $f(w)$:

$$f(w) = A(1-w)^{k-1} - \frac{k}{(n+1)(nk+1)}(1-w)^{k(n+1)}, \qquad A \text{ is any.}$$

4°. There is another way to construct $f(w)$ for which equation (1) admits exact solutions. It involves the following. Let $\bar{w} = \bar{w}(z)$ be a solution of equation (1) with some function $f(w)$. Then $\bar{w} = \bar{w}(z)$ is also a solution of the more complicated equation $(n+1)[F(w)w'_z]'_z + z^n w'_z = 0$ with

$$F(w) = f(w) + Ag(w) \qquad (A \text{ is any}), \tag{3}$$

where the function $g = g(w)$ is defined parametrically by

$$g(w) = \frac{1}{\bar{w}'_z}, \quad w = \bar{w}(z). \tag{4}$$

For example, if $f(w)$ is a power-law function of w, $f(w) = aw^m$, then $\bar{w} = bz^{\frac{n+1}{m}}$ is a solution of equation (1), with b being a constant. It follows from (3) and (4) that $\bar{w}$ is also a solution of equation (1) with $f(w) = aw^m + Aw^{\frac{m-n-1}{n+1}}$.

5°. For $n = -1$, there is an exact solution of the form

$$w = w(\xi), \quad \xi = \ln|x| + \lambda t,$$

where the function $w(\xi)$ is defined implicitly by

$$\int \frac{f(w)\,dw}{\lambda w + F(w) + C_1} = \xi + C_2, \qquad F(w) = \int f(w)\,dw,$$

where λ, C_1, and C_2 are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution.

⊙ *Reference for equation* 5.6.17.16: V. F. Zaitsev and A. D. Polyanin (1996, pp. 439–440).

17. $\dfrac{\partial w}{\partial t} = \dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left[x^n f(w)\dfrac{\partial w}{\partial x}\right] + g(w).$

This is a nonlinear equation of heat and mass transfer in the radial symmetric case ($n = 1$ corresponds to a plane problem and $n = 2$ to a spatial one).

1°. Let $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \left(\frac{a}{f(w)} + b\right)\left(\int f(w)\,dw + c\right),$$

where a, b, and c are arbitrary constants. In this case, there are functional separable solutions defined implicitly by

$$\int f(w)\,dw + c = e^{at} z(x),$$

where the function $z = z(x)$ is described by the linear ordinary differential equation

$$z''_{xx} + \frac{n}{x} z'_x + bz = 0.$$

Its general solution can be expressed in terms of Bessel functions or modified Bessel functions; see Polyanin and Zaitsev (2003).

2°. Let $f(w)$ and $g(w)$ be defined by

$$f(w) = w\varphi'_w(w), \qquad g(w) = a(n+1)w + 2a\frac{\varphi(w)}{\varphi'_w(w)},$$

where $\varphi(w)$ is an arbitrary function. In this case, there is a functional separable solution defined implicitly by

$$\varphi(w) = Ce^{2at} - \tfrac{1}{2}ax^2,$$

where C is an arbitrary constant.

3°. Let $f(w)$ and $g(w)$ be defined as follows:

$$f(w) = a\varphi^{-\frac{n+1}{2}}\varphi' \int \varphi^{\frac{n+1}{2}}\,dw, \qquad g(w) = b\frac{\varphi}{\varphi'},$$

where $\varphi = \varphi(w)$ is an arbitrary function. In this case, there is a functional separable solution defined implicitly by

$$\varphi(w) = \frac{bx^2}{Ce^{-bt} - 4a},$$

where C is an arbitrary constant.

4°. Let $f(w)$ and $g(w)$ be defined as follows:

$$f(w) = A\frac{V(z)}{V'_z(z)}, \qquad g(w) = B\Big[2z^{-\frac{n+1}{2}}V'_z(z) + (n+1)z^{-\frac{n+3}{2}}V(z)\Big], \tag{1}$$

where A and B are arbitrary constants ($AB \neq 0$), $V(z)$ is an arbitrary function, the function $z = z(w)$ defined implicitly by

$$w = \int z^{-\frac{n+1}{2}}V'_z(z)\,dz + C_1; \tag{2}$$

C_1 is an arbitrary constant. In this case, there is a functional separable solution of the form (2) where

$$z = -\frac{x^2}{4At + C_2} + 2Bt + \frac{BC_2}{2A},$$

and C_2 is an arbitrary constant.

Special case 1. For $f(w) = aw^k$ and $g(w) = bw^{1-k}$, which corresponds to the equation

$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n w^k \frac{\partial w}{\partial x}\right) + bw^{1-k},$$

there is the functional separable solution

$$w(x,t) = \left\{\frac{bk}{4a(kn+k+1)}[2a(kn+k+2)t + C] - \frac{kx^2}{2a(kn+k+2)t + C}\right\}^{1/k},$$

where C is an arbitrary constant.

Special case 2. For $f(w) = ae^{\lambda w}$ and $g(w) = be^{-\lambda w}$, which corresponds to the equation

$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n e^{\lambda w}\frac{\partial w}{\partial x}\right) + be^{-\lambda w},$$

there is the functional separable solution

$$w(x,t) = \frac{1}{\lambda}\ln\left\{\frac{b\lambda}{4a(n+1)}[2a(n+1)t + C] - \frac{x^2}{2a(n+1)t + C}\right\},$$

where C is an arbitrary constant.

⊙ *References for equation* 5.6.17.17: V. A. Galaktionov (1994), A. D. Polyanin and V. F. Zaitsev (2004, pp. 119–120), A. D. Polyanin and E. A. Vyazmina (2005), E. A. Vyazmina and A. D. Polyanin (2006).

18. $\dfrac{\partial w}{\partial t} = f(t)\varphi(w)\dfrac{\partial^2 w}{\partial x^2} + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.6.16.3:

$$\frac{\partial w}{\partial \tau} = \varphi(w)\frac{\partial^2 w}{\partial z^2}.$$

19. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left[\varphi(w)\dfrac{\partial w}{\partial x}\right] + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 5.6.15.1:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996, p. 441).

20. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2}{\partial x^2}\big[f(x,w)\big] + \dfrac{g(t)}{f_w(x,w)} + h(x).$

Here, f_w is the partial derivative of f with respect to w.

Functional separable solution in implicit form:

$$f(x,w) = \int g(t)\,dt - \int_{x_0}^{x}(x-\xi)h(\xi)\,d\xi + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants, and x_0 is any number.

21. $f(x)\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[g(x)\varphi(w)\dfrac{\partial w}{\partial x}\right] + h(x)\psi(w)\dfrac{\partial w}{\partial x}.$

For Lie group classification, transformations, and exact solutions to solutions of this form, see the references cited below.

Special cases of this equation are considered in Subsections 5.1.10 and 5.1.15; see also equations 5.1.13.1–5.1.13.4, 5.1.16.1, 5.1.16.4, 5.1.17.9, 5.1.17.10, 5.2.2.1, 5.2.3.1, 5.2.3.7, 5.2.3.10, 5.2.4.2, 5.2.4.6–5.2.4.10, and others.

⊙ *References*: C. Sophocleous (1996, 2000, 2003), R. O. Popovich and N. M. Ivanova (2004, 2005), N. M. Ivanova and C. Sophocleous (2006), N. M. Ivanova (2008).

5.6.18. Equations of the Form $\dfrac{\partial w}{\partial t} = f\left(x, w, \dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial w}{\partial x}\right)^k \dfrac{\partial^2 w}{\partial x^2} + \left[f(t)x + g(t)\right]\dfrac{\partial w}{\partial x} + h(t)w.$

With the transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{k+2}(t)H^k(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial u}{\partial x}\right)^k \frac{\partial^2 u}{\partial z^2}.$$

See equation 5.6.18.4, Special case 1.

2. $\dfrac{\partial w}{\partial t} = f(x)\left(\dfrac{\partial w}{\partial x}\right)^k \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^k t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where the function $\varphi(x)$ is given by

$$\varphi(x) = \int \left[A(k+1)\int \frac{dx}{f(x)} + C_1\right]^{\frac{1}{k+1}} dx + C_2 \quad \text{if } k \neq -1,$$

$$\varphi(x) = C_1 \int \exp\left[A\int \frac{dx}{f(x)}\right] dx + C_2 \quad \text{if } k = -1;$$

A, B, C_1, and C_2 are arbitrary constants.

3°. Generalized separable solution:

$$w(x,t) = (Akt + B)^{-1/k}\Theta(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$f(x)\left(\Theta'_x\right)^k \Theta''_{xx} + A\Theta = 0.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\left(\dfrac{\partial w}{\partial x}\right)^n\right] + g(w).$

Generalized porous medium equation with nonlinear source.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w)(w'_z)^n]'_z - \lambda w'_z + g(w) = 0.$$

3°. Suppose the function $f = f(w)$ is arbitrary and the function $g = g(w)$ is given by

$$g(w) = A[f(w)]^{-1/n} - B,$$

where A and B are some numbers. In this case, there is a functional separable solution determined implicitly by

$$\int [f(w)]^{1/n}\,dw = At + \frac{n}{B(n+1)}(Bx + C_1)^{\frac{n+1}{n}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

4°. Now let the function $g = g(w)$ be arbitrary and let the function $f = f(w)$ be given by

$$f(w) = \frac{A}{[g(w)]^n}\exp\left[Bn\int \frac{dw}{g(w)}\right],$$

where A and B are some numbers. In this case, there is a functional separable solution determined implicitly by

$$\int \frac{dw}{g(w)} = t + \frac{1}{B}\ln|Bx + C_1| + C_2.$$

5°. Let the function $g = g(w)$ be arbitrary and let the function $f = f(w)$ be given by

$$f(w) = \frac{A_1 A_2^n\, w + B}{[g\,(w)]^n} + \frac{A_2^n A_3}{[g\,(w)]^n}\int Z\,dw, \tag{1}$$

$$Z = A_2 \int \frac{dw}{g\,(w)}, \tag{2}$$

where A_1, A_2, and A_3 are some numbers. In this case, there are traveling-wave solutions of the form

$$w = w(Z), \quad Z = \varphi(t)x + \psi(t),$$

where the function $w(Z)$ is determined by inverting relation (2) and the function $\varphi(t)$ and $\psi(t)$ are expressed as

$$\varphi(t) = \left[\frac{1}{C_1 - A_3\,(n+1)\,t}\right]^{\frac{1}{n+1}},$$

$$\psi(t) = \varphi\,(t)\left[A_1\int [\varphi\,(t)]^n\,dt + A_2\int \frac{dt}{\varphi\,(t)} + C_2\right];$$

C_1 and C_2 are arbitrary constants.

6°. Let $g = g(w)$ be arbitrary and let $f = f(w)$ be given by

$$f(w) = \frac{1}{[(g(w)]^n}\left(A_1 w + A_3 \int Z dw\right) \exp\left[nA_4 \int \frac{dw}{g(w)}\right], \tag{3}$$

$$Z = \frac{1}{A_4} \exp\left[A_4 \int \frac{dw}{g(w)}\right] - \frac{A_2}{A_4}, \tag{4}$$

where A_1, A_2, A_3, and A_4 are some numbers ($A_4 \neq 0$). In this case, there are generalized traveling-wave solution of the form

$$w = w(Z), \quad Z = \varphi(t)x + \psi(t),$$

where the function $w(Z)$ is determined by inverting relation (2) and the functions $\varphi(t)$ and $\psi(t)$ are expressed as

$$\varphi(t) = \left[C_1 e^{-(n+1)A_4 t} - \frac{A_3}{A_4}\right]^{-\frac{1}{n+1}},$$

$$\psi(t) = \varphi(t)\left[A_1 \int [\varphi(t)]^n dt + A_2 \int \frac{dt}{\varphi(t)} + C_2\right];$$

C_1 and C_2 are arbitrary constants.

7°. Let the functions $f(w)$ and $g(w)$ be given by

$$f(w) = A\exp\big[-bn\varphi(w)\big][\varphi'_w(w)]^n, \quad g(w) = \frac{a}{\varphi'_w(w)} - Acn\exp\big[-bn\varphi(w)\big],$$

where $\varphi(w)$ is an arbitrary function; A, a, b, and c are any numbers. In this case, there is a functional separable solution given implicitly by

$$\varphi(w) = at + \theta(x),$$

where the function $\theta(x)$ is determined by the autonomous ordinary differential equation

$$(\theta'_x)^{n-1}\theta''_{xx} = b(\theta'_x)^{n+1} + c.$$

The general solution to this equation can be written in parametric form as

$$\theta = \int \frac{u^n\,du}{bu^{n+1} + c} + C_1, \quad x = \int \frac{u^{n-1}\,du}{bu^{n+1} + c} + C_2.$$

8°. Let $f(w)$ and $g(w)$ be given by

$$f(w) = \frac{1}{\lambda^n}\left(a - \frac{n\lambda}{n+1}w\right)\big[\varphi'_w(w)\big]^n, \quad g(w) = \lambda\frac{\varphi(w)}{\varphi'_w(w)} + \frac{n\lambda}{n+1}w - a,$$

where $\varphi(w)$ is an arbitrary function; a and λ are arbitrary numbers. In this case, there is a functional separable solution given implicitly by

$$\varphi(w) = C_1 e^{\lambda t} + \frac{n\lambda}{n+1}(x + C_2)^{\frac{n+1}{n}};$$

C_1 and C_2 are arbitrary constants.

9°. Some equations of this form that admit functional separable solutions of the special form

$$w = F(\xi), \quad \xi = \varphi(x) + \psi(t)$$

were studied by Estévez, Qu, and Zhang (2002). Unlike the equations considered above in Items 3°–8°, the ones considered in this paper do not contain functional arbitrariness.

⊙ *References for equation* 5.6.18.3: A. D. Polyanin and E. A. Vyazmina (2005), E. A. Vyazmina and A. D. Polyanin (2006).

4. $\dfrac{\partial w}{\partial t} = f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}$.

This equation occurs in the nonlinear theory of flows in porous media; it governs also the motion of a nonlinear viscoplastic medium.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1^2 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t, \tag{1}$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the ordinary differential equation

$$k^2 f\left(k\varphi'_z\right)\varphi''_{zz} = \lambda\varphi'_z + A. \tag{2}$$

The general solution of equation (2) can be rewritten in parametric form as

$$\varphi = k\int \frac{uf(u)\,du}{\lambda u + Ak} + C_1, \qquad z = k^2 \int \frac{f(u)\,du}{\lambda u + Ak} + C_2, \tag{3}$$

where C_1 and C_2 are arbitrary constants,

Relations (1) and (3) define a traveling-wave solution for $A = 0$ and an additive separable solution for $\lambda = 0$.

3°. Self-similar solution:

$$w(x,t) = \sqrt{t}\,\Theta(\xi), \qquad \xi = \frac{x}{\sqrt{t}},$$

where the function $\Theta(\xi)$ is determined by the ordinary differential equation

$$2f\left(\Theta'_\xi\right)\Theta''_{\xi\xi} + \xi\Theta'_\xi - \Theta = 0.$$

4°. The original equation can be represented as the system of equations

$$\frac{\partial w}{\partial x} = u, \qquad \frac{\partial w}{\partial t} = f(u)\frac{\partial u}{\partial x}.$$

Eliminating u, we obtain the original equation for $w = w(x,t)$. Eliminating w, we obtain an equation of the form 5.6.15.1:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u)\frac{\partial u}{\partial x}\right]. \tag{4}$$

5°. The hodograph transformation

$$\bar{x} = w(x,t), \qquad \bar{w}(\bar{x},t) = x$$

leads to an equation of the similar form

$$\frac{\partial \bar{w}}{\partial t} = \bar{f}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}, \qquad \bar{f}(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

6°. The transformation

$$\bar{t} = \alpha t + \gamma_1, \qquad \bar{x} = \beta_1 x + \beta_2 w + \gamma_2, \qquad \bar{w} = \beta_3 x + \beta_4 w + \gamma_3,$$

where α, the β_i, and the γ_i are arbitrary constants such that $\alpha \neq 0$ and $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$, takes the original equation to an equation with the same form. We have

$$\bar{f}(\bar{w}_{\bar{x}}) = \frac{1}{\alpha}(\beta_1 + \beta_2 w_x)^2 f(w_x), \qquad w_x = \frac{\beta_1 \bar{w}_{\bar{x}} - \beta_3}{\beta_4 - \beta_2 \bar{w}_{\bar{x}}},$$

where the subscripts x and $\bar{x}$ denote the corresponding partial derivatives.

Special case 1. Equation

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^2 w}{\partial x^2}, \qquad k \neq 0.$$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^k C_2^{k+2} t + C_4) + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = aA^{k+1}t + B + \frac{A}{k+2}\big[(k+1)x + C\big]^{\frac{k+2}{k+1}},$$

where A, B, and C are arbitrary constants.

3°. Generalized separable solution:

$$w(x,t) = (t + C_1)^{-1/k} u(x) + C_2,$$

where C_1 and C_2 are arbitrary constants, and $u(x)$ is determined by the ordinary differential equation $ak(u'_x)^k u''_{xx} + u = 0$, the general solution of which can be written out in the implicit form as

$$\int \left(C_3 - \frac{k+2}{2ak}u^2\right)^{-\frac{1}{k+2}} du = x + C_4.$$

4°. For $k = -2$, the equation in question is reduced to equation (4) with $f(u) = au^{-2}$ and, hence, can be linearized (see equation 5.1.10.3).

Special case 2. Equation

$$\frac{\partial w}{\partial t} = \frac{a}{w_x^2 + b^2}\frac{\partial^2 w}{\partial x^2}, \qquad w_x = \frac{\partial w}{\partial x}.$$

1. Solution:

$$w(x,t) = \pm\sqrt{C_1 - b^2(x + C_2)^2 - 2at} + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2. Solution:

$$w = bx\tan\left(\pm\frac{1}{2}z - \arctan\big(\psi(z)\big) \pm \frac{a}{b^2}t + C\right),$$
$$z = x^2\cos^{-2}\left(\pm\frac{1}{2}z - \arctan\big(\psi(z)\big) \pm \frac{a}{b^2}t + C\right),$$

where C is an arbitrary constant, and the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi'_z = \frac{1}{2}(1 + \psi^2)\left(\pm 1 - \frac{\psi}{z}\right).$$

The function $z = z(x,t)$ in the solution is defined implicitly.

3. Solution:

$$w = bx\tan\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$
$$z = \frac{b^2x^2}{at}\cos^{-2}\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$

where C is an arbitrary constant, and the functions $\varphi(z)$ and $\psi(z)$ are determined by the system of ordinary differential equations

$$\varphi'_z = \frac{\psi}{2z}, \quad \psi'_z = \frac{1}{2}(1 + \psi^2)\left(\frac{C}{2} - \frac{\psi}{2} - \frac{\psi}{z}\right).$$

The function $z = z(x,t)$ in the solution is defined implicitly.

Special case 3. Equation

$$\frac{\partial w}{\partial t} = k\exp\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}.$$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1}w(C_1x + C_2, C_1^2e^{C_3}t + C_4) + C_3x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = 2A\arctan\left(\frac{x+B}{A}\right) - (x+B)\ln\left|\frac{kt+C}{(x+B)^2+A^2}\right| - (2+\ln 2)x + D,$$

where A, B, C, and D are arbitrary constants.

⊙ *References for equation* 5.6.18.4: E. V. Lenskii (1966), I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), J. R. King (1990 *b*), N. H. Ibragimov (1994, pp. 115–118, 129–130), A. D. Polyanin and V. F. Zaitsev (2004, pp. 121–123).

5. $\dfrac{\partial w}{\partial t} = f(x)g(w)h(w_x)\dfrac{\partial^2 w}{\partial x^2}, \qquad w_x = \dfrac{\partial w}{\partial x}.$

The hodograph transformation, according to which x is taken to be the dependent variable and w the independent one,

$$x = x(w,t),$$

leads to the similar equation:

$$\frac{\partial x}{\partial t} = g(w)f(x)\overline{h}(x_w)\frac{\partial^2 x}{\partial w^2}, \qquad \text{where} \quad \overline{h}(z) = z^{-2}h(1/z).$$

6. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^2 = f(t,w)\dfrac{\partial^2 w}{\partial x^2}.$

Let us seek its solution in implicit form $x = x(w,t)$ (it is the *hodograph transformation*). As a result, we obtain the following second-order linear equation:

$$\frac{\partial x}{\partial t} = f(t,w)\frac{\partial^2 x}{\partial w^2}.$$

See Subsection 26.2.1 (Example 26.4) for details.

7. $\dfrac{\partial w}{\partial t} = f(t,w) + g(t,w)\dfrac{\partial w}{\partial x} + \dfrac{h(t,w)}{w_x^2}\dfrac{\partial^2 w}{\partial x^2}, \qquad w_x = \dfrac{\partial w}{\partial x}.$

The Bäcklund transformation

$$w_t = f(t,w) + g(t,w)\eta + h(t,w)\eta^{-2}\eta_x,$$
$$w_x = \eta$$

leads to the equation

$$\eta_t = \eta^2\frac{\partial}{\partial w}\left[\frac{f(t,w)}{\eta} + g(t,w) + h(t,w)\frac{\eta_w}{\eta^2}\right],$$

which is reduced, by the change of variable $\eta = 1/\zeta$, to the linear equation

$$\zeta_t = \frac{\partial}{\partial w}\Big[h(t,w)\zeta_w - f(t,w)\zeta - g(t,w)\Big].$$

Remark. Similar transformations are discussed in detail in Subsection 26.5.2, Item 3°, and are based on using formula (26.5.2.20).

8. $\dfrac{\partial w}{\partial t}+\dfrac{\partial}{\partial x}\left[\dfrac{f(w)}{w_x}\right]=0, \qquad w_x=\dfrac{\partial w}{\partial x}.$

By setting $v=f(w)/w_x$, we represent the equation as the equivalent system of two equations

$$\frac{\partial w}{\partial t}+\frac{\partial v}{\partial x}=0, \qquad v\frac{\partial w}{\partial x}=f(w).$$

Using the *nonclassical hodograph transformation*

$$x=x(w,t), \qquad v=v(w,t)$$

where w and t are treated as independent variables, while x and v are treated as the dependent variables, and expressing the "old" derivatives through the "new" ones

$$w_x=\frac{1}{x_w}, \quad w_t=-\frac{x_t}{x_w}, \quad v_x=\frac{v_w}{x_w}, \quad v_t=v_t-\frac{x_t}{x_w}v_w,$$

we get the linear system of equations

$$\frac{\partial x}{\partial t}-\frac{\partial v}{\partial w}=0, \qquad f(w)\frac{\partial x}{\partial w}=v.$$

Eliminating the function v, we obtain the linear heat equation

$$\frac{\partial x}{\partial t}=\frac{\partial}{\partial w}\left[f(w)\frac{\partial x}{\partial w}\right].$$

Exact solutions to this equation for arbitrary $f(w)$ can be found in the book by Polyanin (2002, p. 141).

9. $\dfrac{\partial w}{\partial t}+\dfrac{\partial}{\partial x}\left[\dfrac{f(t,w)}{w_x}+g(t,w)\right]=0, \qquad w_x=\dfrac{\partial w}{\partial x}.$

By setting $v=f(t,w)/w_x$, we represent the equation as the equivalent system of two equations

$$\frac{\partial w}{\partial t}+\frac{\partial v}{\partial x}+g_w(t,w)w_x=0, \qquad v\frac{\partial w}{\partial x}=f(t,w).$$

Using the nonclassical hodograph transformation

$$x=x(w,t), \qquad v=v(w,t)$$

where w and t are considered as independent variables, while x and v are considered as the dependent variables (for details about this transformation, see the previous equation) we get the linear system of equations

$$\frac{\partial x}{\partial t}-\frac{\partial v}{\partial w}=g_w(t,w), \qquad f(t,w)\frac{\partial x}{\partial w}=v.$$

Eliminating the function v, we obtain the linear equation

$$\frac{\partial x}{\partial t}=\frac{\partial}{\partial w}\left[f(t,w)\frac{\partial x}{\partial w}\right]+g_w(t,w).$$

5.6.19. Evolution Equations Nonlinear in the Second Derivative

1. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x^2} = f\left(t, \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, t) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The *Euler transformation*

$$w(x,t) + u(\xi,\eta) = x\xi, \quad x = \frac{\partial u}{\partial \xi}, \quad t = \eta$$

leads to the linear equation

$$\frac{\partial u}{\partial \eta} = -f(\eta,\xi)\frac{\partial^2 u}{\partial \xi^2}.$$

For details, see Subsection 26.3.4 (Example 26.19).

2. $\dfrac{\partial w}{\partial t} + [a(t)w + b(t)x]\dfrac{\partial w}{\partial x} = \dfrac{f(t, w_x)}{w_{xx}} + g(t, w_x), \qquad w_x = \dfrac{\partial w}{\partial x}, \quad w_{xx} = \dfrac{\partial^2 w}{\partial x^2}.$

This equation can be reduced to a linear equation. For details, see Subsection 26.6.1 (Example 26.43).

3. $\dfrac{\partial w}{\partial t} = f(x)\exp\left(a\dfrac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution:

$$w = \varphi(x) + \psi(x)\theta(t),$$

where

$$\varphi(x) = \frac{1}{2a}A_1 x^2 + C_3 x + C_4 + \frac{1}{a}\int_{x_0}^{x}(x-\xi)\ln\frac{\psi(\xi)}{f(\xi)}\,d\xi,$$

$$\psi(x) = \frac{1}{2}A_2 x^2 + C_1 x + C_2,$$

$$\theta(t) = -\frac{1}{A_2 a}\ln\bigl(C_5 - A_2 a e^{A_1 t}\bigr),$$

and A_1, A_2, C_1, C_2, C_3, C_4, and C_5 are arbitrary constants.

5.6.20. Nonlinear Equations of the Thermal (Diffusion) Boundary Layer

1. $f(x)\dfrac{\partial w}{\partial x} + g(x)y\dfrac{\partial w}{\partial y} = \dfrac{\partial}{\partial y}\left[\varphi(w)\dfrac{\partial w}{\partial y}\right].$

This equation is encountered in nonlinear problems of the steady diffusion boundary layer (mass exchange between drops or bubbles and a flow); the coordinates x and y are reckoned along and normal to the interphase surface, respectively.

The transformation (A and B are arbitrary constants)

$$t = \int \frac{h^2(x)}{f(x)}dx + A, \quad z = yh(x), \qquad \text{where} \quad h(x) = B\exp\left[-\int \frac{g(x)}{f(x)}dx\right],$$

leads to a simpler equation of the form 5.6.15.1:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *References*: A. D. Polyanin (1980, 1982), V. F. Zaitsev and A. D. Polyanin (1996, p. 443).

2. $f(x)y^{n-1}\dfrac{\partial w}{\partial x} + g(x)y^n\dfrac{\partial w}{\partial y} = \dfrac{\partial}{\partial y}\left[\varphi(w)\dfrac{\partial w}{\partial y}\right].$

This equation is encountered in nonlinear problems of the steady diffusion boundary layer (mass exchange between solid particles, drops, or bubbles and the ambient medium; convective diffusion to a flat plate and that in liquid films); the coordinates x and y are reckoned along and normal to the body surface, respectively. The value $n = 2$ corresponds to a solid particle and $n = 1$, to a drop or a bubble.

The transformation (A and B are arbitrary constants)

$$t = \int \frac{h^{n+1}(x)}{f(x)}dx + A, \quad z = yh(x), \qquad \text{where} \quad h(x) = B\exp\left[-\int \frac{g(x)}{f(x)}dx\right],$$

leads to a simpler equation of the form 5.6.17.16:

$$\frac{\partial w}{\partial t} = z^{1-n}\frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *References*: Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985), V. F. Zaitsev and A. D. Polyanin (1996, p. 443).

3. $f\left(\dfrac{y}{\sqrt{x}}\right)\dfrac{\partial w}{\partial x} + \dfrac{1}{\sqrt{x}}g\left(\dfrac{y}{\sqrt{x}}\right)\dfrac{\partial w}{\partial y} = \dfrac{\partial}{\partial y}\left[\varphi(w)\dfrac{\partial w}{\partial y}\right].$

This is a generalization of the linear equation of the thermal boundary layer on a flat plate.

1°. Self-similar solution:

$$w = w(\xi), \quad \xi = \frac{y}{\sqrt{x}}, \tag{1}$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$[\varphi(w)w'_\xi]'_\xi + \left[\tfrac{1}{2}\xi f(\xi) - g(\xi)\right]w'_\xi = 0. \tag{2}$$

2°. Solving the original partial differential equation with simple boundary conditions of the first kind,

$$x = 0,\ w = a; \quad y = 0,\ w = b; \quad y \to \infty,\ w \to a,$$

where a and b are some constants, is reduced to solving equation (2) with the boundary conditions

$$\xi = 0,\ w = b; \quad \xi \to \infty,\ w \to a.$$

Remark. The classical thermal boundary layer equation is defined by

$$f(\xi) = \mathrm{Pr}F'_\xi(\xi), \quad g(\xi) = \tfrac{1}{2}\mathrm{Pr}\left[\xi F'_\xi(\xi) - F(\xi)\right],$$

where $F(\xi)$ is the Blasius solution in the hydrodynamic problem on the longitudinal homogeneous translational flow of a viscid incompressible fluid past a flat plane, and Pr is the Prandtl number (x the coordinate along the plate and y the coordinate normal to the plate surface).

⊙ *References for equation* 5.6.20.3: H. Schlichting (1981), A. D. Polyanin and V. F. Zaitsev (2002, p. 87–88).

5.7. Nonlinear Schrödinger Equations and Related Equations

5.7.1. Equations of the Form $i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + f(|w|)w = 0$ Involving Arbitrary Parameters

▷ *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$.*

1. $i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + k|w|^2 w = 0.$

Schrödinger equation with a cubic nonlinearity. Here, k is a real number. This equation occurs in various chapters of theoretical physics, including nonlinear optics, superconductivity, and plasma physics.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm A_1 w(\pm A_1 x + A_2, A_1^2 t + A_3),$$
$$w_2 = e^{-i(\lambda x + \lambda^2 t + B)} w(x + 2\lambda t, t),$$

where A_1, A_2, A_3, B, and λ are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression of w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,t) = C_1 \exp\{i\,[C_2 x + (kC_1^2 - C_2^2)t + C_3]\},$$
$$w(x,t) = \pm C_1 \sqrt{\frac{2}{k}}\,\frac{\exp[i(C_1^2 t + C_2)]}{\cosh(C_1 x + C_3)},$$
$$w(x,t) = \pm A \sqrt{\frac{2}{k}}\,\frac{\exp[iBx + i(A^2 - B^2)t + iC_1]}{\cosh(Ax - 2ABt + C_2)},$$
$$w(x,t) = \frac{C_1}{\sqrt{t}} \exp\left[i\frac{(x + C_2)^2}{4t} + i(kC_1^2 \ln t + C_3)\right],$$

where A, B, C_1, C_2, and C_3 are arbitrary real constants. The second and third solutions are valid for $k > 0$. The third solution describes the motion of a soliton in a rapidly decaying case.

3°. Solution:

$$w(x,t) = (ax + b)\exp\big[i(\alpha x^2 + \beta x + \gamma)\big],$$

where the functions $a = a(t)$, $b = b(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are determined by the autonomous system of ordinary differential equations

$$a'_t = -6a\alpha,$$
$$b'_t = -2a\beta - 2b\alpha,$$
$$\alpha'_t = ka^2 - 4\alpha^2,$$
$$\beta'_t = 2kab - 4\alpha\beta,$$
$$\gamma'_t = kb^2 - \beta^2.$$

4°. N-soliton solutions for $k > 0$:

$$w(x,t) = \sqrt{\frac{2}{k}}\,\frac{\det \mathbf{R}(x,t)}{\det \mathbf{M}(x,t)}.$$

Here, $\mathbf{M}(x,t)$ is an $N \times N$ matrix with entries

$$M_{n,k}(x,t) = \frac{1 + \overline{g}_n(x,t)g_n(x,t)}{\overline{\lambda}_n - \lambda_k}, \quad g_n(x,t) = \gamma_n e^{i(\lambda_n x - \lambda_n^2 t)}, \qquad n,\, k = 1,\, \ldots,\, N,$$

where the λ_n and γ_n are arbitrary complex numbers that satisfy the constraints $\operatorname{Im}\lambda_n > 0$ ($\lambda_n \neq \lambda_k$ if $n \neq k$) and $\gamma_n \neq 0$; the bar over a symbol denotes the complex conjugate. The square matrix $\mathbf{R}(x,t)$ is of order $N+1$; it is obtained by augmenting $\mathbf{M}(x,t)$ with a column on the right and a row at the bottom. The entries of $\mathbf{R}$ are defined as

$R_{n,k}(x,t) = M_{n,k}(x,t)$	for	$n,k = 1,\, \ldots,\, N$	(bulk of the matrix),
$R_{n,N+1}(x,t) = g_n(x,t)$	for	$n = 1,\, \ldots,\, N$	(rightmost column),
$R_{N+1,n}(x,t) = 1$	for	$n = 1,\, \ldots,\, N$	(bottom row),
$R_{N+1,N+1}(x,t) = 0$			(lower right diagonal entry).

The above solution can be represented, for $t \to \pm\infty$, as the sum of N single-soliton solutions.

5°. Self-similar solution:

$$w(x,t) = \frac{1}{\sqrt{C_1 t + C_2}}\,u(z), \qquad z = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(z)$ is determined by the ordinary differential equation

$$u''_{zz} + k|u|^2 u - \tfrac{1}{2}iC_1(zu'_z + u) = 0.$$

6°. For other exact solutions, see equation 5.7.5.1 with $f(u) = ku^2$.

7°. Auto-Bäcklund transformations preserving the form of the equation (with $k = 1$):

$$\frac{\partial w}{\partial x} - \frac{\partial \widetilde{w}}{\partial x} = iaf_1 - \frac{i}{2}f_2 g_1,$$
$$\frac{\partial w}{\partial t} - \frac{\partial \widetilde{w}}{\partial t} = \frac{1}{2}g_1\left(\frac{\partial w}{\partial x} + \frac{\partial \widetilde{w}}{\partial x}\right) - ag_2 + \frac{i}{4}f_1\bigl(|f_1|^2 + |f_2|^2\bigr).$$

Here,

$$f_1 = w - \widetilde{w}, \quad f_2 = w + \widetilde{w}, \quad g_1 = i\varepsilon\bigl(b - 2|f_1|^2\bigr)^{1/2}, \quad g_2 = i\bigl(af_1 - \tfrac{1}{2}f_2 g_1\bigr),$$

where a and b are arbitrary real constants and $\varepsilon = \pm 1$.

8°. The Schrödinger equation with a cubic nonlinearity admits infinitely many integrals of motion.

The first three integrals for $k = 2$:

$$C_1 = \int_{-\infty}^{\infty} |w|^2\,dx, \quad C_2 = \int_{-\infty}^{\infty} \left(\overline{w}\frac{\partial w}{\partial x} - w\frac{\partial \overline{w}}{\partial x}\right) dx, \quad C_3 = \int_{-\infty}^{\infty} \left(2\left|\frac{\partial w}{\partial x}\right|^2 - |w|^4\right) dx.$$

It is assumed here that the initial distribution $w(x, 0)$ decays quite rapidly as $|x| \to \infty$. The bar over a symbol denotes the complex conjugate.

The first three integrals for $k = -2$:

$$C_1 = \int_{-\infty}^{\infty} \left(1 - |w|^2\right) dx, \quad C_2 = -\int_{-\infty}^{\infty} \left(\overline{w}\frac{\partial w}{\partial x} - w\frac{\partial \overline{w}}{\partial x}\right) dx, \quad C_3 = \int_{-\infty}^{\infty} \left(\left|\frac{\partial w}{\partial x}\right|^2 + |w|^4 - 1\right) dx.$$

9°. The Schrödinger equation with a cubic nonlinearity is integrable by the inverse scattering method; see the literature cited below.

⊙ *References for equation* 5.7.1.1: V. E. Zakharov and A. B. Shabat (1972), G. L. Lamb (1974), M. J. Ablowitz and H. Segur (1981), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), N. H. Ibragimov (1985), L. D. Faddeev and L. A. Takhtadjan (1987), V. E. Korepin, N. N. Bogoliubov, and A. G. Izergin (1993), N. N. Akhmediev and A. Ankiewicz (1997), C. Sulem and P.-L. Sulem (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 88–89; 2004, pp. 125–127).

2. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + (A|w|^2 + B)w = 0.$

Schrödinger equation with a cubic nonlinearity. The numbers A and B are assumed real.

1°. Solutions:

$$w(x,t) = C_1 \exp\left\{i\left[C_2 x + (AC_1^2 + B - C_2^2)t + C_3\right]\right\},$$
$$w(x,t) = \frac{C_1}{\sqrt{t}} \exp\left[i\frac{(x + C_2)^2}{4t} + i(AC_1^2 \ln t + Bt + C_3)\right],$$

where C_1, C_2, and C_3 are arbitrary real constants.

2°. Solution:

$$w(x,t) = (ax + b)\exp\left[i(\alpha x^2 + \beta x + \gamma)\right],$$

where the functions $a = a(t)$, $b = b(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are determined by the autonomous system of ordinary differential equations

$$a'_t = -6a\alpha,$$
$$b'_t = -2a\beta - 2b\alpha,$$
$$\alpha'_t = Aa^2 - 4\alpha^2,$$
$$\beta'_t = 2Aab - 4\alpha\beta,$$
$$\gamma'_t = Ab^2 - \beta^2 + B.$$

3°. For other exact solutions, see equation 5.7.5.1 with $f(u) = Au^2 + B$.

3. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+(A|w|^2+B|w|+C)w=0.$

Schrödinger equation with a cubic nonlinearity. The numbers A, B, and C are assumed real.

1°. There is an exact solution of the form

$$w(x,t)=(ax+b)\exp\bigl[i(\alpha x^2+\beta x+\gamma)\bigr],$$

where $a=a(t)$, $b=b(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are real functions of a real variable.

2°. For other exact solutions, see equation 5.7.5.1 with $f(u)=Au^2+Bu+C$.

4. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+A|w|^{2n}w=0.$

Schrödinger equation with a power-law nonlinearity. The numbers A and n are assumed real.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1=\pm B_1 w(\pm B_1^n x+B_2,B_1^{2n}t+B_3),$$
$$w_2=e^{-i(\lambda x+\lambda^2 t+C)}w(x+2\lambda t,t),$$

where B_1, B_2, B_3, C, and λ are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression of w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,t)=C_1\exp\bigl\{i\,[C_2x+(A|C_1|^{2n}-C_2^2)t+C_3]\bigr\},$$
$$w(x,t)=\pm\left[\frac{(n+1)C_1^2}{A\cosh^2(C_1nx+C_2)}\right]^{\frac{1}{2n}}\exp[i(C_1^2t+C_3)],$$
$$w(x,t)=\frac{C_1}{\sqrt{t}}\exp\left[i\frac{(x+C_2)^2}{4t}+i\left(\frac{AC_1^{2n}}{1-n}t^{1-n}+C_3\right)\right],$$

where C_1, C_2, and C_3 are arbitrary real constants.

3°. Solution:

$$w(x,t)=e^{-i(\lambda x+\lambda^2 t+C)}U(x+2\lambda t),$$

where C and λ are arbitrary constants, and the function $U=U(y)$ is determined by the autonomous ordinary differential equation $U''_{yy}+AU^{2n+1}=0$. Its solution can be represented in implicit form.

4°. Self-similar solution:

$$w(x,t)=(C_1t+C_2)^{-\frac{1}{2n}}u(z),\qquad z=\frac{x+C_3}{\sqrt{C_1t+C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u=u(z)$ is determined by the ordinary differential equation

$$u''_{zz}+k|u|^{2n}u-\frac{1}{2}iC_1\left(zu'_z+\frac{1}{n}u\right)=0.$$

5°. For other exact solutions, see equation 5.7.5.1 with $f(u)=Au^{2n}$.

⊙ *References for equation* 5.7.1.4: M. J. Ablowitz and H. Segur (1981), A. D. Polyanin and V. F. Zaitsev (2004, pp. 127–128).

5.7.2. Equations of the Form $i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + f(|w|)w = 0$ Involving Arbitrary Parameters

▷ *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$. To $n = 1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry, and to $n = 2$, a three-dimensional Schrödinger equation with central symmetry.*

1. $i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + A|w|^2w = 0.$

Schrödinger equation with a cubic nonlinearity.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = C_1 e^{iC_2} w(\pm C_1 x, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + Au^3 = 0.$$

3°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2\int\frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n}u^{-3} + Au^3 = 0.$$

4°. Solution:

$$w(x,t) = C_1(t + C_2)^{-\frac{n+1}{2}}\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{x^2}{4(t + C_2)} - \frac{AC_1^2}{n(t + C_2)^n} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

⊙ *References for equation* 5.7.2.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 90).

2. $i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + (A|w|^2 + B)w = 0.$

Schrödinger equation with a cubic nonlinearity.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + (Au^2 + B)u = 0.$$

2°. Solution:

$$w(x,t) = u(x)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n} u^{-3} + (Au^2 + B)u = 0.$$

3°. Solution:

$$w(x,t) = C_1(t + C_2)^{-\frac{n+1}{2}}\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = \frac{x^2}{4(t + C_2)} - \frac{AC_1^2}{n(t + C_2)^n} + Bt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

3. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + (A|w|^2 + B|w| + C)w = 0.}$

This is a special case of equation 5.7.5.2 with $f(u) = Au^2 + Bu + C$.

4. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + A|w|^k w = 0.}$

Schrödinger equation with power-law nonlinearity.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm C_1^2 e^{iC_2} w(\pm C_1^k x, C_1^{2k} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, are also solutions of the equation. The plus or minus signs are chosen arbitrarily.

2°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + A|u|^k u = 0.$$

3°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n} u^{-3} + A|u|^k u = 0.$$

4°. Solution:

$$w(x,t) = C_1(t+C_2)^{-\frac{n+1}{2}}\exp\bigl[i\varphi(x,t)\bigr],$$
$$\varphi(x,t) = \frac{x^2}{4(t+C_2)} + \frac{2A|C_1|^k}{2-k-nk}(t+C_2)^{\frac{2-k-nk}{2}} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

⊙ *References for equation* 5.7.2.4: A. D. Polyanin and V. F. Zaitsev (2002, p. 91).

5.7.3. Other Equations Involving Arbitrary Parameters

1. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + a\bigl(1 - e^{-k|w|}\bigr)w = 0.}$

This equation is encountered in plasma theory and laser physics. This is a special case of equation 5.7.5.1 with $f(u) = a\bigl(1 - e^{-ku}\bigr)$.

⊙ *Reference*: R. K. Bullough (1977, 1978).

2. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + ia\dfrac{\partial}{\partial x}\bigl(|w|^2 w\bigr) = 0.}$

This equation is encountered in plasma physics (propagation of Alfven and radio waves); a is a real number.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1 = C_1 w(C_1^2 x + C_2, C_1^4 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \qquad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

Here,

$$u = \frac{C_2}{\sqrt{t+C_1}}, \quad \varphi = \frac{1}{4(t+C_1)}, \quad \psi = \frac{C_3 - 2aC_2^2\ln|t+C_1|}{4(t+C_1)}, \quad \chi = -\int\bigl(\psi^2 + a\psi u^2\bigr)\,dt + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

3°. For another solution, see Item 2° of equation 5.7.5.4 with $f(z) = az^2$. See also Calogero and Degasperis (1982).

3. $i\dfrac{\partial w}{\partial t}+a\dfrac{\partial^2}{\partial x^2}\left(\dfrac{w}{\sqrt{1+|w|^2}}\right)=0.$

This is a special case of equation 5.7.5.5 with $f(z)=a(1+z^2)^{-1/2}$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

4. $i\dfrac{\partial w}{\partial t}+(1+ia)\dfrac{\partial^2 w}{\partial x^2}+f(x,t)w|w|^2+g(x,t)w=0.$

Exact solutions of this equation for some specific $f(x,t)$ and $g(x,t)$ are listed in Table 5.2. See also equation 5.7.4.4 with $f_1(t)=1$ and $f_2(t)=a$, and equation 5.7.4.5 $f_1(x)=1$ and $f_2(x)=a$.

TABLE 5.2
Structure of exact solutions for the Schrödinger type equation $i\frac{\partial w}{\partial t}+(1+ia)\frac{\partial^2 w}{\partial x^2}+f(x,t)w|w|^2+g(x,t)w=0$

a	$f(x,t)$	$g(x,t)$	Solution structure $w(x,t)$
0	$1+ib$	0	Solution 1: $w=\varphi(t)\exp\left(\frac{ix^2}{4t}\right)$, Solution 2: $w=\psi(z)/x,\ \ z=x^2/t$
0	$1+ib$	ic/t	Solution 1: $w=\varphi(t)\exp\left(\frac{ix^2}{4t}\right)$, Solution 2: $w=\psi(z)/x,\ \ z=x^2/t$
0	$(1+ib)/x$	$(c_1+ic_2)/x^2$	$w=\psi(z)/\sqrt{x},\ \ z=x^2/t$
0	$(1+ib)\exp\left(\alpha x t^{-1/2}\right)$	$\frac{1}{4}\alpha x t^{-3/2}+\beta t^{-1}$	$w=\psi(z)/x,\ \ z=x^2/t$
0	$\left[f_1(t)+if_2(t)\right]\exp\left[2h(t)x\right]$	$ih_t'(t)x$	$w=\varphi(t)\exp\left[-h(t)x\right]$
arbitrary	$1+ib$	0	$w=\psi(z)/x,\ \ z=x^2/t$
arbitrary	$(1+ib)e^{-x}$	ic	$w=\varphi(t)\exp\left(\frac{1}{2}x\right)$
arbitrary	$(1+ib)e^{-kx}$	$x+ic$	$w=\varphi(t)\exp\left(\frac{1}{2}kx+ixt\right)$
arbitrary	$(1+ib)x^{-k}$	$(c_1+ic_2)x^{-2}$	$w=x^{(k-2)/2}\psi(z),\ \ z=x^2/t$
arbitrary	$1+ib$	$c_1xt^{-3/2}-ic_2t^{-1}$	Solution 1: $w=\varphi(t)\exp\left(-\frac{2ic_1x}{\sqrt{t}}\right)$, Solution 2: $w=\psi(z)/x,\ \ z=x^2/t$

⊙ *Reference*: L. Garnon and P. Winternitz (1993).

5.7.4. Equations with Cubic Nonlinearities Involving Arbitrary Functions

▷ *Throughout this subsection, w is a complex function of real variables x and t; $i^2=-1$.*

1. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\left[f(t)|w|^2+g(t)\right]w=0.$

Schrödinger equation with a cubic nonlinearity. Here, $f(t)$ and $g(t)$ are real functions of a real variable.

1°. Solution:

$$w(x,t) = C_1 \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_2 x - C_2^2 t + \int \bigl[C_1^2 f(t) + g(t)\bigr]\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

2°. Solution:

$$w(x,t) = \frac{C_1}{\sqrt{t}} \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{(x+C_2)^2}{4t} + \int \bigl[C_1^2 f(t) + tg(t)\bigr]\,\frac{dt}{t} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

3°. Solution:

$$w(x,t) = (ax+b)\exp\bigl[i(\alpha x^2 + \beta x + \gamma)\bigr],$$

where the functions $a = a(t)$, $b = b(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
a'_t &= -6a\alpha,\\
b'_t &= -2a\beta - 2b\alpha,\\
\alpha'_t &= f(t)a^2 - 4\alpha^2,\\
\beta'_t &= 2f(t)ab - 4\alpha\beta,\\
\gamma'_t &= f(t)b^2 - \beta^2 + g(t).
\end{aligned}$$

⊙ *References for equation* 5.7.4.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 91).

2. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + [f_1(t) + if_2(t)]w|w|^2 + [g_1(t) + ig_2(t)]w = 0.$

Equations of this form occur in nonlinear optics.

1°. Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)], \qquad \varphi(x,t) = C_1 x - C_1^2 t + \int [f_1(t)u^2(t) + g_1(t)]\,dt + C_2.$$

Here, the function $u = u(t)$ is determined by the Bernoulli equation $u'_t + f_2(t)u^3 + g_2(t)u = 0$, whose general solution is given by

$$u(t) = \left[C_3 e^{G(t)} + 2e^{G(t)} \int e^{-G(t)} f_2(t)\,dt\right]^{-1/2}, \qquad G(t) = 2\int g_2(t)\,dt.$$

2°. Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)], \qquad \varphi(x,t) = \frac{(x+C_1)^2}{4t} + \int [f_1(t)u^2(t) + g_1(t)]\,dt + C_2,$$

where the function $u = u(t)$ is determined by the Bernoulli equation

$$u'_t + f_2(t)u^3 + \left[g_2(t) + \frac{1}{2t}\right]u = 0.$$

Integrating yields

$$u(t) = \left[C_3 e^{G(t)} + 2e^{G(t)} \int e^{-G(t)} f_2(t)\,dt\right]^{-1/2}, \qquad G(t) = \ln t + 2\int g_2(t)\,dt.$$

⊙ *References for equation* 5.7.4.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 92).

3. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+[f_1(x)+if_2(x)]w|w|^2+[g_1(x)+ig_2(x)]w=0.$

Solutions:

$$w(x,t)=\pm u(x)\exp[iC_1t+i\theta(x)],$$

where the functions $u=u(x)$ and $\theta=\theta(x)$ are determined by the system of ordinary differential equations

$$2u'_x\theta'_x+u\theta''_{xx}+f_2(x)u^3+g_2(x)u=0,$$
$$u''_{xx}-C_1u-u(\theta'_x)^2+f_1(x)u^3+g_1(x)u=0.$$

4. $i\dfrac{\partial w}{\partial t}+[f_1(t)+if_2(t)]\dfrac{\partial^2 w}{\partial x^2}+[g_1(t)+ig_2(t)]w|w|^2+[h_1(t)+ih_2(t)]w=0.$

Solutions:

$$w(x,t)=\pm u(t)\exp[i\varphi(x,t)],\quad \varphi(x,t)=C_1x+\int[-C_1^2f_1(t)+g_1(t)u^2(t)+h_1(t)]\,dt+C_2.$$

Here, the function $u=u(t)$ is determined by the Bernoulli equation

$$u'_t+g_2(t)u^3+[h_2(t)-C_1^2f_2(t)]u=0,$$

whose general solution is given by

$$u(t)=\left[C_3e^{F(t)}+2e^{F(t)}\int e^{-F(t)}g_2(t)\,dt\right]^{-1/2},\quad F(t)=2\int\left[h_2(t)-C_1^2f_2(t)\right]dt.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 92).

5. $i\dfrac{\partial w}{\partial t}+[f_1(x)+if_2(x)]\dfrac{\partial^2 w}{\partial x^2}+[g_1(x)+ig_2(x)]w|w|^2+[h_1(x)+ih_2(x)]w=0.$

Solutions:

$$w(x,t)=\pm u(x)\exp[iC_1t+i\theta(x)],$$

where the functions $u=u(x)$ and $\theta=\theta(x)$ are determined by the system of ordinary differential equations

$$2f_1u'_x\theta'_x+f_1u\theta''_{xx}+f_2u''_{xx}-f_2u(\theta'_x)^2+g_2u^3+h_2u=0,$$
$$f_1u''_{xx}-C_1u-f_1u(\theta'_x)^2-2f_2u'_x\theta'_x-f_2u\theta''_{xx}+g_1u^3+h_1u=0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 92).

6. $\dfrac{\partial w}{\partial t}+[f_1(t)+if_2(t)]\dfrac{\partial^2 w}{\partial x^2}+[g_1(t)+ig_2(t)]w|w|^2+[h_1(t)+ih_2(t)]w=0.$

With f_n, g_n, h_n = const, this equation is used for describing two-component reaction-diffusion systems near a bifurcation point.

Solutions:

$$w(x,t)=\pm u(t)\exp[i\varphi(x,t)],$$
$$\varphi(x,t)=C_1x+\int[C_1^2f_2(t)-g_2(t)u^2(t)-h_2(t)]\,dt+C_2.$$

Here, the function $u = u(t)$ is determined by the Bernoulli equation

$$u'_t + g_1(t)u^3 + [h_1(t) - C_1^2 f_1(t)]u = 0,$$

whose general solution is given by

$$u(t) = \left[C_3 e^{F(t)} + 2e^{F(t)} \int e^{-F(t)} g_1(t)\, dt\right]^{-1/2},$$
$$F(t) = 2\int \left[h_1(t) - C_1^2 f_1(t)\right] dt.$$

⊙ *References*: Y. Kuramoto and T. Tsuzuki (1975), A. D. Polyanin and V. F. Zaitsev (2002, p. 93).

7. $\dfrac{\partial w}{\partial t} + [f_1(x) + i f_2(x)]\dfrac{\partial^2 w}{\partial x^2} + [g_1(x) + i g_2(x)]w|w|^2 + [h_1(x) + i h_2(x)]w = 0.$

Solutions:

$$w(x,t) = \pm u(x)\exp[iC_1 t + i\theta(x)],$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$f_1 u''_{xx} - f_1 u(\theta'_x)^2 - f_2 u\theta''_{xx} - 2f_2 u'_x\theta'_x + g_1 u^3 + h_1 u = 0,$$
$$f_2 u''_{xx} + C_1 u - f_2 u(\theta'_x)^2 + f_1 u\theta''_{xx} + 2f_1 u'_x\theta'_x + g_2 u^3 + h_2 u = 0.$$

5.7.5. Equations of General Form Involving Arbitrary Functions of a Single Argument

▷ *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$.*

1. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + f(|w|)w = 0.$

Schrödinger equation of general form; $f(u)$ is a real function of a real variable.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1 = e^{-i(\lambda x + \lambda^2 t + C_1)} w(x + 2\lambda t + C_2, t + C_3),$$

where C_1, C_2, C_3, and λ are arbitrary real constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = C_1 \exp\left[i\varphi(x,t)\right], \qquad \varphi(x,t) = C_2 x - C_2^2 t + f(|C_1|)t + C_3.$$

3°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where the function $u = u(x)$ is defined implicitly by

$$\int \frac{du}{\sqrt{C_1 u^2 - 2F(u) + C_3}} = C_4 \pm x, \qquad F(u) = \int u f(|u|)\, du.$$

Here, $C_1, \ldots, C_4$ are arbitrary real constants.

4°. Solution:

$$w(x,t) = U(\xi)e^{i(Ax+Bt+C)}, \qquad \xi = x - 2At, \tag{1}$$

where the function $U = U(\xi)$ is determined by the autonomous ordinary differential equation $U''_{\xi\xi} + f(|U|)U - (A^2 + B)U = 0$. Integrating yields the general solution in implicit form:

$$\int \frac{dU}{\sqrt{(A^2 + B)U^2 - 2F(U) + C_1}} = C_2 \pm \xi, \qquad F(U) = \int U f(|U|)\, dU. \tag{2}$$

Relations (1) and (2) involve arbitrary real constants A, B, C, C_1, and C_2.

5°. Solution (A, B, and C are arbitrary real constants):

$$w(x,t) = \psi(z)\exp\big[i(Axt - \tfrac{2}{3}A^2t^3 + Bt + C)\big], \qquad z = x - At^2,$$

where the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi''_{zz} + f(|\psi|)\psi - (Az + B)\psi = 0.$$

6°. Solutions:

$$w(x,t) = \pm\frac{1}{\sqrt{C_1 t}}\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = \frac{(x + C_2)^2}{4t} + \int f\big(|C_1 t|^{-1/2}\big)\, dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

7°. Solution:

$$w(x,t) = u(x)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1 t + C_2\int \frac{dx}{u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the autonomous ordinary differential equation

$$u''_{xx} - C_1 u - C_2^2 u^{-3} + f(|u|)u = 0,$$

whose general solution can be written in implicit form.

8°. Solution:

$$w(x,t) = u(z)\exp\big[iAt + i\varphi(z)\big], \qquad z = kx + \lambda t,$$

where A, k, and λ are arbitrary real constants, and the functions $u = u(z)$ and $\varphi = \varphi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned} k^2 u\varphi''_{zz} + 2k^2 u'_z\varphi'_z + \lambda u'_z &= 0,\\ k^2 u''_{zz} - k^2 u(\varphi'_z)^2 - \lambda u\varphi'_z - Au + f(|u|)u &= 0. \end{aligned}$$

⊙ *Reference for equation* 5.7.5.1: A. D. Polyanin and V. F. Zaitsev (2002, pp. 93–94).

2. $i\dfrac{\partial w}{\partial t}+\dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right)+f(|w|)w=0.$

Schrödinger equation of general form; $f(u)$ is a real function of a real variable. To $n=1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry and to $n=2$, a three-dimensional Schrödinger equation with central symmetry.

1°. Multiplicative separable solution:

$$w(x,t)=u(x)e^{i(C_1t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^nu'_x)'_x-C_1u+f(|u|)u=0.$$

2°. Solution:

$$w(x,t)=u(x)\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=C_1t+C_2\int\frac{dx}{x^nu^2(x)}+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^nu'_x)'_x-C_1u-C_2^2x^{-2n}u^{-3}+f(|u|)u=0.$$

3°. Solution:

$$w(x,t)=C_1t^{-\frac{n+1}{2}}\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=\frac{x^2}{4t}+\int f\Bigl(|C_1|t^{-\frac{n+1}{2}}\Bigr)\,dt+C_2,$$

where C_1 and C_2 are arbitrary real constants.

⊙ *Reference for equation* 5.7.5.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 95).

3. $\dfrac{\partial w}{\partial t}=(a+ib)\dfrac{\partial^2w}{\partial x^2}+\bigl[f(|w|)+ig(|w|)\bigr]w.$

Generalized Landau–Ginzburg equation; $f(u)$ and $g(u)$ are real functions of a real variable, a and b are real numbers. Equations of this form are used for studying second-order phase transitions in superconductivity theory (see Landau and Ginzburg, 1950) and to describe two-component reaction-diffusion systems near a point of bifurcation (Kuramoto and Tsuzuki, 1975).

1°. Suppose $w(x,t)$ is a solution of the generalized Landau–Ginzburg equation. Then the function

$$w_1=e^{iC_1}w(x+C_2,t+C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w(x,t)=C_1\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=\pm x\sqrt{\frac{f(|C_1|)}{a}}+t\left[g(|C_1|)-\frac{b}{a}f(|C_1|)\right]+C_2,$$

where C_1 and C_2 are arbitrary real constants.

3°. Solution:

$$w(x,t) = u(t)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 x - C_1^2 bt + \int g(|u|)\,dt + C_2,$$

where $u = u(t)$ is determined by the ordinary differential equation $u_t' = f(|u|)u - aC_1^2 u$, whose general solution can be represented in implicit form as

$$\int \frac{du}{f(|u|)u - aC_1^2 u} = t + C_3.$$

4°. Solution:

$$w(x,t) = U(z)\exp\bigl[iC_1 t + i\theta(z)\bigr], \qquad z = x + \lambda t,$$

where C_1 and λ are arbitrary real constants, and the functions $U = U(z)$ and $\theta = \theta(z)$ are determined by the system of ordinary differential equations

$$aU_{zz}'' - aU(\theta_z')^2 - bU\theta_{zz}'' - 2bU_z'\theta_z' - \lambda U_z' + f(|U|)U = 0,$$
$$aU\theta_{zz}'' - bU(\theta_z')^2 + bU_{zz}'' + 2aU_z'\theta_z' - \lambda U\theta_z' - C_1 U + g(|U|)U = 0.$$

⊙ *References for equation* 5.7.5.3: V. S. Berman and Yu. A. Danilov (1981), A. D. Polyanin and V. F. Zaitsev (2002, pp. 96–97).

4. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + i\dfrac{\partial}{\partial x}\bigl[f(|w|)w\bigr] = 0.$

1°. Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \qquad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u_t' + 2\varphi u = 0,$$
$$\varphi_t' + 4\varphi^2 = 0,$$
$$\psi_t' + 4\varphi\psi + 2\varphi f(u) = 0,$$
$$\chi_t' + \psi^2 + \psi f(u) = 0.$$

Integrating yields

$$u = \frac{C_2}{\sqrt{t + C_1}}, \quad \varphi = \frac{1}{4(t + C_1)}, \quad \psi = -2\varphi\int f(u)\,dt + C_3\varphi, \quad \chi = -\int\bigl[\psi^2 + \psi f(u)\bigr]\,dt + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

2°. Solution:

$$w(x,t) = U(z)\exp[i\beta t + iV(z)], \qquad z = kx + \lambda t,$$

where k, β, and λ are arbitrary real constants, and the functions $U = U(z)$ and $V = V(z)$ are determined by the system of ordinary differential equations

$$\lambda U_z' + k^2(UV_z')_z' + k^2 U_z' V_z' + k[f(U)U]_z' = 0,$$
$$-U(\beta + \lambda V_z') + k^2 U_{zz}'' - k^2 U(V_z')^2 - kf(U)UV_z' = 0.$$

⊙ *Reference for equation* 5.7.5.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 136).

5. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2}{\partial x^2}\big[f(|w|)w\big] = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm e^{iC_1} w(\pm C_2 x + C_3, C_2^2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary real constants, are also solutions of the equation. The plus or minus signs are chosen arbitrarily.

2°. Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \quad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} u_t' + 2u\varphi f(u) &= 0,\\ \varphi_t' + 4\varphi^2 f(u) &= 0,\\ \psi_t' + 4\varphi\psi f(u) &= 0,\\ \chi_t' + \psi^2 f(u) &= 0. \end{aligned}$$

Integrating yields

$$\varphi = C_1 u^2, \quad \psi = C_2 u^2, \quad \chi = -C_2^2 \int u^4 f(u)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(t)$ is defined implicitly as (C_4 is an arbitrary constant)

$$\int \frac{du}{u^3 f(u)} + 2C_1 t + C_4 = 0.$$

3°. There is a solution of the form

$$w(x,t) = U(z)\exp[i\beta t + iV(z)], \quad z = kx + \lambda t,$$

where k, β, and λ are arbitrary real constants, and the functions $U = U(z)$ and $V = V(z)$ are determined by an appropriate system of ordinary differential equations (which is not written out here).

4°. There is a self-similar solution of the form $w(x,t) = V(\xi)$, where $\xi = x^2/t$.

⊙ *Reference for equation* 5.7.5.5: A. D. Polyanin and V. F. Zaitsev (2004, pp. 136–137).

5.7.6. Equations of General Form Involving Arbitrary Functions of Two Arguments

▷ *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$.*

1. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+f(x,|w|)w=0.$

Schrödinger equation of general form; $f(x,u)$ is a real function of two real variables.

1°. Multiplicative separable solution:

$$w(x,t)=u(x)e^{i(C_1t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$u''_{xx}-C_1u+f(x,|u|)u=0.$$

2°. Solution:

$$w(x,t)=u(x)\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=C_1t+C_2\int\frac{dx}{u^2(x)}+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$u''_{xx}-C_1u-C_2^2u^{-3}+f(x,|u|)u=0.$$

⊙ *Reference for equation* 5.7.6.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 94).

2. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+f(t,|w|)w=0.$

Schrödinger equation of general form; $f(t,u)$ is a real function of two real variables.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1=e^{-i(\lambda x+\lambda^2t+C_1)}w(x+2\lambda t+C_2,\ t),$$

where C_1, C_2, and λ are arbitrary real constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t)=C_1\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=C_2x-C_2^2t+\int f(t,|C_1|)\,dt+C_3;$$

$$w(x,t)=C_1t^{-1/2}\exp\bigl[i\psi(x,t)\bigr],\qquad \psi(x,t)=\frac{(x+C_2)^2}{4t}+\int f\bigl(t,|C_1|t^{-1/2}\bigr)\,dt+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

⊙ *Reference for equation* 5.7.6.2: A. D. Polyanin and V. F. Zaitsev (2002, pp. 94–95).

3. $i\dfrac{\partial w}{\partial t}+\dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right)+f(x,|w|)w=0.$

Schrödinger equation of general form; $f(x,u)$ is a real function of two real variables. To $n=1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry, and to $n=2$, a three-dimensional Schrödinger equation with central symmetry.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1u + f(x,|u|)u = 0.$$

2°. Solution:

$$w(x,t) = u(x)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1t + C_2\int\frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1u - C_2^2x^{-2n}u^{-3} + f(x,|u|)u = 0.$$

⊙ *Reference for equation* 5.7.6.3: A. D. Polyanin and V. F. Zaitsev (2002, p. 95).

4. $i\dfrac{\partial w}{\partial t} + \dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + f(t,|w|)w = 0.$

Schrödinger equation of general form; $f(t,u)$ is a real function of two real variables.

Solution:

$$w(x,t) = C_1t^{-\frac{n+1}{2}}\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = \frac{x^2}{4t} + \int f\Big(t,|C_1|t^{-\frac{n+1}{2}}\Big)\,dt + C_2,$$

where C_1 and C_2 are arbitrary real constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 96).

5. $i\dfrac{\partial w}{\partial t} + f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + \Phi(x,|w|)w = 0.$

Schrödinger equation of general form; $\Phi(x,u)$ is a real function of two real variables. The case $g(x) = f'_x(x)$ corresponds to an anisotropic medium.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$f(x)u''_{xx} + g(x)u'_x - C_1u + \Phi(x,|u|)u = 0.$$

2°. Solution:

$$w(x,t) = U(x)\exp\big[i\varphi(x,t)\big],$$

$$\varphi(x,t) = C_1t + C_2\int\frac{R(x)}{U^2(x)}\,dx + C_3, \quad R(x) = \exp\left[-\int\frac{g(x)}{f(x)}\,dx\right],$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $U = U(x)$ is determined by the ordinary differential equation

$$f(x)U''_{xx} + g(x)U'_x - C_1U - C_2^2f(x)R^2(x)U^{-3} + \Phi(x,|U|)U = 0.$$

6. $\dfrac{\partial w}{\partial t} = \big[f_1(t,|w|) + if_2(t,|w|)\big]\dfrac{\partial^2 w}{\partial x^2} + \big[g_1(t,|w|) + ig_2(t,|w|)\big]w.$

Solution:

$$w(x,t) = u(t)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1 x + \int \big[g_2(t,|u|) - C_1^2 f_2(t,|u|)\big]\,dt + C_2,$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u_t' = ug_1(t,|u|) - C_1^2 u f_1(t,|u|).$$

7. $\dfrac{\partial w}{\partial t} = \big[f_1(x,|w|) + if_2(x,|w|)\big]\dfrac{\partial^2 w}{\partial x^2} + \big[g_1(x,|w|) + ig_2(x,|w|)\big]w.$

Solution:

$$w(x,t) = u(x)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1 t + \theta(x),$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$f_1 u_{xx}'' - f_1 u(\theta_x')^2 - f_2 u\theta_{xx}'' - 2f_2 u_x'\theta_x' + g_1 u = 0,$$
$$f_1 u\theta_{xx}'' - f_2 u(\theta_x')^2 + f_2 u_{xx}'' + 2f_1 u_x'\theta_x' - C_1 u + g_2 u = 0.$$

Here, for brevity, the notation $f_n = f_n(x,|u|)$ and $g_n = g_n(x,|u|)$ with $n = 1,\ 2$ is used.

8. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + i\dfrac{\partial}{\partial x}\big[f(t,|w|)w\big] = 0.$

Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \qquad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u_t' + 2\varphi u = 0,$$
$$\varphi_t' + 4\varphi^2 = 0,$$
$$\psi_t' + 4\varphi\psi + 2\varphi f(t,u) = 0,$$
$$\chi_t' + \psi^2 + \psi f(t,u) = 0.$$

Integrating yields

$$u = \frac{C_2}{\sqrt{t+C_1}}, \quad \varphi = \frac{1}{4(t+C_1)}, \quad \psi = -2\varphi\int f(t,u)\,dt + C_3\varphi, \quad \chi = -\int\big[\psi^2 + \psi f(t,u)\big]\,dt + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

9. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + i\dfrac{\partial}{\partial x}\big[f(x,|w|)w\big] = 0.$

Solution:

$$w(x,t) = U(x)\exp[i\beta t + iV(x)],$$

where β is an arbitrary real constant, and the real functions $U = U(x)$ and $V = V(x)$ are determined by the system of ordinary differential equations

$$(UV_x')_x' + U_x'V_x' + [f(x,U)U]_x' = 0,$$
$$-\beta U + U_{xx}'' - U(V_x')^2 - f(x,U)UV_x' = 0.$$

10. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2}{\partial x^2}\big[f(t,|w|)w\big]=0.$

Solution:

$$w(x,t)=u(t)\exp[iv(x,t)],\quad v(x,t)=\varphi(t)x^2+\psi(t)x+\chi(t),$$

where the functions $u=u(t)$, $\varphi=\varphi(t)$, $\psi=\psi(t)$, and $\chi=\chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}u_t'+2u\varphi f(t,u)&=0,\\ \varphi_t'+4\varphi^2 f(t,u)&=0,\\ \psi_t'+4\varphi\psi f(t,u)&=0,\\ \chi_t'+\psi^2 f(t,u)&=0.\end{aligned}$$

Integrating yields

$$\varphi=C_1u^2,\quad \psi=C_2u^2,\quad \chi=-C_2^2\int u^4 f(t,u)\,dt+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and $u=u(t)$ is determined by the ordinary differential equation $u_t'+2C_1u^3f(t,u)=0$.

11. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial}{\partial x}\left[f(t,|w|)\dfrac{\partial w}{\partial x}\right]=0.$

Solution:

$$w(x,t)=u(t)\exp[iv(x,t)],\quad v(x,t)=\varphi(t)x^2+\psi(t)x+\chi(t),$$

where the functions $u=u(t)$, $\varphi=\varphi(t)$, $\psi=\psi(t)$, and $\chi=\chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}u_t'+2u\varphi f(t,u)&=0,\\ \varphi_t'+4\varphi^2 f(t,u)&=0,\\ \psi_t'+4\varphi\psi f(t,u)&=0,\\ \chi_t'+\psi^2 f(t,u)&=0.\end{aligned}$$

Integrating yields

$$\varphi=C_1u^2,\quad \psi=C_2u^2,\quad \chi=-C_2^2\int u^4 f(t,u)\,dt+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and $u=u(t)$ is determined by the ordinary differential equation $u_t'+2C_1u^3f(t,u)=0$.

12. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial}{\partial x}\left[f(x,|w|)\dfrac{\partial w}{\partial x}\right]=0.$

There is a solution of the form

$$w(x,t)=U(x)\exp[i\beta t+iV(x)],$$

where β is an arbitrary real constant, and the functions $U=U(z)$ and $V=V(z)$ are determined by an appropriate system of ordinary differential equations (which is not written out here).

Chapter 6
Second-Order Parabolic Equations with Two or More Space Variables

6.1. Equations with Two Space Variables Involving Power-Law Nonlinearities

6.1.1. Equations of the Form

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + aw^p$$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 6.4.2.1 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ C_1^{p-1}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2};$$

$$w(x, y, t) = t^{\frac{1}{1-p}}V(z_1, z_2), \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = yt^{\frac{1}{m-2}}.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 6.4.2.3 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ y + \frac{1-p}{\lambda}\ln C_1,\ C_1^{p-1}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = t^{\frac{1}{1-p}}V(z_1, z_2), \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 6.4.2.2 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x + \frac{1-p}{\beta}\ln C_1,\ y + \frac{1-p}{\lambda}\ln C_1,\ C_1^{p-1}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = t^{\frac{1}{1-p}}V(z_1, z_2), \quad z_1 = x + \frac{1}{\beta}\ln t, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

6.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 6.1.3.1 with $c = 0$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

Boussinesq equation. It arises in nonlinear heat conduction theory and the theory of unsteady flows through porous media with a free surface (see Polubarinova–Kochina, 1962). This is a special case of equation 6.1.2.4 with $n = 1$.

1°. Solution linear in all independent variables:

$$w(x, y, t) = Ax + By + (A^2 + B^2)t + C,$$

where A, B, and C are arbitrary constants.

2°. Traveling-wave solution (k_1, k_2, and λ are arbitrary constants):

$$w = w(\xi), \quad \xi = k_1 x + k_2 y + \lambda t,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\lambda w'_\xi = (k_1^2 + k_2^2)(ww'_\xi)'_\xi.$$

The solution of this equation can be written out in implicit form:

$$\xi = B + \frac{k_1^2 + k_2^2}{\lambda^2}\big(\lambda w - A\ln|A + \lambda w|\big),$$

where A and B are arbitrary constants.

3°. Generalized separable solution quadratic in the space variables:

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = 6f^2 + 2fh + g^2, \tag{1}$$
$$g'_t = 6(f+h)g, \tag{2}$$
$$h'_t = 6h^2 + 2fh + g^2. \tag{3}$$

It follows from (1) and (3) that $f'_t - h'_t = 6(f+h)(f-h)$. Further, using (2) and assuming $g \not\equiv 0$, we find that $f = h + Ag$, where A is an arbitrary constant. With this relation, we eliminate h from (2) and (3) to obtain a nonlinear ordinary differential equation for $g(t)$:

$$3gg''_{tt} - 5g'^2_t - 36(1+A^2)g^4 = 0.$$

On solving this equation with the change of variable $u(g) = (g'_t)^2$, we obtain (B is an arbitrary constant)

$$g'_t = g\Phi(g), \qquad \Phi(g) = \pm\sqrt{Bg^{4/3} + 36(1+A^2)g^2}, \tag{4}$$
$$h = \tfrac{1}{12}\Phi(g) - \tfrac{1}{2}Ag, \quad f = \tfrac{1}{12}\Phi(g) + \tfrac{1}{2}Ag,$$

where the first equation is separable, and, hence, its solution can be written out in implicit form.

In the special case $B = 0$, the solution can be represented in explicit form (C is an arbitrary constant):

$$f(t) = \frac{\mu + A}{2(C-\mu t)}, \quad g(t) = \frac{1}{C-\mu t}, \quad h(t) = \frac{\mu - A}{2(C-\mu t)}, \qquad \mu = \pm\sqrt{1+A^2}.$$

4°. Generalized separable solution (generalizes the solution of Item 3°):

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t),$$

where the functions $f(t)$, $g(t)$, $h(t)$, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations

$$f'_t = 6f^2 + 2fh + g^2, \quad \varphi'_t = 2(3f+h)\varphi + 2g\psi,$$
$$g'_t = 6(f+h)g, \quad \psi'_t = 2g\varphi + 2(f+3h)\psi,$$
$$h'_t = 6h^2 + 2fh + g^2, \quad \chi'_t = \varphi^2 + \psi^2 + 2(f+h)\chi.$$

The first three equations for f, g, and h can be solved independently (see Item 3°).

Special case. Solution:

$$w(x,t) = -\frac{y^2}{6t} + Cxt^{-1/3} + \frac{3}{2}C^2t^{1/3},$$

where C is an arbitrary constant.

5°. There is a "two-dimensional" solution in multiplicative separable form:

$$w(x,y,t) = (At+B)^{-1}\Theta(x,y),$$

where A and B are arbitrary constants, and the function Θ is determined by the stationary equation written out in Item 4° of equation 6.1.2.4 with $n = \alpha = 1$.

⊙ *References for equation* 6.1.2.2: S. S. Titov and V. A. Ustinov (1985), V. V. Pukhnachov (1995), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

3. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)\right].$

This is a special case of equation 6.1.2.4 with $n = -1$. This equation arises as a model for long van der Waals interactions in thin films of a fluid spreading on a solid surface if fourth-order effects are neglected; see Gennes (1885), Bertozzi and Pugh (1996).

1°. Traveling-wave solutions:

$$w(\xi) = -\frac{\alpha(k_1^2 + k_2^2)}{\lambda(\xi + A)}, \quad \xi = k_1 x + k_2 y + \lambda t,$$

$$w(\xi) = \left\{A + B\exp\left[\frac{\lambda\xi}{\alpha A(k_1^2 + k_2^2)}\right]\right\}^{-1},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

2°. Solutions:

$$w(x, y, t) = \frac{2\alpha t + B}{(\sin y + Ae^x)^2},$$

$$w(x, y, t) = \frac{2A^2\alpha t + C}{e^{2x}\sinh^2\left(Ae^{-x}\sin y + B\right)},$$

$$w(x, y, t) = \frac{C - 2A^2\alpha t}{e^{2x}\cosh^2\left(Ae^{-x}\sin y + B\right)},$$

$$w(x, y, t) = \frac{2A^2\alpha t + C}{e^{2x}\cos^2\left(Ae^{-x}\sin y + B\right)},$$

where A, B, and C are arbitrary constants.

3°. The exact solutions specified in Item 2° are special cases of a more general solution having the form of the product of two functions with different arguments:

$$w(x, y, t) = (A\alpha t + B)e^{\Theta(x,y)},$$

where A and B are arbitrary constants, and the function $\Theta(x, y)$ is a solution of the stationary equation

$$\Delta\Theta - Ae^{\Theta} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which is encountered in combustion theory. For solutions of this equation, see 9.2.1.1.

4°. Other exact solutions:

$$w(x, y, t) = \frac{2\sinh(\alpha t + C)\cosh(\alpha t + C)}{(x + A)^2\sinh^2(\alpha t + C) + (y + B)^2\cosh^2(\alpha t + C)},$$

$$w(x, y, t) = \left[\frac{1}{A + \alpha\mu^2 t} + B(A + \alpha\mu^2 t)e^{\mu x} \pm \frac{\sin(\mu y + \eta_0)}{A + \alpha\mu^2 t}\right]^{-1},$$

$$w(x, y, t) = \left[A\coth\theta(t) + B\sinh\theta(t)\,e^{\mu x} \pm A\frac{\sin(\mu y + \eta_0)}{\sinh\theta(t)}\right]^{-1},$$

$$w(x,y,t)=\left[A\cot\theta(t)+B\sin\theta(t)\,e^{\mu x}\pm A\frac{\sin(\mu y+\eta_0)}{\sin\theta(t)}\right]^{-1},$$

$$w(x,y,t)=\left[\frac{A}{\cos\theta(t)}\pm A\frac{1+\sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x+\xi_0)+sA\frac{1-\sin\theta(t)}{2\cos\theta(t)}\sin(\mu y+\eta_0)\right]^{-1},$$

$$w(x,y,t)=\left[-\frac{A}{\cos\theta(t)}\pm A\frac{1-\sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x+\xi_0)+sA\frac{1+\sin\theta(t)}{2\cos\theta(t)}\sin(\mu y+\eta_0)\right]^{-1},$$

$$w(x,y,t)=\left[\frac{A}{\sinh\theta(t)}\pm A\frac{1-\cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x+\xi_0)+sA\frac{1+\cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y+\eta_0)\right]^{-1},$$

$$w(x,y,t)=\left[-\frac{A}{\sinh\theta(t)}\pm A\frac{1+\cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x+\xi_0)+sA\frac{1-\cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y+\eta_0)\right]^{-1},$$

where $\theta(t)=\alpha\mu^2At+\tau_0$; A, B, μ, ξ_0, η_0, and τ_0 are arbitrary constants; and s is a parameter that admits the values 1 or –1 (the first solution was indicated by Pukhnachov, 1995).

By swapping the variables, $x\rightleftarrows y$, in the above relations, one can obtain another group of solutions (not written out here).

5°. Solutions with axial symmetry:

$$w(r,t)=\frac{\lambda^2r^{\lambda-2}}{r^\lambda+Ce^{\alpha t}},$$

$$w(r,t)=\frac{\lambda\varphi r^{\varphi-2}}{C_1+r^\varphi(\varphi\ln r-1)},\quad \varphi=\frac{\lambda}{\alpha t+C_2},$$

where $r=\sqrt{x^2+y^2}$ and C, C_1, C_2, and λ are arbitrary constants.

6°. The transformation $w=1/U$ leads to an equation of the form 6.1.4.4 with $\beta=0$:

$$\frac{\partial U}{\partial t}=\alpha U\Delta U-\alpha\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2\right].$$

7°. The original equation is invariant under the transformation

$$\bar x=\varphi(x,y),\quad \bar y=\psi(x,y),\quad \bar w=\frac{w}{\varphi_x^2+\varphi_y^2},\tag{1}$$

where φ and ψ are arbitrary independent harmonic conjugate functions that solve the equations

$$\psi_x=\varphi_y,\quad \psi_y=-\varphi_x\qquad(\psi_x\varphi_y-\psi_y\varphi_x\neq0).\tag{2}$$

Applying the transformation (1)–(2) to a solution $w=f(x,y,t)$ we have an infinite family of solutions

$$w(x,y,t)=(\varphi_x^2+\varphi_y^2)f(\varphi(x,y),\psi(x,y),t),$$

containing a pair of harmonic conjugate functions (2). In particular, taking solution $w=f(x,t)$ independent of y, i.e., satisfying the one-dimensional equation 5.1.10.2, $w_t=a(w_x/w)_x$, yields the solution

$$w(x,y,t)=(\varphi_x^2+\varphi_y^2)f(\varphi(x,y),t).$$

⊙ *References for equation* 6.1.2.3: G. A. Nariboli (1970), V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), V. A. Galaktionov and S. A. Posashkov (1989), J. R. King (1992), N. H. Ibragimov (1994), V. V. Pukhnachov (1995), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998, pp. 297–298), S. N. Aristov (1999), A. D. Polyanin and V. F. Zaitsev (2004, pp. 143–145).

4. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right)\right].$

This is a two-dimensional heat and mass transfer equation with power-law temperature-dependent thermal conductivity (diffusion coefficient), where n can be integer, fractional, and negative. This is a special case of equation 6.4.3.3 with $f(w) = \alpha w^n$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2/n} C_2^{1/n} w(\pm C_1 x + C_3, \pm C_1 y + C_4, C_2 t + C_5),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation. The plus or minus signs can be chosen arbitrarily.

2°. Traveling-wave solution:

$$w = \left[\frac{n\lambda(k_1 x + k_2 y + \lambda t + C)}{\alpha(k_1^2 + k_2^2)}\right]^{1/n},$$

where C, k_1, k_2, and λ are arbitrary constants.

3°. Traveling-wave solution in implicit form (generalizes the solution of Item 2°):

$$\alpha(k_1^2 + k_2^2)\int \frac{w^n\,dw}{\lambda w + C_1} = k_1 x + k_2 y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

4°. Multiplicative separable solution:

$$w(x, y, t) = f(t)\Theta(x, y), \qquad f(t) = (A\alpha n t + B)^{-1/n}.$$

Here, A and B are arbitrary constants, and the function $\Theta(x, y)$ is a solution of the two-dimensional stationary equation

$$\frac{\partial}{\partial x}\left(\Theta^n\frac{\partial\Theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Theta^n\frac{\partial\Theta}{\partial y}\right) + A\Theta = 0.$$

If $n \neq -1$, this equation can be reduced to

$$\Delta u + A(n+1)u^{\frac{1}{n+1}} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad u = \Theta^{n+1}.$$

5°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = F(z, t), \qquad z = k_1 x + k_2 y;$$
$$w(x, y, t) = G(r, t), \qquad r = \sqrt{x^2 + y^2};$$
$$w(x, y, t) = H(\xi_1, \xi_2), \qquad \xi_1 = k_1 x + \lambda_1 t, \quad \xi_2 = k_2 y + \lambda_2 t;$$
$$w(x, y, t) = t^\beta U(\eta_1, \eta_2), \qquad \eta_1 = x t^{-\frac{n\beta+1}{2}}, \quad \eta_2 = y t^{-\frac{n\beta+1}{2}};$$
$$w(x, y, t) = e^{2\beta t} V(\zeta_1, \zeta_2), \quad \zeta_1 = x e^{-\beta n t}, \quad \zeta_2 = y e^{-\beta n t},$$

where k_1, k_2, λ_1, λ_2, and β are arbitrary constants.

6°. See also equations 6.5.5.5 and 6.5.5.6 for the case of two space variables.

⊙ *References for equation* 6.1.2.4: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), S. S. Titov and V. A. Ustinov (1985), J. R. King (1993), V. V. Pukhnachov (1995).

5. $\dfrac{\partial w}{\partial t} = a_1 \dfrac{\partial}{\partial x}\left(w^{n_1}\dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^{n_2}\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 6.4.3.4 with $f(w) = a_1 w^{n_1}$ and $g(w) = a_2 w^{n_2}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = A^2 w(\pm A^{-n_1} Bx + C_1, \pm A^{-n_2} By + C_2, B^2 t + C_3),$$

where A, B, C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \frac{a_1 k_1^2 w^{n_1} + a_2 k_2^2 w^{n_2}}{\lambda w + C_1}\, dw = k_1 x + k_2 y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x, y, t) = t^k U(\xi, \eta), \quad \xi = x t^{-\frac{1}{2}(kn_1+1)}, \quad \eta = y t^{-\frac{1}{2}(kn_2+1)},$$

where k is an arbitrary constant and the function $U(\xi, \eta)$ is determined by the differential equation

$$kU - \tfrac{1}{2}(kn_1 + 1)\xi \frac{\partial U}{\partial \xi} - \tfrac{1}{2}(kn_2 + 1)\eta \frac{\partial U}{\partial \eta} = a_1 \frac{\partial}{\partial x}\left(U^{n_1}\frac{\partial U}{\partial x}\right) + a_2 \frac{\partial}{\partial y}\left(U^{n_2}\frac{\partial U}{\partial y}\right).$$

4°. "Two-dimensional" solution:

$$w(x, y, t) = e^{2\beta t} V(z_1, z_2), \quad z_1 = x e^{-\beta n_1 t}, \quad z_2 = y e^{-\beta n_2 t},$$

where β is an arbitrary constant and the function $V(z_1, z_2)$ is determined by the differential equation

$$2\beta V - \beta n_1 z_1 \frac{\partial V}{\partial z_1} - \beta n_2 z_2 \frac{\partial V}{\partial z_2} = a_1 \frac{\partial}{\partial z_1}\left(V^{n_1}\frac{\partial V}{\partial z_1}\right) + a_2 \frac{\partial}{\partial z_2}\left(V^{n_2}\frac{\partial V}{\partial z_2}\right).$$

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = F(\xi_1, \xi_2), \quad \xi_1 = \alpha_1 x + \beta_1 y + \gamma_1 t, \quad \xi_2 = \alpha_2 x + \beta_2 y + \gamma_2 t,$$

where the α_i, β_i, and γ_i are arbitrary constants.

⊙ *References for equation* 6.1.2.5: A. A. Samarskii and I. M. Sobol' (1963), V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994, pp. 155–158).

6.1.3. Equations of the Form $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]+h(w)$

1. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[(bw+c)\dfrac{\partial w}{\partial y}\right]$.

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_2 x + C_3, \pm C_1 C_2 y + C_4, C_2^2 t + C_5) + \frac{c(1-C_1^2)}{bC_1^2},$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$bk_2^2 w + (ak_1^2 + ck_2^2 - C_1 bk_2^2)\ln|w + C_1| = \lambda(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w = u(z) - 4abC_1^2 x^2 - 4abC_1 C_2 x, \qquad z = y + bC_1 x^2 + bC_2 x + C_3 t,$$

where C_1, C_2, and C_3 are arbitrary constants and the function $u(z)$ is determined by the first-order ordinary differential equation

$$(bu + c + ab^2 C_2^2)u'_z + (2abC_1 - C_3)u = 8a^2 bC_1^2 z + C_4.$$

With appropriate translations in both variables, one can reduce this equation to a homogeneous one, which can be integrated by quadrature.

4°. Generalized separable solution linear in y (a degenerate solution):

$$w = F(x,t)y + G(x,t),$$

where the functions F and G are determined by solving the one-dimensional equations

$$\frac{\partial F}{\partial t} - a\frac{\partial^2 F}{\partial x^2} = 0, \tag{1}$$

$$\frac{\partial G}{\partial t} - a\frac{\partial^2 G}{\partial x^2} = bF^2. \tag{2}$$

Equation (1) is a linear homogeneous heat equation. Given $F = F(x,t)$, equation (2) can be treated as a linear nonhomogeneous heat equation. For these equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. Generalized separable solution quadratic in y:

$$w = f(x,t)y^2 + g(x,t)y + h(x,t),$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are determined by the system of differential equations

$$f_t = af_{xx} + 6bf^2,$$
$$g_t = ag_{xx} + 6bfg,$$
$$h_t = ah_{xx} + bg^2 + 2bfh + 2cf.$$

Here, the subscripts denote partial derivatives.

6°. "Two-dimensional" solution:

$$w = |y + C|^{1/2}\theta(x,t) - \frac{c}{b},$$

where the function $\theta(x,t)$ is determined by the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

7°. "Two-dimensional" solution:

$$w = U(\xi,t) - \frac{aC_1^2 + cC_2^2}{bC_2^2}, \qquad \xi = C_1 x + C_2 y,$$

where C_1 and C_2 are arbitrary constants and the function $U(\xi,t)$ is determined by a differential equation of the form 5.10.1.1:

$$\frac{\partial U}{\partial t} = bC_2^2 \frac{\partial}{\partial \xi}\left(U\frac{\partial U}{\partial \xi}\right).$$

8°. "Two-dimensional" solution:

$$w = V(\eta,t) - 4abC_1^2x^2 - 4abC_1C_2x, \qquad \eta = y + bC_1x^2 + bC_2x,$$

where C_1 and C_2 are arbitrary constants and the function $V(\eta,t)$ is determined by the differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial \eta}\left[(bV + c + ab^2C_2^2)\frac{\partial V}{\partial \eta}\right] + 2abC_1\frac{\partial V}{\partial \eta} - 8a^2bC_1^2.$$

9°. Solution:

$$w = U(z) + f(x,t), \qquad z = y + g(x,t).$$

Here,

$$f(x,t) = -\frac{C_2^2}{2\sqrt{a}\,b}xt + \frac{C_2^2}{2b}t^2 - \frac{2\sqrt{a}\,BC_2}{b}t + \frac{C_3}{b} - \frac{1}{b}(C_2\eta + 4aB)\psi'(\eta),$$
$$g(x,t) = \frac{C_2}{8a}\left(x^2 + 2\sqrt{a}\,xt - 3at^2\right) + B(x + \sqrt{a}\,t) + \psi(\eta) \quad \eta = x - t\sqrt{a},$$

where $\psi(\eta)$ is an arbitrary function and the prime denotes a derivative with respect to η.

⊙ *References for equation* 6.1.3.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 147–148), A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov (2005, pp. 101–102).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(\alpha w + \beta)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(\alpha w + \beta)\dfrac{\partial w}{\partial y}\right].$

This is a two-dimensional heat and mass transfer equation with a linear temperature-dependent thermal conductivity (diffusion coefficient).

The substitution $U = \alpha w + \beta$ leads to an equation of the form 6.1.2.2 for $U = U(x,y,t)$.

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{\alpha w+\beta}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{\alpha w+\beta}\dfrac{\partial w}{\partial y}\right).$

This is a two-dimensional heat and mass transfer equation with a hyperbolic temperature-dependent thermal conductivity (diffusion coefficient).

The substitution $U = \alpha w + \beta$ leads to an equation of the form 6.1.2.3 for $U = U(x, y, t)$.

4. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)\right] + \beta w.$

1°. The transformation

$$w(x, y, t) = e^{\beta t}u(x, y, \tau), \quad \tau = C - \frac{1}{\beta}e^{-\beta t},$$

where C is an arbitrary constant, leads to a simpler equation of the form 6.1.2.3:

$$\frac{\partial u}{\partial \tau} = \alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right].$$

2°. In Zhuravlev (2000), a nonlinear superposition principle is presented that allows the construction of complicated multimodal solutions of the original equation; some exact solutions are also specified there.

5. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)\right] + \beta w^2.$

The substitution $w = 1/U$ leads to an equation of the form 6.1.4.4 for $U = U(x, y, t)$.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994, pp. 157–158).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{1/2}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w^{1/2}\dfrac{\partial w}{\partial y}\right) + aw + bw^{1/2}.$

Solution:

$$w = [f(x, y) + g(x, y)e^{at/2}]^2,$$

where the functions $f = f(x, y)$ and $g = g(x, y)$ are described by the system of stationary equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + af^2 + bf = 0, \qquad \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + afg + bg = 0.$$

The former equation does not include g and is solved for f independently (it is a special case of equation 6.4.1.1). The latter equation is linear in g and is easy to solve once f has been found.

⊙ *Reference*: O. V. Kaptsov and I. V. Verevkin (2003).

7. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right)\right] + \beta w.$

The transformation (C is an arbitrary constant)

$$w(x, y, t) = e^{\beta t}U(x, y, \tau), \quad \tau = \frac{1}{\beta n}e^{\beta n t} + C$$

leads to a simpler equation of the form 6.1.2.4:

$$\frac{\partial U}{\partial \tau} = \alpha\left[\frac{\partial}{\partial x}\left(U^n\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(U^n\frac{\partial U}{\partial y}\right)\right].$$

8. $\dfrac{\partial w}{\partial t} = a_1 \dfrac{\partial}{\partial x}\left(w^{n_1}\dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^{n_2}\dfrac{\partial w}{\partial y}\right) + bw^k.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = A^2 w(\pm A^{k-n_1-1}x + B_1, \pm A^{k-n_2-1}y + B_2, A^{2k-2}t + B_3),$$

where A, B_1, B_2, and B_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x, y, t) = u(z), \quad z = t - \lambda_1 x - \lambda_2 y,$$

where λ_1 and λ_2 are arbitrary constants, and the function $u = u(z)$ is determined by the ordinary differential equation

$$u'_z = [(a_1\lambda_1^2 u^{n_1} + a_2\lambda_2^2 u^{n_2})u'_z]'_z + bu^k.$$

3°. "Two-dimensional" solution:

$$w(x, y, t) = (\alpha t + \beta)^{\frac{1}{1-k}} F(\xi, \eta), \quad \xi = x(\alpha t + \beta)^{\frac{n_1-k+1}{2(k-1)}}, \quad \eta = y(\alpha t + \beta)^{\frac{n_2-k+1}{2(k-1)}},$$

where the function $F = F(\xi, \eta)$ is determined by the differential equation

$$\frac{\alpha}{1-k}F + \alpha\frac{n_1-k+1}{2(k-1)}\xi\frac{\partial F}{\partial \xi} + \alpha\frac{n_2-k+1}{2(k-1)}\eta\frac{\partial F}{\partial \eta} = a_1\frac{\partial}{\partial \xi}\left(F^{n_1}\frac{\partial F}{\partial \xi}\right) + a_2\frac{\partial}{\partial \eta}\left(F^{n_2}\frac{\partial F}{\partial \eta}\right) + bF^k.$$

⊙ *References for equation* 6.1.3.8: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), M. I. Bakirova, S. N. Dimova, V. A. Dorodnitsyn, S. P. Kurdyumov, A. A. Samarskii, and S. R. Svirshchevskii (1988), N. H. Ibragimov (1994, pp. 156–158).

6.1.4. Other Equations

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial y}.$

Nonlinear Kolmogorov type equation. This equation arises in mathematical finance, when studying agents' decisions under risk [see Citti, Pascucci, and Polidoro (2001), Pascucci and Polidoro (2003), and Pascucci (2005)].

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = A^{-1}w(\pm B^{1/2}x + D_1, ABy + ABCbt + D_2, Bt + D_3) + C,$$

where A, B, C, D_1, D_2, and D_3 are arbitrary constants, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w = Au(z,t) + B, \qquad z = x + \lambda y + bB\lambda t,$$

where A, B, and λ are arbitrary constants, and the function $u = u(x,t)$ is described by the unnormalized Burgers equation of the form 5.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + Ab\lambda u\frac{\partial u}{\partial z}.$$

3°. Generalized separable solution:

$$w = \frac{u(x,t) - y}{b(t + C)}, \tag{1}$$

where C is an arbitrary constant and the function $u = u(x,t)$ is described by the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

4°. Generalized separable solution (generalizes the solution from Item 3°):

$$w = f(x,t)y + g(x,t), \tag{2}$$

where the functions $f = f(x,t)$ and $g = g(x,t)$ are determined by the system of differential equations

$$f_t = af_{xx} + bf^2,$$
$$g_t = ag_{xx} + bfg.$$

The first equation for f coincides, up to obvious renaming, with equation 5.1.1.1. The second equation has a particular solution $g = Cf$.

Remark. Solutions (1) and (2) are generalizations of the solutions pointed out by Lagno and Stognii (2010):

$$w = -\frac{y}{bt} + \varphi(z), \qquad z = \frac{x}{\sqrt{t}}, \qquad a\varphi''_{zz} + \frac{1}{2}z\varphi'_z - \varphi = 0;$$
$$w = -\frac{y}{bt} + \frac{1}{t}\varphi(z), \qquad z = \frac{x}{\sqrt{t}}, \qquad a\varphi''_{zz} + \frac{1}{2}z\varphi'_z = 0;$$
$$w = -\frac{y}{bt} \pm \frac{\ln t}{bt} + \frac{1}{t}\varphi(z), \qquad z = \frac{x}{\sqrt{t}}, \qquad a\varphi''_{zz} + \frac{1}{2}z\varphi'_z \mp \frac{1}{b} = 0;$$
$$w = \frac{y}{t}\varphi(z), \qquad z = \frac{x}{\sqrt{t}}, \qquad a\varphi''_{zz} + \frac{1}{2}z\varphi'_z + \varphi + b\varphi^2 = 0.$$

The second and third equations for φ are integrable by quadrature using the substitution $\psi(x) = \varphi'_x$. The solution of the first equation is expressed in terms of degenerate hypergeometric functions.

5°. "Two-dimensional" solution:

$$w = t^k\theta(\xi,\eta), \qquad \xi = xt^{-1/2}, \qquad \eta = yt^{-k-1},$$

where k is an arbitrary constant and the function $\theta = \theta(\xi,\eta)$ is determined by the differential equation

$$ku - \tfrac{1}{2}\xi u_\xi - (k+1)\eta u_\eta = au_{\xi\xi} + buu_\eta.$$

2. $\dfrac{\partial w}{\partial t} = aw\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(C_2 x + C_3, C_2 y + C_4, C_1 C_2^2 t + C_5),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x, y, t) = \varphi(t)(C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x + C_5 y + C_6), \qquad \varphi(t) = \frac{1}{C_7 - 2a(C_1 + C_3)t},$$

where $C_1, \ldots, C_7$ are arbitrary constants.

3°. The equation admits a more general solution in the form of the product of functions with different arguments:

$$w(x, y, t) = \frac{\Theta(x, y)}{A + Bt},$$

where A and B are arbitrary constants, and the function $\Theta = \Theta(x, y)$ satisfies the two-dimensional Poisson equation

$$a\Delta\Theta + B = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. Traveling-wave solution in implicit form:

$$a(k_1^2 + k_2^2)\int \frac{dw}{\lambda \ln|w| + C_1} = k_1 x + k_2 y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

5°. The equation has "two-dimensional" solutions with the following forms:

$$w(x, y, t) = F(z, t), \qquad z = k_1 x + k_2 y;$$
$$w(x, y, t) = G(r, t), \qquad r = \sqrt{x^2 + y^2};$$
$$w(x, y, t) = H(\xi_1, \xi_2), \qquad \xi_1 = k_1 x + \lambda_1 t, \quad \xi_2 = k_2 y + \lambda_2 t;$$
$$w(x, y, t) = t^{\beta} U(\eta_1, \eta_2), \quad \eta_1 = x^2 t^{-\beta-1}, \quad \eta_2 = y^2 t^{-\beta-1};$$
$$w(x, y, t) = e^{2t} V(\zeta_1, \zeta_2), \quad \zeta_1 = xe^{-t}, \quad \zeta_2 = ye^{-t},$$

where k_1, k_2, λ_1, λ_2, and β are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = (\alpha + \beta w)\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + \gamma w^2 + \delta w + \varepsilon.$

Generalized separable solution:

$$w(x, y, t) = f(t) + g(t)\Theta(x, y).$$

Here, $\Theta(x, y)$ is any solution of the two-dimensional Helmholtz equation

$$\Delta\Theta + \kappa\Theta = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

where $\kappa = \gamma/\beta$ ($\beta \neq 0$). For solutions of the linear equation (1), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The functions $f(t)$ and $g(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f'_t &= \gamma f^2 + \delta f + \varepsilon, \\ g'_t &= (\gamma f + \delta - \alpha\kappa)g. \end{aligned} \tag{2}$$

The first equation in (2) is independent of $g(t)$ and is separable. On finding $f(t)$, one can solve the second equation in (2), which is linear in $g(t)$.

The functions $f(t)$ and $g(t)$ have different forms depending on the values of the equation parameters. Below are five possible cases; C_1 and C_2 are arbitrary constants.

1°. For $\gamma = \delta = 0$,

$$f(t) = C_1 + \varepsilon t, \quad g(t) = C_2 e^{-\alpha\kappa t}.$$

2°. For $\gamma = 0$ and $\delta \neq 0$,

$$f(t) = C_1 e^{\delta t} - \frac{\varepsilon}{\delta}, \quad g(t) = C_2 e^{(\delta - \alpha\kappa)t}.$$

3°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = \mu^2 > 0$ ($\mu > 0$),

$$f(t) = \frac{s_1 + s_2 C_1 e^{\mu t}}{1 + C_1 e^{\mu t}}, \quad g(t) = \frac{C_2}{1 + C_1 e^{\mu t}} e^{-(\gamma s_2 + \alpha\kappa)t}, \qquad s_{1,2} = \frac{-\delta \pm \mu}{2\gamma}.$$

4°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = 0$,

$$f(t) = -\frac{\delta}{2\gamma} - \frac{1}{C_1 + \gamma t}, \quad g(t) = \frac{C_2}{C_1 + \gamma t} \exp\left[\left(\tfrac{1}{2}\delta - \alpha\kappa\right)t\right].$$

5°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = -\mu^2 < 0$ ($\mu > 0$),

$$f(t) = \frac{\mu}{2\gamma} \tan\left(\tfrac{1}{2}\mu t + C_1\right) - \frac{\delta}{2\gamma}, \quad g(t) = C_2 \frac{\exp\left[\left(\tfrac{1}{2}\delta - \alpha\kappa\right)t\right]}{\cos\left(\tfrac{1}{2}\mu t + C_1\right)}.$$

⊙ *References for equation* 6.1.4.3: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994, pp. 155–158).

4. $\dfrac{\partial w}{\partial t} = \alpha w\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - \alpha\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] - \beta.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 y + C_3, C_1 t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution:

$$w(x, y, t) = C_1 - \beta t + C_2 \exp\left[\alpha(\mu^2 + \nu^2)\left(C_1 t - \tfrac{1}{2}\beta t^2\right)\right] e^{\mu x + \nu y},$$

where μ, ν, C_1, and C_2 are arbitrary constants.

3°. Solutions for $\beta = 0$:

$$w(x, y, t) = \frac{1}{A + \alpha\mu^2 t} + B(A + \alpha\mu^2 t)e^{\mu x} \pm \frac{\sin(\mu y + \eta_0)}{A + \alpha\mu^2 t},$$
$$w(x, y, t) = A\coth\theta(t) + B\sinh\theta(t)\, e^{\mu x} \pm A\frac{\sin(\mu y + \eta_0)}{\sinh\theta(t)},$$
$$w(x, y, t) = A\cot\theta(t) + B\sin\theta(t)\, e^{\mu x} \pm A\frac{\sin(\mu y + \eta_0)}{\sin\theta(t)},$$
$$w(x, y, t) = \frac{A}{\cos\theta(t)} \pm A\frac{1 + \sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 - \sin\theta(t)}{2\cos\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = -\frac{A}{\cos\theta(t)} \pm A\frac{1 - \sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 + \sin\theta(t)}{2\cos\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = \frac{A}{\sinh\theta(t)} \pm A\frac{1 - \cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 + \cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = -\frac{A}{\sinh\theta(t)} \pm A\frac{1 + \cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 - \cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y + \eta_0),$$

where A, B, μ, ξ_0, η_0, and τ_0 are arbitrary constants, $\theta(t) = \alpha\mu^2 At + \tau_0$, and s is a parameter that can assume the values 1 or –1.

By swapping the variables, $x \rightleftarrows y$, in the above relations, one can obtain another group of solutions (not written out here).

4°. There are generalized separable solutions of the form

$$w(x, y, t) = f(t) + g(t)\varphi(x) + h(t)\psi(y). \tag{1}$$

In particular, if $\varphi''_{xx} = \nu\varphi$ and $\psi''_{yy} = -\nu\psi$, where ν is an arbitrary constant, we have

$$\varphi(x) = A_1\cosh\mu x + A_2\sinh\mu x, \quad \psi(y) = B_1\cos\mu y + B_2\sin\mu y \qquad (\nu = \mu^2 > 0),$$
$$\varphi(x) = A_1\cos\mu x + A_2\sin\mu x, \qquad \psi(y) = B_1\cosh\mu y + B_2\sinh\mu y \quad (\nu = -\mu^2 < 0).$$

Here, A_1, A_2, B_1, and B_2 are arbitrary constants. The functions $f(t)$, $g(t)$, and $h(t)$ in (1) are determined by the system of ordinary differential equations

$$f'_t = \alpha\nu(A_1^2 - sA_2^2)g^2 - \alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,$$
$$g'_t = \alpha\nu fg,$$
$$h'_t = -\alpha\nu fh,$$

where $s = \operatorname{sign}\nu$. The order of this system can be reduced by 2; then the system becomes

$$f = \Phi(h), \quad g = C_2/h, \quad h'_t = -\alpha\nu h\Phi(h),$$

where

$$\Phi(h) = \pm\sqrt{C_1 + (A_1^2 - sA_2^2)\frac{C_2^2}{h^2} + \frac{2\beta}{\alpha\nu}\ln|h| + (B_1^2 + sB_2^2)h^2}\,;$$

C_1 and C_2 are arbitrary constants. For $\beta = 0$, solutions in explicit form may be obtained in some cases (see Item 3°).

5°. There are generalized separable solutions of the form

$$w(x, y, t) = f(t) + g(t)\varphi(x) + h(t)\psi(y) + u(t)\theta(x)\chi(y). \tag{2}$$

For $\varphi''_{xx} = 4\nu\varphi$, $\psi''_{yy} = -4\nu\psi$, $\theta''_{xx} = \nu\theta$, and $\chi''_{yy} = -\nu\chi$, where ν is an arbitrary constant, one can set in (2)

for $\nu = \mu^2 > 0$	for $\nu = -\mu^2 < 0$
$\varphi(x) = A_1\cosh 2\mu x + A_2\sinh 2\mu x$	$\varphi(x) = A_1\cos 2\mu x + A_2\sin 2\mu x$
$\psi(y) = B_1\cos 2\mu y + B_2\sin 2\mu y$	$\psi(y) = B_1\cosh 2\mu y + B_2\sinh 2\mu y$
$\theta(x) = C_1\cosh \mu x + C_2\sinh \mu x$	$\theta(x) = C_1\cos \mu x + C_2\sin \mu x$
$\chi(y) = D_1\cos \mu y + D_2\sin \mu y$	$\chi(y) = D_1\cosh \mu y + D_2\sinh \mu y$

The functions $f(t)$, $g(t)$, $h(t)$, and $u(t)$ are determined by the following system of ordinary differential equations ($s = \operatorname{sign}\nu$):

$$\begin{aligned}
f'_t &= -4\alpha\nu(A_1^2 - sA_2^2)g^2 + 4\alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,\\
g'_t &= -4\alpha\nu fg + \alpha\nu a_1(D_1^2 + sD_2^2)u^2,\\
h'_t &= 4\alpha\nu fh - \alpha\nu a_2(C_1^2 - sC_2^2)u^2,\\
u'_t &= -2\alpha\nu(a_3 g - a_4 h)u.
\end{aligned}$$

The arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are related by the two constraints

$$2A_1C_1C_2 = A_2(C_1^2 + sC_2^2), \quad 2B_1D_1D_2 = B_2(D_1^2 - sD_2^2).$$

The coefficients a_1, a_2, a_3, and a_4 are defined by

$$a_1 = \frac{C_1^2 + sC_2^2}{2A_1}, \quad a_2 = \frac{D_1^2 - sD_2^2}{2B_1}, \quad a_3 = A_2\frac{C_1^2 - sC_2^2}{C_1C_2}, \quad a_4 = B_2\frac{D_1^2 + sD_2^2}{D_1D_2},$$

provided $A_1 \neq 0$, $B_1 \neq 0$, $C_1C_2 \neq 0$, and $D_1D_2 \neq 0$.

If $A_1 = 0$ ($A_2 \neq 0$), then one should set $a_1 = C_1C_2/A_2$. If $B_1 = 0$ ($B_2 \neq 0$), then $a_2 = D_1D_2/B_2$. If $C_1 = 0$ ($C_2 \neq 0$), then $a_3 = -A_1$. If $C_2 = 0$ ($C_1 \neq 0$), then $a_3 = A_1$. If $D_1 = 0$ ($D_2 \neq 0$), then $a_4 = -B_1$. If $D_2 = 0$ ($D_1 \neq 0$), then $a_4 = B_1$.

6°. The equation admits a traveling-wave solution: $w = w(k_1x + k_2y + \lambda t)$, where k_1, k_2, and λ are arbitrary constants.

7°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

In the special case $\varphi(t) = \psi(t) \equiv 0$, the functions $f(t)$, $g(t)$, $h(t)$, and $\chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f'_t &= \alpha(2fh - 2f^2 - g^2), \qquad & h'_t &= \alpha(2fh - 2h^2 - g^2),\\ g'_t &= -2\alpha g(f+h), & \chi'_t &= 2\alpha(f+h)\chi - \beta, \end{aligned}$$

which can be completely integrated.

8°. In the special case $\beta = 0$, the original equation is invariant under the transformation

$$\bar{x} = \varphi(x,y), \quad \bar{y} = \psi(x,y), \quad \bar{w} = (\varphi_x^2 + \varphi_y^2)w, \tag{1}$$

where φ and ψ are arbitrary independent harmonic conjugate functions solving the equations

$$\psi_x = \varphi_y, \quad \psi_y = -\varphi_x \qquad (\psi_x\varphi_y - \psi_y\varphi_x \neq 0). \tag{2}$$

Applying the transformation (1)–(2) to a solution $w = f(x,y,t)$ we have an infinite family of solutions

$$w(x,y,t) = \frac{f(\varphi(x,y), \psi(x,y), t)}{\varphi_x^2 + \varphi_y^2},$$

containing a pair of harmonic conjugate functions (2). In particular, taking solution $w = f(x,t)$ independent of y, i.e., satisfying the one-dimensional equation 5.1.9.8 with $b = 0$, $w_t = a(ww_{xx} - w_x^2)$, yields the solution

$$w(x,y,t) = \frac{f(\varphi(x,y), t)}{\varphi_x^2 + \varphi_y^2}.$$

⊙ *References for equation* 6.1.4.4: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998, pp. 293–295), A. D. Polyanin and V. F. Zaitsev (2002, pp. 100–101), V. A. Galaktionov and S. R. Svirshchevskii (2007).

5. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(|\nabla w|\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(|\nabla w|\dfrac{\partial w}{\partial y}\right)\right] + \beta w^2.$

Here, $|\nabla w|^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= C_1 w(\pm x + C_2, \pm y + C_3, C_1 t + C_4),\\ w_2 &= w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t), \end{aligned}$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,\eta), \quad \xi = C_1x + C_1y + C_3t, \quad \eta = C_4x + C_5y + C_6t,$$
$$w(x,y,t) = (C_1t + C_2)^{-1}V(x,y).$$

3°. "Two-dimensional" generalized separable solution:

$$w(x,y,t) = \frac{1}{C_1 - \beta t} + \frac{C_2}{(C_1 - \beta t)^2}F(x,y),$$

where C_1 and C_2 are arbitrary constants, and the function $F(x,y)$ is any solution of the stationary equation

$$\frac{\partial}{\partial x}\left(|\nabla F|\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla F|\frac{\partial F}{\partial y}\right) + \kappa F^2 = 0, \qquad \kappa = \frac{\beta}{\alpha}\operatorname{sign} C_2.$$

4°. "Two-dimensional" generalized separable solution:

$$w(x,y,t) = f(t) + g(t)\Theta(x,y).$$

Here, the functions $f(t)$ and $g(t)$ are given by

$$f(t) = \frac{1}{B - \beta t}, \quad g(t) = \frac{\beta}{(B - \beta t)[A + C(B - \beta t)]},$$

where A, B, and C are arbitrary constants, and the function $\Theta(x,y)$ is any solution of the stationary equation

$$\frac{\partial}{\partial x}\left(|\nabla\Theta|\frac{\partial\Theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla\Theta|\frac{\partial\Theta}{\partial y}\right) \pm \kappa\Theta^2 = \mu\Theta, \qquad \kappa = \frac{\beta}{\alpha}, \quad \mu = \frac{A}{\alpha}.$$

⊙ *References for equation* 6.1.4.5: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994).

6.2. Equations with Two Space Variables Involving Exponential Nonlinearities

6.2.1. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + ae^{\lambda w}$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + ce^{\lambda w}.$

This is a special case of equation 6.4.2.1 with $f(w) = ce^{\lambda w}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{1}{2-n}}x,\ C_1^{\frac{1}{2-m}}y,\ C_1t + C_2\Big) + \frac{1}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(r,t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2};$$
$$w(x,y,t) = V(z_1,z_2) - \frac{1}{\lambda}\ln t, \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = yt^{\frac{1}{m-2}}.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + ce^{\beta w}.$

This is a special case of equation 6.4.2.3 with $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{1}{2-n}}x,\ y - \frac{1}{\lambda}\ln C_1,\ C_1 t + C_2\Big) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = V(z_1, z_2) - \frac{1}{\beta}\ln t, \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + ce^{\mu w}.$

This is a special case of equation 6.4.2.2 with $f(w) = ce^{\mu w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(x - \frac{1}{\beta}\ln C_1,\ y - \frac{1}{\lambda}\ln C_1,\ C_1 t + C_2\Big) + \frac{1}{\mu}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = V(z_1, z_2) - \frac{1}{\mu}\ln t, \quad z_1 = x + \frac{1}{\beta}\ln t, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

6.2.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right) + f(w)$

1. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(e^{\mu w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(e^{\mu w}\dfrac{\partial w}{\partial y}\right)\right].$

This is a two-dimensional nonstationary heat (diffusion) equation with exponential temperature (concentration) dependent thermal diffusivity (diffusion coefficient). This is a special case of equation 6.4.3.3 with $f(w) = \alpha e^{\mu w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(C_1 x + C_3, C_1 y + C_4, C_2 t + C_5) + \frac{1}{\mu}\ln\frac{C_2}{C_1^2},$$

$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$\alpha(k_1^2+k_2^2)\int \frac{e^{\mu w}\,dw}{\lambda w+C_1} = k_1x+k_2y+\lambda t+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. "Two-dimensional" additive separable solution:

$$w(x,y,t) = f(t)+\frac{1}{\mu}\ln\Theta(x,y), \qquad f(t) = -\frac{1}{\mu}\ln(A\alpha t+B).$$

Here, A, B, and μ are arbitrary constants, and the function $\Theta(x,y)$ is any solution of the Poisson equation

$$\Delta\Theta + A = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,t) = F(z,t), && z = k_1x+k_2y;\\
&w(x,y,t) = G(r,t), && r = \sqrt{x^2+y^2};\\
&w(x,y,t) = G(\xi_1,\xi_2), && \xi_1 = k_1x+\lambda_1t,\ \ \xi_2 = k_2y+\lambda_2t;\\
&w(x,y,t) = H(\eta_1,\eta_2), && \eta_1 = x^2/t,\quad \eta_2 = y^2/t;\\
&w(x,y,t) = \frac{2}{\mu}t+U(\zeta_1,\zeta_2), && \zeta_1 = xe^{-t},\quad \zeta_2 = ye^{-t},
\end{aligned}$$

where k_1, k_2, λ_1, and λ_2 are arbitrary constants.

⊙ *Reference for equation* 6.2.2.1: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1C_2^{\beta}x+C_3, \pm C_1C_2^{\lambda}y+C_4, C_1^2t+C_5)-2\ln|C_2|,$$

where $C_1,\ldots,C_5$ are arbitrary constants, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w(x,y,t) = u(\theta,t), \quad \theta = c_1x+c_2y,$$

where c_1 and c_2 are arbitrary constants, and the function $u = u(\theta,t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial\theta}\left[\left(ac_1^2e^{\beta w}+bc_2^2e^{\lambda w}\right)\frac{\partial w}{\partial\theta}\right].$$

3°. "Two-dimensional" solution:

$$w(x,y,t) = u(\zeta_1,\zeta_2), \quad \zeta_1 = k_1x+m_1t, \quad \zeta_2 = k_2y+m_2t,$$

where k_1, k_2, m_1, and m_2 are arbitrary constants, and the function $u = u(\zeta_1, \zeta_2)$ is determined by the differential equation

$$m_1\frac{\partial u}{\partial \zeta_1} + m_2\frac{\partial u}{\partial \zeta_2} = ak_1^2\frac{\partial}{\partial \zeta_1}\left(e^{\beta w}\frac{\partial w}{\partial \zeta_1}\right) + bk_2^2\frac{\partial}{\partial \zeta_2}\left(e^{\lambda w}\frac{\partial w}{\partial \zeta_2}\right).$$

4°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, \eta) + 2kt, \quad \xi = xe^{-k\beta t}, \quad \eta = ye^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U = U(\xi, \eta)$ is determined by the differential equation

$$2k - k\beta\xi\frac{\partial U}{\partial \xi} - k\lambda\eta\frac{\partial U}{\partial \eta} = a\frac{\partial}{\partial \xi}\left(e^{\beta U}\frac{\partial U}{\partial \xi}\right) + b\frac{\partial}{\partial \eta}\left(e^{\lambda U}\frac{\partial U}{\partial \eta}\right).$$

5°. "Two-dimensional" solution:

$$w(x, y, t) = V(z_1, z_2), \quad z_1 = \frac{x + C_1}{\sqrt{t + C_3}}, \quad z_2 = \frac{y + C_2}{\sqrt{t + C_3}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $V = V(z_1, z_2)$ is determined by the differential equation

$$-\frac{1}{2}z_1\frac{\partial V}{\partial z_1} - \frac{1}{2}z_2\frac{\partial V}{\partial z_2} = a\frac{\partial}{\partial z_1}\left(e^{\beta w}\frac{\partial w}{\partial z_1}\right) + b\frac{\partial}{\partial z_2}\left(e^{\lambda w}\frac{\partial w}{\partial z_2}\right).$$

6°. "Two-dimensional" solution ($\beta = 1$):

$$w(x, y, t) = u(z, t) + 2\ln|x|, \quad z = x^{-\lambda}y,$$

where the function $u = u(z, t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t} = \left(a\lambda^2 z^2 e^u + be^{\lambda u}\right)\frac{\partial^2 u}{\partial z^2} + \lambda\left(a\lambda z^2 e^u + be^{\lambda u}\right)\left(\frac{\partial u}{\partial z}\right)^2 + a\lambda(\lambda - 3)ze^u\frac{\partial u}{\partial z} + 2ae^u.$$

⊙ *References for equation* 6.2.2.2: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994, pp. 155–156).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + ce^{\beta w}.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1}x + C_2, \pm C_1^{\beta-\lambda_2}y + C_3, C_1^{2\beta}t + C_4) + 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = U(\xi, \eta) - \frac{1}{\beta}\ln t, \quad \xi = xt^{\frac{\lambda_1-\beta}{2\beta}}, \quad \eta = yt^{\frac{\lambda_2-\beta}{2\beta}}.$$

⊙ *References for equation* 6.2.2.3: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994, pp. 155–156).

4. $\dfrac{\partial w}{\partial t} = \alpha\left[\dfrac{\partial}{\partial x}\left(e^{\mu w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(e^{\mu w}\dfrac{\partial w}{\partial y}\right)\right] + \beta e^{\mu w} + \gamma + \delta e^{-\mu w}.$

The substitution $w = \dfrac{1}{\mu}\ln U$ leads to an equation of the form 6.1.4.3:

$$\frac{\partial U}{\partial t} = \alpha U\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \beta\mu U^2 + \mu\gamma U + \mu\delta.$$

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994, pp. 155–156).

6.3. Other Equations with Two Space Variables Involving Arbitrary Parameters

6.3.1. Equations with Logarithmic Nonlinearities

1. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - kw\ln w.$

This is a nonstationary equation with a logarithmic source arising in heat and mass transfer theory and combustion theory. This is a special case of equation 6.4.1.1 with $f(w) = -kw\ln w$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{-kt})w(x + C_2, \pm y + C_3, t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. "Two-dimensional" multiplicative separable solution:

$$w(x, y, t) = \exp(C_1 e^{-kt})\Theta(x, y),$$

where the function $\Theta(x, y)$ is a solution of the stationary equation

$$a\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) - k\Theta\ln\Theta = 0.$$

This equation has a particular solution of the form

$$\Theta = \exp(A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6),$$

where the coefficients A_k are determined by an algebraic system of equations.

3°. "Two-dimensional" solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x, y, t) = \varphi(x, t)\psi(y, t),$$

where the functions $\varphi(x, t)$ and $\psi(y, t)$ are determined by solving two independent one-dimensional nonlinear parabolic differential equations:

$$\frac{\partial\varphi}{\partial t} = a\frac{\partial^2\varphi}{\partial x^2} - k\varphi\ln\varphi,$$
$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial y^2} - k\psi\ln\psi.$$

For solutions of these equations, see 5.6.1.4 with $f(t) = 0$.

4°. There are exact solutions in the form of the product of functions representing two independent traveling waves:

$$w(x, y, t) = \varphi(\xi)\psi(\eta), \qquad \xi = a_1x + b_1t, \qquad \eta = a_2y + b_2t,$$

where a_1, b_1, a_2, and b_2 are arbitrary constants. This solution is a special case of the solution presented in Item 3°.

⊙ *References for equation* 6.3.1.1: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, p. 98).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + cw\ln w + sw.$

This is a special case of equation 6.4.2.1 with $f(w) = cw\ln w + sw$ and a special case of equation 6.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by ax^n and by^m, respectively.

There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw\ln w + sw.$

This is a special case of equation 6.4.2.2 with $f(w) = cw\ln w + sw$ and a special case of equation 6.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by $e^{\beta x}$ and $be^{\lambda y}$, respectively.

There are exact solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw\ln w + sw.$

This is a special case of equation 6.4.2.3 with $f(w) = cw\ln w + sw$ and a special case of equation 6.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by ax^n and $be^{\lambda y}$, respectively.

There are exact solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

6.3.2. Equations with Trigonometrical Nonlinearities

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + c\sin^k(pw+q).$

This is a special case of equation 6.4.2.1 with $f(w) = c\sin^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m}.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\sin^k(pw+q).$

This is a special case of equation 6.4.2.2 with $f(w) = c\sin^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\sin^k(pw+q).$

This is a special case of equation 6.4.2.3 with $f(w) = c\sin^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + c\tan^k(pw+q).$

This is a special case of equation 6.4.2.1 with $f(w) = c\tan^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m}.$$

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\tan^k(pw+q).$

This is a special case of equation 6.4.2.2 with $f(w) = c\tan^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\tan^k(pw+q).$

This is a special case of equation 6.4.2.3 with $f(w) = c\tan^k(pw+q)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

6.4. Equations Involving Arbitrary Functions

6.4.1. Heat and Mass Transfer Equations in Quiescent or Moving Media with Chemical Reactions

1. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer or combustion in a quiescent medium.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, t + C_3),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 can be chosen independently of each other).

2°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = Ax + By + \lambda t,$$

where A, B, and λ are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$a(A^2 + B^2)w''_{\xi\xi} - \lambda w'_\xi + f(w) = 0.$$

For solutions of this equation, see Polyanin and Zaitsev (1995, 2003).

3°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = F(z, t), \qquad z = k_1 x + k_2 y;$$
$$w(x, y, t) = G(r, t), \qquad r = \sqrt{x^2 + y^2};$$
$$w(x, y, t) = H(\xi_1, \xi_2), \quad \xi_1 = k_1 x + \lambda_1 t, \quad \xi_2 = k_2 y + \lambda_2 t,$$

where k_1, k_2, λ_1, and λ_2 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer with a volume chemical reaction in a steady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \qquad \xi = x - at, \qquad \eta = y - bt$$

leads to a simpler equation of the form 6.4.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} - f(U).$$

3. $\dfrac{\partial w}{\partial t} + (a_1 x + b_1 y + c_1)\dfrac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer with a volume chemical reaction in a steady translational-shear fluid flow.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(x + Cb_1 e^{\lambda t}, y + C(\lambda - a_1)e^{\lambda t}, t\big),$$

where C is an arbitrary constant and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0, \tag{1}$$

are also solutions of the equation.

2°. Solutions:

$$w = w(z), \qquad z = a_2x + (\lambda - a_1)y + Ce^{\lambda t},$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\big[\lambda z + a_2c_1 + (\lambda - a_1)c_2\big]w'_z = \big[a_2^2 + (\lambda - a_1)^2\big]w''_{zz} - f(w).$$

3°. "Two-dimensional" solutions:

$$w = U(\zeta, t), \qquad \zeta = a_2x + (\lambda - a_1)y,$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + \big[\lambda\zeta + a_2c_1 + (\lambda - a_1)c_2\big]\frac{\partial U}{\partial \zeta} = \big[a_2^2 + (\lambda - a_1)^2\big]\frac{\partial^2 U}{\partial \zeta^2} - f(U).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 = 0$.

⊙ *Reference for equation* 6.4.1.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 160).

4. $\dfrac{\partial w}{\partial t} + f_1(t)\dfrac{\partial w}{\partial x} + f_2(t)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - g(w).$

This equation describes mass transfer with volume chemical reaction in an unsteady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \qquad \xi = x - \int f_1(t)\,dt, \qquad \eta = y - \int f_2(t)\,dt,$$

leads to a simpler equation of the form 6.4.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} - g(U).$$

6.4.2. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + h(w)$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + f(w).$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with power-law coordinate-dependent principal thermal diffusivities (diffusion coefficients).

Solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi, t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m},$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{4 - nm}{(2-n)(2-m)}.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 5.1.1 to 5.1.3 and equations 5.2.1.1 to 5.2.1.3, 5.4.1.2, 5.4.1.3, 5.4.1.7, and 5.4.1.8.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 108).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + f(w).$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with exponential coordinate-dependent principal thermal diffusivities (diffusion coefficients).

Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y},$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 108).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + f(w).$

Solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y},$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{n}{2-n}\frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 108).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + aw\ln w + bw.$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with arbitrary coordinate-dependent principal thermal diffusivities (diffusion coefficients) and a logarithmic source.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = \exp(C_1 e^{at})w(x, y, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" multiplicative separable solution:

$$w(x, y, t) = \exp(C_1 e^{at})U(x, y),$$

where the function $U(x, y)$ is determined by the stationary equation

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial U}{\partial y}\right] + aU\ln U + bU = 0.$$

3°. "Two-dimensional" solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x,y,t)=\varphi(x,t)\psi(y,t),$$

where the functions $\varphi(x,t)$ and $\psi(y,t)$ are determined from the two independent one-dimensional nonlinear parabolic differential equations

$$\frac{\partial\varphi}{\partial t}=\frac{\partial}{\partial x}\left[f(x)\frac{\partial\varphi}{\partial x}\right]+a\varphi\ln\varphi+C(t)\varphi,$$
$$\frac{\partial\psi}{\partial t}=\frac{\partial}{\partial y}\left[g(y)\frac{\partial\psi}{\partial y}\right]+a\psi\ln\psi+b\psi-C(t)\psi,$$

and $C(t)$ is an arbitrary function.

⊙ *Reference for equation* 6.4.2.4: A. D. Polyanin (2000).

6.4.3. Equations of the Form $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]+h(t,w)$

1. $\dfrac{\partial w}{\partial t}=a\left[\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right)\right]+f(t)w.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=C_1^{-2}w\big(\pm C_1^n x+C_2,\ \pm C_1^n y+C_3,\ t\big),$$
$$w_2=w(x\cos\beta-y\sin\beta,\ x\sin\beta+y\cos\beta,\ t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 can be chosen independently of each other).

2°. Multiplicative separable solution:

$$w(x,y,t)=\exp\left[\int f(t)\,dt\right]\big[\Theta(x,y)\big]^{\frac{1}{n+1}},\tag{1}$$

where the function $\Theta(x,y)$ is a solution of the Laplace equation

$$\Delta\Theta=0,\qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

For solutions of this linear stationary equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Multiplicative separable solution:

$$w(x,y,t)=\varphi(t)\big[\Theta(x,y)\big]^{\frac{1}{n+1}},\tag{2}$$

where the function $\varphi(t)$ is a solution of the Bernoulli equation

$$\varphi_t'-f(t)\varphi+Aa\varphi^{n+1}=0,\tag{3}$$

A is an arbitrary constant, and the function $\Theta(x,y)$ is determined by the stationary equation

$$\Delta\Theta+A(n+1)\Theta^{\frac{1}{n+1}}=0,\qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The general solution of equation (3) is given by

$$\varphi(t)=\exp\big[F(t)\big]\left\{Aan\int\exp\big[nF(t)\big]\,dt+B\right\}^{-1/n},\quad F(t)=\int f(t)\,dt,$$

where B is an arbitrary constant.

4°. The transformation

$$w(x,y,t)=F(t)U(x,y,\tau),\quad \tau=\int F^n(t)\,dt,\qquad F(t)=\exp\left[\int f(t)\,dt\right]$$

leads to a simpler equation of the form 6.1.2.4:

$$\frac{\partial U}{\partial \tau}=a\left[\frac{\partial}{\partial x}\left(U^n\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(U^n\frac{\partial U}{\partial y}\right)\right].$$

2. $\dfrac{\partial w}{\partial t}=a\left[\dfrac{\partial}{\partial x}\left(e^{\mu w}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(e^{\mu w}\dfrac{\partial w}{\partial y}\right)\right]+f(t).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w\big(C_1x+C_2,\ \pm C_1y+C_3,\ t\big)-\frac{2}{\mu}\ln|C_1|,$$
$$w_2=w(x\cos\beta-y\sin\beta,\ x\sin\beta+y\cos\beta,\ t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y,t)=\varphi(t)+\frac{1}{\mu}\ln\Theta(x,y),$$

where the function $\varphi=\varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t+A(a/\mu)\exp(\mu\varphi)-f(t)=0, \tag{1}$$

and the function $\Theta(x,y)$ is a solution of the two-dimensional Poisson equation

$$\Delta\Theta+A=0,\qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{2}$$

The general solution of equation (1) is given by

$$\varphi(t)=F(t)-\frac{1}{\mu}\ln\left\{B+Aa\int\exp\big[\mu F(t)\big]\,dt\right\},\qquad F(t)=\int f(t)\,dt. \tag{3}$$

For solutions of the linear stationary equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Note that equations (1), (2) and relation (3) involve arbitrary constants A and B.

3°. The transformation

$$w(x,y,t)=U(x,y,\tau)+F(t),\quad \tau=\int\exp[\mu F(t)]\,dt,\qquad F(t)=\int f(t)\,dt,$$

leads to a simpler equation of the form 6.2.2.1:

$$\frac{\partial U}{\partial \tau}=a\left[\frac{\partial}{\partial x}\left(e^{\mu U}\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\mu U}\frac{\partial U}{\partial y}\right)\right].$$

⊙ *Reference for equation* 6.4.3.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 110).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right].$

This is a two-dimensional nonlinear heat and mass transfer equation for an anisotropic medium.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(C_1x + C_2, C_1y + C_3, C_1^2t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$(k_1^2 + k_2^2)\int \frac{f(w)\,dw}{\lambda w + C_1} = k_1x + k_2y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w(x, y, t) = U(\xi), \qquad \xi = \frac{x^2 + y^2}{t},$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$[\xi f(U)U'_\xi]'_\xi + \tfrac{1}{4}\xi U'_\xi = 0.$$

4°. "Two-dimensional" solutions (for the axisymmetric problems):

$$w(x, y, t) = V(r, t), \qquad r = \sqrt{x^2 + y^2},$$

where the function $V = V(r, t)$ is determined by the differential equation

$$\frac{\partial V}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rf(V)\frac{\partial V}{\partial r}\right].$$

5°. For other "two-dimensional" solutions, see equation 6.4.3.4 with $g(w) = f(w)$.

⊙ *Reference for equation* 6.4.3.3: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983).

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right].$

This is a two-dimensional unsteady heat (mass) transfer equation in an anisotropic case with arbitrary dependence of the principal thermal diffusivities (diffusion coefficients) on temperature (concentration).

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(C_1x + C_2, \pm C_1y + C_3, C_1^2t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$\int \frac{k_1^2 f(w) + k_2^2 g(w)}{\lambda w + C_1}\, dw = k_1 x + k_2 y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(z, t), \qquad z = k_1 x + k_2 y,$$

where the function $U = U(z, t)$ is determined by a differential equation of the form 5.6.15.1:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\left[\varphi(U)\frac{\partial U}{\partial z}\right], \qquad \varphi(U) = k_1^2 f(U) + k_2^2 g(U).$$

4°. There are more complicated "two-dimensional" solutions of the form

$$w(x, y, t) = V(\zeta_1, \zeta_2), \qquad \zeta_1 = a_1 x + a_2 y + a_3 t, \qquad \zeta_2 = b_1 x + b_2 y + b_3 t.$$

5°. "Two-dimensional" solution:

$$w(x, y, t) = W(\xi, \eta), \qquad \xi = \frac{x}{\sqrt{at}}, \qquad \eta = \frac{y}{\sqrt{at}},$$

where $a \neq 0$ is any number and the function $W = W(\xi, \eta)$ is determined by the differential equation

$$\frac{\partial}{\partial \xi}\left[f(W)\frac{\partial W}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[g(W)\frac{\partial W}{\partial \eta}\right] + \frac{a}{2}\xi\frac{\partial W}{\partial \xi} + \frac{a}{2}\eta\frac{\partial W}{\partial \eta} = 0.$$

6°. For group classification of the equation in question, see Dorodnitsyn, Knyazeva, and Svirshchevskii (1983).

5. $\dfrac{\partial w}{\partial t} = \Delta f(w) + \dfrac{a f(w) + b}{f'(w)} + c[a f(w) + b], \qquad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}.$

1°. Solution in implicit form with $a \neq 0$:

$$f(w) = \frac{1}{a}[\rho(x, y)e^{at} - b],$$

where $\rho = \rho(x, y)$ is an arbitrary solution of the linear two-dimensional *Helmholtz equation*

$$\Delta\rho + ac\rho = 0.$$

2°. Solution in implicit form with $a = 0$:

$$f(w) = \rho(x, y) + bt,$$

where $\rho = \rho(x, y)$ is an arbitrary solution of the linear two-dimensional *Poisson equation*

$$\Delta\rho + bc = 0.$$

⊙ *Reference for equation* 6.4.3.5: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 391).

6.4.4. Other Equations Linear in the Highest Derivatives

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial y}.$

Nonlinear Kolmogorov type equation. This equation arises in mathematical finance [see Pascucci and Polidoro (2003), and Pascucci (2005)].

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(Ax + B, A^2y + C, A^2t + D),$$

where A, B, C, and D are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" solution:

$$w = u(z,t), \qquad z = x + \lambda y + \mu t,$$

where λ and μ are arbitrary constants, and the function $u = u(x,t)$ is described by the generalized Burgers equation of the form 5.6.3.8:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + [\lambda f(u) - \mu]\frac{\partial u}{\partial z}.$$

3°. "Two-dimensional" solution:

$$w = \theta(\xi,\eta), \qquad \xi = xt^{-1/2}, \qquad \eta = yt^{-1},$$

where k is an arbitrary constant and the function $\theta = \theta(\xi,\eta)$ is determined by the differential equation

$$-\tfrac{1}{2}\xi\theta_\xi - \eta\theta_\eta = a\theta_{\xi\xi} + f(\theta)\theta_\eta.$$

⊙ *Reference for equation* 6.4.4.1: V. I. Lagno and V. I. Stognii (2010).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + f(w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

The substitution

$$U = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\dfrac{\partial w}{\partial t} = \big[aw + f(t)\big]\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + bw^2 + g(t)w + h(t), \qquad a \neq 0.$

"Two-dimensional" generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)\Theta(x, y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$
$$\psi'_t = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x, y)$ is any solution of the two-dimensional Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

The first equation (1) is independent of ψ and is a Riccati equation for φ. In Polyanin and Zaitsev (2003), many exact solutions of equation (1) for various $g(t)$ and $h(t)$ are presented. Solving equation (1) followed by substituting the expression of $\varphi = \varphi(t)$ into (2), we arrive at a linear equation for $\psi = \psi(t)$, which is easy to integrate.

In the special case $B = 0$, a solution of system (1), (2) is given by

$$\varphi(t) = \exp\big[G(t)\big]\left\{A + \int h(t)\exp\big[-G(t)\big]\,dt\right\}, \quad G(t) = \int g(t)\,dt,$$
$$\psi(t) = B\exp\big[G(t) - \beta F(t)\big], \quad F(t) = \int f(t)\,dt,$$

where A and B are arbitrary constants.

For solutions of the linear stationary equation (3), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4. $\dfrac{\partial w}{\partial t} = aw\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - a\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] + f(t).$

1°. Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)e^{\beta x + \gamma y},$$

where β and γ are arbitrary constants and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = f(t), \qquad \psi'_t = a(\beta^2 + \gamma^2)\varphi\psi.$$

Solving this system yields the solution

$$w(x, y, t) = \varphi(t) + A\exp\left[\beta x + \gamma y + a(\beta^2 + \gamma^2)\int \varphi(t)\,dt\right], \quad \varphi(t) = \int f(t)\,dt + B,$$

where A and B are arbitrary constants.

2°. There are generalized separable solutions of the following forms:

$$w(x,y,t)=\varphi(t)+\psi(t)(A_1\cosh\mu x+A_2\sinh\mu x)+\chi(t)(B_1\cos\mu y+B_2\sin\mu y),$$
$$w(x,y,t)=\varphi(t)+\psi(t)(A_1\cos\mu x+A_2\sin\mu x)+\chi(t)(B_1\cosh\mu y+B_2\sinh\mu y),$$

where A_1, A_2, B_1, B_2, and μ are arbitrary constants, and the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of ordinary differential equations (not written out here).

3°. There is a generalized separable solution of the form

$$w(x,y,t)=\varphi(t)+\psi(t)F(x)+\chi(t)G(y)+\eta(t)H(x)P(y),$$

where

$$F(x)=A_1\cos 2\mu x+A_2\sin 2\mu x,\qquad G(y)=B_1\cosh 2\mu y+B_2\sinh 2\mu y,$$
$$H(x)=C_1\cos\mu x+C_2\sin\mu x,\qquad P(y)=D_1\cosh\mu y+D_2\sinh\mu y,$$

where the constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, and μ are related by two constraints, and the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$, and $\eta(t)$ are determined by a system of ordinary differential equations (not written out here).

5. $$\frac{\partial w}{\partial t}+(a_1x+b_1y+c_1)\frac{\partial w}{\partial x}+(a_2x+b_2y+c_2)\frac{\partial w}{\partial y}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right].$$

This equation describes unsteady anisotropic heat/mass transfer in a steady translational-shear fluid flow.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_{1,2}=w\big(x+Cb_1e^{\lambda t},y+C(\lambda-a_1)e^{\lambda t},t\big),$$

where C is an arbitrary constant and $\lambda=\lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2-(a_1+b_2)\lambda+a_1b_2-a_2b_1=0, \tag{1}$$

are also solutions of the equation.

2°. Solutions:

$$w=w(z),\qquad z=a_2x+(\lambda-a_1)y+Ce^{\lambda t}, \tag{2}$$

where $\lambda=\lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\big[\lambda z+a_2c_1+(\lambda-a_1)c_2\big]w'_z=[\varphi(w)w'_z]'_z,\qquad \varphi(w)=a_2^2f(w)+(\lambda-a_1)^2g(w).$$

3°. "Two-dimensional" solutions:

$$w=U(\zeta,t),\qquad \zeta=a_2x+(\lambda-a_1)y, \tag{3}$$

where $\lambda=\lambda_{1,2}$ are roots of the quadratic equation (1), and the function $U(\zeta,t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t}+[\lambda\zeta+a_2c_1+(\lambda-a_1)c_2]\frac{\partial U}{\partial\zeta}=\frac{\partial}{\partial\zeta}\left[\varphi(U)\frac{\partial U}{\partial\zeta}\right],\qquad \varphi(U)=a_2^2f(U)+(\lambda-a_1)^2g(U).$$

Remark 1. A more general equation, with an additional term $h(w)$ on the right-hand side, where h is an arbitrary function, also has solutions of the forms (2) and (3).

Remark 2. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1+b_2=0$.

⊙ *Reference for equation* 6.4.4.5: A. D. Polyanin and V. F. Zaitsev (2004, p. 166).

6. $\dfrac{\partial w}{\partial t} = f(w)\mathbf{L}[w] + g(t)w + h(t).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator with respect to the space variables x, y (the operator is independent of t).

"Two-dimensional" generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)\Theta(x, y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = e^{G(t)}\left[A + \int h(t)e^{-G(t)}\,dt\right], \quad \psi(t) = Be^{G(t)}, \qquad G(t) = \int g(t)\,dt,$$

A is an arbitrary constant, and the function $\Theta(x, y)$ is a solution of the linear stationary equation

$$\mathbf{L}[\Theta] = 0.$$

Remark 1. In the equation under consideration, the order of the linear operator $\mathbf{L}$ and the number of space variables can be any. The coefficients of $\mathbf{L}$ can be dependent on the space variables.

Remark 2. The above remains valid if $f(w)$ in the equation is substituted by a function $f(x, y, t, w)$. In the special case $f(x, y, t, w) = f_1(t) + \alpha w$, $\mathbf{L}[w] = \Delta w + \beta w$, where Δ is the Laplace operator, α and β are some constants, we obtain an equation of the form 6.4.4.3.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x, y)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(x, y)\dfrac{\partial w}{\partial y}\right] + kw\ln w.$

This is an equation of unsteady heat (mass) transfer or combustion in an anisotropic case with arbitrary coordinate-dependent principal thermal diffusivities (diffusion coefficients) and a logarithmic source.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = \exp(C_1e^{kt})w(x, y, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x, y, t) = \exp(C_1e^{kt})\Theta(x, y),$$

where the function $U(x, y)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x, y)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x, y)\frac{\partial U}{\partial y}\right] + kU\ln U = 0.$$

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f_1(x, t)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(y, t)\dfrac{\partial w}{\partial y}\right] + \big[g_1(x, t) + g_2(y, t)\big]w + h(t)w\ln w.$

Exact solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x, y, t) = \varphi(x, t)\psi(y, t).$$

Here, the functions $\varphi(x, t)$ and $\psi(y, t)$ are determined from the two one-dimensional nonlinear parabolic differential equations

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial x}\left[f_1(x, t)\frac{\partial \varphi}{\partial x}\right] + g_1(x, t)\varphi + h(t)\varphi\ln\varphi + C(t)\varphi,$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial y}\left[f_2(y, t)\frac{\partial \psi}{\partial y}\right] + g_2(y, t)\psi + h(t)\psi\ln\psi - C(t)\psi,$$

where $C(t)$ is an arbitrary function.

9. $$\frac{\partial w}{\partial t} = f_1(x,y)\frac{\partial^2 w}{\partial x^2} + f_2(x,y)\frac{\partial^2 w}{\partial x \partial y} + f_3(x,y)\frac{\partial^2 w}{\partial y^2} + g_1(x,y)\frac{\partial w}{\partial x} + g_2(x,y)\frac{\partial w}{\partial y} + \big[h(x,y) + s(t)\big]w + kw\ln w.$$

Multiplicative separable solution:

$$w(x,y,t) = \exp\left[Ae^{kt} + e^{kt}\int e^{-kt}s(t)\,dt\right]\Theta(x,y),$$

where A is an arbitrary constant, and the function $\Theta(x,y)$ is a solution of the stationary equation

$$f_1(x,y)\frac{\partial^2\Theta}{\partial x^2} + f_2(x,y)\frac{\partial^2\Theta}{\partial x\partial y} + f_3(x,y)\frac{\partial^2\Theta}{\partial y^2} + g_1(x,y)\frac{\partial\Theta}{\partial x} + g_2(x,y)\frac{\partial\Theta}{\partial y} + h(x,y)\Theta + k\Theta\ln\Theta = 0.$$

10. $$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x,t)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left\{\big[g(x,t)w + h(x,t)\big]\frac{\partial w}{\partial y}\right\}.$$

There are "two-dimensional" generalized separable solutions linear and quadratic in y:

$$w(x,y,t) = \varphi(x,t)y + \psi(x,t),$$
$$w(x,y,t) = \varphi(x,t)y^2 + \psi(x,t)y + \chi(x,t).$$

6.4.5. Nonlinear Diffusion Boundary Layer Equations

1. $$\frac{\partial w}{\partial t} + f(x,t)\frac{\partial w}{\partial x} + g(x,t)y\frac{\partial w}{\partial y} = h(x,t)\frac{\partial}{\partial y}\left[k(w)\frac{\partial w}{\partial y}\right].$$

This equation arises in nonlinear problems of the unsteady diffusion boundary layer (mass exchange of drops and bubbles with a flow, convective diffusion in fluid films), where the coordinates x and y are longitudinal and normal to the interphase surface, respectively.

The transformation

$$w = U(\zeta,\tau,\psi), \quad \zeta = y\varphi(x,t), \quad \tau = \tau(x,t), \quad \psi = \psi(x,t),$$

where the functions $\varphi(x,t)$, $\tau(x,t)$, and $\psi(x,t)$ are determined by the system of first-order partial differential equations

$$\begin{aligned}
&\frac{\partial\varphi}{\partial t} + f(x,t)\frac{\partial\varphi}{\partial x} = -g(x,t)\varphi,\\
&\frac{\partial\tau}{\partial t} + f(x,t)\frac{\partial\tau}{\partial x} = h(x,t)\varphi^2,\\
&\frac{\partial\psi}{\partial t} + f(x,t)\frac{\partial\psi}{\partial x} = 0,
\end{aligned} \tag{1}$$

leads to a simpler equation of the form 5.6.15.1:

$$\frac{\partial U}{\partial\tau} = \frac{\partial}{\partial\zeta}\left[k(U)\frac{\partial U}{\partial\zeta}\right]. \tag{2}$$

The cyclic variable ψ does not appear in equation (2); however, it can be involved in the transformed initial and boundary conditions as a parameter.

Integrating system (1) is reduced to solving a single ordinary differential equation: $x'_t = f(x,t)$. In particular, if the functions f, g, and h are only dependent on x, the general solution of system (1) is given by

$$\varphi = \Phi_1(z)E(x), \quad \tau = \Phi_1^2(z)\int \frac{h(x)}{f(x)}E^2(x)\,dx + \Phi_2(z), \quad \psi = \Phi_3(z),$$

where $\Phi_1(z)$, $\Phi_2(z)$, and $\Phi_3(z)$ are arbitrary functions, and

$$z = t - \int \frac{dx}{f(x)}, \quad E(x) = \exp\left[-\int \frac{g(x)}{f(x)}\,dx\right].$$

⊙ *References*: A. D. Polyanin (1982), A. D. Polyanin and V. F. Zaitsev (2004, pp. 168–169).

2. $\boldsymbol{f(x,y)z^{n-1}\dfrac{\partial w}{\partial x} + g(x,y)z^{n-1}\dfrac{\partial w}{\partial y} + h(x,y)z^n\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial z}\left[k(w)\dfrac{\partial w}{\partial z}\right].}$

This equation arises in nonlinear problems of the steady three-dimensional diffusion boundary layer (mass exchange of solid particles, drops, and bubbles with a flow, convective diffusion in fluid films), where z is a normal coordinate to the particle surface. To a solid particle there corresponds $n = 2$ and to drops and bubbles, $n = 1$.

The transformation

$$w = U(\zeta,\tau,\psi), \quad \zeta = z\varphi(x,y), \quad \tau = \tau(x,y), \quad \psi = \psi(x,y),$$

where the functions $\varphi(x,y)$, $\tau(x,y)$, and $\psi(x,y)$ are determined by the system of first-order partial differential equations

$$f(x,y)\frac{\partial\varphi}{\partial x} + g(x,y)\frac{\partial\varphi}{\partial y} = -h(x,y)\varphi, \tag{1}$$

$$f(x,y)\frac{\partial\tau}{\partial x} + g(x,y)\frac{\partial\tau}{\partial y} = \varphi^2, \tag{2}$$

$$f(x,y)\frac{\partial\psi}{\partial x} + g(x,y)\frac{\partial\psi}{\partial y} = 0, \tag{3}$$

leads to a simpler equation of the form 5.6.17.16:

$$\frac{\partial U}{\partial\tau} = \zeta^{1-n}\frac{\partial}{\partial\zeta}\left[k(U)\frac{\partial U}{\partial\zeta}\right]. \tag{4}$$

The cyclic variable ψ does not enter into equation (4); however, it can appear in the transformed initial and boundary conditions as a parameter.

Suppose an integral of the ordinary differential equation $f(x,y)y'_x = g(x,y)$ has the form

$$\Xi(x,y) = C.$$

Then the general solution of equation (3) is given by $\psi = F(\Xi)$, where F is an arbitrary function. On passing in (1)–(2) from x, y to the new variables x, Ξ, one arrives at ordinary differential equations with independent variable x where Ξ appears as a parameter.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 169).

6.4.6. Equations Nonlinear in the Highest Derivatives

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - \left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = \frac{D}{(A_1B_2 - A_2B_1)^2} w(A_1x + B_1y + C_1, A_2x + B_2y + C_2, Dt + C_3) + C_4x + C_5y + C_6,$$

where A_k, B_k, C_n, and D are arbitrary constants, are also solutions of the equation ($A_1B_2 - A_2B_1 \neq 0$).

2°. There are generalized separable solutions of the form:

$$w = \sum_{i,j=0}^{i+j\le 4} \varphi_{i,j}(t) x^i y^j.$$

Special case 1. There is a solution that only contains even powers:

$$w = \psi_1(t) + \psi_2(t)x^2 + \psi_3(t)y^2 + \psi_4(t)x^2y^2 + \psi_5(t)x^4 + \psi_6(t)y^4,$$

where the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1' &= 4\psi_2\psi_3,\\
\psi_2' &= 4\psi_2\psi_4 + 24\psi_3\psi_5,\\
\psi_3' &= 24\psi_2\psi_6 + 4\psi_3\psi_4,\\
\psi_4' &= -12\psi_4^2 + 144\psi_5\psi_6,\\
\psi_5' &= 24\psi_4\psi_5,\\
\psi_6' &= 24\psi_4\psi_6.
\end{aligned}$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 306).

3°. Additive separable solution:

$$w = f(x, y) - At - B,$$

where A and B are arbitrary constants and the function $f = f(x, y)$ satisfies a Monge–Ampère equation of the form 11.2.2.2:

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = A.$$

4°. Multiplicative separable solution:

$$w = \frac{g(x, y)}{t + C},$$

where C is an arbitrary constant and the function $f = f(x, y)$ is determined by the differential equation

$$\left(\frac{\partial^2 g}{\partial x \partial y}\right)^2 - \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2} = g.$$

This equation is a special case of equation 11.2.3.1 with $f(x) = 1$, $g(x) = 0$ and a special case of equation 11.2.3.5 with $f(w) = w$.

5°. Generalized separable solution:

$$w = f(x,t)y^2 + g(x,t)y + h(x,t),$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are determined by the system of differential equations

$$f_t = 2ff_x - 4f_x^2,$$
$$g_t = 2fg_x - 4f_xg_x,$$
$$h_t = 2fh_x - g_x^2.$$

6°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z,t)$ is determined by the differential equation

$$u_t = 2\big[(4ac - b^2)z + as^2 + ck^2 - bks\big]u_zu_{zz} + (4ac - b^2)(u_z)^2.$$

Special case 2. In particular, for $b = c = k = 0$, we have the equation $u_t = 2as^2u_zu_{zz}$, which admits exact solutions of the form

$$u = \varphi_0(t) + \varphi_1(t)z + \varphi_2(t)z^2 + \varphi_3(t)z^3,$$
$$u = \phi_1(t) + \psi_2(t)z^{3/2} + \psi_3(t)z^3.$$

7°. "Two-dimensional" solution:

$$w = t\Phi(\xi,\eta), \quad \xi = xt^{-k}, \quad \eta = yt^{k-1},$$

where k is an arbitrary constant and the function $\Phi = \Phi(\xi,\eta)$ is determined by the differential equation

$$\Phi - k\xi\Phi_\xi + (k-1)\eta\Phi_\eta = \Phi_{\xi\xi}\Phi_{\eta\eta} - \Phi_{\xi\eta}^2.$$

2. $\dfrac{\partial w}{\partial t} = f(U), \quad U = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - \left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2.$

This kind of equation arises in various geometric problems and applications, such as logarithmic Gauss and Hessian curvature flows, the Minkowski problem, the Weyl problem, and many others; see Taylor (1996), Gilbarg and Trudinger (2001), Gutiérres (2001).

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = \frac{1}{D}w(A_1x + B_1y + C_1, A_2x + B_2y + C_2, Dt + C_3) + C_4x + C_5y + C_6,$$

where $D = \pm(A_1B_2 - A_2B_1) \neq 0$ and A_k, B_k, and C_n are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution:

$$w = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6 + f(4C_1C_3 - C_2^2)t,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 305).

3°. Additive separable solution:

$$w = f(C_1)t + C_2 + u(x, y),$$

where C_1 and C_2 are arbitrary constants and the function $u = u(x, y)$ is determined by an equation of the form 11.2.2.2:

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = -C_1.$$

4°. "Two-dimensional" solution:

$$w(x, y, t) = u(z, t), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z, t)$ is determined by the differential equation

$$u_t = f\Big(2\big[(4ac - b^2)z + as^2 + ck^2 - bks\big]u_z u_{zz} + (4ac - b^2)u_z^2\Big).$$

5°. "Two-dimensional" solution:

$$w = t\Phi(\xi, \eta), \quad \xi = xt^{-k}, \quad \eta = yt^{k-1},$$

where k is an arbitrary constant and the function $\Phi = \Phi(\xi, \eta)$ is determined by the differential equation

$$\Phi - k\xi\Phi_\xi + (k-1)\eta\Phi_\eta = f\big(\Phi_{\xi\xi}\Phi_{\eta\eta} - \Phi_{\xi\eta}^2\big).$$

3. $\dfrac{\partial w}{\partial t} = f(w, U), \quad U = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - \left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2.$

"Two-dimensional" solution:

$$w(x, y, t) = u(z, t), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z, t)$ is determined by the differential equation

$$u_t = f\big(u,\ 2[(4ac - b^2)z + as^2 + ck^2 - bks]u_z u_{zz} + (4ac - b^2)u_z^2\big).$$

6.5. Equations with Three or More Space Variables

6.5.1. Equations of Mass Transfer in Quiescent or Moving Media with Chemical Reactions

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} - f(w).$

This equation describes unsteady mass or heat transfer with a volume reaction in a quiescent medium.

The equation admits translations in any of the variables x, y, z, t.

1°. There is a traveling-wave solution, $w = w(k_1 x + k_2 y + k_3 z + \lambda t)$.

2°. For the axisymmetric case, the Laplace operator on the right-hand side of the equation takes the following forms in cylindrical and spherical coordinates, respectively:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right)+\frac{\partial^2 w}{\partial z^2}, \qquad \rho=\sqrt{x^2+y^2};$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right), \quad r=\sqrt{x^2+y^2+z^2}.$$

Consequently, there are solutions of the forms $w = w(\rho, t)$ and $w = w(r, t)$.

3°. "Three-dimensional" solution:

$$w = u(\xi,\eta,t), \qquad \xi = y+\frac{x}{C}, \quad \eta = (C^2-1)x^2-2Cxy+C^2z^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $u = u(\xi, \eta, t)$ is determined by the equation

$$\frac{\partial u}{\partial t}=\left(1+\frac{1}{C^2}\right)\frac{\partial^2 u}{\partial \xi^2}-4\xi\frac{\partial^2 u}{\partial \xi\partial \eta}+4C^2(\xi^2+\eta)\frac{\partial^2 u}{\partial \eta^2}+2(2C^2-1)\frac{\partial u}{\partial \eta}-f(u).$$

Remark. The solution specified in Item 3° can be used to obtain other "three-dimensional" solutions by means of the cyclic permutations of the space variables.

4°. "Three-dimensional" solution:

$$w = u(\xi,\eta,t), \qquad \xi = Ax+By+Cz, \quad \eta=\sqrt{(Bx-Ay)^2+(Cy-Bz)^2+(Az-Cx)^2},$$

where A, B, and C are arbitrary constants and the function $u = u(\xi, \eta, t)$ is determined by the equation

$$\frac{\partial u}{\partial t}=(A^2+B^2+C^2)\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right)-f(u).$$

2. $\dfrac{\partial w}{\partial t}=a\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\right)+f(t)w\ln w+g(t)w.$

1°. There is a functional separable solution of the form

$$w(x,y,z,t)=\exp\left[\sum_{n,m=1}^{3}\varphi_{nm}(t)x_nx_m+\sum_{n=1}^{3}\psi_n(t)x_n+\chi(t)\right], \qquad x_1=x,\ \ x_2=y,\ \ x_3=z.$$

2°. There is an incomplete separable solution of the form

$$w(x,y,z,t)=\Phi_1(x,t)\Phi_2(y,t)\Phi_3(z,t).$$

3°. For $f(t) = b = \text{const}$, the equation also has a multiplicative separable solution of the form

$$w(x,y,z,t)=\varphi(t)\Theta(x,y,z),$$

where $\varphi(t)$ is given by

$$\varphi(t)=\exp\left[Ae^{bt}+e^{bt}\int e^{-bt}g(t)\,dt\right],$$

A is an arbitrary constant, and $\Theta(x, y, z)$ is a solution of the stationary equation

$$a\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}+\frac{\partial^2\Theta}{\partial z^2}\right)+b\Theta\ln\Theta=0.$$

3. $\dfrac{\partial w}{\partial t} + a_1\dfrac{\partial w}{\partial x} + a_2\dfrac{\partial w}{\partial y} + a_3\dfrac{\partial w}{\partial z} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} - f(w).$

This equation describes unsteady mass transfer with a volume chemical reaction in a steady translational fluid flow.

The transformation

$$w = U(\xi, \eta, \zeta, t), \quad \xi = x - a_1 t, \quad \eta = y - a_2 t, \quad \zeta = z - a_3 t$$

leads to a simpler equation of the form 6.5.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} - f(U).$$

4. $\dfrac{\partial w}{\partial t} + f_1(t)\dfrac{\partial w}{\partial x} + f_2(t)\dfrac{\partial w}{\partial y} + f_3(t)\dfrac{\partial w}{\partial z} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} - g(w).$

This equation describes unsteady mass transfer with a volume chemical reaction in an unsteady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \quad \xi = x - \int f_1(t)\,dt, \quad \eta = y - \int f_2(t)\,dt, \quad \zeta = z - \int f_3(t)\,dt$$

leads to a simpler equation of the form 6.5.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} - g(U).$$

5. $\dfrac{\partial w}{\partial t} + (a_1x + b_1y + c_1z + d_1)\dfrac{\partial w}{\partial x} + (a_2x + b_2y + c_2z + d_2)\dfrac{\partial w}{\partial y}$
$$+ (a_3x + b_3y + c_3z + d_3)\frac{\partial w}{\partial z} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - f(w).$$

This equation describes unsteady mass transfer with a volume chemical reaction in a three-dimensional steady translational-shear fluid flow.

1°. Let λ be a root of the cubic equation

$$\begin{vmatrix} a_1 - \lambda & a_2 & a_3 \\ b_1 & b_2 - \lambda & b_3 \\ c_1 & c_2 & c_3 - \lambda \end{vmatrix} = 0, \tag{1}$$

and let the constants A_1, A_2, and A_3 solve the degenerate system of linear algebraic equations

$$\begin{aligned} (a_1 - \lambda)A_1 + a_2A_2 + a_3A_3 &= 0, \\ b_1A_1 + (b_2 - \lambda)A_2 + b_3A_3 &= 0, \\ c_1A_1 + c_2A_2 + (c_3 - \lambda)A_3 &= 0. \end{aligned} \tag{2}$$

One of these equations can be omitted, since it is a consequence of the other two.

Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + A_1Ce^{\lambda t}, y + A_2Ce^{\lambda t}, z + A_3Ce^{\lambda t}, t\big),$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), is also a solution of the equation.

2°. Solution:

$$w = w(\xi), \qquad \xi = A_1x + A_2y + A_3z + Ce^{\lambda t},$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\lambda\xi + A_1d_1 + A_2d_2 + A_3d_3)w'_\xi = (A_1^2 + A_2^2 + A_3^2)w''_{\xi\xi} - f(w).$$

3°. Let λ be a root of the cubic equation (1) and let A_1, A_2, and A_3 be the corresponding solution of the algebraic system (2).

"Two-dimensional" solution:

$$w = U(\zeta, t), \qquad \zeta = A_1x + A_2y + A_3z,$$

where the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + (\lambda\zeta + A_1d_1 + A_2d_2 + A_3d_3)\frac{\partial U}{\partial \zeta} = (A_1^2 + A_2^2 + A_3^2)\frac{\partial^2 U}{\partial \zeta^2} - f(U).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

6.5.2. Heat Equations with Power-Law or Exponential Temperature-Dependent Thermal Diffusivity

▷ *Throughout this subsection, the symbols* div, ∇, *and* Δ *stand for the divergence operator, gradient operator, and Laplace operator in Cartesian coordinates* x, y, z *(cylindrical, spherical, and other three-dimensional orthogonal systems of coordinates can be used instead of the Cartesian coordinates).*

1. $\dfrac{\partial w}{\partial t} = \Delta(w^m).$

This is a special case of equation 6.5.5.6.

2. $\dfrac{\partial w}{\partial t} = \alpha\,\mathrm{div}\big(w^n\nabla w\big) + f(t)w.$

1°. Multiplicative separable solution:

$$w(x, y, z, t) = \exp\left[\int f(t)\,dt\right]\big[\Theta(x, y, z)\big]^{\frac{1}{n+1}}, \tag{1}$$

where the function $\Theta(x, y, z)$ satisfies the Laplace equation

$$\Delta\Theta = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. Multiplicative separable solution:

$$w(x,y,z,t)=\varphi(t)\big[\Theta(x,y,z)\big]^{\frac{1}{n+1}}, \tag{2}$$

where the function $\varphi(t)$ is determined by the Bernoulli equation

$$\varphi_t'-f(t)\varphi+A\alpha\varphi^{n+1}=0. \tag{3}$$

Here, A is an arbitrary constant, and $\Theta(x,y,z)$ is a solution of the stationary equation

$$\Delta\Theta+A(n+1)\Theta^{\frac{1}{n+1}}=0.$$

The general solution of equation (3) is given by

$$\varphi(t)=\exp\big[F(t)\big]\left\{A\alpha n\int\exp\big[nF(t)\big]\,dt+B\right\}^{-1/n},\quad F(t)=\int f(t)\,dt,$$

where B is an arbitrary constant.

3°. Using the transformation

$$w(x,y,z,t)=F(t)U(x,y,z,\tau),\quad \tau=\int F^n(t)\,dt,\qquad F(t)=\exp\left[\int f(t)\,dt\right],$$

one arrives at the simpler equation $\dfrac{\partial U}{\partial\tau}=\alpha\operatorname{div}(U^n\nabla U)$.

3. $\dfrac{\partial w}{\partial t}=\Delta(w^{1/2})+aw+bw^{1/2}$.

Solution:

$$w=[f(x,y,z)+g(x,y,z)e^{at/2}]^2,$$

where functions $f=f(x,y,z)$ and $g=g(x,y,z)$ satisfy the system of two stationary equations

$$\Delta f+af^2+bf=0,\qquad \Delta g+afg+bg=0.$$

The former equation is solved independently for f. The latter equation is linear in g and is easy to solve once f has been found.

Remark. The original equation and the above solution admit a generalization to the case where the Laplace operator depends on n spatial coordinates.

⊙ *Reference*: O. V. Kaptsov and I. V. Verevkin (2003).

4. $\dfrac{\partial w}{\partial t}=\alpha\operatorname{div}(w^n\nabla w)+f(t)w+g(t)w^{1-n}$.

The substitution $U=w^n$ leads to a special case of equation 6.5.4.4:

$$\frac{\partial U}{\partial t}=\alpha U\Delta U+\frac{\alpha}{n}|\nabla U|^2+nf(t)U+ng(t).$$

5. $\dfrac{\partial w}{\partial t} = \alpha \operatorname{div}\left(e^{\mu w}\nabla w\right) + f(t).$

1°. Additive separable solution:

$$w(x, y, z, t) = \int f(t)\,dt + \frac{1}{\mu}\ln\Theta(x, y, z),$$

where the function $\Theta = \Theta(x, y, z)$ is any solution of the Laplace equation $\Delta\Theta = 0$.

2°. Additive separable solution:

$$w(x, y, z, t) = \varphi(t) + \frac{1}{\mu}\ln\Theta(x, y, z),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t + A(\alpha/\mu)\exp(\mu\varphi) - f(t) = 0. \tag{1}$$

Here, A is an arbitrary constant, and the function $\Theta = \Theta(x, y, z)$ is a solution of the Poisson equation

$$\Delta\Theta + A = 0. \tag{2}$$

The general solution of equation (1) is given by

$$\varphi(t) = F(t) - \frac{1}{\mu}\ln\left\{B + A\alpha\int \exp\left[\mu F(t)\right]dt\right\}, \qquad F(t) = \int f(t)\,dt. \tag{3}$$

For solutions of the linear stationary equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Note that equations (1), (2) and relation (3) contain arbitrary constants A and B.

3°. Using the transformation

$$w(x, y, z, t) = U(x, y, z, \tau) + F(t), \quad \tau = \int \exp[\mu F(t)]\,dt, \qquad F(t) = \int f(t)\,dt,$$

one arrives at the simpler equation $\dfrac{\partial U}{\partial \tau} = \alpha \operatorname{div}(e^{\mu U}\nabla U)$.

6. $\dfrac{\partial w}{\partial t} = a \operatorname{div}\left(e^{\mu w}\nabla w\right) + be^{\mu w} + g(t) + h(t)e^{-\mu w}.$

The substitution $U = e^{\mu w}$ leads to an equation of the form 6.5.4.5 for $U = U(x, y, z, t)$:

$$\frac{\partial U}{\partial t} = aU\Delta U + b\mu U^2 + \mu g(t)U + \mu h(t).$$

Hence, the original equation has solutions of the form

$$w(x, y, z, t) = \frac{1}{\mu}\ln[\varphi(t) + \psi(t)\Theta(x, y, z)].$$

Note that, with $g(t) \equiv$ const and $h(t) \equiv$ const, the original equation was studied in Galaktionov and Posashkov (1989) and Ibragimov (1994, pp. 158–163).

6.5.3. Equations of Heat and Mass Transfer in Anisotropic Media

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^n\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $m \neq 2$ and $n \neq 2$:

$$w = w(\xi, t), \qquad \xi^2 = \frac{x^2}{a} + \frac{4\,y^{2-m}}{b(2-m)^2} + \frac{4\,z^{2-n}}{c(2-n)^2},$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{2(4-m-n)}{(2-m)(2-n)}.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 5.1.1 to 5.1.3 and equations 5.2.1.1 to 5.2.1.3, 5.4.1.2, 5.4.1.3, 5.4.1.7, and 5.4.1.8.

2°. Solution for $m \neq 2$ and $n \neq 2$:

$$w = w(x, \xi, t), \qquad \xi^2 = \frac{4\,y^{2-m}}{b(2-m)^2} + \frac{4\,z^{2-n}}{c(2-n)^2},$$

where the function $w(x, \xi)$ is determined by the two-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{4-mn}{(2-m)(2-n)}.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^l\dfrac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $m \neq 2$, and $l \neq 2$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-l}}{c(2-l)^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = 2\left(\frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-l}\right) - 1.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 5.1.1 to 5.1.3 and equations 5.2.1.1 to 5.2.1.3, 5.4.1.2, 5.4.1.3, 5.4.1.7, and 5.4.1.8.

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right),$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{4-nm}{(2-n)(2-m)}.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 5.1.1 to 5.1.3 and equations 5.2.1.1 to 5.2.1.3, 5.4.1.2, 5.4.1.3, 5.4.1.7, and 5.4.1.8.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{n}{2-n}\frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right] + g(w).$

For group classification and exact solutions of this equation for some $f_n(w)$ and $g(w)$, see Dorodnitsyn, Knyazeva, and Svirshchevskii (1983).

7. $\dfrac{\partial w}{\partial t} + (a_1x + b_1y + c_1z + d_1)\dfrac{\partial w}{\partial x} + (a_2x + b_2y + c_2z + d_2)\dfrac{\partial w}{\partial y} + (a_3x + b_3y + c_3z + d_3)\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right].$

This equation describes unsteady anisotropic heat or mass transfer in a three-dimensional steady translational-shear fluid flow.

1°. Let λ be a root of the cubic equation

$$\begin{vmatrix} a_1-\lambda & a_2 & a_3 \\ b_1 & b_2-\lambda & b_3 \\ c_1 & c_2 & c_3-\lambda \end{vmatrix} = 0, \tag{1}$$

and let the constants A_1, A_2, and A_3 solve the degenerate system of linear algebraic equations

$$\begin{aligned} (a_1-\lambda)A_1 + a_2A_2 + a_3A_3 &= 0, \\ b_1A_1 + (b_2-\lambda)A_2 + b_3A_3 &= 0, \\ c_1A_1 + c_2A_2 + (c_3-\lambda)A_3 &= 0. \end{aligned} \tag{2}$$

One of these equations is redundant and can be omitted.

Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + A_1 C e^{\lambda t}, y + A_2 C e^{\lambda t}, z + A_3 C e^{\lambda t}, t\big),$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), is also a solution of the equation.

2°. Solution:

$$w = w(\xi), \qquad \xi = A_1 x + A_2 y + A_3 z + C e^{\lambda t}, \tag{3}$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\lambda\xi + A_1 d_1 + A_2 d_2 + A_3 d_3) w'_\xi = [\varphi(w) w'_\xi]'_\xi,$$
$$\varphi(w) = A_1^2 f_1(w) + A_2^2 f_2(w) + A_3^2 f_3(w).$$

3°. Let λ be a root of the cubic equation (1) and let A_1, A_2, and A_3 be the corresponding solution of the algebraic system (2).

"Two-dimensional" solutions:

$$w = U(\zeta, t), \qquad \zeta = A_1 x + A_2 y + A_3 z, \tag{4}$$

where the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + (\lambda\zeta + A_1 d_1 + A_2 d_2 + A_3 d_3)\frac{\partial U}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left[\varphi(U)\frac{\partial U}{\partial \zeta}\right],$$
$$\varphi(U) = A_1^2 f_1(U) + A_2^2 f_2(U) + A_3^2 f_3(U).$$

Remark 1. A more general equation, with an additional term $g(w)$ on the right-hand side, where $g(w)$ is an arbitrary function, also has solutions of the forms (3) and (4).

Remark 2. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

6.5.4. Other Equations with Three Space Variables

▷ *Throughout this subsection, the symbols* div, ∇, *and* Δ *stand for the divergence operator, gradient operator, and Laplace operator in Cartesian coordinates* x, y, z; *cylindrical, spherical, and other three-dimensional orthogonal systems of coordinates can be used instead of the Cartesian coordinates.*

1. $\dfrac{\partial w}{\partial t} = a\Delta w + f(t)|\nabla w|^2 + g(t)w + h(t).$

There is a generalized separable solution of the form

$$w(x_1, x_2, x_3, t) = \sum_{k,l=1}^{3} \varphi_{kl}(t) x_k x_l + \sum_{k=1}^{3} \psi_k(t) x_k + \chi(t).$$

Remark. The more general equation

$$\frac{\partial w}{\partial t} = \sum_{n,m=1}^{3} a_{nm}(t)\frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} b_n(t)\left(\frac{\partial w}{\partial x_n}\right)^2 + \sum_{n=1}^{3} c_n(t)\frac{\partial w}{\partial x_n} + g(t)w + h(t)$$

has solutions of the same form.

2. $\dfrac{\partial w}{\partial t} = \Delta w + f(w)|\nabla w|^2.$

The substitution

$$U = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \Delta U.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\dfrac{\partial w}{\partial t} = \alpha w \Delta w - \alpha|\nabla w|^2 - \beta.$

1°. Solutions:

$$w(x,y,z,t) = Ax + By + Cz - [\alpha(A^2 + B^2 + C^2) + \beta]t + D,$$
$$w(x,y,z,t) = A - \beta t + B\exp\left[\alpha(\kappa^2 + \mu^2 + \nu^2)\left(At - \tfrac{1}{2}\beta t^2\right)\right]e^{\kappa x + \mu y + \nu z},$$

where A, B, C, D, κ, μ, and ν are arbitrary constants.

2°. See 6.5.4.4 with $f(t) = -\alpha$, $g(t) = 0$, and $h(t) = -\beta$.

4. $\dfrac{\partial w}{\partial t} = \alpha w \Delta w + f(t)|\nabla w|^2 + g(t)w + h(t).$

There are generalized separable solutions of the form

$$w(x_1, x_2, x_3, t) = \sum_{k,l=1}^{3} \varphi_{kl}(t)x_k x_l + \sum_{k=1}^{3} \psi_k(t)x_k + \chi(t).$$

Remark. The more general equation

$$\frac{\partial w}{\partial t} = \sum_{n,m=1}^{3} \left[a_{nm}(t)w + b_{nm}(t)\right]\frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} c_n(t)\left(\frac{\partial w}{\partial x_n}\right)^2 + \sum_{n=1}^{3} s_n(t)\frac{\partial w}{\partial x_n} + g(t)w + h(t)$$

has solutions of the same form.

5. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\Delta w + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$). This is a special case of equation 6.5.4.6 with $\mathbf{L}[w] \equiv \Delta w$.

Note that, with $f(t) \equiv \text{const}$, $g(t) \equiv \text{const}$, and $h(t) \equiv \text{const}$, this equation was studied in Galaktionov and Posashkov (1989) and Ibragimov (1994, pp. 158–163).

6. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\mathbf{L}[w] + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$); $\mathbf{L}[w]$ is an arbitrary linear differential operator of the second (or any) order that depends on the space variables $x_1 = x$, $x_2 = y$, $x_3 = z$ only and satisfies the condition $\mathbf{L}[\text{const}] \equiv 0$:

$$\mathbf{L}[w] \equiv \sum_{n,m=1}^{3} p_{nm}(\mathbf{x})\frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} q_n(\mathbf{x})\frac{\partial w}{\partial x_n}, \qquad \mathbf{x} = \{x_1, x_2, x_3\}.$$

There is a generalized separable solution of the form

$$w(x_1, x_2, x_3, t) = \varphi(t) + \psi(t)\Theta(x_1, x_2, x_3),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$

$$\psi_t' = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x_1, x_2, x_3)$ is a solution of the linear stationary equation

$$\mathbf{L}[\Theta] + \beta\Theta = 0. \tag{3}$$

Equation (1) is independent of ψ and represents a Riccati equation for φ. A large number of exact solutions to equation (1) for various $g(t)$ and $h(t)$ can be found in Polyanin and Zaitsev (2003). On solving (1) and substituting the resulting $\varphi = \varphi(t)$ into (2), one obtains a linear equation for $\psi = \psi(t)$, which is easy to integrate.

In the special case $b = 0$, the solution of system (1), (2) is given by

$$\varphi(t) = \exp\big[G(t)\big]\left\{A + \int h(t)\exp\big[-G(t)\big]\,dt\right\}, \quad G(t) = \int g(t)\,dt,$$

$$\psi(t) = B\exp\big[G(t) - \beta F(t)\big], \quad F(t) = \int f(t)\,dt,$$

where A and B are arbitrary constants.

In the special case $\mathbf{L} \equiv \Delta$, see Tikhonov and Samarskii (1990) and Polyanin (2002) for solutions of the linear stationary equation (3).

7. $\dfrac{\partial w}{\partial t} = f(t)\mathbf{N}_\beta[w] + g(t)w.$

Here, $\mathbf{N}_\beta[w]$ is an arbitrary homogeneous nonlinear differential operator of degree β with respect to w (i.e., $\mathbf{N}_\beta[\alpha w] = \alpha^\beta \mathbf{N}_\beta[w]$, $\alpha = \text{const}$) that depends on the space variables x, y, z only (and is independent of t).

Using the transformation

$$w(x, y, z, t) = G(t)U(x, y, z, \tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives at the simpler equation

$$\frac{\partial U}{\partial \tau} = \mathbf{N}_\beta[U], \tag{1}$$

which has a multiplicative separable solution $U = \varphi(\tau)\Theta(x, y, z)$.

Remark 1. The order of the nonlinear operator $\mathbf{N}_\beta$ (with respect to the derivatives) and the number of the space variables in the original equation can be any. The coefficients of $\mathbf{N}_\beta$ can be dependent on the space variables.

Remark 2. If $\mathbf{N}_\beta$ is independent explicitly of the space variables, then equation (1) has also a traveling-wave solution, $U = U(\xi)$, where $\xi = k_1x + k_2y + k_3z + \lambda\tau$. Below are two examples of such operators:

$$\mathbf{N}_\beta[w] = a\,\mathrm{div}(w^{\beta-1}\nabla w) + b|\nabla w|^\beta + cw^\beta,$$
$$\mathbf{N}_\beta[w] = a\,\mathrm{div}(|\nabla w|^{\beta-1}\nabla w) + bw^\mu|\nabla w|^{\beta-\mu},$$

where a, b, c, and μ are some constants.

8. $\dfrac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w = \Delta w + f(w)|\nabla w|^2.$

This is a special case of equation 6.5.5.8 with $n = 3$.

9. $\dfrac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w = a\Delta w + a|\nabla w|^2 + f(\mathbf{x}, t).$

This is a special case of equation 6.5.5.9 with $n = 3$.

10. $\dfrac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{w} = a\Delta\mathbf{w}.$

Vector Burgers equation; $\mathbf{w} = \{w_1, w_2, w_3\}$ and $w_n = w_n(x_1, x_2, x_3)$. The Hamilton operator ∇ and Laplace operator Δ can be represented in any orthogonal system of coordinates.

Solution:

$$\mathbf{w} = -\frac{2a}{\theta}\nabla\theta,$$

where θ is a solution of the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\Delta\theta.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: S. Nerney, E. J. Schmahl, and Z. E. Musielak (1996).

6.5.5. Equations with n Space Variables

▷ *Notation:* $\mathbf{x} = (x_1, \ldots, x_n)$, $\Delta w = \sum\limits_{k=1}^{n} \dfrac{\partial^2 w}{\partial x_k^2}$, $|\nabla w|^2 = \sum\limits_{k=1}^{n} \left(\dfrac{\partial w}{\partial x_k}\right)^2$, $(\mathbf{v}\cdot\nabla)w = \sum\limits_{k=1}^{n} v_k \dfrac{\partial w}{\partial x_k}$, $\nabla\cdot\mathbf{v} = \sum\limits_{k=1}^{n} \dfrac{\partial v_k}{\partial x_k}$.

1. $\dfrac{\partial w}{\partial t} = \Delta w + f(w)|\nabla w|^2.$

The substitution $U = \int F(w)\,dw$, where $F(w) = \exp\left[\int f(w)\,dw\right]$, leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \Delta U.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2. $\dfrac{\partial w}{\partial t} = f(t)\Delta w + g(t)w\ln w + h(t)w.$

There is a functional separable solution of the form

$$w(x_1,\dots,x_n,t) = \exp\left[\sum_{i,j=1}^{n}\varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n}\psi_i(t)x_i + \chi(t)\right].$$

Special case 1. Let $f(t) = 1$, $g(t) = 1$, and $h(t) = 0$. Solutions in the radially symmetric case:

$$w = \exp\left(\frac{n}{2} - \frac{1}{4}r^2 + Be^t\right), \quad r = \sqrt{x_1^2 + \cdots + x_n^2},$$

$$w = \exp\left\{-\frac{1}{4}r^2(1 - Ae^{-t})^{-1} + e^t\left[B - \frac{n}{2A}\ln(1 - Ae^{-t})\right]\right\},$$

where A and B are arbitrary constants, $A < 1$. The first solution is a special case of the second solution as $A \to 0$.

Special case 2. Let $f(t) = 1$, $g(t) = -1$, and $h(t) = 0$. Solution in radially symmetric case:

$$w = \exp\left\{-\frac{1}{4}r^2(Ae^t - 1)^{-1} + e^{-t}\left[B - \frac{n}{2A}\ln(Ae^t - 1)\right]\right\},$$

where A and B are arbitrary constants, $A > 1$.

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

3. $\dfrac{\partial w}{\partial t} = f_1(t)\Delta w + f_2(t)|\nabla w|^2 + f_3(t)w + \sum\limits_{i,j=1}^{n} g_{ij}(t)x_ix_j + \sum\limits_{i=1}^{n} h_i(t)x_i + p(t).$

There are exact solutions of the following forms:

$$w(x_1,\dots,x_n,t) = \sum_{i,j=1}^{n}\varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n}\psi_i(t)x_i + \chi(t).$$

4. $\dfrac{\partial w}{\partial t} = f_1(t)w\Delta w + f_2(t)|\nabla w|^2 + f_3(t)w + \sum\limits_{i,j=1}^{n} g_{ij}(t)x_ix_j + \sum\limits_{i=1}^{n} h_i(t)x_i + p(t).$

There are exact solutions of the following forms:

$$w(x_1,\dots,x_n,t) = \sum_{i,j=1}^{n}\varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n}\psi_i(t)x_i + \chi(t).$$

5. $\dfrac{\partial w}{\partial t} = a\nabla\cdot(w^m\nabla w).$

For $m > 1$, this equation describes the flow of a polytropic gas through a homogeneous porous medium (w is the gas density).

1°. In the radially symmetric case the equation is written as

$$\frac{\partial w}{\partial t} = \frac{a}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}w^m\frac{\partial w}{\partial r}\right), \qquad r = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Its exact solutions are given in 5.1.15.13, where n should be substituted by $n - 1$.

2°. Solution of the instantaneous source type for $a = 1$:

$$w = \begin{cases} t^{-n/(nm+2)}\left[\dfrac{m}{2(nm+2)}\left(K_0^2 - \dfrac{r^2}{t^{2/(nm+2)}}\right)\right]^{1/m} & \text{if } r \le K_0 t^{1/(nm+2)}, \\ 0 & \text{if } r > K_0 t^{1/(nm+2)}, \end{cases}$$

where

$$K_0 = \left\{\pi^{-n/2}\left[\frac{2(nm+2)}{m}\right]^{1/m}\frac{\Gamma(n/m+1+1/m)}{\Gamma(1/m+1)}E_0\right\}^{m/(nm+2)}, \qquad E_0 = \text{const}.$$

This is the solution of the initial-value problem with initial function

$$w(\mathbf{x}, 0) = E_0\delta(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

satisfying the condition of constant energy:

$$\int_{\mathbb{R}^n} w(\mathbf{x}, t)\, d\mathbf{x} = E_0 = \text{const} > 0.$$

3°. See also equation 6.5.5.6, in which m should be substituted by $m + 1$.

⊙ *References for equation* 6.5.5.5: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952).

6. $\dfrac{\partial w}{\partial t} = \Delta(w^m)$.

For $m > 1$, this equation describes the flow of a polytropic gas through a homogeneous porous medium (w is the gas density). It can be rewritten in the form of equation 6.5.5.5:

$$\frac{\partial w}{\partial t} = m\nabla \cdot (w^{m-1}\nabla w).$$

1°. Solution for $m > 1$:

$$w = \left(\prod_{k=1}^{n}\varphi_k\right)^{-1}\left(A - \sum_{k=1}^{n}\frac{x_k^2}{\varphi_k^2}\right)^{\frac{1}{m-1}},$$

where A is an arbitrary constant ($A > 0$), and the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\varphi_1\frac{d\varphi_1}{dt} = \cdots = \varphi_n\frac{d\varphi_n}{dt} = \frac{2m}{m-1}\left(\prod_{k=1}^{n}\varphi_k\right)^{1-m}. \tag{1}$$

System (1) admits $n - 1$ first integrals:

$$\varphi_j^2 = \varphi_n^2 + C_j, \qquad j = 1, 2, \ldots, n-1, \tag{2}$$

where the C_j are arbitrary constants.

The function $\varphi_n = \varphi_n(t)$ is defined in implicit form by (the C_j are assumed to be positive)

$$\int_B^{\varphi_n} z^m\left(\prod_{j=1}^{n-1}(z^2 + C_j)\right)^{\frac{m-1}{2}} dz = \frac{2mt}{m-1},$$

where B is an arbitrary constant, and the remaining $\varphi_j(t)$ are determined by the positive roots of the quadratic equations (2).

2°. Solution for $0 < m < 1$:

$$w = \left(\prod_{k=1}^{n} \varphi_k\right)^{-1} \left(A + \sum_{k=1}^{n} \frac{x_k^2}{\varphi_k^2}\right)^{\frac{1}{m-1}},$$

where A is an arbitrary constant, and the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations (1).

3°. There is an exact solution of the form

$$w = \left(\sum_{i,j=1}^{n} a_{ij}(t)x_i x_j + \sum_{i=1}^{n} b_i(t)x_i + c(t)\right)^{\frac{1}{m-1}}.$$

⊙ *References for equation* 6.5.5.6: S. S. Titov and V. A. Ustinov (1985), J. R. King (1993), V. V. Pukhnachov (1995), G. A. Rudykh and E. I. Semenov (2000; other exact solutions are also given there).

7. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\Delta w + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$).

There is a generalized separable solution of the form

$$w(x_1, \ldots, x_n, t) = \varphi(t) + \psi(t)\Theta(x_1, \ldots, x_n),$$

where the functions $\varphi(t)$, $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$

$$\psi_t' = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x_1, \ldots, x_n)$ is a solution of the Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0. \tag{3}$$

Equation (1) is independent of ψ and represents a Riccati equation for φ. A large number of exact solutions to equation (1) for various $g(t)$ and $h(t)$ can be found in Polyanin and Zaitsev (2003). On solving (1) and substituting the resulting $\varphi = \varphi(t)$ into (2), one obtains a linear equation for $\psi = \psi(t)$, which is easy to integrate.

For solutions of the linear stationary equation (3), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

8. $\dfrac{\partial w}{\partial t} + (\mathbf{v} \cdot \nabla)w = \Delta w + f(w)|\nabla w|^2.$

Here, $\mathbf{v}$ is a prescribed vector function dependent on the space coordinates and time (but independent of w).

The substitution

$$\Theta = \int F(w)\, dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\, dw\right],$$

leads to a linear convective heat and mass transfer equation for $\Theta = \Theta(x_1, \ldots, x_n, t)$:

$$\frac{\partial \Theta}{\partial t} + (\mathbf{v} \cdot \nabla)\Theta = \Delta\Theta.$$

9. $\dfrac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w = a\Delta w + a|\nabla w|^2 + f(\mathbf{x},t).$

Here, $\mathbf{v}$ is a prescribed vector function dependent on the space coordinates and time (but independent of w).

The substitution $\Theta = e^w$ leads to the linear equation

$$\frac{\partial \Theta}{\partial t} + (\mathbf{v}\cdot\nabla)\Theta = a\Delta\Theta + f(\mathbf{x},t)\Theta.$$

10. $\dfrac{\partial w}{\partial t} = \alpha\nabla\cdot\left(w^m\nabla w\right) + f(t)w.$

1°. Multiplicative separable solution:

$$w(\mathbf{x},t) = \exp\left[\int f(t)\,dt\right]\left[\Theta(\mathbf{x})\right]^{\frac{1}{m+1}}, \tag{1}$$

where the function $\Theta(\mathbf{x})$ satisfies the Laplace equation

$$\Delta\Theta = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. Multiplicative separable solution:

$$w(\mathbf{x},t) = \varphi(t)\left[\Theta(\mathbf{x})\right]^{\frac{1}{m+1}}, \tag{2}$$

where the function $\varphi(t)$ is determined by the Bernoulli equation

$$\varphi_t' - f(t)\varphi + A\alpha\varphi^{m+1} = 0. \tag{3}$$

Here, A is an arbitrary constant and the function $\Theta(\mathbf{x})$ satisfies the stationary equation

$$\Delta\Theta + A(m+1)\Theta^{\frac{1}{m+1}} = 0.$$

The general solution of equation (3) is given by

$$\varphi(t) = \exp\left[F(t)\right]\left\{A\alpha m\int \exp\left[mF(t)\right]dt + B\right\}^{-1/m}, \quad F(t) = \int f(t)\,dt,$$

where B is an arbitrary constant.

3°. The transformation

$$w(\mathbf{x},t) = F(t)U(\mathbf{x},\tau), \quad \tau = \int F^m(t)\,dt, \qquad F(t) = \exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation: $\dfrac{\partial U}{\partial \tau} = \alpha\nabla\cdot(U^m\nabla w)$.

Special case. For $\alpha = 1$, $f(t) = -\beta < 0$, we have

$$\frac{\partial w}{\partial t} = \nabla \cdot (w^m \nabla w) - \beta w.$$

Solution in the radially symmetric case:

$$w = \begin{cases} e^{-\beta t/m}[g(t)]^{-n/(nm+2)}\left[\dfrac{m}{2(nm+2)}\left(\eta_0^2 - \dfrac{r^2}{[g(t)]^{2/(nm+2)}}\right)\right]^{1/m} & \text{if } r \le r_*(t), \\ 0 & \text{if } r > r_*(t), \end{cases}$$

where

$$r = \sqrt{x_1^2 + \cdots + x_n^2}, \quad g(t) = 1 + \frac{1 - e^{-\beta m t}}{\beta m}, \quad r_*(t) = \eta_0\left(1 + \frac{1 - e^{-\beta m t}}{\beta m}\right)^{1/(nm+2)}.$$

The diameter of the support of this solution is monotonically increasing but is bounded now by the constant

$$L = \lim_{t\to\infty} |r_*(t)| = \eta_0\left(1 + \frac{1}{\beta m}\right)^{1/(nm+2)} < \infty.$$

The perturbation is localized in a ball of radius L.

⊙ *References for equation* 6.5.5.10: L. K. Martinson and K. B. Pavlov (1972), A. D. Polyanin and V. F. Zaitsev (2002, pp. 116–117).

11. $\dfrac{\partial w}{\partial t} = \nabla \cdot (w^m \nabla w) - w^{1-m}$.

Solution in the radially symmetric case for $0 < m < 1$:

$$w = \begin{cases} \left[\dfrac{2(nm+2)}{m}t\right]^{-1/m} V^{1/m} & \text{if } V \ge 0, \\ 0 & \text{if } V < 0, \end{cases}$$

where

$$V = At^{2/(nm+2)} - \frac{(nm+2)^2}{nm+1}t^2 - r^2, \qquad r = \sqrt{x_1^2 + \cdots + x_n^2};$$

A is an arbitrary constant ($A > 0$). The solution has a compact support. The diameter of the support increases with t on the time interval $(0, t_*)$, where

$$t_* = \left[\frac{A(nm+1)}{(nm+2)^3}\right]^{\frac{nm+2}{nm+1}}$$

and decreases on the interval (t_*, T_0), where

$$T_0 = \left[\frac{A(nm+1)}{(nm+2)^2}\right]^{\frac{nm+2}{2(nm+1)}}.$$

The solution vanishes at $t = T_0$.

⊙ *References*: R. Kersner (1978), L. K. Martinson (1979).

12. $\dfrac{\partial w}{\partial t} = a\nabla\cdot(e^{\lambda w}\nabla w) + be^{\lambda w} + f(t) + g(t)e^{-\lambda w}.$

Functional separable solution:

$$w(\mathbf{x},t) = \frac{1}{\lambda}\ln\big[\varphi(t) + \psi(t)\Theta(\mathbf{x})\big], \qquad \psi(t) = \exp\left\{\lambda\int\big[b\varphi(t) + f(t)\big]\,dt\right\},$$

where the function $\varphi(t)$ is determined by the Riccati equation

$$\varphi_t' = b\lambda\varphi^2 + \lambda f(t)\varphi + \lambda g(t), \tag{1}$$

and the function $\Theta = \Theta(\mathbf{x})$ is a solution of the Helmholtz equation

$$a\Delta\Theta + b\lambda\Theta = 0. \tag{2}$$

For details about the Riccati equation (1), see Kamke (1977) and Polyanin and Zaitsev (2003). For solutions of the linear equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

13. $\dfrac{\partial w}{\partial t} = \nabla\cdot[f(w)\nabla w] + \dfrac{a}{f(w)} + b.$

Solution in implicit form:

$$\int f(w)\,dw = at + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the Poisson equation

$$\Delta U + b = 0.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: V. A. Galaktionov (1994).

14. $\dfrac{\partial w}{\partial t} = \nabla\cdot[f(w)\nabla w] + \dfrac{g(t)}{f(w)} + h(\mathbf{x}).$

Solution in implicit form:

$$\int f(w)\,dw = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the Poisson equation

$$\Delta U + h(\mathbf{x}) = 0.$$

15. $\dfrac{\partial w}{\partial t} = \Delta f(w) + \dfrac{af(w) + b}{f'(w)} + c[af(w) + b].$

Solution in implicit form:

$$f(w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the Helmholtz equation

$$\Delta U + acU = 0.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: V. A. Galaktionov (1994).

16. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(w)] + \dfrac{g(t)}{f'(w)} + h(\mathbf{x}).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(w) = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + h(\mathbf{x}) = 0.$$

17. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(w)] + \dfrac{af(w) + b}{f'(w)} + g(\mathbf{x})[af(w) + b].$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + ag(\mathbf{x})U = 0.$$

18. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(x, w)] + \dfrac{g(t)}{f_w(x, w)} + h(\mathbf{x}).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$; and f_w stands for the partial derivative of f with respect to w.

Solution in implicit form:

$$f(x, w) = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + h(\mathbf{x}) = 0.$$

19. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(x, w)] + \dfrac{af(x, w) + b}{f_w(x, w)} + g(\mathbf{x})[af(x, w) + b].$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(x, w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + ag(\mathbf{x})U = 0.$$

20. $\dfrac{\partial w}{\partial t} = a\nabla\cdot(|\nabla w|\nabla w) + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(\mathbf{x},t) = \varphi(t) + \exp\left\{\int\left[2b\varphi(t) + f(t)\right]dt\right\}\Theta(\mathbf{x}),$$

where the function $\varphi(t)$ is determined by the Riccati equation

$$\varphi_t' = b\varphi^2 + f(t)\varphi + g(t),$$

and the function $\Theta = \Theta(\mathbf{x})$ is a solution of the stationary equation

$$a\nabla\cdot(|\nabla\Theta|\nabla\Theta) + b\Theta^2 = 0.$$

6.6. Nonlinear Schrödinger Equations

6.6.1. Two-Dimensional Equations

1. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + A|w|^2 w = 0.$

Two-dimensional Schrödinger equation with a cubic nonlinearity. This is a special case of equation 6.6.1.3 with $f(u) = Au^2$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm C_1 w(\pm C_1 x + C_2, \pm C_1 y + C_3, C_1^2 t + C_4),$$
$$w_2 = e^{-i[\lambda_1 x + \lambda_2 y + (\lambda_1^2+\lambda_2^2)t + C_5]} w(x + 2\lambda_1 t, y + 2\lambda_2 t, t),$$
$$w_3 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_5$, λ_1, λ_2, and β are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,y,t) = C_1\exp\left\{i\left[C_2 x + C_3 y + (AC_1^2 - C_2^2 - C_3^2)t + C_4\right]\right\},$$
$$w(x,y,t) = \frac{C_1}{t}\exp\left[i\frac{(x+C_2)^2 + (y+C_3)^2 - 4AC_1^2}{4t} + iC_4\right],$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

3°. "Two-dimensional" solution:

$$w(x,y,t) = e^{i(C_1 t + C_2)}u(x,y),$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x,y)$ is determined by the stationary equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Au^3 - C_1 u = 0.$$

4°. Solution:

$$w(x,y,t) = (f_1x + f_2y + f_3)\exp\bigl[i(g_1x^2 + g_2xy + g_3y^2 + h_1x + h_2y + h_3)\bigr],$$

where the functions $f_k = f_k(t)$, $g_k = g_k(t)$, and $h_k = h_k(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
f_1' + 2(3g_1 + g_3)f_1 + 2f_2g_2 &= 0,\\
f_2' + 2(g_1 + 3g_3)f_2 + 2f_1g_2 &= 0,\\
f_3' + 2(g_1 + g_3)f_3 + 2(f_1h_1 + f_2h_2) &= 0,\\
g_1' + 4g_1^2 + g_2^2 - Af_1^2 &= 0,\\
g_2' + 4(g_1 + g_3)g_2 - 2Af_1f_2 &= 0,\\
g_3' + g_2^2 + 4g_3^2 - Af_2^2 &= 0,\\
h_1' + 4g_1h_1 + 2g_2h_2 - 2Af_1f_3 &= 0,\\
h_2' + 2g_2h_1 + 4g_3h_2 - 2Af_2f_3 &= 0,\\
h_3' + h_1^2 + h_2^2 - Af_3^2 &= 0.
\end{aligned}$$

The prime denotes a derivative with respect to t.

5°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi_1,\xi_2)e^{i(k_1x+k_2y+at+b)}, \quad \xi_1 = x - 2k_1t, \quad \xi_2 = y - 2k_2t,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $U = U(\xi_1,\xi_2)$ is determined by a differential equation of the form 9.4.1.1:

$$\frac{\partial^2 U}{\partial\xi_1^2} + \frac{\partial^2 U}{\partial\xi_2^2} + A|U|^2U - (k_1^2 + k_2^2 + a)U = 0.$$

For solutions of this equation, see also equation 9.1.1.2 with $n = 3$.

6°. "Two-dimensional" solution:

$$w(x,y,t) = \Phi(z_1,z_2)\exp\bigl[i(k_1xt + k_2yt - \tfrac{2}{3}k_1^2t^3 - \tfrac{2}{3}k_2^2t^3 + at + b)\bigr], \quad z_1 = x - k_1t^2, \quad z_2 = y - k_2t^2,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $\Phi = \Phi(z_1,z_2)$ is determined by a differential equation of the form 9.4.1.1:

$$\frac{\partial^2 \Phi}{\partial z_1^2} + \frac{\partial^2 \Phi}{\partial z_2^2} + A|\Phi|^2\Phi - (k_1z_1 + k_2z_2 + a)\Phi = 0.$$

7°. "Two-dimensional" solution:

$$w(x,y,t) = \frac{1}{\sqrt{C_1t + C_2}}u(\xi,\eta), \quad \xi = \frac{x + C_3}{\sqrt{C_1t + C_2}}, \quad \eta = \frac{y + C_4}{\sqrt{C_1t + C_2}},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $u = u(\xi,\eta)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} - \frac{1}{2}iC_1\left(\xi\frac{\partial u}{\partial\xi} + \eta\frac{\partial u}{\partial\eta} + u\right) + A|u|^2u = 0.$$

⊙ *References for equation* 6.6.1.1: L. Gagnon, P. Winternitz (1988, 1989), N. H. Ibragimov (1995, pp. 437–438), A. M. Vinogradov and I. S. Krasil'shchik (1997), A. D. Polyanin and V. F. Zaitsev (2004, pp. 186–187).

2. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+A|w|^{2n}w=0.$

Two-dimensional Schrödinger equation with a power-law nonlinearity; A and n are real numbers. This is a special case of equation 6.6.1.3 with $f(u)=Au^{2n}$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1=\pm C_1 w(\pm C_1^n x+C_2,\pm C_1^n y+C_3,C_1^{2n}t+C_4),$$
$$w_2=e^{-i[\lambda_1 x+\lambda_2 y+(\lambda_1^2+\lambda_2^2)t+C_5]}w(x+2\lambda_1 t,y+2\lambda_2 t,t),$$
$$w_3=w(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta,t),$$

where $C_1,\ldots,C_5$, β, λ_1, and λ_2 are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,y,t)=C_1\exp\bigl\{i\,[C_2x+C_3y+(A|C_1|^{2n}-C_2^2-C_3^2)t+C_4]\bigr\},$$
$$w(x,y,t)=\frac{C_1}{t}\exp\left[i\frac{(x+C_2)^2+(y+C_3)^2}{4t}+i\frac{AC_1^{2n}}{1-2n}t^{1-2n}+iC_4\right],$$

where $C_1,\ldots,C_4$ are arbitrary real constants.

3°. For other exact solutions, see equation 6.6.1.3 with $f(w)=Aw^{2n}$.

3. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+f(|w|)w=0.$

Two-dimensional nonlinear Schrödinger equation of general form.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1=e^{-i[\lambda_1 x+\lambda_2 y+(\lambda_1^2+\lambda_2^2)t+A]}w(x+2\lambda_1 t+C_1,y+2\lambda_2 t+C_2,t+C_3),$$
$$w_2=w(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta,t),$$

where A, C_1, C_2, C_3, λ_1, λ_2, and β are arbitrary real constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w(x,y,t)=C_1\exp\bigl[i\,(C_2x+C_3y+\lambda t+C_4)\bigr],\qquad \lambda=f(|C_1|)-C_2^2-C_3^2,$$

where $C_1,\ldots,C_4$ are arbitrary real constants.

3°. Exact solutions depending only on the radial variable $r=\sqrt{x^2+y^2}$ and time t are determined by the equation

$$i\frac{\partial w}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+f(|w|)w=0,$$

which is a special case of equation 5.7.5.2 with $n=1$.

4°. "Two-dimensional" solution:

$$w(x,y,t) = e^{i(At+B)}u(x,y),$$

where A and B are arbitrary real constants, and the function $u = u(x,y)$ is determined by a stationary equation of the form 9.4.1.1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(|u|)u - Au = 0.$$

5°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi,\eta)e^{i(A_1x+A_2y+Bt+C)}, \quad \xi = x - 2A_1t, \quad \eta = y - 2A_2t,$$

where A_1, A_2, B, and C are arbitrary constants, and the function $U = U(\xi,\eta)$ is determined by a differential equation of the form 9.4.1.1:

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + f(|U|)U - (A_1^2 + A_2^2 + B)U = 0.$$

6°. "Two-dimensional" solution:

$$w(x,y,t) = \Phi(z_1,z_2)\exp\left[i(k_1xt + k_2yt - \tfrac{2}{3}k_1^2t^3 - \tfrac{2}{3}k_2^2t^3 + at + b)\right], \quad z_1 = x - k_1t^2, \quad z_2 = y - k_2t^2,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $\Phi = \Phi(z_1,z_2)$ is determined by the differential equation of the form 9.4.1.1:

$$\frac{\partial^2 \Phi}{\partial z_1^2} + \frac{\partial^2 \Phi}{\partial z_2^2} + f(|\Phi|)\Phi - (k_1z_1 + k_2z_2 + a)\Phi = 0.$$

7°. There is a "two-dimensional" solution of the form

$$w(x,y,t) = U(z_1,z_2), \quad z_1 = a_1x + b_1y + c_1t, \quad z_2 = a_2x + b_2y + c_2t.$$

8°. For group classification of the original equation, see Gagnon and Winternitz (1988) and Ibragimov (1995, pp. 437–438).

6.6.2. Three and n-Dimensional Equations

1. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} + A|w|^2w = 0.}$

Three-dimensional Schrödinger equation with a cubic nonlinearity. This is a special case of equation 6.6.2.2 with $f(u) = Au^2$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm C_1 w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1z + C_4, C_1^2t + C_5),$$
$$w_2 = e^{-i[\lambda_1x + \lambda_2y + \lambda_3z + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)t + C_6]}w(x + 2\lambda_1t, y + 2\lambda_2t, z + 2\lambda_3t, t),$$

where $C_1, \ldots, C_6$, λ_1, λ_2, and λ_3 are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. There is an exact solution of the form

$$w=(f_1x+f_2y+f_3z+f_4)\exp\left[i(g_1x^2+g_2y^2+g_3z^2+g_4xy+g_5xz+g_6yz+h_1x+h_2y+h_3z+h_4)\right],$$

where the functions $f_k = f_k(t)$, $g_k = g_k(t)$, and $h_k = h_k(t)$ are determined by an appropriate system of ordinary differential equations (which is not written out here).

3°. Solution:

$$w(x,y,z,t)=U(\xi_1,\xi_2,\xi_3)e^{i(k_1x+k_2y+k_3z+at+b)},\quad \xi_1=x-2k_1t,\quad \xi_2=y-2k_2t,\quad \xi_3=z-2k_3t,$$

where k_1, k_2, k_3, a, and b are arbitrary constants, and the function $U = U(\xi_1, \xi_2, \xi_3)$ is determined by the differential equation

$$\frac{\partial^2U}{\partial\xi_1^2}+\frac{\partial^2U}{\partial\xi_2^2}+\frac{\partial^2U}{\partial\xi_3^2}+A|U|^2U-(k_1^2+k_2^2+k_3^2+a)U=0.$$

4°. "Three-dimensional" solution:

$$w(x,y,z,t)=\frac{1}{\sqrt{C_1t+C_2}}u(\xi,\eta,\zeta),\quad \xi=\frac{x+C_3}{\sqrt{C_1t+C_2}},\quad \eta=\frac{y+C_4}{\sqrt{C_1t+C_2}},\quad \zeta=\frac{z+C_5}{\sqrt{C_1t+C_2}},$$

where $C_1, \ldots, C_5$ are arbitrary constants, and the function $u = u(\xi, \eta, \zeta)$ is determined by the differential equation

$$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}+\frac{\partial^2u}{\partial\zeta^2}-\frac{1}{2}iC_1\left(\xi\frac{\partial u}{\partial\xi}+\eta\frac{\partial u}{\partial\eta}+\zeta\frac{\partial u}{\partial\zeta}+u\right)+A|u|^2u=0.$$

⊙ *References for equation* 6.6.2.1: L. Gagnon, P. Winternitz (1988, 1989), N. H. Ibragimov (1995, pp. 437–438), A. M. Vinogradov and I. S. Krasil'shchik (1997), A. D. Polyanin and V. F. Zaitsev (2004, p. 189).

2. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}+f(|w|)w=0.$

Three-dimensional nonlinear Schrödinger equation of general form. It admits translations in any of the independent variables.

1°. Suppose $w(x, y, z, t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1=e^{-i[\lambda_1x+\lambda_2y+\lambda_3z+(\lambda_1^2+\lambda_2^2+\lambda_3^2)t+A]}w(x+2\lambda_1t+C_1,y+2\lambda_2t+C_2,z+2\lambda_3t+C_3,t+C_4),$$

where A, $C_1, \ldots, C_4$, λ_1, λ_2, and λ_3 are arbitrary real constants, is also a solution of the equation.

2°. Exact solutions depending only on the radial variable $r=\sqrt{x^2+y^2+z^2}$ and time t are determined by the equation

$$i\frac{\partial w}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+f(|w|)w=0,$$

which is a special case of equation 5.7.5.2 with $n = 2$.

3°. "Three-dimensional" solution:

$$w(x, y, z, t) = e^{i(At+B)}u(x, y, z),$$

where A and B are arbitrary real constants, and the function $u = u(x, y, z)$ is determined by the stationary equation

$$\Delta u + f(|u|)u - Au = 0.$$

4°. Axisymmetric solutions in cylindrical and spherical coordinates are determined by equations where the Laplace operator has the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right) + \frac{\partial^2 w}{\partial z^2}, \qquad \rho = \sqrt{x^2 + y^2};$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right), \quad r = \sqrt{x^2 + y^2 + z^2},$$

respectively. Consequently, there are solutions of the forms $w = w(\rho, t)$ and $w = w(r, t)$.

5°. "Three-dimensional" solution:

$$w = U(\xi, \eta, t), \qquad \xi = y + \frac{x}{C}, \quad \eta = (C^2 - 1)x^2 - 2Cxy + C^2z^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta, t)$ is determined by the differential equation

$$i\frac{\partial U}{\partial t} + \left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi\frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} + f(|U|)U = 0.$$

6°. "Three-dimensional" solution:

$$w = V(\xi, \eta, t), \qquad \xi = Ax + By + Cz, \quad \eta = \sqrt{(Bx - Ay)^2 + (Cy - Bz)^2 + (Az - Cx)^2},$$

where A, B, and C are arbitrary constants and the function $V = V(\xi, \eta, t)$ is determined by the equation

$$i\frac{\partial V}{\partial t} + (A^2 + B^2 + C^2)\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial V}{\partial \eta}\right) + f(|V|)V = 0.$$

⊙ *References for equation* 6.6.2.2: L. Gagnon, P. Winternitz (1988, 1989), N. H. Ibragimov (1995, pp. 437–438), A. D. Polyanin and V. F. Zaitsev (2004, pp. 189–190).

3. $i\dfrac{\partial w}{\partial t} + \Delta w + A|w|^2 w = 0.$

This is an *n-dimensional Schrödinger equation with a cubic nonlinearity*.

1°. Suppose $w(x_1, \ldots, x_n, t)$ is a solution to the original Schrödinger equation. Then the functions

$$w_1 = \pm C_1 w(\pm C_1 x_1 + C_2, \ \ldots, \ \pm C_1 x_n + C_{n+1}, C_1^2 t + C_{n+2}),$$

$$w_2 = e^{-i[\lambda_1 x_1 + \cdots + \lambda_n x_n + (\lambda_1^2 + \cdots + \lambda_n^2)t + C_{n+3}]}w(x_1 + 2\lambda_1 t, \ \ldots, \ x_n + 2\lambda_n t, t),$$

where $C_1, \ \ldots, \ C_{n+3}, \ \lambda_1, \ \ldots, \ \lambda_n$ are arbitrary real constants, are also solutions of the equation. The plus and minus signs in the expression of w_1 can be chosen arbitrarily.

2°. There is an exact solution of the form

$$w = \left(\sum_{k=1}^{n} f_k x_k + f_{n+1}\right) \exp\left[i\left(\sum_{k,m=1}^{n} g_{km} x_k x_m + \sum_{k=1}^{n} h_k x_k + p\right)\right],$$

where $f_k = f_k(t)$, $g_{km} = g_{km}(t)$, $h_k = h_k(t)$, and $p = p(t)$.

3°. Solution:

$$w(x_1, \ldots, x_n, t) = U(\xi_1, \ldots, \xi_n) e^{i(k_1 x_1 + \cdots + k_n x_n + at + b)}, \quad \xi_1 = x_1 - 2k_1 t, \quad \ldots, \quad \xi_n = x_n - 2k_n t,$$

where $k_1, \ldots, k_n$, a, and b are arbitrary constants, and the function $U = U(\xi_1, \ldots, \xi_n)$ is described by the stationary equation

$$\frac{\partial^2 U}{\partial \xi_1^2} + \cdots + \frac{\partial^2 U}{\partial \xi_n^2} + A|U|^2 U - (k_1^2 + \cdots + k_n^2 + a)U = 0.$$

4. $i\dfrac{\partial w}{\partial t} = \Delta w + |w|^2 w.$

This is an *n-dimensional modified Schrödinger equation with a cubic nonlinearity*.

Conservation laws:

$$\left(|w|^2\right)_t + i\nabla \cdot \left(\bar{w}\nabla w - w\nabla \bar{w}\right)_x = 0,$$
$$\left(|\nabla w|^2 - \tfrac{1}{2}|w|^4\right)_t + i\nabla \cdot \left[(\Delta w + |w|^2 w)\nabla \bar{w} - (\Delta \bar{w} + |w|^2 \bar{w})\nabla w\right]_x = 0.$$

The bar over a symbol denotes the complex conjugate.

⊙ *Reference*: A. M. Vinogradov and I. S. Krasil'shchik (1997, p. 293).

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Chapter 7
Second-Order Hyperbolic Equations with One Space Variable

7.1. Equations with Power-Law Nonlinearities

7.1.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{2n-1}$

▷ *The general properties of equations of this type are outlined in 7.4.1.1; traveling-wave solutions and some other solutions are also presented there.*

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw^n.$

This is a special case of equation 7.4.1.1 with $f(w) = aw^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(\pm C_1^{n-1}x + C_2,\ \pm C_1^{n-1}t + C_3\big),$$
$$w_2 = w(x\cosh\lambda + t\sinh\lambda,\ x\sinh\lambda + t\cosh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,t) = b\big(x + C_1 t + C_2\big)^{\frac{2}{1-n}}, \qquad b = \left[\frac{2(1+n)(C_1^2-1)}{a(1-n)^2}\right]^{\frac{1}{n-1}};$$
$$w(x,t) = \big[k(t+C_1)^2 - k(x+C_2)^2\big]^{\frac{1}{1-n}}, \qquad k = \tfrac{1}{4}a(1-n)^2,$$

where C_1 and C_2 are arbitrary constants.

3°. The solutions of Item 2° are special cases of solutions of the following forms:

$$w(x,t) = F(z), \quad z = x + C_1 t + C_2;$$
$$w(x,t) = G(\xi), \quad \xi = (t+C_1)^2 - (x+C_2)^2.$$

4°. Self-similar solution:

$$w(x,t) = (t+C_1)^{\frac{2}{1-n}}\,u(\xi), \qquad \xi = \frac{x+C_2}{t+C_1},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$(1-\xi^2)u''_{\xi\xi} + \frac{2(1+n)}{1-n}\xi u'_\xi - \frac{2(1+n)}{(1-n)^2}u + au^n = 0.$$

The transformation

$$u = (\cosh\theta)^{\frac{2}{n-1}}U(\theta), \qquad \xi = \tanh\theta$$

brings this equation to the autonomous form

$$U''_{\theta\theta} - \frac{4}{(1-n)^2}U + aU^n = 0.$$

Integrating yields the general solution in implicit form

$$\int\left[\frac{4}{(n-1)^2}U^2 - \frac{2a}{n+1}U^{n+1} + C_3\right]^{-1/2} dU = C_4 \pm \theta,$$

where C_3 and C_4 are arbitrary constants.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^3.$

This is a special case of equation 7.4.1.1 with $f(w) = aw + bw^3$.

1°. Traveling-wave solutions:

$$w = \pm\sqrt{-\frac{a}{b}}\,\tanh\left[\sqrt{\frac{a}{2(1-C_1^2)}}\,(x + C_1 t + C_2)\right],$$

$$w = \pm\sqrt{\frac{a}{b}}\,\tan\left[\sqrt{\frac{a}{2(C_1^2-1)}}\,(x + C_1 t + C_2)\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: M. Dehghan and A. Shokri (2009).

2°. See also equation 7.1.1.3 with $n = 3$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n.$

This is a special case of equation 7.4.1.1 with $f(w) = aw + bw^n$.

1°. Traveling-wave solutions for $a > 0$:

$$w = \left[\frac{2b\sinh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2 \quad \text{if } b(n+1) > 0,$$

$$w = \left[-\frac{2b\cosh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \quad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2 \quad \text{if } b(n+1) < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solutions for $a < 0$ and $b(n+1) > 0$:

$$w = \left[-\frac{2b\cos^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \quad z = \tfrac{1}{2}\sqrt{|a|}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw^n + bw^{2n-1}.$

This is a special case of equation 7.4.1.1 with $f(w) = aw^n + bw^{2n-1}$.

Solutions:

$$w(x,t) = \left[\frac{a(1-n)^2}{2(n+1)}(x\sinh C_1 \pm t\cosh C_1 + C_2)^2 - \frac{b(n+1)}{2an}\right]^{\frac{1}{1-n}},$$

$$w(x,t) = \left\{\frac{1}{4}a(1-n)^2\left[(t+C_1)^2 - (x+C_2)^2\right] - \frac{b}{an}\right\}^{\frac{1}{1-n}},$$

where C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw - a(n+1)w^n + bw^{2n-1}.$

1°. Traveling-wave solutions:

$$w(x,t) = \left(\lambda + C_1\exp z\right)^{\frac{1}{1-n}}, \qquad z = \sqrt{a}\,(1-n)(x\sinh C_2 \pm t\cosh C_2),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $a\lambda^2 - a(n+1)\lambda + b = 0$, and C_1 and C_2 are arbitrary constants.

2°. See also equation 7.1.1.6, in which b should be renamed $-a(n+1)$ and c renamed b.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{2n-1}.$

This is a special case of equation 7.4.1.1 with $f(w) = aw + bw^n + cw^{2n-1}$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x,t) = (A + B\cosh z)^{\frac{1}{1-n}}, \quad z = \sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2};$$

$$w(x,t) = (A + B\sinh z)^{\frac{1}{1-n}}, \quad z = \sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{c}{an} - \frac{b^2}{a^2(n+1)^2}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants (the expressions in square brackets must be nonnegative).

2°. Traveling-wave solutions for $a < 0$:

$$w(x,t) = (A + B\cos z)^{\frac{1}{1-n}}, \quad z = \sqrt{|a|}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

3°. The substitution $u = w^{1-n}$ leads to an equation with a quadratic nonlinearity:

$$u\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{n}{1-n}\left[\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right] = a(1-n)u^2 + b(1-n)u + c(1-n).$$

7.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)w^k.$

This is a special case of equation 7.4.1.2 with $f(w) = aw^k$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + c(x + bt)^n w^k.$

This is a special case of equation 7.4.1.4 with $f(z,w) = cz^n w^k$. For $b = \pm 1$, see also equations 7.4.1.13 and 7.4.1.14 with $f(\xi) = c\xi^n$, $g(w) = w^k$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)(xt)^n w^k.$

This is a special case of equation 7.4.1.5 with $f(z,w) = az^n w^k$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta t} w^k.$

This is a special case of equation 7.4.1.7 with $f(w) = aw^k$.

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{4(k+1)}(t \pm x)\right] + \frac{\sqrt{a}}{\beta}(1-k)e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{4(k+1)}(t \pm x)\right] - \frac{\sqrt{a}}{\beta}(1-k)e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + be^{\beta t} w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{k-1}{4\beta(k+1)}\left(\pm\sqrt{[\beta^2 - (k-1)^2 a][\beta^2 - (k+3)^2 a]}\,x + [\beta^2 + (k-1)(k+3)a]\,t\right)\right] + (k-1)\sqrt{\frac{b}{\beta^2 - (k-1)^2 a}}\,e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{k-1}{4\beta(k+1)}\left(\pm\sqrt{[\beta^2 - (k-1)^2 a][\beta^2 - (k+3)^2 a]}\,x + [\beta^2 + (k-1)(k+3)a]\,t\right)\right] - (k-1)\sqrt{\frac{b}{\beta^2 - (k-1)^2 a}}\,e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{\beta t}(a + be^{\beta t})w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{2(k+1)}(t \pm x)\right] + \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta t}\right]\right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{2(k+1)}(t \pm x)\right] - \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta t}\right]\right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\beta^2}{k^2+4}w + \left(a^2e^{2\beta t} + abke^{\beta t} - b^2\right)w^{-3}, \qquad k \neq 0.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2+4}}\right) + \frac{\sqrt{k^2+4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2+4}}\right) + \frac{\sqrt{k^2+4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2+4}}\right) - \frac{\sqrt{k^2+4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2+4}}\right) - \frac{\sqrt{k^2+4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\beta^2}{k^2-4}w + \left(a^2e^{2\beta t} + abke^{\beta t} + b^2\right)w^{-3}, \quad |k| > 2.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2-4}}\right) + \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2-4}}\right) + \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2-4}}\right) - \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2-4}}\right) - \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - ae^{\beta x} w^k.$

This is a special case of equation 7.4.1.6 with $f(w) = -aw^k$.

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{4(k+1)}(x \pm t)\right] + \frac{\sqrt{a}}{\beta}(1-k)e^{\beta x/2} \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{4(k+1)}(x \pm t)\right] - \frac{\sqrt{a}}{\beta}(1-k)e^{\beta x/2} \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - aw - be^{\beta x} w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{k-1}{4\beta(k+1)}\left(\pm\sqrt{[\beta^2-(k-1)^2a][\beta^2-(k+3)^2a]}\,t \right.\right.\right.$$
$$\left.\left.\left. + [\beta^2+(k-1)(k+3)a]\,x\right)\right] + (k-1)\sqrt{\frac{b}{\beta^2-(k-1)^2a}}\,e^{\beta x/2} \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{k-1}{4\beta(k+1)}\left(\pm\sqrt{[\beta^2-(k-1)^2a][\beta^2-(k+3)^2a]}\,t \right.\right.\right.$$
$$\left.\left.\left. + [\beta^2+(k-1)(k+3)a]\,x\right)\right] - (k-1)\sqrt{\frac{b}{\beta^2-(k-1)^2a}}\,e^{\beta x/2} \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - e^{\beta x}(a + be^{\beta x})w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{2(k+1)}(x \pm t)\right] + \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta x}\right] \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C\exp\left[\frac{\beta(k-1)}{2(k+1)}(x \pm t)\right] - \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta x}\right] \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

12. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ce^{ax+bt} w^k.$

This is a special case of equation 7.4.1.8 with $f(w) = cw^k$.

7.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + bw^m, \qquad a > 0.$

This equation can be rewritten in the equivalent form

$$\frac{\partial^2 w}{\partial t^2} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{n}{x}\frac{\partial w}{\partial x}\right) + bw^m.$$

For $n = 1$ and $n = 2$, this equation describes nonlinear waves with axial and central symmetry, respectively.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(\pm C_1^{k-1} x,\ \pm C_1^{k-1} t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Functional separable solution:

$$w(x, t) = \left\{\frac{b(1-m)^2}{2a(2+n-nm)}\big[a(t+C)^2 - x^2\big]\right\}^{\frac{1}{1-m}}, \tag{1}$$

where C is an arbitrary constant.

3°. Solution (1) is a special case of the wider family of exact solutions

$$w = w(r), \qquad r^2 = A\big[a(t+C)^2 - x^2\big],$$

where the sign of A must coincide with that of the expression in square brackets, and the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{n+1}{r}w'_r = \frac{b}{Aa}w^m. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (2) for specific values of the parameters n and m.

4°. There is a self-similar solution of the form $w = t^{\frac{2}{1-m}} f(\xi)$, where $\xi = x/t$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw^m\dfrac{\partial w}{\partial x}.$

This is a special case of equation 7.4.2.3 with $f(w) = bw^m$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\big(C_1^m x + C_2,\ \pm C_1^m t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w(x,t)=\left[\frac{bm(x+\lambda t+C)}{(m+1)(a-\lambda^2)}\right]^{-1/m}, \tag{1}$$

where C and λ are arbitrary constants.

Solution (1) is a special case of the wider class of traveling-wave solutions

$$\int \frac{dw}{A+bw^{m+1}}=\frac{x+\lambda t+C}{(m+1)(\lambda^2-a)},$$

where A, C, and λ are arbitrary constants.

3°. There is a self-similar solution of the form $w=t^{-1/m}f(x/t)$.

3. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^2 w}{\partial x^2}+b\left(\dfrac{\partial w}{\partial x}\right)^2+cw+s.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm x+C_1,\pm t+C_2)+C_3\cosh(kt)+C_4\sinh(kt) \quad \text{if } c=k^2>0,$$
$$w_2=w(\pm x+C_1,\pm t+C_2)+C_3\cos(kt)+C_4\sin(kt) \quad \text{if } c=-k^2<0,$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Generalized separable solution quadratic in x:

$$w(x,t)=-\frac{c}{4b}(\pm x+C_1t+C_2)^2+\frac{1}{2b}(a-C_1^2)-\frac{s}{c}+U(t),$$
$$U(t)=\begin{cases}C_3\cosh(kt)+C_4\sinh(kt) & \text{if } c=k^2>0,\\ C_3\cos(kt)+C_4\sin(kt) & \text{if } c=-k^2<0.\end{cases}$$

3°. For other solutions, see 7.4.2.4 with $f(t)=s$.

4. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^2 w}{\partial x^2}+b\left(\dfrac{\partial w}{\partial x}\right)^2+cw+st^n.$

This is a special case of equation 7.4.2.4 with $f(t)=st^n$.

5. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^2 w}{\partial x^2}+b\left(\dfrac{\partial w}{\partial x}\right)^2+cw+sx^n.$

This is a special case of equation 7.4.2.5 with $f(x)=sx^n$.

6. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^2 w}{\partial x^2}+c\left(\dfrac{\partial w}{\partial x}\right)^2+bcw^2+kw+s.$

This is a special case of equation 7.4.2.10 with $f(t)=c$, $g(t)=k$, and $h(t)=s$.

Let A be a root of the quadratic equation $bcA^2 + kA + s = 0$.

1°. Suppose that $2Abc + k - ab = \sigma^2 > 0$. Then there are generalized separable solutions

$$w(x,t) = A + \big[C_1 \exp(\sigma t) + C_2 \exp(-\sigma t)\big] \exp\big(\pm x\sqrt{-b}\,\big),$$

where C_1 and C_2 are arbitrary constants.

2°. If $2Abc + k - ab = -\sigma^2 < 0$, there are generalized separable solutions

$$w(x,t) = A + \big[C_1 \cos(\sigma t) + C_2 \sin(\sigma t)\big] \exp\big(\pm x\sqrt{-b}\,\big).$$

For more complicated solutions, see 7.4.2.10.

⊙ *References for equation* 7.1.3.6: V. A. Galaktionov (1995, the case $a = c$ was considered), V. F. Zaitsev and A. D. Polyanin (1996, p. 448).

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bx^n\left(\dfrac{\partial w}{\partial x}\right)^2 + cx^m + st^k.$

This is a special case of equation 7.4.2.8 with $f(x) = bx^n$, $g(x) = cx^m$, and $h(t) = st^k$.

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + ct^n\left(\dfrac{\partial w}{\partial x}\right)^2 + bct^n w^2 + st^m w + pt^k.$

This is a special case of equation 7.4.2.10 with $f(t) = ct^n$, $g(t) = st^m$, and $h(t) = pt^k$.

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n\left(\dfrac{\partial w}{\partial x}\right)^2 + ct^k x\dfrac{\partial w}{\partial x}.$

There is a generalized separable solution quadratic in x:

$$w = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Multiplicative separable solutions with $bc < 0$:

$$w = \begin{cases} \left[C_1 \exp(kt) + C_2 \exp(-kt)\right] \exp\left(\pm\sqrt{-\dfrac{c}{b}}\,x\right) & \text{if } s - \dfrac{ac}{b} = k^2 > 0, \\ \left[C_1 \cos(kt) + C_2 \sin(kt)\right] \exp\left(\pm\sqrt{-\dfrac{c}{b}}\,x\right) & \text{if } s - \dfrac{ac}{b} = -k^2 < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

Remark. The more general equation

$$w_{tt} = aw_{xx} + f(w)\big(bw_x^2 + cw^2\big) + sw,$$

where $f(w)$ is an arbitrary function, has the same solutions.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\bigl(x\sqrt{|c/b|}\,\bigr) + \psi(t)\exp\bigl(-x\sqrt{|c/b|}\,\bigr),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = \left(s - \frac{ac}{b}\right)\varphi + 4c\varphi^2\psi,$$
$$\psi''_{tt} = \left(s - \frac{ac}{b}\right)\psi + 4c\varphi\psi^2.$$

3°. Generalized separable solution with $bc > 0$:

$$w = \varphi(t)\cos\bigl(x\sqrt{c/b}\,\bigr) + \psi(t)\sin\bigl(x\sqrt{c/b}\,\bigr),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = \left(s - \frac{ac}{b}\right)\varphi + c(\varphi^2 + \psi^2)\varphi,$$
$$\psi''_{tt} = \left(s - \frac{ac}{b}\right)\psi + c(\varphi^2 + \psi^2)\psi.$$

7.1.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a(x+\beta)^n \dfrac{\partial^2 w}{\partial x^2} + bw^m, \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Functional separable solution for $n \neq 2$:

$$w(x,t) = \Bigl\{ s\bigl[a(2-n)^2(t+C)^2 - 4(x+\beta)^{2-n}\bigr]\Bigr\}^{\frac{1}{1-m}}, \qquad s = \frac{b(1-m)^2}{2a(2-n)(nm-3n+4)},$$

where C is an arbitrary constant.

2°. Functional separable solution (generalizes the solution of Item 1°):

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(1-n)}{2-n}\frac{1}{r}w'_r = \frac{4b}{k}w^m. \tag{1}$$

The substitution $\xi = r^{\frac{n}{2-n}}$ leads to the Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4b(2-n)^2}{kn^2}\xi^{\frac{4(1-n)}{n}}w^m. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents more than 20 exact solutions to equation (2) for specific values of the parameters n and m.

Special case. For $n = 1$, the general solution of equation (1) is written in explicit form as

$$\int \left[C_1 + \frac{8b}{k(m+1)} w^{m+1}\right]^{-1/2} dw = \pm r + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x + \beta|,$$

where A and B are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} - aBw'_y + bw^m = 0. \tag{3}$$

The solution of equation (3) with $A = \pm B\sqrt{a}$ is given by

$$w(y) = \left[\frac{b(1-m)}{aB} y + C\right]^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

For $A \neq \pm B\sqrt{a}$, the substitution $U(w) = \dfrac{aB^2 - A^2}{aB} w'_y$ brings (3) to the Abel equation

$$UU'_w - U = \frac{b(A^2 - aB^2)}{a^2B^2} w^m,$$

whose general solutions for $m = -2, -1, -\frac{1}{2}, 0, 1$ can be found in Polyanin and Zaitsev (2003).

4°. There is a self-similar solution of the form $w = t^{\frac{2}{1-m}} f(\xi)$, where $\xi = (x + \beta)t^{\frac{2}{n-2}}$.

⊙ *Reference for equation* 7.1.4.1: V. F. Zaitsev and A. D. Polyanin (1996, p. 444).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + bw^m, \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\left(C_1^{\frac{m-1}{2-n}} x, \pm C_1^{\frac{m-1}{2}} t + C_2\right),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w(x,t) = \left\{s\left[a(2-n)^2(t+C)^2 - 4x^{2-n}\right]\right\}^{\frac{1}{1-m}}, \qquad s = \frac{b(1-m)^2}{2a(2-n)(4-n-nm)},$$

where C is an arbitrary constant.

3°. Functional separable solution (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{x^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n}\frac{1}{r}w'_r = \frac{4b}{k}w^m.$$

The substitution $\xi = r^{\frac{n}{n-2}}$ leads to the Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4b(2-n)^2}{kn^2}\xi^{-\frac{4}{n}}w^m. \tag{1}$$

The book by Polyanin and Zaitsev (2003) presents more than 20 exact solutions to equation (1) for specific values of the parameters n and m.

4°. Solution for $n = 2$:

$$w = w(z), \quad z = At + B\ln|x|,$$

where A and B are arbitrary constants, and the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{zz} + aBw'_z + bw^m = 0. \tag{2}$$

The solution of equation (2) with $A = \pm B\sqrt{a}$ is given by

$$w(z) = \left[\frac{b(m-1)}{aB}z + C\right]^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

For $A \neq \pm B\sqrt{a}$, the substitution $U(w) = \dfrac{A^2 - aB^2}{aB}w'_z$ brings (2) to the Abel equation

$$UU'_w - U = \frac{b(A^2 - aB^2)}{a^2B^2}w^m,$$

whose exact solutions for $m = -2, -1, -\frac{1}{2}, 0, 1$ can be found in Polyanin and Zaitsev (1995, 2003).

5°. There is a self-similar solution of the form $w = t^{\frac{2}{1-m}}f(\xi)$, where $\xi = xt^{\frac{2}{n-2}}$.

⊙ *Reference for equation* 7.1.4.2: V. F. Zaitsev and A. D. Polyanin (1996, pp. 445–446).

3. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}w^m\dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 7.4.3.5 with $f(w) = bw^m$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(C_1^2 x,\ \pm C_1^{2-n} t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w = w(z), \quad z = \left|a(2-n)^2(t+C)^2 - 4x^{2-n}\right|^{1/2},$$

where C is an arbitrary constant, and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{a(2-n)z}\left[a(1-n) + bw^m\right]w'_z = 0. \tag{1}$$

The substitution $u(w) = zw'_z$ leads equation (1) to a first-order separable ordinary differential equation. Integrating yields a solution in implicit form:

$$\int \frac{dw}{anw - \dfrac{2b}{m+1}w^{m+1} + C_1} = \frac{1}{a(2-n)}\ln z + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. There is a self-similar solution of the form

$$w = U(\zeta), \quad \zeta = xt^{\frac{2}{n-2}}.$$

4°. Solution for $n = 2$:

$$w = w(\xi), \quad z = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w = w(\xi)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{\xi\xi} + B(bw^m - a)w'_\xi = 0.$$

Integrating yields

$$\int \frac{dw}{bw^{m+1} - a(m+1)w + C_1} = -\frac{B\xi}{(m+1)(aB^2 - A^2)}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}w^m\dfrac{\partial w}{\partial x} + cw^k, \qquad a > 0.$

This is a special case of equation 7.4.3.6 with $f(w) = bw^m$ and $g(w) = cw^k$.

5. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + cw^m, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. This is a special case of equation 7.4.3.9 with $b = 0$ and $f(w) = cw^m$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + cw^m.$

This is a special case of equation 7.4.3.9 with $b = a\lambda$ and $f(w) = cw^m$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Big(x + \frac{1-m}{\lambda}\ln C_1,\ \pm C_1^{\frac{m-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $m \neq \pm 1$ and $\lambda \neq 0$:

$$w = \left[-\frac{c(m-1)^2}{2k(1+m)}(r + C_1)^2\right]^{\frac{1}{1-m}}, \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t + C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. Functional separable solution for $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}w^m = 0.$$

Integrating yields the general solution in implicit form

$$\int\left[C_1 - \frac{2c}{k(m+1)}w^{m+1}\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

4°. There is an exact solution of the form

$$w(x,t) = |t|^{\frac{2}{1-m}} F(z), \qquad z = x + \frac{2}{\lambda}\ln|t|.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x}\dfrac{\partial w}{\partial x} + cw^m, \qquad a > 0.$

Functional separable solution:

$$w = w(z), \qquad z = \left[4ke^{-\lambda x} - ak\lambda^2(t + C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda}\frac{1}{z}w'_z + \frac{c}{ak\lambda^2}w^m = 0. \tag{1}$$

This equation has the exact solution

$$w(z) = \left\{ \frac{2k\lambda\big[a\lambda(m-3) + 2b(1-m)\big]}{c(1-m)^2 z^2} \right\}^{\frac{1}{m-1}}.$$

For $b = a\lambda$, the general solution of equation (1) is given in implicit form by

$$\int \left[C_1 - \frac{2c}{ak\lambda^2(m+1)} w^{m+1} \right]^{-1/2} dw = \pm z + C_2,$$

where C_1 and C_2 are arbitrary constants.

For $b \neq \frac{1}{2}a\lambda$, the substitution $\xi = z^{\frac{2b-a\lambda}{a\lambda}}$ brings (1) to the Emden–Fowler equation

$$w''_{\xi\xi} + \frac{ac}{k(2b-a\lambda)^2} \xi^{\frac{4(a\lambda-b)}{2b-a\lambda}} w^m = 0. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents more than 20 exact solutions to equation (2) for specific values of the parameter m.

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x} \dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x} w^n \dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 7.4.3.10 with $f(w) = bw^n$.

7.1.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = aw^n \dfrac{\partial^2 w}{\partial x^2} + f(x, w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw \dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 7.1.5.5 with $n = 1$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 C_2^2 w(\pm C_1^{-1} x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w = C_1 xt + C_2 x + C_3 t + C_4,$$

$$w = \frac{3x^2 + C_1 x + C_2}{a(t + C_3)^2} + C_4 (x + C_5)(t + C_3)^3,$$

where $C_1, \ldots, C_5$ are arbitrary constants. The first solution is degenerate and the second one is a generalized separable solution.

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1 C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(U - aC_2^2) U''_{zz} - 2C_1 U'_z = 8C_1^2.$$

4°. Generalized separable solution:

$$w = (x^2 + C_1 x + C_2) f(t) + (C_3 x + C_4) f(t) \int \frac{dt}{f^2(t)},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $f = f(t)$ is determined by the autonomous ordinary differential equation $f''_{tt} = 2af^2$.

5°. For other solutions, see 7.1.5.5 with $n = 1$.

⊙ *Reference for equation* 7.1.5.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 129).

2. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-2} w(\pm C_1^2 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w = (C_1 t + C_2)x + \tfrac{1}{2}bt^2 + C_3 t + C_4,$$
$$w = \frac{3x^2}{at^2} + \Big(C_1 t^3 + \frac{C_2}{t^2}\Big)x + C_3 t^3 + \frac{C_4}{t^2} - \frac{1}{4}bt^2.$$

The first solution is degenerate and the second one is a generalized separable solution (another arbitrary constant can be added, since the equation is invariant under translation in t).

3°. Self-similar solution:

$$w(x,t) = t^2 u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u'_\xi + 4\xi^2 u''_{\xi\xi} = auu''_{\xi\xi} + b.$$

4°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1 C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2 C_2^2)U''_{zz} - 2aC_1 U'_z = 8aC_1^2 - b.$$

5°. The second solution in Item 2° is a special case of the generalized separable solution

$$w(x,t) = f(t)x^2 + g(t)x + h(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + bx^n w^5.$

This is a special case of equation 7.4.4.2 with $f(x) = bx^n$.

4. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x} w^5.$

This is a special case of equation 7.4.4.2 with $f(x) = be^{\lambda x}$.

5. $\dfrac{\partial^2 w}{\partial t^2} = aw^n \dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = (C_2/C_1)^{2/n} w(\pm C_1 x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x,t) = k(x + C_1)^{\frac{2}{n}} (At + C_2)^{-\frac{2}{n}}, \qquad k = \left[\frac{A^2(n+2)}{a(2-n)}\right]^{\frac{1}{n}}, \tag{1}$$

where A, C_1, and C_2 are arbitrary constants.

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x)g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - a\lambda g^{n+1} = 0, \tag{2}$$
$$f''_{xx} - \lambda f^{1-n} = 0. \tag{3}$$

The general solutions of equations (2) and (3) can be written out in implicit form:

$$\int \left(C_1 + \frac{2a\lambda}{n+2} g^{n+2}\right)^{-1/2} dg = C_2 \pm t,$$
$$\int \left(C_3 + \frac{2\lambda}{2-n} f^{2-n}\right)^{-1/2} df = C_4 \pm x,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$g(t) = (At + C)^{-2/n}, \qquad A = \pm\sqrt{\frac{a\lambda n^2}{2(n+2)}}.$$

4°. There are also solutions with the following forms:

$$w(x,t) = (t + A)^{-\frac{2k+2}{n}} F(z), \quad z = (x + B)(t + A)^k;$$
$$w(x,t) = e^{-2\lambda t} U(y), \quad y = (x + A)e^{\lambda n t};$$
$$w(x,t) = (At + B)^{-2/n} V(\xi), \quad \xi = x + k\ln(At + B) + C,$$

where A, B, C, k, and λ are arbitrary constants.

7.1.6. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 7.1.6.4 with $n = 1$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 C_2^2 w(\pm C_1^{-1}x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,t) = \tfrac{1}{2}aA^2t^2 + Bt + Ax + C,$$
$$w(x,t) = \tfrac{1}{12}aA^{-2}(At + B)^4 + Ct + D + x(At + B),$$
$$w(x,t) = \frac{1}{a}\left(\frac{x + A}{t + B}\right)^2,$$
$$w(x,t) = (At + B)\sqrt{Cx + D},$$
$$w(x,t) = \pm\sqrt{A(x + a\lambda t) + B} + a\lambda^2,$$

where A, B, C, D, and λ are arbitrary constants.

3°. Generalized separable solution quadratic in x:

$$w(x,t) = f(t)x^2 + g(t)x + h(t),$$

where the functions $f = f(t)$, $g = g(t)$, and $h = h(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 6af^2,$$
$$g''_{tt} = 6afg,$$
$$h''_{tt} = 2afh + ag^2.$$

A particular solution of this system is given by

$$f = \frac{1}{at^2}, \quad g = \frac{C_1}{t^2} + C_2t^3, \quad h = \frac{aC_1^2}{4t^2} + \frac{C_3}{t} + C_4t^2 + \frac{1}{2}aC_1C_2t^3 + \frac{1}{54}aC_2^2t^8,$$

where $C_1, \ldots, C_4$ are arbitrary constants. Another arbitrary constant can be inserted in this solution through the shift in t.

4°. Solution:

$$w = U(z) + 4aC_1^2t^2 + 4aC_1C_2t, \qquad z = x + aC_1t^2 + aC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the first-order ordinary differential equation

$$(U - aC_2^2)U'_z - 2C_1U = 8C_1^2z + C_3.$$

By appropriate translations in both variables, the equation can be made homogeneous, and, hence, the equation is integrable by quadrature.

5°. Solution in implicit form:

$$x^2 = at^2w + C\Big(\frac{t}{aw}\Big)^{4/5}, \qquad t^2 = \frac{x^2}{aw} + C\Big(\frac{x}{a^2w^2}\Big)^{8/7},$$

where C is an arbitrary constant.

6°. For other exact solutions and linearization of this equation, see equation 7.4.4.6 with $f(w) = aw$.

⊙ *References for equation* 7.1.6.1: S. Tomotika and K. Tamada (1950), P. J. Olver and P. Rosenau (1986), M. F. Foursov and E. M. Vorob'ev (1996), G. I. Burde (2001), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002, p. 129).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 7.1.6.4 with $n = -1$.

1°. Multiplicative separable solutions:

$$w(x,t) = (At + B)e^{Cx},$$
$$w(x,t) = (at^2 + At + B)(x + C)^{-2},$$
$$w(x,t) = (-aA^2t^2 + Bt + C)\cosh^{-2}(Ax + D),$$
$$w(x,t) = (aA^2t^2 + Bt + C)\sinh^{-2}(Ax + D),$$
$$w(x,t) = (aA^2t^2 + Bt + C)\cos^{-2}(Ax + D),$$

where A, B, C, and D are arbitrary constants.

2°. Traveling-wave solution in implicit form:

$$\lambda^2 w = ak^2 \ln|w| + C_1(kx + \lambda t) + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. For other exact solutions and linearization of this equation, see equation 7.4.4.6 with $f(w) = a/w$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 7.1.6.4 with $n = -1/2$.

1°. Solutions:

$$w(x,t) = \big(\tfrac{1}{2}A^2ax^2 + Bx + Aat + C\big)^2,$$
$$w(x,t) = \big[\tfrac{1}{12}A^{-2}a^{-1}(Ax + B)^4 + Cx + D + t(Ax + B)\big]^2,$$
$$w(x,t) = a^2\left(\frac{t + A}{x + B}\right)^4,$$
$$w(x,t) = (Ax + B)^2(Ct + D),$$
$$w(x,t) = \big[\pm\sqrt{A(t + \lambda x) + B} + a\lambda^2\big]^2,$$

where A, B, C, D, and λ are arbitrary constants.

2°. The substitution $w = u^2$ leads to an equation of the form 7.1.6.1:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial t}\right).$$

3°. For other exact solutions and linearization of this equation, see equation 7.4.4.6 with $f(w) = aw^{-1/2}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right)$.

This equation is encountered in wave and gas dynamics. This is a special case of equation 7.4.4.6 with $f(w) = a^2 w^n$; consequently, the equation can be linearized.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = (C_2/C_1)^{2/n} w(\pm C_1 x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Degenerate solution:

$$w = (At + B)(Cx + D)^{\frac{1}{n+1}},$$

where A, B, C, and D are arbitrary constants.

3°. Multiplicative separable solution:

$$w = f(x)g(t),$$

where $f = f(x)$ and $g = g(t)$ are defined implicitly by

$$\int\left(C_1 + \frac{2\lambda}{n+2}f^{n+2}\right)^{-1/2} f^n\,df = C_2 \pm x, \tag{1}$$

$$\int\left(C_3 + \frac{2a^2\lambda}{n+2}g^{n+2}\right)^{-1/2} dg = C_4 \pm t, \tag{2}$$

and $C_1, \ldots, C_4$ and λ are arbitrary constants.

The functions $f = f(x)$ and $g = g(t)$ defined by (1) and (2) can be represented in explicit form if $C_1 = 0$ and $C_3 = 0$. To the special case $C_1 = C_3 = 0$ there corresponds

$$w(x, t) = \left(\frac{\pm bx + c}{abt + s}\right)^{2/n}, \tag{3}$$

where b, c, and s are arbitrary constants.

4°. Traveling-wave solution:

$$w = w(z), \quad z = x \pm \lambda t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda^2 w - \frac{a^2}{n+1}w^{n+1} = Az + B. \tag{4}$$

If $n = -\frac{1}{2}$, 1, 2, or 3, equation (4) can be solved for w to give an explicit expression of $w = w(z)$.

5°. Self-similar solution:

$$w = w(\xi), \quad \xi = \frac{x + A}{t + B},$$

where the function $w(\xi)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant):

$$\left(\xi^2 - a^2 w^n\right) w'_\xi = C. \tag{5}$$

To the special case $C = 0$ there corresponds the solution $w = (\xi/a)^{2/n}$; see formula (3). If $C \neq 0$, by treating w in (5) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C\xi'_w = \xi^2 - a^2 w^n. \tag{6}$$

The general solution of equation (6) is expressed in terms of Bessel functions; see Kamke (1977) and Polyanin and Zaitsev (1995, 2003).

6°. There are more complicated self-similar solutions of the form

$$w = (t + \beta)^{2k} F(z), \quad z = \frac{x + \alpha}{(t + \beta)^{nk+1}},$$

where α, β, and k are arbitrary constants, and the function $F = F(z)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$2k(2k-1)F + (nk+1)(nk-4k+2)zF'_z + (nk+1)^2 z^2 F''_{zz} = a^2 (F^n F'_z)'_z.$$

Its order can be reduced.

7°. Generalized self-similar solution (μ is an arbitrary constant):

$$w = e^{-2\mu t}\varphi(y), \quad y = xe^{\mu n t},$$

where the function $\varphi = \varphi(y)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$4\mu^2\varphi + \mu^2 n(n-4)y\varphi'_y + (\mu n)^2 y^2 \varphi''_{yy} = a^2(\varphi^n \varphi'_y)'_y.$$

Its order can be reduced.

8°. Solution (A, b, and c are arbitrary constants):

$$w = (\pm t + A)^{-2/n}\psi(u), \quad u = x + b\ln(\pm t + A) + c,$$

where the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$\frac{2(n+2)}{n^2}\psi - \frac{b(n+4)}{n}\psi'_u + b^2\psi''_{uu} = a^2(\psi^n \psi'_u)'_u. \tag{7}$$

Note two special cases where the equation obtained is integrable by quadrature. For $n = -2$, equation (7) admits a first integral that represents a separable equation. For $n = -4$, with the change of variable $G(\psi) = (\psi'_u)^2$, equation (7) can be reduced to a first-order linear equation.

In the general case, the change of variable $H(\psi) = \psi'_u$ brings (7) to a first-order equation.

9°. For $n \neq -1$, the transformation

$$\tau = x, \quad \zeta = t, \quad V = w^{n+1}$$

brings the original equation to an equation of the similar form

$$\frac{\partial^2 V}{\partial \tau^2} = a^{-2} \frac{\partial}{\partial \zeta} \left(V^{-\frac{n}{n+1}} \frac{\partial V}{\partial \zeta} \right).$$

For $n = -1$, the transformation

$$\tau = x, \quad \zeta = t, \quad V = \ln w$$

brings the original equation to an equation of the form 7.2.4.3:

$$\frac{\partial^2 V}{\partial \tau^2} = a^{-2} \frac{\partial}{\partial \zeta} \left(e^V \frac{\partial V}{\partial \zeta} \right).$$

10°. For other exact solutions of this equation, see equation 7.4.4.6 with $f(w) = aw^n$.

⊙ *References for equation* 7.1.6.4: W. F. Ames, R. J. Lohner, and E. Adams (1981), N. H. Ibragimov (1994, pp. 211–212), V. F. Zaitsev and A. D. Polyanin (1996, pp. 131–132; 2001).

7.1.7. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a \dfrac{\partial}{\partial x} \left(w^n \dfrac{\partial w}{\partial x} \right) + bw^k$

1. $\dfrac{\partial^2 w}{\partial t^2} = a \dfrac{\partial}{\partial x} \left(w \dfrac{\partial w}{\partial x} \right) + b.$

This is a special case of equation 7.1.7.6 with $n = 1$, $k = 0$ and is a special case of equation 12.2.1.3 with $F(u, v) = au^2 + b$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^2 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$ak^2 w^2 - 2\lambda^2 w = -b(kx + \lambda t)^2 + C_1(kx + \lambda t) + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. Self-similar solution:

$$w(x, t) = t^2 u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u'_\xi + 4\xi^2 u''_{\xi\xi} = a(uu'_\xi)'_\xi + b.$$

4°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1 C_2 t, \quad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the first-order ordinary differential equation

$$(aU - a^2 C_2^2) U'_z - 2aC_1 U = (8aC_1^2 - b)z + C_3.$$

By appropriate translations in both variables, the equation can be made homogeneous, and, hence, the equation is integrable by quadrature.

5°. Generalized separable solution quadratic in x:

$$w(x,t) = f(t)x^2 + g(t)x + h(t),$$

where the functions $f = f(t)$, $g = g(t)$, and $h = h(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 6af^2,$$
$$g''_{tt} = 6afg,$$
$$h''_{tt} = 2afh + ag^2 + b.$$

A particular solution of this system is given by

$$f = \frac{1}{at^2}, \quad g = \frac{C_1}{t^2} + C_2 t^3, \quad h = \frac{aC_1^2}{4t^2} + \frac{C_3}{t} + C_4 t^2 + \frac{1}{2}aC_1C_2t^3 + \frac{1}{54}aC_2^2t^8 + \frac{1}{9}bt^2\big(3\ln|t| - 1\big),$$

where $C_1, \ldots, C_4$ are arbitrary constants. Another arbitrary constant, C_5, can be inserted in the solution, by substituting $t + C_5$ for t, since the system is translation invariant in t.

⊙ *References for equation* 7.1.7.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 205–206), V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 252).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw.$

This is a special case of equation 7.1.7.6 with $n = k = 1$.

1°. Generalized separable solution:

$$w = \varphi(t)(x + C_1)^2 + \psi(t),$$

where C_1 is an arbitrary constant and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'' = 6a\varphi^2 + b\varphi,$$
$$\psi'' = 2a\varphi\psi + b\psi.$$

A particular solution of this system is

$$\varphi = -\frac{b}{6a}, \quad \psi = \begin{cases} C_2\cosh\big(\sqrt{\frac{2}{3}b}\,t\big) + C_3\sinh\big(\sqrt{\frac{2}{3}b}\,t\big) & \text{if } b > 0, \\ C_2\cos\big(\sqrt{\frac{2}{3}|b|}\,t\big) + C_3\sin\big(\sqrt{\frac{2}{3}|b|}\,t\big) & \text{if } b < 0. \end{cases}$$

2°. Generalized separable solution:

$$w = \varphi(t)(x + C_1)^2 + \psi(t)\sqrt{x + C_1},$$

where C_1 is an arbitrary constant and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'' = 6a\varphi^2 + b\varphi,$$
$$\psi'' = \tfrac{15}{4}a\varphi\psi + b\psi.$$

A particular solution of this system is

$$\varphi = -\frac{b}{6a}, \quad \psi = \begin{cases} C_2\cosh\big(\sqrt{\frac{3}{8}b}\,t\big) + C_3\sinh\big(\sqrt{\frac{3}{8}b}\,t\big) & \text{if } b > 0, \\ C_2\cos\big(\sqrt{\frac{3}{8}|b|}\,t\big) + C_3\sin\big(\sqrt{\frac{3}{8}|b|}\,t\big) & \text{if } b < 0. \end{cases}$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b\sqrt{w}.$

This is a special case of equation 7.1.7.6 with $n = -\frac{1}{2}$, $k = \frac{1}{2}$ and is a special case of equation 7.1.8.4 with $n = -\frac{1}{2}$ and $c = 0$.

1°. Generalized separable solution:

$$w = [f(x)t^2 + g(x)t + h(x)]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} 2af''_{xx} + bf - 12f^2 &= 0,\\ 2ag''_{xx} + bg - 12fg &= 0,\\ 2ah''_{xx} + bh - 4fh - 2g^2 &= 0. \end{aligned}$$

The first equation has two constant particular solutions, $f = 0$ and $f = \frac{1}{12}c$. In these cases, the other two equations are linear and have constant coefficients and, hence, are easy to integrate—their solutions are expressed in terms of exponentials or sines and cosines.

2°. Multiplicative separable solution:

$$w = \varphi(t)\psi^2(x),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = C_1\sqrt{\varphi}, \qquad 2a\psi''_{xx} + b\psi = C_1\psi^2,$$

with C_1 being an arbitrary constant. The general solutions to the equations for the functions φ and ψ can be written out in implicit form. A particular solution is $\varphi = \frac{1}{144}C_1^2(t + C_2)^4$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + bw.$

This is a special case of equation 7.1.7.6 with $k = 1$.

1°. Multiplicative separable solutions:

$$\begin{aligned} w &= \left(A_1e^{\lambda t} + A_2e^{-\lambda t}\right)\left(B_1x + B_2\right)^{\frac{1}{n+1}} && \text{if} \quad b = \lambda^2 > 0,\\ w &= \left[A_1\cos(\lambda t) + A_2\sin(\lambda t)\right]\left(B_1x + B_2\right)^{\frac{1}{n+1}} && \text{if} \quad b = -\lambda^2 < 0, \end{aligned}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

2°. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the autonomous ordinary differential equations (K is an arbitrary constant)

$$\begin{aligned} a(\varphi^n\varphi'_x)'_x + K\varphi &= 0,\\ \psi''_{tt} - b\psi + K\psi^{n+1} &= 0. \end{aligned}$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + bw^{n+1}.$

This is a special case of equation 7.1.7.6 with $k = n + 1$.

1°. Multiplicative separable solutions:

$$w = \big(A_1 t + A_2\big)\big[B_1\cos(kx) + B_2\sin(kx)\big]^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = k^2 > 0,$$
$$w = \big(A_1 t + A_2\big)\big(B_1 e^{kx} + B_2 e^{-kx}\big)^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = -k^2 < 0,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

2°. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the autonomous ordinary differential equations (K is an arbitrary constant)

$$a(\varphi^n\varphi'_x)'_x + b\varphi^{n+1} + K\varphi = 0,$$
$$\psi''_{tt} + K\psi^{n+1} = 0.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + bw^k.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-n-1}x + C_2, \pm C_1^{k-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x, t) = w(z), \quad z = \beta x + \lambda t,$$

where β and λ are arbitrary constants and the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$[(a\beta^2 w^n - \lambda^2)w'_z]'_z + bw^k = 0,$$

whose general solution can be represented in implicit form.

3°. Self-similar solution with $k \neq 1$:

$$w(x, t) = t^{\frac{2}{1-k}}U(\xi), \quad \xi = xt^{\frac{k-n-1}{1-k}},$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$\frac{2(1+k)}{(1-k)^2}U + \frac{(k-n-1)(2k-n+2)}{(1-k)^2}\xi U'_\xi + \frac{(k-n-1)^2}{(1-k)^2}\xi^2 U''_{\xi\xi} = a(U^n U'_\xi)'_\xi + bU^k.$$

4°. For $k = 1$ and $k = n + 1$ see also equations 7.1.7.4 and 7.1.7.5.

7.1.8. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^n \frac{\partial w}{\partial x}\right) + b_1 w^{k_1} + b_2 w^{k_2} + b_3 w^{k_3}$$

▷ *Equations of this form admit traveling-wave solutions* $w = w(\beta x + \lambda t)$.

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b + c\sqrt{w}.$

Generalized separable solution:

$$w = [f(x)t^2 + g(x)t + h(x)]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} 2af''_{xx} + cf - 12f^2 &= 0,\\ 2ag''_{xx} + cg - 12fg &= 0,\\ 2ah''_{xx} + ch + b - 4fh - 2g^2 &= 0. \end{aligned}$$

The first equation has two constant particular solutions, $f = 0$ and $f = \frac{1}{12}c$. In these cases, the other two equations are linear and have constant coefficients and, hence, are easy to integrate—their solutions are expressed in terms of exponentials or sines and cosines.

Remark. The constants a, b, and c in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b + cw + d\sqrt{w}.$

1°. Generalized separable solution for $c > 0$:

$$w = \left[f(x)\exp\left(\tfrac{1}{2}\sqrt{c}\,t\right) + g(x)\exp\left(-\tfrac{1}{2}\sqrt{c}\,t\right) + h(x)\right]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} 2af''_{xx} + \tfrac{3}{2}cfh + df &= 0,\\ 2ag''_{xx} + \tfrac{3}{2}cgh + dg &= 0,\\ 2ah''_{xx} + ch^2 + dh + b + 2cfg &= 0. \end{aligned}$$

2°. Generalized separable solution for $c < 0$:

$$w = \left[f(x)\cos\left(\tfrac{1}{2}\sqrt{|c|}\,t\right) + g(x)\sin\left(\tfrac{1}{2}\sqrt{|c|}\,t\right) + h(x)\right]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} 2af''_{xx} + \tfrac{3}{2}cfh + df &= 0,\\ 2ag''_{xx} + \tfrac{3}{2}cgh + dg &= 0,\\ 2ah''_{xx} + ch^2 + dh + b + \tfrac{1}{2}c(\varphi^2 + \psi^2) &= 0. \end{aligned}$$

Remark. The constants a, b, and d in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $d = d(x)$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^{-2/3}\dfrac{\partial w}{\partial x}\right) + b + cw^{1/3}.$

Generalized separable solution:

$$w = [f(x)t + g(x)]^3,$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the system of ordinary differential equations

$$3af''_{xx} + cf - 6f^3 = 0,$$
$$3ag''_{xx} + cg + b - 6f^2 g = 0.$$

Remark. The constants a, b, and c in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 282).

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + bw^{n+1} + cw.$

1°. Multiplicative separable solutions for $c = \lambda^2 > 0$:

$$w = \left(A_1 e^{\lambda t} + A_2 e^{-\lambda t}\right)\left[B_1\cos(kx) + B_2\sin(kx)\right]^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = k^2 > 0,$$
$$w = \left(A_1 e^{\lambda t} + A_2 e^{-\lambda t}\right)\left(B_1 e^{kx} + B_2 e^{-kx}\right)^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = -k^2 < 0,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

2°. Multiplicative separable solutions for $c = -\lambda^2 < 0$:

$$w = \left[A_1\cos(\lambda t) + A_2\sin(\lambda t)\right]\left[B_1\cos(kx) + B_2\sin(kx)\right]^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = k^2 > 0,$$
$$w = \left[A_1\cos(\lambda t) + A_2\sin(\lambda t)\right]\left(B_1 e^{kx} + B_2 e^{-kx}\right)^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = -k^2 < 0,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

3°. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the autonomous ordinary differential equations (K is an arbitrary constant)

$$a(\varphi^n\varphi'_x)'_x + b\varphi^{n+1} + K\varphi = 0,$$
$$\psi''_{tt} - c\psi + K\psi^{n+1} = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) - bw^{-2n-1} + c.$

1°. Functional separable solutions with $n \neq -1$:

$$\frac{a}{n+1}w^{n+1} = \pm a\sqrt{\frac{b}{n}}\,t - \frac{c}{2}x^2 + Ax + B,$$

where A and B are arbitrary constants and $bn > 0$.

2°. Functional separable solutions with $n = -1$ and $b < 0$:

$$a\ln|w| = \pm a\sqrt{-b}\,t - \frac{c}{2}x^2 + Ax + B,$$

7.1.9. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + d.$

1°. Generalized separable solution:

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi_0' = 2a\varphi_0\varphi_2 + (a+b)\varphi_1^2 + c\varphi_0 + d,$$
$$\varphi_1' = 2(3a+2b)\varphi_1\varphi_2 + c\varphi_1,$$
$$\varphi_2' = 2(3a+2b)\varphi_2^2 + c\varphi_2.$$

This system is easy to integrate in reverse order, starting from the last equation.

2°. For $d = 0$, there is the multiplicative separable solution:

$$w = \varphi(t)\psi(x),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations

$$\varphi'' - c\varphi + A\varphi^2 = 0,$$
$$a\psi\psi'' + (a+b)(\psi')^2 + A\psi = 0,$$

with A being an arbitrary constant.

3°. There is the traveling-wave solution, $w = w(kx + \lambda t)$.

4°. For $c = 0$, see equation 7.1.9.2 with $n = 2$, where the constants b and c must be renamed.

⊙ *Reference for equation* 7.1.9.1: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 256).

2. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^n + c.$

This is a special case of equation 12.2.1.3 with $F(u, v) = bu^n + c$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^2 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x, t) = t^2 u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u_\xi' + 4\xi^2 u_{\xi\xi}'' = a(uu_\xi')_\xi' + b(u_\xi')^n + c.$$

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \quad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U_{zz}'' + b(U_z')^n - 2aC_1U_z' = 8aC_1^2 - c.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = w^2\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}w\left(\dfrac{\partial w}{\partial x}\right)^2 + aw + b.$

1°. Generalized separable solution:

$$w = \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi_2'' &= \varphi_2\left(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\right) + a\varphi_2,\\
\varphi_1'' &= \varphi_1\left(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\right) + a\varphi_1,\\
\varphi_0'' &= \varphi_0\left(2\varphi_0\varphi_2 - \tfrac{1}{2}\varphi_1^2\right) + a\varphi_0 + b.
\end{aligned}$$

Remark. The parameters a and b in the equation concerned can be replaced by arbitrary functions, $a = a(t)$ and $b = b(t)$.

2°. There is the traveling-wave solution, $w = w(kx + \lambda t)$.

⊙ *Reference for equation* 7.1.9.3: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 22).

4. $\dfrac{\partial^2 w}{\partial t^2} = aw^2\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}aw\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{2}aw^3 + bw + c.$

For $a = 2$, $b = c = 0$, this equation describes short-wave excitations of a nonlinear model, where each atom in a lattice interacts with its neighbors by anharmonic forces.

1°. Generalized separable solution:

$$w = \varphi_1(t) + \varphi_2(t)\cos x + \varphi_3(t)\sin x,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1'' &= \tfrac{1}{2}a\varphi_1(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + b\varphi_1 + c,\\
\varphi_2'' &= \tfrac{1}{2}a\varphi_2(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + b\varphi_2,\\
\varphi_3'' &= \tfrac{1}{2}a\varphi_3(\varphi_1^2 - \varphi_2^2 - \varphi_3^2) + b\varphi_3.
\end{aligned}$$

Remark. The parameters a, b, and c in the equation concerned can be replaced by arbitrary functions, $a = a(t)$, $b = b(t)$, and $c = c(t)$.

2°. There is the traveling-wave solution, $w = w(kx + \lambda t)$.

3°. For $c = 0$, there is the multiplicative separable solution

$$w = \varphi(t)\psi(x),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
\varphi_{tt}'' - b\varphi + aC\varphi^3 &= 0,\\
\psi^2\psi_{xx}'' - \tfrac{1}{2}\psi(\psi_x')^2 + \tfrac{1}{2}\psi^3 + C\psi &= 0,
\end{aligned}$$

with C being an arbitrary constant.

⊙ *References for equation* 7.1.9.4: Y. Kivshar (1994), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 251–252).

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(a_0 + a_1 w + a_2 w^2)\dfrac{\partial w}{\partial x}\right] + b_0 + b_1 w.$

Generalized separable solution (degenerate solution):

$$w = \varphi(t)x + \psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = 2a_2\varphi^3 + b_1\varphi,$$
$$\psi''_{tt} = 2a_2\varphi^2\psi + b_1\psi + a_1\varphi^2 + b_0.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 373).

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\left(aw + \dfrac{b}{\sqrt{w}}\right)\dfrac{\partial w}{\partial x}\right] + cw.$

Multiplicative separable solution:

$$w = \varphi(t)x^2,$$

where the function $\varphi = \varphi(t)$ is determined by the second-order autonomous ordinary differential equation

$$\varphi''_{tt} = 6a\varphi^2 + c\varphi.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 373).

7. $\dfrac{\partial^2 w}{\partial t^2} = ax^n w^m \dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\big(C_2^2 x,\ \pm C_1^m C_2^{2-n} t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = kx^{\frac{2-n}{m}}(C_1 t + C_2)^{-\frac{2}{m}}, \qquad k = \left[\frac{2C_1^2(m+2)}{a(2-n)(2-n-m)}\right]^{\frac{1}{m}}. \tag{1}$$

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x)g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{2}$$
$$f''_{xx} - (\lambda/a)x^{-n} f^{1-m} = 0. \tag{3}$$

The general solution of equation (2) can be written out in implicit form as

$$\int\left(C_1 + \frac{2\lambda}{m+2}g^{m+2}\right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$g(t) = (At + C)^{-2/m}, \qquad A = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to the Emden–Fowler equation (3) for specific values of the parameter m.

4°. There is a self-similar solution of the form

$$w = t^{\frac{(n-2)k-2}{m}} F(y), \quad y = xt^k,$$

where k is an arbitrary constant.

5°. The transformation

$$u(z,t) = \frac{1}{x} w(x,t), \quad z = \frac{1}{z}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = az^{4-n-m} u^m \frac{\partial^2 u}{\partial z^2}. \tag{4}$$

In the special case $n = 4 - m$, equation (4) is greatly simplified to become

$$\frac{\partial^2 u}{\partial t^2} = au^m \frac{\partial^2 u}{\partial z^2}$$

and admits a traveling-wave solution $u = u(kz + \mu t)$; see also equation 7.1.5.5.

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x} w^m \dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\Big(x + \frac{1}{\lambda} \ln \frac{C_1^m}{C_2^2},\ C_2 t + C_3\Big),$$

where C_1, C_2, C_3, and λ are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = U(z), \quad z = x + \frac{2}{\lambda} \ln|t|,$$

where the function $U(z)$ is determined by the ordinary differential equation

$$(a\lambda^2 e^{\lambda z} U^m - 4)U''_{zz} + 2\lambda U'_z = 0.$$

3°. For other solutions, see equation 7.4.5.1 with $f(x) = ae^{\lambda x}$.

9. $\dfrac{\partial^2 w}{\partial t^2} = a \dfrac{\partial}{\partial x}\left(x^n w^m \dfrac{\partial w}{\partial x}\right), \qquad a > 0.$

This is a special case of equation 7.4.5.2 with $f(x) = ax^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(C_2^2 x,\ \pm C_1^m C_2^{2-n} t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = kx^{\frac{2-n}{m}}(C_1 t + C_2)^{-\frac{2}{m}}, \qquad k = \left[\frac{2C_1^2(m+2)}{a(2-n)(m-n+2)}\right]^{\frac{1}{m}}. \tag{1}$$

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x)g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{2}$$

$$a(x^n f^{m+1} f'_x)'_x - \lambda f = 0. \tag{3}$$

The general solution of equation (2) can be written out in implicit form as

$$\int \left(C_1 + \frac{2\lambda}{m+2} f^{m+2}\right)^{-1/2} df = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$f(t) = (At + C)^{-2/m}, \qquad A = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

For $n \neq 1$ and $m \neq -1$, the transformation

$$z = x^{1-n}, \quad \varphi = u^{m+1}$$

brings (3) to the Emden–Fowler equation

$$\varphi''_{zz} = \frac{\lambda(m+1)}{a(1-n)^2} z^{\frac{n}{1-n}} \varphi^{\frac{1}{m+1}}. \tag{4}$$

The book by Polyanin and Zaitsev (2003) presents more than 20 exact solutions to equation (4) for specific values of the parameter m.

4°. There is a self-similar solution of the form

$$w = (t+b)^{\frac{(n-2)k-2}{m}} F(y), \quad y = xt^k,$$

where b and k are arbitrary constants.

5°. Suppose $m \neq -1$ and $2m - 2n - nm + 3 \neq 0$. The transformation

$$w(x,t) = x^{\frac{1-n}{m+1}} u(\xi, t), \quad \xi = x^{\frac{2m-2n-nm+3}{m+1}}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = A\frac{\partial}{\partial \xi}\left(\xi^{\frac{3m-3n-2nm+4}{2m-2n-nm+3}} u^m \frac{\partial u}{\partial \xi}\right), \tag{5}$$

where $A = a\left(\dfrac{2m-2n-nm+3}{m+1}\right)^2$.

In the special case $n = \dfrac{3m+4}{2m+3}$, equation (5) is greatly simplified and coincides, up to notation, with equation 7.1.6.4:

$$\frac{\partial^2 u}{\partial t^2} = A\frac{\partial}{\partial \xi}\left(u^m \frac{\partial u}{\partial \xi}\right).$$

10. $\dfrac{\partial^2 w}{\partial t^2} = k(ax^2+bx+c)^m w^{4-2m}\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 7.4.5.4 with $f(u) = ku^{-2m}$.

1°. The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

leads to an equation of the form 7.4.4.8:

$$\frac{\partial^2 u}{\partial t^2} = ku^{4-2m}\frac{\partial^2 u}{\partial z^2} + k(ac-\tfrac{1}{4}b^2)u^{5-2m},$$

which has a traveling-wave solution $u = u(z+\lambda t)$ and a multiplicative separable solution $u = f(t)g(z)$.

2°. By the transformation

$$w(x,t) = \big[v(\xi,t)\big]^{\frac{1}{2m+3}}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m} \tag{1}$$

the original equation can be reduced to the divergence form

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial \xi}\left[F(\xi)v^{\frac{4-2m}{2m-3}}\frac{\partial v}{\partial \xi}\right], \tag{2}$$

where the function $F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{k}{(ax^2+bx+c)^m}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}. \tag{3}$$

Note some special cases of equation (2) where $F = F(\xi)$ of (3) can be represented in explicit form:

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{\cos^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1,\ a=1,\ b=0,\ c=1;$$

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{\cosh^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1,\ a=-1,\ b=0,\ c=1;$$

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{v^{-3/2}}{\cos\xi}\frac{\partial v}{\partial \xi}\right), \qquad m=\tfrac{1}{2},\ a=-1,\ b=0,\ c=1.$$

11. $w\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1\exp(C_2e^{-ax})w(x+C_3, t+C_4),$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w = C\exp\bigl(kx \pm \sqrt{ak+b}\,t\bigr),$$

where C and k are arbitrary constants ($ak + b \ge 0$).

3°. Multiplicative separable solution:

$$w = \varphi(t)\psi(x),$$

where

$$\varphi(t) = \begin{cases} C_1\exp(\sqrt{K}\,t) + C_2\exp(-\sqrt{K}\,t) & \text{if } K > 0, \\ C_1\cos(\sqrt{|K|}\,t) + C_2\cos(\sqrt{|K|}\,t) & \text{if } K < 0, \\ C_1 t + C_2 & \text{if } K = 0, \end{cases}$$

$$\psi(x) = C_3\exp\left(C_4 e^{-ax} + \frac{K-b}{a}x\right),$$

where C_1, C_2, C_3, C_4, and K are arbitrary constants.

4°. Solution:

$$w = e^{kx+\lambda t}u(z), \qquad z = x + \beta t,$$

where k, β, and λ are arbitrary constants and the function $u = u(z)$ is determined by the second-order autonomous ordinary differential equation

$$(1-\beta^2)uu''_{zz} - (u'_z)^2 + (a - 2\beta\lambda)uu'_z + (b + ak - \lambda^2)u^2 = 0.$$

To $k = \lambda = 0$ there corresponds a traveling-wave solution.

5°. The substitution

$$w = e^{\lambda x}u$$

leads to an equation of the similar form

$$u\frac{\partial^2 u}{\partial t^2} = u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 + au\frac{\partial u}{\partial x} + (b + a\lambda)u^2.$$

Consequently, if $w = f(x, t; a, b)$ is a solution of the original equation, then the function

$$w = e^{\lambda x}f(x, t; a, b + a\lambda),$$

where λ is an arbitrary constant, is also a solution of the equation.

6°. The substitution

$$w = A\exp\left(-\frac{b}{a}x + v\right), \qquad v = v(x, t)$$

leads to the equation

$$\frac{\partial^2 v}{\partial t^2} + \left(\frac{\partial v}{\partial t}\right)^2 = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x}.$$

This equation has a generalized separable solution of the form

$$v = f(x)t^2 + g(x)t + h(x),$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + af'_x - 4f^2 = 0,$$
$$g''_{xx} + ag'_x - 4fg = 0,$$
$$h''_{xx} + ah'_x - 2f - g^2 = 0.$$

12. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + cw^2 + pw + q.$

Let λ be a root of the quadratic equation

$$c\lambda^2 + p\lambda + q = 0$$

and

$$D = b^2 - 4a\left(2c + \frac{p}{\lambda}\right)$$

is the discriminant of the quadratic equation

$$a\mu^2 + b\mu + 2c + \frac{p}{\lambda} = 0.$$

Generalized separable solutions:

$$w = \lambda + f(x)g(t),$$

where

$$f(x) = \begin{cases} \exp\Big(-\dfrac{b}{2a}\,x\Big)\Big[C_1\exp\Big(\dfrac{\sqrt{D}}{2a}\,x\Big) + C_2\exp\Big(\dfrac{-\sqrt{D}}{2a}\,x\Big)\Big] & \text{if } D > 0; \\ \exp\Big(-\dfrac{b}{2a}\,x\Big)(C_1 x + C_2) & \text{if } D = 0; \\ \exp\Big(-\dfrac{b}{2a}\,x\Big)\Big[C_1\cos\Big(\dfrac{\sqrt{|D|}}{2a}\,x\Big) + C_2\sin\Big(\dfrac{\sqrt{|D|}}{2a}\,x\Big)\Big] & \text{if } D < 0, \end{cases}$$

C_1 and C_2 are arbitrary constants, and the function $g = g(t)$ is determined by the ordinary differential equation $g_t' g_{tt}'' = -\big(2c + \frac{p}{\lambda}\big)\,g^2$. A particular solution of the last equation is $g = \exp[-(2c + p/\lambda)^{1/3}t]$ and the general solution is

$$\int [C_3 - (2c + p/\lambda)g^3]^{-1/3}dg = t + C_4.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 358).

13. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial t^2} = -2aw\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw^2 + c.$

Additive separable solution:

$$w = f(x) + C_1 g(t),$$

where

$$f(x) = \begin{cases} \sqrt{\dfrac{c}{b}}\sin\left(\sqrt{-\dfrac{b}{a}}\,x + C_2\right) & \text{if } ab < 0 \text{ and } bc > 0, \\ C_2\exp\left(\sqrt{\dfrac{b}{a}}\,x\right) + \dfrac{c}{4bC_2}\exp\left(-\sqrt{\dfrac{b}{a}}\,x\right) & \text{if } ab > 0, \end{cases}$$

C_1 and C_2 are arbitrary constants, and the function $g = g(t)$ is determined by the ordinary differential equation $g_t' g_{tt}'' = bg^2$. A particular solution of the last equation is $g = \exp(b^{1/3}t)$ and the general solution is

$$\int (bg^3 + C_3)^{-1/3}dg = t + C_4.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 358–359).

14. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 \exp(C_2 e^{-ax}) w(x + C_3, t + C_4),$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w = C_1 \exp\big[kx + (ak + b)^{1/3} t\big] \quad \text{if} \quad b \text{ is any number},$$
$$w = C_1 \exp\big(-ax - a^{2/3} t\big) + C_2 \quad \text{if} \quad b = 0,$$

where C_1, C_2, and k are arbitrary constants.

3°. Generalized separable solution:

$$w = \exp\left(-\frac{a^2 + b}{a} x + C_1 e^{-ax}\right)\left(C_2 e^{ax} + C_3 \varphi(t)\right),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t \varphi''_{tt} = -a^2 \varphi^2$. A particular solution of the last equation is $\varphi = \exp(-a^{2/3} t)$ and the general solution is

$$\int (C_3 - a^2 \varphi^3)^{-1/3} d\varphi = t + C_4.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 357).

4°. Multiplicative separable solution:

$$w = \varphi(t)\psi(x),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(x)$ are determined by the second-order autonomous ordinary differential equations

$$\varphi'_t \varphi''_{tt} - K\varphi^2 = 0, \tag{1}$$
$$\psi\psi''_{xx} - (\psi'_x)^2 + a\psi\psi'_x + (b - K)\psi^2 = 0, \tag{2}$$

with K being an arbitrary constant.

For $K = 0$, we have the degenerate solution with $\varphi = C_1 t + C_2$. For $K \neq 0$, a particular solution of equation (1) is $\varphi = e^{K^{1/3} t}$ and the general solution is

$$\int (K\varphi^3 + C_1)^{-1/3} d\varphi = t + C_2,$$

where C_1 and C_2 are arbitrary constants.

The general solution of equation (2) is

$$\psi = C_3 \exp\left(C_4 e^{-ax} + \frac{K - b}{a} x\right),$$

where C_3 and C_4 are arbitrary constants.

5°. Solution:

$$w = e^{kx+\lambda t}u(z), \qquad z = x + \beta t,$$

where k, β, and λ are arbitrary constants and the function $u = u(z)$ is determined by the second-order autonomous ordinary differential equation

$$[\beta^3 u'_z + (\beta^2\lambda - 1)u]u''_{zz} = -(2\beta^2\lambda + 1)(u'_z)^2 + (a - 3\beta\lambda^2)uu'_z + (b + ak - \lambda^3)u^2.$$

To $k = \lambda = 0$ there corresponds a traveling-wave solution.

6°. The substitution

$$w = e^{\lambda x}u$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} = u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 + au\frac{\partial u}{\partial x} + (b + a\lambda)u^2.$$

Consequently, if $w = f(x, t; a, b)$ is a solution of the original equation, then the function

$$w = e^{\lambda x}f(x, t; a, b + a\lambda),$$

where λ is an arbitrary constant, is also a solution of the equation.

By applying the above property with $\lambda = -b/a$ to the second equation from Item 2°, we get the solution

$$w = C_1 \exp\left[(\lambda - a)x - a^{2/3}t\right] + C_2 \exp(\lambda x), \qquad \lambda = -b/a,$$

where C_1 and C_2 are arbitrary constants.

7°. The substitution

$$w = A\exp\left(-\frac{b}{a}x + v\right), \qquad v = v(x, t)$$

leads to the equation

$$\frac{\partial v}{\partial t}\left[\frac{\partial^2 v}{\partial t^2} + \left(\frac{\partial v}{\partial t}\right)^2\right] = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x}.$$

This equation has the generalized separable solution

$$v = (C_1 t + C_2)e^{-ax} + \frac{C_1^3}{6a^2}e^{-3ax}.$$

7.2. Equations with Exponential Nonlinearities

7.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\beta w} + ce^{\gamma w}$

1. $\dfrac{\partial^2 w}{\partial t^2} = a^2\dfrac{\partial^2 w}{\partial x^2} + be^{\beta w}.$

This is a special case of equation 7.4.1.1 with $f(w) = be^{\beta w}$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2,\ \pm C_1 t + C_3) + \frac{2}{\beta}\ln|C_1|,$$

$$w_2 = w(x\cosh\lambda + at\sinh\lambda,\ t\cosh\lambda + a^{-1}x\sinh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta(Ax + Bt + C)^2}\right],$$

$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(a^2A^2 - B^2)}{b\beta\cosh^2(Ax + Bt + C)}\right],$$

$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta\sinh^2(Ax + Bt + C)}\right],$$

$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta\cos^2(Ax + Bt + C)}\right],$$

where A, B, and C are arbitrary constants.

3°. Functional separable solutions:

$$w(x,t) = \frac{1}{\beta}\ln\left(\frac{8a^2C}{b\beta}\right) - \frac{2}{\beta}\ln\left|(x + A)^2 - a^2(t + B)^2 + C\right|,$$

$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{\lambda x} \pm \frac{\sqrt{2b\beta}}{2a\lambda}\sinh(a\lambda t + C_2)\right],$$

$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{\lambda x} \pm \frac{\sqrt{-2b\beta}}{2a\lambda}\cosh(a\lambda t + C_2)\right],$$

$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{a\lambda t} \pm \frac{\sqrt{-2b\beta}}{2a\lambda}\sinh(\lambda x + C_2)\right],$$

$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{a\lambda t} \pm \frac{\sqrt{2b\beta}}{2a\lambda}\cosh(\lambda x + C_2)\right],$$

where A, B, C, C_1, C_2, and λ are arbitrary constants.

4°. The change of the independent variables

$$z = x - at, \quad y = x + at$$

leads to the Liouville equation 7.5.1.2:

$$\frac{\partial^2 w}{\partial z\partial y} = -\tfrac{1}{4}a^{-2}b\exp(\beta w).$$

Hence, the general solution of the original equation is expressed as

$$w(x,t) = \frac{1}{\beta}\big[f(z) + g(y)\big] - \frac{2}{\beta}\ln\left|k\int\exp\big[f(z)\big]\,dz - \frac{b\beta}{8a^2k}\int\exp\big[g(y)\big]\,dy\right|,$$
$$z = x - at, \qquad y = x + at,$$

where $f = f(z)$ and $g = g(y)$ are arbitrary functions and k is an arbitrary constant.

⊙ *References for equation* 7.2.1.1: J. Liouville (1853), R. K. Bullough and P. J. Caudrey (1980), V. F. Zaitsev and A. D. Polyanin (1996, p. 435).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta w} + be^{2\beta w}.$

1°. Traveling-wave solution for $b\beta > 0$:

$$w(x,t) = -\frac{1}{\beta}\ln\left\{-\frac{b}{a} + C_1\exp\left[a\sqrt{\frac{\beta}{b}}\,\bigl(x\sinh C_2 \pm t\cosh C_2\bigr)\right]\right\},$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solution (generalizes the solution of Item 1°):

$$w(x,t) = -\frac{1}{\beta}\ln\left[\frac{a\beta}{C_1^2 - C_2^2} + C_3\exp(C_1x + C_2t) + \frac{a^2\beta^2 + b\beta(C_1^2 - C_2^2)}{4C_3(C_1^2 - C_2^2)^2}\exp(-C_1x - C_2t)\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Traveling-wave solution:

$$w(x,t) = -\frac{1}{\beta}\ln\left[\frac{a\beta}{C_2^2 - C_1^2} + \frac{\sqrt{a^2\beta^2 + b\beta(C_2^2 - C_1^2)}}{C_2^2 - C_1^2}\,\sin(C_1x + C_2t + C_3)\right].$$

⊙ *Reference for equation* 7.2.1.2: A. D. Polyanin and V. F. Zaitsev (2004, p. 214).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta w} - be^{-\beta w}.$

The substitution

$$w(x,t) = u(x,t) + k, \quad k = \frac{1}{2\beta}\ln\frac{b}{a}$$

leads to an equation of the form 7.3.1.1:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2\sqrt{ab}\,\sinh(\beta u).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta w} - be^{-2\beta w}.$

1°. Functional separable solution:

$$w(x,t) = \frac{1}{\beta}\ln[\varphi(x) + \psi(t)],$$

where the functions $\psi(t)$ and $\varphi(x)$ are determined by the first-order autonomous ordinary differential equations

$$(\varphi_x')^2 = -2a\beta\varphi^3 + C_1\varphi^2 - C_2\varphi + C_3 - b\beta,$$
$$(\psi_t')^2 = 2a\beta\psi^3 + C_1\psi^2 + C_2\psi + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants. Solving these equations for the derivatives, one obtains separable equations.

2°. The transformation

$$t = (a^2b\beta^3)^{-1/6}(\xi + \eta), \quad x = (a^2b\beta^3)^{-1/6}(\xi - \eta), \quad w = \frac{1}{\beta}U + \frac{1}{3\beta}\ln\frac{b}{a}$$

leads to an equation of the form 7.5.1.3:

$$\frac{\partial^2 U}{\partial\xi\partial\eta} = e^U - e^{-2U}.$$

3°. The equation can be integrated with the inverse scattering method.

⊙ *References for equation* 7.2.1.4: A. V. Mikhailov (1979), A. P. Fordy and J. A. Gibbons (1980), F. Calogero and A. Degasperis (1982), A. M. Grundland and E. Infeld (1992), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

7.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x, t, w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta t}e^{\lambda w}.$

This is a special case of equation 7.4.1.7 with $f(w) = ae^{\lambda w}$.

1°. Solutions:

$$w(x,t) = -\frac{\beta}{\lambda}t - \frac{2}{\lambda}\ln\left(C_1 + C_2 x \pm \sqrt{C_2^2 + \tfrac{1}{2}\lambda a}\, t\right),$$
$$w(x,t) = -\frac{\beta}{\lambda}t - \frac{2}{\lambda}\ln\left(C_1 e^{-\sigma t} + C_2 e^{\sigma x} - \frac{\lambda a}{8\sigma^2 C_1}e^{\sigma t}\right),$$

where C_1, C_2, and σ are arbitrary constants.

2°. The substitution $\lambda U = \lambda w + \beta t$ leads to an equation of the form 7.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + a\lambda e^{\lambda U}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta x}e^{\lambda w}.$

This is a special case of equation 7.4.1.6 with $f(w) = ae^{\lambda w}$.

1°. Solutions:

$$w(x,t) = -\frac{\beta}{\lambda}x - \frac{2}{\lambda}\ln\left(C_1 + C_2 t \pm \sqrt{C_2^2 - \tfrac{1}{2}\lambda a}\, x\right),$$
$$w(x,t) = -\frac{\beta}{\lambda}x - \frac{2}{\lambda}\ln\left(C_1 e^{-\sigma x} + C_2 e^{\sigma t} + \frac{\lambda a}{8\sigma^2 C_1}e^{\sigma x}\right),$$

where C_1, C_2, and σ are arbitrary constants.

2°. The substitution $\lambda U = \lambda w + \beta x$ leads to an equation of the form 7.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + a\lambda e^{\lambda U}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ce^{ax+bt}e^{\lambda w}.$

This is a special case of equation 7.4.1.8 with $f(w) = ce^{\lambda w}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha t + \gamma)e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left\{C\exp\left[-\frac{\beta^2\lambda}{2\alpha}(t \pm x)\right] - \frac{1}{\beta}(\alpha t + \gamma) + \frac{\alpha^2}{\beta^3\lambda}\right\},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha x + \gamma)e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left\{C\exp\left[\frac{\beta^2\lambda}{2\alpha}(x \pm t)\right] - \frac{1}{\beta}(\alpha x + \gamma) - \frac{\alpha^2}{\beta^3\lambda}\right\},$$

where C is an arbitrary constant.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha e^{kt} + \gamma)e^{2\lambda w}.$

1°. Functional separable solutions for $k^2\gamma - \beta^2\lambda \neq 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{k^2\gamma - \beta^2\lambda}{2k\gamma}x + \frac{k^2\gamma + \beta^2\lambda}{2k\gamma}t\right) + \frac{\alpha\beta\lambda}{k^2\gamma - \beta^2\lambda}e^{kt} - \frac{\gamma}{\beta}\right],$$

where C is an arbitrary constant.

2°. Generalized traveling-wave solutions for $k^2\gamma - \beta^2\lambda = 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[Ce^{kt} + \frac{\alpha k}{2\beta}(t \pm x)e^{kt} - \frac{\lambda\beta}{k^2}\right].$$

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + k\beta e^{\lambda w} + (\alpha e^{kt} + \lambda\beta^2)e^{2\lambda w}.$

Generalized traveling-wave solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[Ce^{kt} + \frac{\alpha}{2\beta}(t \pm x)e^{kt} - \frac{\lambda\beta}{k}\right],$$

where C is an arbitrary constant.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha e^{kx} + \gamma) e^{2\lambda w}.$

1°. Functional separable solutions for $k^2\gamma + \beta^2\lambda \neq 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{k^2\gamma+\beta^2\lambda}{2k\gamma}t + \frac{k^2\gamma-\beta^2\lambda}{2k\gamma}x\right) - \frac{\alpha\beta\lambda}{k^2\gamma+\beta^2\lambda}e^{kx} - \frac{\gamma}{\beta}\right],$$

where C is an arbitrary constant.

2°. Generalized traveling-wave solutions for $k^2\gamma + \beta^2\lambda = 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[Ce^{kx} + \frac{\alpha k}{2\beta}(x \pm t)e^{kx} + \frac{\lambda\beta}{k^2}\right].$$

⊙ *Reference for equation* 7.2.2.8: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - k\beta e^{\lambda w} - (\alpha e^{kx} + \lambda\beta^2) e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[Ce^{kx} + \frac{\alpha}{2\beta}(x \pm t)e^{kx} - \frac{\lambda\beta}{k}\right],$$

where C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{kt} e^{\lambda w} + (\alpha e^{2kt} + \gamma) e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{4k^2\alpha-\beta^2\lambda}{4k\alpha}x - \frac{\beta^2\lambda}{4k\alpha}t\right) + \frac{\beta\gamma\lambda}{4k^2\alpha-\beta^2\lambda}e^{-kt} - \frac{\alpha}{\beta}e^{kt}\right],$$

where C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{kx} e^{\lambda w} + (\alpha e^{2kx} + \gamma) e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{4k^2\alpha+\beta^2\lambda}{4k\alpha}t + \frac{\beta^2\lambda}{4k\alpha}x\right) - \frac{\beta\gamma\lambda}{4k^2\alpha+\beta^2\lambda}e^{-kx} - \frac{\alpha}{\beta}e^{kx}\right],$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7.2.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 7.4.2.3 with $f(w) = be^{\lambda w}$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(C_1 x + C_2,\ \pm C_1 t + C_3\big) + \frac{1}{\lambda}\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = -\frac{1}{\lambda}\ln\left[\frac{\exp(Ax + A\mu t + B) - b}{A(a-\mu^2)}\right],$$

where μ, A, and B are arbitrary constants.

3°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{1}{\lambda}\ln|t|, \quad z = \frac{x}{t}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + ce^{\lambda w}, \qquad a > 0.$

This is a special case of equation 7.2.3.5 with $b = an$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(C_1^{\frac{2}{2-n}} x,\ \pm C_1 t + C_2\big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{2c\lambda(2-n)}{n}\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right]\right\}.$$

3°. Functional separable solution for $n \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + ck^{-1}e^{\lambda w} = 0, \qquad A = \frac{2}{2-n}.$$

4°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{2}{\lambda}\ln|t|, \quad z = x|t|^{\frac{2}{n-2}}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + ce^{\lambda w}, \qquad a > 0.$

For $n = 1$ and $n = 2$, the equation describes the propagation of nonlinear waves with axial and central symmetry, respectively.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(C_1 x,\ \pm C_1 t + C_2\big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 0$ and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{c\lambda}{2an}\left[x^2 - a(t + C)^2\right]\right\}.$$

3°. Functional separable solution (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = k\left[x^2 - a(t + C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{n+1}{r}w'_r + \frac{c}{ak}e^{\lambda w} = 0.$$

4°. There is an exact solution of the form

$$w(x, t) = F(z) - \frac{2}{\lambda}\ln|t|, \quad z = \frac{x}{t}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a(x + \beta)^n \dfrac{\partial^2 w}{\partial x^2} + ce^{\lambda w}, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. The substitution $z = x + \beta$ leads to a special case of equation 7.2.3.5 with $b = 0$:

$$\frac{\partial^2 w}{\partial t^2} = az^n \frac{\partial^2 w}{\partial z^2} + ce^{\lambda w}.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}\dfrac{\partial w}{\partial x} + ce^{\lambda w}, \qquad a > 0.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(C_1^2 x,\ \pm C_1^{2-n} t + C_2\big) + \frac{4 - 2n}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}a(2-n)^2(t+C)^2 - x^{2-n}.$$

Here, C is an arbitrary constant, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + Aw'_\xi = Be^{\lambda w}, \tag{1}$$

where

$$A = \frac{a(4-3n)+2b}{2a(2-n)}, \quad B = \frac{c}{a(2-n)^2}.$$

For $A \neq 1$, an exact solution of equation (1) is given by

$$w(\xi) = \frac{1}{\lambda}\ln\left(\frac{1-A}{\lambda B\xi}\right).$$

For $A = 1$, which corresponds to $b = \frac{1}{2}an$, exact solutions of equation (1) are expressed as

$$w(\xi) = \frac{1}{\lambda}\ln\left[\frac{2a(2-n)^2}{c\lambda\xi(\ln|\xi|+q)^2}\right],$$
$$w(\xi) = \frac{1}{\lambda}\ln\left[\frac{2ap^2(2-n)^2}{c\lambda\xi\cos^2(p\ln|\xi|+q)}\right],$$
$$w(\xi) = \frac{1}{\lambda}\ln\left[\frac{-2ap^2(2-n)^2}{c\lambda\xi\cosh^2(p\ln|\xi|+q)}\right],$$

where p and q are arbitrary constants.

For $A \neq 1$, the substitution $\xi = kz^{\frac{1}{1-A}}$ $(k = \pm 1)$ brings (1) to the generalized Emden–Fowler equation

$$w''_{zz} = \frac{kB}{(1-A)^2}z^{\frac{2A-1}{1-A}}e^{\lambda w}. \tag{2}$$

In the special case $A = \frac{1}{2}$, which corresponds to $b = a(n-1)$, solutions of equation (2) are given by

$$w(z) = \frac{1}{\lambda}\ln\left[\frac{-a(2-n)^2}{2kc\lambda(z+q)^2}\right],$$
$$w(z) = \frac{1}{\lambda}\ln\left[\frac{ap^2(2-n)^2}{2kc\lambda\cosh^2(pz+q)}\right],$$
$$w(z) = \frac{1}{\lambda}\ln\left[\frac{-ap^2(2-n)^2}{2kc\lambda\cos^2(pz+q)}\right],$$

where p and q are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + (b-a)Bw'_y + ce^{\lambda w} = 0. \tag{3}$$

Solution of equation (3) with $A = \pm B\sqrt{a}$:

$$w(y) = -\frac{1}{\lambda}\ln\left[\frac{c\lambda}{B(b-a)}y + C_1\right].$$

Solutions of equation (3) with $b = a$:

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2(A^2 - aB^2)}{c\lambda(y+q)^2}\right],$$

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2p^2(aB^2 - A^2)}{c\lambda\cosh^2(py+q)}\right],$$

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2p^2(A^2 - aB^2)}{c\lambda\cos^2(py+q)}\right],$$

where p and q are arbitrary constants.

6. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}e^{\lambda w}\dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 7.4.3.5 with $f(w) = be^{\lambda w}$.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + ce^{\mu w}, \qquad a > 0.$

This is a special case of equation 7.2.3.9 with $b = a\lambda$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\left(x - \frac{2}{\lambda}\ln|C_1|,\ \pm C_1 t + C_2\right) + \frac{2}{\mu}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t + C_1)^2\right],$$

where C_1 and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}e^{\mu w} = 0.$$

Its general solution is expressed as

$$w = \begin{cases} -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2k}(r + C_3)^2\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sin^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sinh^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[\dfrac{c\mu}{2kC_2^2}\cosh^2(C_2 r + C_3)\right] & \text{if } ck\mu > 0, \end{cases}$$

where C_2 and C_3 are arbitrary constants.

3°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{2}{\mu}\ln|t|, \qquad z = x + \frac{2}{\lambda}\ln|t|.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + ce^{\mu w}, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. This is a special case of equation 7.2.3.9 with $b = 0$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x}\dfrac{\partial w}{\partial x} + ce^{\mu w}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\Big(x - \frac{2}{\lambda}\ln|C_1|,\ \pm C_1 t + C_2\Big) + \frac{2}{\mu}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution:

$$w = w(z), \quad z = \big[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\big]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda}\frac{1}{z}w'_z + \frac{c}{ak\lambda^2}e^{\mu w} = 0. \tag{1}$$

A solution of equation (1) has the form

$$w(z) = \frac{1}{\mu}\ln\left[\frac{2k\lambda(a\lambda - 2b)}{c\mu z^2}\right].$$

Note some other exact solutions of equation (1):

$$w(z) = \frac{1}{\mu}\ln\left[\frac{-2ak\lambda^2}{c\mu(z+B)^2}\right] \qquad \text{if } b = a\lambda,$$

$$w(z) = \frac{1}{\mu}\ln\left[\frac{2aA^2k\lambda^2}{c\mu\cosh^2(Az+B)}\right] \qquad \text{if } b = a\lambda,$$

$$w(z) = \frac{1}{\mu}\ln\left[\frac{-2aA^2k\lambda^2}{c\mu\sinh^2(Az+B)}\right] \qquad \text{if } b = a\lambda,$$

$$w(z) = \frac{1}{\mu}\ln\left[\frac{-2aA^2k\lambda^2}{c\mu\cos^2(Az+B)}\right] \qquad \text{if } b = a\lambda,$$

$$w(z) = \frac{1}{\mu}\ln\left[\frac{8ABak\lambda^2}{c\mu(Az^2+B)^2}\right] \qquad \text{if } b = \tfrac{1}{2}a\lambda,$$

where A and B are arbitrary constants.

10. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x+\mu w}\dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 7.4.3.10 with $f(w) = be^{\mu w}$.

11. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x+\mu w}\dfrac{\partial w}{\partial x} + ce^{\beta w}, \qquad a > 0.$

This is a special case of equation 7.4.3.11 with $f(w) = be^{\mu w}$ and $g(w) = ce^{\beta w}$.

7.2.4. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 C_2^{\lambda} x + C_3, \pm C_1 t + C_4,) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = Axt + Bx + Ct + D,$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{B^2}{a}\frac{(x+A)^2}{\cosh^2(Bt+C)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{1}{aA^2}\frac{\cosh^2(Ax+B)}{(t+C)^2}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ax+B)}{\cosh^2(Ct+D)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cosh^2(Ax+B)}{\sinh^2(Ct+D)}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ax+B)}{\cosh^2(Ct+D)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cosh^2(Ax+B)}{\cos^2(Ct+D)}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{4BC\beta^2(x+A)^2}{a(Be^{\beta t}+Ce^{-\beta t})^2}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{(Ae^{\beta x}+Be^{-\beta x})^2}{4aAB\beta^2(t+C)^2}\right],$$

where A, B, C, D, and β are arbitrary constants. The first solution is degenerate, while the others are representable as the sum of functions with different arguments.

3°. Self-similar solution:

$$w = w(z), \quad z = \frac{x+A}{t+B},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(ae^{\lambda w} - z^2)w''_{zz} - zw'_z = 0,$$

whose order can be reduced with the transformation $\xi = z^{-2}e^{\lambda w}$, $U(\xi) = zw'_z$.

4°. Solution:

$$w = \frac{2(k-1)}{\lambda}\ln(t+C_1) + f(\zeta), \quad \zeta = \frac{x+C_2}{(t+C_1)^k},$$

where C_1, C_2, and k are arbitrary constants, and the function $f = f(\zeta)$ is determined by the ordinary differential equation

$$k^2\zeta^2 f''_{\zeta\zeta} + k(k+1)\zeta f'_\zeta - \frac{2(k-1)}{\lambda} = ae^{\lambda f} f''_{\zeta\zeta}.$$

5°. There are exact solutions of the following forms:

$$w(x,t) = F(\eta) - \frac{2}{\lambda}\ln|t|, \quad \eta = x + k\ln|t|;$$

$$w(x,t) = H(\rho) - \frac{2}{\lambda}t, \quad \rho = xe^t,$$

where k is an arbitrary constant.

⊙ *Reference for equation* 7.2.4.1: V. F. Zaitsev and A. D. Polyanin (1996, pp. 458–459).

2. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2} + be^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda}x + C_2, \pm C_1^{\beta}t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_2x + k_1t,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$(k_1^2 - ak_2^2e^{\lambda u})u''_{zz} = be^{\beta u}.$$

Its solution can be written out in implicit form as

$$\int \frac{du}{\sqrt{F(u)}} = C_1 \pm z, \qquad F(u) = 2b\int \frac{e^{\beta u}\,du}{k_1^2 - ak_2^2e^{\lambda u}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w = U(\xi) - \frac{2}{\beta}\ln|t|, \qquad \xi = x|t|^{\frac{\lambda-\beta}{\beta}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\beta} + \frac{(\lambda-\beta)(\lambda-2\beta)}{\beta^2}\xi U'_\xi + \frac{(\lambda-\beta)^2}{\beta^2}\xi^2U''_{\xi\xi} = ae^{\lambda U}U''_{\xi\xi} + be^{\beta U}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda w}\dfrac{\partial w}{\partial x}\right), \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1C_2^{\lambda}x + C_3, \pm C_1t + C_4,) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solutions:

$$w(x,t) = \frac{1}{\lambda}\ln|Ax + B| + Ct + D, \tag{1}$$

$$w(x,t) = \frac{2}{\lambda}\ln|Ax + B| - \frac{2}{\lambda}\ln|\pm A\sqrt{a}\,t + C|, \tag{2}$$

$$w(x,t) = \frac{1}{\lambda}\ln(aA^2x^2 + Bx + C) - \frac{2}{\lambda}\ln(aAt + D), \tag{3}$$

$$w(x,t) = \frac{1}{\lambda}\ln(Ax^2 + Bx + C) + \frac{1}{\lambda}\ln\left[\frac{p^2}{aA\cos^2(pt+q)}\right], \tag{4}$$

$$w(x,t) = \frac{1}{\lambda}\ln(Ax^2 + Bx + C) + \frac{1}{\lambda}\ln\left[\frac{p^2}{aA\sinh^2(pt+q)}\right], \tag{5}$$

$$w(x,t) = \frac{1}{\lambda}\ln(Ax^2 + Bx + C) + \frac{1}{\lambda}\ln\left[\frac{-p^2}{aA\cosh^2(pt+q)}\right], \tag{6}$$

where A, B, C, D, p, and q are arbitrary constants. Expressions (1) to (6) exhaust all solutions that can be representable in the form of the sum of functions with different arguments.

3°. Traveling-wave solution:

$$w = w(z), \quad z = x \pm \mu t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda\mu^2 w - ae^{\lambda w} = Az + B.$$

4°. Self-similar solution:

$$w = u(\xi), \quad \xi = \frac{x + A}{t + B}.$$

Here, A and B are arbitrary constants, and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 u'_\xi)'_\xi = (ae^{\lambda u} u'_\xi)'_\xi,$$

which admits the first integral

$$\left(\xi^2 - ae^{\lambda u}\right) u'_\xi = C. \tag{7}$$

To the special case $C = 0$ there corresponds a solution of the form (2). For $C \neq 0$, treating u in (7) as the independent variable, one obtains a Riccati equation for $\xi = \xi(u)$,

$$C\xi'_u = \xi^2 - ae^{\lambda u},$$

which is considered in the book by Polyanin and Zaitsev (2003).

5°. Solution:

$$w = \frac{2(k-1)}{\lambda} \ln(t + C_1) + f(\zeta), \quad \zeta = \frac{x + C_2}{(t + C_1)^k},$$

where C_1, C_2, and k are arbitrary constants, and the function $f = f(\zeta)$ is determined by the ordinary differential equation

$$k^2\zeta^2 f''_{\zeta\zeta} + k(k+1)\zeta f'_\zeta - \frac{2(k-1)}{\lambda} = a(e^{\lambda f} f'_\zeta)'_\zeta.$$

6°. There are exact solutions of the following forms:

$$w(x,t) = F(\eta) - \frac{2}{\lambda} \ln|t|, \quad \eta = x + k \ln|t|;$$
$$w(x,t) = H(\zeta) - \frac{2}{\lambda} t, \quad \eta = xe^t,$$

where k is an arbitrary constant.

7°. For other solutions, see equation 7.4.4.6 with $f(w) = ae^{\lambda w}$.

⊙ *References for equation* 7.2.4.3: W. F. Ames, R. J. Lohner, and E. Adams (1981), A. D. Polyanin and V. F. Zaitsev (2002, p. 141).

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + ae^{\lambda w}.$

This is a special case of equation 7.2.4.5 with $\beta = \lambda$. For $a = 0$ see equation 7.2.4.3.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1,\ C_2 t + C_3) + \frac{1}{\lambda}\ln C_2^2,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Degenerate solutions in additive form:

$$w = \theta(x) + C_1 t,$$

where

$$\begin{aligned}
&\theta = \frac{1}{\lambda}\ln\left|C_2\cos\left(\sqrt{a\lambda}\,x\right) + C_3\sin\left(\sqrt{a\lambda}\,x\right)\right| && \text{if } a\lambda > 0,\\
&\theta = \frac{1}{\lambda}\ln\left|C_2\exp\left(\sqrt{|a\lambda|}\,x\right) + C_3\exp\left(-\sqrt{|a\lambda|}\,x\right)\right| && \text{if } a\lambda < 0,\\
&\theta = \frac{1}{\lambda}\ln|x + C_2| + C_3 && \text{if } a = 0,\\
&\theta = \tfrac{1}{2}ax^2 + C_2 x + C_3 && \text{if } \lambda = 0;
\end{aligned}$$

C_1, C_2, and C_3 are arbitrary constants.

3°. Nondegenerate solutions in additive form:

$$w = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are expressed as

$$\varphi(x) = \begin{cases}
\dfrac{1}{\lambda}\ln\left|\dfrac{C}{a} + A_1\cos\left(\sqrt{a\lambda}\,x\right) + A_2\sin\left(\sqrt{a\lambda}\,x\right)\right| & \text{if } a\lambda > 0,\\
\dfrac{1}{\lambda}\ln\left|\dfrac{C}{a} + A_1\exp\left(\sqrt{|a\lambda|}\,x\right) + A_2\exp\left(-\sqrt{|a\lambda|}\,x\right)\right| & \text{if } a\lambda < 0,
\end{cases}$$

$$\psi(t) = \begin{cases}
-\dfrac{1}{\lambda}\ln\left[\dfrac{C\lambda}{2B_1^2}\sin^2(B_1 t + B_2)\right] & \text{if } C\lambda > 0,\\
-\dfrac{1}{\lambda}\ln\left[\dfrac{C\lambda}{2B_1^2}\sinh^2(B_1 t + B_2)\right] & \text{if } C\lambda > 0,\\
-\dfrac{1}{\lambda}\ln\left[\dfrac{C\lambda}{2}(t + B_2)^2\right] & \text{if } C\lambda > 0,\\
-\dfrac{1}{\lambda}\ln\left[-\dfrac{C\lambda}{2B_1^2}\cosh^2(B_1 t + B_2)\right] & \text{if } C\lambda < 0,
\end{cases}$$

where A_1, A_2, B_1, B_2, and C are arbitrary constants.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda}x + C_2, \pm C_1^{\beta}t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_2x + k_1t,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$k_1^2u''_{zz} - ak_2^2(e^{\lambda u}u'_z)'_z = be^{\beta u}.$$

The substitution $\Theta(u) = (u'_z)^2$ leads to the first-order linear equation

$$(k_1^2 - ak_2^2e^{\lambda u})\Theta'_u - 2ak_2^2\lambda e^{\lambda u}\Theta = 2be^{\beta u}.$$

3°. Solution:

$$w = U(\xi) - \frac{2}{\beta}\ln|t|, \qquad \xi = x|t|^{\frac{\lambda-\beta}{\beta}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\beta} + \frac{(\lambda-\beta)(\lambda-2\beta)}{\beta^2}\xi U'_\xi + \frac{(\lambda-\beta)^2}{\beta^2}\xi^2U''_{\xi\xi} = (ae^{\lambda U}U'_\xi)'_\xi + be^{\beta U}.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b - ce^{-2\lambda w}.$

Functional separable solution:

$$w = \frac{1}{\lambda}\ln\left(\sqrt{c\lambda}\,t - \frac{b\lambda}{2a}x^2 + C_1x + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(ae^{\lambda w} + be^{2\lambda w})\dfrac{\partial w}{\partial x}\right] + c.$

Additive separable solution:

$$w = \frac{1}{\lambda}\ln x + \varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the second-order autonomous ordinary differential equation

$$\varphi''_{tt} = \frac{b}{\lambda}e^{2\lambda\varphi} + c.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 373).

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x+\mu t+\beta w}\dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

The substitution $\beta u = \lambda x + \mu t + \beta w$ leads to an equation of the form 7.2.4.1:

$$\frac{\partial^2 u}{\partial t^2} = ae^{\beta u}\frac{\partial^2 u}{\partial x^2}.$$

7.3. Other Equations Involving Arbitrary Parameters

7.3.1. Equations with Hyperbolic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sinh(\lambda w).$

Sinh-Gordon equation (also known as *hyperbolic sine–Gordon equation*). It arises in some areas of physics. This is a special case of equation 7.4.1.1 with $f(w) = b\sinh(\lambda w)$.

1°. Traveling-wave solutions:

$$w(x,t) = \pm\frac{2}{\lambda}\ln\left[\tan\frac{b\lambda(kx+\mu t+\theta_0)}{2\sqrt{b\lambda(\mu^2-ak^2)}}\right],$$
$$w(x,t) = \pm\frac{4}{\lambda}\operatorname{arctanh}\left[\exp\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2-ak^2)}}\right],$$

where k, μ, and θ_0 are arbitrary constants. It is assumed that $b\lambda(\mu^2 - ak^2) > 0$ in both formulas.

2°. Functional separable solution:

$$w(x,t) = \frac{4}{\lambda}\operatorname{arctanh}\big[f(t)g(x)\big], \qquad \operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z},$$

where the functions $f = f(t)$ and $g = g(x)$ are determined by the first-order autonomous ordinary differential equations

$$\big(f_t'\big)^2 = Af^4 + Bf^2 + C,$$
$$a\big(g_x'\big)^2 = Cg^4 + (B - b\lambda)g^2 + A,$$

where A, B, and C are arbitrary constants.

3°. For other exact solutions of this equation, see equation 7.4.1.1 with $f(w) = b\sinh(\lambda w)$, Items 2° and 3°.

⊙ *References for equation* 7.3.1.1: A. Grauel (1985), A. M. Grundland and E. Infeld (1992), M. Musette and R. Conte (1994), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), C. Hoenselaers (2007).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a\sinh(\beta w) + b\sinh(2\beta w).$

Double sinh-Gordon equation. Denote: $k = \dfrac{a}{2b}$.

Traveling-wave solutions:

$$w = \pm\frac{1}{\beta}\operatorname{arccosh}\frac{1-k\sin z}{\sin z - k}, \qquad z = \sqrt{2b\beta(1-k^2)}\,(x\sinh C_1 \pm t\cosh C_1 + C_2)$$
if $|k| < 1$;

$$w = \pm\frac{2}{\beta}\operatorname{arctanh}\left(\sqrt{\frac{k+1}{k-1}}\,\tanh\frac{\xi}{2}\right), \quad \xi = \sqrt{2b\beta(k^2-1)}\,(x\sinh C_1 \pm t\cosh C_1 + C_2)$$
if $|k| > 1$,

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t}\sinh^k(\lambda w).$

This is a special case of equation 7.4.1.7 with $f(w) = b\sinh^k(\lambda w)$. Hence, for $k = 1$, this equation is reduced to a simpler equation of 7.3.1.1.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x}\sinh^k(\lambda w).$

This is a special case of equation 7.4.1.6 with $f(w) = b\sinh^k(\lambda w)$. Hence, for $k = 1$, this equation is reduced to a simpler equation of 7.3.1.1.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + k\sinh(\lambda w).$

This is a special case of equation 7.4.2.1 with $f(w) = k\sinh(\lambda w)$ and $n = b/a$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\cosh(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 7.4.4.6 with $f(w) = a\cosh(\lambda w)$.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\sinh(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 7.4.4.6 with $f(w) = a\sinh(\lambda w)$.

7.3.2. Equations with Logarithmic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + kw.$

This is a special case of equation 7.4.1.1 with $f(w) = bw\ln w + kw$.

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the autonomous ordinary differential equations

$$\varphi''_{tt} - b\varphi\ln\varphi - k\varphi = 0,$$
$$a\psi''_{xx} + b\psi\ln\psi = 0,$$

whose general solutions can be represented in implicit form.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + \left(cx^k + st^n\right)w.$

This is a special case of equation 7.4.1.10 with $f(x) = cx^k$ and $g(t) = st^n$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k\ln w.$

This is a special case of equation 7.4.1.1 with $f(w) = bw^k\ln w$. For $k = 1$, see also equation 7.4.1.9 with $f(t) = 0$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)\ln^k(\lambda w).$

This is a special case of equation 7.4.1.2 with $f(w) = a\ln^k(\lambda w)$.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t} w \ln w.$

The transformation

$$w = U(z,\tau), \quad z = \tfrac{1}{2}\beta \exp\left(\tfrac{1}{2}\beta t\right)\sinh\left(\tfrac{1}{2}\beta x\right), \quad \tau = \tfrac{1}{2}\beta \exp\left(\tfrac{1}{2}\beta t\right)\cosh\left(\tfrac{1}{2}\beta x\right)$$

leads to a simpler equation of the form 7.3.2.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + bU\ln U.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x} w \ln w.$

The transformation

$$w = U(z,\tau), \quad z = \tfrac{1}{2}\exp\left(\tfrac{1}{2}\beta x\right)\cosh\left(\tfrac{1}{2}\beta t\right), \quad \tau = \tfrac{1}{2}\exp\left(\tfrac{1}{2}\beta x\right)\sinh\left(\tfrac{1}{2}\beta t\right)$$

leads to a simpler equation of the form 7.3.2.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + bU\ln U.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + cw^k \ln w.$

This is a special case of equation 7.4.2.1 with $f(w) = cw^k \ln w$ and $b = an$.

8. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + b\ln^k(\lambda w).$

This is a special case of equation 7.4.3.2 with $\beta = 0$ and $f(w) = b\ln^k(\lambda w)$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}\ln^k(\lambda w)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 7.4.3.5 with $f(w) = b\ln^k(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[\left(a + \dfrac{b}{w}\right)\dfrac{\partial w}{\partial x}\right] + cw\ln w + kw + \dfrac{bc}{2a}.$

Multiplicative separable solution:

$$w = \varphi(t)\exp\left[-\frac{c}{4a}(x - x_0)^2\right],$$

where x_0 is an arbitrary constant and the function $\varphi = \varphi(t)$ is determined by the second-order autonomous ordinary differential equation

$$\varphi''_{tt} = c\varphi\ln\varphi + (k - \tfrac{1}{2}c)\varphi.$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 373).

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left[\ln^k(\lambda w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 7.4.4.6 with $f(w) = a\ln^k(\lambda w)$.

12. $w\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2 + cw^2\ln w.$

The substitution

$$w = e^u$$

leads to the equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu + b.$$

This equation has additive separable and generalized separable solutions.

1°. Additive separable solution:

$$u = f(x) + \ln|g(t)|,$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by the second-order constant-coefficient ordinary differential equations

$$f''_{xx} + af'_x + cf + b = K,$$
$$g''_{tt} = Kg,$$

with K being an arbitrary constant.

2°. Generalized separable solution:

$$u = f(x)t^2 + g(x)t + h(x),$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + af'_x + cf - 4f^2 = 0,$$
$$g''_{xx} + ag'_x + cg - 4fg = 0,$$
$$h''_{xx} + ah'_x + ch + b - 2f - g^2 = 0.$$

Remark. The parameters a, b, and c in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

13. $w\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2 + cw^2\ln w + sw^2\ln^2 w.$

The substitution

$$w = e^u$$

leads to the equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + su^2 + cu + b.$$

This equation has generalized separable solutions.

1°. Generalized separable solution with $s > 0$:

$$w = f(x)\exp(\sqrt{s}\,t) + g(x)\exp(-\sqrt{s}\,t) + h(x),$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f''_{xx} + af'_x + (c-s)f + 2sfh &= 0,\\ g''_{xx} + ag'_x + (c-s)g + 2sgh &= 0,\\ h''_{xx} + ah'_x + ch + b + s(4fg + h^2) &= 0. \end{aligned}$$

2°. Generalized separable solution with $s < 0$:

$$w = f(x)\cos(\sqrt{|s|}\,t) + g(x)\sin(\sqrt{|s|}\,t) + h(x),$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f''_{xx} + af'_x + (c-s)f + 2sfh &= 0,\\ g''_{xx} + ag'_x + (c-s)g + 2sgh &= 0,\\ h''_{xx} + ah'_x + ch + b + s(f^2 + g^2 + h^2) &= 0. \end{aligned}$$

Remark. The parameters a, b, and c in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

14. $w\ln(kw)\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2 + cw^2\ln w.$

The substitution

$$w = e^u$$

leads to the equation

$$(u + \ln k)\left[\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t}\right)^2\right] = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + cu + b.$$

Generalized separable solution:

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f''_{xx} + af'_x + cf &= f^3,\\ g''_{xx} + ag'_x + cg &= (g + \ln k)f^2. \end{aligned}$$

Remark. The parameters a, b, and c in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, and $c = c(x)$.

15. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + aw\dfrac{\partial w}{\partial x} + bw^2 + cw^2\ln w.$

The substitution

$$w = e^u$$

leads to the equation

$$\frac{\partial u}{\partial t}\left[\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t}\right)^2\right] = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + b + cu.$$

This equation has additive separable and generalized separable solutions.

1°. Additive separable solution:

$$u = f(x) + \ln|g(t)|,$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by the system of ordinary differential equations

$$f''_{xx} + af'_x + cf + b = K,$$
$$g'_t g''_{tt} = Kg^2;$$

K is an arbitrary constant. The first equation is a second-order constant-coefficient ordinary differential equation for f and, hence, is easy to integrate. A particular solution of the second equation is $g = C_1\exp(K^{1/3}t)$ and the general solution is

$$\int (Kg^3 + C_1)^{-1/3}dg = t + C_2,$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution:

$$u = f(x)t + g(x),$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the autonomous system of ordinary differential equations

$$f''_{xx} + af'_x + cf = 0,$$
$$g''_{xx} + ag'_x + cg = f^3 - b.$$

This system is easy to integrate, since the former equation is a second-order linear homogeneous ordinary differential equation and the latter equation is a second-order linear nonhomogeneous ordinary differential equation.

Remark. The parameters a, b, c, and s in the original equation can be replaced by arbitrary functions $a = a(x)$, $b = b(x)$, $c = c(x)$, and $s = s(x)$.

7.3.3. Sine-Gordon Equation and Other Equations with Trigonometric Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin(\lambda w).$

Sine-Gordon equation. It arises in differential geometry and various areas of physics (superconductivity, dislocations in crystals, waves in ferromagnetic materials, laser pulses in two-phase media, meson theory of nuclear forces and others).

Remark. This equation is sometimes called the *Enneper equation*; see historical aspects in Seeger and Wesolowski (1981).

1°. Suppose $w = \varphi(x,t)$ is a solution of the sine-Gordon equation. Then the functions

$$w_1 = \frac{2\pi n}{\lambda} \pm \varphi(C_1 \pm x, C_2 \pm t), \qquad n = 0, \pm 1, \pm 2, \ldots;$$
$$w_2 = \pm\varphi\left(x\cosh\sigma + t\sqrt{a}\,\sinh\sigma,\; x\frac{\sinh\sigma}{\sqrt{a}} + t\cosh\sigma\right),$$

where C_1, C_2, and σ are arbitrary constants, are also solutions of the equation. The plus or minus signs in the first expression are chosen in any sequence.

2°. Traveling-wave solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2-ak^2)}}\right]\right\} \qquad \text{if} \quad b\lambda(\mu^2-ak^2) > 0,$$
$$w(x,t) = -\frac{\pi}{\lambda} + \frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(ak^2-\mu^2)}}\right]\right\} \quad \text{if} \quad b\lambda(\mu^2-ak^2) < 0,$$

where k, μ, and θ_0 are arbitrary constants. The first expression corresponds to a *single-soliton solution*.

3°. Functional separable solution:

$$w(x,t) = \frac{4}{\lambda}\arctan\big[f(x)g(t)\big], \tag{1}$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by the first-order autonomous separable ordinary differential equations

$$\begin{aligned} \big(f'_x\big)^2 &= Af^4 + Bf^2 + C, \\ \big(g'_t\big)^2 &= -aCg^4 + (aB+b\lambda)g^2 - aA, \end{aligned} \tag{2}$$

where A, B, and C are arbitrary constants. Note some exact solutions that follow from (1) and (2).

3.1. For $A = 0$, $B = k^2 > 0$, and $C > 0$, we have

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sinh(kx+A_1)}{k\sqrt{a}\cosh(\mu t+B_1)}\right], \quad \mu^2 = ak^2 + b\lambda > 0, \tag{3}$$

where k, A_1, and B_1 are arbitrary constants. Formula (3) corresponds to the *two-soliton solution* of Perring–Skyrme (1962).

3.2. For $A = 0$, $B = -k^2 < 0$, and $C > 0$,

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sin(kx+A_1)}{k\sqrt{a}\cosh(\mu t+B_1)}\right], \quad \mu^2 = b\lambda - ak^2 > 0,$$

where k, A_1, and B_1 are arbitrary constants.

3.3. For $A = k^2 > 0$, $B = k^2\gamma^2 > 0$, and $C = 0$,

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\gamma}{\mu}\frac{e^{\mu(t+A_1)} - ak^2e^{-\mu(t+A_1)}}{e^{k\gamma(x+B_1)} + e^{-k\gamma(x+B_1)}}\right], \quad \mu^2 = ak^2\gamma^2 + b\lambda > 0,$$

where k, A_1, B_1, and γ are arbitrary constants.

4°. Two-soliton periodic solution, called a *breather*, is given by ($a = 1$, $b = -1$, and $\lambda = 1$):

$$w = 4\arctan\left[\frac{\sqrt{1-\omega^2}}{\omega\cosh\left(\sqrt{1-\omega^2}\,x\sin\omega t\right)}\right],$$

where ω is an arbitrary constant ($0 < \omega < 1$).

5°. An N-soliton solution is given by ($a = 1$, $b = -1$, and $\lambda = 1$)

$$w(x,t) = \arccos\left[1 - 2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)(\ln F)\right],$$

$$F = \det\left[M_{ij}\right], \quad M_{ij} = \frac{2}{a_i + a_j}\cosh\left(\frac{z_i + z_j}{2}\right),$$

$$z_i = \pm\frac{x - \mu_i t + C_i}{\sqrt{1-\mu_i^2}}, \quad a_i = \pm\sqrt{\frac{1-\mu_i}{1+\mu_i}},$$

where μ_i and C_i are arbitrary constants.

6°. For other exact solutions of the original equation, see equation 7.4.1.1 with $f(w) = b\sin(\lambda w)$, Item 3°.

7°. The sine-Gordon equation is integrated by the inverse scattering method; see the book by Novikov, Manakov, Pitaevskii, and Zakharov (1984). Belokolos (1995) obtained a general formula for the solution of the sine-Gordon equation with arbitrary initial and boundary conditions.

8°. The transformation

$$z = x - at, \quad y = x + at$$

leads to an equation of the form 7.5.1.5: $\partial_{zy} w = -\frac{1}{4}a^{-2}\sin w$.

⊙ *References for equation* 7.3.3.1: A. Enneper (1870), R. Steuerwald (1936), A. Seeger, H. Donth, and A. Kochendörfer (1953), J. K. Perring, T. H. R. Skyrme (1962), M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur (1973), V. E. Zakharov, L. A. Takhtajan, and L. D. Faddeev (1974), G. B. Whitham (1974), G. Liebbrandt (1978), R. K. Bullough and P. J. Caudrey (1980), I. M. Krichever (1980), G. L. Lamb (1980), M. J. Ablowitz and H. Segur (1981), A. Seeger and Z. Wesolowski (1981), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), J. Weiss (1984), M. J. Ablowitz and P. A. Clarkson (1991), R. Beutler (1993), M. Musette and R. Conte (1994).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin(\lambda w) + c\sin\left(\frac{1}{2}\lambda w\right)$.

Double sine-Gordon equation. It arises in nonlinear optics (propagation of ultrashort pulses in a resonance degenerate medium) and low temperature physics (propagation of spin waves in anisotropic spin liquids).

1°. Traveling-wave solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\sqrt{4b^2-c^2}}{2b-c}\,\tanh\frac{\lambda\sqrt{4b^2-c^2}(kx+\mu t+\theta_0)}{4\sqrt{b\lambda(ak^2-\mu^2)}}\right] \quad \text{if} \quad c^2 < 4b^2,$$

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\sqrt{c^2-4b^2}}{c-2b}\,\tan\frac{\lambda\sqrt{c^2-4b^2}(kx+\mu t+\theta_0)}{4\sqrt{b\lambda(ak^2-\mu^2)}}\right] \quad \text{if} \quad c^2 > 4b^2.$$

Here, k, μ, and θ_0 are arbitrary constants. It is assumed that $b\lambda(ak^2 - \mu^2) > 0$ in both formulas.

2°. Traveling-wave solutions:

$$w(x,t) = A + \frac{4}{\lambda}\arctan\bigl(B_1 e^{\theta} + C_1\bigr) + \frac{4}{\lambda}\arctan\bigl(B_2 e^{\theta} + C_2\bigr), \quad \theta = \mu t \pm kx + \theta_0,$$

where the parameters A, B_1, B_2, C_1, C_2, μ, and k are related by algebraic constraints with the parameters a, b, c, and λ of the original equation; θ_0 is an arbitrary constant.

Note some special cases of interest that arise in applications.

2.1. For $a = 1$, $b = -1$, $c = -\frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 4\arctan\bigl(e^{\theta-\Delta}\bigr) + 4\arctan\bigl(e^{\theta+\Delta}\bigr); \qquad \Delta = \ln\bigl(\sqrt{5}+2\bigr), \quad k = \mu + \tfrac{5}{4}\mu^{-1}.$$

2.2. For $a = 1$, $b = -1$, $c = -\frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 2\pi + 4\arctan\bigl(e^{\theta-\Delta}\bigr) - 4\arctan\bigl(e^{\theta+\Delta}\bigr); \qquad \Delta = \ln\bigl(\sqrt{3}+2\bigr), \quad k = \mu + \tfrac{3}{4}\mu^{-1}.$$

2.3. For $a = 1$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = \delta - 2\pi + 4\arctan\left(\frac{4}{\sqrt{15}}e^{\theta} + \frac{1}{\sqrt{15}}\right); \qquad \delta \text{ is any}, \quad k = \mu + \tfrac{15}{16}\mu^{-1}.$$

2.4. For $a = 1$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 2\pi - \delta + 4\arctan\left(\frac{4}{\sqrt{15}}e^{\theta} - \frac{1}{\sqrt{15}}\right); \qquad \delta \text{ is any}, \quad k = \mu + \tfrac{15}{16}\mu^{-1}.$$

⊙ *References for equation* 7.3.3.2: P. B. Burt (1978), R. K. Bullough and P. J. Caudrey (1980), F. Calogero and A. Degasperis (1982).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\cos(\lambda w).$

The substitution $w = u + \dfrac{\pi}{2\lambda}$ leads to an equation of the form 7.3.3.1:

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} - b\sin(\lambda u).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t}\sin^k(\lambda w).$

This is a special case of equation 7.4.1.7 with $f(w) = b\sin^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 7.3.3.1.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x}\sin^k(\lambda w).$

This is a special case of equation 7.4.1.6 with $f(w) = b\sin^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 7.3.3.1.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t}\cos^k(\lambda w).$

This is a special case of equation 7.4.1.7 with $f(w) = b\cos^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 7.3.3.3.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x}\cos^k(\lambda w).$

This is a special case of equation 7.4.1.6 with $f(w) = b\cos^k(\lambda w)$. Therefore, for $k = 1$, the equation reduced to a simpler equation of 7.3.3.3.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + k\sin(\lambda w).$

This is a special case of equation 7.4.2.1 with $f(w) = k\sin(\lambda w)$ and $b = an$.

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\cos^n(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 7.4.4.6 with $f(w) = a\cos^n(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\sin^n(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 7.4.4.6 with $f(w) = a\sin^n(\lambda w)$.

7.3.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]$

1. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right).$

The transformation

$$\tau = t + \ln|w|, \quad dz = aw^{-2}w_x dt + (w + w_t)dx, \quad u = 1/w \qquad (dz = z_t\,dt + z_x\,dx),$$

where the subscripts denote the corresponding partial derivatives, leads to the linear *telegraph equation*

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2}.$$

⊙ *References*: C. Rogers and T. Ruggeri (1985), C. Rogers and W. F. Ames (1989).

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[a(w + b)^n\dfrac{\partial w}{\partial x}\right].$

1°. Solution for $n \neq -1$:

$$w(x, t) = (x + C_2)^{1/(1+n)}(C_1 e^{-kt} + C_2) - b,$$

where C_1 and C_2 are arbitrary constants.

2°. Solution:

$$w(x,t) = (x+C)^{2/n}u(t) - b,$$

where C is an arbitrary constant, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + ku'_t = \frac{2a(n+2)}{n^2}u^{n+1}.$$

This equation is easy to integrate for $n = -2$ and $n = -1$. For $n = -3/2$ and -3, its exact solutions are given in the handbook by Polyanin and Zaitsev (2003).

3°. Traveling-wave solution in implicit form:

$$\int_0^{w+b} \frac{bu^n - \lambda^2}{k\lambda u + C_1}\,dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

4°. Solution for $n = -1$:

$$w(x,t) = \frac{2at + C_1e^{-kt} + C_2}{k(x+C_3)^2} - b,$$

where C_1, C_2, and C_3 are arbitrary constants.

5°. Generalized separable solution for $n = 1$:

$$w(x,t) = f(t)x^2 + g(t)x + h(t) - b,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the system of ordinary differential equations

$$f''_{tt} + kf'_t = 6af^2,$$
$$g''_{tt} + kg'_t = 6afg,$$
$$h''_{tt} + kh'_t = 2afh + ag^2.$$

⊙ *References for equation* 7.3.4.2: N. H. Ibragimov (1994, pp. 169–170), A. D. Polyanin and V. F. Zaitsev (2002, p. 147).

3. $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(be^{\lambda w}\dfrac{\partial w}{\partial x}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(C_1x + C_2,\ t + C_3\big) - \frac{2}{\lambda}\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(C_1x + C_2) + C_3e^{-at} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(\lambda C_1 x^2 + C_2 x + C_3) + u(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + au'_t = 2bC_1 e^{\lambda u}.$$

4°. Traveling-wave solution in implicit form:

$$\int \frac{be^{\lambda w} - \lambda^2}{a\lambda w + C_1}\,dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

⊙ *References for equation* 7.3.4.3: N. H. Ibragimov (1994, pp. 169–170), A. D. Polyanin and V. F. Zaitsev (2002, p. 146).

7.3.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + f(w)\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[g(w)\dfrac{\partial w}{\partial x}\right]$

▷ *Equations of this form admit traveling-wave solutions $w = w(kx + \lambda t)$; to $k = 0$ there corresponds a homogeneous solution dependent on t alone and to $\lambda = 0$, a stationary solution dependent on x alone. For $g(w) =$ const, such equations are encountered in the theory of electric field and nonlinear Ohm laws, where w is the electric field strength.*

1. $\dfrac{\partial^2 w}{\partial t^2} + w^n\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(\pm C_1^n x + C_2,\ C_1^n t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions in implicit form:

$$\int \frac{(n+1)dw}{w^{n+1} + C_1} = -\frac{a(x-at)}{1-a^2} + C_2 \quad \text{if} \quad n \neq -1,$$
$$\int \frac{dw}{\ln|w| + C_1} = -\frac{a(x-at)}{1-a^2} + C_2 \quad \text{if} \quad n = -1,$$

where C_1, C_2, and a are arbitrary constants ($a \neq 0, \pm 1$).

Special case 1. Traveling-wave solution for $n \neq 0, -1$:

$$w = \left(\frac{n}{n+1}\xi + C\right)^{-1/n}, \quad \xi = \frac{a(x-at)}{1-a^2},$$

where C and a are arbitrary constants ($a \neq 0, \pm 1$).

Special case 2. Traveling-wave solutions for $n = 1$:

$$w = 2C_1 \tanh(C_1\xi + C_2), \qquad w = -2C_1 \tan(C_1\xi + C_2), \qquad \xi = \frac{a(x - at)}{1 - a^2},$$

where C_1, C_2, and a are arbitrary constants; $a \neq 0, \pm 1$.

3°. Self-similar solution:

$$w = t^{-1/n}\varphi(\xi), \quad \xi = \frac{x}{t},$$

where the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$(1 - \xi^2)\varphi''_{\xi\xi} + \left[\varphi^n - \frac{2(n+1)}{n}\right]\xi\varphi'_\xi + \frac{1}{n}\varphi^{n+1} + \frac{n+1}{n^2}\varphi = 0.$$

4°. Generalized separable solution for $n = 1$:

$$w = \varphi(x)\left[t + C_1 + C_2\int \frac{dx}{\varphi^2(x)}\right].$$

Here, the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation $\varphi''_{xx} = \varphi^2$, which has a particular solution $\varphi = 6(x + C)^{-2}$.

⊙ *References for equation* 7.3.5.1: Y. P. Emech and V. B. Taranov (1972), N. H. Ibragimov (1994, pp. 243–244; 1995, pp. 334–337), A. D. Polyanin and V. F. Zaitsev (2002, p. 148).

2. $\dfrac{\partial^2 w}{\partial t^2} + aw^n\dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2}$.

1°. Self-similar solution:

$$w(x, t) = u(z)t^{-1/n}, \quad z = xt^{-1},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$n^2(z^2 - b)u''_{zz} + nz(2n + 2 - nau^n)u'_z + u(1 + n - nau^n) = 0.$$

2°. Passing to the new independent variables $\tau = at$ and $z = ab^{-1/2}x$, we arrive at an equation of the form 7.3.5.1:

$$\frac{\partial^2 w}{\partial \tau^2} + w^n\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} + aw^n\dfrac{\partial w}{\partial t} = b\dfrac{\partial}{\partial x}\left(w^k\dfrac{\partial w}{\partial x}\right)$.

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{2n-k}x + C_2,\ C_1^{2n}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x, t) = u(z)t^{-1/n}, \quad z = xt^{(k-2n)/(2n)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\left[4bn^2u^k - (2n - k)^2z^2\right]u''_{zz} + 4bkn^2u^{k-1}\left(u'_z\right)^2 + (2n - k)(k - 4 - 4n + 2nau^n)zu'_z = 4u(1 + n - anu^n).$$

⊙ *Reference for equation* 7.3.5.3: N. H. Ibragimov (1994, pp. 243–244).

4. $\dfrac{\partial^2 w}{\partial t^2} + e^w \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2,\ C_1 t + C_3) + \ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions:

$$w(x,t) = -\ln\left\{\frac{1}{C_1}\left[\exp\left(\frac{C_1}{a^2-1}(ax-t)+C_2\right)-1\right]\right\},$$
$$w(x,t) = -\ln\left\{\frac{1}{a^2-1}(ax-t)+C_1\right\},$$

where C_1, C_2, and a are arbitrary constants ($a \neq \pm 1$).

3°. Solutions:

$$w(x,t) = -\ln\left(\frac{t+Cx}{1-C^2} + 2a\sqrt{|t^2-x^2|}\left|\frac{x+t}{x-t}\right|^a\right),$$
$$w(x,t) = -\ln\left(\mp\frac{x}{2} + C(x\pm t) + \frac{1}{4}(x\pm t)\ln\left|\frac{x+t}{x-t}\right|\right),$$

where C and a are arbitrary constants.

⊙ *References for equation* 7.3.5.4: Y. P. Emech and V. B. Taranov (1973), N. H. Ibragimov (1994, pp. 243–244; 1995, pp. 334–335).

5. $\dfrac{\partial^2 w}{\partial t^2} + ae^{\lambda w}\dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2}.$

1°. Solution:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \quad z = xt^{-1},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\lambda(z^2-b)u''_{zz} + \lambda z(2-ae^{\lambda u})u'_z + 1 - ae^{\lambda u} = 0.$$

2°. Passing to the new variables $\tau = at$, $z = a\beta^{-1/2}x$, and $u = \lambda w$, we arrive at an equation of the form 7.3.5.4:

$$\frac{\partial^2 u}{\partial \tau^2} + e^u \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2}.$$

6. $\dfrac{\partial^2 w}{\partial t^2} + ae^{\lambda w}\dfrac{\partial w}{\partial t} = b\dfrac{\partial}{\partial x}\left(e^{\mu w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{2\lambda-\mu}x + C_2,\ C_1^{2\lambda}t + C_3) + 2\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \quad z = xt^{(\lambda-2\mu)/(2\mu)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\left[(\mu-2\lambda)^2 z^2 - 4b\lambda^2 e^{\mu u}\right]u''_{zz} - 4b\mu\lambda^2 e^{\mu u}(u'_z)^2 + (\mu-2\lambda)(\mu-4\lambda+2a\lambda e^{\lambda u})zu'_z + 4\lambda(1-ae^{\lambda u}) = 0.$$

⊙ *Reference for equation* 7.3.5.6: N. H. Ibragimov (1994, pp. 243–244).

7.4. Equations Involving Arbitrary Functions

7.4.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \alpha\dfrac{\partial^2 w}{\partial x^2} + f(w)$.

Nonlinear Klein–Gordon equation.

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm t + C_2),$$
$$w_2 = w\big(x\cosh\beta + t\alpha^{1/2}\sinh\beta,\ t\cosh\beta + x\alpha^{-1/2}\sinh\beta\big),$$

where C_1, C_2, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\left[C_1 + \frac{2}{\lambda^2-\alpha k^2}\int f(w)\,dw\right]^{-1/2} dw = kx + \lambda t + C_2, \tag{1}$$

where C_1, C_2, k, and λ are arbitrary constants.

Nesterov (1978) indicated several cases where solution (1) can be written out in explicit form ($\alpha = \mu = 1$):

$$f(w) = -b^2\frac{\tanh w}{\cosh^2 w}, \quad w(z) = \operatorname{arcsinh}\left[\sinh k\,\sin\left(\frac{bz+c}{\cosh k\sqrt{\lambda^2-1}}\right)\right],$$
$$f(w) = -b^2\frac{\tan w}{\cos^2 w}, \quad w(z) = \arcsin\left[\sin k\,\sin\left(\frac{bz+c}{\cos k\sqrt{\lambda^2-1}}\right)\right],$$

where k and c are arbitrary constants. In these cases, the following relationships between the wave speed, λ, and the amplitude, b, correspond to periodic solutions in z with period 2π:

$$\lambda^2 = 1 + b^2\cosh^{-2}k,$$
$$\lambda^2 = 1 + b^2\cos^{-2}k.$$

3°. Functional separable solution:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}\alpha(t+C_1)^2 - \tfrac{1}{4}(x+C_2)^2,$$

where C_1 and C_2 are arbitrary constants, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + w'_\xi - \frac{1}{\alpha} f(w) = 0.$$

4°. For exact solutions of the nonlinear Klein–Gordon equation with $f(w) = bw^m$, $f(w) = be^{\beta w}$, $f(w) = b\sinh(\lambda w)$, $f(w) = bw\ln w$, and $f(w) = b\sin(\lambda w)$, see equations 7.1.1.1, 7.2.1.1, 7.3.1.1, 7.3.2.1, and 7.3.3.1, respectively. For solutions of the original equation with some other $f = f(w)$, see Example 30.10 in Subsection 30.3.2.

⊙ *References for equation* 7.4.1.1: P. A. Clarkson, J. B. McLeod, P. J. Olver, and R. Ramani (1986), A. M. Grundland and E. Infeld (1992), R. Z. Zhdanov (1994), A. D. Polyanin and V. F. Zaitsev (2002, pp. 149–150).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + (x^2 - t^2) f(w)$.

1°. Suppose $w = w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x, \pm t),$$
$$w_2 = w(x\cosh\beta + t\sinh\beta,\ x\sinh\beta + t\cosh\beta),$$

where β is an arbitrary constant, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Functional separable solution:

$$w = w(\xi), \qquad \xi = \tfrac{1}{2}\left(x^2 - t^2\right),$$

where the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + w'_\xi + \xi f(w) = 0.$$

3°. Self-similar solution:

$$w = w(\tau), \qquad \tau = xt.$$

Here, the function $w = w(\tau)$ is determined by the autonomous ordinary differential equation

$$w''_{\tau\tau} = f(w),$$

whose general solution can be represented in implicit form as

$$\int \left[C_1 + 2F(w)\right]^{-1/2} dw = C_2 \pm \tau, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

Remark. This solution is a nonclassical (noninvariant) self-similar solution, since the equation concerned is not invariant to the scaling transformation.

4°. Functional separable solution:

$$w = w(z), \qquad z = \tfrac{1}{2}(x^2 + t^2).$$

Here, the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} + f(w) = 0,$$

whose general solution can be represented in implicit form as

$$\int \left[C_1 - 2F(w)\right]^{-1/2} dw = C_2 \pm z, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

5°. Functional separable solution (generalizes the solution of Items 3° and 4°):

$$w = w(r), \qquad r = C_1x^2 + C_2xt + C_1t^2 + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $w = w(r)$ is determined by the autonomous ordinary differential equation

$$(C_2^2 - 4C_1^2)w''_{rr} = f(w).$$

6°. The transformation

$$w = U(z,\tau), \quad z = \tfrac{1}{2}(x^2 + t^2), \quad \tau = xt$$

leads to a simpler equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + f(U).$$

⊙ *Reference for equation* 7.4.1.2: A. D. Polyanin and V. F. Zaitsev (2002, pp. 150–151).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + (t^2 - x^2)^n f(w), \qquad n = 2, 3, \ldots.$

This is a special case of equation 7.4.1.16 with $f(y) = y^n$ and $g(z) = z^n$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x + bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(a - b^2)w''_{\xi\xi} + f(\xi, w) = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + (x^2 - t^2) f(xt, w).$

1°. Self-similar solution:

$$w = w(\tau), \qquad \tau = xt.$$

Here, the function $w = w(\tau)$ is determined by the ordinary differential equation

$$w''_{\tau\tau} = f(\tau, w).$$

2°. The transformation

$$z = \tfrac{1}{2}(x^2 + t^2), \quad \tau = xt$$

leads to the simpler equation

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial z^2} + f(\tau, w).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{\beta x} f(w).$

The transformation

$$w = U(z,\tau), \quad z = \exp\left(\tfrac{1}{2}\beta x\right)\cosh\left(\tfrac{1}{2}\beta t\right), \quad \tau = \exp\left(\tfrac{1}{2}\beta x\right)\sinh\left(\tfrac{1}{2}\beta t\right)$$

leads to a simpler equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + 4\beta^{-2} f(U).$$

For arbitrary $f = f(U)$, this equation admits a traveling-wave solution $U = U(kz + \lambda\tau)$ and a solution of the form $U = U(z^2 - \tau^2)$.

⊙ *Reference for equation* 7.4.1.6: A. D. Polyanin and V. F. Zaitsev (2002, p. 152).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{\beta t} f(w).$

The transformation

$$w = U(z,\tau), \quad z = \exp\left(\tfrac{1}{2}\beta t\right)\sinh\left(\tfrac{1}{2}\beta x\right), \quad \tau = \exp\left(\tfrac{1}{2}\beta t\right)\cosh\left(\tfrac{1}{2}\beta x\right)$$

leads to a simpler equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + 4\beta^{-2} f(U).$$

For arbitrary $f = f(U)$, this equation admits a traveling-wave solution $U = U(kz + \lambda\tau)$ and a solution of the form $U = U(z^2 - \tau^2)$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 152).

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{ax+bt} f(w).$

1°. There is a solution of the form $w = w(ax + bt)$.

2°. For $b \neq \pm a$, the transformation

$$\xi = ax + bt, \quad \tau = bx + at$$

leads to an equation of the form 7.4.1.6:

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{a^2 - b^2} e^{\xi} f(w).$$

3°. For $b = a$, see equation 7.4.1.13 with $f(z) = e^{az}$, and for $b = -a$, see equation 7.4.1.14 with $f(z) = e^{-az}$.

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + f(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}\varphi_{tt}'' - \big[b\ln\varphi + f(t) + C\big]\varphi &= 0,\\ a\psi_{xx}'' + \big(b\ln\psi - C\big)\psi &= 0,\end{aligned}$$

and C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + \big[f(x) + g(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}\varphi_{tt}'' - \big[b\ln\varphi + g(t) + C\big]\varphi &= 0,\\ a\psi_{xx}'' + \big[b\ln\psi + f(x) - C\big]\psi &= 0,\end{aligned}$$

and C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + \big[bf(t)x + g(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bx}\varphi(t),$$

where the function $\varphi(t)$ is determined by the ordinary differential equation

$$\varphi_{tt}'' = f(t)\varphi\ln\varphi + \big[g(t) + ab^2\big]\varphi.$$

12. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_{xx}'' + f(x)\varphi\ln\varphi + \big[g(x) - b^2\big]\varphi = 0.$$

13. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(w).$

The transformation

$$w = U(\xi,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}(t-x), \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}(t-x),$$

where a is an arbitrary constant, leads to an equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + g(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 153).

14. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t-x)g(w).$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}(t+x) - \frac{1}{2}\int_a^{t-x} f(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}(t+x) + \frac{1}{2}\int_a^{t-x} f(\sigma)\,d\sigma,$$

where a is an arbitrary constant, leads to an equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + g(U).$$

15. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(t-x)e^{\beta w}.$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma,$$

where a and b are arbitrary constants, leads to an equation of the form 7.2.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + e^{\beta U}.$$

16. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(t-x)h(w).$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma,$$

where a and b are arbitrary constants, leads to an equation of the form 7.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + h(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 153).

7.4.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{b}{x}\dfrac{\partial w}{\partial x} + f(w).$

1°. This is a special case of equation 7.4.3.4 with $n = 0$. This equation can be rewritten in the form

$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^m}\frac{\partial}{\partial x}\left(x^m \frac{\partial w}{\partial x}\right) + f(w), \qquad m = \frac{b}{a}.$$

To $m = 1$ and $m = 2$ there correspond nonlinear waves with axial and central symmetry, respectively.

2°. Functional separable solution:

$$w = w(\xi), \quad \xi = \sqrt{ak(t + C)^2 - kx^2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{a + b}{a\xi} w'_\xi = \frac{1}{ak} f(w).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + bw \ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x, t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b \ln \varphi + h(t) + C\big]\varphi = 0,$$
$$a\psi''_{xx} + f(x)\psi'_x + \big[b \ln \psi + g(x) - C\big]\psi = 0,$$

and C is an arbitrary constant.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x}.$

Traveling-wave solution in implicit form:

$$(\lambda^2 - a)\int \frac{dw}{F(w) + C_1} = x + \lambda t + C_2, \qquad F(w) = \int f(w)\, dw,$$

where C_1, C_2, and λ are arbitrary constants. To the stationary solution there correspond $\lambda = 0$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 \cosh(kt) + C_3 \sinh(kt) \quad \text{if } c = k^2 > 0,$$
$$w_2 = w(\pm x + C_1, t) + C_2 \cos(kt) + C_3 \sin(kt) \quad \text{if } c = -k^2 < 0,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = U(t) + \Theta(\xi), \quad \xi = \pm x + \lambda t,$$

where λ is an arbitrary constant, and the functions $U(t)$ and $\Theta(\xi)$ are determined by the ordinary differential equations

$$U''_{tt} - cU - f(t) = 0, \tag{1}$$

$$(a-\lambda^2)\Theta''_{\xi\xi} + b\big(\Theta'_\xi\big)^2 + c\Theta = 0. \tag{2}$$

The solution of equation (1) is given by

$$\begin{aligned} U(t) &= C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad c = k^2 > 0, \\ U(t) &= C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad c = -k^2 < 0, \end{aligned} \tag{3}$$

where C_1 and C_2 are arbitrary constants.

Equation (2) can be solved with the change of variable $z(\Theta) = \big(\Theta'_\xi\big)^2$, which leads to first-order linear equation.

Particular solution of equation (2):

$$\Theta = -\frac{c}{4b}(\xi + C_3)^2 + \frac{1}{2b}(a-\lambda^2).$$

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t), \tag{4}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 4b\varphi^2 + c\varphi, \tag{5}$$

$$\psi''_{tt} = (4b\varphi + c)\psi, \tag{6}$$

$$\chi''_{tt} = c\chi + b\psi^2 + 2a\varphi + f(t). \tag{7}$$

Equation (5) has the trivial particular solution $\varphi(t) \equiv 0$, to which there corresponds a solution of (4) linear in x. Another particular solution to equation (5) is given by $\varphi = -\frac{1}{4}c/b$.

The general solution of the autonomous equation (5) can be represented in implicit form:

$$\int \big(\tfrac{8}{3}b\varphi^3 + c\varphi^2 + C_1\big)^{-1/2} d\varphi = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

The functions $\psi = \psi(t)$ and $\chi = \chi(t)$ can be found by successively integrating equations (6) and (7), which are linear in ψ and χ, respectively.

Note that equation (6) has a particular solution $\psi = \bar{\varphi}(t)$, where $\bar{\varphi}(t)$ is any nontrivial particular solution to (5). Hence, the general solution to (6) is expressed as

$$\psi(t) = C_3\bar{\varphi}(t) + C_4\bar{\varphi}(t)\int \frac{dt}{\bar{\varphi}^2(t)},$$

where C_3 and C_4 are arbitrary constants.

4°. The substitution

$$w = z(x,t) + U(t),$$

where the function $U(t)$ is defined by formula (3), leads to the simpler equation

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial^2 z}{\partial x^2} + b\left(\frac{\partial z}{\partial x}\right)^2 + cz.$$

⊙ *Reference for equation* 7.4.2.4: A. D. Polyanin and V. F. Zaitsev (2002, pp. 154–155).

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(x).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x, \pm t + C_1) + C_2\cosh(kt) + C_3\sinh(kt) \quad \text{if } c = k^2 > 0,$$
$$w_2 = w(x, \pm t + C_1) + C_2\cos(kt) + C_3\sin(kt) \quad \text{if } c = -k^2 < 0,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t).$$

Here,

$$\psi(t) = \begin{cases} C_1\cosh(kt) + C_2\sinh(kt) & \text{if } c = k^2 > 0, \\ C_1\cos(kt) + C_2\sin(kt) & \text{if } c = -k^2 < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b\left(\varphi'_x\right)^2 + c\varphi + f(x) = 0.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = k\varphi^2 + f(t)\varphi + g(t), \tag{1}$$
$$\psi''_{tt} = \left[(c\lambda + 2k)\varphi + f(t) + a\lambda^2\right]\psi. \tag{2}$$

In the special case $f(t) = \text{const}$, $g(t) = \text{const}$, equation (1) has exact solutions of the form $\varphi = \text{const}$ and, due to its autonomy, can be integrated by quadrature. Equation (2) is linear in ψ and, hence, with $\varphi = \text{const}$, the general solution to (6) is expressed in terms of exponentials or sine and cosine.

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t), \tag{1}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients (the arguments of f, g, and h are not specified)

$$\varphi''_{tt} = 4f\varphi^2 + g\varphi, \tag{2}$$
$$\psi''_{tt} = (4f\varphi + g)\psi, \tag{3}$$
$$\chi''_{tt} = g\chi + f\psi^2 + h + 2a\varphi. \tag{4}$$

Equation (2) has the trivial particular solution $\varphi(t) \equiv 0$, to which there corresponds a solution of (1) linear in x.

If a solution $\varphi = \varphi(t)$ of the nonlinear equation (2) has been found, the functions $\psi = \psi(t)$ and $\chi = \chi(t)$ can be obtained by successively solving equations (3) and (4), which are linear in ψ and χ, respectively.

Note that equation (3) has a particular solution $\psi = \bar{\varphi}(t)$, where $\bar{\varphi}(t)$ is any nontrivial particular solution to (2). Hence, the general solution to (3) is expressed as

$$\psi(t) = C_1\bar{\varphi}(t) + C_2\bar{\varphi}(t)\int \frac{dt}{\bar{\varphi}^2(t)},$$

where C_1 and C_2 are arbitrary constants. If the functions f and g are proportional to each other, then a particular solution to equation (2) is given by $\varphi = -\frac{1}{4}g/f = \text{const}$.

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)h(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\big(\varphi'_x\big)^2 + g(x) - A = 0.$$

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - h(t) = 0,$$
$$a\psi''_{xx} + f(x)(\psi'_x)^2 + b\psi + g(x) = 0.$$

The solution of the first equation for $\varphi(t)$ is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t h(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$

$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t h(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions:

$$w(x,t) = \varphi(t) + \psi(t)\exp\left(\pm x\sqrt{-b}\right), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients (the arguments of f, g, and h are not specified)

$$\varphi''_{tt} = bf\varphi^2 + g\varphi + h, \tag{2}$$

$$\psi''_{tt} = (2bf\varphi + g - ab)\psi. \tag{3}$$

If a solution $\varphi = \varphi(t)$ to equation (2) has been found, the function $\psi = \psi(t)$ can be obtained by solving equation (3) linear in ψ.

If the functions f, g, and h are proportional to each other,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

particular solutions to equation (2) are expressed as

$$\varphi = k_1, \quad \varphi = k_2, \tag{4}$$

where k_1 and k_2 are roots of the quadratic equation $bk^2 + \alpha k + \beta = 0$. In this case, equation (3) can be rewritten in the form

$$\psi''_{tt} = \left[(2bk_n + \alpha)f - ab\right]\psi, \qquad n = 1,\ 2. \tag{5}$$

Kamke (1977) and Polyanin and Zaitsev (2003) present many exact solutions of the linear equation (5) for various $f = f(t)$. In the special case $f = \text{const}$, the general solution of equation (5) is the sum of exponentials (or sine and cosine).

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t) = \varphi(t) + \psi(t)\left[A\exp\left(x\sqrt{-b}\right) + B\exp\left(-x\sqrt{-b}\right)\right], \qquad b < 0, \tag{6}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations with variable coefficients

$$\varphi''_{tt} = bf\left(\varphi^2 + 4AB\psi^2\right) + g\varphi + h, \tag{7}$$

$$\psi''_{tt} = \left(2bf\varphi + g - ab\right)\psi. \tag{8}$$

We express φ from (8) in terms of ψ and then substitute into (7) to obtain a nonlinear fourth-order equation for ψ; with f, g, $h = \text{const}$, the equation is autonomous and, hence, its order can be reduced.

Note two special cases of solutions (6) that can be expressed in terms of hyperbolic functions:

$$w(x,t) = \varphi(t) + \psi(t)\cosh\left(x\sqrt{-b}\right), \qquad A = \tfrac{1}{2},\ B = \tfrac{1}{2},$$

$$w(x,t) = \varphi(t) + \psi(t)\sinh\left(x\sqrt{-b}\right), \qquad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos\left(x\sqrt{b} + c\right), \qquad b > 0, \tag{9}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients

$$\varphi''_{tt} = bf\left(\varphi^2 + \psi^2\right) + g\varphi + h, \tag{10}$$

$$\psi''_{tt} = \left(2bf\varphi + g - ab\right)\psi. \tag{11}$$

⊙ *Reference for equation* 7.4.2.10: V. A. Galaktionov (1995, the case of $f = a$, $g = \text{const}$, and $h = \text{const}$ was considered), V. F. Zaitsev and A. D. Polyanin (1996, pp. 467–468).

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[g_1(t)x + g_0(t)\big]\dfrac{\partial w}{\partial x} + h(t)w + p_2(t)x^2 + p_1(t)x + p_0(t).$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

12. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(x)\dfrac{\partial w}{\partial x} + bw + h_1(t) + h_2(x).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t).$$

Here,

$$\psi(t) = \begin{cases} C_1\cosh(kt) + C_2\sinh(kt) + \dfrac{1}{k}\displaystyle\int_0^t \sinh\big[k(t-\tau)\big]h_1(\tau)\,d\tau & \text{if} \quad b = k^2 > 0, \\ C_1\cos(kt) + C_2\sin(kt) + \dfrac{1}{k}\displaystyle\int_0^t \sin\big[k(t-\tau)\big]h_1(\tau)\,d\tau & \text{if} \quad b = -k^2 < 0, \\ C_1 + C_2 t + \displaystyle\int_0^t (t-\tau)h_1(\tau)\,d\tau & \text{if} \quad b = 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\big(\varphi'_x\big)^k + g(x)\varphi'_x + b\varphi + h_2(x) = 0.$$

13. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)g(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f\big(x, \varphi'_x\big) - A = 0.$$

14. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi''_{xx} + f\big(x, \psi'_x\big) + b\psi = 0.$$

The solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$

$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

15. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \big[a\lambda^2 + f(t,\lambda)\big]\varphi.$$

7.4.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = (ax^2 + b)\dfrac{\partial^2 w}{\partial x^2} + ax\dfrac{\partial w}{\partial x} + f(w).$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2 + b}}$ leads to an equation of the form 7.4.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a(x+\beta)^n\dfrac{\partial^2 w}{\partial x^2} + f(w), \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium. For $n = 0$ see equation 7.4.1.1.

1°. The substitution $y = x + \beta$ leads to a special case of equation 7.4.3.4 with $b = 0$.

2°. Functional separable solution for $n \neq 2$:

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(1-n)}{2-n}\frac{1}{r}w'_r = \frac{4}{k}f(w). \tag{1}$$

The substitution $\xi = r^{\frac{n}{2-n}}$ leads to the generalized Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4(2-n)^2}{kn^2}\xi^{\frac{4(1-n)}{n}}f(w). \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents a number of exact solutions to equation (2) for various $f = f(w)$.

Special case. For $n = 1$, the general solution of equation (1) is written out in implicit form as

$$\int \left[C_1 + \frac{8}{k}F(w)\right]^{-1/2} dw = \pm r + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x+\beta|,$$

where A and B are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} - aBw'_y + f(w) = 0. \tag{3}$$

Solution of equation (3) with $A = \pm B\sqrt{a}$ in implicit form:

$$aB\int \frac{dw}{f(w)} = y + C,$$

where C is an arbitrary constant.

⊙ *Reference for equation* 7.4.3.2: V. F. Zaitsev and A. D. Polyanin (1996, pp. 462–463).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a(x+\beta)^n \dfrac{\partial w}{\partial x}\right] + f(w), \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Functional separable solution for $n \neq 2$:

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n}\frac{1}{r}w'_r = \frac{4}{k}f(w).$$

The substitution $\xi = r^{\frac{n}{n-2}}$ leads to the generalized Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4(2-n)^2}{kn^2}\xi^{-\frac{4}{n}} f(w). \tag{1}$$

The book by Polyanin and Zaitsev (2003) presents a number of exact solutions to equation (1) for various $f = f(w)$.

2°. Solution for $n = 2$:

$$w = w(z), \quad z = At + B\ln|x+\beta|,$$

where A and B are arbitrary constants, and the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{zz} + aBw'_z + f(w) = 0. \tag{2}$$

Solution of equation (2) with $A = \pm B\sqrt{a}$ in implicit form:

$$aB\int \frac{dw}{f(w)} = -z + C,$$

where C is an arbitrary constant.

⊙ *Reference for equation* 7.4.3.3: V. F. Zaitsev and A. D. Polyanin (1996, p. 463).

4. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}\dfrac{\partial w}{\partial x} + f(w), \qquad a > 0.$

1°. Functional separable solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}a(2-n)^2(t+C)^2 - x^{2-n}.$$

Here, C is an arbitrary constant, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + Aw'_\xi - Bf(w) = 0, \tag{1}$$

where

$$A = \frac{a(4-3n)+2b}{2a(2-n)}, \quad B = \frac{1}{a(2-n)^2}.$$

For $A \neq 1$, the substitution $\xi = kz^{\frac{1}{1-A}}$ $(k = \pm 1)$ brings (1) to the generalized Emden–Fowler equation

$$w''_{zz} - \frac{kB}{(1-A)^2} z^{\frac{2A-1}{1-A}} f(w) = 0. \tag{2}$$

In the special case $A = \frac{1}{2}$, which corresponds to $b = a(n-1)$, the general solution of equation (2) is expressed as

$$\int \left[C_1 + 8kBF(w)\right]^{-1/2} dw = \pm z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

The books by Polyanin and Zaitsev (1995, 2003) present a number of exact solutions to equation (2) for some $f = f(w)$.

2°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + (b-a)Bw'_y + f(w) = 0. \tag{3}$$

Solutions of equation (3) with $A = \pm B\sqrt{a}$ in implicit form:

$$(b-a)B \int \frac{dw}{f(w)} = -y + C_1.$$

Solutions of equation (3) with $b = a$:

$$\int \left[C_1 + \frac{2}{A^2 - aB^2} F(w)\right]^{-1/2} dw = \pm y + C_2, \qquad F(w) = \int f(w)\,dw.$$

For $A \neq \pm B\sqrt{a}$ and $b \neq a$, the substitution $u(w) = \dfrac{aB^2 - A^2}{B(a-b)} w'_y$ brings (3) to the Abel equation

$$uu'_w - u = \frac{A^2 - aB^2}{B^2(a-b)^2} f(w),$$

whose exact solutions for various $f = f(w)$ can be found in Polyanin and Zaitsev (2003).

5. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x^{n-1} f(w) \dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(C_1^2 x,\ \pm C_1^{2-n} t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w = w(z), \quad z = \big|a(2-n)^2(t+C)^2 - 4x^{2-n}\big|^{1/2},$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{a(2-n)z}\big[a(1-n) + f(w)\big]w'_z = 0. \tag{1}$$

The substitution $u(w) = zw'_z$ brings (1) to a separable first-order equation, the integration of which yields the general solution in implicit form:

$$\int \frac{dw}{anw - 2F(w) + C_1} = \frac{1}{a(2-n)} \ln z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = x|t|^{\frac{2}{n-2}},$$

where the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\left[a\xi^{n-1} - \frac{4}{(n-2)^2}\xi\right]w''_{\xi\xi} + \left[\xi^{n-2} f(w) + \frac{2(n-4)}{(n-2)^2}\right]w'_\xi = 0.$$

4°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + B\big[f(w) - a\big]w'_y = 0,$$

whose solution with $A \neq \pm B\sqrt{a}$ is given by

$$\frac{aB^2 - A^2}{B}\int \frac{dw}{F(w) - aw + C_1} = -y, \qquad F(w) = \int f(w)\,dw.$$

⊙ *Reference for equation* 7.4.3.5: V. F. Zaitsev and A. D. Polyanin (1996, pp. 465–466).

6. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x^{n-1} f(w) \dfrac{\partial w}{\partial x} + g(w).$

1°. Functional separable solution for $n \neq 2$:

$$w = w(z), \quad z = \left[ka(2-n)^2(t+C)^2 - 4kx^{2-n}\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{a(2-n)}\left[a(1-n) + f(w)\right]\frac{1}{z}w'_z - \frac{1}{ak(2-n)^2}g(w) = 0.$$

2°. Solution for $n = 2$:

$$w = w(\xi), \quad \xi = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{\xi\xi} + B\left[f(w) - a\right]w'_\xi + g(w) = 0. \tag{1}$$

Solution of equation (1) with $A = \pm B\sqrt{a}$:

$$B\int \left[f(w) - a\right]\frac{dw}{g(w)} = -\xi + C_1.$$

In the general case, the change of variable $u(w) = w'_\xi$ brings (1) to an Abel equation, whose exact solutions for various $f = f(w)$ and $g = g(w)$ can be found in Polyanin and Zaitsev (2003).

⊙ *Reference for equation* 7.4.3.6: V. F. Zaitsev and A. D. Polyanin (1996, p. 466).

7. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + f(w), \qquad a > 0.$

This is a special case of equation 7.4.3.9 with $b = 0$.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + f(w), \qquad a > 0.$

This is a special case of equation 7.4.3.9 with $b = a\lambda$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x}\dfrac{\partial w}{\partial x} + f(w), \qquad a > 0.$

Functional separable solution:

$$w = w(z), \quad z = \left[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda}\frac{1}{z}w'_z + \frac{1}{ak\lambda^2}f(w) = 0. \tag{1}$$

For $b = a\lambda$, the solution of equation (1) is expressed as

$$\int \left[C_1 - \frac{2}{ak\lambda^2}F(w)\right]^{-1/2} dw = \pm z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

For $b \neq \frac{1}{2}a\lambda$, the substitution $\xi = z^{\frac{2b-a\lambda}{a\lambda}}$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\xi\xi} + \frac{a}{k(2b-a\lambda)^2}\xi^{\frac{4(a\lambda-b)}{2b-a\lambda}} f(w) = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a number of exact solutions to equation (2) for some $f = f(w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + e^{\lambda x} f(w)\dfrac{\partial w}{\partial x}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(x + 2C_1,\ \pm e^{\lambda C_1}t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $\lambda \neq 0$:

$$w = w(z), \quad z = \left|4e^{-\lambda x} - a\lambda^2(t+C)^2\right|^{1/2},$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{z}\left[1 - \frac{1}{a\lambda}f(w)\right]w'_z = 0. \tag{1}$$

The substitution $u(w) = zw'_z$ brings (1) to a separable first-order equation, the integration of which yields the general solution in implicit form:

$$\int \frac{dw}{2F(w) - a\lambda w + C_1} = \frac{1}{a\lambda}\ln z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

3°. Generalized self-similar solution:

$$w = w(\xi), \quad \xi = t^2 e^{\lambda x},$$

where the function $w = w(z)$ is determined by the ordinary differential equation

$$(a\lambda^2\xi^2 - 4\xi)w''_{\xi\xi} + \big[\lambda\xi f(w) + a\lambda^2\xi - 2\big]w'_\xi = 0.$$

4°. For $\lambda = 0$, see equation 7.4.2.3.

⊙ *Reference for equation* 7.4.3.10: V. F. Zaitsev and A. D. Polyanin (1996, pp. 466–467).

11. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + e^{\lambda x} f(w)\dfrac{\partial w}{\partial x} + g(w).$

1°. Functional separable solution for $\lambda \neq 0$:

$$w = w(z), \quad z = \left[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{z}\left[1 - \frac{1}{a\lambda}f(w)\right]w'_z + \frac{1}{ak\lambda^2}g(w) = 0.$$

2°. For $\lambda = 0$, there is a traveling-wave solution: $w = w(\alpha x + \beta t)$.

12. $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + aw\ln w + \big[h(x) + p(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + a\varphi\ln\varphi + \big[C + h(x)\big]\varphi = 0,$$
$$\psi''_{tt} - a\psi\ln\psi + \big[C - p(t)\big]\psi = 0.$$

7.4.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 2[2f(t) + a]\varphi^2 + g(t)\varphi + h_2(t),$$
$$\psi''_{tt} = 2[2f(t) + a]\varphi\psi + g(t)\psi + h_1(t),$$
$$\chi''_{tt} = 2a\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + f(x)w^5.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\big(x, \pm C_1^2 t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Suppose $u = u(x)$ is any nontrivial solution of the second-order linear ordinary differential equation

$$au''_{xx} + f(x)u = 0. \tag{1}$$

The transformation

$$\xi = \int \frac{dx}{u^2}, \quad z = \frac{w}{u}$$

brings the original equation to a simpler equation of the form 7.1.5.5 with $n = 4$:

$$\frac{\partial^2 z}{\partial t^2} = az^4 \frac{\partial^2 z}{\partial \xi^2}.$$

For example, this equation has the following solutions (A, B, C, D, and λ are arbitrary constants):

$$z(\xi, t) = A\xi t + B\xi + Ct + D,$$

$$z(\xi, t) = \lambda^{-1/4}(t + C)^{-1/2}\left[\frac{3\lambda}{4A^2 a} + (A\xi + B)^2\right]^{1/2}.$$

The first solution is degenerate and the second one is a special case of a multiplicative separable solution $z(\xi, t) = f(\xi)g(t)$. There is also a traveling-wave solution, $z = z(\alpha\xi + \beta t)$, and a self-similar solution of the form

$$z = t^k \varphi(\zeta), \quad \zeta = \xi t^{-2k-1},$$

where k is an arbitrary constant.

3°. Multiplicative separable solution:

$$w(x, t) = (\pm 2\lambda t + C)^{-1/2} g(x),$$

where C is an arbitrary constant, and the function $g = g(x)$ is determined by the Yermakov's equation

$$ag''_{xx} + f(x)g - 3\lambda^2 g^{-3} = 0. \tag{2}$$

Given a particular solution $u = u(x)$ of the linear equation (1), the general solution of the nonlinear equation (2) is expressed as (e.g., see Polyanin and Zaitsev, 2003):

$$Ag^2 = \frac{3\lambda^2}{a}u^2 + u^2\left(B + A\int \frac{dx}{u^2}\right)^2,$$

where A and B are arbitrary constants ($A \neq 0$).

⊙ *Reference for equation* 7.4.4.2: V. F. Zaitsev and A. D. Polyanin (1996, pp. 468–469).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + f(x)w^{-1/3}, \quad a > 0.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^3 w\big(x, \pm C_1^{-2}t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Suppose $u = u(x)$ is any nontrivial particular solution of the second-order linear ordinary differential equation

$$au''_{xx} - \tfrac{1}{3}f(x)u = 0. \tag{1}$$

The transformation

$$\xi = \int \frac{dx}{u^2}, \quad z = wu^3 \tag{2}$$

brings the original equation to a simpler equation of the form 7.1.6.4 with $n = -4/3$:

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right).$$

3°. For $f = b = \text{const}$, the auxiliary equation (1) employed to determine the transformation (2) has the following solution:

$$u(x) = \begin{cases} C_1 \exp(\lambda x) + C_2 \exp(-\lambda x) & \text{if } ab > 0, \\ C_1 \cos(\lambda x) + C_2 \sin(\lambda x) & \text{if } ab < 0, \end{cases}$$

where $\lambda = \left|\frac{1}{3}b/a\right|^{1/2}$ and C_1 and C_2 are arbitrary constants.

For $f(x) = bx^m$ or $f(x) = be^{\beta x}$, the solutions of equation (1) are expressed in terms of Bessel functions.

⊙ *Reference for equation* 7.4.4.3: V. F. Zaitsev and A. D. Polyanin (1996, p. 469).

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + f(x)w^{n+1} + g(t)w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a(\varphi^n\varphi'_x)'_x + f(x)\varphi^{n+1} + C\varphi = 0,$$
$$\psi''_{tt} - g(t)\psi + C\psi^{n+1} = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(x)e^{\lambda w} + g(t).$

Additive separable solution:

$$w = \varphi(x) + \psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a(e^{\lambda\varphi}\varphi'_x)'_x + f(x)e^{\lambda\varphi} + C = 0,$$
$$\psi''_{tt} - g(t) + Ce^{\lambda\psi} = 0.$$

By the change of variable $U = e^{\lambda\varphi}$ the first equation is reduced to the linear equation $aU''_{xx} + \lambda f(x)U + \lambda C = 0$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

This equation is encountered in wave and gas dynamics.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$x = \tau, \quad t = z, \quad u = \int f(w)\,dw$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial}{\partial z}\left[g(u)\frac{\partial w}{\partial z}\right],$$

where the function $g = g(u)$ is defined parametrically as

$$u = \int f(w)\,dw, \quad g(u) = \frac{1}{f(w)}.$$

3°. Traveling-wave solution:

$$w = w(z), \quad z = x \pm \lambda t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda^2 w - \int f(w)\,dw = Az + B.$$

4°. Self-similar solution:

$$w = w(\xi), \quad \xi = \frac{x+a}{t+b},$$

where a and b are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$\big(\xi^2 w'_\xi\big)'_\xi = \big[f(w)w'_\xi\big]'_\xi,$$

which admits the first integral

$$\big[\xi^2 - f(w)\big]w'_\xi = C. \tag{1}$$

To the special case $C = 0$ there corresponds the solution (in implicit form):

$$\xi^2 = f(w).$$

For $C \neq 0$, treating w in (1) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C\xi'_w = \xi^2 - f(w). \tag{2}$$

The handbook by Polyanin and Zaitsev (2003) presents a large number of solutions to equation (2) for various $f = f(w)$.

By the change of variable $\xi = -Cy'_w/y$, equation (1) is reduced to the second-order linear equation $y''_{ww} = C^{-2}f(w)y$.

5°. Solutions in implicit form:

$$x + t\sqrt{f(w)} = \varphi(w),$$
$$x - t\sqrt{f(w)} = \psi(w),$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions.

6°. Solution in parametric form:

$$x = C_1 v^2 + C_2 v + \int f(w)(2C_1 w + C_3)\,dw + C_4,$$
$$t = (2C_1 w + C_3)v + C_2 w + C_5.$$

Here and henceforth, $C_1, \ldots, C_5$ are arbitrary constants.

7°. Solution in parametric form:

$$x = [C_1 F(w) + C_2]v + C_3 F(w) + C_4, \quad F(w) = \int f(w)\,dw,$$
$$t = \frac{1}{2}C_1 v^2 + C_3 v + \int [C_1 F(w) + C_2]\,dw + C_5.$$

8°. Solution in parametric form:

$$x = [C_1 F(w) + C_2]v^2 + C_3 F(w) + C_4 + 2\int \left\{ f(w) \int [C_1 F(w) + C_2]\,dw \right\} dw,$$
$$t = \frac{1}{3}C_1 v^3 + C_3 v + 2v \int [C_1 F(w) + C_2]\,dw + C_5.$$

9°. Solution in parametric form:

$$x = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_3,$$
$$t = \frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})\frac{1}{f(w)}H'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[H] - \lambda^2 H = 0$, and the differential operator $\mathbf{L}_f$ is expressed as

$$\mathbf{L}_f[\varphi] \equiv \frac{d}{dw}\left[\frac{1}{f(w)}\frac{d\varphi}{dw}\right]. \tag{3}$$

10°. Solution in parametric form:

$$x = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_3,$$
$$t = \frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]\frac{1}{f(w)}Z'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[Z] + \lambda^2 Z = 0$, and the differential operator $\mathbf{L}_f$ is defined by (3).

11°. Solution in parametric form:

$$x = [2C_1F(w) + C_3]v + C_2F(w) + C_5, \quad F(w) = \int f(w)\,dw,$$
$$t = C_1v^2 + C_2v + \int [2C_1F(w) + C_3]\,dw + C_4.$$

12°. Solution in parametric form:

$$x = \frac{1}{2}C_1v^2 + C_3v + \int f(w)(C_1w + C_2)\,dw + C_5,$$
$$t = (C_1w + C_2)v + C_3w + C_4.$$

13°. Solution in parametric form:

$$x = \frac{1}{3}C_1v^3 + C_3v + 2v\int f(w)(C_1w + C_2)\,dw + C_5,$$
$$t = (C_1w + C_2)v^2 + C_3w + C_4 + 2\int\left\{\int f(w)(C_1w + C_2)\,dw\right\}dw.$$

14°. Solution in parametric form:

$$x = \frac{1}{\lambda}(C_1e^{\lambda v} - C_2e^{-\lambda v})H'_w(w) + C_3,$$
$$t = (C_1e^{\lambda v} + C_2e^{-\lambda v})H(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $H = H(w)$ is determined by the ordinary differential equation $H''_{ww} - \lambda^2 f(w)H = 0$.

15°. Solution in parametric form:

$$x = \frac{1}{\lambda}[C_2\sin(\lambda v) - C_1\cos(\lambda v)]Z'_w(w) + C_3,$$
$$t = [C_1\sin(\lambda v) + C_2\cos(\lambda v)]Z(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $Z = Z(w)$ is determined by the ordinary differential equation $Z''_{ww} + \lambda^2 f(w)Z = 0$.

16°. The original equation can be represented as the system of equations

$$f(w)\frac{\partial w}{\partial x} = \frac{\partial v}{\partial t}, \quad \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x}. \tag{4}$$

The hodograph transformation

$$x = x(w, v), \quad t = t(w, v), \tag{5}$$

where w and v are treated as the independent variables and x and t as the dependent ones, brings (4) to the linear system

$$f(w)\frac{\partial t}{\partial v} = \frac{\partial x}{\partial w}, \quad \frac{\partial x}{\partial v} = \frac{\partial t}{\partial w}. \tag{6}$$

Eliminating t, we obtain a linear equation for $x = x(w, v)$:

$$\frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial x}{\partial w}\right] - \frac{\partial^2 x}{\partial v^2} = 0. \tag{7}$$

Likewise, from system (6) we obtain another linear equation for $t = t(w, v)$:

$$\frac{\partial^2 t}{\partial w^2} - f(w)\frac{\partial^2 t}{\partial v^2} = 0. \tag{8}$$

The procedure for constructing exact solutions of the original nonlinear equation consists of the following two stages. First, one finds an exact solution of the linear equation (7) for $x = x(w, v)$. Further this solution is substituted into the linear system (6), which is then solved to obtain $t = t(w, v)$ in the form

$$t = \int_{v_0}^{v} \frac{1}{f(w)}\frac{\partial x}{\partial w}(w, \xi)\, d\xi + \int_{w_0}^{w} \frac{\partial x}{\partial v}(\eta, v_0)\, d\eta, \tag{9}$$

where w_0 and v_0 are any numbers. The thus obtained expressions of (5) will give an exact solution of the original equation in parametric form.

Likewise, one can first construct an exact solution to the linear equation (8) for $t = t(w, v)$ and then determine $x = x(w, v)$ from (6).

17°. Solutions of equation (7) with even powers of v:

$$x = \sum_{k=0}^{n} \varphi_k(w) v^{2k}, \tag{10}$$

where the functions $\varphi_k = \varphi_k(w)$ are determined by the recurrence relations

$$\varphi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\varphi_{k-1}(w) = A_k F(w) + B_k + 2k(2k-1)\int f(w)\left\{\int \varphi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

The dependence $t = t(w, v)$ is defined by (9) and, together with (10), gives a solution of the original nonlinear equation in parametric form.

18°. Solutions of equation (7) with odd powers of v:

$$x = \sum_{k=0}^{n} \psi_k(w) v^{2k+1}, \tag{11}$$

where the functions $\psi_k = \psi_k(w)$ are determined by the recurrence relations

$$\psi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\psi_{k-1}(w) = A_k F(w) + B_k + 2k(2k+1)\int f(w)\left\{\int \psi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

The dependence $t = t(w, v)$ is defined by (9) and, together with (11), gives a solution of the original nonlinear equation in parametric form.

⊙ *References for equation* 7.4.4.6: W. F. Ames, R. J. Lohner, and E. Adams (1981), N. H. Ibragimov (1994, pp. 208–211), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002, pp. 163–166).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w).$

1°. Traveling-wave solution:

$$w = w(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$\{[k^2 f(w) - \lambda^2]w'_z\}'_z + g(w) = 0.$$

The substitution $U(w) = [k^2 f(w) - \lambda^2]w'_z$ leads to the first-order separable equation

$$UU'_w = -[k^2 f(w) - \lambda^2]g(w).$$

2°. Suppose the function $f = f(w)$ is arbitrary and the function $g = g(w)$ is defined as

$$g(w) = -a^2\frac{f'(w)}{f^3(w)} + b,$$

where a and b are some numbers. Then there is a functional separable solution that can be written in implicit form

$$\int f(w)\,dw = at - \frac{1}{2}bx^2 + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Suppose $f(w)$ and $g(w)$ are defined parametrically as

$$f = \frac{Ae^{bz}}{\varphi'_z(z)},$$
$$g = a^2\varphi''_{zz}(z) + Ace^{bz},$$
$$w = \varphi(z),$$

where $\varphi(z)$ is an arbitrary function, z is the parameter, and A, a, b, and c are some numbers. Then the equation has an exact solution

$$w = \varphi(z), \quad z = at + \theta(x),$$

where

$$\theta = \frac{1}{b}\ln\left|C_1\cos\left(\sqrt{bc}\,x\right) + C_2\sin\left(\sqrt{bc}\,x\right)\right| \qquad \text{if } bc > 0,$$
$$\theta = \frac{1}{b}\ln\left|C_1\exp\left(\sqrt{|bc|}\,x\right) + C_2\exp\left(-\sqrt{|bc|}\,x\right)\right| \qquad \text{if } bc < 0,$$
$$\theta = \frac{1}{b}\ln|x + C_1| + C_2 \qquad \text{if } c = 0,$$
$$\theta = \tfrac{1}{2}cx^2 + C_1 x + C_2 \qquad \text{if } b = 0,$$

with C_1 and C_2 being arbitrary constants.

⊙ *Reference for equation* 7.4.4.7: A. D. Polyanin and V. F. Zaitsev (2004, p. 256).

8. $\dfrac{\partial^2 w}{\partial t^2} = f(w)\dfrac{\partial^2 w}{\partial x^2} + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w) - \lambda^2]w''_{zz} + g(w)w'_z + h(w) = 0.$$

By the change variable $u(w) = w'_z$ this equation is reduced to the Abel equation

$$[f(w) - \lambda^2]uu'_w + g(w)u + h(w) = 0. \tag{1}$$

The substitution $\xi = -\displaystyle\int \frac{g(w)\,dw}{f(w) - \lambda^2}$ brings (1) to the canonical form

$$uu'_\xi - u = F(\xi), \tag{2}$$

where the function $F = F(\xi)$ is defined parametrically as

$$F(\xi) = \frac{h(w)}{g(w)}, \qquad \xi = -\int \frac{g(w)\,dw}{f(w) - \lambda^2}.$$

A large number of exact solutions to the Abel equation (2) for various $F = F(\xi)$ can be found in Polyanin and Zaitsev (2003).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$\{[f(w) - \lambda^2]w'_z\}'_z + g(w)w'_z + h(w) = 0.$$

With the change of variable

$$u(w) = [f(w) - \lambda^2]w'_z,$$

this equation is reduced to the Abel equation

$$uu'_w + g(w)u + h(w)[f(w) - \lambda^2] = 0. \tag{1}$$

The substitution $\xi = -\displaystyle\int g(w)\,dw$ brings (1) to the canonical form

$$uu'_\xi - u = F(\xi), \tag{2}$$

where the function $F = F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{h(w)}{g(w)}[f(w) - \lambda^2], \qquad \xi = -\int g(w)\,dw.$$

A large number of exact solutions to the Abel equation (2) for various $F = F(\xi)$ can be found in Polyanin and Zaitsev (2003).

10. $\dfrac{\partial^2 w}{\partial t^2} = f(w)\dfrac{\partial^2 w}{\partial x^2} + a f'_w(w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

Equations of this form are encountered in the theory of liquid crystals and other applications.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. In the general case, the equation has exact solutions of the form

$$w(x,t) = w(z), \quad z = kx + \lambda t \quad \text{(traveling-wave solution)},$$
$$w(x,t) = w(\xi), \quad \xi = \frac{x+b}{t+c} \quad \text{(self-similar solution)},$$

where k, λ, b, and c are arbitrary constants.

3°. The structure of other exact solutions for some specific $f(w)$:

$$f(w) = Aw + B, \qquad w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t);$$
$$f(w) = Aw^k, \qquad w(x,t) = \varphi(x)\psi(t);$$
$$f(w) = Ae^{\beta w}, \qquad w(x,t) = \varphi(x) + \psi(t).$$

4°. A qualitative analysis of the structure of solutions to the original equation was undertaken in Glassey, Hunter, and Zheng (1997) and Melikyan (1998).

7.4.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x,w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = f(x)w^m\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\big(x, \pm C_1^m t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = g(t)h(x),$$

where the functions $g = g(t)$ and $h = h(x)$ are determined by the ordinary differential equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{1}$$
$$h''_{xx} - \lambda\big[f(x)\big]^{-1}h^{1-m} = 0, \tag{2}$$

where λ is an arbitrary constant.

The general solution of equation (1) is written out in implicit form:

$$\int \left(C_1 + \frac{2\lambda}{m+2} g^{m+2} \right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, if $C_1 = 0$, it follows that

$$g(t) = (at + C)^{-2/m}, \qquad a = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

For $m = 1$, the general solution of equation (2) is expressed as

$$h(x) = \lambda \int_{x_0}^{x} \frac{(x-\xi)}{f(\xi)} d\xi + Ax + B,$$

where A, B, and x_0 are arbitrary constants.

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of exact solutions to the generalized Emden–Fowler equation (2) for various $f = f(x)$.

3°. The transformation

$$u(z,t) = \frac{1}{x} w(x,t), \quad z = \frac{1}{x}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = z^{4-m} f\left(\frac{1}{z}\right) u^m \frac{\partial^2 u}{\partial z^2}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x) w^m \dfrac{\partial w}{\partial x} \right]$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\big(x, \pm C_1^m t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = g(t)h(x),$$

where the functions $g = g(t)$ and $h = h(x)$ are determined by the ordinary differential equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{1}$$
$$[f(x)h^m h'_x]'_x - \lambda h = 0, \tag{2}$$

and λ is an arbitrary constant.

The general solution of equation (1) is written out in implicit form:

$$\int \left(C_1 + \frac{2\lambda}{m+2} g^{m+2} \right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, if $C_1 = 0$, it follows that

$$g(t) = (at + C)^{-2/m}, \qquad a = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

The transformation

$$z = \int \frac{dx}{f(x)}, \quad \Phi = h^{m+1}$$

brings (2) to the generalized Emden–Fowler equation

$$\Phi''_{zz} - F(z)\Phi^{\frac{1}{m+1}} = 0, \tag{3}$$

where the function $F = F(z)$ is defined parametrically by

$$F = \lambda(m+1)f(x), \qquad z = \int \frac{dx}{f(x)}.$$

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of exact solutions to equation (2) for various $F = F(z)$.

3°. The transformation

$$w(x,t) = \big[\psi(x)\big]^{\frac{1}{m+1}} u(\xi,t), \quad \xi = \int \big[\psi(x)\big]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)},$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[\mathcal{F}(\xi)u^m \frac{\partial u}{\partial \xi}\right],$$

where the function $\mathcal{F} = \mathcal{F}(\xi)$ is defined parametrically by

$$\mathcal{F} = f(x)\big[\psi(x)\big]^{\frac{3m+4}{m+1}}, \quad \xi = \int \big[\psi(x)\big]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = w^4 f\left(\dfrac{w}{x}\right)\dfrac{\partial^2 w}{\partial x^2}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\big(C_1^{-1}x, \pm C_1 t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. With the transformation

$$u(z,t) = \frac{1}{x}w(x,t), \quad z = \frac{1}{x}$$

one arrives at the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = u^4 f(u)\frac{\partial^2 u}{\partial z^2},$$

which has a traveling-wave solution $u = u(kz + \lambda t)$ and self-similar solutions of the form $u = u(z/t)$.

4. $\dfrac{\partial^2 w}{\partial t^2} = w^4 f\left(\dfrac{w}{\sqrt{ax^2+bx+c}}\right)\dfrac{\partial^2 w}{\partial x^2}.$

The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

leads to an equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = u^4 f(u)\frac{\partial^2 u}{\partial z^2} + (ac - \tfrac{1}{4}b^2)u^5 f(u),$$

which has a traveling-wave solution $u = u(kz + \lambda t)$.

7.4.6. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(t,w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w\big(C_1 x + C_2,\ t\big),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solutions:

$$w(x,t) = (C_1 t + C_2)(C_3 x + C_4)^{1/2},$$
$$w(x,t) = (C_1 t + C_2)x + \int_a^t (t-\tau)(C_1\tau + C_2)^2 f(\tau)\,d\tau + C_3 t + C_4,$$

where $C_1, \ldots, C_4$ and a are arbitrary constants.

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6f(t)\varphi^2,$$
$$\psi''_{tt} = 6f(t)\varphi\psi,$$
$$\chi''_{tt} = 2f(t)\varphi\chi + f(t)\psi^2.$$

4°. Multiplicative separable solution:

$$w(x,t) = \Phi(t)\Psi(x),$$

where the functions $\Phi = \Phi(t)$ and $\Psi = \Psi(x)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$\Phi''_{tt} = Cf(t)\Phi^2,$$
$$(\Psi\Psi'_x)'_x = C\Psi.$$

The last equation is autonomous and has a particular solution $\Psi = \frac{1}{6}Cx^2$; in the general case, it is integrable by quadrature.

2. $\dfrac{\partial^2 w}{\partial t^2} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi''_{tt} &= 6f(t)\varphi^2 + g(t)\varphi + h_2(t),\\
\psi''_{tt} &= 6f(t)\varphi\psi + g(t)\psi + h_1(t),\\
\chi''_{tt} &= 2f(t)\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).
\end{aligned}$$

3. $\dfrac{\partial^2 w}{\partial t^2} = [a(t)w + b(t)]\dfrac{\partial^2 w}{\partial x^2} + c(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + d(t)w + e(t)x^2 + f(t)x + g(t).$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

⊙ *Reference*: V. A. Galaktionov (1995); the case of $a = 1$, $b = e = f = 0$, and $c = \text{const}$ was considered.

7.4.7. Other Equations Linear in the Highest Derivatives

1. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(C_1x + C_2,\ t\big) - \frac{2}{\lambda}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(C_1x + C_2) + C_3\int F(t)\,dt + C_4, \qquad F(t) = \exp\left[-\int f(t)\,dt\right],$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(\lambda C_1x^2 + C_2x + C_3) + u(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + f(t)u'_t = 2C_1g(t)e^{\lambda u}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{f(w)}{t}\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + g(w).$

Solution:

$$w = w(\xi), \quad \xi = (x + C)^2 - at^2,$$

where C is an arbitrary constant and the function $w(\xi)$ is determined by the ordinary differential equation

$$4a\xi w''_{\xi\xi} + 2a[f(w) + 2]w'_\xi + g(w) = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} + f(w)\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[g(w)\dfrac{\partial w}{\partial x}\right].$

1°. Traveling-wave solution in implicit form:

$$\int \frac{k^2 g(w) - \lambda^2}{\lambda F(w) + C_1}\,dw = kx + \lambda t + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1, C_2, k, and λ are arbitrary constants.

2°. For exact solutions of this equation for some specific $f(w)$ and $g(w)$, see Baikov, Gazizov, and Ibragimov (1989) and Ibragimov (1994).

4. $\left[a + b\left(\dfrac{\partial w}{\partial t}\right)^k\right]\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2}.$

This equation arises in nonlinear optics and is a special case of equation 7.4.7.10, in which the variables x and t must be renamed.

1°. Additive separable solution:

$$w = C_1 x + C_2 x^2 + \varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\begin{aligned} &a\varphi'_t + \frac{b}{k+1}(\varphi'_t)^{k+1} = 2C_2 t + C_3 \quad \text{if} \quad k \neq -1,\\ &a\varphi'_t + \ln|\varphi'_t| = 2C_2 t + C_3 \qquad\quad \text{if} \quad k = -1. \end{aligned}$$

Here, C_1, C_2, and C_3 are arbitrary constants. These equations can be integrated; see Item 2° with $\lambda = 0$.

2°. Solution of the more general form

$$w = C_1 x + C_2 x^2 + \varphi(z), \qquad z = t + \lambda x,$$

where C_1, C_2, and k are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation $2C_2 = \left[b(\varphi'_z)^k + a - \lambda^2\right]\varphi''_{zz}$. Its general solution can be represented in parametric form

$$z = \frac{b}{2C_2(k+1)}\xi^{k+1} + \frac{a-\lambda^2}{2C_2}\xi + C_3, \qquad \varphi = \frac{b}{2C_2(k+2)}\xi^{k+2} + \frac{a-\lambda^2}{4C_2}\xi^2 + C_4,$$

where C_2 and C_4 are arbitrary constants ($k \neq -1, -2$). If $k \neq -1$ or $k = -2$, the term $\frac{1}{k+1}\xi^{k+1}$ or, respectively, $\frac{1}{k+2}\xi^{k+2}$ in the solution must be replaced by $\ln|\xi|$.

3°. Self-similar solution:

$$w = t\psi(y), \quad y = t/x,$$

where the function $\psi = \psi(y)$ is determined by the ordinary differential equation

$$[a + b(y\psi'_y + \psi)^k - y^2](y\psi''_{yy} + 2\psi'_y) = 0.$$

Equating the expression in square brackets to zero, we have

$$a + b(y\psi'_y + \psi)^k - y^2 = 0.$$

The general solution of this equation is

$$\psi = \frac{C}{y} + \frac{1}{b^{1/k}y}\int (y^2 - a)^{1/k}\,dy,$$

where C is an arbitrary constant.

4°. The Legendre transformation

$$u(\xi, \eta) = x\xi + t\eta - w(x, t), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial t},$$

where u is the new dependent variable, and ξ and η are the new independent variables, leads to the linear equation

$$(a + b\eta^k)\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{\partial \eta^2}.$$

5°. Generalized separable solution for $k = 2$:

$$w = \varphi_0(x) + \varphi_1(x)t + \varphi_2(x)t^2,$$

where the functions $\varphi_n(x)$ are determined by the system of ordinary differential equations:

$$\varphi''_0 = 2\varphi_2(a + b\varphi_1^2),$$
$$\varphi''_1 = 8b\varphi_1\varphi_2^2,$$
$$\varphi''_2 = 8b\varphi_2^3.$$

⊙ *References for equation* 7.4.7.4: G. Webb, M. P. Sorensen, M. Brio, A. R. Zakharian, and J. V. Moloney (2004), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 285–286).

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = w(x, t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } b = k^2 > 0,$$
$$w_2 = w(x, t) + C_1\cos(kt) + C_2\sin(kt) \quad \text{if } b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi''_{xx} + f\left(x, \psi'_x\right) + b\psi = 0.$$

The solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

Special case. For $f(x, w_x) = f(w_x)$, there are more complicated solutions of the form $w(x,t) = \varphi(t) + \psi(z)$, where $z = x + \lambda t$.

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g\left(t, \dfrac{\partial w}{\partial t}\right).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + f\left(x, \varphi'_x\right) = C,$$
$$\psi''_{tt} - g\left(t, \psi'_t\right) = C.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g\left(t, \dfrac{\partial w}{\partial t}\right) + bw.$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + f\left(x, \varphi'_x\right) + b\varphi = C,$$
$$\psi''_{tt} - g\left(t, \psi'_t\right) - b\psi = C.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + wg\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}\right) + bw\ln w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + \varphi f\left(x, \varphi'_x/\varphi\right) + b\varphi\ln\varphi + C\varphi = 0,$$
$$\psi''_{tt} - \psi g\left(t, \psi'_t/\psi\right) - b\psi\ln\psi + C\psi = 0.$$

9. $\dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = F\left(x, t, \dfrac{\partial w}{\partial t} + a\dfrac{\partial w}{\partial x}\right).$

The transformation

$$\xi = x + at, \quad \eta = x - at, \quad w = w(\xi, \eta)$$

leads to the equation

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = -\frac{1}{4a^2} F\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2a}, 2a\frac{\partial w}{\partial \xi}\right),$$

whose order can be reduced with the substitution $u = w_\xi$:

$$\frac{\partial u}{\partial \eta} = -\frac{1}{4a^2} F\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2a}, 2au\right).$$

Special case. For $F(x, t, U) = F(U)$, the general solution of the last equation can be represented in implicit form:

$$\int \frac{du}{F(2au)} = -\frac{1}{4a^2}\eta + \varphi(\xi),$$

where $\varphi(\xi)$ is an arbitrary function.

10. $\dfrac{\partial^2 w}{\partial t^2} = f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

Model equation for dynamics of nonlinear strings (Zabusky, 1962). For $f(z) = -z$, this equation is encountered in aerodynamics (theory of transonic gas flows).

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 t + C_3) + C_4 t + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x, t) = Axt + Bx + Ct + D,$$

where A, B, C, and D are arbitrary constants.

3°. Additive separable solution:

$$w(x, t) = At^2 + Bt + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $2A = f(\varphi'_x)\varphi''_{xx}$. Its general solution can be represented in parametric form

$$x = \frac{1}{2A}\int f(\xi)\, d\xi + C_1, \quad \varphi = \frac{1}{2A}\int \xi f(\xi)\, d\xi + C_2,$$

where C_1 and C_2 are arbitrary constants.

4°. Solution of the more general form

$$w(x,t) = At^2 + Bt + \varphi(z), \qquad z = x + \lambda t,$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation $2A = \left[f(\varphi'_z) - \lambda^2\right]\varphi''_{zz}$. Its general solution can be represented in parametric form

$$z = \frac{1}{2A}\int f(\xi)\,d\xi - \frac{\lambda^2}{2A}\xi + C_1, \qquad \varphi = \frac{1}{2A}\int \xi f(\xi)\,d\xi - \frac{\lambda^2}{4A}\xi^2 + C_2,$$

where C_1 and C_2 are arbitrary constants.

5°. Self-similar solution:

$$w = x\psi(y), \qquad y = x/t,$$

where the function $\psi = \psi(y)$ is determined by the ordinary differential equation

$$[f(y\psi'_y + \psi) - y^2](y\psi''_{yy} + 2\psi'_y) = 0.$$

Equating the expression in square brackets to zero, we have

$$f(y\psi'_y + \psi) - y^2 = 0.$$

The general solution of this equation in parametric form:

$$y = \pm\sqrt{f(\tau)}, \qquad \psi = \frac{1}{2\sqrt{f(\tau)}}\int \frac{\tau f'_\tau(\tau)}{\sqrt{f(\tau)}}\,d\tau + C.$$

6°. The Legendre transformation

$$u(z,\tau) = tz + x\tau - w(x,t), \qquad z = \frac{\partial w}{\partial t}, \qquad \tau = \frac{\partial w}{\partial x},$$

where u is the new dependent variable, and z and τ are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial \tau^2} = f(\tau)\frac{\partial^2 u}{\partial z^2}.$$

7°. The substitution $v(x,t) = \dfrac{\partial w}{\partial x}$ leads to an equation of the form 7.4.4.6:

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left[f(v)\frac{\partial v}{\partial x}\right].$$

8°. Below are exact solutions of the equation for some specific $f = f(U)$.

Special case 1. Let $f(U) = aU$.

1. Generalized separable solution quadratic in x:

$$w = (C_1t + C_2)x^2 + \left[\tfrac{1}{3}aC_1^{-2}(C_1t + C_2)^4 + C_3t + C_4\right]x + \tfrac{1}{63}a^2C_1^{-4}(C_1t + C_2)^7 + \tfrac{1}{6}aC_1C_3t^4 + \tfrac{1}{3}a(C_1C_4 + C_2C_3)t^3 + aC_2C_4t^2 + C_5t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

2. Generalized separable solution cubic in x:

$$w = f(t)x^3 + g(t)x^2 + h(t)x + p(t),$$

where the functions $f = f(t)$, $g = g(t)$, $h = h(t)$, $p = p(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 18af^2,$$
$$g''_{tt} = 18afg,$$
$$h''_{tt} = 6afh + 4ag^2,$$
$$p''_{tt} = 2agh.$$

A particular solution of the system of the first three equations is given by

$$f = \frac{1}{3a(t + C_1)^2}, \quad g = \frac{C_2}{(t + C_1)^2} + C_3(t + C_1)^3,$$
$$h = \frac{C_4}{t + C_1} + C_5(t + C_1)^2 + \frac{aC_2^2}{(t + C_1)^2} + 2aC_2C_3(t + C_1)^3 + \frac{2aC_3^2}{27}(t + C_1)^8,$$

where $C_1, \ldots, C_5$ are arbitrary constants. The function $p = p(t)$ is determined from the last equation by integrating the right-hand side twice.

3. There is the solution in multiplicative separable form: $w = \varphi(x)\psi(t)$.

4. For other solutions of the original equation, see equation 9.1.6.10, where y and a must be renamed t and $-a$, respectively.

Special case 2. Let $f(U) = aU^k$.

1+. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the autonomous ordinary differential equations

$$(\psi'_t)^2 = \frac{2aC_1}{k + 2}\psi^{k+2} + C_2, \qquad \frac{2}{k + 2}(\varphi'_x)^{k+2} = C_1\varphi^2 + C_3,$$

and C_1, C_2, and C_3 are arbitrary constants. The general solutions to these equations can be written out in implicit form. Below are exact solutions representable in explicit form:

$$\psi(t) = A_1t^{-2/k} \quad \text{if} \quad C_2 = 0, \qquad \varphi(x) = A_2x^{(k+2)/k} \quad \text{if} \quad C_3 = 0.$$

The coefficients A_1 and A_2 are determined by substituting these expressions into the above equations.

2+. Self-similar solution:

$$w = t^\sigma u(\zeta), \quad \zeta = t^\beta x, \quad \sigma = -(k\beta + 2\beta + 2)/k,$$

where β is an arbitrary constant, and the function $u(\zeta)$ is determined by the ordinary differential equation

$$\sigma(\sigma - 1)u + \beta(2\sigma + \beta - 1)\zeta u'_\zeta + \beta^2\zeta^2u''_{\zeta\zeta} = a(u'_\zeta)^k u''_{\zeta\zeta}.$$

3. Conservation laws for $a = 1$:

$$D_t\left(w_t\right) + D_x\left(-\frac{1}{k + 1}w_x^{k+1}\right) = 0,$$
$$D_t\left(w_tw_x\right) + D_x\left(-\frac{1}{k + 2}w_x^{k+2} - \frac{1}{2}w_t^2\right) = 0,$$
$$D_t\left(\frac{1}{2}w_t^2 + \frac{1}{(k + 1)(k + 2)}w_x^{k+2}\right) + D_x\left(-\frac{1}{k + 1}w_tw_x^{k+1}\right) = 0,$$
$$D_t\left(tw_t - w\right) + D_x\left(-\frac{1}{k + 1}tw_x^{k+1}\right) = 0,$$

where $D_t = \frac{\partial}{\partial t}$ and $D_x = \frac{\partial}{\partial x}$.

Special case 3. Let $f(U) = a\exp(\lambda U)$.

Generalized separable solution:

$$w = (x + C_1)\varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} = aC_2\exp(\lambda\varphi), \tag{1}$$

$$\exp(\lambda\psi'_x)\psi''_{xx} = C_2(x + C_1), \tag{2}$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (1) is given by

$$\varphi(t) = -\frac{1}{\lambda}\ln\left[\frac{aC_2\lambda}{2\beta^2}\cos^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda > 0,$$

$$\varphi(t) = -\frac{1}{\lambda}\ln\left[\frac{aC_2\lambda}{2\beta^2}\sinh^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda > 0,$$

$$\varphi(t) = -\frac{1}{\lambda}\ln\left[-\frac{aC_2\lambda}{2\beta^2}\cosh^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda < 0,$$

where C_3 and β are arbitrary constants. The general solution of equation (2) is expressed as

$$\psi(x) = \int \ln\left(\tfrac{1}{2}C_2x^2 + C_1C_2x + C_4\right)dx + \frac{\ln\lambda}{\lambda}x + C_5,$$

where C_4 and C_5 are arbitrary constants.

⊙ *References for equation* 7.4.7.10: V. A. Vinokurov and I. G. Nurgalieva (1985), N. H. Ibragimov (1994, pp. 212–218), A. D. Polyanin and V. F. Zaitsev (2002, pp. 172–173; 2004, pp. 262–264).

11. $a\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial w}{\partial t} = f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}, \qquad a \neq 0.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1(x, t) = w(t + C_1, x + C_2) + C_3e^{-t/a} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x, t) = U(z) + C_1e^{-t/a} + C_2, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$\lambda U'_z = \left[f(U'_z) - a\lambda^2\right]U''_{zz}.$$

Integrating yields its solution in parametric form:

$$U = \frac{1}{\lambda}\int f(\tau)\,d\tau - a\lambda\tau + C_3, \qquad z = \frac{1}{\lambda}\int \frac{f(\tau)}{\tau}\,d\tau - a\lambda\ln|\tau| + C_4,$$

where C_3 and C_4 are arbitrary constants (C_3 can be set equal to zero).

3°. Additive separable solution:

$$w(x, t) = C_1t + C_2 + C_3e^{-t/a} + \varphi(x),$$

where C_1, C_2, and C_3 are arbitrary constants, the function $\varphi(x)$ is determined by the autonomous ordinary differential equation $f(\varphi'_x)\varphi''_{xx} = C_1$. Integrating yields its solution in parametric form:

$$\varphi = \frac{1}{C_1}\int \xi f(\xi)\,d\xi + C_4, \qquad x = \frac{1}{C_1}\int f(\xi)\,d\xi + C_5,$$

where C_4 and C_5 are arbitrary constants (C_4 can be set equal to zero).

4°. The solutions of Items 2° and 3° are special cases of the more general solution

$$w(x,t) = C_1 t + C_2 + C_3 e^{-t/a} + \varphi(z), \qquad z = x + \lambda t,$$

where the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$\lambda \varphi_z' + C_1 = \left[f(\varphi_z') - a\lambda^2 \right] \varphi_{zz}''.$$

5°. The contact transformation

$$\bar{t} = t + a \ln|w_x|, \quad \bar{x} = w + a w_t, \quad \bar{w} = x + a w_t / w_x, \quad \bar{w}_{\bar{x}} = 1/w_x, \quad \bar{w}_{\bar{t}} = -w_t/w_x \quad (1)$$

leads to an equation of the similar form

$$a \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial \bar{w}}{\partial \bar{t}} = F\left(\frac{\partial \bar{w}}{\partial \bar{x}} \right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}, \qquad \text{where} \qquad F(u) = \frac{1}{u^2} f\left(\frac{1}{u} \right).$$

Transformation (1) has an inverse; it is given by

$$t = \bar{t} + a \ln|\bar{w}_{\bar{x}}|, \quad x = \bar{w} + a \bar{w}_{\bar{t}}, \quad w = \bar{x} + a \bar{w}_{\bar{t}} / \bar{w}_{\bar{x}}, \quad w_x = 1/\bar{w}_{\bar{x}}, \quad w_t = -\bar{w}_{\bar{t}} / \bar{w}_{\bar{x}}. \quad (2)$$

The formulas of (2) can be used if the Jacobian function

$$J = \left[(\bar{w}_{\bar{x}} + a \bar{w}_{\bar{x}\bar{t}})^2 - a \bar{w}_{\bar{x}\bar{x}} (\bar{w}_{\bar{t}} + a \bar{w}_{\bar{t}\bar{t}}) \right]$$

is nonzero.

Special case 1. For $f(w_x) = b(w_x)^{-2}$, transformation (1) leads to the linear telegraph equation

$$a \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial \bar{w}}{\partial \bar{t}} = b \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}.$$

6°. Conservation laws:

$$D_t\big(a w_t + w\big) + D_x\big[-\Psi'(w_x)\big] = 0,$$
$$D_t\big(a e^{t/a} w_t\big) + D_x\big[-e^{t/a} \Psi'(w_x)\big] = 0,$$
$$D_t\big(a e^{t/a} w_t w_x\big) + D_x\big\{ e^{t/a} \big[\Psi(w_x) - w_x \Psi'(w_x) - \tfrac{1}{2} a (w_t)^2 \big] \big\} = 0,$$

where the prime stands for the differentiation,

$$D_t = \frac{\partial}{\partial t}, \qquad D_x = \frac{\partial}{\partial x}, \qquad \Psi(u) = \int_0^u (u - \zeta) f(\zeta)\, d\zeta + C_1 u + C_2,$$

and C_1 and C_2 are arbitrary constants.

Special case 2. For $f(w_x) = b w_x^n$ $(n \neq 0, -2)$, there is an additional conservation law:

$$D_t\left\{ a e^{t/a} \left[\frac{3n+4}{2} a w_t^2 + w_t\big((n+2)w - n x w_x\big) + (3n+4)\Psi \right] \right\}$$
$$+ D_t\left\{ e^{t/a} \left[\frac{n}{2} a x w_t^2 + \frac{d\Psi}{dw_x} \big(n x w_x - a(3n+4) w_t - (n+2) w \big) - n x \Psi + \Phi \right] \right\} = 0,$$

where

$$\Psi = \frac{b}{(n+1)(n+2)} w_x^{n+2}, \quad \Phi = 0 \qquad \text{if } n \neq -1;$$
$$\Psi = b w_x (\ln|w_x| - 1), \qquad \Phi = 2bw \quad \text{if } n = -1.$$

Special case 3. For $f(w_x) = b e^{k w_x}$, there is an additional conservation law $(k \neq 0)$:

$$D_t\left\{ a e^{t/a} \left[\frac{3}{2} a k w_t^2 + w_t(k w + 2x - k x w_x) + \frac{3b}{k} e^{k w_x} \right] \right\}$$
$$+ D_x\left\{ e^{t/a} \left[\frac{1}{2} a k x w_t^2 - b \left(w + \frac{3}{k} x - x w_x + 3 a w_t \right) e^{k w_x} \right] \right\} = 0.$$

⊙ *References for equation* 7.4.7.11: S. R. Svirshchevskii (1986, 1988), N. H. Ibragimov (1994, pp. 165–166, 168), A. D. Polyanin and V. F. Zaitsev (2004, pp. 264–265).

12. $\left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial t^2} = \left(\dfrac{\partial w}{\partial t}\right)^2 \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(C_2 x + C_3, C_4 t + C_5) + C_6,$$

where C_1, C_2, C_3, C_4, C_5, and C_6 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w = \Phi(kx + \lambda t),$$
$$w = \Psi(xt),$$

where $\Phi(z)$ and $\Psi(\xi)$ are arbitrary functions and k and λ are arbitrary constants.

3°. General solution:

$$w = F(u), \quad u = \frac{(xZ - G(Z) + t)^2}{Z},$$

where $F(u)$ and $G(Z)$ are arbitrary functions, and $Z = Z(x,t)$ is any solution of the transcendental equation

$$G(Z) - 2ZG'(Z) + xZ - t = 0.$$

⊙ *Reference for equation* 7.4.7.12: Yu. N. Kosovtsov (2008 *a*).

13. $\left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial t^2} = \left(\dfrac{\partial w}{\partial t}\right)^2 \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^4.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$x + t\left(\int f(w)\,dw + C_1\right)^{1/2} = \varphi(w),$$
$$x - t\left(\int f(w)\,dw + C_2\right)^{1/2} = \psi(w),$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions and C_1 and C_2 are arbitrary constants.

14. $\dfrac{\partial^2 w}{\partial t^2} = f\left(\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w\big(C_1 x + C_2,\ C_1 t + C_3\big) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. The Legendre transformation

$$u(z,\tau) = tz + x\tau - w(x,t), \quad z = \frac{\partial w}{\partial t}, \quad \tau = \frac{\partial w}{\partial x},$$

where u is the new dependent variable, and z and τ are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial \tau^2} = f(z,\tau)\frac{\partial^2 u}{\partial z^2}.$$

Exact solutions of this equation for some specific $f(z,\tau)$ can be found in Polyanin (2002).

7.5. Equations of the Form $\dfrac{\partial^2 w}{\partial x \partial y} = F\left(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)$

7.5.1. Equations Involving Arbitrary Parameters of the Form $\dfrac{\partial^2 w}{\partial x \partial y} = f(w)$

1. $\dfrac{\partial^2 w}{\partial x \partial y} = a w^n.$

This is a special case of equation 7.5.3.1 with $f(w) = aw^n$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = (C_1 C_2)^{\frac{1}{n-1}} w\big(C_1 x + C_3,\, C_2 y + C_4\big),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x, y) = \left[\frac{a(1-n)^2}{2(1+n)}\right]^{\frac{1}{1-n}} \left(C_1 x + \frac{y}{C_1} + C_2\right)^{\frac{2}{1-n}},$$

$$w(x, y) = \big[a(1-n)^2\big]^{\frac{1}{1-n}} \big(xy + C_1 x + C_2 y + C_1 C_2\big)^{\frac{1}{1-n}}.$$

3°. Traveling-wave solution in implicit form (generalizes the first solution of Item 2°):

$$\int \left(C_2 + \frac{2a}{n+1} w^{n+1}\right)^{-1/2} dw = C_1 x + \frac{y}{C_1} + C_3.$$

4°. Self-similar solution:

$$w = x^{\frac{\beta-1}{n-1}} U(\xi), \quad \xi = y x^{\beta},$$

where β is an arbitrary constant, and the function $U(\xi)$ is determined by the modified Emden–Fowler equation

$$\beta \xi U''_{\xi\xi} + \frac{n\beta - 1}{n-1} U'_{\xi} = a U^n.$$

For exact solutions of this equation, see the book by Polyanin and Zaitsev (2003).

2. $\dfrac{\partial^2 w}{\partial x \partial y} = a e^{\lambda w}.$

Liouville equation. This is a special case of equation 7.5.3.1 with $f(w) = ae^{\lambda w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(C_1 x + C_2,\, C_3 y + C_4\big) + \frac{1}{\lambda} \ln(C_1 C_3),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. General solution:

$$w = \frac{1}{\lambda}\big[f(x) + g(y)\big] - \frac{2}{\lambda}\ln\left|k\int \exp\big[f(x)\big]\,dx + \frac{a\lambda}{2k}\int \exp\big[g(y)\big]\,dy\right|,$$

where $f = f(x)$ and $g = g(y)$ are arbitrary functions and k is an arbitrary constant.

3°. The Liouville equation is related to the linear equation $\partial_{xy}u = 0$ by the Bäcklund transformation

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} + \frac{2k}{\lambda}\exp\left[\frac{1}{2}\lambda(w+u)\right],$$
$$\frac{\partial u}{\partial y} = -\frac{\partial w}{\partial y} - \frac{a}{k}\exp\left[\frac{1}{2}\lambda(w-u)\right].$$

4°. The original equation can also be linearized with either of the differential substitutions

$$w = \frac{1}{\lambda}\ln\left(\frac{2}{v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right), \qquad v = v(x,y);$$
$$w = \frac{1}{\lambda}\ln\left(\frac{2}{\cos^2 z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right), \quad z = z(x,y).$$

5°. Solutions (for $a = \lambda = 1$):

$$w = \ln\left[f(x)g(y)\cosh^{-2}\left(C_1 + C_2\int g(y)\,dy - \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$
$$w = \ln\left[f(x)g(y)\sinh^{-2}\left(C_1 + C_2\int g(y)\,dy + \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$
$$w = \ln\left[f(x)g(y)\cos^{-2}\left(C_1 + C_2\int g(y)\,dy + \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$

where $f(x)$ and $g(y)$ are arbitrary functions, and C_1 and C_2 are arbitrary constants.

⊙ *References for equation* 7.5.1.2: J. Liouville (1853), R. K. Bullough and P. J. Caudrey (1980), S. V. Khabirov (1990), N. H. Ibragimov (1994, pp. 204–206).

3. $\dfrac{\partial^2 w}{\partial x \partial y} = e^{w} - e^{-2w}$.

Tzitzéica equation arises in differential geometry and physics. This is a special case of equation 7.5.3.1 with $f(w) = e^{w} - e^{-2w}$.

1°. Solutions:

$$w = \ln\left[1 - 2\frac{\partial^2(\ln\zeta_k)}{\partial x \partial y}\right], \tag{1}$$

where

$$\zeta_1 = 1 + A\exp\left(kx + \frac{3}{k}y\right),$$

$$\zeta_2 = 1 + A_1\exp\left(k_1x + \frac{3}{k_1}y\right) + A_2\exp\left(k_2x + \frac{3}{k_2}y\right) + A_1A_2\frac{(k_1-k_2)^2(k_1^2-k_1k_2+k_2^2)}{(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}\exp\left[(k_1+k_2)x + \left(\frac{3}{k_1}+\frac{3}{k_2}\right)y\right],$$

$$\zeta_3 = 1 + A(k^2x - 3y)\exp\left(kx + \frac{3}{k}y\right) - \frac{A^2k^2}{12}\exp\left(2kx + \frac{6}{k}y\right),$$

$$\zeta_4 = \sin\left(kx - \frac{3}{k}y\right) + \sqrt{3}\left(kx + \frac{3}{k}y\right),$$

and A, A_1, A_2, k, k_1, and k_2 are arbitrary constants.

2°. On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 7.2.1.4:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + e^w - e^{-2w}.$$

3°. The Tzitzéica equation is often written in the form

$$(\ln u)_{xy} = u - u^{-2}, \qquad u = e^w,$$

and can be treated as a consistency condition for the following system of three linear equations for ψ:

$$\psi_{xy} = u\psi, \quad \psi_{xx} = \frac{u_x\psi_x + \lambda\psi_y}{u}, \quad \psi_{yy} = \frac{u_y\psi_y + \lambda^{-1}\psi_x}{u},$$

where λ is an arbitrary parameter.

⊙ *References for equation* 7.5.1.3: G. Tzitzéica (1910), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and Morris (1982), S. S. Safin and R. A. Sharipov (1993), O. V. Kaptsov and Yu. V. Shan'ko (1999, other exact solutions are also given there).

4. $\dfrac{\partial^2 w}{\partial x\partial y} = a\sinh w.$

Sinh-Gordon equation. On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 7.3.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + a\sinh w.$$

⊙ *References*: S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), A. Grauel (1985).

5. $\dfrac{\partial^2 w}{\partial x \partial y} = a \sin w.$

Sine-Gordon equation. This is a special case of equation 7.5.3.1 with $f(w) = a \sin w$.

1°. Traveling-wave solution:

$$w(x,y) = \begin{cases} 4 \arctan\left[\exp\left(\sqrt{\dfrac{a}{AB}}\,(Ax + By + C)\right)\right] & \text{if } aAB > 0, \\ 4 \operatorname{arctanh}\left[\exp\left(\sqrt{-\dfrac{a}{AB}}\,(Ax + By + C)\right)\right] & \text{if } aAB < 0, \end{cases}$$

where A, B, and C are arbitrary constants.

2°. Solution:

$$w(x,y) = 4 \arctan\left[\frac{C_1 + C_2}{C_1 - C_2} \frac{\sinh(v_1 - v_2)}{\cosh(v_1 + v_2)}\right], \quad v_k = \frac{1}{2}\left(C_k x - \frac{a}{C_k} y\right), \quad k = 1, 2,$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution:

$$w = U(\xi), \quad \xi = xy,$$

where the function $U = U(\xi)$ is determined by the second-order ordinary differential equation $\xi U''_{\xi\xi} + U'_\xi = a \sin U$.

4°. The Bäcklund transformation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial w}{\partial x} + 2k \sin\left(\frac{w+u}{2}\right), \\ \frac{\partial u}{\partial y} &= -\frac{\partial w}{\partial y} - \frac{2a}{k} \sin\left(\frac{w-u}{2}\right) \end{aligned} \tag{1}$$

brings the original equation to the identical equation

$$\frac{\partial^2 u}{\partial x \partial y} = a \sin u.$$

Given a single exact solution, the formulas of (1) allow us to successively generate other solutions of the sine-Gordon equation.

5°. The sine-Gordon equation has infinitely many conservation laws. The first three of them read as follows:

$$\begin{aligned} &D_x\left(w_y^2\right) + D_y\left(2a \cos w\right) = 0, \\ &D_x\left(w_y^4 - 4w_{yy}^2\right) + D_y\left(4a w_y^2 \cos w\right) = 0, \\ &D_x\left(3w_y^6 - 12 w_y^2 w_{yy}^2 + 16 w_y^3 w_{yyy} + 24 w_{yyy}^2\right) + D_y\left[a(2w_y^4 - 24 w_{yy}^2)\cos w\right] = 0, \end{aligned}$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$ (analogous laws can be obtained by swapping the independent variables $x \rightleftarrows y$).

6°. The equation in question is related to the equation

$$\frac{\partial^2 z}{\partial x \partial y} = z\sqrt{a^2 - \left(\frac{\partial z}{\partial y}\right)^2}$$

by the transformation

$$z = \frac{\partial w}{\partial x}, \quad \frac{\partial z}{\partial y} = a \sin w.$$

⊙ *References for equation* 7.5.1.5: R. Steuerwald (1936), A. C. Scott, F. Y. Chu, and D. W. McLaughlin (1973), J. L. Lamb (1974), R. K. Dodd and R. K. Bullough (1977), I. M. Krichever (1980), R. K. Bullough and P. J. Caudrey (1980), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), N. H. Ibragimov (1994, pp. 206–208).

6. $\dfrac{\partial^2 w}{\partial x \partial y} = a \sin w + b \sin\left(\frac{1}{2} w\right).$

On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 7.3.3.2:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + a \sin w + b \sin\left(\tfrac{1}{2} w\right).$$

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

7.5.2. Other Equations Involving Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial x \partial y} = w \dfrac{\partial w}{\partial x}.$

General solution:

$$w = \frac{g''_{yy}(y)}{g'_y(y)} - \frac{2g'_y(y)}{f(x) + g(y)},$$

where $f(x)$ and $g(y)$ are arbitrary functions.

⊙ *References*: A. R. Forsyth (1906), O. V. Kaptsov (2009).

2. $\dfrac{\partial^2 w}{\partial x \partial y} = a \dfrac{\partial w}{\partial x} \dfrac{\partial w}{\partial y}.$

General solution:

$$w(x, y) = -\frac{1}{a} \ln\big[f(x) + g(y)\big],$$

where $f(x)$ and $g(y)$ are arbitrary functions.

3. $\dfrac{\partial^2 w}{\partial x \partial y} + a \dfrac{\partial w}{\partial x} \dfrac{\partial w}{\partial y} + b \dfrac{\partial w}{\partial x} + c \dfrac{\partial w}{\partial y} = 0.$

Thomas equation. This equation arises in some problems of chemical engineering and chromatography. The substitution $u = e^{aw}$ leads to the linear equation

$$\frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial u}{\partial x} + c \frac{\partial u}{\partial y} = 0.$$

⊙ *Reference*: H. C. Thomas (1944), G. B. Whitham (1972).

4. $$w\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

General solution:

$$w(x, y) = f(x)g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions.

5. $$w\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + aw^2.$$

General solution:

$$w(x, y) = e^{axy} f(x)g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

6. $$\frac{\partial^2 w}{\partial x \partial y} = aw^n \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

This is a special case of equation 7.5.3.8 with $f(w) = aw^n$.

7. $$\frac{\partial^2 w}{\partial x \partial y} = ae^{\beta w}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

This is a special case of equation 7.5.3.8 with $f(w) = ae^{\beta w}$.

General solution in implicit form:

$$\int \exp\left(-\frac{a}{\beta}e^{\beta w}\right) dw = \varphi(x) + \psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

8. $$\frac{\partial^2 w}{\partial x \partial y} = \frac{2}{w}\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y} + 1\right).$$

General solution:

$$w(x, y) = -\frac{F(y)}{F'_y(y)} + \left\{[F'_y(y)]^2\left[G(x) - \int \frac{F''_{yy}(y)\,dy}{F(y)[F'_y(y)]^2}\right]\right\}^{-1},$$

where $F(y)$ and $G(x)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

9. $$\frac{\partial^2 w}{\partial x \partial y} = \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial w}{\partial y}.$$

This is a special case of equation 7.5.3.12 with $f(z) = z^2$.

General solution:

$$w(x, y) = -\frac{4F'_y(y)}{F(y) + G(x)} + \int \left(\frac{F''_{yy}(y)}{F'_y(y)}\right)^2 dy,$$

where $F(y)$ and $G(x)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

10. $\dfrac{\partial^2 w}{\partial x \partial y} = a\sqrt{\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}.$

This is a special case of equation 7.5.3.9 with $f(x, y) = \frac{1}{4}a^2$ (the original equation is reduced to a linear one).

11. $w\dfrac{\partial^2 w}{\partial x \partial y} = \sqrt{1 - \left(\dfrac{\partial w}{\partial x}\right)^2}\sqrt{1 - \left(\dfrac{\partial w}{\partial y}\right)^2}.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w\big(C_1 x + C_2,\ C_1 y + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the following forms:

$$w = w(k_1 x + k_2 y) \quad \text{(traveling-wave solution)},$$
$$w = xU(y/x) \quad \text{(self-similar solution)}.$$

3°. For this and some other integrable nonlinear hyperbolic equations, see Zhiber and Sokolov (2001).

7.5.3. Equations Involving Arbitrary Functions

1. $\dfrac{\partial^2 w}{\partial x \partial y} = f(w).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w\big(C_1 x + C_2,\ C_1^{-1} y + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = ax + by,$$

where a and b are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation $abw''_{zz} = f(w)$.

3°. Self-similar solution:

$$w = w(\xi), \qquad \xi = xy,$$

where the function $w(\xi)$ is determined by the second-order ordinary differential equation $\xi w''_{\xi\xi} + w'_\xi = f(w)$.

4°. Solution (generalizes the solutions of Items 2° and 3°):

$$w = w(\zeta), \qquad \zeta = axy + bx + cy,$$

where the function $w(\zeta)$ is determined by the second-order ordinary differential equation

$$(a\zeta + bc)w''_{\zeta\zeta} + aw'_\zeta = f(w).$$

5°. On passing to the new independent variables $z = x - y$ and $t = x + y$, we obtain an equation of the form 7.4.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

6°. Conservation laws:

$$D_x\left(\tfrac{1}{2}w_y^2\right) + D_y\left[-F(w)\right] = 0,$$
$$D_x\left[-F(w)\right] + D_y\left(\tfrac{1}{2}w_x^2\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(w) = \displaystyle\int f(w)\,dw$.

2. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x)g(y)e^{\beta w}$.

The transformation

$$\xi = \int f(x)\,dx, \quad \eta = \int g(y)\,dy$$

leads to an equation of the form 7.5.1.2:

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = e^{\beta w}.$$

3. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x)g(y)h(w)$.

The transformation

$$\xi = \int f(x)\,dx, \quad \eta = \int g(y)\,dy$$

leads to an equation of the form 7.5.3.1:

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = h(w).$$

4. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x)g(w)\dfrac{\partial w}{\partial y}$.

1°. Functional separable solution in implicit form:

$$\int \frac{dw}{G(w)} = \varphi(y) + \int f(x)\,dx, \qquad \text{where} \quad G(w) = \int g(w)\,dw.$$

Here, $\varphi(y)$ is an arbitrary function.

2°. Integrating the original equation with respect to y, we arrive at a first-order partial differential equation:

$$\frac{\partial w}{\partial x} = f(x)\int g(w)\,dw + \psi(x),$$

where $\psi(x)$ is an arbitrary function.

5. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x,w)\dfrac{\partial w}{\partial y} + g(x,y).$

Integrating the original equation with respect to y, we arrive at a first-order partial differential equation:

$$\frac{\partial w}{\partial x} = \int_a^w f(x,\tau)\,d\tau + \int_b^y g(x,s)\,ds + \psi(x),$$

where $\psi(x)$ is an arbitrary function, and a and b are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 177).

6. $\dfrac{\partial^2 w}{\partial x \partial y} = a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} + h(x,y).$

The substitution $u = e^{-aw}$ leads to the linear equation

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y)\frac{\partial u}{\partial x} + g(x,y)\frac{\partial u}{\partial y} - ah(x,y)u.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 272).

7. $\dfrac{\partial^2 w}{\partial x \partial y} + \dfrac{\partial}{\partial x}\Big[A(x,y)e^{w}\Big] - \dfrac{\partial}{\partial y}\Big[B(x,y)e^{-w}\Big] + C(x,y) = 0.$

Consider the auxiliary first-order system

$$e^{-u} = (\ln z)_x + b, \tag{1}$$
$$u_y + ke^{u} = (\ln z)_y + a, \tag{2}$$

where

$$k = b_y + ab - c, \tag{3}$$

and a, b, and c are some functions of two variables x and y.

Eliminating u from (1)–(2), one obtains the linear equation

$$z_{xy} + az_x + bz_y + cz = 0. \tag{4}$$

Eliminating z from (1)–(2) yields the linear equation

$$u_{xy} + (ke^{u})_x - (e^{-u})_y + k - h = 0, \tag{5}$$

where

$$h = a_x + ab - c, \qquad k - h = b_y - a_x. \tag{6}$$

Whenever a solution to equation (4) is known, one gets a solution to equation (5) by formula (1). Conversely, a solution to the linear equation (4) is obtained from a solution to equation (5) by integrating the first-order equations (1)–(2) with respect to the unknown z.

With the change of variable $w = u + \ln B$, the original equation can be reduced to

$$u_{xy} + (ABe^{u})_x - (e^{-u})_y + C + (\ln B)_{xy} = 0.$$

With the notation $k = AB$ and $h = AB - C - (\ln B)_{xy}$, one gets equation (5). Thus, in view of (3) and (6), the functional coefficients of the original nonlinear equation and the coefficients of the linear equation (4) are related by the two constraints

$$\begin{aligned} AB &= b_y + ab - c, \\ AB - C - (\ln B)_{xy} &= a_x + ab - c, \end{aligned} \tag{7}$$

which can be treated, for example, as a linear system for the functions b and c with given functions A, B, and C. The function a is set arbitrarily; in particular, one can set $a \equiv 0$ and in this case, system (7) is easily solvable starting from the second equation.

⊙ *References*: M. E. Goursat (1898), O. V. Kaptsov (2009).

8. $\dfrac{\partial^2 w}{\partial x \partial y} = f(w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}$.

1°. The substitution

$$u = \int F(w)\,dw, \qquad F(w) = \exp\left[-\int f(w)\,dw\right]$$

leads to the constant coefficient linear equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

2°. General solution in implicit form:

$$\int \exp\left[-\int f(w)\,dw\right] dw = \varphi(x) + \psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

⊙ *Reference for equation* 7.5.3.8: A. D. Polyanin and V. F. Zaitsev (2004, p. 272).

9. $\dfrac{\partial^2 w}{\partial x \partial y} = 2\sqrt{f(x,y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}$.

Goursat equation. Introduce functions $u = u(x, y)$ and $v = v(x, y)$ by the differential relations

$$u = \sqrt{\frac{\partial w}{\partial x}}, \qquad v = \sqrt{\frac{\partial w}{\partial y}}.$$

On differentiating these relations with respect to y and x, respectively, and eliminating w using the original equation, one arrives at the system

$$\frac{\partial u}{\partial y} = v\sqrt{f(x,y)}, \qquad \frac{\partial v}{\partial x} = u\sqrt{f(x,y)}.$$

Eliminating v yields a linear equation for $u = u(x, y)$:

$$\frac{\partial^2 u}{\partial x \partial y} = g(x,y)\frac{\partial u}{\partial y} + f(x,y)u, \qquad \text{where} \quad g(x,y) = \frac{1}{2}\frac{\partial}{\partial x}\ln f(x,y).$$

⊙ *Reference*: E. I. Ganzha (2000).

10. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y} = f(x)g(w)\dfrac{\partial w}{\partial y}.$

1°. Functional separable solution in implicit form:

$$\int \frac{dw}{\sqrt{G(w)}} = \varphi(y) \pm \int \sqrt{2f(x)}\,dx, \qquad \text{where} \quad G(w) = \int g(w)\,dw.$$

Here, $\varphi(y)$ is an arbitrary function.

2°. Integrating the original equation with respect to y, we have

$$\left(\frac{\partial w}{\partial x}\right)^2 = 2f(x)\int g(w)\,dw + \psi(x),$$

where $\psi(x)$ is an arbitrary function. The equation obtained can be treated as a first-order ordinary differential equation in x for which the constant of integration will be dependent on y.

⊙ *Reference for equation* 7.5.3.10: A. D. Polyanin and V. F. Zaitsev (2002, p. 178).

11. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y} = f(x,w)\dfrac{\partial w}{\partial y} + g(x,y).$

Integrating the original equation with respect to y, one arrives at a first-order partial differential equation:

$$\left(\frac{\partial w}{\partial x}\right)^2 = 2\int_a^w f(x,\tau)\,d\tau + 2\int_b^y g(x,s)\,ds + \psi(x),$$

where $\psi(x)$ is an arbitrary function and a and b are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 178).

12. $\dfrac{\partial^2 w}{\partial x\partial y} = f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial w}{\partial y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + C_1,\ \varphi(y)\big) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(y)$ is an arbitrary function, is also a solution of the equation.

2°. Functional separable solution:

$$w = U(z), \qquad z = x + \varphi(y),$$

where $\varphi(y)$ is an arbitrary function and the function $U(z)$ is defined parametrically by

$$z = \int \frac{d\xi}{\xi f(\xi)} + C_1, \quad U = \int \frac{d\xi}{f(\xi)} + C_2,$$

and C_1 and C_2 are arbitrary constants.

3°. First integral:

$$\int_a^{w_x} \frac{du}{f(u)} = w + \theta(x),$$

where $w_x = \frac{\partial w}{\partial x}$.

13. $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial w}{\partial x} f\left(\dfrac{\partial w}{\partial x} + aw\right).$

Generalized separable solution:

$$w(x, y) = \varphi(y) + C \exp\left[-ax + \int f(a\varphi(y))\, dy\right],$$

where $\varphi(y)$ is an arbitrary function and C is an arbitrary constant.

⊙ *Reference*: I. Tsyfra (1997).

14. $\dfrac{\partial^2 w}{\partial x \partial y} = f\left(\dfrac{\partial w}{\partial x} + aw\right) \dfrac{\partial^2 w}{\partial x^2}.$

Generalized separable solution:

$$w(x, y) = \varphi(y) + C \exp\left[-ax - a \int f(a\varphi(y))\, dy\right],$$

where $\varphi(y)$ is an arbitrary function and C is an arbitrary constant.

15. $w \dfrac{\partial^2 w}{\partial x \partial y} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w\big(C_1 x + C_2,\ C_1 y + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the following forms:

$$\begin{aligned} &w = w(k_1 x + k_2 y) && \text{(traveling-wave solution)},\\ &w = xU(y/x) && \text{(self-similar solution)}. \end{aligned}$$

16. $f\left(x, \dfrac{\partial w}{\partial x}\right) \dfrac{\partial^2 w}{\partial x \partial y} = g(x, w) \dfrac{\partial w}{\partial y} + h(x, y).$

Integrating the original equation with respect to y, one arrives at a first-order partial differential equation:

$$\int_a^{w_x} f(x, \lambda)\, d\lambda = \int_b^w g(x, \tau)\, d\tau + \int_c^y h(x, s)\, ds + \psi(x),$$

where w_x is the partial derivative of w with respect to x, $\psi(x)$ is an arbitrary function, and a, b, and c are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 178).

▷ *Some more complicated equations of hyperbolic type are considered in Chapter 11.*

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Chapter 8
Second-Order Hyperbolic Equations with Two or More Space Variables

8.1. Equations with Two Space Variables Involving Power-Law Nonlinearities

8.1.1. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + aw^p$$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} + cw^p.$

This is a special case of equation 8.4.1.1 with $f(w) = cw^p$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 8.4.1.2 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ \pm C_1^{\frac{p-1}{2}}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $p \neq 1$:

$$w = \left[\frac{1}{2c(p-1)}\left(\frac{1+p}{1-p} + \frac{2}{2-n} + \frac{2}{2-m}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. Solution for $n \neq 2$ and $m \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + ck^{-1}w^p = 0, \qquad A = \frac{2(4-n-m)}{(2-n)(2-m)}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$
$$w(x,y,t) = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = W(y,\zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = |t|^{\frac{2}{1-p}} F(z_1,z_2), \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y|t|^{\frac{2}{m-2}}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 8.4.1.3 with $f(w) = cw^p$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}} x,\ y + \frac{1-p}{\lambda}\ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\lambda \neq 0$, and $p \neq 1$:

$$w = \left[\frac{1}{2c(p-1)}\left(\frac{1+p}{1-p} + \frac{2}{2-n}\right)\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. Solution for $n \neq 2$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + ck^{-1}w^p = 0, \qquad A = \frac{2}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$
$$w(x,y,t) = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = W(y,\zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = |t|^{\frac{2}{1-p}} F(z_1,z_2), \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 8.4.1.4 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Big(x + \frac{1-p}{\beta}\ln C_1,\ y + \frac{1-p}{\lambda}\ln C_1,\ \pm C_1^{\frac{p-1}{2}}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $p \neq \pm 1$, $\beta \neq 0$, and $\lambda \neq 0$:

$$w = \left[-\frac{c(p-1)^2}{2k(1+p)}(r + C_1)^2\right]^{\frac{1}{1-p}}, \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. Solution for $\beta \neq 0$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}w^p = 0.$$

Integrating yields its general solution in implicit form:

$$\int\left[C_1 - \frac{2c}{k(p+1)}w^{p+1}\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

4°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, t), \quad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$

$$w(x, y, t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = |t|^{\frac{2}{1-p}}F(z_1, z_2), \quad z_1 = x + \frac{2}{\beta}\ln|t|, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

8.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 8.1.3.1 with $c = 0$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_2^2 w(\pm C_1 x + C_3, \pm C_1 y + C_4, \pm C_1 C_2 t + C_5),$$
$$w_2 = w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w = \frac{\lambda^2 \pm \sqrt{A(k_1 x + k_2 y + \lambda t) + B}}{k_1^2 + k_2^2},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Generalized separable solution linear in space variables:

$$w(x, y, t) = (A_1 t + B_1)x + (A_2 t + B_2)y + \tfrac{1}{12}(A_1^2 + A_2^2)t^4 + \tfrac{1}{3}(A_1 B_1 + A_2 B_2)t^3 + \tfrac{1}{2}(B_1^2 + B_2^2)t^2 + Ct + D,$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. Solutions:

$$w(x, y, t) = \tfrac{3}{4}t^{-2}\left[(x + C_1)^2 + (y + C_2)^2\right],$$
$$w(x, y, t) = t^{-2}\left(x\sin\lambda + y\cos\lambda + C_1\right)^2,$$
$$w(x, y, t) = \frac{1}{C_1^2 + C_2^2}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2,$$
$$w(x, y, t) = \frac{C_2^2 (x + C_4)^2}{(C_1 y + C_2 t + C_3)^2 + C_1^2 (x + C_4)^2},$$
$$w(x, y, t) = t\left[C_1 \ln\left(x^2 + y^2\right) + C_2\right]^{1/2},$$
$$w(x, y, t) = t\left[C_1 \exp\left(\lambda x\right)\sin\left(\lambda y + C_2\right) + C_3\right]^{1/2},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the last two solutions of Item 4°):

$$w(x, y, t) = (C_1 t + C_2)\sqrt{|U(x, y)|},$$

where the function $U = U(x, y)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

6°. There is a generalized separable solution quadratic in space variables:

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 6f^2 + 2fh + g^2, \tag{1}$$
$$g''_{tt} = 6(f+h)g, \tag{2}$$
$$h''_{tt} = 6h^2 + 2fh + g^2. \tag{3}$$

A particular solution of system (1)–(3) is given by

$$h(t) = f(t), \quad g(t) = \pm 2f(t), \quad \text{where} \quad f''_{tt} = 12f^2$$

(the general solution for f can be written out in implicit form).

7°. There is a generalized separable solution of the form (generalizes the solution of Item 6°)

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t),$$

where the functions $f(t)$, $g(t)$, $h(t)$, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 6f^2 + 2fh + g^2, \quad \varphi''_{tt} = 2(3f+h)\varphi + 2g\psi,$$
$$g''_{tt} = 6(f+h)g, \quad \psi''_{tt} = 2g\varphi + 2(f+3h)\psi,$$
$$h''_{tt} = 6h^2 + 2fh + g^2, \quad \chi''_{tt} = \varphi^2 + \psi^2 + 2(f+h)\chi.$$

The first three equations for f, g, and h are solved independently of the other three (see Item 6°).

8°. There is a "two-dimensional" solution in multiplicative separable form

$$w(x,y,t) = (At+B)^{-2}\Theta(x,y).$$

9°. For other solutions, see equation 8.1.2.6 with $a = b = n = 1$ and equation 8.1.2.7 with $a = b = n = m = 1$.

⊙ *Reference for equation* 8.1.2.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 181).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 8.1.2.7 with $n = k = -1/2$ and equation 8.4.2.3 with $f(w) = aw^{-1/2}$ and $g(w) = bw^{-1/2}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^4 w(\pm C_1C_2x + C_3, \pm C_1C_2y + C_4, \pm C_2t + C_5),$$
$$w_2 = w(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,y,t)=\left(\frac{aC_1^2+bC_2^2}{C_3^2}\pm\sqrt{C_1x+C_2y+C_3t+C_4}\right)^2,$$

where $C_1,\ldots,C_4$ are arbitrary constants.

3°. Solutions:

$$w(x,y,t)=t^4\left(\frac{\sin\lambda}{\sqrt{a}}x+\frac{\cos\lambda}{\sqrt{b}}y+C_1\right)^{-4},$$

$$w(x,y,t)=\left(\tfrac{2}{3}ab\right)^2t^4(bx^2+ay^2)^{-2},$$

$$w(x,y,t)=(aC_1^2+bC_2^2)^2\left(\frac{t+C_4}{C_1x+C_2y+C_3}\right)^4,$$

$$w(x,y,t)=\frac{\left[a(C_1y+C_2t+C_3)^2+bC_1^2(x+C_4)^2\right]^2}{C_2^4(x+C_4)^4},$$

$$w(x,y,t)=t\left[C_1\ln\left(bx^2+ay^2\right)+C_2\right]^2,$$

$$w(x,y,t)=t\left[C_1\exp\left(\lambda\sqrt{b}\,x\right)\sin\left(\lambda\sqrt{a}\,y+C_2\right)+C_3\right]^2,$$

where $C_1,\ldots,C_4$ and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the last two solutions of Item 3°):

$$w(x,y,t)=(C_1t+C_2)U^2(\xi,\eta),\quad \xi=\frac{x}{\sqrt{a}},\quad \eta=\frac{y}{\sqrt{b}},$$

where the function $U=U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2U}{\partial\xi^2}+\frac{\partial^2U}{\partial\eta^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" generalized separable solution quadratic in t:

$$w(x,y,t)=\left[f(\xi,\eta)t+g(\xi,\eta)\right]^2,\quad \xi=\frac{x}{\sqrt{a}},\quad \eta=\frac{y}{\sqrt{b}},$$

where the functions $f=f(\xi,\eta)$ and $g=g(\xi,\eta)$ are determined by the system of differential equations

$$\frac{\partial^2f}{\partial\xi^2}+\frac{\partial^2f}{\partial\eta^2}=0, \tag{1}$$

$$\frac{\partial^2g}{\partial\xi^2}+\frac{\partial^2g}{\partial\eta^2}=f^2. \tag{2}$$

Equation (1) is the Laplace equation, and (2) is a Helmholtz equation (wherever f is known). For these linear equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

6°. There is a "two-dimensional" generalized separable solution of the form

$$w(x,y,t)=\left[f_2(x,y)t^2+f_1(x,y)t+f_0(x,y)\right]^2.$$

7°. For other solutions, see equation 8.1.2.6 with $n=-1/2$ and equation 8.1.2.7 with $n=m=-1/2$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\left[\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)\right].$

This is a special case of equation 8.1.2.7 with $a = b$, $n = k = -1$ and equation 8.4.2.3 with $f(w) = a/w$ and $g(w) = b/w$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1 C_2 x + C_3, \pm C_1 C_2 y + C_4, \pm C_2 t + C_5),$$
$$w_2 = w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$a(k_1^2 + k_2^2)\ln|w| - \lambda^2 w = A(k_1 x + k_2 y + \lambda t) + B,$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Solutions:

$$w(x, y, t) = (C_1 t + C_2)e^{Ax+By},$$
$$w(x, y, t) = (C_1 t + C_2)\exp\big[A(x^2 - y^2)\big],$$
$$w(x, y, t) = (C_1 t + C_2)\exp\big[Ae^{\lambda x}\sin(\lambda y + B)\big],$$
$$w(x, y, t) = \frac{a[(Ay + Bt + C_1)^2 + A^2(x + C_2)^2]}{B^2(x + C_2)^2},$$
$$w(x, y, t) = \frac{at^2 + At + B}{(x\sin\lambda + y\cos\lambda + C)^2},$$
$$w(x, y, t) = \frac{at^2 + At + B}{(\sin y + Ce^x)^2},$$
$$w(x, y, t) = \frac{C_1^2(at^2 + At + B)}{e^{2x}\sinh^2(C_1 e^{-x}\sin y + C_2)},$$
$$w(x, y, t) = \frac{C_1^2(-at^2 + At + B)}{e^{2x}\cosh^2(C_1 e^{-x}\sin y + C_2)},$$
$$w(x, y, t) = \frac{C_1^2(at^2 + At + B)}{e^{2x}\cos^2(C_1 e^{-x}\sin y + C_2)},$$

where A, B, C, C_1, C_2, and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the first three solutions of Item 3°):

$$w(x, y, t) = (C_1 t + C_2)e^{U(x,y)},$$

where the function $U = U(x, y)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the last four solutions of Item 3°):

$$w(x,y,t) = (\tfrac{1}{2}Aat^2 + Bt + C)e^{\Theta(x,y)},$$

where A, B, and C are arbitrary constants, and the function $\Theta(x,y)$ is a solution of the stationary equation

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} = Ae^{\Theta},$$

which occurs in combustion theory. For solutions of this equation, see 9.2.1.1.

6°. For other solutions, see equation 8.1.2.6 with $a = b$, $n = -1$ and equation 8.1.2.7 with $a = b$, $n = m = -1$.

⊙ *References for equation* 8.1.2.4: V. A. Baikov (1990), N. Ibragimov (1994, p. 225), A. D. Polyanin and V. F. Zaitsev (2002, p. 182).

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right)$.

This is a special case of equation 8.4.2.1 with $g(w) = bw^n$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2x + C_3, \pm C_1^nC_2y + C_4, \pm C_2t + C_5),$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,t) = y^{\frac{1}{n+1}}\left[\varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right],$$
$$w(x,y,t) = \left[y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\right]^{\frac{1}{n+1}},$$
$$w(x,y,t) = \left[y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right]^{\frac{1}{n+1}},$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x,y,t) = y^{\frac{2}{n}}\left(C_1x \pm t\sqrt{aC_1^2 + b} + C_2\right)^{-\frac{2}{n}},$$
$$w(x,y,t) = \left[\frac{2a}{b(n+2)}\right]^{\frac{1}{n}} y^{\frac{2}{n}}\left[a(t + C_1)^2 - (x + C_2)^2\right]^{-\frac{1}{n}},$$
$$w(x,y,t) = \left[\frac{1}{bC_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2 - \frac{aC_1^2}{bC_2^2}\right]^{\frac{1}{n}},$$
$$w(x,y,t) = \left[\frac{C_2^2}{bC_1^2} - \frac{a}{bC_1^2}\left(\frac{C_1y + C_2t + C_3}{x + C_4}\right)\right]^{\frac{1}{n}},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y+\lambda t)+(t\sqrt{a}\pm x)(bw^n-\lambda^2)=\psi(w),$$

where $\psi(w)$ is an arbitrary function and λ is an arbitrary constant.

5°. Solution:

$$w(x,y,t)=V(z)y^{2/n},\quad z=x^2-at^2,$$

where the function $V=V(z)$ is determined by the ordinary differential equation

$$2an^2(zV_z''+V_z')+b(n+2)V^{n+1}=0.$$

6°. "Two-dimensional" solution in multiplicative separable form:

$$w(x,y,t)=u(x,t)y^{2/n},$$

where the function $u=u(x,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2}=a\frac{\partial^2 u}{\partial x^2}+\frac{2b(n+2)}{n^2}u^{n+1}.$$

For $n=-1$ and $n=-2$, this equation is linear.

Remark. The first solution of Item 2°, the first two solutions of Item 3°, and the solutions of Items 5° and 6° are special cases of a multiplicative separable solution $w=u(x,t)\theta(y)$, where $\theta=\theta(y)$ is determined by the autonomous ordinary differential equation $(\theta^n\theta_y')_y'=C\theta$.

7°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t)=F(y,r),\quad r=x^2-at^2;$$
$$w(x,y,t)=|t|^{2\lambda}G(\xi,\eta),\quad \xi=\frac{x}{t},\quad \eta=\frac{y}{|t|^{n\lambda+1}};$$
$$w(x,y,t)=|t|^{-2/n}H(y,z),\quad z=x/t;$$
$$w(x,y,t)=|y|^{2/n}U(z_1,z_2),\quad z_1=t+k_1\ln|y|,\quad z_2=x+k_2\ln|y|;$$
$$w(x,y,t)=\exp\Big(-\frac{2y}{n+1}\Big)V(\rho_1,\rho_2),\quad \rho_1=t\exp\Big(-\frac{ny}{n+1}\Big),\quad \rho_2=x\exp\Big(-\frac{ny}{n+1}\Big),$$

where k_1, k_2, and λ are arbitrary constants.

8°. There is an exact solution of the form

$$w(x,y,t)=W(z),\quad z=(x^2-at^2)y^{-2}.$$

9°. For other solutions, see equation 8.1.2.7, in which n should be set equal to zero and k should be renamed n.

⊙ *References for equation* 8.1.2.5: N. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2002, pp. 281–282).

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 8.1.2.7 with $n = k$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = (C_2/C_1)^{2/n} w(\pm C_1 x + C_3, \pm C_1 y + C_4, \pm C_2 t + C_5),$$
$$w_2 = w(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x, y, t) = t^{-\frac{2}{n}}\left(\frac{\sin\lambda}{\sqrt{a}}x + \frac{\cos\lambda}{\sqrt{b}}y + C_1\right)^{\frac{2}{n}},$$

$$w(x, y, t) = \left[\frac{n+2}{2ab(n+1)}\right]^{\frac{1}{n}} t^{-\frac{2}{n}}(bx^2 + ay^2)^{\frac{1}{n}},$$

$$w(x, y, t) = \frac{1}{(aC_1^2 + bC_2^2)^{1/n}}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^{2/n},$$

$$w(x, y, t) = \frac{C_2^{2/n}(x + C_4)^{2/n}}{\left[a(C_1 y + C_2 t + C_3)^2 + bC_1^2(x + C_4)^2\right]^{1/n}},$$

$$w(x, y, t) = t\left[C_1 \ln\left(bx^2 + ay^2\right) + C_2\right]^{\frac{1}{n+1}},$$

$$w(x, y, t) = t\left[C_1 \exp\left(\lambda\sqrt{b}\,x\right)\sin\left(\lambda\sqrt{a}\,y + C_2\right) + C_3\right]^{\frac{1}{n+1}},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$\frac{ak_1^2 + bk_2^2}{n+1}w^{n+1} - \lambda^2 w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the fifth and sixth solutions of Item 2°):

$$w(x, y, t) = (C_1 t + C_2)\left[U(\xi, \eta)\right]^{\frac{1}{n+1}}, \quad \xi = \sqrt{b}\,x, \quad \eta = \sqrt{a}\,y,$$

where the function $U = U(\xi, \eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the first and second solutions of Item 2°):

$$w(x,y,t) = f(t)\Theta(x,y),$$

where the function $f(t)$ is determined by the autonomous ordinary differential equation

$$f''_{tt} = \lambda f^{n+1}, \tag{1}$$

λ is an arbitrary constant, and the function $\Theta = \Theta(x,y)$ is a solution of the two-dimensional stationary equation

$$a\frac{\partial}{\partial x}\left(\Theta^n\frac{\partial\Theta}{\partial x}\right) + b\frac{\partial}{\partial y}\left(\Theta^n\frac{\partial\Theta}{\partial y}\right) - \lambda\Theta = 0. \tag{2}$$

A particular solution to equation (1) is given by (C is an arbitrary constant):

$$f = (C \pm kt)^{-2/n}, \quad k = n\sqrt{\frac{\lambda}{2(n+2)}}.$$

6°. There are solutions of the following forms:

$w(x,y,t) = F(r,t), \quad r = bx^2 + ay^2$ "two-dimensional" solution;

$w(x,y,t) = t^{2\lambda}G(\xi,\eta), \quad \xi = \dfrac{x}{t^{n\lambda+1}}, \quad \eta = \dfrac{y}{t^{n\lambda+1}}$ "two-dimensional" solution;

$w(x,y,t) = y^{2/n}H(z,t), \quad z = y/x$ "two-dimensional" solution;

$w(x,y,t) = |t|^{-2/n}U(z_1,z_2), \quad z_1 = x + k_1\ln|t|, \quad z_2 = y + k_2\ln|t|$ "two-dimensional" solution;

$w(x,y,t) = e^{-2t}V(\rho_1,\rho_2), \quad \rho_1 = xe^{nt}, \quad \rho_2 = ye^{nt}$ "two-dimensional" solution;

$w(x,y,t) = W(\theta), \quad \theta = (bx^2 + ay^2)t^{-2}$ "one-dimensional" solution,

where k_1, k_2, and λ are arbitrary constants.

7°. For other solutions, see equation 8.1.2.7 with $k = n$.

⊙ *References for equation* 8.1.2.6: N. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2002, p. 183).

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)$.

This is a special case of equation 8.4.2.3 with $f(w) = aw^n$ and $g(w) = bw^k$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^n C_2 x + C_3, \pm C_1^k C_2 y + C_4, \pm C_2 t + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\frac{a\beta_1^2}{n+1}w^{n+1}+\frac{b\beta_2^2}{k+1}w^{k+1}-\lambda^2 w=C_1(\beta_1 x+\beta_2 y+\lambda t)+C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1x+C_2y+C_3}{t+C_4}\right)^2=aC_1^2w^n+bC_2^2w^k,$$
$$a\left(\frac{C_1y+C_2t+C_3}{x+C_4}\right)^2w^n+bC_1^2w^k=C_2^2,$$
$$b\left(\frac{C_1x+C_2t+C_3}{y+C_4}\right)^2w^k+aC_1^2w^n=C_2^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,t)=u(z,t),\quad z=c_1x+c_2y,$$

where the function $u=u(z,t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2u}{\partial t^2}=\frac{\partial}{\partial z}\left[\varphi(u)\frac{\partial u}{\partial z}\right],\quad \varphi(u)=ac_1^2u^n+bc_2^2u^k,$$

which can be reduced to a linear equation.

5°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,t)=v(x,\xi),\quad \xi=s_1y+s_2t,$$

where the function $v=v(x,\xi)$ is determined by a differential equation of the form 9.4.4.8:

$$a\frac{\partial}{\partial x}\left(v^n\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left[\psi(v)\frac{\partial v}{\partial z}\right]=0,\quad \psi(v)=bs_1^2v^k-s_2^2,$$

which can be reduced to a linear equation.

6°. There is a "two-dimensional" solution of the form (generalizes the solutions of Items 3° and 4°):

$$w(x,y,t)=U(z_1,z_2),\quad z_1=a_1x+b_1y+c_1t,\quad z_2=a_2x+b_2y+c_2t.$$

7°. There are exact solutions of the following forms:

$w(x,y,t)=t^{2\lambda}F(\xi,\eta),\quad \xi=\dfrac{x}{t^{n\lambda+1}},\quad \eta=\dfrac{y}{t^{k\lambda+1}}$ "two-dimensional" solution;

$w(x,y,t)=x^{2/n}G(\zeta,t),\quad \zeta=x^{-k/n}y$ "two-dimensional" solution;

$w(x,y,t)=e^{-2t}H(z_1,z_2),\quad z_1=xe^{nt},\quad z_2=ye^{kt}$ "two-dimensional" solution;

$w(x,y,t)=(x/t)^{2/n}U(\theta),\quad \theta=x^{-k/n}yt^{k/n-1}$ "one-dimensional" solution;

where λ is an arbitrary constant.

⊙ *References for equation* 8.1.2.7: N. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2002, pp. 284–285).

8.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(bw + c)\dfrac{\partial w}{\partial y}\right].$

This is a special case of equation 8.4.2.1 with $g(w) = bw + c$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_2 x + C_3, \pm C_1 C_2 y + C_4, \pm C_2 t + C_5) + \frac{c(1 - C_1^2)}{bC_1^2},$$

$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x, y, t) = |y|^{1/2}\left[\varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right] - \frac{c}{b},$$

$$w(x, y, t) = \left|y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\right|^{1/2} - \frac{c}{b},$$

$$w(x, y, t) = \left|y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right|^{1/2} - \frac{c}{b},$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x, y, t) = A\sqrt{C_1 x + C_2 y + C_3 t + C_4} + \frac{C_3^2 - aC_1^2}{bC_2^2} - \frac{c}{b},$$

$$w(x, y, t) = \frac{(y + C_1)^2}{\left(C_2 x \pm t\sqrt{aC_2^2 + b} + C_3\right)^2} - \frac{c}{b},$$

$$w(x, y, t) = \frac{2a(y + C_1)^2}{3b\left[a(t + C_2)^2 - (x + C_3)^2\right]} - \frac{c}{b},$$

$$w(x, y, t) = \frac{1}{bC_2^2}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 - \frac{aC_1^2}{bC_2^2} - \frac{c}{b},$$

$$w(x, y, t) = \frac{C_2^2 - aC_1^2}{b}\left(\frac{y + C_4}{C_1 x + C_2 t + C_3}\right)^2 - \frac{c}{b},$$

$$w(x, y, t) = \frac{C_2^2}{bC_1^2} - \frac{a}{bC_1^2}\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)^2 - \frac{c}{b},$$

where $A, C_1, \ldots, C_4$ are arbitrary constants (the first solution is of the traveling-wave type).

4°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y + \lambda t) + (t\sqrt{a} \pm x)(bw + c - \lambda^2) = \varphi(w),$$

where $\varphi(w)$ is an arbitrary function and λ is an arbitrary constant.

5°. Solution:

$$w = u(z) - 4abC_1^2x^2, \qquad z = y + bC_1x^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $u(z)$ is determined by the first-order ordinary differential equation

$$(bu + c - C_2^2)u'_z + 2abC_1u = 8a^2bC_1^2z + C_3.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

6°. Solution:

$$w = v(r) - 4abC_1^2x^2 + 4bC_2^2t^2, \qquad r = y + bC_1x^2 + bC_2t^2,$$

where C_1 and C_2 are arbitrary constants and the function $v(r)$ is determined by the first-order ordinary differential equation

$$(bv + c)v'_r + 2b(aC_1 - C_2)v = 8b(a^2C_1^2 + C_2^2)r + C_3.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

7°. Solution (generalizes the solutions of Items 5° and 6°):

$$w = U(\xi) + A_1x^2 + A_2t^2 + A_3xt + A_4x + A_5t, \qquad \xi = y + b(B_1x^2 + B_2t^2 + B_3xt + B_4x + B_5t),$$

where B_1, B_2, B_3, B_4, and B_5 are arbitrary constants, and the coefficients A_n are expressed in terms of B_n as

$$\begin{aligned} A_1 &= b(B_3^2 - 4aB_1^2),\\ A_2 &= b(4B_2^2 - aB_3^2),\\ A_3 &= 4bB_3(B_2 - aB_1),\\ A_4 &= 2b(B_3B_5 - 2aB_1B_4),\\ A_5 &= 2b(2B_2B_5 - aB_3B_4), \end{aligned}$$

and the function $U(\xi)$ is determined by the first-order ordinary differential equation

$$(bU + c + ab^2B_4^2 - b^2B_5^2)U'_\xi + 2b(aB_1 - B_2)U = 2(A_2 - aA_1)\xi + C_1.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

8°. Generalized separable solution linear in y:

$$w = F(x,t)y + G(x,t),$$

where the functions F and G are determined by the system of differential equations

$$\frac{\partial^2F}{\partial t^2} - a\frac{\partial^2F}{\partial x^2} = 0, \tag{1}$$

$$\frac{\partial^2G}{\partial t^2} - a\frac{\partial^2G}{\partial x^2} = bF^2. \tag{2}$$

Equation (1) is a linear homogeneous wave equation. Given $F = F(x,t)$, (2) represents a linear nonhomogeneous wave equation.

The general solution of system (1)–(2) is given by

$$F(x,t) = \varphi_1(\xi) + \varphi_2(\eta),$$

$$G(x,t) = \psi_1(\xi) + \psi_2(\eta) - \frac{b}{4a}\eta\int\varphi_1^2(\xi)\,d\xi - \frac{b}{4a}\xi\int\varphi_2^2(\eta)\,d\eta - \frac{b}{2a}\int\varphi_1(\xi)\,d\xi\int\varphi_2(\eta)\,d\eta,$$

$$\xi = x + t\sqrt{a}, \quad \eta = x - t\sqrt{a},$$

where $\varphi_1(\xi)$, $\varphi_2(\eta)$, $\psi_1(\xi)$, and $\psi_2(\eta)$ are arbitrary functions.

9°. "Two-dimensional" generalized separable solution quadratic in y (generalizes the second to sixth solutions of Item 3°):

$$w = f(x,t)y^2 + g(x,t)y + h(x,t),$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are determined by the system of differential equations

$$\begin{aligned} f_{tt} &= af_{xx} + 6bf^2, \\ g_{tt} &= ag_{xx} + 6bfg, \\ h_{tt} &= ah_{xx} + bg^2 + 2bfh + 2cf. \end{aligned}$$

Here, the subscripts denote partial derivatives.

10°. "Two-dimensional" solution:

$$w = V(\eta,t) - 4abC_1^2x^2 - 4abC_1C_2x, \qquad \eta = y + bC_1x^2 + bC_2x,$$

where C_1 and C_2 are arbitrary constants and the function $V(\eta,t)$ is determined by the differential equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial}{\partial \eta}\left[(bV + c + ab^2C_2^2)\frac{\partial V}{\partial \eta}\right] + 2abC_1\frac{\partial V}{\partial \eta} - 8a^2bC_1^2.$$

11°. "Two-dimensional" solution:

$$w = W(x,\zeta) + 4bC_1^2t^2 + 4bC_1C_2t, \qquad \zeta = y + bC_1t^2 + bC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $W(\zeta,t)$ is determined by the differential equation

$$a\frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial \zeta}\left[(bW + c - b^2C_2^2)\frac{\partial W}{\partial \zeta}\right] - 2bC_1\frac{\partial W}{\partial \zeta} - 8bC_1^2 = 0.$$

12°. Solution:

$$w = R(\rho) - 4aC_1\varphi(\xi), \qquad \rho = y + bC_1\left(x - t\sqrt{a}\right) + \int \varphi(\xi)\,d\xi, \qquad \xi = x + t\sqrt{a},$$

where C_1 is an arbitrary constant, $\varphi(\xi)$ is an arbitrary function, and the function $R(\rho)$ is determined by the simple ordinary differential equation $[(bR + c)R'_\rho]'_\rho = 0$. Integrating yields a solution of the original equation in the form

$$b(w + 4aC_1\varphi)^2 + 2c(w + 4aC_1\varphi) = C_2y + bC_1C_2\left(x - t\sqrt{a}\right) + C_2\int \varphi\,d\xi + C_3, \qquad \varphi = \varphi(\xi).$$

13°. Solution (obtained in the same way as in Item 12°):

$$b(w + 4aC_1\psi)^2 + 2c(w + 4aC_1\psi) = C_2y + bC_1C_2\left(x + t\sqrt{a}\right) + C_2\int \psi\,d\eta + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, $\psi = \psi(\eta)$ is an arbitrary function, and $\eta = x - t\sqrt{a}$.

14°. Solution:

$$w = U(z) - \frac{A^2}{2\sqrt{a}\,b}xt + \frac{A^2}{2b}t^2 - \frac{2\sqrt{a}\,AB}{b}t - \frac{1}{b}(A\eta + 4aB)\psi(\eta),$$
$$z = y + \frac{A}{8a}\left(x^2 + 2\sqrt{a}\,xt - 3at^2\right) + B(x + \sqrt{a}\,t) + \int \psi(\eta)\,d\eta, \quad \eta = x - t\sqrt{a}, \tag{3}$$

where A and B are arbitrary constants, $\psi(\eta)$ is an arbitrary function, and the function $U(z)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$(bU + c)U'_z + AU - \frac{A^2}{b}z + C = 0.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means it is integrable by quadrature.

Another solution can be obtained by substituting $-t$ for t in (3).

15°. There are solutions of the following forms:

$w(x, y, t) = F(y, r), \quad r = x^2 - at^2$ "two-dimensional" solution;

$w(x, y, t) = t^{2\lambda}G(\xi, \eta) - \dfrac{c}{b}, \quad \xi = \dfrac{x}{t}, \quad \eta = \dfrac{y}{t^{\lambda+1}}$ "two-dimensional" solution;

$w(x, y, t) = H(z), \quad z = (x^2 - at^2)y^{-2}$ "one-dimensional" solution;

where λ is an arbitrary constant.

16°. The substitution $u = w + (c/b)$ leads to a special case of equation 8.1.2.5 with $n = 1$.

17°. For other solutions, see equation 8.4.2.3 with $f(w) = a$ and $g(w) = bw + c$.

⊙ *References for equation* 8.1.3.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 285–288).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(aw + b)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(aw + b)\dfrac{\partial w}{\partial y}\right].$

The substitution $U = aw + b$ leads to an equation of the form 8.1.2.2:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial U}{\partial y}\right).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(a_1 w + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2 w + b_2)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x, y, t) = A\sqrt{k_1 x + k_2 y + \lambda t + B} + \frac{\lambda^2 - b_1 k_1^2 - b_2 k_2^2}{a_1 k_1^2 + a_2 k_2^2},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Generalized separable solution linear in space variables:

$$w(x,y,t) = (A_1t + B_1)x + (A_2t + B_2)y + \tfrac{1}{12}(a_1A_1^2 + a_2A_2^2)t^4 + \tfrac{1}{3}(a_1A_1B_1 + a_2A_2B_2)t^3 + \tfrac{1}{2}(a_1B_1^2 + a_2B_2^2)t^2 + Ct + D,$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. Solutions:

$$w(x,y,t) = \frac{1}{a_1C_1^2 + a_2C_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2 - \frac{b_1C_1^2 + b_2C_2^2}{a_1C_1^2 + a_2C_2^2},$$

$$w(x,y,t) = \frac{(C_2^2 - b_2C_1^2)(x + C_4)^2 - b_1(C_1y + C_2t + C_3)^2}{a_2C_1^2(x + C_4)^2 + a_1(C_1y + C_2t + C_3)^2},$$

$$w(x,y,t) = \frac{(C_2^2 - b_1C_1^2)(y + C_4)^2 - b_2(C_1x + C_2t + C_3)^2}{a_1C_1^2(y + C_4)^2 + a_2(C_1x + C_2t + C_3)^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

5°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

6°. For other solutions, see equation 8.4.2.3 with $f(w) = a_1w + b_1$ and $g(w) = a_2w + b_2$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(\dfrac{1}{aw + b}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{aw + b}\dfrac{\partial w}{\partial y}\right).$

The substitution $U = aw + b$ leads to an equation of the form 8.1.2.4:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{U}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{U}\frac{\partial U}{\partial y}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(a_1w^n + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2w^n + b_2)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,y,t) = \big(C_1x + C_2y + \lambda t + C_3\big)^{\frac{1}{n+1}}, \qquad \lambda = \pm\sqrt{b_1C_1^2 + b_2C_2^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x,y,t)=\left[\frac{1}{a_1C_1^2+a_2C_2^2}\left(\frac{C_1x+C_2y+C_3}{t+C_4}\right)^2-\frac{b_1C_1^2+b_2C_2^2}{a_1C_1^2+a_2C_2^2}\right]^{1/n},$$

$$w(x,y,t)=\left[\frac{(C_2^2-b_2C_1^2)(x+C_4)^2-b_1(C_1y+C_2t+C_3)^2}{a_2C_1^2(x+C_4)^2+a_1(C_1y+C_2t+C_3)^2}\right]^{1/n},$$

$$w(x,y,t)=\left[\frac{(C_2^2-b_1C_1^2)(y+C_4)^2-b_2(C_1x+C_2t+C_3)^2}{a_1C_1^2(y+C_4)^2+a_2(C_1x+C_2t+C_3)^2}\right]^{1/n},$$

where $C_1,\ldots,C_4$ are arbitrary constants.

4°. For other solutions, see equation 8.4.2.3 with $f(w)=a_1w^n+b_1$ and $g(w)=a_2w^n+b_2$.

⊙ *Reference for equation* 8.1.3.5: A. D. Polyanin, V. F. Zaitsev (2004, p. 289).

8.1.4. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2}=(\alpha+\beta w)\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)+\gamma w^2+\delta w+\varepsilon.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm x+C_1,\pm y+C_2,\pm t+C_3),$$
$$w_2=w(x\cos\beta+y\sin\beta,-x\sin\beta+y\cos\beta,\,t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" generalized separable solution:

$$w(x,y,t)=f(t)+g(t)\Theta(x,y). \tag{1}$$

Here, the function $\Theta(x,y)$ satisfies the two-dimensional Helmholtz equation

$$\Delta\Theta+\kappa\Theta=0,\qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

where $\kappa=\gamma/\beta$ ($\beta\neq0$). For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002). The functions $f(t)$ and $g(t)$ in (1) are determined from the autonomous system of nonlinear ordinary differential equations

$$f''_{tt}=\gamma f^2+\delta f+\varepsilon, \tag{2}$$
$$g''_{tt}=(\gamma f+\delta-\alpha\kappa)g. \tag{3}$$

Equation (2) is independent of $g(t)$. Particular solutions of the equation are given by $f=\text{const}$, where f satisfies the quadratic equation $\gamma f^2+\delta f+\varepsilon=0$. For $\gamma=0$, (2) is a constant-coefficient linear equation. For $\gamma\neq0$, the general solution of (2) can be written out in implicit form as

$$\int\frac{df}{\sqrt{\frac{2}{3}\gamma f^3+\delta f^2+2\varepsilon f+C_1}}=C_2\pm t,$$

where C_1 and C_2 are arbitrary constants. Equation (3) is linear in $g(t)$. For particular solutions of the form $f=\text{const}$, it is a constant-coefficient linear equation.

3°. There is a "two-dimensional" solution of the form

$$w(x,y,t)=U(z_1,z_2),\quad z_1=a_1x+b_1y+c_1t,\quad z_2=a_2x+b_2y+c_2t,$$

where the a_n, b_n, and c_n are arbitrary constants ($n=1,\,2$). To the special case $U=U(z_1)$ there corresponds a traveling wave solution.

2. $\dfrac{\partial^2 w}{\partial t^2} = \alpha w\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - \alpha\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] - \beta.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^2 x + C_2, \pm C_1^2 y + C_3, C_1 t + C_4),$$
$$w_2 = w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are generalized separable solutions of the form

$$w(x, y, t) = f(t) + g(t)\varphi(x) + h(t)\psi(y).$$

In particular, if $\varphi''_{xx} = \nu\varphi$ and $\psi''_{yy} = -\nu\psi$, where ν is an arbitrary constant, we have (A_1, A_2, B_1, and B_2 are arbitrary constants)

$$\varphi(x) = A_1\cosh\mu x + A_2\sinh\mu x, \quad \psi(y) = B_1\cos\mu y + B_2\sin\mu y \qquad (\nu = \mu^2 > 0),$$
$$\varphi(x) = A_1\cos\mu x + A_2\sin\mu x, \qquad \psi(y) = B_1\cosh\mu y + B_2\sinh\mu y \quad (\nu = -\mu^2 < 0).$$

The functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = \alpha\nu(A_1^2 - sA_2^2)g^2 - \alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,$$
$$g''_{tt} = \alpha\nu f g,$$
$$h''_{tt} = -\alpha\nu f h,$$

where $s = \operatorname{sign}\nu$.

3°. There are generalized separable solutions of the form

$$w(x, y, t) = f(t) + g(t)\varphi(x) + h(t)\psi(y) + u(t)\theta(x)\chi(y). \tag{1}$$

For $\varphi''_{xx} = 4\nu\varphi$, $\psi''_{yy} = -4\nu\psi$, $\theta''_{xx} = \nu\theta$, and $\chi''_{yy} = -\nu\chi$, where ν is an arbitrary constant, one should set in (1):

if $\nu = \mu^2 > 0$	if $\nu = -\mu^2 < 0$
$\varphi(x) = A_1\cosh 2\mu x + A_2\sinh 2\mu x$	$\varphi(x) = A_1\cos 2\mu x + A_2\sin 2\mu x$
$\psi(y) = B_1\cos 2\mu y + B_2\sin 2\mu y$	$\psi(y) = B_1\cosh 2\mu y + B_2\sinh 2\mu y$
$\theta(x) = C_1\cosh\mu x + C_2\sinh\mu x$	$\theta(x) = C_1\cos\mu x + C_2\sin\mu x$
$\chi(y) = D_1\cos\mu y + D_2\sin\mu y$	$\chi(y) = D_1\cosh\mu y + D_2\sinh\mu y$

The functions $f(t)$, $g(t)$, $h(t)$, and $u(t)$ are determined by the following system of ordinary differential equations ($s = \operatorname{sign}\nu$):

$$f''_{tt} = -4\alpha\nu(A_1^2 - sA_2^2)g^2 + 4\alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,$$
$$g''_{tt} = -4\alpha\nu f g + \alpha\nu a_1(D_1^2 + sD_2^2)u^2,$$
$$h''_{tt} = 4\alpha\nu f h - \alpha\nu a_2(C_1^2 - sC_2^2)u^2,$$
$$u''_{tt} = -2\alpha\nu(a_3 g - a_4 h)u.$$

The arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are related by the two constraints

$$2A_1C_1C_2 = A_2(C_1^2 + sC_2^2), \quad 2B_1D_1D_2 = B_2(D_1^2 - sD_2^2).$$

The coefficients a_1, a_2, a_3, and a_4 are expressed as

$$a_1 = \frac{C_1^2 + sC_2^2}{2A_1}, \quad a_2 = \frac{D_1^2 - sD_2^2}{2B_1}, \quad a_3 = A_2\frac{C_1^2 - sC_2^2}{C_1C_2}, \quad a_4 = B_2\frac{D_1^2 + sD_2^2}{D_1D_2},$$

with $A_1 \neq 0$, $B_1 \neq 0$, $C_1C_2 \neq 0$, and $D_1D_2 \neq 0$.

If $A_1 = 0$ ($A_2 \neq 0$), then one should set $a_1 = C_1C_2/A_2$. If $B_1 = 0$ ($B_2 \neq 0$), then $a_2 = D_1D_2/B_2$. If $C_1 = 0$ ($C_2 \neq 0$), then $a_3 = -A_1$. If $C_2 = 0$ ($C_1 \neq 0$), then $a_3 = A_1$. If $D_1 = 0$ ($D_2 \neq 0$), then $a_4 = -B_1$. If $D_2 = 0$ ($D_1 \neq 0$), then $a_4 = B_1$.

4°. There is a generalized separable solution of the form

$$w(x, y, t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

In the special case $\varphi(t) = \psi(t) \equiv 0$, the functions $f(t)$, $g(t)$, $h(t)$, and $\chi(t)$ are determined by the autonomous system of ordinary differential equations

$$f_{tt}'' = \alpha(2fh - 2f^2 - g^2), \quad h_{tt}'' = \alpha(2fh - 2h^2 - g^2),$$
$$g_{tt}'' = -2\alpha g(f + h), \quad \chi_{tt}'' = 2\alpha(f + h)\chi - \beta.$$

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = U(z_1, z_2), \quad z_1 = a_1x + b_1y + c_1t, \quad z_2 = a_2x + b_2y + c_2t,$$

where the a_n, b_n, and c_n are arbitrary constants ($n = 1, 2$). To the special case $U = U(z_1)$ there corresponds a traveling wave solution.

3. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right) + bw^p.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{p-n-1}x + C_2, \pm C_1^{p-k-1}y + C_3, \pm C_1^{p-1}t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = F(\xi, \eta), \quad \xi = \alpha_1x + \alpha_2y + \alpha_3t, \quad \eta = \beta_1x + \beta_2y + \beta_3t;$$
$$w(x, y, t) = t^{\frac{2}{1-p}}U(z_1, z_2), \quad z_1 = xt^{\frac{p-n-1}{1-p}}, \quad z_2 = yt^{\frac{p-k-1}{1-p}}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - \left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = \frac{D^2}{(A_1B_2 - A_2B_1)^2}w(A_1x + B_1y + C_1, A_2x + B_2y + C_2, Dt + C_3) + C_4x + C_5y + C_6t + C_7,$$

where A_k, B_k, C_n, and D are arbitrary constants, are also solutions of the equation ($A_1B_2 - A_2B_1 \neq 0$).

2°. There are generalized separable solutions of the form:

$$w = \sum_{i,j=0}^{i+j\le 4} \varphi_{i,j}(t)x^i y^j.$$

Special case 1. There is a solution containing only even powers:

$$w = \psi_1(t) + \psi_2(t)x^2 + \psi_3(t)y^2 + \psi_4(t)x^2y^2 + \psi_5(t)x^4 + \psi_6(t)y^4,$$

where the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1'' &= 4\psi_2\psi_3,\\
\psi_2'' &= 4\psi_2\psi_4 + 24\psi_3\psi_5,\\
\psi_3'' &= 24\psi_2\psi_6 + 4\psi_3\psi_4,\\
\psi_4'' &= -12\psi_4^2 + 144\psi_5\psi_6,\\
\psi_5'' &= 24\psi_4\psi_5,\\
\psi_6'' &= 24\psi_4\psi_6.
\end{aligned}$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 306).

3°. Additive separable solution:

$$w = f(x,y) - \frac{1}{2}At^2 - Bt - C,$$

where A, B, and C are arbitrary constants and the function $f = f(x,y)$ is determined by an equation of the form 11.2.2.2:

$$\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = A.$$

4°. Multiplicative separable solution:

$$w = \frac{\theta(x,y)}{(t+C)^2},$$

where C is an arbitrary constant and the function $\theta = \theta(x,y)$ is determined by an equation of the form 11.2.3.1 with $f(x) = -6$ and $g(x) = 0$:

$$\left(\frac{\partial^2 \theta}{\partial x\partial y}\right)^2 - \frac{\partial^2 \theta}{\partial x^2}\frac{\partial^2 \theta}{\partial y^2} = -6\theta.$$

This equation is a special case of equation 11.2.3.5.

5°. Generalized separable solution:

$$w = f(x,t)y^2 + g(x,t)y + h(x,t),$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are determined by the system of differential equations

$$\begin{aligned}
f_{tt} &= 2ff_x - 4f_x^2,\\
g_{tt} &= 2fg_x - 4f_xg_x,\\
h_{tt} &= 2fh_x - g_x^2.
\end{aligned}$$

6°. Solution:

$$w(x,y,t) = u(z,t), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z,t)$ is determined by the differential equation

$$u_{tt} = 2\big[(4ac-b^2)z + as^2 + ck^2 - bks\big]u_z u_{zz} + (4ac-b^2)(u_z)^2.$$

Special case 2. In particular, if $b = c = k = 0$, we have the equation $u_{tt} = 2as^2 u_z u_{zz}$, which admits exact solutions of the forms

$$u = \varphi_0(t) + \varphi_1(t)z + \varphi_2(t)z^2 + \varphi_3(t)z^3,$$
$$u = \psi_1(t) + \psi_2(t)z^{3/2} + \psi_3(t)z^3.$$

8.2. Equations with Two Space Variables Involving Exponential Nonlinearities

8.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + ae^{\lambda w}$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} + ce^{\lambda w}.$

This is a special case of equation 8.4.1.1 with $f(w) = ce^{\lambda w}$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + ce^{\lambda w}.$

This is a special case of equation 8.4.1.2 with $f(w) = ce^{\lambda w}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ \pm C_1 t + C_2\Big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{2c\lambda(2-n)(2-m)}{4-nm}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right]\right\}.$$

3°. Solution for $n \neq 2$ and $m \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w_{rr}'' + \frac{A}{r}w_r' + ck^{-1}e^{\lambda w} = 0, \qquad A = \frac{2(4-n-m)}{(2-n)(2-m)}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$
$$w(x,y,t) = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = W(y,\zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = F(z_1,z_2) - \frac{2}{\lambda}\ln|t|, \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y|t|^{\frac{2}{m-2}}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + ce^{\beta w}.$

This is a special case of equation 8.4.1.3 with $f(w) = ce^{\beta w}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(C_1^{\frac{2}{2-n}}x,\ y - \frac{2}{\lambda}\ln C_1,\ \pm C_1 t + C_2\Big) + \frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\beta \neq 0$, and $\lambda \neq 0$:

$$w = -\frac{1}{\beta}\ln\left\{\frac{2c\beta(2-n)}{n}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right]\right\}.$$

3°. Solution for $n \neq 2$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w_{rr}'' + \frac{A}{r}w_r' + ck^{-1}e^{\beta w} = 0, \qquad A = \frac{2}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$
$$w(x,y,t) = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = W(y,\zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x,y,t) = F(z_1,z_2) - \frac{2}{\beta}\ln|t|, \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + ce^{\mu w}.$

This is a special case of equation 8.4.1.4 with $f(w) = ce^{\mu w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(x - \frac{2}{\beta}\ln C_1,\ y - \frac{2}{\lambda}\ln C_1,\ \pm C_1 t + C_2\Big) + \frac{2}{\mu}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C_1)^2\right],$$

where C_1 and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}e^{\mu w} = 0.$$

Its general solution is given by

$$w = \begin{cases} -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2k}(r + C_3)^2\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sin^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sinh^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[\dfrac{c\mu}{2kC_2^2}\cosh^2(C_2 r + C_3)\right] & \text{if } ck\mu > 0, \end{cases}$$

where C_2 and C_3 are arbitrary constants.

3°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, t), \quad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$

$$w(x, y, t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = F(z_1, z_2) - \frac{2}{\mu}\ln|t|, \quad z_1 = x + \frac{2}{\beta}\ln|t|, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

8.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(e^{w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 x + C_3, C_2 y + C_4, \pm C_1 t + C_5) + \ln\frac{C_1^2}{C_2^2},$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x, y, t) = \varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,) + \ln|C_1 y + C_2|,$$
$$w(x, y, t) = \ln\big[y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\big],$$
$$w(x, y, t) = \ln\big[y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\big],$$

where C_1 and C_2 are arbitrary constants and $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x, y, t) = \ln\left[\frac{(B^2 - aA^2)(y + D)^2}{b(Ax + Bt + C)^2}\right],$$
$$w(x, y, t) = \ln\left[\frac{(B^2 - aA^2)(y + D)^2}{2\cos^2(Ax + Bt + C)}\right],$$
$$w(x, y, t) = \ln\left[\frac{(aA^2 - B^2)(y + D)^2}{b\cosh^2(Ax + Bt + C)}\right],$$
$$w(x, y, t) = \ln\left[\frac{(B^2 - aA^2)(y + D)^2}{b\sinh^2(Ax + Bt + C)}\right],$$
$$w(x, y, t) = \ln\left(\frac{4aC}{b}\right) - 2\ln\big|(x + A)^2 - a(t + B)^2 + C\big| + 2\ln|y + D|,$$
$$w(x, y, t) = \ln\left[\frac{1}{bB^2}\left(\frac{Ax + By + C}{t + D}\right)^2 - \frac{aA^2}{bB^2}\right],$$
$$w(x, y, t) = \ln\left[\frac{B^2}{bA^2} - \frac{a}{bA^2}\left(\frac{Ay + Bt + C}{x + D}\right)^2\right],$$

where A, B, C, and D are arbitrary constants.

4°. "Two-dimensional" solution:

$$w(x, y, t) = U(x, t) + 2\ln|y + C|,$$

where the function $U = U(x, t)$ is determined by a differential equation of the form 7.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^2 U}{\partial x^2} + 2be^U.$$

Integrating yields a solution of the original equation in the form

$$w(x,y,t) = f(\xi) + g(\eta) - 2\ln\left|k\int e^{f(\xi)}\,d\xi - \frac{b}{4ak}\int e^{g(\eta)}\,d\eta\right| + 2\ln|y+C|,$$
$$\xi = x - \sqrt{a}\,t, \qquad \eta = x + \sqrt{a}\,t,$$

where $f = f(\xi)$ and $g = g(\eta)$ are arbitrary functions and k is an arbitrary constant.

5°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y+\lambda t) + (t\sqrt{a} \pm x)(be^w - \lambda^2) = \varphi(w),$$

where $\varphi(w)$ is an arbitrary function and λ is an arbitrary constant.

6°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = F(y,r), \quad r = x^2 - at^2;$$
$$w(x,y,t) = G(\xi,\eta) - 2k\ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{k-1};$$
$$w(x,y,t) = H(\zeta_1,\zeta_2) + 2\ln|y|, \quad \zeta_1 = t + k_1\ln|y|, \quad \zeta_2 = x + k_2\ln|y|;$$
$$w(x,y,t) = U(\rho_1,\rho_2) + 2y, \quad \rho_1 = te^y, \quad \rho_2 = xe^y;$$
$$w(x,y,t) = V(\chi) + 2\ln|y/t|, \quad \chi = x/t,$$

where k, k_1, and k_2 are arbitrary constants.

7°. There is an exact solution of the form

$$w(x,y,t) = W(z), \quad z = (x^2 - at^2)y^{-2}.$$

8°. For other exact solutions, see equation 8.4.2.3 with $f(w) = a$ and $g(w) = be^w$.

⊙ *References for equation* 8.2.2.1: N. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2002, pp. 294–296).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1x + C_3, \pm C_1y + C_4, C_2t + C_5) + \frac{1}{\lambda}\ln\frac{C_2^2}{C_1^2},$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln(C_3x + C_4y + C_5);$$
$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln\big[C_3(bx^2 - ay^2) + C_4xy + C_5\big];$$
$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln\big[C_3\ln(bx^2 + ay^2) + C_4\big];$$

$$w(x,y,t) = C_1 t + C_2 + \sqrt{b}\,C_3 x + \frac{1}{\lambda}\ln\cos\bigl(\sqrt{a}\,C_3\lambda y + C_4\bigr);$$

$$w(x,y,t) = C_1 t + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\exp\bigl(\sqrt{b}\,C_4 x\bigr)\cos\bigl(\sqrt{a}\,C_4 y + C_5\bigr) + C_6\bigr];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{1}{aC_1^2 + bC_2^2}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{C_2^2(x + C_4)^2}{a(C_1 y + C_2 t + C_3)^2 + bC_1^2(x + C_4)^2}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy + Ky^2 + C_3x + C_4y + C_5}{\cos^2(C_1t + C_6)}\right], \quad K = \frac{C_1^2}{b} - aC_1;$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy + Ky^2 + C_3x + C_4y + C_5}{\sinh^2(C_1t + C_6)}\right], \quad K = \frac{C_1^2}{b} - aC_1;$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy - Ky^2 + C_3x + C_4y + C_5}{\cosh^2(C_1t + C_6)}\right], \quad K = \frac{C_1^2}{b} + aC_1;$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cos^2(aC_1t + C_5)}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cos^2(bC_1t + C_5)}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\sinh^2(aC_1t + C_5)}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\sinh^2(bC_1t + C_5)}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{-aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cosh^2(aC_1t + C_5)}\right];$$

$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{-bC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cosh^2(bC_1t + C_5)}\right];$$

where $C_1, \ldots, C_6$ are arbitrary constants; the first five solutions are degenerate.

3°. "Two-dimensional" solution (generalizes the first five solutions of Item 2°):

$$w(x,y,t) = C_1 t + C_2 + \frac{1}{\lambda}\ln U(\xi,\eta), \qquad \xi = \frac{x}{\sqrt{a}}, \qquad \eta = \frac{y}{\sqrt{b}},$$

where C_1 and C_2 are arbitrary constants and the function $U = U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,t) = f(t) + \frac{1}{\lambda}\ln V(\xi,\eta), \qquad \xi = \frac{x}{\sqrt{a}}, \qquad \eta = \frac{y}{\sqrt{b}},$$

where the function $f = f(t)$ is determined by the autonomous ordinary differential equation

$$f''_{tt} = e^{\lambda f}, \tag{1}$$

and the function $V = V(\xi, \eta)$ is a solution of the Poisson equation

$$\Delta V - \lambda = 0, \qquad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}. \tag{2}$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The general solution of equation (1) is given by

$$f(t) = \begin{cases} -\dfrac{1}{\lambda} \ln\left[\frac{1}{2}\lambda(t + C_1)^2\right] & \text{if } \lambda > 0, \\ -\dfrac{1}{\lambda} \ln\left[\dfrac{\lambda}{2C_1^2} \cos^2(C_1 t + C_2)\right] & \text{if } \lambda > 0, \\ -\dfrac{1}{\lambda} \ln\left[\dfrac{\lambda}{2C_1^2} \sinh^2(C_1 t + C_2)\right] & \text{if } \lambda > 0, \\ -\dfrac{1}{\lambda} \ln\left[-\dfrac{\lambda}{2C_1^2} \cosh^2(C_1 t + C_2)\right] & \text{if } \lambda < 0. \end{cases}$$

5°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = F(z, t) + \frac{2}{\lambda} \ln x, \quad z = \frac{y}{x};$$
$$w(x, y, t) = G(r, t), \quad r = bx^2 + ay^2;$$
$$w(x, y, t) = H(z_1, z_2) - \frac{2k}{\lambda} \ln|t|, \quad z_1 = x|t|^{k-1}, \quad z_2 = y|t|^{k-1};$$
$$w(x, y, t) = U(\xi, \eta) - \frac{2}{\lambda} \ln|t|, \quad \xi = x + k_1 \ln|t|, \quad \eta = y + k_2 \ln|t|;$$
$$w(x, y, t) = V(\rho_1, \rho_2) - \frac{2}{\lambda} t, \quad \rho_1 = xe^t, \quad \rho_2 = ye^t,$$

where k, k_1, and k_2 are arbitrary constants.

6°. There are "one-dimensional" solutions of the following forms:

$$w(x, y, t) = W(z) + \frac{2}{\lambda} \ln\left|\frac{x}{t}\right|, \quad z = \frac{y}{x};$$
$$w(x, y, t) = R(\zeta), \quad \zeta = \frac{bx^2 + ay^2}{t^2}.$$

7°. For other exact solutions, see equation 8.4.2.3 with $f(w) = ae^{\lambda w}$ and $g(w) = be^{\lambda w}$.

⊙ *References for equation* 8.2.2.2: N. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2002, pp. 296–297).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 C_2 x + C_3, \pm C_1 C_2^{\lambda} y + C_4, \pm C_1 t + C_5) - 2\ln|C_2|,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$ak_1^2 e^w + bk_2^2 \lambda^{-1} e^{\lambda w} - \beta^2 w = C_1(k_1 x + k_2 y + \beta t) + C_2,$$

where C_1, C_2, k_1, k_2, and β are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 = aC_1^2 e^w + bC_2^2 e^{\lambda w},$$
$$a\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)^2 e^w + bC_1^2 e^{\lambda w} = C_2^2,$$
$$b\left(\frac{C_1 x + C_2 t + C_3}{y + C_4}\right)^2 e^{\lambda w} + aC_1^2 e^w = C_2^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x, y, t) = u(z, t), \quad z = c_1 x + c_2 y,$$

where the function $u = u(z, t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left[\varphi(u)\frac{\partial u}{\partial z}\right], \quad \varphi(u) = ac_1^2 e^u + bc_2^2 e^{\lambda u},$$

which can be reduced to a linear equation.

5°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x, y, t) = v(x, \xi), \quad \xi = s_1 y + s_2 t,$$

where the function $v = v(x, \xi)$ is determined by a differential equation of the form 9.4.4.8:

$$a\frac{\partial}{\partial x}\left(e^v \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left[\psi(v)\frac{\partial v}{\partial z}\right] = 0, \quad \psi(v) = bs_1^2 e^{\lambda v} - s_2^2,$$

which can be reduced to a linear equation.

6°. There is a "two-dimensional" solution of the form (generalizes the solutions of Items 3° to 5°):

$$w(x,y,t)=U(z_1,z_2),\quad z_1=a_1x+b_1y+c_1t,\quad z_2=a_2x+b_2y+c_2t.$$

7°. Solution:

$$w(x,y,t)=U(\xi)+2\ln(x/t),\quad \xi=x^{-\lambda}yt^{\lambda-1},$$

where the function $U=U(\xi)$ is determined by the ordinary differential equation

$$\big[a\lambda^2\xi^2e^U+be^{\lambda U}-(\lambda-1)^2\xi^2\big]U''_{\xi\xi}+\lambda\big(a\lambda\xi^2e^U+be^{\lambda U}\big)(U'_\xi)^2 +\xi\big[a\lambda(\lambda-3)e^U-(\lambda-1)(\lambda-2)\big]U'_\xi+2(ae^U-1)=0.$$

8°. "Two-dimensional" solution:

$$w(x,y,t)=u(z,t)+2\ln x,\quad z=x^{-\lambda}y,$$

where the function $u=u(z,t)$ is determined by the differential equation

$$\frac{\partial^2u}{\partial t^2}=\big(a\lambda^2z^2e^u+be^{\lambda u}\big)\frac{\partial^2u}{\partial z^2}+\lambda\big(a\lambda z^2e^u+be^{\lambda u}\big)\left(\frac{\partial u}{\partial z}\right)^2+a\lambda(\lambda-3)ze^u\frac{\partial u}{\partial z}+2ae^u.$$

9°. For other exact solutions, see equation 8.4.2.3 with $f(w)=ae^w$ and $g(w)=be^{\lambda w}$.

⊙ *References for equation* 8.2.2.3: N. H. Ibragimov (1994, pp. 222–225), A. D. Polyanin and V. F. Zaitsev (2004, p. 298).

8.2.3. Other Equations

1. $\dfrac{\partial^2w}{\partial t^2}=\dfrac{\partial}{\partial x}\left[(a_1e^{\lambda w}+b_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2e^{\lambda w}+b_2)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1x+C_2,\pm C_1y+C_3,\pm C_1t+C_4),$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,y,t)=\frac{1}{\lambda}\ln\big(C_1x+C_2y+\beta t+C_3\big),\quad \beta=\pm\sqrt{b_1C_1^2+b_2C_2^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x,y,t)=\frac{1}{\lambda}\ln\left[\frac{1}{a_1C_1^2+a_2C_2^2}\left(\frac{C_1x+C_2y+C_3}{t+C_4}\right)^2-\frac{b_1C_1^2+b_2C_2^2}{a_1C_1^2+a_2C_2^2}\right],$$

$$w(x,y,t)=\frac{1}{\lambda}\ln\left[\frac{(C_2^2-b_2C_1^2)(x+C_4)^2-b_1(C_1y+C_2t+C_3)^2}{a_2C_1^2(x+C_4)^2+a_1(C_1y+C_2t+C_3)^2}\right],$$

$$w(x,y,t)=\frac{1}{\lambda}\ln\left[\frac{(C_2^2-b_1C_1^2)(y+C_4)^2-b_2(C_1x+C_2t+C_3)^2}{a_1C_1^2(y+C_4)^2+a_2(C_1x+C_2t+C_3)^2}\right],$$

where $C_1,\ldots,C_4$ are arbitrary constants.

4°. For other solutions, see equation 8.4.2.3 with $f(w)=a_1e^{\lambda w}+b_1$ and $g(w)=a_2e^{\lambda w}+b_2$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + ce^{\beta w}.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1}x + C_2, \pm C_1^{\beta-\lambda_2}y + C_3, \pm C_1^{\beta}t + C_4) + 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, \eta) - \frac{2}{\beta}\ln|t|, \quad \xi = x|t|^{\frac{\lambda_1-\beta}{\beta}}, \quad \eta = y|t|^{\frac{\lambda_2-\beta}{\beta}};$$
$$w(x, y, t) = V(\eta_1, \eta_2), \quad \eta_1 = a_1x + b_1y + c_1t, \quad \eta_2 = a_2x + b_2y + c_2t.$$

8.3. Nonlinear Telegraph Equations with Two Space Variables

8.3.1. Equations Involving Power-Law Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(bw + c)\dfrac{\partial w}{\partial y}\right].$

1°. "Two-dimensional" generalized separable solution linear in y:

$$w = f(x, t)y + g(x, t),$$

where the functions f and g are determined by the one-dimensional equations

$$\frac{\partial^2 f}{\partial t^2} + k\frac{\partial f}{\partial t} = a\frac{\partial^2 f}{\partial x^2},$$
$$\frac{\partial^2 g}{\partial t^2} + k\frac{\partial g}{\partial t} = a\frac{\partial^2 g}{\partial x^2} + bf^2.$$

The first equation is a linear homogeneous telegraph equation. Given $f = f(x, t)$, the second one represents a linear nonhomogeneous telegraph equation. For these equations, see the book by Polyanin (2002).

2°. There is a "two-dimensional" generalized separable solution quadratic in y:

$$w = f(x, t)y^2 + g(x, t)y + h(x, t).$$

3°. The substitution $u = w + (c/b)$ leads to the special case of equation 8.3.1.4 with $m = 1$.

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a_1w + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2w + b_2)\dfrac{\partial w}{\partial y}\right].$

1°. Additive separable solution:

$$w(x, y, t) = Akx + Bky + Ce^{-kt} + k(A^2a_1 + B^2a_2)t + D,$$

where A, B, C, and D are arbitrary constants.

2°. Generalized separable solution linear in the space variables:

$$w(x,y,t) = (A_1e^{-kt} + B_1)x + (A_2e^{-kt} + B_2)y + \frac{1}{2k^2}(a_1A_1^2 + a_2A_2^2)e^{-2kt} - \frac{2}{k}(a_1A_1B_1 + a_2A_2B_2)te^{-kt} + C_1e^{-kt} + \frac{1}{k}(a_1B_1^2 + a_2B_2^2)t + C_2,$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants.

3°. Traveling-wave solution in implicit form ($k \neq 0$):

$$k\lambda(a_1\beta_1^2 + a_2\beta_2^2)w + [k\lambda(b_1\beta_1^2 + b_2\beta_2^2 - \lambda^2) - C_1(a_1\beta_1^2 + a_2\beta_2^2)]\ln(k\lambda w + C_1) = k^2\lambda^2(\beta_1x + \beta_2y + \lambda t) + C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

4°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\left[\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)\right].$

This is a special case of equation 8.3.1.6 with $n = m = -1$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1x + C_2, \pm C_1y + C_3, t + C_4),$$
$$w_2 = w(x\cos\beta + y\sin\beta,\ -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y,t) = \frac{2at + A + Be^{-kt}}{k(\sin y + Ce^x)^2},$$
$$w(x,y,t) = \frac{C_1^2(2at + A + Be^{-kt})}{ke^{2x}\sinh^2(C_1e^{-x}\sin y + C_2)},$$
$$w(x,y,t) = \frac{C_1^2(-2at + A + Be^{-kt})}{ke^{2x}\cosh^2(C_1e^{-x}\sin y + C_2)},$$
$$w(x,y,t) = \frac{C_1^2(2at + A + Be^{-kt})}{ke^{2x}\cos^2(C_1e^{-x}\sin y + C_2)},$$

where A, B, C, C_1, and C_2 are arbitrary constants.

3°. The exact solutions specified in Item 2° are special cases of a more general solution in the form of the product of functions with different arguments:

$$w(x,y,t) = (Aat + B + Ce^{-kt})e^{\Theta(x,y)},$$

where A, B, and C are arbitrary constants and the function $\Theta(x,y)$ is a solution of the stationary equation

$$\Delta\Theta - Ake^{\Theta} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which occurs in combustion theory. For solutions of this equation, see 9.2.1.1.

⊙ *Reference for equation* 8.3.1.3: N. H. Ibragimov (1994, p. 245).

4. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 8.3.1.6 with $n = 0$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm x + C_2, \pm C_1^m y + C_3, t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" solution:

$$w(x, y, t) = u(x, t) y^{2/m},$$

where the function $u(x, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{2b(m+2)}{m^2} u^{m+1}.$$

For $m = -2$ and $m = -1$, this equation is linear.

3°. "Two-dimensional" multiplicative separable solution:

$$w(x, y, t) = \begin{cases} U(x, t)|y + C|^{1/(m+1)} & \text{if } m \neq -1, \\ U(x, t)\exp(Cy) & \text{if } m = -1, \end{cases}$$

where C is an arbitrary constant and the function $U(x, t)$ is determined by the telegraph equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2}.$$

For solutions of this linear equation, see the book by Polyanin (2002).

⊙ *Reference for equation* 8.3.1.4: N. H. Ibragimov (1994, pp. 245–246).

5. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^n \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n x + C_2, \pm C_1^n y + C_3, t + C_4),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x, y, t) = F(t)\Phi(x, y),$$

where the function $F(t)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$F_{tt}'' + kF_t' = CF^{n+1}, \tag{1}$$

and the function $\Phi(x, y)$ satisfies the stationary equation

$$a\frac{\partial}{\partial x}\left(\Phi^n\frac{\partial\Phi}{\partial x}\right)+b\frac{\partial}{\partial y}\left(\Phi^n\frac{\partial\Phi}{\partial y}\right)=C\Phi. \tag{2}$$

Special case. For $C = 0$, it follows from (1) that $F = Ae^{-kt} + B$, where A and B are arbitrary constants. For $C = 0$, equation (2) is reduced to the Laplace equation

$$\frac{\partial^2\Psi}{\partial\bar{x}^2}+\frac{\partial^2\Psi}{\partial\bar{y}^2}=0, \qquad \text{where } \Psi=\Phi^{n+1},\ \ \bar{x}=\frac{x}{\sqrt{a}},\ \ \bar{y}=\frac{y}{\sqrt{b}}.$$

3°. "Two-dimensional" solution:

$$w(x, y, t) = u(r, t), \quad r=\sqrt{bx^2+ay^2},$$

where the function $u(r, t)$ is determined by the differential equation

$$\frac{\partial^2u}{\partial t^2}+k\frac{\partial u}{\partial t}=\frac{ab}{r}\frac{\partial}{\partial r}\left(ru^n\frac{\partial u}{\partial r}\right).$$

4°. Solution:

$$w(x, y, t) = U(t)(bx^2 + ay^2)^{1/n},$$

where the function $U(t)$ is determined by the autonomous ordinary differential equation

$$U''_{tt}+kU'_t=\frac{4ab(n+1)}{n^2}U^{n+1}.$$

⊙ *Reference for equation* 8.3.1.5: N. H. Ibragimov (1994, p. 245).

6. $\dfrac{\partial^2 w}{\partial t^2}+k\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right)$.

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^n x + C_2, \pm C_1^m y + C_3, t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\frac{a_1\beta_1^2w^n+a_2\beta_2^2w^m-\lambda^2}{k\lambda w+C_1}\,dw=\beta_1x+\beta_2y+\lambda t+C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, t), \quad \xi=\beta x+\mu y,$$

where β and μ are arbitrary constants, and the function $U = U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2U}{\partial t^2}+k\frac{\partial U}{\partial t}=\frac{\partial}{\partial\xi}\left[\left(a\beta^2U^n+b\mu^2U^m\right)\frac{\partial U}{\partial\xi}\right].$$

Remark. There is a more general, "two-dimensional" solution of the form

$$w(x, y, t) = V(\xi_1, \xi_2), \quad \xi_1=\beta_1x+\mu_1y+\lambda_1t, \quad \xi_2=\beta_2x+\mu_2y+\lambda_2t,$$

where the β_i, μ_i, and λ_i are arbitrary constants.

4°. "Two-dimensional" solution:

$$w(x,y,t) = y^{2/m}u(z,t), \quad z = xy^{-n/m},$$

where the function $u = u(z,t)$ is determined by the differential equation

$$m^2\left(\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t}\right) = \left(am^2u^n + bn^2z^2u^m\right)\frac{\partial^2 u}{\partial z^2}$$
$$+ nm\left(amu^{n-1} + bnz^2u^{m-1}\right)\left(\frac{\partial u}{\partial z}\right)^2 + bn(n-3m-4)zu^m\frac{\partial u}{\partial z} + 2b(m+2)u^{m+1}.$$

⊙ *Reference for equation* 8.3.1.6: N. H. Ibragimov (1994, pp. 245–246).

8.3.2. Equations Involving Exponential Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(e^w\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 8.4.3.10 with $f(t) = k$, $g(t) = a$, $h(t) = b$, and $\lambda = 1$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_2, \pm C_1 y + C_3, t + C_4) - 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" solution:

$$w(x,y,t) = u(x,t) + \ln|y + C|,$$

where C is an arbitrary constant and the function $u(x,t)$ is determined by the equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

For solutions of this linear equation, see the book by Polyanin (2002).

3°. "Two-dimensional" solution:

$$w(x,y,t) = U(x,t) + 2\ln|y + C|,$$

where C is an arbitrary constant and the function $U(x,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + 2be^U.$$

⊙ *Reference for equation* 8.3.2.1: N. H. Ibragimov (1994, p. 245).

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^w \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, t + C_4) - 2\ln|C_1|,$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" additive separable solution:

$$w(x, y, t) = \varphi(t) + \ln[\psi(x, y)],$$

where the function $\varphi(t)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$\varphi''_{tt} + k\varphi'_t = Ce^{\varphi},$$

and the function $\psi(x, y)$ satisfies the Poisson equation

$$\frac{\partial^2 \psi}{\partial \bar{x}^2} + \frac{\partial^2 \psi}{\partial \bar{y}^2} = C, \qquad \text{where } \bar{x} = \frac{x}{\sqrt{a}},\ \ \bar{y} = \frac{y}{\sqrt{b}}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. "Two-dimensional" solution:

$$w(x, y, t) = u(r, t), \quad r = \sqrt{bx^2 + ay^2},$$

where the function $u(r, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = \frac{ab}{r}\frac{\partial}{\partial r}\left(re^u \frac{\partial u}{\partial r}\right).$$

4°. Solution:

$$w(x, y, t) = u(t) + \ln(bx^2 + ay^2),$$

where the function $u(t)$ is determined by the autonomous ordinary differential equation

$$u''_{tt} + ku'_t = 4abe^u.$$

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = u(z, t) + 2\ln|x|, \quad z = y/x.$$

⊙ *Reference for equation* 8.3.2.2: N. H. Ibragimov (1994, pp. 245–246).

3. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^w\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1^{\lambda} y + C_3, t + C_4) - 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \frac{a_1\beta_1^2 e^w + a_2\beta_2^2 e^{\lambda w} - \gamma^2}{k\gamma w + C_1}\,dw = \beta_1 x + \beta_2 y + \gamma t + C_2,$$

where C_1, C_2, β_1, β_2, and γ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, t), \quad \xi = \beta x + \mu y,$$

where β and μ are arbitrary constants, and the function $U = U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = \frac{\partial}{\partial \xi}\left[\left(a\beta^2 e^w + b\mu^2 e^{\lambda w}\right)\frac{\partial U}{\partial \xi}\right].$$

Remark. There is a more general, "two-dimensional" solution of the form

$$w(x, y, t) = V(\xi_1, \xi_2), \quad \xi_1 = \beta_1 x + \mu_1 y + \sigma_1 t, \quad \xi_2 = \beta_2 x + \mu_2 y + \sigma_2 t,$$

where the β_i, μ_i, and σ_i are arbitrary constants.

4°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = u(z, t) + 2\ln|x|, \quad z = y|x|^{-\lambda}.$$

⊙ *Reference for equation* 8.3.2.3: N. H. Ibragimov (1994, pp. 245–246).

8.4. Equations with Two Space Variables Involving Arbitrary Functions

8.4.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + h(w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, \pm t + C_3),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta,\ -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),$$
$$w_3 = w\big(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$
$$w_4 = w\big(x,\ y\cosh\lambda + tb^{1/2}\sinh\lambda,\ yb^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$

where C_1, C_2, C_3, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[C_1 + \frac{2}{\lambda^2 - ak_1^2 - bk_2^2} F(w)\right]^{-1/2} dw = k_1 x + k_2 y + \lambda t + C_2, \qquad F(w) = \int f(w)\, dw,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution (C_1, C_2, and C_3 are arbitrary constants):

$$w = w(r), \qquad r^2 = A\left[\frac{(x + C_1)^2}{a} + \frac{(y + C_2)^2}{b} - (t + C_3)^2\right].$$

Here, A and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + 2r^{-1} w'_r + A^{-1} f(w) = 0.$$

4°. "Two-dimensional" solution:

$$w = U(\xi, \eta), \quad \xi = \frac{x}{\sqrt{a}C} + \frac{y}{\sqrt{b}}, \quad \eta = (C^2 - 1)\frac{x^2}{a} - 2C\frac{xy}{\sqrt{ab}} - C^2 t^2, \tag{1}$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi \frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} + f(U) = 0. \tag{2}$$

Remark. Relations (1) and equation (2) can be used to obtain other "two-dimensional" solutions by means of the following renaming: $(x, a) \rightleftarrows (y, b)$.

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = u(z_1, z_2), \qquad z_1 = C_1 x + C_2 y + \lambda_1 t, \qquad z_2 = C_3 x + C_4 y + \lambda_2 t.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) + f(w)$.

1°. Solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t + C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + k^{-1} f(w) = 0, \qquad A = \frac{2(4 - n - m)}{(2 - n)(2 - m)}.$$

2°. "Two-dimensional" solution for $n \neq 2$ and $m \neq 2$:

$$w = U(\xi, t), \qquad \xi^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{B_1}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U), \qquad B_1 = \frac{4 - nm}{(2-n)(2-m)}.$$

3°. "Two-dimensional" solution for $m \neq 2$:

$$w = V(x, \eta), \qquad \eta^2 = 4k\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $V(x, \eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + k\left(\frac{\partial^2 V}{\partial \eta^2} + \frac{B_2}{\eta}\frac{\partial V}{\partial \eta}\right) + f(V) = 0, \qquad B_2 = \frac{2}{2-m}.$$

4°. "Two-dimensional" solution for $n \neq 2$:

$$w = W(y, \zeta), \qquad \zeta^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $W(y, \zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + k\left(\frac{\partial^2 W}{\partial \zeta^2} + \frac{B_3}{\zeta}\frac{\partial W}{\partial \zeta}\right) + f(W) = 0, \qquad B_3 = \frac{2}{2-n}.$$

⊙ *Reference for equation* 8.4.1.2: A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin and V. F. Zaitsev (2004, pp. 306).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y} \dfrac{\partial w}{\partial y}\right) + f(w).$

1°. Solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + k^{-1} f(w) = 0, \qquad A = \frac{2}{2-n}.$$

2°. "Two-dimensional" solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = U(\xi, t), \qquad \xi^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{B}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U), \qquad B = \frac{n}{2-n}.$$

3°. "Two-dimensional" solution for $\lambda \neq 0$:

$$w = V(x,\eta), \qquad \eta^2 = 4k\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $V(x,\eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + k\frac{\partial^2 V}{\partial \eta^2} + f(V) = 0.$$

4°. "Two-dimensional" solution for $n \neq 2$:

$$w = W(y,\zeta), \qquad \zeta^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $W(y,\zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + k\left(\frac{\partial^2 W}{\partial \zeta^2} + \frac{A}{\zeta}\frac{\partial W}{\partial \zeta}\right) + f(W) = 0, \qquad A = \frac{2}{2-n}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + f(w).$

1°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + k^{-1}f(w) = 0.$$

Integrating yields its general solution in implicit form:

$$\int\left[C_1 + 2k^{-1}\int f(w)\,dw\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

2°. "Two-dimensional" solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = U(\xi,t), \qquad \xi^2 = 4k\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$

where the function $U(\xi,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U).$$

3°. "Two-dimensional" solution for $\lambda \neq 0$:

$$w = V(x,\eta), \qquad \eta^2 = 4k\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $V(x,\eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial w}{\partial x}\right)+k\frac{\partial^2 V}{\partial \eta^2}+f(V)=0.$$

4°. "Two-dimensional" solution for $\beta\neq 0$:

$$w=W(y,\zeta),\qquad \zeta^2=4k\left[\frac{e^{-\beta x}}{a\beta^2}-\frac{1}{4}(t+C)^2\right],$$

where the function $W(y,\zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right)+k\frac{\partial^2 W}{\partial \zeta^2}+f(W)=0.$$

⊙ *References*: A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin and V. F. Zaitsev (2002).

5. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]+aw\ln w+bw.$

This is a special case of equation 8.4.3.6 with $g(t)=b$ and $h_1(x)=h_2(y)=0$ and a special case of equation 8.4.3.7 with $f(x,y)=f(x)$ and $g(x,y)=g(y)$.

8.4.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]+h(w)$

1. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1x+C_2,\pm C_1y+C_3,\pm C_1t+C_4),$$
$$w_2=w(x\cosh\lambda+ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda+t\cosh\lambda),$$

where $C_1,\ldots,C_4$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$k_2^2\int g(w)\,dw+(ak_1^2-\lambda^2)w=C_1(k_1x+k_2y+\lambda t)+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\int g(w)\,dw=y\varphi_1\big(x\pm t\sqrt{a}\,\big)+\varphi_2\big(x\pm t\sqrt{a}\,\big);$$
$$2\lambda\sqrt{a}\,(y+\lambda t)+(t\sqrt{a}\pm x)[g(w)-\lambda^2]=\psi(w),$$

where $\varphi_1(z)$, $\varphi_2(z)$, and $\psi(w)$ are arbitrary functions, and λ is an arbitrary constant.

4°. "Two-dimensional" solutions (generalize the solutions of Item 3°):

$$w(x,y,t) = U(\xi,\eta), \quad \xi = y + \lambda t, \quad \eta = x \pm t\sqrt{a},$$

where λ is an arbitrary constant and the function $U = U(\xi,\eta)$ is determined by the first-order partial differential equation

$$\big[g(U) - \lambda^2\big]\frac{\partial U}{\partial \xi} \mp 2\lambda\sqrt{a}\,\frac{\partial U}{\partial \eta} = \varphi(\eta), \tag{1}$$

with $\varphi(\eta)$ being an arbitrary function.

In the special case $\lambda = 0$, equation (1) is an ordinary differential equation in ξ that can be readily integrated to obtain the first group of solutions specified in Item 3°.

In the general case, the characteristic system corresponding to equation (1) has the form (Polyanin, Zaitsev, and Moussiaux, 2002)

$$\frac{d\xi}{g(U) - \lambda^2} = \mp\frac{d\eta}{2\lambda\sqrt{a}} = \frac{dU}{\varphi(\eta)}.$$

Its independent integrals are given by

$$U \pm \Phi(\eta) = C_1, \quad \xi \pm \frac{1}{2\lambda\sqrt{a}}\int g\big(C_1 \mp \Phi(\eta)\big)\,d\eta \mp \frac{\lambda}{2\sqrt{a}}\eta = C_2, \tag{2}$$

where

$$\Phi(\eta) = \frac{1}{2\lambda\sqrt{a}}\int \varphi(\eta)\,d\eta.$$

We first calculate the integral in the second relation of (2) and then, in the resulting expression, substitute the left-hand side of the first relation of (2) for C_1.

The general solution of equation (1) has the form

$$F(C_1, C_2) = 0,$$

where $F(C_1, C_2)$ is an arbitrary function of two variables, and C_1 and C_2 are determined by (2).

To the special case $\varphi(\eta) = 0$ in (1) there corresponds the second group of solutions specified in Item 3°.

5°. "Two-dimensional" solution:

$$w(x,y,t) = u(y,z), \quad z = x^2 - at^2,$$

where the function $u = u(y,z)$ is determined by the differential equation

$$4a\left(z\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] = 0.$$

6°. Solution:

$$w(x,y,t) = v(\zeta), \quad \zeta = (x^2 - at^2)y^{-2},$$

where the function $v = v(\zeta)$ is determined by the ordinary differential equation

$$2a\zeta v''_{\zeta\zeta} + 2av'_\zeta + 2\zeta^2[g(v)v'_\zeta]'_\zeta + 3\zeta g(v)v'_\zeta = 0.$$

7°. For other exact solutions, see equation 8.4.2.3 with $f(w) = a$.

⊙ *References for equation* 8.4.2.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 308–309).

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$(k_1^2 + k_2^2)\int f(w)\,dw - \lambda^2 w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w(x, y, t) = U(\zeta), \quad \zeta = (x^2 + y^2)t^{-2},$$

where the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$2\zeta^2 U''_{\zeta\zeta} + 3\zeta U'_\zeta = 2[\zeta f(U)U'_\zeta]'_\zeta.$$

4°. "Two-dimensional" solution with axial symmetry:

$$w(x, y, t) = u(r, t), \quad r = \sqrt{x^2 + y^2},$$

where the function $u = u(r, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[r f(u)\frac{\partial u}{\partial r}\right].$$

5°. For other exact solutions, see equation 8.4.2.3 with $f(w) = g(w)$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the function

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[k_1^2 f(w) + k_2^2 g(w)\right] dw - \lambda^2 w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1x+C_2y+C_3}{t+C_4}\right)^2 = C_1^2 f(w) + C_2^2 g(w),$$
$$\left(\frac{C_1y+C_2t+C_3}{x+C_4}\right)^2 f(w) + C_1^2 g(w) = C_2^2,$$
$$\left(\frac{C_1x+C_2t+C_3}{y+C_4}\right)^2 g(w) + C_1^2 f(w) = C_2^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. Solutions in implicit form:

$$x\frac{\sin\varphi_1(w)}{\sqrt{f(w)}} + y\frac{\cos\varphi_1(w)}{\sqrt{g(w)}} + t = \psi_1(w),$$
$$x\frac{\sin\varphi_2(w)}{\sqrt{f(w)}} + y\frac{\cos\varphi_2(w)}{\sqrt{g(w)}} - t = \psi_2(w),$$

where $\varphi_1(w)$, $\varphi_2(w)$, $\psi_1(w)$, and $\psi_2(w)$ are arbitrary functions.

5°. Solution:

$$w = w(\xi), \quad \xi = \frac{C_1x+C_2y+C_3}{t+C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 w'_\xi)'_\xi = [\varphi(w)w'_\xi]'_\xi, \qquad \varphi(w) = C_1^2 f(w) + C_2^2 g(w),$$

which admits the first integral

$$\left[\xi^2 - C_1^2 f(w) - C_2^2 g(w)\right] w'_\xi = C_5. \tag{1}$$

To the special case $C_5 = 0$ there corresponds the first solution of Item 3°.

For $C_5 \neq 0$, treating w in (1) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C_5\xi'_w = \xi^2 - C_1^2 f(w) - C_2^2 g(w). \tag{2}$$

For exact solutions of equation (2), which is reduced to a second-order linear equation, see the book by Polyanin and Zaitsev (2003).

6°. Solution:

$$w = u(\eta), \quad \eta = \frac{C_1y+C_2t+C_3}{x+C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $u(\eta)$ is determined by the ordinary differential equation

$$C_2^2 u''_{\eta\eta} = [\eta^2 f(u)u'_\eta]'_\eta + C_1^2[g(u)u'_\eta]'_\eta,$$

which admits the first integral

$$[\eta^2 f(u) + C_1^2 g(u) - C_2^2]u'_\eta = C_5. \tag{3}$$

To the special case $C_5 = 0$ there corresponds the second solution of Item 3°.

For $C_5 \neq 0$, treating u in (3) as the independent variable, one obtains a Riccati equation for $\eta = \eta(u)$:

$$C_5\eta'_u = \eta^2 f(u) + C_1^2 g(u) - C_2^2. \tag{4}$$

For exact solutions of equation (4), which is reduced to a second-order linear equation, see the book by Polyanin and Zaitsev (2003).

7°. Solution:

$$w = v(\zeta), \quad \zeta = \frac{C_1x + C_2t + C_3}{y + C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $v(\zeta)$ is determined by the first-order ordinary differential equation

$$[\zeta^2 g(v) + C_1^2 f(v) - C_2^2]v'_\zeta = C_5.$$

To the special case $C_5 = 0$, there corresponds the third solution of Item 3°. The inverse function $\zeta = \zeta(v)$ is determined by the Riccati equation that can be obtained from (4) with the renaming $u \to v$, $\eta \to \zeta$, and $f \rightleftarrows g$.

8°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x, y, t) = U(z, t), \quad z = ax + by,$$

where the function $U = U(z, t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial z}\left[\varphi(U)\frac{\partial U}{\partial z}\right], \quad \varphi(U) = a^2 f(U) + b^2 g(U),$$

which can be reduced to a linear equation.

9°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x, y, t) = V(x, \xi), \quad \xi = ay + bt,$$

where the function $V = V(x, \xi)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial x}\left[f(V)\frac{\partial V}{\partial x}\right] + \frac{\partial}{\partial \xi}\left[\psi(V)\frac{\partial V}{\partial \xi}\right] = 0, \quad \psi(V) = a^2 g(V) - b^2,$$

which can be reduced to a linear equation.

10°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x, y, t) = W(y, \eta), \quad \eta = ax + bt,$$

where the function $W = W(y, \eta)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial y}\left[g(W)\frac{\partial W}{\partial x}\right] + \frac{\partial}{\partial \eta}\left[\chi(W)\frac{\partial W}{\partial \eta}\right] = 0, \quad \chi(W) = a^2 f(W) - b^2,$$

which can be reduced to a linear equation.

11°. There is a "two-dimensional" solution of the form (generalizes the solutions of Items 8° to 10°):

$$w(x, y, t) = Q(z_1, z_2), \quad z_1 = a_1x + b_1y + c_1t, \quad z_2 = a_2x + b_2y + c_2t.$$

12°. "Two-dimensional" solution:

$$w(x, y, t) = R(\xi, \eta), \quad \xi = xt^{-1}, \quad \eta = yt^{-1},$$

where the function $R = R(\xi, \eta)$ is determined by the differential equation

$$\xi^2\frac{\partial^2 R}{\partial \xi^2} + 2\xi\eta\frac{\partial^2 R}{\partial \xi \partial \eta} + \eta^2\frac{\partial^2 R}{\partial \eta^2} + 2\xi\frac{\partial R}{\partial \xi} + 2\eta\frac{\partial R}{\partial \eta} = \frac{\partial}{\partial \xi}\left[f(R)\frac{\partial R}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[g(R)\frac{\partial R}{\partial \eta}\right].$$

13°. For results of the group analysis of the original equation, see Ibragimov (1994).

⊙ *References for equation* 8.4.2.3: A. D. Polyanin and V. F. Zaitsev (2004, pp. 310–312), A. D. Polyanin (2006, http://eqworld.ipmnet.ru/eqarchive/view.php?id=44).

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + g(w).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, \pm t + C_3),$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int h(w)\left[C_1 - 2\int h(w)g(w)\,dw\right]^{-1/2} dw = k_1x + k_2y + \lambda t + C_2, \qquad h(w) = k_2^2 f(w) + ak_1^2 - \lambda^2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\int f(w)\left[\varphi\big(x + t\sqrt{a}\,\big) - 2\int f(w)g(w)\,dw\right]^{-1/2} dw = \psi\big(x + t\sqrt{a}\,\big) \pm y,$$
$$\int f(w)\left[\varphi\big(x - t\sqrt{a}\,\big) - 2\int f(w)g(w)\,dw\right]^{-1/2} dw = \psi\big(x - t\sqrt{a}\,\big) \pm y,$$

where $\varphi(z)$ and $\psi(z)$ are arbitrary functions.

4°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(y, z), \quad z = x^2 - at^2;$$
$$w(x, y, t) = V(\xi, \eta), \quad \xi = A_1x + B_1y + C_1t, \quad \eta = A_2x + B_2y + C_2t.$$

⊙ *References for equation* 8.4.2.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 312).

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + g(w).$

1°. Suppose the function $f = f(w)$ is arbitrary and the function $g = g(w)$ is defined as

$$g(w) = -a^2\frac{f'(w)}{f^3(w)} + b,$$

where a and b are some numbers. Then there is a functional separable solution, which can be represented in the implicit form

$$\int f(w)\,dw = at + U(x, y),$$

where the function $U = U(x, y)$ is determined by the Poisson equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + b = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The constant b in the expression of g can be replaced by an arbitrary function $b = b(x, y)$.

2°. Suppose that the determining functions are defined parametrically as

$$f = \frac{Ae^{-bz}}{\varphi'_z(z)},$$
$$g = a^2\varphi''_{zz}(z) - Ace^{-bz},$$
$$w = \varphi(z),$$

where $\varphi(z)$ is an arbitrary function, z is the parameter, and A, a, b, and c are some numbers. Then the equation in question has a functional separable solution of the form

$$w = \varphi(z), \quad z = at + \theta(x, y),$$

where θ is a function that satisfies the equation

$$\Delta\theta = b|\nabla\theta|^2 + c.$$

The substitution

$$\theta = -\frac{1}{b}\ln|u|$$

reduces this equation to the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + bcu = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

8.4.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + kf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m}\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$
$$w(x, y, t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$
$$w(x, y, t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right].$$

2. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\lambda y} \dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + kf(w) = 0, \qquad A = \frac{2(3-n)}{2-n}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$
$$w = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$
$$w = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right].$$

3. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\beta x} \dfrac{\partial^2 w}{\partial x^2} + be^{\lambda y} \dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + 4r^{-1} w'_r + kf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \quad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$
$$w = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$
$$w = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t+C)^2\right].$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \big[aw + f(t)\big]\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + bw^2 + g(t)w + h(t), \qquad a \neq 0.$

Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)\Theta(x, y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = b\varphi^2 + g(t)\varphi + h(t),$$
$$\psi'_{tt} = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a,$$

and the function $\Theta = \Theta(x, y)$ satisfies the two-dimensional Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

5. $\dfrac{\partial^2 w}{\partial t^2} = aw\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - a\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] + f(t).$

1°. Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)e^{\beta x + \gamma y},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = f(t), \qquad \psi''_{tt} = a(\beta^2 + \gamma^2)\varphi\psi.$$

The solution of the first equation is expressed as (C_1 and C_2 are arbitrary constants)

$$\varphi(t) = \int_0^t (t - \tau)f(\tau)\,d\tau + C_1 t + C_2.$$

The solution of the second equation, which is linear in ψ, can be found in Kamke (1977) and Polyanin and Zaitsev (1995, 2003) for many $f(t)$.

2°. There are generalized separable solutions of the following forms:

$$w(x, y, t) = \varphi(t) + \psi(t)(A_1 \cosh \mu x + A_2 \sinh \mu x) + \chi(t)(B_1 \cos \mu y + B_2 \sin \mu y),$$
$$w(x, y, t) = \varphi(t) + \psi(t)(A_1 \cos \mu x + A_2 \sin \mu x) + \chi(t)(B_1 \cosh \mu y + B_2 \sinh \mu y),$$

where A_1, A_2, B_1, B_2, and μ are arbitrary constants, and the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by solving an appropriate system of second-order ordinary differential equations (not written out here).

3°. There are generalized separable solutions of the form

$$w(x, y, t) = \varphi(t) + \psi(t)F(x) + \chi(t)G(y) + \eta(t)H(x)P(y),$$

where

$$F(x) = A_1 \cos 2\mu x + A_2 \sin 2\mu x, \qquad G(y) = B_1 \cosh 2\mu y + B_2 \sinh 2\mu y,$$
$$H(x) = C_1 \cos \mu x + C_2 \sin \mu x, \qquad P(y) = D_1 \cosh \mu y + D_2 \sinh \mu y.$$

The arbitrary constants $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$, and μ are related by two constraints, and the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$, and $\eta(t)$ satisfy a system of second-order ordinary differential equations (not written out here).

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f_1(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(y)\dfrac{\partial w}{\partial y}\right] + aw\ln w + \big[g(t) + h_1(x) + h_2(y)\big]w.$

Multiplicative separable solution:

$$w(x, y, t) = \varphi(x)\psi(y)\chi(t),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(y)$, and $\chi = \chi(t)$ are determined by the ordinary differential equations

$$[f_1(x)\varphi_x']_x' + a\varphi\ln\varphi + [h_1(x) + C_1]\varphi = 0,$$
$$[f_2(y)\psi_y']_y' + a\psi\ln\psi + [h_2(y) + C_2]\psi = 0,$$
$$\chi_{tt}'' - a\chi\ln\chi - [g(t) - C_1 - C_2]\chi = 0,$$

and C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x, y)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(x, y)\dfrac{\partial w}{\partial y}\right] + kw\ln w.$

Multiplicative separable solution:

$$w(x, y, t) = \varphi(t)\Theta(x, y),$$

where the function $\varphi(t)$ is determined by the ordinary differential equation

$$\varphi_{tt}'' - k\varphi\ln\varphi - A\varphi = 0, \tag{1}$$

A is an arbitrary constant, and the function $\Theta(x, y)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x, y)\frac{\partial\Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x, y)\frac{\partial\Theta}{\partial y}\right] + k\Theta\ln\Theta - A\Theta = 0.$$

A particular solution of equation (1) is given by (B is an arbitrary constant)

$$\varphi(t) = \exp\left[\frac{k}{4}(t + B)^2 + \frac{k - 2A}{2k}\right],$$

and the general solution can be written out in implicit form as (B and C are arbitrary constants)

$$\int\left[k\varphi^2\ln\varphi + (A - \tfrac{1}{2}k)\varphi^2 + B\right]^{-1/2} d\varphi = C \pm t.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = f_1(x, y)\dfrac{\partial^2 w}{\partial x^2} + f_2(x, y)\dfrac{\partial^2 w}{\partial x\partial y} + f_3(x, y)\dfrac{\partial^2 w}{\partial y^2}$
$+ g_1(x, y)\dfrac{\partial w}{\partial x} + g_2(x, y)\dfrac{\partial w}{\partial y} + \big[h(x, y) + s(t)\big]w + kw\ln w.$

Multiplicative separable solution:

$$w(x, y, t) = \varphi(t)\theta(x, y),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi_{tt}'' - k\varphi\ln\varphi - \big[s(t) + C\big]\varphi = 0,$$

and the function $\theta = \theta(x, y)$ satisfies the stationary equation

$$f_1(x, y)\frac{\partial^2\theta}{\partial x^2} + f_2(x, y)\frac{\partial^2\theta}{\partial x\partial y} + f_3(x, y)\frac{\partial^2\theta}{\partial y^2}$$
$$+ g_1(x, y)\frac{\partial\theta}{\partial x} + g_2(x, y)\frac{\partial\theta}{\partial y} + \big[h(x, y) - C\big]\theta + k\theta\ln\theta = 0.$$

9. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right).$

1°. "Two-dimensional" solution:

$$w(x,y,t) = \begin{cases} u(x,t)|y+C|^{1/(m+1)} & \text{if } m \neq -1, \\ u(x,t)\exp(Cy) & \text{if } m = -1, \end{cases}$$

where C is an arbitrary constant and the function $u(x,t)$ is determined by the linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + f(t)\frac{\partial u}{\partial t} = g(t)\frac{\partial^2 u}{\partial x^2}.$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,t)|y+C|^{2/m},$$

where the function $v(x,t)$ is determined by the differential equation

$$\frac{\partial^2 v}{\partial t^2} + f(t)\frac{\partial v}{\partial t} = g(t)\frac{\partial^2 v}{\partial x^2} + \frac{2(m+2)}{m^2}h(t)v^{m+1}.$$

For $m = -2$ and $m = -1$, this equation is linear.

10. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(x,t) + \frac{1}{\lambda}\ln|y+C|$$

where C is an arbitrary constant and the function $u(x,t)$ is determined by the linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + f(t)\frac{\partial u}{\partial t} = g(t)\frac{\partial^2 u}{\partial x^2}.$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,t) + \frac{2}{\lambda}\ln|y+C|,$$

where C is an arbitrary constant and the function $v(x,t)$ is determined by the differential equation

$$\frac{\partial^2 v}{\partial t^2} + f(t)\frac{\partial v}{\partial t} = g(t)\frac{\partial^2 v}{\partial x^2} + \frac{2}{\lambda}h(t)e^{\lambda v}.$$

8.5. Equations with Three Space Variables Involving Arbitrary Parameters

8.5.1. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] + aw^p$$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ C_1^{\frac{p-1}{2-k}}z,\ \pm C_1^{\frac{p-1}{2}}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $k \neq 2$, and $p \neq 1$:

$$w = \left[\frac{A}{2s(p-1)}\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}},$$
$$A = \frac{1+p}{1-p} + \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}}F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z|t|^{\frac{2}{k-2}}.$$

4°. For other exact solutions, see equation 8.6.1.2 with $f(w) = sw^p$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(x + \frac{1-p}{\lambda}\ln C_1,\ y + \frac{1-p}{\mu}\ln C_1,\ z + \frac{1-p}{\nu}\ln C_1,\ \pm C_1^{\frac{p-1}{2}}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $p \neq \pm 1$, $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = \left[-\frac{s(p-1)^2}{2k(1+p)}(r + C_1)^2\right]^{\frac{1}{1-p}}, \quad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}}F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x + \frac{2}{\lambda}\ln|t|, \quad \rho_2 = y + \frac{2}{\mu}\ln|t|, \quad \rho_3 = z + \frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.3 with $f(w) = sw^p$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}} x,\ C_1^{\frac{p-1}{2-m}} y,\ z + \frac{1-p}{\nu}\ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $\nu \neq 0$, and $p \neq 1$:

$$w = \left[\frac{A}{2s(p-1)}\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}},$$
$$A = \frac{1+p}{1-p} + \frac{2}{2-n} + \frac{2}{2-m}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}} F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z + \frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.4 with $f(w) = sw^p$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}} x,\ y + \frac{1-p}{\mu}\ln C_1,\ z + \frac{1-p}{\nu}\ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\mu \neq 0$, $\nu \neq 0$, and $p \neq 1$:

$$w = \left[\frac{1}{2s(p-1)}\left(\frac{1+p}{1-p} + \frac{2}{2-n}\right)\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}} F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y + \frac{2}{\mu}\ln|t|, \quad \rho_3 = z + \frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.5 with $f(w) = sw^p$.

8.5.2. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] + ae^{\lambda w}$$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right) + se^{\lambda w}.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w\Big(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ C_1^{\frac{2}{2-k}}z,\ \pm C_1 t + C_2\Big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $k \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{2s\lambda}{A}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right]\right\},$$
$$A = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k} - 1.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = F(\rho_1, \rho_2, \rho_3) - \frac{2}{\lambda}\ln|t|, \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z|t|^{\frac{2}{k-2}}.$$

4°. For other exact solutions, see equation 8.6.1.2 with $f(w) = se^{\lambda w}$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + se^{\beta w}.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w\Big(x - \frac{2}{\lambda}\ln C_1,\ y - \frac{2}{\mu}\ln C_1,\ z - \frac{2}{\nu}\ln C_1,\ \pm C_1 t + C_2\Big) + \frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solutions for $\lambda \neq 0$, $\mu \neq 0$, $\nu \neq 0$, and $\beta \neq 0$:

$$w(x, y, z, t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2k}(r + C_1)^2\right] \quad \text{if} \quad sk\beta < 0;$$
$$w(x, y, z, t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2kC_1^2}\sin^2(C_1 r + C_2)\right] \quad \text{if} \quad sk\beta < 0;$$
$$w(x, y, z, t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2kC_1^2}\sinh^2(C_1 r + C_2)\right] \quad \text{if} \quad sk\beta < 0;$$
$$w(x, y, z, t) = -\frac{1}{\beta}\ln\left[\frac{s\beta}{2kC_1^2}\cosh^2(C_1 r + C_2)\right] \quad \text{if} \quad sk\beta > 0,$$

where C_1, C_2, and k are arbitrary constants and

$$r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C_3)^2\right],$$

where k and the expression in square brackets must have like signs.

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t)=F(\rho_1,\rho_2,\rho_3)-\frac{2}{\beta}\ln|t|,\quad \rho_1=x+\frac{2}{\lambda}\ln|t|,\quad \rho_2=y+\frac{2}{\mu}\ln|t|,\quad \rho_3=z+\frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.3 with $f(w)=se^{\beta w}$.

3. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right)+se^{\beta w}.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1=w\Big(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ z-\frac{2}{\nu}\ln C_1,\ \pm C_1 t+C_2\Big)+\frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, $\nu\neq 0$, and $\beta\neq 0$:

$$w=-\frac{1}{\beta}\ln\left\{\frac{2s\beta}{A}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\nu z}}{c\nu^2}-\frac{1}{4}(t+C)^2\right]\right\},$$
$$A=\frac{2}{2-n}+\frac{2}{2-m}-1.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t)=F(\rho_1,\rho_2,\rho_3)-\frac{2}{\beta}\ln|t|,\quad \rho_1=x|t|^{\frac{2}{n-2}},\quad \rho_2=y|t|^{\frac{2}{m-2}},\quad \rho_3=z+\frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.4 with $f(w)=se^{\beta w}$.

4. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right)+se^{\beta w}.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1=w\Big(C_1^{\frac{2}{2-n}}x,\ y-\frac{2}{\mu}\ln C_1,\ z-\frac{2}{\nu}\ln C_1,\ \pm C_1 t+C_2\Big)+\frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n\neq 2$, $\mu\neq 0$, $\nu\neq 0$, and $\beta\neq 0$:

$$w=-\frac{1}{\beta}\ln\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}-\frac{1}{4}(t+C)^2\right]+\frac{1}{\beta}\ln\frac{n}{2s\beta(2-n)}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t)=F(\rho_1,\rho_2,\rho_3)-\frac{2}{\beta}\ln|t|,\quad \rho_1=x|t|^{\frac{2}{n-2}},\quad \rho_2=y+\frac{2}{\mu}\ln|t|,\quad \rho_3=z+\frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 8.6.1.5 with $f(w)=se^{\beta w}$.

8.5.3. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + b\frac{\partial}{\partial y}\left(w^m\frac{\partial w}{\partial y}\right) + c\frac{\partial}{\partial z}\left(w^k\frac{\partial w}{\partial z}\right) + sw^p$$

1. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= C_1^{-2}w(\pm C_2x + C_3, \pm C_2y + C_4, \pm C_1^kC_2z + C_5, \pm C_2t + C_6),\\
w_2 &= w(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t),\\
w_3 &= w(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda + t\cosh\lambda),\\
w_4 &= w(x,\ y\cosh\mu + ta_2^{1/2}\sinh\mu,\ z,\ ya_2^{-1/2}\sinh\mu + t\cosh\mu),
\end{aligned}$$

where $C_1, \ldots, C_6$, β, λ, and μ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$\begin{aligned}
w(x, y, z, t) &= |z|^{\frac{1}{k+1}}\big[Ax^2 + By^2 + (a_1A + a_2B)t^2 + C_1xy + C_2xt + C_3yt\\
&\quad + C_4x + C_5y + C_6t + C_7\big];\\
w(x, y, z, t) &= |z|^{\frac{1}{k+1}}\big[A(a_2x^2 + a_1y^2 - a_1a_2t^2)^{-1/2} + B\big];\\
w(x, y, z, t) &= A|z|^{\frac{1}{k+1}}\exp\big(\lambda_1x + \lambda_2y \pm \gamma t\big) + B, \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};\\
w(x, y, z, t) &= A|z|^{\frac{1}{k+1}}\sin\big(\lambda_1x + C_1\big)\sin\big(\lambda_2y + C_2\big)\sin\big(\gamma t + C_3\big), \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};\\
w(x, y, z, t) &= \left[\frac{1}{a_3C_3^2}\left(\frac{C_1x + C_2y + C_3z + C_4}{t + C_5}\right)^2 - \frac{a_1C_1^2 + a_2C_2^2}{a_3C_3^2}\right]^{1/k};\\
w(x, y, z, t) &= \left[\frac{C_3^2 - a_2C_1^2}{a_3C_2^2} - \frac{a_1}{a_3C_2^2}\left(\frac{C_1y + C_2z + C_3t + C_4}{x + C_5}\right)^2\right]^{1/k};\\
w(x, y, z, t) &= \left[\frac{C_3^2 - a_1C_1^2}{a_3C_2^2} - \frac{a_2}{a_3C_2^2}\left(\frac{C_1x + C_2z + C_3t + C_4}{y + C_5}\right)^2\right]^{1/k};\\
w(x, y, z, t) &= \left[\frac{C_3^2 - a_1C_1^2 - a_2C_2^2}{a_3}\left(\frac{z + C_5}{C_1x + C_2y + C_3t + C_4}\right)^2\right]^{1/k},
\end{aligned}$$

where A, B, $C_1, \ldots, C_7$, λ_1, and λ_2 are arbitrary constants.

3°. Solutions:

$$w = \big|z\varphi(\xi) + \psi(\xi)\big|^{\frac{1}{k+1}}, \qquad \xi = C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2},$$

where C_1 and C_2 are arbitrary constants, $\varphi(\xi)$ and $\psi(\xi)$ are arbitrary functions.

4°. "Three-dimensional" solution (generalizes the first four solutions of Item 2°):

$$w(x,y,z,t)=|z|^{\frac{1}{k+1}}u(\widehat{x},\widehat{y},t),\quad \widehat{x}=a_1^{-1/2}x,\quad \widehat{y}=a_2^{-1/2}y,$$

where the function $u=u(\widehat{x},\widehat{y},t)$ is determined by the linear wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial \widehat{x}^2}+\frac{\partial^2 u}{\partial \widehat{y}^2}.$$

For this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. Solutions in implicit form:

$$2\lambda\sqrt{a_1C_1^2+a_2C_2^2}\,\big(z+\lambda t\big)\pm\big(C_1x+C_2y\pm t\sqrt{a_1C_1^2+a_2C_2^2}\,\big)\big(a_3w^k-\lambda^2\big)=\Phi(w),$$

where $\Phi(w)$ is an arbitrary function, C_1, C_2, and λ are arbitrary constants.

6°. There are "three-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,z,t)=|z|^{2/k}F(x,y,t);\\
&w(x,y,z,t)=|t|^{2\lambda}G(\xi,\eta,\zeta),\quad \xi=xt^{-1},\quad \eta=yt^{-1},\quad \zeta=z|t|^{-k\lambda-1};\\
&w(x,y,z,t)=H(r,z,t),\quad r=a_2x^2+a_1y^2;\\
&w(x,y,z,t)=U(\xi,y,z),\quad \xi=x^2-a_1t^2.
\end{aligned}$$

7°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,z,t)=|t|^{2\lambda}V(\rho,\zeta),\quad \rho=t^{-1}\sqrt{a_2x^2+a_1y^2},\quad \zeta=z|t|^{-k\lambda-1};\\
&w(x,y,z,t)=W(\zeta,z),\quad \zeta=a_2x^2+a_1y^2-a_1a_2t^2;\\
&w(x,y,z,t)=R(\eta),\quad \eta=(a_2x^2+a_1y^2-a_1a_2t^2)z^{-2};\\
&w(x,y,z,t)=|z|^{2/k}Q(p),\quad p=a_2x^2+a_1y^2-a_1a_2t^2,
\end{aligned}$$

where λ is an arbitrary constant.

8°. For other exact solutions, see equation 8.5.3.6 with $n=m=0$.

⊙ *References for equation* 8.5.3.1: N. H. Ibragimov (1994, p. 234), A. D. Polyanin and V. F. Zaitsev (2004, pp. 320–321).

2. $\dfrac{\partial^2 w}{\partial t^2}=a_1\dfrac{\partial^2 w}{\partial x^2}+a_2\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)+a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right)$.

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
&w_1=C_1^{-2}w(\pm C_2x+C_3,\pm C_1^kC_2y+C_4,\pm C_1^kC_2z+C_5,\pm C_2t+C_6),\\
&w_2=w\big(x,\ y\cos\beta+z\sqrt{a_2/a_3}\,\sin\beta,-y\sqrt{a_3/a_2}\,\sin\beta+z\cos\beta,\ t\big),\\
&w_3=w\big(x\cosh\lambda+ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda+t\cosh\lambda\big),
\end{aligned}$$

where $C_1,\ldots,C_6$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z,t)=\left[\frac{(C_1x+C_2y+C_3z+C_4)^2-a_1C_1^2(t+C_5)^2}{(a_2C_2^2+a_3C_3^2)(t+C_5)^2}\right]^{1/k},$$

$$w(x,y,z,t)=\left[\frac{C_3^2(x+C_5)^2-a_1(C_1y+C_2z+C_3t+C_4)^2}{(a_2C_1^2+a_3C_2^2)(x+C_5)^2}\right]^{1/k},$$

$$w(x,y,z,t)=\left[\frac{(C_3^2-a_1C_1^2)(y+C_5)^2}{a_2(C_1x+C_2z+C_3t+C_4)^2+a_3C_2^2(y+C_5)^2}\right]^{1/k},$$

$$w(x,y,z,t)=\left[\frac{(C_3^2-a_1C_1^2)(z+C_5)^2}{a_3(C_1x+C_2y+C_3t+C_4)^2+a_2C_2^2(z+C_5)^2}\right]^{1/k},$$

where $C_1,\ldots,C_5$ are arbitrary constants.

3°. Solution:

$$w(x,y,z,t)=\big[\varphi(x+t\sqrt{a_1}\,)+\psi(x-t\sqrt{a_1}\,)\big]u^{\frac{1}{k+1}}(\widehat{y},\widehat{z}),\quad \widehat{y}=a_2^{-1/2}y,\quad \widehat{z}=a_3^{-1/2}z,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and the function $u(\widehat{y},\widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2u}{\partial\widehat{y}^2}+\frac{\partial^2u}{\partial\widehat{z}^2}=0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Three-dimensional" solutions:

$$w=\big|v(\widehat{y},\widehat{z},\zeta)\big|^{\frac{1}{k+1}},\quad \widehat{y}=a_2^{-1/2}y,\quad \widehat{z}=a_3^{-1/2}z,\quad \zeta=x\pm t\sqrt{a_1},$$

where the function $v(\widehat{y},\widehat{z},\zeta)$ is determined by the Laplace equation

$$\frac{\partial^2v}{\partial\widehat{y}^2}+\frac{\partial^2v}{\partial\widehat{z}^2}=0,$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution are arbitrary functions of ζ).

5°. Multiplicative separable solution (generalizes the solution of Item 3°):

$$w(x,y,z,t)=R(x,t)Q(y,z),$$

where the functions $R=R(x,t)$ and $Q=Q(y,z)$ are determined by the differential equations

$$\frac{\partial^2R}{\partial t^2}=a_1\frac{\partial^2R}{\partial x^2}+AR^{k+1},$$
$$a_2\frac{\partial}{\partial y}\left(Q^k\frac{\partial Q}{\partial y}\right)+a_3\frac{\partial}{\partial z}\left(Q^k\frac{\partial Q}{\partial z}\right)=AQ,$$

and A is an arbitrary constant.

6°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = t^{2\lambda}F(\xi,\eta,\zeta), \quad \xi = xt^{-1}, \ \ \eta = yt^{-k\lambda-1}, \ \ \zeta = zt^{-k\lambda-1};$$
$$w(x,y,z,t) = G(x,r,t), \qquad r = a_3y^2 + a_2z^2;$$
$$w(x,y,z,t) = H(\xi,y,z), \qquad \xi = x^2 - a_1t^2,$$

where λ is an arbitrary constant.

7°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z,t) = U(\xi,\rho), \quad \xi = xt^{-1}, \ \ \rho = t^{-k\lambda-1}\sqrt{a_3y^2 + a_2z^2};$$
$$w(x,y,z,t) = V(r,\xi), \quad r = a_3y^2 + a_2z^2, \quad \xi = x^2 - a_1t^2;$$
$$w(x,y,z,t) = W(p,q), \quad p = (a_3y^2 + a_2z^2)t^{-2}, \quad q = xt^{-1};$$
$$w(x,y,z,t) = \Theta(\eta), \qquad \eta = (a_3y^2 + a_2z^2)(x^2 - a_1t^2)^{-1},$$

where λ is an arbitrary constant.

8°. For other exact solutions, see equation 8.5.3.6 with $n = 0$ and $m = k$.

⊙ *References for equation* 8.5.3.2: N. H. Ibragimov (1994, p. 233), A. D. Polyanin and V. F. Zaitsev (2004, pp. 322–323).

3. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2x + C_3, \pm C_1^mC_2y + C_4, \pm C_1^kC_2z + C_5, \pm C_2t + C_6),$$
$$w_2 = w(x\cosh\lambda + ta_1^{1/2}\sinh\lambda, \ y, \ z, \ xa_1^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_6$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x,y,z,t) = \left[\varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,)\right]|y + C_1|^{\frac{1}{m+1}}|z + C_2|^{\frac{1}{k+1}},$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and C_1 and C_2 are arbitrary constants.

3°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = t^{2\lambda}F(\xi,\eta,\zeta), \ \ \xi = xt^{-1}, \ \ \eta = yt^{-m\lambda-1}, \ \ \zeta = zt^{-k\lambda-1};$$
$$w(x,y,z,t) = G(r,y,z), \quad r = x^2 - a_1t^2;$$
$$w(x,y,z,t) = y^{2/m}H(x,s,t), \quad s = zy^{-k/m},$$

where λ is an arbitrary constant.

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z,t) = U(p,q), \quad p = (x^2 - a_1t^2)y^{-2}, \quad q = zy^{-1};$$
$$w(x,y,z,t) = y^{2/m}V(r,s), \quad r = x^2 - a_1t^2, \quad s = zy^{-k/m}.$$

5°. For other exact solutions, see equation 8.5.3.6 with $n = 0$.

⊙ *References for equation* 8.5.3.3: N. H. Ibragimov (1994, p. 235), A. D. Polyanin, V. F. Zaitsev (2004, p. 323).

4. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(w^k\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^k C_2 x + C_3, \pm C_1^k C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x, y, z, t) = (C_1 t + C_2)\big[a_3 C_3 x^2 + a_3 C_4 y^2 - (a_1 C_3 + a_2 C_4) z^2 + C_5\big]^{\frac{1}{k+1}},$$

$$w(x, y, z, t) = (C_1 t + C_2)\left(\frac{C_3}{\sqrt{a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2}} + C_4\right)^{\frac{1}{k+1}},$$

$$w(x, y, z, t) = \left[\frac{1}{a_1 C_1^2 + a_2 C_2^2 + a_3 C_3^2}\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2\right]^{1/k},$$

$$w(x, y, z, t) = \left[\frac{C_3^2 (x + C_5)^2}{a_1 (C_1 y + C_2 z + C_3 t + C_4)^2 + (a_2 C_1^2 + a_3 C_2^2)(x + C_5)^2}\right]^{1/k},$$

$$w(x, y, z, t) = \left[\frac{C_3^2 (y + C_5)^2}{a_2 (C_1 x + C_2 z + C_3 t + C_4)^2 + (a_1 C_1^2 + a_3 C_2^2)(y + C_5)^2}\right]^{1/k},$$

$$w(x, y, z, t) = \left[\frac{C_3^2 (z + C_5)^2}{a_3 (C_1 x + C_2 y + C_3 t + C_4)^2 + (a_1 C_1^2 + a_2 C_2^2)(z + C_5)^2}\right]^{1/k},$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x, y, z, t) = (C_1 t + C_2)\big[\Theta(\widehat{x}, \widehat{y}, \widehat{z})\big]^{\frac{1}{k+1}}, \quad \widehat{x} = a_1^{-1/2} x, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where the function $\Theta = \Theta(\widehat{x}, \widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 \Theta}{\partial \widehat{x}^2} + \frac{\partial^2 \Theta}{\partial \widehat{y}^2} + \frac{\partial^2 \Theta}{\partial \widehat{z}^2} = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = |t|^{2\lambda} F(\xi, \eta, \zeta), \quad \xi = x|t|^{-k\lambda-1}, \quad \eta = y|t|^{-k\lambda-1}, \quad \zeta = z|t|^{-k\lambda-1};$$
$$w(x, y, z, t) = |z|^{2/k} G(p, q, t), \quad p = xz^{-1}, \quad q = yz^{-1};$$
$$w(x, y, z, t) = H(\rho, z, t), \quad \rho = a_2 x^2 + a_1 y^2,$$

where λ is an arbitrary constant.

5°. There are solutions of the following forms:

$w(x,y,z,t) = U(r,t), \quad r = a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2$ "two-dimensional" solution;
$w(x,y,z,t) = V(\chi), \quad \chi = (a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2)t^{-2}$ "one-dimensional" solution.

6°. For other exact solutions, see equation 8.5.3.6 with $n = m = k$.

⊙ *References for equation* 8.5.3.4: N. H. Ibragimov (1994, p. 232), A. D. Polyanin and V. F. Zaitsev (2004, p. 324).

5. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^n C_2 x + C_3, \pm C_1^n C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = t^{2\lambda}F(\xi,\eta,\zeta), \quad \xi = xt^{-n\lambda-1}, \quad \eta = yt^{-n\lambda-1}, \quad \zeta = zt^{-k\lambda-1};$$
$$w(x,y,z,t) = G(r,z,t), \quad r = a_2x^2 + a_1y^2;$$
$$w(x,y,z,t) = z^{2/k}H(p,q,t), \quad p = xz^{-n/k}, \quad q = yz^{-n/k},$$

where λ is an arbitrary constant.

3°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z,t) = U(r,s), \quad r = (a_2x^2 + a_1y^2)t^{-2}, \quad s = zt^{-1};$$
$$w(x,y,z,t) = t^{-2/k}z^{2/k}V(\chi), \quad \chi = (a_2x^2 + a_1y^2)t^{2n/k-2}z^{-2n/k}.$$

4°. For other exact solutions, see equation 8.5.3.6 with $m = n$.

6. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right).$

This is a special case of equation 8.6.2.6 with $f(w) = a_1w^n$, $g(w) = a_2w^m$, and $h(w) = a_3w^k$.

1°. Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^n C_2 x + C_3, \pm C_1^m C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x,y,z,t) = (C_1 t + C_2)|x + C_3|^{\frac{1}{n+1}}|y + C_4|^{\frac{1}{m+1}}|z + C_5|^{\frac{1}{k+1}}.$$

3°. Traveling-wave solution in implicit form:

$$\frac{a_1 b_1^2}{n+1}w^{n+1} + \frac{a_2 b_2^2}{m+1}w^{m+1} + \frac{a_3 b_3^2}{k+1}w^{k+1} - \lambda^2 w = C_1(b_1 x + b_2 y + b_3 z + \lambda t) + C_2,$$

where C_1, C_2, b_1, b_2, b_3, and λ are arbitrary constants.

4°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2 = a_1 C_1^2 w^n + a_2 C_2^2 w^m + a_3 C_3^2 w^k,$$

$$a_1 w^n \left(\frac{C_1 y + C_2 z + C_3 t + C_4}{x + C_5}\right)^2 + a_2 C_1^2 w^m + a_3 C_2^2 w^k = C_3^2,$$

$$a_2 w^m \left(\frac{C_1 x + C_2 z + C_3 t + C_4}{y + C_5}\right)^2 + a_1 C_1^2 w^n + a_3 C_2^2 w^k = C_3^2,$$

$$a_3 w^k \left(\frac{C_1 x + C_2 y + C_3 t + C_4}{z + C_5}\right)^2 + a_1 C_1^2 w^n + a_2 C_2^2 w^m = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. "Two-dimensional" solution (b_1, b_2, and b_3 are arbitrary constants):

$$w(x,y,z,t) = u(\xi,t), \quad \xi = b_1 x + b_2 y + b_3 z,$$

where the function $u = u(\xi,t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[(a_1 b_1^2 u^n + a_2 b_2^2 u^m + a_3 b_3^2 u^k)\frac{\partial u}{\partial \xi}\right],$$

which can be reduced to a linear equation.

6°. "Two-dimensional" solution (c_1, c_2, and c_3 are arbitrary constants):

$$w(x,y,z,t) = v(x,\eta), \quad \eta = c_1 t + c_2 y + c_3 z, \tag{1}$$

where the function $v = v(x,\eta)$ is determined by a differential equation of the form 9.4.4.8:

$$a_1 \frac{\partial}{\partial x}\left(v^n \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial \eta}\left[\left(a_2 c_2^2 v^m + a_3 c_3^2 v^k - c_1^2\right)\frac{\partial v}{\partial \eta}\right] = 0, \tag{2}$$

which can be reduced to a linear equation.

Formula (1) and equation (2) can be used to obtain two other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a_1, n) & \\ \nearrow & & \searrow \\ (z, a_3, k) & \longleftarrow & (y, a_2, m) \end{array}$$

7°. "Two-dimensional" solution (the b_n and c_n are arbitrary constants):

$$w(x,y,z,t) = U(\zeta,\rho), \quad \zeta = b_1 t + b_2 x, \quad \rho = c_1 y + c_2 z,$$

where the function $U = U(\zeta,\rho)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial \zeta}\left[\Phi(U)\frac{\partial U}{\partial \zeta}\right] + \frac{\partial}{\partial \rho}\left[\Psi(U)\frac{\partial U}{\partial \rho}\right] = 0, \quad \Phi(U) = a_1 b_2^2 U^n - b_1^2, \quad \Psi(U) = a_2 c_1^2 U^m + a_3 c_2^2 U^k,$$

which can be reduced to a linear equation.

Remark. The solution specified in Item 7° can be used to obtain other "two-dimensional" solutions by means of cyclic permutations of variables and determining parameters, as shown in Item 6°.

8°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = t^{2\lambda}F(\xi,\eta,\zeta), \quad \xi = xt^{-n\lambda-1}, \quad \eta = yt^{-m\lambda-1}, \quad \zeta = zt^{-k\lambda-1};$$
$$w(x,y,z,t) = x^{2/n}G(r,s,t), \quad r = yx^{-m/n}, \quad s = zx^{-k/n},$$

where λ is an arbitrary constant.

9°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z,t) = H(p,q), \quad p = b_1x + b_2y + b_3z + b_4t, \quad q = c_1x + c_2y + c_3z + c_4t;$$
$$w(x,y,z,t) = t^{-2/n}x^{2/n}U(\rho,\chi), \quad \rho = x^{-m/n}yt^{(m-n)/n}, \quad \chi = x^{-k/n}zt^{(k-n)/n},$$

where the b_n and c_n are arbitrary constants.

⊙ *References for equation* 8.5.3.6: N. H. Ibragimov (1994, p. 236), A. D. Polyanin and V. F. Zaitsev (2004, pp. 325–326).

7. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right) + bw^p.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{p-n-1}x + C_2, \pm C_1^{p-m-1}y + C_3, \pm C_1^{p-k-1}z + C_4, \pm C_1^{p-1}t + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = t^{\frac{2}{1-p}}U(\xi,\eta,\zeta), \quad \xi = xt^{\frac{p-n-1}{1-p}}, \quad \eta = yt^{\frac{p-m-1}{1-p}}, \quad \zeta = zt^{\frac{p-k-1}{1-p}}.$$

8.5.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + c\dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right) + se^{\beta w}$

1. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + a_3\dfrac{\partial}{\partial z}\left(e^w\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_3, \pm C_1y + C_4, \pm C_1C_2z + C_5, \pm C_1t + C_6) - 2\ln|C_2|,$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t\big),$$
$$w_3 = w\big(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$
$$w_4 = w\big(x,\ y\cosh\mu + ta_2^{1/2}\sinh\mu,\ z,\ ya_2^{-1/2}\sinh\mu + t\cosh\mu\big),$$

where $C_1, \ldots, C_6$, β, λ, and μ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z,t) = C_1x^2 + C_2y^2 + (a_1C_1 + a_2C_2)t^2 + C_3xy + C_4xt + C_5yt + C_6x + C_7y + C_8t + C_9 + \ln|z|;$$

$$w(x,y,z,t) = C_1(a_2x^2 + a_1y^2 - a_1a_2t^2)^{-1/2} + C_2 + \ln|z|;$$

$$w(x,y,z,t) = C_1\exp\big(\lambda_1x + \lambda_2y \pm \gamma t\big) + C_2 + \ln|z|, \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};$$

$$w(x,y,z,t) = C_1\sin\big(\lambda_1x + C_2\big)\sin\big(\lambda_2y + C_3\big)\sin\big(\gamma t + C_4\big) + \ln|z|, \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};$$

$$w(x,y,z,t) = \ln\left[\frac{1}{a_3C_3^2}\left(\frac{C_1x + C_2y + C_3z + C_4}{t + C_5}\right)^2 - \frac{a_1C_1^2 + a_2C_2^2}{a_3C_3^2}\right];$$

$$w(x,y,z,t) = \ln\left[\frac{C_3^2 - a_2C_1^2}{a_3C_2^2} - \frac{a_1}{a_3C_2^2}\left(\frac{C_1y + C_2z + C_3t + C_4}{x + C_5}\right)^2\right];$$

$$w(x,y,z,t) = \ln\left[\frac{C_3^2 - a_1C_1^2}{a_3C_2^2} - \frac{a_2}{a_3C_2^2}\left(\frac{C_1x + C_2z + C_3t + C_4}{y + C_5}\right)^2\right];$$

$$w(x,y,z,t) = \ln\left[\frac{C_3^2 - a_1C_1^2 - a_2C_2^2}{a_3}\left(\frac{z + C_5}{C_1x + C_2y + C_3t + C_4}\right)^2\right],$$

where the C_n are arbitrary constants.

3°. Solutions:

$$w = \ln|z\varphi(\xi) + \psi(\xi)|, \qquad \xi = C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2},$$

where C_1 and C_2 are arbitrary constants, and $\varphi(\xi)$ and $\psi(\xi)$ are arbitrary functions.

4°. "Three-dimensional" solution (generalizes the first four solutions of Item 2°):

$$w(x,y,z,t) = u(\widehat{x},\widehat{y},t) + \ln|z|, \quad \widehat{x} = a_1^{-1/2}x, \quad \widehat{y} = a_2^{-1/2}y,$$

where the function $u = u(\widehat{x},\widehat{y},t)$ is determined by the linear wave equation

$$\frac{\partial^2u}{\partial t^2} = \frac{\partial^2u}{\partial\widehat{x}^2} + \frac{\partial^2u}{\partial\widehat{y}^2}.$$

For this equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution:

$$w(x,y,z,t) = U(\xi,t) + 2\ln|z|, \quad \xi = C_1x + C_2y,$$

where C_1 and C_2 are arbitrary constants and the function $U = U(\xi,t)$ is determined by a solvable equation of the form 7.2.1.1:

$$\frac{\partial^2U}{\partial t^2} = (a_1C_1^2 + a_2C_2^2)\frac{\partial^2U}{\partial\xi^2} + 2a_3e^U.$$

6°. "Two-dimensional" solution:

$$w(x,y,z,t) = v(x,\eta) + 2\ln|z|, \quad \eta = C_1y + C_2t, \tag{1}$$

where C_1 and C_2 are arbitrary constants and the function $v = v(\eta, t)$ is determined by the equation

$$(C_1^2 - a_2C_2^2)\frac{\partial^2 v}{\partial \eta^2} = a_1\frac{\partial^2 v}{\partial x^2} + 2a_3e^v. \tag{2}$$

For $\sigma = C_1^2 - a_2C_2^2 > 0$, on dividing by σ, one obtains a solvable equation of the form 7.2.1.1. For $\sigma = C_1^2 - a_2C_2^2 < 0$, the transformation $\eta = \widetilde{\eta}\sqrt{|\sigma|}$, $x = \widetilde{x}\sqrt{a_1}$ leads to a solvable equation of the form 9.2.1.1:

$$\frac{\partial^2 v}{\partial \widetilde{x}^2} + \frac{\partial^2 v}{\partial \widetilde{\eta}^2} = -2a_3e^v.$$

Remark. Relations (1) and equation (2) can be used to obtain another "two-dimensional" solution by means of the following renaming: $(x, a_1) \rightleftarrows (y, a_2)$.

7°. Solutions in implicit form:

$$2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,\big(z + \lambda t\big) \pm \big(C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2}\,\big)\big(a_3e^w - \lambda^2\big) = \Phi(w),$$

where $\Phi(w)$ is an arbitrary function, and C_1, C_2, and λ are arbitrary constants.

8°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = F(x, y, t) + 2\ln|z|;$$
$$w(x, y, z, t) = G(\xi_1, \xi_2, \xi_3) - 2k\ln|t|, \quad \xi_1 = xt^{-1}, \quad \xi_2 = yt^{-1}, \quad \xi_3 = z|t|^{k-1};$$
$$w(x, y, z, t) = H(\eta_1, \eta_2, \eta_3) + 2\ln|z|, \quad \eta_1 = t + k_1\ln|z|, \quad \eta_2 = x + k_2\ln|z|, \quad \eta_3 = y + k_3\ln|z|;$$
$$w(x, y, z, t) = E(\zeta_1, \zeta_2, \zeta_3) + 2z, \quad \zeta_1 = te^z, \quad \zeta_2 = xe^z, \quad \zeta_3 = ye^z;$$
$$w(x, y, z, t) = P(r, z, t), \quad r = a_2x^2 + a_1y^2;$$
$$w(x, y, z, t) = Q(\rho, y, z), \quad \rho = x^2 - a_1t^2,$$

where k, k_1, k_2, and k_3 are arbitrary constants.

9°. There are solutions of the following forms:

$$w(x, y, z, t) = U(r, t) + 2\ln|z|, \quad r = a_2x^2 + a_1y^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x, y, z, t) = V(\rho, y) + 2\ln|z|, \quad \rho = x^2 - a_1t^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x, y, z, t) = W(\theta, z), \quad \theta = a_2x^2 + a_1y^2 - a_1a_2t^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x, y, z, t) = R(\xi_1, \xi_2) + 2\ln|z/t|, \quad \xi_1 = xt^{-1}, \quad \xi_2 = yt^{-1} \qquad \text{"two-dimensional" solution;}$$
$$w(x, y, z, t) = S(\theta) + 2\ln|z|, \quad \theta = a_2x^2 + a_1y^2 - a_1a_2t^2 \qquad \text{"one-dimensional" solution;}$$
$$w(x, y, z, t) = T(\chi), \quad \chi = (a_2x^2 + a_1y^2 - a_1a_2t^2)z^{-2} \qquad \text{"one-dimensional" solution.}$$

10°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$, $g(w) = a_2$, and $h(w) = a_3e^w$.

⊙ *References for equation* 8.5.4.1: N. H. Ibragimov (1994, p. 235), A. D. Polyanin and V. F. Zaitsev (2004, pp. 327–328).

2. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial}{\partial y}\left(e^w\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(e^w\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_3, \pm C_1 C_2 y + C_4, \pm C_1 C_2 z + C_5, \pm C_1 t + C_6) - 2\ln|C_2|,$$
$$w_2 = w\big(x, y\cos\beta + z\sqrt{a_2/a_3}\,\sin\beta, -y\sqrt{a_3/a_2}\,\sin\beta + z\cos\beta, t\big),$$
$$w_3 = w\big(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z, xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$

where $C_1, \ldots, C_6$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z,t) = \varphi(x+t\sqrt{a_1}\,) + \psi(x-t\sqrt{a_1}\,) + \ln(a_3C_1y^2 + C_2yz - a_2C_1z^2 + C_3y + C_4z + C_5),$$
$$w(x,y,z,t) = \varphi(x+t\sqrt{a_1}\,) + \psi(x-t\sqrt{a_1}\,) + \ln\big[C_1\exp(C_2\sqrt{a_3}\,y)\sin(C_2\sqrt{a_2}\,z + C_3) + C_4\big],$$
$$w(x,y,z,t) = \varphi(x+t\sqrt{a_1}\,) + \psi(x-t\sqrt{a_1}\,) + \ln\big[C_1\exp(C_2\sqrt{a_2}\,z)\sin(C_2\sqrt{a_3}\,y + C_3) + C_4\big],$$
$$w(x,y,z,t) = \ln\left[\frac{(C_1x + C_2y + C_3z + C_4)^2 - a_1C_1^2(t+C_5)^2}{(a_2C_2^2 + a_3C_3^2)(t+C_5)^2}\right],$$
$$w(x,y,z,t) = \ln\left[\frac{C_3^2(x+C_5)^2 - a_1(C_1y + C_2z + C_3t + C_4)^2}{(a_2C_1^2 + a_3C_2^2)(x+C_5)^2}\right],$$
$$w(x,y,z,t) = \ln\left[\frac{(C_3^2 - a_1C_1^2)(y+C_5)^2}{a_2(C_1x + C_2z + C_3t + C_4)^2 + a_3C_2^2(y+C_5)^2}\right],$$
$$w(x,y,z,t) = \ln\left[\frac{(C_3^2 - a_1C_1^2)(z+C_5)^2}{a_3(C_1x + C_2y + C_3t + C_4)^2 + a_2C_2^2(z+C_5)^2}\right],$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and $C_1, \ldots, C_5$ are arbitrary constants.

3°. Solution (generalizes the first three solutions of Item 2°):

$$w(x,y,z,t) = \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln u(\widehat{y}, \widehat{z}), \quad \widehat{y} = a_2^{-1/2}y, \quad \widehat{z} = a_3^{-1/2}z,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and the function $u(\widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 u}{\partial \widehat{y}^2} + \frac{\partial^2 u}{\partial \widehat{z}^2} = 0. \tag{1}$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Three-dimensional" solutions:

$$w = \ln\big|v(\widehat{y}, \widehat{z}, \zeta)\big|, \quad \widehat{y} = a_2^{-1/2}y, \quad \widehat{z} = a_3^{-1/2}z, \quad \zeta = x \pm t\sqrt{a_1},$$

where the function $v(\widehat{y}, \widehat{z}, \zeta)$ is determined by the Laplace equation

$$\frac{\partial^2 v}{\partial \widehat{y}^2} + \frac{\partial^2 v}{\partial \widehat{z}^2} = 0,$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

5°. Additive separable solution (generalizes the solution of Item 3°):

$$w(x,y,z,t) = R(\widehat{x},t) + \ln Q(\widehat{y},\widehat{z}), \quad \widehat{x} = a_1^{-1/2}x, \quad \widehat{y} = a_2^{-1/2}y, \quad \widehat{z} = a_3^{-1/2}z,$$

where the functions $R = R(\widehat{x},t)$ and $Q = Q(\widehat{y},\widehat{z})$ are determined by the differential equations

$$\frac{\partial^2 R}{\partial t^2} = \frac{\partial^2 R}{\partial \widehat{x}^2} + Ae^R, \tag{2}$$

$$\frac{\partial^2 Q}{\partial \widehat{y}^2} + \frac{\partial^2 Q}{\partial \widehat{z}^2} = A, \tag{3}$$

and A is an arbitrary constant. The general solution of equation (2) is given in 7.2.1.1. By the substitution $Q = \frac{1}{2}A\widehat{y}^2 + u$, the Helmholtz equation (3) can be reduced to the Laplace equation (1).

6°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = F(\xi,\eta,\zeta) - 2\lambda \ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{\lambda-1}, \quad \zeta = z|t|^{\lambda-1};$$
$$w(x,y,z,t) = G(x,r,t), \quad r = a_3y^2 + a_2z^2;$$
$$w(x,y,z,t) = H(\rho,y,z), \quad \rho = x^2 - a_1t^2,$$

where λ is an arbitrary constant.

7°. There are solutions of the following forms:

$w(x,y,z,t) = E(r,\rho), \quad r = a_3y^2 + a_2z^2, \quad \rho = x^2 - a_1t^2$ "two-dimensional" solution;
$w(x,y,z,t) = U(\chi_1,\chi_2) + 2\ln|y/t|, \quad \chi_1 = x/t, \quad \chi_2 = z/y$ "two-dimensional" solution;
$w(x,y,z,t) = V(p,q), \quad p = (a_3y^2 + a_2z^2)t^{-2}, \quad q = xt^{-1}$ "two-dimensional" solution;
$w(x,y,z,t) = W(\eta), \quad \eta = (a_3y^2 + a_2z^2)(x^2 - a_1t^2)^{-1}$ "one-dimensional" solution.

8°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$, $g(w) = a_2e^w$, and $h(w) = a_3e^w$.

⊙ *References for equation* 8.5.4.2: N. H. Ibragimov (1994, p. 235), A. D. Polyanin and V. F. Zaitsev (2004, pp. 329–330).

3. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial}{\partial y}\left(e^w\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(e^{kw}\dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_3, \pm C_1C_2y + C_4, \pm C_1C_2^kz + C_5, \pm C_1t + C_6) - \ln C_2^2,$$
$$w_2 = w(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_6$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Additive separable solution:

$$w(x,y,z,t) = \varphi(x + t\sqrt{a_1}) + \psi(x - t\sqrt{a_1}) + \ln|y + C_1| + \frac{1}{k}\ln|z + C_2|,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and C_1 and C_2 are arbitrary constants.

3°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = F(\xi,\eta,\zeta) - 2\beta\ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{\beta-1}, \quad \zeta = z|t|^{k\beta-1};$$
$$w(x,y,z,t) = G(r,y,z), \quad r = x^2 - a_1t^2,$$

where β is an arbitrary constant.

4°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z,t) = F(\rho_1,\rho_2) + 2\ln\left|\frac{y}{t}\right|, \quad \rho_1 = xt^{-1}, \quad \rho_2 = |t|^{k-1}|y|^{-k}z;$$
$$w(x,y,z,t) = U(p,q), \quad p = (x^2 - a_1t^2)y^{-2}, \quad q = zy^{-1};$$
$$w(x,y,z,t) = V(r,s) + 2\ln|y|, \quad r = x^2 - a_1t^2, \quad s = z|y|^{-k};$$
$$w(x,y,z,t) = W(\chi) - \frac{2}{k-1}\ln\left|\frac{y}{z}\right|, \quad \chi = |x^2 - a_1t^2|^{k-1}|y|^{-2k}z^2.$$

5°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$, $g(w) = a_2e^w$, and $h(w) = a_3e^{kw}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(e^w\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(e^w\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(e^w\dfrac{\partial w}{\partial z}\right).$

This is a special case of equation 8.6.2.4 with $f(w) = e^w$.

1°. Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1C_2x + C_3, \pm C_1C_2y + C_4, \pm C_1C_2z + C_5, \pm C_1t + C_6) - \ln C_2^2,$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t\big),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z,t) = C_1t + C_2 + \ln\big[a_3C_3x^2 + a_3C_4y^2 - (a_1C_3 + a_2C_4)z^2 + C_5\big],$$
$$w(x,y,z,t) = C_1t + C_2 + \ln\big[C_3(a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2)^{-1/2} + C_4\big],$$
$$w(x,y,z,t) = \ln\left[\frac{1}{a_1C_1^2 + a_2C_2^2 + a_3C_3^2}\left(\frac{C_1x + C_2y + C_3z + C_4}{t + C_5}\right)^2\right],$$
$$w(x,y,z,t) = \ln\left[\frac{C_3^2(x + C_5)^2}{a_1(C_1y + C_2z + C_3t + C_4)^2 + (a_2C_1^2 + a_3C_2^2)(x + C_5)^2}\right],$$
$$w(x,y,z,t) = \ln\left[\frac{C_3^2(y + C_5)^2}{a_2(C_1x + C_2z + C_3t + C_4)^2 + (a_1C_1^2 + a_3C_2^2)(y + C_5)^2}\right],$$
$$w(x,y,z,t) = \ln\left[\frac{C_3^2(z + C_5)^2}{a_3(C_1x + C_2y + C_3t + C_4)^2 + (a_1C_1^2 + a_2C_2^2)(z + C_5)^2}\right],$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Additive separable solution:

$$w(x,y,z,t) = C_1 t + C_2 + \ln\Theta(\widehat{x},\widehat{y},\widehat{z}), \quad \widehat{x} = a_1^{-1/2}x, \quad \widehat{y} = a_2^{-1/2}y, \quad \widehat{z} = a_3^{-1/2}z,$$

where the function $\Theta = \Theta(\widehat{x},\widehat{y},\widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2\Theta}{\partial\widehat{x}^2} + \frac{\partial^2\Theta}{\partial\widehat{y}^2} + \frac{\partial^2\Theta}{\partial\widehat{z}^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = F(\xi,\eta,\zeta) - 2\beta\ln|t|, \quad \xi = x|t|^{\beta-1}, \quad \eta = y|t|^{\beta-1}, \quad \zeta = z|t|^{\beta-1};$$
$$w(x,y,z,t) = G(\rho,z,t), \quad \rho = a_2x^2 + a_1y^2;$$
$$w(x,y,z,t) = H(p,q,t) + 2\ln|z|, \quad p = x/z, \quad q = y/z,$$

where β is an arbitrary constant.

5°. There are solutions of the following forms:

$w(x,y,z,t) = U(r,t)$, $\quad r = a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2$ "two-dimensional" solution;
$w(x,y,z,t) = V(\chi)$, $\quad \chi = (a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2)t^{-2}$ "one-dimensional" solution.

6°. For other exact solutions, see equation 8.5.4.6 with $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

⊙ *References for equation* 8.5.4.4: N. H. Ibragimov (1994, p. 232), A. D. Polyanin and V. F. Zaitsev (2004, p. 331).

5. $$\frac{\partial^2 w}{\partial t^2} = a_1\frac{\partial}{\partial x}\left(e^w\frac{\partial w}{\partial x}\right) + a_2\frac{\partial}{\partial y}\left(e^w\frac{\partial w}{\partial y}\right) + a_3\frac{\partial}{\partial z}\left(e^{kw}\frac{\partial w}{\partial z}\right).$$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1C_2x + C_3, \pm C_1C_2y + C_4, \pm C_1C_2^kz + C_5, \pm C_1t + C_6) - \ln C_2^2,$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta,\ -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t\big),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Additive separable solution:

$$w(x,y,z,t) = C_1t + C_2 + \frac{1}{k}\ln|z + C_3| + \ln\Theta(\widehat{x},\widehat{y}), \quad \widehat{x} = a_1^{-1/2}x, \quad \widehat{y} = a_2^{-1/2}y,$$

where the function $\Theta = \Theta(\widehat{x},\widehat{y})$ is determined by the Laplace equation

$$\frac{\partial^2\Theta}{\partial\widehat{x}^2} + \frac{\partial^2\Theta}{\partial\widehat{y}^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. There are "three-dimensional" solutions of the following forms:

$$w = F(\xi,\eta,\zeta) - 2\beta\ln|t|,\quad \xi = x|t|^{\beta-1},\quad \eta = y|t|^{\beta-1},\quad \zeta = z|t|^{k\beta-1};$$
$$w = G(r,z,t),\quad r = a_2x^2 + a_1y^2,$$

where β is an arbitrary constant.

4°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w = H(\rho_1,\rho_2) + 2\ln|x/t|,\quad \rho_1 = y/x,\quad \rho_2 = |t|^{k-1}|x|^{-k}z;$$
$$w = U(\chi) + \ln\bigl[(a_2x^2 + a_1y^2)t^{-2}\bigr],\quad \chi = (a_2x^2 + a_1y^2)|z|^{-2/k}|t|^{2/k-2}.$$

5°. For other exact solutions, see equation 8.5.4.6 with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = k$.

⊙ *References for equation* 8.5.4.5: N. H. Ibragimov (1994, p. 233), A. D. Polyanin and V. F. Zaitsev (2004, p. 332).

6. $$\frac{\partial^2 w}{\partial t^2} = a_1\frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right) + a_2\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) + a_3\frac{\partial}{\partial z}\left(e^{\lambda_3 w}\frac{\partial w}{\partial z}\right).$$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1C_2^{\lambda_1}x + C_3, \pm C_1C_2^{\lambda_2}y + C_4, \pm C_1C_2^{\lambda_3}z + C_5, \pm C_1t + C_6) - \ln C_2^2,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Additive separable solution:

$$w(x,y,z,t) = C_1t + C_2 + \frac{1}{\lambda_1}\ln|x + C_3| + \frac{1}{\lambda_2}\ln|y + C_4| + \frac{1}{\lambda_3}\ln|z + C_5|.$$

3°. Traveling-wave solution in implicit form:

$$\frac{a_1k_1^2}{\lambda_1}e^{\lambda_1 w} + \frac{a_2k_2^2}{\lambda_2}e^{\lambda_2 w} + \frac{a_3k_3^2}{\lambda_3}e^{\lambda_3 w} - \beta^2 w = C_1(k_1x + k_2y + k_3z + \beta t) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and β are arbitrary constants.

4°. Solutions in implicit form:

$$\left(\frac{C_1x + C_2y + C_3z + C_4}{t + C_5}\right)^2 = a_1C_1^2e^{\lambda_1 w} + a_2C_2^2e^{\lambda_2 w} + a_3C_3^2e^{\lambda_3 w},$$
$$a_1e^{\lambda_1 w}\left(\frac{C_1y + C_2z + C_3t + C_4}{x + C_5}\right)^2 + a_2C_1^2e^{\lambda_2 w} + a_3C_2^2e^{\lambda_3 w} = C_3^2,$$
$$a_2e^{\lambda_2 w}\left(\frac{C_1x + C_2z + C_3t + C_4}{y + C_5}\right)^2 + a_1C_1^2e^{\lambda_1 w} + a_3C_2^2e^{\lambda_3 w} = C_3^2,$$
$$a_3e^{\lambda_3 w}\left(\frac{C_1x + C_2y + C_3t + C_4}{z + C_5}\right)^2 + a_1C_1^2e^{\lambda_1 w} + a_2C_2^2e^{\lambda_2 w} = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. "Two-dimensional" solution (k_1, k_2, and k_3 are arbitrary constants):

$$w(x,y,z,t)=u(\xi,t),\quad \xi=k_1x+k_2y+k_3z,$$

where the function $u=u(\xi,t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial\xi}\left[\varphi(u)\frac{\partial u}{\partial\xi}\right],\quad \varphi(u)=a_1k_1^2e^{\lambda_1 w}+a_2k_2^2e^{\lambda_2 w}+a_3k_3^2e^{\lambda_3 w},$$

which can be reduced to a linear equation.

6°. "Two-dimensional" solution (b_1, b_2, and b_3 are arbitrary constants):

$$w(x,y,z,t)=v(x,\eta),\quad \eta=b_1y+b_2z+b_3t, \tag{1}$$

where the function $v=v(x,\eta)$ is determined by a differential equation of the form 9.4.4.8:

$$a_1\frac{\partial}{\partial x}\left(e^{\lambda_1 v}\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial\eta}\left[\psi(v)\frac{\partial v}{\partial\eta}\right]=0,\quad \psi(v)=a_2b_1^2e^{\lambda_2 v}+a_3b_2^2e^{\lambda_3 v}-b_3^2, \tag{2}$$

which can be reduced to a linear equation.

Relations (1) and equation (2) can be used to obtain two other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a_1,\lambda_1) & \\ \nearrow & & \searrow \\ (z,a_3,\lambda_3) & \longleftarrow & (y,a_2,\lambda_2) \end{array}$$

7°. "Two-dimensional" solution (b_n and c_n are arbitrary constants):

$$w(x,y,z,t)=U(\zeta,\rho),\quad \zeta=b_1t+b_2x,\quad \rho=c_1y+c_2z,$$

where the function $U=U(\zeta,\rho)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial\zeta}\left[\Phi(U)\frac{\partial U}{\partial\zeta}\right]+\frac{\partial}{\partial\rho}\left[\Psi(U)\frac{\partial U}{\partial\rho}\right]=0,\quad \Phi(U)=a_1b_2^2e^{\lambda_1 U}-b_1^2,\quad \Psi(U)=a_2c_1^2e^{\lambda_2 U}+a_3c_2^2e^{\lambda_3 U},$$

which can be reduced to a linear equation.

Remark. The solution specified in Item 7° can be used to obtain other "two-dimensional" solutions by means of cyclic permutations of variables and determining parameters as shown in Item 6°.

8°. There are more complicated "two-dimensional" solutions of the form

$$w(x,y,z,t)=V(z_1,z_2),\quad z_1=b_1x+b_2y+b_3z+b_4t,\quad z_2=c_1x+c_2y+c_3z+c_4t.$$

9°. There is a "two-dimensional" solution of the form

$$w(x,y,z,t)=W(\rho_1,\rho_2)+\frac{2}{\lambda_1}\ln\left|\frac{x}{t}\right|,\quad \rho_1=|t|^{\lambda_2/\lambda_1-1}|x|^{-\lambda_2/\lambda_1}y,\quad \rho_2=|t|^{\lambda_3/\lambda_1-1}|x|^{-\lambda_3/\lambda_1}z.$$

10°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t)=F(\xi,\eta,\zeta)-2\beta\ln|t|,\quad \xi=x|t|^{\beta\lambda_1-1},\quad \eta=y|t|^{\beta\lambda_2-1},\quad \zeta=z|t|^{\beta\lambda_3-1},$$

where β is an arbitrary constant.

⊙ *References for equation* 8.5.4.6: N. H. Ibragimov (1994, p. 235), A. D. Polyanin and V. F. Zaitsev (2004, pp. 332–333).

7. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right) + be^{\beta w}.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1}x + C_2, \pm C_1^{\beta-\lambda_2}y + C_3, \pm C_1^{\beta-\lambda_3}z + C_4, \pm C_1^{\beta}t + C_5) + 2\ln|C_1|,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = U(\xi, \eta, \zeta) - \frac{2}{\beta}\ln|t|, \quad \xi = x|t|^{\frac{\lambda_1-\beta}{\beta}}, \quad \eta = y|t|^{\frac{\lambda_2-\beta}{\beta}}, \quad \zeta = z|t|^{\frac{\lambda_3-\beta}{\beta}},$$

$$w(x, y, z, t) = V(\eta_1, \eta_2, \eta_3), \quad \eta_n = a_n x + b_n y + c_n z + d_n t \quad (n = 1, 2, 3).$$

8.6. Equations with Three or More Space Variables Involving Arbitrary Functions

8.6.1. Equations of the Form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[f_3(z)\frac{\partial w}{\partial z}\right] + g(w)$$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} + f(w).$

The equation admits translations in any of the variables x, y, z, and t and is invariant under rotations in the three-dimensional space x, y, z as well as under linear transformations preserving the quantity $t^2 - x^2 - y^2 - z^2$.

1°. Traveling-wave solution in implicit form:

$$\int\left[C_1 + \frac{2}{\lambda^2 - k_1^2 - k_2^2 - k_3^2}\int f(w)\,dw\right]^{-1/2} dw = k_1x + k_2y + k_3z + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

2°. Solution:

$$w(x, y, z, t) = w(\rho), \quad \rho^2 = A\big[(x + C_1)^2 + (y + C_2)^2 + (z + C_3)^2 - (t + C_4)^2\big],$$

where the arbitrary constant A and the expression in square brackets must have like signs, and the function $w(\rho)$ is determined by the ordinary differential equation

$$w''_{\rho\rho} + 3\rho^{-1}w'_\rho + A^{-1}f(w) = 0.$$

3°. For the case of axisymmetric solutions, the Laplace operator on the right-hand side of the equation is expressed in cylindrical and spherical coordinates as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}, \qquad r = \sqrt{x^2 + y^2};$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

4°. "Three-dimensional" solution:

$$w = u(\xi,\eta,t), \quad \xi = y + \frac{x}{C}, \quad \eta = (C^2-1)x^2 - 2Cxy + C^2z^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $u = u(\xi,\eta,t)$ is determined by the equation

$$\frac{\partial^2 u}{\partial t^2} = \left(1+\frac{1}{C^2}\right)\frac{\partial^2 u}{\partial \xi^2} - 4\xi\frac{\partial^2 u}{\partial \xi \partial \eta} + 4C^2(\xi^2+\eta)\frac{\partial^2 u}{\partial \eta^2} + 2(2C^2-1)\frac{\partial u}{\partial \eta} + f(u).$$

5°. "Three-dimensional" solution:

$$w = v(z,\xi,\zeta), \quad \xi = y + \frac{x}{C}, \quad \zeta = (C^2-1)x^2 - 2Cxy - C^2t^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $v = v(z,\xi,\zeta)$ is determined by the equation

$$\frac{\partial^2 v}{\partial z^2} + \left(1+\frac{1}{C^2}\right)\frac{\partial^2 v}{\partial \xi^2} - 4\xi\frac{\partial^2 v}{\partial \xi \partial \zeta} + 4C^2(\xi^2+\zeta)\frac{\partial^2 v}{\partial \zeta^2} + 2(2C^2-1)\frac{\partial v}{\partial \zeta} + f(v) = 0.$$

Remark. The solutions specified in Items 4° and 5° can be used to obtain other "three-dimensional" solutions by means of the cyclic permutations of the space variables.

6°. "Three-dimensional" solution:

$$w = U(\xi,\eta,t), \quad \xi = Ax + By + Cz, \quad \eta = \sqrt{(Bx-Ay)^2 + (Cy-Bz)^2 + (Az-Cx)^2},$$

where A, B, and C are arbitrary constants and the function $U = U(\xi,\eta,t)$ is determined by the equation

$$\frac{\partial^2 U}{\partial t^2} = (A^2+B^2+C^2)\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial U}{\partial \eta}\right) + f(U).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants (B and the expression in square brackets must have like signs), and the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2 w}{dr^2} + \frac{A}{r}\frac{dw}{dr} + Bf(w) = 0, \qquad A = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right];$$

$$w = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W(\zeta,\rho), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right].$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a,n) & \\ \nearrow & & \searrow \\ (z,c,k) & \longleftarrow & (y,b,m) \end{array}$$

⊙ *Reference for equation* 8.6.1.2: A. D. Polyanin and A. I. Zhurov (1998).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_1)^2\right],$$

where B and C_1 are arbitrary constants and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + Bf(w) = 0.$$

Integrating yields its solution in implicit form:

$$\int \left[C_2 - 2B\int f(w)\,dw\right]^{-1/2} dw = C_3 \pm r,$$

where C_2 and C_3 are arbitrary constants.

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi,t), \quad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right);$$

$$w = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W(\zeta,\rho), \quad \zeta^2 = \pm 4\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a,\lambda) & \\ \nearrow & & \searrow \\ (z,c,\nu) & \longleftarrow & (y,b,\mu) \end{array}$$

⊙ *Reference for equation* 8.6.1.3: A. D. Polyanin and A. I. Zhurov (1998).

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_1)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = \frac{2}{2-n} + \frac{2}{2-m}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$
$$w = V_1(x, \eta_1), \quad \eta_1^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$
$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right];$$
$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$
$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$
$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n}\frac{1}{r}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2}\right].$$

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(z)\dfrac{\partial w}{\partial z}\right] + aw\ln w + bw.$

1°. Multiplicative separable solution:

$$w(x, y, z, t) = X(x)Y(y)Z(z)\varphi(t),$$

where the functions $X(x)$, $Y(y)$, $Z(z)$, and $\varphi(t)$ are determined by the ordinary differential equations

$$[f(x)X'_x]'_x + aX\ln X + C_1X = 0,$$
$$[g(y)Y'_y]'_y + aY\ln Y + C_2Y = 0,$$
$$[h(z)Z'_z]'_z + aZ\ln Z + C_3Z = 0,$$
$$\varphi''_{tt} - a\varphi\ln\varphi + (C_1 + C_2 + C_3 - b)\varphi = 0,$$

where C_1, C_2, and C_3 are arbitrary constants. A particular and the general solutions of the last equations can be obtained from the formulas of Item 2°, where A should be set equal to $b - C_1 - C_2 - C_3$.

2°. Multiplicative separable solution:

$$w(x, y, z, t) = \varphi(t)\Theta(x, y, z).$$

Here, the function $\varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi''_{tt} - a\varphi\ln\varphi - A\varphi = 0, \tag{1}$$

where A is an arbitrary constant, and the function $\Theta(x, y, z)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial\Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial\Theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial\Theta}{\partial z}\right] + a\Theta\ln\Theta + (b - A)\Theta = 0.$$

A particular solution of equation (1) is given by

$$\varphi(t) = \exp\left[\frac{a}{4}(t+B)^2 + \frac{a-2A}{2a}\right],$$

where B is an arbitrary constant, and the general solution can be written out in implicit form (C is an arbitrary constant):

$$\int \left[a\varphi^2 \ln\varphi + (A - \tfrac{1}{2}a)\varphi^2 + B\right]^{-1/2} d\varphi = C \pm t.$$

8.6.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right] + g(w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$
$$w_3 = w\big(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z, xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$

where $C_1, \ldots, C_5$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions in implicit form:

$$\int h(w)\,dw = z\varphi(\eta) + \psi(\eta), \qquad \eta = C_1 x + C_2 y \pm t\sqrt{a_1 C_1^2 + a_2 C_2^2},$$

where C_1 and C_2 are arbitrary constants and $\varphi(\eta)$ and $\psi(\eta)$ are arbitrary functions.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x, y, z, t) = U(\xi, \eta), \quad \xi = z + \lambda t, \quad \eta = C_1 x + C_2 y \pm t\sqrt{a_1 C_1^2 + a_2 C_2^2},$$

where C_1, C_2, and λ are arbitrary constants, and the function $U = U(\xi, \eta)$ is determined by the first-order partial differential equation

$$\big[h(U) - \lambda^2\big]\frac{\partial U}{\partial \xi} \mp 2\lambda\sqrt{a_1 C_1^2 + a_2 C_2^2}\,\frac{\partial U}{\partial \eta} = \varphi(\eta), \tag{1}$$

and $\varphi(\eta)$ is an arbitrary function.

In the special case $\lambda = 0$, equation (1) is an ordinary differential equation in ξ and can be easily integrated to obtain solutions of Item 2°.

In the general case, equation (1) can be solved using a characteristic system of ordinary differential equations (see Section 23.1). In the special case $\varphi(\eta) = 0$, the general solution of equation (1) can be written out in implicit form:

$$2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,\xi \pm \eta[h(U) - \lambda^2] = \Phi(U),$$

where $\Phi(U)$ is an arbitrary function.

4°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1}, \tag{2}$$

where the function $u(y, z, \zeta)$ is determined by a differential equation of the form 9.4.4.8:

$$a_2\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial z}\left[h(u)\frac{\partial u}{\partial z}\right] = 0, \tag{3}$$

which can be reduced to a linear equation. Equation (3) is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

Remark 1. Relations (2) and equation (3) can be used to obtain another "three-dimensional" solution by means of the following renaming: $(x, a_1) \rightleftarrows (y, a_2)$.

5°. "Three-dimensional" solution:

$$w = v(z, \xi, \eta), \quad \xi = \frac{x}{\sqrt{a_1}C} + \frac{y}{\sqrt{a_2}}, \quad \eta = (C^2 - 1)\frac{x^2}{a_1} - 2C\frac{xy}{\sqrt{a_1a_2}} - C^2t^2, \tag{4}$$

where C is an arbitrary constant ($C \neq 0$), and the function $v = v(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 v}{\partial \xi^2} - 4\xi\frac{\partial^2 v}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 v}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial v}{\partial \eta} + \frac{\partial}{\partial z}\left[h(v)\frac{\partial v}{\partial z}\right] = 0. \tag{5}$$

Remark 2. Relations (4) and equation (5) can be used to obtain another "three-dimensional" solution by means of the following renaming: $(x, a_1) \rightleftarrows (y, a_2)$.

6°. There are solutions of the following forms:

$w(x, y, z, t) = F(r, z, t)$,	$r = a_2x^2 + a_1y^2$	"three-dimensional" solution;
$w(x, y, z, t) = G(\xi, y, z)$,	$\xi = x^2 - a_1t^2$	"three-dimensional" solution;
$w(x, y, z, t) = H(\zeta, z)$,	$\zeta = a_2x^2 + a_1y^2 - a_1a_2t^2$	"two-dimensional" solution;
$w(x, y, z, t) = U(\eta)$,	$\eta = (a_2x^2 + a_1y^2 - a_1a_2t^2)z^{-2}$	"one-dimensional" solution.

7°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$ and $g(w) = a_2$.

⊙ *References for equation* 8.6.2.1: N. H. Ibragimov (1994, p. 234), A. D. Polyanin and V. F. Zaitsev (2004, pp. 338–339).

2. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right] + a_3\dfrac{\partial}{\partial z}\left[g(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w\big(x, y\cos\beta + z\sqrt{a_2/a_3}\,\sin\beta, -y\sqrt{a_3/a_2}\,\sin\beta + z\cos\beta, t\big),$$
$$w_3 = w\big(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z, xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$

where $C_1, \ldots, C_5$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1},$$

where the function $u(y, z, \zeta)$ is determined by the differential equation

$$a_2\frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] + a_3\frac{\partial}{\partial z}\left[g(u)\frac{\partial u}{\partial z}\right] = 0, \tag{1}$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution are arbitrary functions of ζ). The transformation

$$v = \int g(u)\,du, \qquad \overline{y} = \frac{y}{\sqrt{a_2}}, \qquad \overline{z} = \frac{z}{\sqrt{a_3}}$$

brings (1) to the Laplace equation

$$\frac{\partial^2 v}{\partial \overline{y}^2} + \frac{\partial^2 v}{\partial \overline{z}^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. There are solutions of the following forms:

$w(x, y, z, t) = F(x, r, t)$,	$r = a_3y^2 + a_2z^2$	"three-dimensional" solution;
$w(x, y, z, t) = G(\xi, y, z)$,	$\xi = x^2 - a_1t^2$	"three-dimensional" solution;
$w(x, y, z, t) = H(r, \xi)$,	$r = a_3y^2 + a_2z^2$, $\quad \xi = x^2 - a_1t^2$	"two-dimensional" solution;
$w(x, y, z, t) = U(p, q)$,	$p = (a_3y^2 + a_2z^2)t^{-2}$, $\quad q = xt^{-1}$	"two-dimensional" solution;
$w(x, y, z, t) = V(\eta)$,	$\eta = (a_3y^2 + a_2z^2)(x^2 - a_1t^2)^{-1}$	"one-dimensional" solution.

4°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$, in which $g(w)$ should be renamed $a_2g(w)$ and $h(w)$ renamed $a_3g(w)$.

⊙ *References for equation* 8.6.2.2: N. H. Ibragimov (1994, p. 235), A. D. Polyanin and V. F. Zaitsev (2004, pp. 339–340).

3. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w(x\cosh\lambda + t a_1^{1/2}\sinh\lambda,\ y,\ z,\ x a_1^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1},$$

where the function $u(y, z, \zeta)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(u)\frac{\partial u}{\partial z}\right] = 0,$$

which can be reduced to a linear equation. The equation obtained is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

3°. There are solutions of the following forms:

$w(x, y, z, t) = W(\xi, y, z), \quad \xi = x^2 - a_1 t^2$ "three-dimensional" solution;
$w(x, y, z, t) = U(p, q), \quad p = (x^2 - a_1 t^2)y^{-2}, \quad q = zy^{-1}$ "two-dimensional" solution.

4°. For other exact solutions, see equation 8.6.2.6 with $f(w) = a_1$.

⊙ *References for equation* 8.6.2.3: N. H. Ibragimov (1994, p. 235), A. D. Polyanin and V. F. Zaitsev (2004, p. 340).

4. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + a_2\dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + a_3\dfrac{\partial}{\partial z}\left[f(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. There are "three-dimensional" solutions of the following forms:

$$w = W(\rho, z, t), \quad \rho = a_2 x^2 + a_1 y^2;$$
$$w = U(\xi, \eta, t), \quad \xi = \frac{x}{\sqrt{a_1}C} + \frac{y}{\sqrt{a_2}}, \quad \eta = (C^2 - 1)\frac{x^2}{a_1} - 2C\frac{xy}{\sqrt{a_1 a_2}} + C^2\frac{z^2}{a_3};$$
$$w = V(\zeta, \theta, t), \quad \zeta = \frac{Ax}{\sqrt{a_1}} + \frac{By}{\sqrt{a_2}} + \frac{Cz}{\sqrt{a_3}},$$
$$\theta = \left(\frac{Bx}{\sqrt{a_1}} - \frac{Ay}{\sqrt{a_2}}\right)^2 + \left(\frac{Cy}{\sqrt{a_2}} - \frac{Bz}{\sqrt{a_3}}\right)^2 + \left(\frac{Az}{\sqrt{a_3}} - \frac{Cx}{\sqrt{a_1}}\right)^2,$$

where A, B, and C are arbitrary constants.

Remark. The first and second solutions above can be used to obtain other "three-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a_1) & \\ \nearrow & & \searrow \\ (z, a_3) & \longleftarrow & (y, a_2) \end{array}$$

3°. There are exact solutions of the following forms:

$w(x,y,z,t) = \Phi(r,t), \quad r = a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2$ "two-dimensional" solution;

$w(x,y,z,t) = \Psi(\chi), \quad \chi = (a_2a_3x^2 + a_1a_3y^2 + a_1a_2z^2)t^{-2}$ "one-dimensional" solution.

4°. For other exact solutions, see equation 8.6.2.6, in which $f(w)$, $g(w)$, and $h(w)$ should be renamed $a_1f(w)$, $a_2f(w)$, and $a_3f(w)$, respectively.

⊙ *Reference for equation* 8.6.2.4: N. H. Ibragimov (1994, p. 232).

5. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + a_2\dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1z + C_4, \pm C_1t + C_5),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. There are solutions of the following forms:

$w(x,y,z,t) = W(\xi,z,t), \quad \xi = a_2x^2 + a_1y^2$ "three-dimensional" solution;

$w(x,y,z,t) = U(p,q), \quad p = (a_2x^2 + a_1y^2)t^{-2}, \quad q = zt^{-1}$ "two-dimensional" solution.

3°. For other exact solutions, see equation 8.6.2.6, in which $f(w)$ should be renamed $a_1f(w)$ and $g(w)$ renamed $a_2f(w)$.

⊙ *Reference for equation* 8.6.2.5: N. H. Ibragimov (1994, p. 233).

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1z + C_4, \pm C_1t + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \big[k_1^2 f(w) + k_2^2 g(w) + k_3^2 h(w)\big]\,dw - \lambda^2 w = C_1(k_1x + k_2y + k_3z + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1x+C_2y+C_3z+C_4}{t+C_5}\right)^2 = C_1^2f(w)+C_2^2g(w)+C_3^2h(w),$$
$$\left(\frac{C_1y+C_2z+C_3t+C_4}{x+C_5}\right)^2 f(w)+C_1^2g(w)+C_2^2h(w) = C_3^2,$$
$$\left(\frac{C_1x+C_2z+C_3t+C_4}{y+C_5}\right)^2 g(w)+C_1^2f(w)+C_2^2h(w) = C_3^2,$$
$$\left(\frac{C_1x+C_2y+C_3t+C_4}{z+C_5}\right)^2 h(w)+C_1^2f(w)+C_2^2g(w) = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

4°. Solutions in implicit form:

$$x\varphi_1(w)+y\varphi_2(w)+z\varphi_3(w) = \psi(w)+t,$$
$$x\varphi_1(w)+y\varphi_2(w)+z\varphi_3(w) = \psi(w)-t,$$

where $\varphi_1(w)$, $\varphi_2(w)$, and $\psi(w)$ are arbitrary functions, and the function $\varphi_3(w)$ is determined by the normality condition

$$f(w)\varphi_1^2(w)+g(w)\varphi_2^2(w)+h(w)\varphi_3^2(w) = 1.$$

5°. Solution:

$$w = w(\xi), \quad \xi = \frac{C_1x+C_2y+C_3z+C_4}{t+C_5},$$

where $C_1, \ldots, C_5$ are arbitrary constants, and the function $u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2w'_\xi)'_\xi = [\varphi(w)w'_\xi]'_\xi, \qquad \varphi(w) = C_1^2f(w)+C_2^2g(w)+C_3h(w),$$

which admits the first integral

$$\left[\xi^2 - C_1^2f(w) - C_2^2g(w) - C_3^2h(w)\right]w'_\xi = C_6. \tag{1}$$

To the special case $C_6 = 0$ there corresponds the first solution in Item 3°.

For $C_6 \neq 0$, treating w in (1) as the independent variable, we obtain a Riccati equation for $\xi = \xi(w)$:

$$C_6\xi'_w = \xi^2 - C_1^2f(w) - C_2^2g(w) - C_3^2h(w). \tag{2}$$

For exact solutions of equation (2), which can be reduced to a second-order linear equation, see Polyanin and Zaitsev (2003).

6°. Solution:

$$w = u(\eta), \quad \eta = \frac{C_1y+C_2z+C_3t+C_4}{x+C_5}, \tag{3}$$

where $C_1, \ldots, C_5$ are arbitrary constants, and the function $u(\eta)$ is determined by the ordinary differential equation

$$C_3^2u''_{\eta\eta} = [\eta^2f(u)u'_\eta]'_\eta + C_1^2[g(u)u'_\eta]'_\eta + C_2^2[h(u)u'_\eta]'_\eta,$$

which admits the first integral

$$[\eta^2 f(u) + C_1^2 g(u) + C_2^2 h(u) - C_3^2]u'_\eta = C_6. \tag{4}$$

To the special case $C_6 = 0$ there corresponds the second solution in Item 3°.

For $C_6 \neq 0$, treating u in (4) as the independent variable, we obtain a Riccati equation for $\eta = \eta(u)$:

$$C_6\eta'_u = \eta^2 f(u) + C_1^2 g(u) + C_2^2 h(u) - C_3^2. \tag{5}$$

For exact solutions of equation (5), which can be reduced to a second-order linear equation, see Polyanin and Zaitsev (2003).

Formula (3) and equation (5) can be used to obtain two other "one-dimensional" solutions by means of the following cyclic permutations of variables and determining functions:

$$\begin{array}{ccc} & (x, f) & \\ \nearrow & & \searrow \\ (z, h) & \longleftarrow & (y, g) \end{array}$$

7°. "Two-dimensional" solution (k_1, k_2, and k_3 are arbitrary constants):

$$w(x, y, z, t) = u(\xi, t), \quad \xi = k_1 x + k_2 y + k_3 z,$$

where the function $u = u(\xi, t)$ is determined by a differential equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[\varphi(u)\frac{\partial u}{\partial \xi}\right], \quad \varphi(u) = k_1^2 f(u) + k_2^2 g(u) + k_3^2 h(u),$$

which can be reduced to a linear equation.

8°. "Two-dimensional" solution (a, b, and c are arbitrary constants):

$$w(x, y, z, t) = v(x, \eta), \quad \eta = ay + bz + ct,$$

where the function $v = v(x, \eta)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial x}\left[f(v)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right] = 0, \quad \psi(v) = a^2 g(v) + b^2 h(v) - c^2,$$

which can be reduced to a linear equation.

9°. "Two-dimensional" solution (the a_n and b_n are arbitrary constants):

$$w(x, y, z, t) = U(\zeta, \rho), \quad \zeta = a_1 t + a_2 x, \quad \rho = b_1 y + b_2 z,$$

where the function $U = U(\zeta, \rho)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial \zeta}\left[\Phi(U)\frac{\partial U}{\partial \zeta}\right] + \frac{\partial}{\partial \rho}\left[\Psi(U)\frac{\partial U}{\partial \rho}\right] = 0, \quad \Phi(U) = a_2^2 f(U) - a_1^2, \quad \Psi(U) = b_1^2 g(U) + b_2^2 h(U),$$

which can be reduced to a linear equation.

Remark. The solutions specified in Items 7° and 8° can be used to obtain other "two-dimensional" solutions by means of the cyclic permutations of variables and determining functions as shown in Item 5°.

10°. There are more complicated "two-dimensional" solutions of the form

$$w(x,y,z,t) = V(z_1, z_2), \quad z_1 = a_1x + a_2y + a_3z + a_4t, \quad z_2 = b_1x + b_2y + b_3z + b_4t.$$

11°. "Three-dimensional" solution:

$$w(x,y,z,t) = \Theta(p,q,s), \quad p = x/t, \quad q = y/t, \quad s = z/t,$$

where the function $\Theta = \Theta(p,q,s)$ is determined by the differential equation

$$p^2\frac{\partial^2\Theta}{\partial p^2} + q^2\frac{\partial^2\Theta}{\partial q^2} + r^2\frac{\partial^2\Theta}{\partial r^2} + 2pq\frac{\partial^2\Theta}{\partial p\partial q} + 2pr\frac{\partial^2\Theta}{\partial p\partial r} + 2rq\frac{\partial^2\Theta}{\partial r\partial q}$$
$$+ 2p\frac{\partial\Theta}{\partial p} + 2q\frac{\partial\Theta}{\partial q} + 2r\frac{\partial\Theta}{\partial r} = \frac{\partial}{\partial p}\left[f(\Theta)\frac{\partial\Theta}{\partial p}\right] + \frac{\partial}{\partial q}\left[g(\Theta)\frac{\partial\Theta}{\partial q}\right] + \frac{\partial}{\partial r}\left[h(\Theta)\frac{\partial\Theta}{\partial r}\right].$$

12°. For results of the group analysis of the original equation, see Ibragimov (1994).

⊙ *References for equation* 8.6.2.6: A. D. Polyanin and V. F. Zaitsev (2004, pp. 342–343), A. D. Polyanin (2006, http://eqworld.ipmnet.ru/eqarchive/view.php?id=45).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f(w)\dfrac{\partial w}{\partial z}\right] + g(w).$

1°. Suppose the function $f = f(w)$ is arbitrary and the function $g = g(w)$ is defined as

$$g(w) = -a^2\frac{f'(w)}{f^3(w)} + b,$$

where a and b are some numbers. Then there is a functional separable solution, which can be written in the implicit form

$$\int f(w)\,dw = at + U(x,y,z),$$

where the function $U = U(x,y,z)$ is determined by the Poisson equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + b = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The constant b in the expression of g can be replaced by an arbitrary function $b = b(x,y,z)$.

2°. Suppose that the determining functions are defined parametrically as

$$f = \frac{Ae^{-b\zeta}}{\varphi'_\zeta(\zeta)},$$
$$g = a^2\varphi''_{\zeta\zeta}(\zeta) - Ace^{-b\zeta},$$
$$w = \varphi(\zeta),$$

where $\varphi(\zeta)$ is an arbitrary function, ζ is the parameter, and A, a, b, and c are some numbers. Then the equation in question has a functional separable solution of the form

$$w = \varphi(\zeta), \quad \zeta = at + \theta(x, y, z),$$

where the function θ is described by the equation

$$\Delta\theta = b|\nabla\theta|^2 + c.$$

The substitution

$$\theta = -\frac{1}{b}\ln|u|$$

reduces this equation to the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + bcu = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

8.6.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + cz^k\dfrac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and B are arbitrary constants ($B \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m} + \frac{1-k}{2-k}\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right];$$

$$w = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W(\zeta, \rho), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right].$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, n) & \\ \nearrow & & \searrow \\ (z, c, k) & \longleftarrow & (y, b, m) \end{array}$$

2. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\mu y}\dfrac{\partial^2 w}{\partial y^2} + ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + 6r^{-1}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \quad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right);$$
$$w = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$
$$w = W(\zeta, \rho), \quad \zeta^2 = \pm 4\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, \lambda) & \\ \nearrow & & \searrow \\ (z, c, \nu) & \longleftarrow & (y, b, \mu) \end{array}$$

3. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_1)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m} + 1\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

4. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\mu y} \dfrac{\partial^2 w}{\partial y^2} + ce^{\nu z} \dfrac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(5-3n)}{2-n}\frac{1}{r}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2}\right].$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \sum\limits_{k=1}^{n} \dfrac{\partial}{\partial x_k}\left[f_k(w)\dfrac{\partial w}{\partial x_k}\right].$

Exact solutions in implicit form:

$$\sum_{k=1}^{n} x_k \varphi_k(w) = \psi_1(w) + t,$$
$$\sum_{k=1}^{n} x_k \varphi_k(w) = \psi_2(w) - t,$$

where $\varphi_1(w)$, $\ldots$, $\varphi_{n-1}(w)$, $\psi_1(w)$, and $\psi_2(w)$ are arbitrary functions, and the function $\varphi_n(w)$ is determined by the normalization condition

$$\sum_{k=1}^{n} f_k(w)\varphi_k^2(w) = 1.$$

⊙ *Reference*: A. D. Polyanin (2006, http://eqworld.ipmnet.ru/eqarchive/view.php?id=45).

Chapter 9
Second-Order Elliptic Equations with Two Space Variables

9.1. Equations with Power-Law Nonlinearities

9.1.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = aw + bw^n + cw^{2n-1}$

▷ *The general properties of this type of equation are listed in 9.4.1.1; traveling-wave solutions and solutions with central symmetry are also treated there.*

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = kw^n.$

This is a steady heat and mass transfer equation with an nth-order volume reaction in two dimensions. This equation arises also in combustion theory and is a special case of equation 9.4.1.1 with $f(w) = kw^n$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{n-1}x + C_2, \pm C_1^{n-1}y + C_3),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x, y) = (Ax + By + C)^{\frac{2}{1-n}}, \quad B = \pm\sqrt{\frac{k(n-1)^2}{2(n+1)} - A^2};$$
$$w(x, y) = s\left[(x + C_1)^2 + (y + C_2)^2\right]^{\frac{1}{1-n}}, \quad s = \left[\tfrac{1}{4}k(1-n)^2\right]^{\frac{1}{1-n}},$$

where A, C, C_1, and C_2 are arbitrary constants.

3°. Traveling-wave solution in implicit form (generalizes the first solution of Item 2°):

$$\int \left[D + \frac{2kw^{n+1}}{(n+1)(A^2 + B^2)}\right]^{-1/2} dw = Ax + By + C,$$

where A, B, C, and D are arbitrary constants ($n \neq -1$).

4°. Solution (generalizes the second solution of Item 2°):

$$w = w(r), \qquad r = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r}w'_r = kw^n.$$

5°. Self-similar solution:

$$w(x, y) = (x + C_1)^{\frac{2}{1-n}} u(\xi), \qquad \xi = \frac{y + C_2}{x + C_1},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$(1 + \xi^2)u''_{\xi\xi} - \frac{2(1 + n)}{1 - n}\xi u'_\xi + \frac{2(1 + n)}{(1 - n)^2}u - ku^n = 0.$$

6°. Multiplicative separable solution in polar coordinates (another representation of the solution of Item 5°):

$$w(x, y) = r^{\frac{2}{1-n}} U(\theta), \qquad r = \sqrt{(x + C_1)^2 + (y + C_2)^2}, \quad \tan\theta = \frac{y + C_2}{x + C_1},$$

where the function $U = U(\theta)$ is determined by the autonomous ordinary differential equation

$$U''_{\theta\theta} + \frac{4}{(1 - n)^2}U = kU^n.$$

Integrating yields the general solution in implicit form:

$$\int \left[\frac{2k}{n + 1}U^{n+1} - \frac{4}{(n - 1)^2}U^2 + C_3\right]^{-1/2} dU = C_4 \pm \theta,$$

where C_3 and C_4 are arbitrary constants.

⊙ *Reference for equation* 9.1.1.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 190).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw + bw^n.$

This is a special case of equation 9.4.1.1 with $f(w) = aw + bw^n$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x, y) = \left[\frac{2b\sinh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2 \quad \text{if } b(n+1) > 0,$$

$$w(x, y) = \left[-\frac{2b\cosh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2 \quad \text{if } b(n+1) < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solutions for $a < 0$ and $b(n + 1) > 0$:

$$w(x, y) = \left[-\frac{2b\cos^2 z}{a(n + 1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{|a|}\,(1 - n)(x\sin C_1 + y\cos C_1) + C_2.$$

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw^n+bw^{2n-1}.$

This is a special case of equation 9.4.1.1 with $f(w)=aw^n+bw^{2n-1}$.

Solutions:

$$w(x,y)=\left[\frac{a(1-n)^2}{2(n+1)}(x\sin C_1+y\cos C_1+C_2)^2-\frac{b(n+1)}{2an}\right]^{\frac{1}{1-n}},$$

$$w(x,y)=\left\{\frac{1}{4}a(1-n)^2\left[(x+C_1)^2+(y+C_2)^2\right]-\frac{b}{an}\right\}^{\frac{1}{1-n}},$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw-a(n+1)w^n+bw^{2n-1}.$

1°. Traveling-wave solutions:

$$w(x,y)=\big(\lambda+C_1\exp z\big)^{\frac{1}{1-n}},\qquad z=\sqrt{a}\,(1-n)(x\sin C_2+y\cos C_2),$$

where $\lambda=\lambda_{1,2}$ are roots of the quadratic equation $a\lambda^2-a(n+1)\lambda+b=0$, and C_1 and C_2 are arbitrary constants.

2°. See also equation 9.1.1.5, where the following renaming should be made: $b\to -a(n+1)$ and $c\to b$.

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw+bw^n+cw^{2n-1}.$

This is a special case of equation 9.4.1.1 with $f(w)=aw+bw^n+cw^{2n-1}$.

1°. Traveling-wave solutions for $a>0$:

$$w(x,y)=(A+B\cosh z)^{\frac{1}{1-n}},\quad z=\sqrt{a}\,(1-n)(x\sin C_1+y\cos C_1)+C_2,$$
$$A=-\frac{b}{a(n+1)},\quad B=\pm\left[\frac{b^2}{a^2(n+1)^2}-\frac{c}{an}\right]^{1/2};$$

$$w(x,y)=(A+B\sinh z)^{\frac{1}{1-n}},\quad z=\sqrt{a}\,(1-n)(x\sin C_1+y\cos C_1)+C_2,$$
$$A=-\frac{b}{a(n+1)},\quad B=\pm\left[\frac{c}{an}-\frac{b^2}{a^2(n+1)^2}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants (the expressions in square brackets must be nonnegative).

2°. Traveling-wave solutions for $a<0$:

$$w(x,y)=(A+B\cos z)^{\frac{1}{1-n}},\quad z=\sqrt{|a|}\,(1-n)(x\sin C_1+y\cos C_1)+C_2,$$
$$A=-\frac{b}{a(n+1)},\quad B=\pm\left[\frac{b^2}{a^2(n+1)^2}-\frac{c}{an}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

3°. The substitution $u = w^{1-n}$ leads to an equation of the form 9.1.6.7:

$$u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{n}{1-n}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=a(1-n)u^2+b(1-n)u+c(1-n).$$

9.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(x,y,w)$

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a(x^2+y^2)w^n.$

This is a special case of equation 9.4.1.3 with $f(w)=aw^n$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^4 w(\pm C_1^{n-1}x,\ \pm C_1^{n-1}y),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1 and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. The transformation

$$w = U(z,\zeta),\quad z=\tfrac{1}{2}(x^2-y^2),\quad \zeta = xy$$

leads to a simpler equation of the form 9.1.1.1:

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial \zeta^2}=aU^n.$$

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=c(ax+by)^k w^n.$

This is a special case of equation 9.4.1.11 with $f(z,w)=cz^k w^n$.

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a(x^2+y^2)^k w^n.$

This is a special case of equation 9.4.1.4 with $f(w)=aw^n$.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a(x^2+y^2)(xy)^k w^n.$

This is a special case of equation 9.4.1.13 with $f(z,w)=az^k w^n$.

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta x}w^n.$

This is a special case of equation 9.4.1.5 with $f(w)=aw^n$.

6. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ke^{ax-by}w^n.$

This is a special case of equation 9.4.1.6 with $f(w)=kw^n$.

7. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=k\big(w+A_{11}x^2+A_{12}xy+A_{22}y^2+B_1x+B_2y\big)^n.$

This is a special case of equation 9.4.1.15 with $f(u)=ku^n$.

9.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=F\Big(x,y,w,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\Big)$

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}+(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}+kw^n.$

This is a special case of equation 9.4.2.3 with $f(w)=-kw^n$.

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{a}{x}\dfrac{\partial w}{\partial x}+\dfrac{b}{y}\dfrac{\partial w}{\partial y}=kw^n.$

This is a special case of equation 9.4.2.6 with $f(\xi,w)=kw^n$.

3. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=b\left(\dfrac{\partial w}{\partial y}\right)^2+cw+sx^n.$

This is a special case of equation 9.4.2.9 with $f(x)=b$, $g(x)=c$, and $h(x)=sx^n$.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\left(\dfrac{\partial w}{\partial y}\right)^2+\beta x^n y^2+\gamma x^m y+\mu x^k.$

This is a special case of equation 9.4.2.11 with $a=b=1$, $f(x)=\alpha$, $g(x)=h_1(x)=h_0(x)=p(x)=0$, $q_2(x)=\beta x^n$, $q_1(x)=\gamma x^m$, and $q_0(x)=\mu x^k$.

5. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=c\left(\dfrac{\partial w}{\partial y}\right)^2+bcw^2+kw+s.$

Let A be a root of the quadratic equation $bcA^2+kA+s=0$.

1°. Suppose the inequality $2Abc+k+ab=\sigma^2>0$ holds. Then the equation has the generalized separable solutions

$$w(x,y)=A+\big[C_1\exp(\sigma x)+C_2\exp(-\sigma x)\big]\exp\big(\pm y\sqrt{-b}\,\big),$$

where C_1 and C_2 are arbitrary constants.

2°. If $2Abc+k+ab=-\sigma^2<0$, then the equation has the generalized separable solutions

$$w(x,y)=A+\big[C_1\cos(\sigma x)+C_2\sin(\sigma x)\big]\exp\big(\pm y\sqrt{-b}\,\big).$$

3°. For more complicated solutions, see equation 9.4.2.10 with $f(x)=c$, $g(x)=k$, and $h(x)=s$.

6. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=cx^n\left(\dfrac{\partial w}{\partial y}\right)^2+bcx^n w^2+kx^m w+sx^l.$

This is a special case of equation 9.4.2.10 with $f(x)=cx^n$, $g(x)=kx^m$, and $h(x)=sx^l$.

7. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=ce^{\beta x}\left(\dfrac{\partial w}{\partial y}\right)^2+bce^{\beta x}w^2+ke^{\mu x}w+se^{\nu x}.$

This is a special case of equation 9.4.2.10 with $f(x)=ce^{\beta x}$, $g(x)=ke^{\mu x}$, and $h(x)=se^{\nu x}$.

8. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw^n\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 9.4.2.3 with $f(w)=aw^n$. The substitution

$$U=\int \exp\left(-\frac{a}{n+1}w^{n+1}\right)dw$$

leads to the two-dimensional Laplace equation for $U=U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

9. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\left(\dfrac{\partial w}{\partial x}\right)^n+\beta\left(\dfrac{\partial w}{\partial y}\right)^m+kw.$

This is a special case of equation 9.4.2.13 with $a=b=1$, $f(x)=\alpha$, and $g(y)=\beta$.

10. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(a_1x+b_1y+c_1)\left(\dfrac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\dfrac{\partial w}{\partial y}\right)^k.$

This is a special case of equation 9.4.2.15 with $f(w,u,v)=0$.

9.1.4. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(x,y)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(x,y)\dfrac{\partial w}{\partial y}\right]=g(w)$

▷ *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, f_1 and f_2 are the principal thermal diffusivities (diffusion coefficients) dependent on the space coordinates x and y, and $g=g(w)$ is a source function that defines the law of heat (substance) release or absorption.*

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=cw^k.$

This is a special case of equation 9.4.3.1 with $f(w)=cw^k$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{k-1}{2-n}} x,\ C_1^{\frac{k-1}{2-m}} y\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2 x^{2-n} + a(2-n)^2 y^{2-m}\big]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi} w'_\xi = B w^k, \tag{1}$$

where

$$A = \frac{4-nm}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. Equation (1) admits an exact solution of the form

$$w = \left[\frac{2(1+k+A-Ak)}{B(1-k)^2}\right]^{\frac{1}{k-1}} \xi^{\frac{2}{1-k}}$$

with $k \neq 1$.

3.2. For $m = 4/n$, the exact solution can be represented in implicit form as

$$\int \left[C_1 + \frac{2cn^2 w^{k+1}}{ab(k+1)(2-n)^4}\right]^{-1/2} dw = C_2 \pm \xi,$$

where C_1 and C_2 are arbitrary constants.

3.3. The substitution $\zeta = \xi^{1-A}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \frac{B}{(1-A)^2} \zeta^{\frac{2A}{1-A}} w^k. \tag{2}$$

Over 20 exact solutions to equation (2) for various values of k can be found in Polyanin and Zaitsev (2003).

2. $\dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) = cw^m.$

This is a special case of equation 9.4.3.8 with $f(w) = cw^m$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{m-1}{2-n}} x,\ y + \frac{1-m}{\mu} \ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big[b\mu^2 x^{2-n} + a(2-n)^2 e^{-\mu y}\big]^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{n}{2-n} \frac{1}{\xi} w'_\xi = \frac{4c}{ab\mu^2(2-n)^2} w^m.$$

3. $\dfrac{\partial}{\partial x}\left(ae^{\beta x} \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) = cw^m.$

This is a special case of equation 9.4.3.6 with $f(w) = cw^m$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x + \frac{1-m}{\beta}\ln C_1,\ y + \frac{1-m}{\mu}\ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} - \frac{1}{\xi}w'_\xi = Aw^m, \qquad A = \frac{4c}{ab\beta^2\mu^2}. \tag{1}$$

3°. Below are some exact solutions of equation (1).

3.1. Equation (1) admits a solution of the form

$$w = \left[\frac{abm\beta^2\mu^2}{c(1-m)^2}\right]^{\frac{1}{m-1}} \xi^{\frac{2}{1-m}}.$$

3.2. The substitution $\zeta = \xi^2$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \tfrac{1}{4}A\zeta^{-1}w^m,$$

whose solutions with $m = -1$ and $m = -2$ can be found in Polyanin and Zaitsev (2003).

4. $\dfrac{\partial}{\partial x}\left[(ay+c)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(bx+s)\dfrac{\partial w}{\partial y}\right] = kw^n.$

This is a special case of equation 9.4.4.1 with $f(w) = kw^n$.

The equation can be rewritten in the simpler form

$$(ay+c)\frac{\partial^2 w}{\partial x^2} + (bx+s)\frac{\partial^2 w}{\partial y^2} = kw^n.$$

5. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right] = kw^n.$

This is a special case of equation 9.4.4.2 with $f(w) = kw^n$.

9.1.5. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] = g(w)$

▷ *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, $f_1 = f_1(w)$ and $f_2 = f_2(w)$ are the temperature (concentration) dependent principal thermal diffusivities (diffusion coefficients), and $g = g(w)$ is a source function that defines the law of heat (substance) release or absorption. Simple solutions dependent on a single space variable, $w = w(x)$ and $w = w(y)$, are not considered in this subsection.*

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(\alpha w + \beta)\dfrac{\partial w}{\partial y}\right] = 0.$

Stationary Khokhlov–Zabolotskaya equation (for $\alpha = 1$ and $\beta = 0$). It arises in acoustics, nonlinear mechanics, and heat and mass transfer theory. This is a special case of equation 9.4.4.8 with $f(w) = 1$ and $g(w) = \alpha w + \beta$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = \frac{C_1^2}{C_2^2} w(C_1 x + C_3, C_2 y + C_4) + \frac{\beta}{\alpha}\left(\frac{C_1^2}{C_2^2} - 1\right),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x, y) = Ay - \tfrac{1}{2}A^2\alpha x^2 + C_1 x + C_2,$$
$$w(x, y) = (Ax + B)y - \frac{\alpha}{12A^2}(Ax + B)^4 + C_1 x + C_2,$$
$$w(x, y) = -\frac{1}{\alpha}\left(\frac{y + A}{x + B}\right)^2 + \frac{C_1}{x + B} + C_2(x + B)^2 - \frac{\beta}{\alpha},$$
$$w(x, y) = -\frac{1}{\alpha}\left[\beta + \lambda^2 \pm \sqrt{A(y + \lambda x) + B}\,\right],$$
$$w(x, y) = (Ax + B)\sqrt{C_1 y + C_2} - \frac{\beta}{\alpha},$$

where A, B, C_1, C_2, and λ are arbitrary constants. The first two solutions are linear in y, the third is quadratic in y, and the fourth one is a traveling-wave solution.

3°. Generalized separable solution quadratic in y (generalizes the third solution of Item 2°):

$$w(x, y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} + 6\alpha\varphi^2 = 0, \tag{1}$$
$$\psi''_{xx} + 6\alpha\varphi\psi = 0, \tag{2}$$
$$\chi''_{xx} + 2\alpha\varphi\chi = -2\beta\varphi - \alpha\psi^2. \tag{3}$$

The nonlinear autonomous equation (1) is independent of the others; its solution can be expressed in terms of elliptic integrals. Equations (2) and (3) are solved successively (these are linear in the unknowns ψ and χ, respectively).

System (1)–(3) admits the following five-parameter family of solutions:

$$\varphi(x) = -\frac{1}{\alpha(x + A)^2},$$
$$\psi(x) = \frac{B_1}{(x + A)^2} + B_2(x + A)^3,$$
$$\chi(x) = \frac{C_1}{x + A} + C_2(x + A)^2 - \frac{\beta}{\alpha} - \frac{\alpha B_1^2}{4(x + A)^2} - \frac{1}{2}\alpha B_1 B_2(x + A)^3 - \frac{1}{54}\alpha B_2^2(x + A)^8,$$

where A, B_1, B_2, C_1, and C_2 are arbitrary constants.

4°. Solution in parametric form:

$$x = C_1 wt + C_2 w + C_3 t + C_4,$$
$$y = \tfrac{1}{2}C_1 t^2 + C_2 t - \tfrac{1}{3}\alpha C_1 w^3 - \tfrac{1}{2}(\alpha C_3 + \beta C_1)w^2 - \beta C_3 w + C_5.$$

5°. Solution in parametric form:

$$x = C_1 t^2 + C_2 wt + C_3 t + C_4 w - C_1\left(\tfrac{1}{3}\alpha w^3 + \beta w^2\right) + C_5,$$
$$y = \tfrac{1}{2}C_2 t^2 + C_4 t - C_1 t\left(\alpha w^2 + 2\beta w\right) - \tfrac{1}{3}\alpha C_2 w^3 - \tfrac{1}{2}(\alpha C_3 + \beta C_2)w^2 - \beta C_3 w + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

6°. Self-similar solution (A and B are arbitrary constants):

$$w = w(\zeta), \quad \zeta = \frac{x + A}{y + B}$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} + [\zeta^2(\alpha w + \beta)w'_\zeta]'_\zeta = 0.$$

On integrating the equation once and taking w to be the independent variable, one obtains a Riccati equation for $\zeta = \zeta(w)$:

$$C\zeta'_w = (\alpha w + \beta)\zeta^2 + 1,$$

where C is an arbitrary constant. The general solution to this equation can be expressed in terms of Bessel functions; see Polyanin and Zaitsev (2003).

7°. Solution (generalizes the last solution of Item 2°):

$$w(x, y) = \frac{1}{\alpha} f(x)g(y) - \frac{\beta}{\alpha}.$$

The functions $f(x)$ and $g(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$f''_{xx} = Af^2, \qquad (gg'_y)'_y = -Ag, \tag{4}$$

which are independent. Integrating the equations of (4) yields their general solutions in implicit form:

$$C_1 \pm x = \int \left(\tfrac{2}{3}Af^3 + B_1\right)^{-1/2} df,$$
$$C_2 \pm y = \int g\left(-\tfrac{2}{3}Ag^3 + B_2\right)^{-1/2} dg,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants.

8°. Solution (A, B, and k are arbitrary constants):

$$w = \frac{1}{\alpha}(x + A)^{2k} F(z) - \frac{\beta}{\alpha}, \quad z = \frac{y + B}{(x + A)^{k+1}},$$

where the function $F = F(z)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$2k(2k - 1)F - (k + 1)(3k - 2)zF'_z + (k + 1)^2 z^2 F''_{zz} + (FF'_z)'_z = 0.$$

Its order can be reduced.

9°. Solution (A and λ are arbitrary constants):

$$w = \frac{1}{\alpha}e^{-2\lambda x}\Phi(u) - \frac{\beta}{\alpha}, \quad u = (y + A)e^{\lambda x},$$

where the function $\Phi = \Phi(u)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$4\lambda^2\Phi - 3\lambda^2 u\Phi'_u + \lambda^2 u^2\Phi''_{uu} + (\Phi\Phi'_u)'_u = 0.$$

Its order can be reduced.

10°. Solution (A, B, and C are arbitrary constants):

$$w = \frac{1}{\alpha}(\pm x + A)^{-2}\Psi(\xi) - \frac{\beta}{\alpha}, \quad \xi = y + B\ln(\pm x + A) + C,$$

where the function $\Psi = \Psi(\xi)$ is determined by the autonomous ordinary differential equation

$$6\Psi - 5B\Psi'_\xi + B^2\Psi''_{\xi\xi} + (\Psi\Psi'_\xi)'_\xi = 0.$$

Its order can be reduced.

11°. Solution:

$$w = U(\eta) - 4\alpha C_1^2 x^2 - 4\alpha C_1 C_2 x, \quad \eta = y + \alpha C_1 x^2 + \alpha C_2 x,$$

where C_1 and C_2 are arbitrary constants and the function $U(\eta)$ is determined by the first-order ordinary differential equation

$$(\alpha U + \beta + \alpha^2 C_2^2)U'_\eta + 2\alpha C_1 U = 8\alpha C_1^2\eta + C_3.$$

Through appropriate translations in both variables, one can make the equation homogeneous, which means that the equation is integrable by quadrature.

12°. The original equation can be rewritten as the system of equations

$$\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \quad -(\alpha w + \beta)\frac{\partial w}{\partial y} = \frac{\partial v}{\partial x}.$$

The hodograph transformation $x = x(w, v)$, $y = y(w, v)$ (w and v treated as the independent variables, and x and y, as the dependent ones) brings it to the linear system

$$\frac{\partial y}{\partial v} = \frac{\partial x}{\partial w}, \quad -(\alpha w + \beta)\frac{\partial x}{\partial v} = \frac{\partial y}{\partial w}.$$

On eliminating y, one obtains a linear equation for $x = x(w, v)$:

$$\frac{\partial^2 x}{\partial w^2} + (\alpha w + \beta)\frac{\partial^2 x}{\partial v^2} = 0.$$

13°. Let $w(x, y)$ be any solution of the Khokhlov–Zabolotskaya equation (with $\alpha = 1$ and $\beta = 0$). Then the ordinary differential equation

$$u''_{tt} = F(t, u), \qquad F(t, u) = \frac{1}{9\varphi}\left(\frac{\partial v}{\partial u} + 3\varphi''_{tt}u + 3\psi'_t\right),$$

where

$$v = -\varphi^{1/3}w(x, y) - \varphi^{-1}(\varphi'_t u + \psi)^2, \quad x = \frac{1}{3}\int \varphi^{-2/3}\,dt, \quad y = \varphi^{-1/3}u - \frac{1}{3}\int \varphi^{-4/3}\psi\,dt,$$

and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, has a first integral cubic in u'_t.

⊙ *References for equation* 9.1.5.1: Y. Kodama and J. Gibbons (1989), V. V. Kozlov (1995, pp. 379–381), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002, pp. 193–195).

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(\dfrac{1}{\alpha w+\beta}\dfrac{\partial w}{\partial y}\right)=0.$

1°. Solutions:

$$w(x,y)=\frac{-A^2x^2+Bx+C}{\alpha(Ay+D)^2}-\frac{\beta}{\alpha},$$

$$w(x,y)=\frac{p^2}{A\alpha}\frac{Ax^2+Bx+C}{\cosh^2(py+q)}-\frac{\beta}{\alpha},$$

$$w(x,y)=-\frac{p^2}{A\alpha}\frac{Ax^2+Bx+C}{\sinh^2(py+q)}-\frac{\beta}{\alpha},$$

$$w(x,y)=-\frac{p^2}{A\alpha}\frac{Ax^2+Bx+C}{\cos^2(py+q)}-\frac{\beta}{\alpha},$$

where A, B, C, D, p, and q are arbitrary constants.

2°. Solution in parametric form:

$$x=C_1wt+C_2w+C_3t+C_4,$$
$$y=\frac{1}{2}C_1t^2+C_2t-\frac{C_1}{\alpha}w-\frac{1}{\alpha^2}(\alpha C_3-\beta C_1)\ln|\alpha w+\beta|+C_5,$$

where $C_1,\ldots,C_5$ are arbitrary constants.

3°. For other exact solutions, see equation 9.4.4.8 with $f(w)=1$ and $g(w)=(\alpha w+\beta)^{-1}$.

4°. The substitution $\alpha w+\beta=e^U$ leads to an equation of the form 9.2.4.1 (with swapped variables, $x\rightleftarrows y$):

$$\frac{\partial}{\partial x}\left(e^U\frac{\partial U}{\partial x}\right)+\frac{\partial^2 U}{\partial y^2}=0.$$

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(\dfrac{\alpha}{\sqrt{w+\beta}}\dfrac{\partial w}{\partial y}\right)=0.$

The substitution $U=\dfrac{1}{\alpha}\sqrt{w+\beta}$ leads to the equation

$$\frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right)+\frac{\partial^2 U}{\partial y^2}=0.$$

Up to the swap of the coordinates ($x\rightleftarrows y$) and renaming the unknown function, this equation coincides with a special case of 9.1.5.1.

4. $\dfrac{\partial}{\partial x}\left[(\alpha_1 w+\beta_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(\alpha_2 w+\beta_2)\dfrac{\partial w}{\partial y}\right]=\gamma.$

1°. Traveling-wave solutions linear in the coordinates:

$$w(x,y)=Ax\pm\sqrt{\frac{\gamma-A^2\alpha_1}{\alpha_2}}\,y+B,$$

where A and B are arbitrary constants.

2°. Traveling-wave solution in implicit form:

$$(A^2\alpha_1 + B^2\alpha_2)w^2 + 2(A^2\beta_1 + B^2\beta_2)w = \gamma(Ax + By)^2 + C_1(Ax + By) + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants.

3°. For other solutions with $\gamma = 0$, see 9.4.4.8 with $f(w) = \alpha_1 w + \beta_1$ and $g(w) = \alpha_2 w + \beta_2$.

5. $\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right) = \alpha w^n.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{n-m-1}x + C_2, \pm C_1^{n-m-1}y + C_3),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. For $m \neq -1$, the substitution $U = w^{m+1}$ leads to an equation of the form 9.1.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \alpha(m+1)U^{\frac{n}{m+1}}.$$

3°. For $m = -1$, the substitution $w = e^V$ leads to an equation of the form 9.2.1.1:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \alpha e^{nV}.$$

6. $\dfrac{\partial}{\partial x}\left(aw^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(bw^m\dfrac{\partial w}{\partial y}\right) = 0.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n C_2 x + C_3, \pm C_1^m C_2 y + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x, y) = f(x)g(y). \tag{1}$$

The functions $f(x)$ and $g(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$(f^n f'_x)'_x = Abf^{m+1}, \qquad (g^m g'_y)'_y = -Aag^{n+1}, \tag{2}$$

which are independent. Integrating the equations of (2) yields their general solutions in implicit form:

$$\int f^n\left(\frac{2Ab}{n+m+2}f^{n+m+2} + B_1\right)^{-1/2} df = C_1 \pm x,$$
$$\int g^m\left(-\frac{2Aa}{n+m+2}g^{n+m+2} + B_2\right)^{-1/2} dg = C_2 \pm y,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants; $n + m + 2 \neq 0$.

3°. There are exact solutions of the following forms:

$$w(x,y) = x^{-2k}F(z), \qquad z = yx^{mk-nk-1},$$
$$w(x,y) = x^{\frac{2}{n-m}}G(\xi), \quad \xi = y + k\ln x,$$
$$w(x,y) = e^{2x}H(\eta), \qquad \eta = ye^{(n-m)x},$$

where k is an arbitrary constant.

4°. For other exact solutions of the original equation, see 9.4.4.8 with $f(w) = aw^n$ and $g(w) = bw^m$.

7. $\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right) = bw^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-n-1}x + C_2, \pm C_1^{k-m-1}y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are exact solutions of the following forms:

$w(x,y) = F(\xi), \quad \xi = \alpha_1 x + \alpha_2 y$ traveling-wave solution;
$w(x,y) = x^{\frac{2}{n-k+1}}U(z), \quad z = yx^{\frac{k-m-1}{n-k+1}}$ self-similar solution.

9.1.6. Other Equations Involving Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial x^2} + aw^4\dfrac{\partial^2 w}{\partial y^2} = by^n w^5.$

This is a special case of equation 9.4.5.1 with $f(y) = by^n$.

2. $\dfrac{\partial^2 w}{\partial x^2} + aw^4\dfrac{\partial^2 w}{\partial y^2} = be^{\beta y}w^5.$

This is a special case of equation 9.4.5.1 with $f(y) = be^{\beta y}$.

3. $ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} = cw^k.$

This is a special case of equation 9.4.5.5 with $k = s = 0$ and $f(w) = cw^k$.

1°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \left[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = Bw^k, \tag{1}$$

where

$$A = \frac{3nm - 4n - 4m + 4}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

2°. Below are some exact solutions of equation (1).

2.1. For $k \neq 1$, equation (1) admits an exact solution of the form

$$w = \left[\frac{2(1+k+A-Ak)}{B(1-k)^2}\right]^{\frac{1}{k-1}} \xi^{\frac{2}{1-k}}.$$

2.2. For $m = \dfrac{4n-4}{3n-4}$, the general solution of (1) is written out in implicit form as

$$\int \left[C_1 + \frac{2c(3n-4)^2 w^{k+1}}{ab(k+1)(2-n)^4}\right]^{-1/2} dw = C_2 \pm \xi,$$

where C_1 and C_2 are arbitrary constants.

2.3. The substitution $\zeta = \xi^{1-A}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \frac{B}{(1-A)^2} \zeta^{\frac{2A}{1-A}} w^k. \tag{2}$$

Over 20 exact solutions to equation (2) for various values of k can be found in Polyanin and Zaitsev (2003).

4. $ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\beta y} \dfrac{\partial^2 w}{\partial y^2} = cw^m.$

This is a special case of equation 9.4.5.9 with $k = s = 0$ and $f(w) = cw^m$.

5. $ae^{\beta x} \dfrac{\partial^2 w}{\partial x^2} + be^{\mu y} \dfrac{\partial^2 w}{\partial y^2} = cw^m.$

This is a special case of equation 9.4.5.7 with $k = s = 0$ and $f(w) = cw^m$.

1°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \left(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{3}{\xi} w'_\xi = Aw^m, \qquad A = \frac{4c}{ab\beta^2\mu^2}. \tag{1}$$

2°. Below are some exact solutions of equation (1).

2.1. There is a solution of the form

$$w(\xi) = \left[\frac{ab(2-m)\beta^2\mu^2}{c(1-m)^2}\right]^{\frac{1}{m-1}} \xi^{\frac{2}{1-m}}.$$

2.2. The substitution $\zeta = \xi^{-2}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \tfrac{1}{4} A \zeta^{-3} w^m,$$

whose solution for $m = 3$ can be found in Polyanin and Zaitsev (2003).

6. $w\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)-\left(\dfrac{\partial w}{\partial x}\right)^2-\left(\dfrac{\partial w}{\partial y}\right)^2=\alpha w^\beta.$

The substitution $w=e^U$ leads to an equation of the form 9.2.1.1:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\alpha e^{(\beta-2)U}.$$

7. $w\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)+\sigma\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right]=\alpha w^2+\beta w+\gamma.$

1°. Traveling-wave solutions for $\alpha(1+\sigma)>0$:

$$w(x,y)=A_1+B_1\cosh z,\quad z=\sqrt{\frac{\alpha}{1+\sigma}}\,\frac{k_1x+k_2y}{\sqrt{k_1^2+k_2^2}}+C,$$

$$A_1=-\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma},\quad B_1=\pm\sqrt{\frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2}-\frac{\gamma(1+\sigma)}{\alpha\sigma}};$$

$$w(x,y)=A_2+B_2\sinh z,\quad z=\sqrt{\frac{\alpha}{1+\sigma}}\,\frac{k_1x+k_2y}{\sqrt{k_1^2+k_2^2}}+C,$$

$$A_2=-\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma},\quad B_2=\pm\sqrt{\frac{\gamma(1+\sigma)}{\alpha\sigma}-\frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2}},$$

where k_1, k_2, and C are arbitrary constants.

2°. Traveling-wave solutions for $\alpha(1+\sigma)<0$:

$$w(x,y)=A+B\cos z,\quad z=\sqrt{-\frac{\alpha}{1+\sigma}}\,\frac{k_1x+k_2y}{\sqrt{k_1^2+k_2^2}}+C,$$

$$A=-\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma},\quad B=\pm\sqrt{\frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2}-\frac{\gamma(1+\sigma)}{\alpha\sigma}},$$

where k_1, k_2, and C are arbitrary constants.

3°. Solution:

$$w=w(r),\qquad r=\sqrt{(x+C_1)^2+(y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$ww_{rr}''+\frac{1}{r}ww_r'+\sigma\left(w_r'\right)^2=\alpha w^2+\beta w+\gamma.$$

4°. For $\gamma=0$, apart from the solutions presented in Items 1° to 3°, other solutions can be constructed. To this end, we apply the change of variable $w=u^2$ to the original equation to obtain

$$u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(1+2\sigma)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=\tfrac{1}{2}\alpha u^2+\tfrac{1}{2}\beta.$$

This equation is a special case of the original one. It follows that its solution can be obtained with the formulas given in Items 1° and 2°, where variables and parameters should be renamed as follows: $\sigma\to 1+2\sigma$, $\alpha\to\frac{1}{2}\alpha$, $\beta\to 0$, and $\gamma\to\frac{1}{2}\beta$.

5°. Solutions for $\alpha = 0$:

$$w(x,y) = \frac{\beta}{2(1+2\sigma)}\left(\frac{k_1x+k_2y}{\sqrt{k_1^2+k_2^2}}+C\right)^2 - \frac{\gamma(1+2\sigma)}{2\beta\sigma},$$

$$w(x,y) = \frac{\beta}{4(1+\sigma)}\left[(x+C_1)^2+(y+C_2)^2\right] - \frac{\gamma(1+\sigma)}{\beta\sigma},$$

where k_1, k_2, C, C_1, and C_2 are arbitrary constants.

⊙ *Reference for equation* 9.1.6.7: A. D. Polyanin and V. F. Zaitsev (2002, pp. 199–200).

8. $$\frac{\partial}{\partial x}\left[(a_1x+b_1y+c_1w+k_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2x+b_2y+c_2w+k_2)\frac{\partial w}{\partial y}\right] = 0.$$

This is a special case of equation 9.4.4.11 with $f(w) = c_1w + k_1$ and $g(w) = c_2w + k_2$.

9. $$\frac{\partial}{\partial x}\left[(a_1x+b_1y+c_1w^n)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2x+b_2y+c_2w^k)\frac{\partial w}{\partial y}\right] = 0.$$

This is a special case of equation 9.4.4.11 with $f(w) = c_1w^n$ and $g(w) = c_2w^k$.

10. $$a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

This is an *equation of steady transonic gas flow* (with $a < 0$).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-3}C_2^2 w(C_1x+C_3, C_2y+C_4) + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

⊙ *Reference*: N. H. Ibragimov (1985).

2°. Solutions:

$$w(x,y) = C_1xy + C_2x + C_3y + C_4,$$

$$w(x,y) = -\frac{(x+C_1)^3}{3a(y+C_2)^2} + C_3y + C_4,$$

$$w(x,y) = \frac{a^2C_1^3}{39}(y+A)^{13} + \frac{2}{3}aC_1^2(y+A)^8(x+B) + 3C_1(y+A)^3(x+B)^2 - \frac{(x+B)^3}{3a(y+A)^2},$$

$$w(x,y) = -aC_1y^2 + C_2y + C_3 \pm \frac{4}{3C_1}(C_1x+C_4)^{3/2},$$

$$w(x,y) = -aA^3y^2 - \frac{B^2}{aA^2}x + C_1y + C_2 \pm \frac{4}{3}(Ax+By+C_3)^{3/2},$$

$$w(x,y) = \frac{1}{3}(Ay+B)(2C_1x+C_2)^{3/2} - \frac{aC_1^3}{12A^2}(Ay+B)^4 + C_3y + C_4,$$

$$w(x,y) = -\frac{9aA^2}{y+C_1} + 4A\left(\frac{x+C_2}{y+C_1}\right)^{3/2} - \frac{(x+C_2)^3}{3a(y+C_1)^2} + C_3y + C_4,$$

$$w(x,y) = -\frac{3}{7}aA^2(y+C_1)^7 + 4A(x+C_2)^{3/2}(y+C_1)^{5/2} - \frac{(x+C_2)^3}{3a(y+C_1)^2} + C_3y + C_4,$$

where A, B, $C_1, \ldots, C_4$ are arbitrary constants (the first solution is degenerate).

3°. Self-similar solution:

$$w(x,y) = y^{-3k-2}U(z), \qquad z = xy^k,$$

where k is an arbitrary constant, and the function $U = U(z)$ is determined by the ordinary differential equation

$$aU_z'U_{zz}'' + k^2z^2U_{zz}'' - 5k(k+1)zU_z' + 3(k+1)(3k+2)U = 0.$$

4°. Generalized separable solution:

$$w(x,y) = \varphi_1(y) + \varphi_2(y)x^{3/2} + \varphi_3(y)x^3,$$

where the functions $\varphi_k = \varphi_k(y)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
&\varphi_1'' + \tfrac{9}{8}a\varphi_2^2 = 0,\\
&\varphi_2'' + \tfrac{45}{4}a\varphi_2\varphi_3 = 0,\\
&\varphi_3'' + 18a\varphi_3^2 = 0,
\end{aligned}$$

where the prime stands for the differentiation with respect to y. The general solution of the third equation can be written out in implicit form (it is expressed in terms of the Weierstrass function).

5°. Generalized separable solution cubic in x:

$$w(x,y) = \psi_1(y) + \psi_2(y)x + \psi_3(y)x^2 + \psi_4(y)x^3,$$

where the functions $\psi_k = \psi_k(y)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
&\psi_1'' + 2a\psi_2\psi_3 = 0,\\
&\psi_2'' + 2a(2\psi_3^2 + 3\psi_2\psi_4) = 0,\\
&\psi_3'' + 18a\psi_3\psi_4 = 0,\\
&\psi_4'' + 18a\psi_4^2 = 0.
\end{aligned}$$

A particular solution of the system is given by

$$\begin{aligned}
\psi_1(y) &= -2a\int_{y_0}^{y}(y-t)\psi_2(t)\psi_3(t)\,dt + B_1y + B_2,\\
\psi_2(y) &= C_3(y+A)^{-1} + C_4(y+A)^2 - aC_1^2(y+A)^{-2} - 2aC_1C_2(y+A)^3 - \frac{2}{27}aC_2^2(y+A)^8,\\
\psi_3(y) &= C_1(y+A)^{-2} + C_2(y+A)^3, \qquad \psi_4(y) = -\frac{1}{3a}(y+A)^{-2},
\end{aligned}$$

where A, B_1, B_2, C_1, …, C_4 are arbitrary constants and y_0 is any number.

6°. Generalized separable solution:

$$w(x,y) = \eta(y)\theta(x) - aC_1\int_0^y(y-t)\eta^2(t)\,dt + C_2y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $\eta(y)$ and $\theta(x)$ satisfy the autonomous ordinary differential equations (C_4 is an arbitrary constant)

$$\eta''_{yy} + aC_4\eta^2 = 0, \tag{1}$$
$$\theta'_x\theta''_{xx} = C_4\theta + C_1. \tag{2}$$

The solutions to equations (1) and (2) can be written out in implicit form:

$$\int \left(C_5 - \tfrac{2}{3}aC_4\eta^3\right)^{-1/2} d\eta = C_6 \pm y,$$
$$\int \left(\tfrac{3}{2}C_4\theta^2 + 3C_1\theta + C_7\right)^{-1/3} d\theta = x + C_8,$$

where C_5, C_6, C_7, and C_8 are arbitrary constants.

7°. The Legendre transformation

$$u(\xi,\eta) = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where u is the new dependent variable and ξ and η are the new independent variables, leads to *Tricomi's equation*

$$a\xi\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2} = 0.$$

Exact solutions to this linear equation can be found in the book by Polyanin (2002).

⊙ *References for equation* 9.1.6.10: K. G. Guderley (1962), N. H. Ibragimov (1985), S. S. Titov (1988), S. R. Svirshchevskii (1995), A. D. Polyanin and V. F. Zaitsev (2004, pp. 361–362), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 3–4).

11. $\dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{y}\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} = 0.$

For $b < 0$, this equation describes a transonic gas flow.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-3}C_2^2 w(C_1x + C_3, C_2y) + C_4y^{1-a} + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,y) = -\frac{bC_1}{4(a+1)}y^2 + C_2y^{1-a} + C_3 \pm \frac{2}{3C_1}(C_1x + C_4)^{3/2},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Generalized separable solutions:

$$w(x,y) = -\frac{9A^2b}{16(n+1)(2n+1+a)}y^{2n+2} + Ay^n(x+C)^{3/2} + \frac{a-3}{9b}\frac{(x+C)^3}{y^2},$$

where A and C are arbitrary constants, and the $n = n_{1,2}$ are roots of the quadratic equation

$$n^2 + (a-1)n + \tfrac{5}{4}(a-3) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = (Ay^{1-a} + B)(2C_1x + C_2)^{3/2} + 9bC_1^3\theta(y),$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\theta = \theta(y)$ is determined by the second-order linear ordinary differential equation

$$\theta''_{yy} + \frac{a}{y}\theta'_y + (Ay^{1-a} + B)^2 = 0.$$

Integrating yields

$$\theta(y) = -\frac{B^2}{2(a+1)}y^2 - \frac{AB}{3-a}y^{3-a} - \frac{A^2}{2(2-a)(3-a)}y^{4-2a} + C_3y^{1-a} + C_4.$$

5°. Self-similar solution:

$$w(x,y) = y^{-3k-2}U(z), \qquad z = xy^k,$$

where k is an arbitrary constant, and the function $U = U(z)$ is determined by the ordinary differential equation

$$bU'_zU''_{zz} + k^2z^2U''_{zz} + k(a-5k-5)zU'_z + (3k+2)(3k+3-a)U = 0.$$

6°. Generalized separable solution:

$$w(x,y) = \varphi_1(y) + \varphi_2(y)x^{3/2} + \varphi_3(y)x^3,$$

where the functions $\varphi_k = \varphi_k(y)$ are determined by the system of ordinary differential equations

$$\varphi''_1 + \frac{a}{y}\varphi'_1 + \frac{9}{8}b\varphi_2^2 = 0,$$
$$\varphi''_2 + \frac{a}{y}\varphi'_2 + \frac{45}{4}b\varphi_2\varphi_3 = 0,$$
$$\varphi''_3 + \frac{a}{y}\varphi'_3 + 18b\varphi_3^2 = 0,$$

where the prime stands for the differentiation with respect to y.

7°. Generalized separable solution cubic in x:

$$w(x,y) = \psi_1(y) + \psi_2(y)x + \psi_3(y)x^2 + \psi_4(y)x^3,$$

where the functions $\psi_k = \psi_k(y)$ are determined by the system of ordinary differential equations

$$\psi''_1 + \frac{a}{y}\psi'_1 + 2b\psi_2\psi_3 = 0,$$
$$\psi''_2 + \frac{a}{y}\psi'_2 + 2b(2\psi_3^2 + 3\psi_2\psi_4) = 0,$$
$$\psi''_3 + \frac{a}{y}\psi'_3 + 18b\psi_3\psi_4 = 0,$$
$$\psi''_4 + \frac{a}{y}\psi'_4 + 18b\psi_4^2 = 0.$$

8°. Generalized separable solution:

$$w(x,y) = \xi(y) + \eta(y)\theta(x).$$

Here, the functions $\xi(y)$ and $\eta(y)$ are determined by the system of ordinary differential equations

$$\eta''_{yy} + \frac{a}{y}\eta'_y + bC_1\eta^2 = 0,$$
$$\xi''_{yy} + \frac{a}{y}\xi'_y + bC_2\eta^2 = 0,$$

where C_1 and C_2 are arbitrary constants, and the function $\theta = \theta(x)$ is determined by the autonomous ordinary differential equation

$$\theta'_x\theta''_{xx} = C_1\theta + C_2.$$

Its solution can be written out in implicit form:

$$\int \left(\tfrac{3}{2}C_1\theta^2 + 3C_2\theta + C_3\right)^{-1/3} d\theta = x + C_4,$$

where C_3 and C_4 are arbitrary constants.

⊙ *References for equation* 9.1.6.11: K. G. Guderley (1962), S. S. Titov (1988), S. R. Svirshchevskii (1995), A. D. Polyanin and V. F. Zaitsev (2004, pp. 363–364), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 3–4).

12. $\dfrac{\partial^2 w}{\partial x^2} = (ay^k + b)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 9.1.6.15 with $f(y) = ay^k + b$.

13. $\dfrac{\partial^2 w}{\partial x^2} = (ay + b)^k\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 9.1.6.15 with $f(y) = (ay + b)^k$.

14. $\dfrac{\partial^2 w}{\partial x^2} = (ae^{\lambda y} + c)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}.$

This is a special case of equation 9.1.6.15 with $f(y) = ae^{\lambda y} + c$.

15. $\dfrac{\partial^2 w}{\partial x^2} = f(y)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}.$

1°. Generalized separable solution:

$$w = \frac{1}{12}A^2x^4 + \frac{1}{3}ABx^3 + \frac{1}{2}B^2x^2 + Cx + D + (Ax + B)\int\left[2\int\frac{dy}{f(y)}\right]^{1/2} dy,$$

where A, B, C, and D are arbitrary constants.

2°. Generalized separable solution:

$$w = Ax + B + C(x + D)^{-2}\varphi(y),$$

where A, B, C, and D are arbitrary constants, and the function $\varphi = \varphi(y)$ is determined by the ordinary differential equation

$$f(y)\varphi'_y\varphi''_{yy} = 6\varphi.$$

⊙ *Reference*: E. A. Vyazmina (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=261).

9.2. Equations with Exponential Nonlinearities

9.2.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = a + be^{\beta w} + ce^{\gamma w}$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta w}.$

This equation occurs in combustion theory and is a special case of equation 9.4.1.1 with $f(w) = ae^{\beta w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1^{\beta} x + C_2, \pm C_1^{\beta} y + C_3) + 2\ln|C_1|,$$
$$w_2 = w(x\cos\lambda - y\sin\lambda,\ x\sin\lambda + y\cos\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta(Ax+By+C)^2}\right] \quad \text{if } a\beta > 0,$$
$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta\sinh^2(Ax+By+C)}\right] \quad \text{if } a\beta > 0,$$
$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{-2(A^2+B^2)}{a\beta\cosh^2(Ax+By+C)}\right] \quad \text{if } a\beta < 0,$$
$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta\cos^2(Ax+By+C)}\right] \quad \text{if } a\beta > 0,$$
$$w(x,y) = \frac{1}{\beta}\ln\left(\frac{8C}{a\beta}\right) - \frac{2}{\beta}\ln\left|(x+A)^2 + (y+B)^2 - C\right|,$$

where A, B, and C are arbitrary constants. The first four solutions are of traveling-wave type and the last one is a radial symmetric solution with center at the point $(-A, -B)$.

Special case. For $a = \beta = 1$, the boundary value problem for the circle $r = \sqrt{x^2+y^2} \le 1$ with the boundary condition $w\big|_{r=1} = 0$ has the following two solutions (see the last solution in Item 2° with $a = \beta = 1$, $A = B = 0$, and $C = k$):

$$w(r) = \ln\frac{8k}{(k-r^2)^2}, \qquad k = 5 \pm 2\sqrt{6}.$$

The first solution is bounded at every point inside the circle, $r \le 1$, and the second one has a singularity at the circumference $r = \sqrt{k}$.

3°. Functional separable solutions:

$$w(x,y) = -\frac{2}{\beta}\ln\left[C_1 e^{ky} \pm \frac{\sqrt{2a\beta}}{2k}\cos(kx + C_2)\right],$$
$$w(x,y) = \frac{1}{\beta}\ln\frac{2k^2(B^2 - A^2)}{a\beta[A\cosh(kx + C_1) + B\sin(ky + C_2)]^2},$$
$$w(x,y) = \frac{1}{\beta}\ln\frac{2k^2(A^2 + B^2)}{a\beta[A\sinh(kx + C_1) + B\cos(ky + C_2)]^2},$$

where A, B, C_1, C_2, and k are arbitrary constants (x and y can be swapped to give another three solutions).

4°. General solution:

$$w(x,y)=-\frac{2}{\beta}\ln\frac{\left|1-2a\beta\Phi(z)\overline{\Phi(z)}\,\right|}{4|\Phi'_z(z)|},$$

where $\Phi=\Phi(z)$ is an arbitrary analytic (holomorphic) function of the complex variable $z=x+iy$ with nonzero derivative, and the bar over a symbol denotes the complex conjugate.

5°. The original equation is related to the linear equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \tag{1}$$

by the Bäcklund transformation

$$\frac{\partial U}{\partial x}+\tfrac{1}{2}\beta\frac{\partial w}{\partial y}=\left(\tfrac{1}{2}a\beta\right)^{1/2}\exp\left(\tfrac{1}{2}\beta w\right)\sin U, \tag{2}$$

$$\frac{\partial U}{\partial y}-\tfrac{1}{2}\beta\frac{\partial w}{\partial x}=\left(\tfrac{1}{2}a\beta\right)^{1/2}\exp\left(\tfrac{1}{2}\beta w\right)\cos U. \tag{3}$$

Suppose there is a (particular) solution $U=U(x,y)$ of the Laplace equation (1). Then (2) can be treated as a first-order ordinary differential equation for $w=w(y)$ with parameter x, which can be reduced to a linear equation with the help of the change of variable $z=\exp\left(-\frac{1}{2}\beta w\right)$. Finally, we have

$$w=-\frac{2}{\beta}F-\frac{2}{\beta}\ln\left[\Psi(x)-k\int e^{-F}\sin U\,dy\right],\qquad F=\int\left(\frac{\partial U}{\partial x}\right)dy,$$

where $k=\left(\frac{1}{2}a\beta\right)^{1/2}$; in the integration x is treated as a parameter. The function $\Psi(x)$ is determined after substituting this expression for w into equation (3).

⊙ *References for equation* 9.2.1.1: I. N. Vekua (1960), R. K. Bullough and P. J. Caudrey (1980), D. A. Frank-Kamenetskii (1987), V. F. Zaitsev and A. D. Polyanin (1996, p. 476), S. N. Aristov (1999), I. Kh. Sabitov (2001).

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta w}+be^{2\beta w}$.

1°. Traveling-wave solution for $b\beta>0$:

$$w(x,y)=-\frac{1}{\beta}\ln\left\{-\frac{b}{a}+C_1\exp\left[a\sqrt{\frac{\beta}{b}}\,\left(x\sin C_2+y\cos C_2\right)\right]\right\},$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solution (generalizes the solution of Item 1°):

$$w(x,y)=-\frac{1}{\beta}\ln\left[-\frac{a\beta}{C_1^2+C_2^2}+C_3\exp(C_1x+C_2y)+\frac{a^2\beta^2-b\beta(C_1^2+C_2^2)}{4C_3(C_1^2+C_2^2)^2}\exp(-C_1x-C_2y)\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Traveling-wave solution:

$$w(x,y)=-\frac{1}{\beta}\ln\left[\frac{a\beta}{C_1^2+C_2^2}+\frac{\sqrt{a^2\beta^2+b\beta(C_1^2+C_2^2)}}{C_1^2+C_2^2}\,\sin(C_1x+C_2y+C_3)\right].$$

⊙ *References for equation* 9.2.1.2: A. D. Polyanin and V. F. Zaitsev (2004, p. 366).

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta w}-be^{-\beta w}.$

The transformation

$$w(x,y)=u(x,y)+k,\quad k=\frac{1}{2\beta}\ln\frac{b}{a}$$

leads to an equation of the form 9.3.1.1:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2\sqrt{ab}\,\sinh(\beta u).$$

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta w}+be^{-2\beta w}.$

Functional separable solution:

$$w(x,y)=\frac{1}{\beta}\ln[\varphi(x)+\psi(y)],$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the first-order autonomous ordinary differential equations

$$(\varphi'_x)^2=2a\beta\varphi^3+C_1\varphi^2+C_2\varphi+C_3,$$
$$(\psi'_y)^2=2a\beta\psi^3-C_1\psi^2+C_2\psi-C_3-b\beta,$$

where C_1, C_2, and C_3 are arbitrary constants. Solving these equations for the derivatives, one obtains separable equations.

⊙ *References*: A. M. Grundland and E. Infeld (1992), W. Miller (Jr.) and L. A. Rubel (1993), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a+be^{\beta w}+ce^{2\beta w}.$

The substitution $u=e^{-\beta w}$ leads to a equation with a quadratic nonlinearity of the form 9.1.6.7:

$$u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial u}{\partial y}\right)^2+a\beta u^2+b\beta u+c\beta=0.$$

9.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(x,y,w)$

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=Ae^{\alpha x+\beta y}e^{\mu w}.$

The substitution $U=\alpha x+\beta y+\mu w$ leads to an equation of the form 9.2.1.1:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=A\mu e^{U}.$$

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=Ae^{\alpha xy+\beta x+\gamma y}e^{\mu w}.$

The substitution $U=\alpha xy+\beta x+\gamma y+\mu w$ leads to an equation of the form 9.2.1.1:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=A\mu e^{U}.$$

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=A(x^2+y^2)e^{\beta w}.$

The transformation

$$w=U(z,\zeta),\quad z=\tfrac{1}{2}(x^2-y^2),\quad \zeta=xy$$

leads to a simpler equation of the form 9.2.1.1:

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial \zeta^2}=Ae^{\beta U}.$$

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=A(x^2+y^2)^k e^{\beta w}.$

This is a special case of equation 9.4.1.4 with $f(w)=Ae^{\beta w}$.

9.2.3. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(x,y)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(x,y)\dfrac{\partial w}{\partial y}\right]=g(w)$

▷ *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, f_1 and f_2 are the principal thermal diffusivities (diffusion coefficients) dependent on the space coordinates x and y, and $g=g(w)$ is a source function that defines the law of heat (substance) release or absorption.*

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=ce^{\beta w}.$

This is a special case of equation 9.4.3.1 with $f(w)=ce^{\beta w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(C^{\frac{2}{2-n}}x,\ C^{\frac{2}{2-m}}y\Big)+\frac{2}{\beta}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n\neq 2$ and $m\neq 2$:

$$w=w(\xi),\qquad \xi=\big[b(2-m)^2x^{2-n}+a(2-n)^2y^{2-m}\big]^{1/2}.$$

Here, the function $w=w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{A}{\xi}w'_\xi=Be^{\beta w},\tag{1}$$

where

$$A=\frac{4-nm}{(2-n)(2-m)},\quad B=\frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. For $A\neq 1$, there is a solution of the form

$$w(\xi)=-\frac{1}{\beta}\ln\left[\frac{B\beta}{2(1-A)}\xi^2\right].$$

3.2. For $A=0$, which corresponds to $m=\dfrac{4}{n}$ and $B=\dfrac{cn^2}{ab(2-n)^4}$, we obtain from (1) several more families of exact solutions to the original equation:

$$w(\xi)=\frac{1}{\beta}\ln\left[\frac{2}{\beta B(\xi+C)^2}\right] \qquad \text{if } \beta B>0,$$

$$w(\xi)=\frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\cos^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B>0,$$

$$w(\xi)=\frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\sinh^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B>0,$$

$$w(\xi)=\frac{1}{\beta}\ln\left[\frac{-2\lambda^2}{\beta B\cosh^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B<0,$$

$$w(\xi)=\frac{1}{\beta}\ln\left[\frac{-8\lambda^2C_1C_2}{\beta B\left(C_1e^{\lambda\xi}+C_2e^{-\lambda\xi}\right)^2}\right],$$

where λ, C, C_1, and C_2 are arbitrary constants.

3.3. For $A=1$, which corresponds to $m=\dfrac{n}{n-1}$, another family of exact solutions follows from (1):

$$w(\xi)=\frac{1}{\beta}\ln\left(-\frac{8C}{\beta B}\right)-\frac{2}{\beta}\ln(\xi^2+C),\qquad B=\frac{4c(n-1)^2}{ab(2-n)^4},$$

where C is an arbitrary constant.

4°. There is an exact solution of the form

$$w(x,y)=U(z)+\frac{n-2}{\beta}\ln x,\quad z=yx^{\frac{n-2}{2-m}}.$$

2. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=ce^{\lambda w}$.

This is a special case of equation 9.4.3.6 with $f(w)=ce^{\lambda w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(x-\frac{2}{\beta}\ln C,\ y-\frac{2}{\mu}\ln C\Big)+\frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution:

$$w = -\frac{1}{\lambda}\ln\left[c\lambda\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\mu y}}{b\mu^2}\right)\right].$$

3°. Functional separable solution for $\beta\mu \neq 0$ (generalizes the solution of Item 2°):

$$w = w(\xi), \qquad \xi = \left(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\right)^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} - \frac{1}{\xi}w'_\xi = Ae^{\lambda w}, \qquad A = \frac{4c}{ab\beta^2\mu^2}.$$

4°. There is an exact solution of the form

$$w(x, y) = U(z) + \frac{\beta}{\lambda}x, \qquad z = y - \frac{\beta}{\mu}x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right) = ce^{\lambda w}.$

This is a special case of equation 9.4.3.8 with $f(w) = ce^{\lambda w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\left(C^{\frac{2}{2-n}}x,\ y - \frac{2}{\beta}\ln C\right) + \frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{c\lambda(2-n)}{(1-n)}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2}\right]\right\}.$$

3°. Functional separable solution for $n \neq 2$ and $\beta \neq 0$ (generalizes the solution of Item 2°):

$$w = w(\xi), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2}\right],$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = ce^{\lambda w}, \qquad A = \frac{n}{2-n}.$$

4°. There is an exact solution of the form

$$w(x, y) = U(z) + \frac{n-2}{\lambda}\ln x, \qquad z = y + \frac{2-n}{\beta}\ln x.$$

4. $\dfrac{\partial}{\partial x}\left[(ay + c)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(bx + s)\dfrac{\partial w}{\partial y}\right] = ke^{\lambda w}.$

This is a special case of equation 9.4.4.1 with $f(w) = ke^{\lambda w}$.

The equation can be rewritten in the form

$$(ay + c)\frac{\partial^2 w}{\partial x^2} + (bx + s)\frac{\partial^2 w}{\partial y^2} = ke^{\lambda w}.$$

5. $\dfrac{\partial}{\partial x}\left[(a_1x + b_1y + c_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2x + b_2y + c_2)\dfrac{\partial w}{\partial y}\right] = ke^{\lambda w}.$

This is a special case of equation 9.4.4.2 with $f(w) = ke^{\lambda w}$.

9.2.4. Equations of the Form $\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(w)\frac{\partial w}{\partial y}\right]=g(w)$

▷ *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, $f_1 = f_1(w)$ and $f_2 = f_2(w)$ are the principal thermal diffusivities (diffusion coefficients) dependent on the temperature (concentration) w, and $g = g(w)$ is a source function that defines the law of heat (substance) release or absorption. Simple solutions dependent on a single coordinate, $w = w(x)$ and $w = w(y)$, are not treated in this subsection.*

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(ae^{\beta w}\dfrac{\partial w}{\partial y}\right)=0.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1x + C_3, \pm C_1C_2^{\beta}y + C_4) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solutions:

$$w(x,y) = \frac{1}{\beta}\ln(Ay + B) + Cx + D,$$

$$w(x,y) = \frac{1}{\beta}\ln(-aA^2y^2 + By + C) - \frac{2}{\beta}\ln(-aAx + D),$$

$$w(x,y) = \frac{1}{\beta}\ln(Ay^2 + By + C) + \frac{1}{\beta}\ln\left[\frac{p^2}{aA\cosh^2(px+q)}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln(Ay^2 + By + C) + \frac{1}{\beta}\ln\left[\frac{p^2}{-aA\cos^2(px+q)}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln(Ay^2 + By + C) + \frac{1}{\beta}\ln\left[\frac{p^2}{-aA\sinh^2(px+q)}\right],$$

where A, B, C, D, p, and q are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$k_1^2w + \frac{ak_2^2}{\beta}e^{\beta w} = C_1(k_1x + k_2y) + C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

4°. Self-similar solution (A and B are arbitrary constants):

$$w = u(z), \quad z = \frac{x + A}{y + B},$$

where the function $u(z)$ is determined by the ordinary differential equation

$$(z^2u'_z)'_z + (ae^{\beta u}u'_z)'_z = 0.$$

This equation admits the first integral

$$\left(z^2 + ae^{\beta u}\right)u'_z = C.$$

Treating u as the independent variable, we get a Riccati equation for $z = z(u)$,

$$Cz'_u = z^2 + ae^{\beta u},$$

whose solution is expressed in terms of Bessel functions (see Polyanin and Zaitsev, 2003).

5°. Solution (generalizes the solution of Item 4° with $A = B = 0$):

$$w = U(\xi) - \frac{2(k+1)}{\beta}\ln|x|, \qquad \xi = y|x|^k,$$

where k is an arbitrary constant and the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2(k+1)}{\beta} + k(k-1)\xi U'_\xi + k^2\xi^2 U''_{\xi\xi} + (ae^{\beta U}U'_\xi)'_\xi = 0.$$

6°. There are exact solutions of the following forms:

$$w(x,y) = F(\eta) - \frac{2}{\beta}\ln|x|, \qquad \eta = y + k\ln|x|;$$
$$w(x,y) = H(\zeta) - \frac{2}{\beta}x, \qquad \zeta = ye^x,$$

where k is an arbitrary constant.

7°. For other solutions, see equation 9.4.4.8 with $f(w) = 1$ and $g(w) = ae^{\beta w}$.

⊙ *Reference for equation* 9.2.4.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 370–371).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left(ae^{\beta w}\dfrac{\partial w}{\partial y}\right) = be^{\lambda w}$.

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^\lambda x + C_2, \pm C_1^{\lambda-\beta}y + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_1x + k_2y,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$k_1^2u''_{zz} + ak_2^2(e^{\beta u}u'_z)'_z = be^{\lambda u}.$$

The substitution $\Theta(u) = (u'_z)^2$ leads to the first-order linear equation

$$(k_1^2 + ak_2^2e^{\beta u})\Theta'_u + 2ak_2^2\beta e^{\beta u}\Theta = 2be^{\lambda u}.$$

3°. Solution:

$$w = U(\xi) - \frac{2}{\lambda}\ln|x|, \qquad \xi = yx^{\frac{\beta-\lambda}{\lambda}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\lambda} + \frac{(\beta-\lambda)(\beta-2\lambda)}{\lambda^2}\xi U'_\xi + \frac{(\beta-\lambda)^2}{\lambda^2}\xi^2U''_{\xi\xi} + (ae^{\beta U}U'_\xi)'_\xi = be^{\lambda U}.$$

3. $\dfrac{\partial}{\partial x}\left(ae^{\beta w}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma w}\dfrac{\partial w}{\partial y}\right)=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 C_2^{\beta} x + C_3, \pm C_1 C_2^{\gamma} y + C_4) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$ak_1^2\beta^{-1}e^{\beta w} + bk_2^2\gamma^{-1}e^{\gamma w} = C_1(k_1x + k_2y) + C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

3°. Additive separable solution:

$$w(x,y) = \varphi(x) + \psi(y). \tag{1}$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$\begin{aligned} \varphi''_{xx} + \beta(\varphi'_x)^2 &= Abe^{(\gamma-\beta)\varphi}, \\ \psi''_{yy} + \gamma(\psi'_y)^2 &= -Aae^{(\beta-\gamma)\psi}, \end{aligned} \tag{2}$$

which are independent of each other.

Integrating yields the general solutions to the equations of (2) in implicit form:

$$\int e^{\beta\varphi}\left[\frac{2Ab}{\beta+\gamma}e^{(\beta+\gamma)\varphi} + B_1\right]^{-1/2} d\varphi = C_1 \pm x,$$
$$\int e^{\gamma\psi}\left[-\frac{2Aa}{\beta+\gamma}e^{(\beta+\gamma)\psi} + B_2\right]^{-1/2} d\psi = C_2 \pm y,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants; $\beta+\gamma \neq 0$.

Remark. Particular solutions to equations (2) are given by

$$\varphi(x) = \frac{1}{\beta-\gamma}\ln\left[\frac{Ab(\beta-\gamma)^2}{2(\beta+\gamma)}(x+C_3)^2\right],$$
$$\psi(y) = \frac{1}{\gamma-\beta}\ln\left[-\frac{Aa(\beta-\gamma)^2}{2(\beta+\gamma)}(y+C_4)^2\right],$$

where C_3 and C_4 are arbitrary constants.

4°. There are exact solutions of the following forms:

$$w(x,y) = F(z) + \frac{2k}{\beta-\gamma}\ln|x|, \qquad z = y|x|^{k-1};$$
$$w(x,y) = G(\xi) + \frac{2}{\beta-\gamma}\ln|x|, \qquad \xi = y + k\ln|x|;$$
$$w(x,y) = H(\eta) + 2x, \qquad \eta = ye^{(\beta-\gamma)x},$$

where k is an arbitrary constant.

5°. For other exact solutions of the original equation, see 9.4.4.8 with $f(w) = ae^{\beta w}$ and $g(w) = be^{\gamma w}$.

4. $a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial}{\partial y}\left(e^{\gamma w}\dfrac{\partial w}{\partial y}\right)=ce^{\lambda w}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-\beta}x + C_2, \pm C_1^{\lambda-\gamma}y + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are exact solutions of the following forms:

$$w(x,y) = F(z), \qquad z = k_1x + k_2y;$$
$$w(x,y) = G(\xi) + \frac{2}{\beta-\lambda}\ln|x|, \qquad \xi = y|x|^{\frac{\lambda-\gamma}{\beta-\lambda}}.$$

9.2.5. Other Equations Involving Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{a}{x}\dfrac{\partial w}{\partial x}+\dfrac{b}{y}\dfrac{\partial w}{\partial y}=ce^{\beta w}.$

This is a special case of equation 9.4.2.6 with $f(\xi, w) = ce^{\beta w}$.

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta w}\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 9.4.2.12 with $f(w) = ae^{\beta w}$.

The substitution $U = \int \exp\left(-\dfrac{a}{\beta}e^{\beta w}\right) dw$ leads to the two-dimensional Laplace equation for $U = U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\dfrac{\partial^2 w}{\partial x^2}+ae^{\beta w}\dfrac{\partial^2 w}{\partial y^2}=0, \qquad a>0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_\pm = w(C_1x + C_3, \pm C_1C_2^{\beta}y + C_4,) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y) = Axy + By + Cx + D,$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{B^2}{a}\frac{(y+A)^2}{\sinh^2(Bx+C)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{1}{aA^2}\frac{\sinh^2(Ay+B)}{(x+C)^2}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{B^2}{a}\frac{(y+A)^2}{\cos^2(Bx+C)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{1}{aA^2}\frac{\cos^2(Ay+B)}{(x+C)^2}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ay+B)}{\sinh^2(Cx+D)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ay+B)}{\cos^2(Cx+D)}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ay+B)}{\sinh^2(Cx+D)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ay+B)}{\cos^2(Cx+D)}\right],$$

where A, B, C, and D are arbitrary constants. The first solution is degenerate and the others are representable as the sum of functions with different arguments.

3°. Self-similar solution:

$$w = w(z), \qquad z = y/x,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(z^2 + ae^{\beta w})w''_{zz} + 2zw'_z = 0.$$

4°. Solution (generalizes the solution of Item 3°):

$$w = U(\zeta) - \frac{2(k+1)}{\beta}\ln|x + C_1|, \qquad \zeta = (y + C_2)(x + C_1)^k,$$

where C_1, C_2, and k are arbitrary constants, and the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$(k^2\zeta^2 + ae^{\beta U})U''_{\zeta\zeta} + k(k-1)\zeta U'_\zeta + \frac{2(k+1)}{\beta} = 0.$$

4. $\dfrac{\partial^2 w}{\partial x^2} + ae^{\beta w}\dfrac{\partial^2 w}{\partial y^2} = be^{\lambda w}$.

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-\beta}y + C_2, \pm C_1^{\lambda}x + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_1x + k_2y,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$(k_1^2 + ak_2^2e^{\beta u})u''_{zz} = be^{\lambda u}.$$

Its solution can be written out in implicit form as

$$\int \frac{du}{\sqrt{F(u)}} = C_1 \pm z, \qquad F(u) = 2b\int \frac{e^{\lambda u}\,du}{k_1^2 + ak_2^2e^{\beta u}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w = U(\xi) - \frac{2}{\lambda}\ln|x|, \qquad \xi = y|x|^{\frac{\beta-\lambda}{\lambda}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\lambda} + \frac{(\beta-\lambda)(\beta-2\lambda)}{\lambda^2}\xi U'_\xi + \frac{(\beta-\lambda)^2}{\lambda^2}\xi^2 U''_{\xi\xi} + ae^{\beta U}U''_{\xi\xi} = be^{\lambda U}.$$

5. $ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} = ce^{\beta w}$.

This is a special case of equation 9.4.5.5 with $k = s = 0$ and $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C^{\frac{2}{2-n}}x,\, C^{\frac{2}{2-m}}y\Big) + \frac{2}{\beta}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\big]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = Be^{\beta w}, \tag{1}$$

where

$$A = \frac{3nm - 4n - 4m + 4}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. For $A \neq 1$, equation (1) admits an exact solution of the form

$$w(\xi) = -\frac{1}{\beta}\ln\left[\frac{B\beta}{2(1-A)}\xi^2\right].$$

3.2. For $A = 0$, which corresponds to $m = \dfrac{4n-4}{3n-4}$ and $B = \dfrac{c(3n-4)^2}{ab(2-n)^4}$, we obtain from (1) several more families of exact solutions to the original equation:

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\cos^2(\lambda\xi + C)}\right] \quad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\sinh^2(\lambda\xi + C)}\right] \quad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{-2\lambda^2}{\beta B\cosh^2(\lambda\xi + C)}\right] \quad \text{if } \beta B < 0,$$

where λ and C are arbitrary constants.

3.3. For $A = 1$, which corresponds to $m = \dfrac{n}{n-1}$, another family of exact solutions follows from (1):

$$w(\xi) = \frac{1}{\beta}\ln\left(-\frac{8C}{\beta B}\right) - \frac{2}{\beta}\ln(\xi^2 + C), \qquad B = \frac{4c(n-1)^2}{ab(2-n)^4},$$

where C is an arbitrary constant.

6. $ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + be^{\mu y}\dfrac{\partial^2 w}{\partial y^2} = ce^{\lambda w}.$

This is a special case of equation 9.4.5.7 with $k = s = 0$ and $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(x - \frac{2}{\beta}\ln C,\ y - \frac{2}{\mu}\ln C\Big) + \frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{3}{\xi}w'_\xi = Ae^{\lambda w}, \qquad A = \frac{4c}{ab\beta^2\mu^2},$$

which admits the exact solution

$$w = -\frac{1}{\lambda}\ln\left(-\frac{1}{4}A\lambda\xi^2\right).$$

7. $ax^n\dfrac{\partial^2 w}{\partial x^2} + be^{\beta y}\dfrac{\partial^2 w}{\partial y^2} = ce^{\lambda w}.$

This is a special case of equation 9.4.5.9 with $k = s = 0$ and $f(w) = ce^{\lambda w}$.

8. $\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial w}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 w}{\partial\varphi^2} = -2 + Ae^{-w}.$

The left-hand side of the equation includes the two-dimensional Laplace operator written for the surface of a unit radius sphere; θ and φ are the spherical coordinates.

The transformation

$$x = \varphi, \quad y = \ln\Big|\tan\frac{\theta}{2}\Big|, \quad w = 2\ln|\sin\theta| + U(x, y),$$

leads to the equation of the form 9.2.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = Ae^{-U}.$$

⊙ *Reference*: S. N. Aristov (1998).

9.3. Equations Involving Other Nonlinearities

9.3.1. Equations with Hyperbolic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\sinh(\beta w).$

This is a special case of equation 9.4.1.1 with $f(w) = a\sinh(\beta w)$.

1°. Traveling-wave solution in implicit form:

$$\int \left[D + \frac{2a\cosh(\beta w)}{\beta(A^2 + B^2)}\right]^{-1/2} dw = Ax + By + C,$$

where A, B, C, and D are arbitrary constants.

2°. Solution with central symmetry about the point $(-C_1, -C_2)$:

$$w = w(\xi), \qquad \xi = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{1}{\xi} w'_\xi = a\sinh(\beta w).$$

3°. Functional separable solution:

$$w(x, y) = \frac{4}{\beta}\operatorname{arctanh}\big[f(x)g(y)\big], \qquad \operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z},$$

where the functions $f = f(x)$ and $g = g(y)$ are determined by the first-order autonomous ordinary differential equations

$$\big(f'_x\big)^2 = Af^4 + Bf^2 + C,$$
$$\big(g'_y\big)^2 = -Cg^4 + (a\beta - B)g^2 - A,$$

and A, B, and C are arbitrary constants.

4°. The original equation is related to an equation of the form 9.3.3.1,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = a\sin(\beta U),$$

by the Bäcklund transformation

$$\frac{\partial U}{\partial x} + \frac{\partial w}{\partial y} = 2\sqrt{\frac{a}{\beta}}\sin\big(\tfrac{1}{2}\beta U\big)\cosh\big(\tfrac{1}{2}\beta w\big),$$
$$\frac{\partial U}{\partial y} - \frac{\partial w}{\partial x} = 2\sqrt{\frac{a}{\beta}}\cos\big(\tfrac{1}{2}\beta U\big)\sinh\big(\tfrac{1}{2}\beta w\big).$$

⊙ *References for equation* 9.3.1.1: R. K. Bullough and P. J. Caudrey (1980), A. C. Wing, H. H. Cheb, and Y. C. Lee (1987), J. Miller (Jr.) and L. A. Rubel (1993), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a\sinh(\beta w)+b\sinh(2\beta w).$

Denote $k=\dfrac{a}{2b}$.

Traveling-wave solutions:

$$w=\pm\frac{1}{\beta}\operatorname{arccosh}\frac{1-k\sin z}{\sin z-k},\qquad z=\sqrt{2b\beta(1-k^2)}\,(x\sin C_1+y\cos C_1+C_2)$$

if $|k|<1$;

$$w=\pm\frac{2}{\beta}\operatorname{arctanh}\left(\sqrt{\frac{k+1}{k-1}}\tanh\frac{\xi}{2}\right),\qquad \xi=\sqrt{2b\beta(k^2-1)}\,(x\sin C_1+y\cos C_1+C_2)$$

if $|k|>1$,

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a(x^2+y^2)\sinh(\beta w).$

This is a special case of equation 9.4.1.3 with $f(w)=a\sinh(\beta w)$.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a(x^2+y^2)\cosh(\beta w).$

This is a special case of equation 9.4.1.3 with $f(w)=a\cosh(\beta w)$.

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta x}\sinh(\lambda w).$

This is a special case of equation 9.4.1.5 with $f(w)=a\sinh(\lambda w)$.

6. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a\cosh^n(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 9.4.2.12 with $f(w)=a\cosh^n(\beta w)$.

7. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=k\sinh(\beta w).$

This is a special case of equation 9.4.3.1 with $f(w)=k\sinh(\beta w)$.

8. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=k\sinh(\lambda w).$

This is a special case of equation 9.4.3.6 with $f(w)=k\sinh(\lambda w)$.

9. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[a\cosh(\beta w)\dfrac{\partial w}{\partial y}\right]=0.$

This is a special case of equation 9.4.4.8 with $f(w)=1$ and $g(w)=a\cosh(\beta w)$.

9.3.2. Equations with Logarithmic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha w\ln(\beta w).$

This is a special case of equation 9.4.1.1 with $f(w)=\alpha w\ln(\beta w)$.

On making the change of variable $U=\ln(\beta w)$, one obtains an equation with a quadratic nonlinearity:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2=\alpha U. \tag{1}$$

1°. Equation (1) has exact solutions quadratic in the independent variables:

$$U(x,y)=\tfrac{1}{4}\alpha(x+A)^2+\tfrac{1}{4}\alpha(y+B)^2+1,$$
$$U(x,y)=A(x+B)^2\pm\sqrt{A\alpha-4A^2}\,(x+B)(y+C)+(\tfrac{1}{4}\alpha-A)(y+C)^2+\tfrac{1}{2},$$

where A, B, and C are arbitrary constants.

2°. Equation (1) has a traveling-wave solution:

$$U(x,y)=F(\xi),\qquad \xi=Ax+By+C.$$

Here, the function $F=F(\xi)$ is defined implicitly by

$$\xi=\int\left[De^{-2F}+\frac{\alpha}{A^2+B^2}(F-\tfrac{1}{2})\right]^{-1/2}dF,$$

where A, B, C, and D are arbitrary constants.

3°. Equation (1) has a solution in the form of the sum of functions with different arguments:

$$U(x,y)=f(x)+g(y).$$

Here, the functions $f=f(x)$ and $g=g(y)$ are defined implicitly by

$$A_1\pm x=\int\left(B_1e^{-2f}+\alpha f-\tfrac{1}{2}\alpha\right)^{-1/2}df,$$
$$A_2\pm y=\int\left(B_2e^{-2g}+\alpha g-\tfrac{1}{2}\alpha\right)^{-1/2}dg,$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

4°. The original equation admits exact solutions of the form

$$w=w(\zeta),\qquad \zeta=\sqrt{(x+C_1)^2+(y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta}+\frac{1}{\zeta}w'_\zeta=\alpha w\ln(\beta w).$$

⊙ *References for equation* 9.3.2.1: J. A. Shercliff (1977), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

2. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw\ln w+(bx^n+cy^k)w.$

This is a special case of equation 9.4.1.9 with $f(x)=bx^n$ and $g(y)=cy^k$.

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha(x^2+y^2)\ln(\beta w).$

This is a special case of equation 9.4.1.3 with $f(w)=\alpha\ln(\beta w)$.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta x}\ln(\lambda w).$

This is a special case of equation 9.4.1.5 with $f(w)=a\ln(\lambda w)$.

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=axw+bw\ln|w|.$

This equation is used for describing some flows of ideal stratified fluids. It is a special case of equation 9.3.2.6 with $k=a_2=a_0=0$.

1°. Multiplicative separable solution:

$$w(x,y)=\exp\left[-\frac{a}{b}x+\frac{b}{4}(y+C)^2+\frac{a^2}{b^3}+\frac{1}{2}\right],$$

where C is an arbitrary constant.

2°. Multiplicative separable solution (generalizes the solution of Item 1°):

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi=\varphi(x)$ and $\psi=\psi(y)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx}=b\varphi\ln|\varphi|+(ax+C)\varphi,$$
$$\psi''_{yy}=b\psi\ln|\psi|-C\psi,$$

C is an arbitrary constant.

⊙ *Reference for equation* 9.3.2.5: V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

6. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{k}{x}\dfrac{\partial w}{\partial x}=(a_2x^2+a_1x+a_0)w+bw\ln|w|.$

Grad–Shafranov equation (with $k=-1$ and $a_1=a_0=0$). This equation is used to describe some steady-state axisymmetric (swirling) flows of ideal fluids. It also occurs in plasma physics.

1°. Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi=\varphi(x)$ and $\psi=\psi(y)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx}+\frac{k}{x}\varphi'_x=b\varphi\ln|\varphi|+(a_2x^2+a_1x+a_0+C)\varphi,$$
$$\psi''_{yy}=b\psi\ln|\psi|-C\psi,$$

with C being an arbitrary constant.

2°. Solutions for $a_1 = 0$:

$$w(x,y) = \exp\left[Ax^2 + \frac{b}{4}(y+B)^2 + \frac{2}{b}A(k+1) - \frac{a_0}{b} + \frac{1}{2}\right], \quad A = \frac{1}{8}\left(b \pm \sqrt{b^2 + 16a_2}\right),$$

where B is an arbitrary constant.

3°. Solution for $a_1 = a_2 = 0$:

$$w = w(r), \quad r = \sqrt{x^2 + y^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{k+1}{r}w'_r = a_0 w + bw\ln|w|.$$

⊙ *Reference for equation* 9.3.2.6: G. Rosen (1969), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{x}\dfrac{\partial w}{\partial x} + \dfrac{b}{y}\dfrac{\partial w}{\partial y} = cw^n \ln(\beta w).$

This is a special case of equation 9.4.2.6 with $f(\xi, w) = cw^n \ln(\beta w)$.

8. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\ln^n(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 9.4.2.12 with $f(w) = a\ln^n(\beta w)$.

9. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[a\ln^n(\beta w)\dfrac{\partial w}{\partial y}\right] = 0.$

This is a special case of equation 9.4.4.8 with $f(w) = 1$ and $g(w) = a\ln^n(\beta w)$.

10. $\dfrac{\partial}{\partial x}\left[(a_1 x + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2 y + b_2)\dfrac{\partial w}{\partial y}\right] = kw\ln(\beta w).$

1°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = \frac{x}{a_1} + \frac{y}{a_2} + \frac{b_1}{a_1^2} + \frac{b_2}{a_2^2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(\xi w'_\xi)'_\xi = kw\ln(\beta w).$$

Remark. The above solution is a nonclassical (noninvariant) traveling-wave solution, since the equation in question is not invariant under translation transformations in the independent variables.

2°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$[(a_1 x + b_1)\varphi'_x]'_x - k\varphi\ln(\beta\varphi) + C\varphi = 0,$$
$$[(a_2 y + b_2)\psi'_y]'_y - k\psi\ln\psi - C\psi = 0,$$

with C being an arbitrary constant.

11. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right]=kw\ln w.$

This is a special case of equation 9.4.4.2 with $f(w)=kw\ln w$.

12. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=k\ln(\beta w).$

This is a special case of equation 9.4.3.1 with $f(w)=k\ln(\beta w)$.

13. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=kw\ln w.$

This is a special case of equation 9.4.3.1 with $f(w)=kw\ln w$ and a special case of equation 9.4.3.9 with $f(x)=ax^n$ and $g(y)=by^m$.

14. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=k\ln(\lambda w).$

This is a special case of equation 9.4.3.6 with $f(w)=k\ln(\lambda w)$.

15. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=kw\ln w.$

This is a special case of equation 9.4.3.6 with $f(w)=kw\ln w$ and a special case of equation 9.4.3.9 with $f(x)=ae^{\beta x}$ and $g(y)=be^{\mu y}$.

9.3.3. Equations with Trigonometric Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\sin(\beta w).$

This is a special case of equation 9.4.1.1 with $f(w)=\alpha\sin(\beta w)$.

1°. Traveling-wave solution:

$$w=w(z),\qquad z=Ax+By+C,$$

where $w(z)$ is defined implicitly by

$$\int\left[D-\frac{2\alpha\cos(\beta w)}{\beta(A^2+B^2)}\right]^{-1/2}dw=z,$$

and A, B, C, and D are arbitrary constants.

2°. Solution with central symmetry about the point $(-C_1,-C_2)$:

$$w=w(\xi),\qquad \xi=\sqrt{(x+C_1)^2+(y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w=w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{1}{\xi}w'_\xi=\alpha\sin(\beta w).$$

3°. Functional separable solution for $\alpha = \beta = 1$:

$$w(x,y) = 4\arctan\left(\cot A\frac{\cosh F}{\cosh G}\right),$$
$$F = \frac{\cos A}{\sqrt{1+B^2}}(x - By), \quad G = \frac{\sin A}{\sqrt{1+B^2}}(y + Bx),$$

where A and B are arbitrary constants.

4°. Functional separable solution:

$$w(x,y) = \frac{4}{\beta}\arctan\big[f(x)g(y)\big],$$

where the functions $f = f(x)$ and $g = g(y)$ are determined by the first-order autonomous ordinary differential equations

$$\big(f'_x\big)^2 = Af^4 + Bf^2 + C,$$
$$\big(g'_y\big)^2 = Cg^4 + (\alpha\beta - B)g^2 + A,$$

and A, B, and C are arbitrary constants.

5°. Auto-Bäcklund transformation ($\alpha = \beta = 1$):

$$\frac{\partial\widetilde{w}}{\partial x} = -i\frac{\partial w}{\partial y} + k\sin\frac{\widetilde{w}+w}{2} + \frac{1}{k}\sin\frac{\widetilde{w}-w}{2},$$
$$-i\frac{\partial\widetilde{w}}{\partial y} = \frac{\partial w}{\partial x} + k\sin\frac{\widetilde{w}+w}{2} - \frac{1}{k}\sin\frac{\widetilde{w}-w}{2},$$

where $i^2 = -1$. With this transformation, the original equation is reduced to exactly the same equation.

⊙ *References for equation* 9.3.3.1: R. K. Bullough and P. J. Caudrey (1980), W. Miller (Jr.) and L. A. Rubel (1993), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\sin(\beta w) + b\sin(2\beta w).$

Denote $k = \dfrac{a}{2b}$.

Traveling-wave solutions:

$$w = \pm\frac{2}{\beta}\arctan\left(\frac{k+1}{\sqrt{1-k^2}}\coth\frac{z}{2}\right), \quad z = \sqrt{2b\beta(1-k^2)}\,(x\sin C_1 + y\cos C_1 + C_2)$$
if $|k| < 1$;
$$w = \pm\frac{2}{\beta}\arctan\left(\frac{k+1}{\sqrt{k^2-1}}\tan\frac{\xi}{2}\right), \quad \xi = \sqrt{2b\beta(k^2-1)}\,(x\sin C_1 + y\cos C_1 + C_2)$$
if $|k| > 1$,

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\cos(\beta w).$

The substitution $\beta w=\beta u+\frac{1}{2}\pi$ leads to an equation of the form 9.3.3.1:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-\alpha\sin(\beta u).$$

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha(x^2+y^2)\sin(\beta w).$

This is a special case of equation 9.4.1.3 with $f(w)=\alpha\sin(\beta w)$.

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha(x^2+y^2)\cos(\beta w).$

This is a special case of equation 9.4.1.3 with $f(w)=\alpha\cos(\beta w)$.

6. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=ae^{\beta x}\sin(\lambda w).$

This is a special case of equation 9.4.1.5 with $f(w)=a\sin(\lambda w)$.

7. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=a\cos(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 9.4.2.12 with $f(w)=a\cos(\beta w)$.

8. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=k\sin(\beta w).$

This is a special case of equation 9.4.3.1 with $f(w)=k\sin(\beta w)$.

9. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=k\sin(\lambda w).$

This is a special case of equation 9.4.3.6 with $f(w)=k\sin(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[a\cos^n(\beta w)\dfrac{\partial w}{\partial y}\right]=0.$

This is a special case of equation 9.4.4.8 with $f(w)=1$ and $g(w)=a\cos^n(\beta w)$.

9.4. Equations Involving Arbitrary Functions

9.4.1. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=F(x,y,w)$

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(w).$

This is a *stationary heat equation with a nonlinear source*.

1°. Suppose $w = w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1, C_2, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[C + \frac{2}{A^2 + B^2}F(w)\right]^{-1/2} dw = Ax + By + D, \qquad F(w) = \int f(w)\,dw,$$

where A, B, C, and D are arbitrary constants.

3°. Solution with central symmetry about the point $(-C_1, -C_2)$:

$$w = w(\zeta), \qquad \zeta = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w = w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} + \frac{1}{\zeta}w'_\zeta = f(w).$$

4°. For exact solutions of the original equation for some $f(w)$, see Subsections 9.1.1 and 9.2.1 and equations 9.3.1.1, 9.3.2.1, and 9.3.3.1 (see also Example 30.11 in Subsection 30.3.2).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = yf(w) + g(w).$

This equation describes a steady planar flow of an inviscid stratified fluid.

For $f(w) \equiv 0$, see equation 9.4.1.1.

1°. The translation transformation $y = \bar{y} + C$ leads to an equation of the same form, in which the function $g(w)$ must be replaced by $g(w) + Cf(w)$.

2°. Let $f(w) = aw^{3k-1}$ and $g(w) = bw^{2k+1} + cw^{3k-1}$, where a, b, c, and k are arbitrary constants ($ak \neq 0$).

Solution:

$$w = x^{-\frac{1}{k}}u(\eta), \qquad \eta = \frac{y}{x} + \frac{c}{ax},$$

where the function $u = u(\eta)$ is determined by the ordinary differential equation

$$(\eta^2 + 1)u''_{\eta\eta} + \frac{2(k+1)}{k}\eta u'_\eta + \frac{k+1}{k^2}u = a\eta u^{3k-1} + bu^{2k+1}.$$

To $c = 0$ there corresponds a self-similar solution.

3°. Let $f(w) = ae^{3\lambda w}$ and $g(w) = be^{2\lambda w} + ce^{3\lambda w}$.

Solution:

$$w = U(\eta) - \frac{1}{\lambda}\ln x, \qquad \eta = \frac{y}{x} + \frac{c}{ax},$$

where the function $U = U(\eta)$ is determined by the ordinary differential equation

$$(\eta^2 + 1)U''_{\eta\eta} + 2\eta U'_\eta + \frac{1}{\lambda} = a\eta e^{3\lambda U} + be^{2\lambda U}.$$

4°. Let $f(w) = aw$ and $g(w) = bw\ln w + cw$.

Multiplicative separable solution:

$$w = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the second-order ordinary differential equations

$$\varphi''_{xx} - (b\ln\varphi + K)\varphi = 0,$$
$$\psi''_{yy} - (b\ln\psi + ay + c - K)\psi = 0,$$

and K is an arbitrary constant.

Remark. For linear functions $f(w) = a$ and $g(w) = bw + c$, the substitution $w = \varphi(y) + u(x, y)$, where

$$\varphi(y) = \begin{cases} \frac{1}{6}ay^3 + \frac{1}{2}cy^2 & \text{if } b = 0, \\ -\dfrac{ay + c}{b} & \text{if } b \neq 0, \end{cases}$$

leads to the Helmholtz equation $\Delta u = bu$.

5°. Let

$$f(w) = -\frac{2\varphi''}{(\varphi')^3}, \qquad g(w) = \frac{2\varphi\varphi''}{(\varphi')^3} - \frac{1}{\varphi'},$$

where $\varphi = \varphi(w)$ is an arbitrary function.

Solution in implicit form:

$$\varphi(w) = y - \frac{(x + C)^2 + 1}{2},$$

where C is an arbitrary constant.

6°. Let

$$f(w) = k\big[\cos w + \cos\big(\tfrac{1}{2}w\big)\big], \qquad g(w) = \sin w.$$

Functional separable solution:

$$w = 4\arctan\left(\frac{y}{R(x)}\right),$$

where $R = R(x)$ satisfies the equation for elliptic functions

$$(R'_x)^2 = 1 + R^2 + kR^3.$$

7°. Let

$$f(w) = k\big[\cosh w \pm \cosh\big(\tfrac{1}{2}w\big)\big], \qquad g(w) = \sinh w.$$

Functional separable solutions:

$$w = 4\operatorname{arctanh}\left(\frac{y}{R(x)}\right)$$

for the plus signin the expression of $f(w)$ and

$$w = 4\operatorname{arctanh}\left(\frac{R(x)}{y}\right)$$

for the minus sign; the function $R = R(x)$ satisfies the equation

$$(R'_x)^2 = -1 + R^2 + kR^3.$$

⊙ *References for equation* 9.4.1.2: C.-S. Yih (1980, p. 144), O. V. Kaptsov (1990), Yu. V. Shan'ko (2001).

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2) f(w).$

1°. Suppose $w = w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w = w(r), \qquad r = \sqrt{x^2 + y^2},$$

where the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r}w'_r = r^2 f(w).$$

3°. Self-similar solution:

$$w = w(\zeta), \qquad \zeta = xy.$$

Here, the function $w = w(\zeta)$ is determined by the autonomous ordinary differential equation

$$w''_{\zeta\zeta} = f(w),$$

whose general solution can be written out in implicit form as

$$\int \left[C_1 + 2F(w)\right]^{-1/2} dw = C_2 \pm \zeta, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

Remark. The above solution is a nonclassical (noninvariant) self-similar solution, since the equation in question is not invariant under scaling transformations.

4°. Functional separable solution:

$$w = w(z), \qquad z = \tfrac{1}{2}(x^2 - y^2).$$

Here, the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} = f(w),$$

whose general solution can be written out in implicit form as

$$\int \big[C_1 + 2F(w)\big]^{-1/2} dw = C_2 \pm z, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

5°. The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to a simpler equation of the form 9.4.1.1:

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = f(U).$$

⊙ *References for equation* 9.4.1.3: A. D. Polyanin and V. F. Zaitsev (2002, p. 212), A. D. Polyanin (2004 *a*).

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2)^k f(w).$

1°. Suppose $w = w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w = w(r), \qquad r = \sqrt{x^2 + y^2},$$

where the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r} w'_r = r^{2k} f(w).$$

3°. Suppose $k = \pm 1, \pm 2, \ldots$. The transformation

$$\begin{aligned}
&z = \tfrac{1}{2}(x^2 - y^2), && \zeta = xy && \text{for } k = 1,\\
&z = \tfrac{1}{3}(x^3 - 3xy^2), && \zeta = \tfrac{1}{3}(3x^2 y - y^3) && \text{for } k = 2,\\
&z = \tfrac{1}{2}\ln(x^2 + y^2), && \zeta = \arctan\frac{y}{x} && \text{for } k = -1,\\
&z = -\frac{x}{x^2 + y^2}, && \zeta = \frac{y}{x^2 + y^2} && \text{for } k = -2
\end{aligned}$$

leads to a simpler equation of the form 9.4.1.1:

$$\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial \zeta^2}=f(w). \tag{1}$$

For arbitrary $f=f(w)$, this equation admits a traveling-wave solution $w=w(Az+B\zeta)$, where A and B are arbitrary constants, and a solution of the form $w=w(z^2+\zeta^2)$.

In the general case, for any integer $k\neq -1$, the transformation

$$z=\frac{(x+iy)^{k+1}+(x-iy)^{k+1}}{2(k+1)},\quad \zeta=\frac{(x+iy)^{k+1}-(x-iy)^{k+1}}{2(k+1)i},\quad i^2=-1 \tag{2}$$

leads to equation (1). It follows from (2) that

$$z^2+\zeta^2=\frac{1}{(k+1)^2}(x^2+y^2)^{k+1}.$$

4°. Suppose k is an arbitrary constant ($k\neq -1$). The transformation

$$z=\frac{1}{k+1}r^{k+1}\cos\big[(k+1)\varphi\big],\quad \zeta=\frac{1}{k+1}r^{k+1}\sin\big[(k+1)\varphi\big], \tag{3}$$

where $x=r\cos\varphi$ and $y=r\sin\varphi$, leads to the simpler equation (1). For $k=\pm1,\pm2,\ldots$, transformation (3) coincides with transformation (2).

⊙ *Reference for equation* 9.4.1.4: A. D. Polyanin and V. F. Zaitsev (2002, pp. 212–213).

5. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=e^{\beta x}f(w).$

The transformation

$$w=W(u,v),\quad u=\exp\big(\tfrac{1}{2}\beta x\big)\cos\big(\tfrac{1}{2}\beta y\big),\quad v=\exp\big(\tfrac{1}{2}\beta x\big)\sin\big(\tfrac{1}{2}\beta y\big)$$

leads to a simpler equation of the form 9.4.1.1:

$$\frac{\partial^2 W}{\partial u^2}+\frac{\partial^2 W}{\partial v^2}=4\beta^{-2}f(W).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 213).

6. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=e^{ax-by}f(w).$

Let us represent the exponent of e in the form

$$ax-by=\beta(x\cos\sigma-y\sin\sigma);\quad \beta=\sqrt{a^2+b^2},\quad \cos\sigma=a/\beta,\quad \sin\sigma=b/\beta.$$

The transformation

$$\xi=x\cos\sigma-y\sin\sigma,\quad \eta=x\sin\sigma+y\cos\sigma,$$

leads to an equation of the form 9.4.1.5:

$$\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \eta^2}=e^{\beta\xi}f(w).$$

7. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(x,y)e^{\beta w}.$

Suppose $f(x,y)=a|F(z)|^2$, where $F=F(z)$ is a prescribed analytic function of the complex variable $z=x+iy$.

General solution:

$$w(x,y)=-\frac{2}{\beta}\ln\frac{|F(z)||1-2a\beta\Phi(z)\overline{\Phi(z)}|}{4|\Phi'_z(z)|},$$

where $\Phi=\Phi(z)$ is an arbitrary analytic (holomorphic) function of the complex variable $z=x+iy$ with nonzero derivative; the bar over a symbol denotes the complex conjugate.

⊙ *Reference*: I. Kh. Sabitov (2001).

8. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw\ln w+f(x)w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi''_{xx}-\big[a\ln\varphi+f(x)+C\big]\varphi=0,$$
$$\psi''_{yy}-\big(a\ln\psi-C\big)\psi=0,$$

and C is an arbitrary constant.

9. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw\ln w+\big[f(x)+g(y)\big]w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi''_{xx}-\big[a\ln\varphi+f(x)+C\big]\varphi=0,$$
$$\psi''_{yy}-\big[a\ln\psi+g(y)-C\big]\psi=0,$$

and C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(x)w\ln w+\big[af(x)y+g(x)\big]w.$

Multiplicative separable solution:

$$w(x,y)=e^{-ay}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi''_{xx}=f(x)\varphi\ln\varphi+\big[g(x)-a^2\big]\varphi.$$

11. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(ax+by,w).$

Solution:

$$w=w(\xi),\qquad \xi=ax+by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(a^2+b^2)w''_{\xi\xi}=f(\xi,w).$$

12. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(x^2+y^2,w).$

1°. Suppose $w=w(x,y)$ is a solution of this equation. Then the function

$$w_1=w(x\cos\beta-y\sin\beta,\ x\sin\beta+y\cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w=w(\xi),\qquad \xi=\left(x^2+y^2\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{1}{\xi}w'_\xi=f(\xi^2,w).$$

13. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(x^2+y^2)f(xy,\ w).$

1°. Self-similar solution:

$$w=w(\zeta),\qquad \zeta=xy,$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta}=f(\zeta,w).$$

Remark. The above solution is a nonclassical (noninvariant) self-similar solution, since the equation in question is not invariant under scaling transformations.

2°. The transformation

$$w=U(z,\zeta),\quad z=\tfrac{1}{2}(x^2-y^2),\quad \zeta=xy$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial \zeta^2}=f(\zeta,U).$$

⊙ *Reference for equation* 9.4.1.13: A. D. Polyanin and V. F. Zaitsev (2002, p. 215).

14. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(x^2+y^2)f(x^2-y^2,\ w).$

1°. Functional separable solution:

$$w=w(z),\qquad z=\tfrac{1}{2}(x^2-y^2),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz}=f(2z,w).$$

2°. The transformation

$$w=U(z,\zeta),\quad z=\tfrac{1}{2}(x^2-y^2),\quad \zeta=xy$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial \zeta^2}=f(2z,U).$$

15. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f\big(w+A_{11}x^2+A_{12}xy+A_{22}y^2+B_1x+B_2y\big).$

The substitution $U=w+A_{11}x^2+A_{12}xy+A_{22}y^2+B_1x+B_2y$ leads to an equation of the form 9.4.1.1:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=f(U)+2A_{11}+2A_{22}.$$

9.4.2. Equations of the Form $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=F\Big(x,y,w,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\Big)$

1. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f(w).$

This equation describes steady-state processes of heat/mass transfer or combustion in anisotropic media. Here, a and b are the principal thermal diffusivities (diffusion coefficients) and $f=f(w)$ is a kinetic function that defines the law of heat (substance) release.

The transformation $\xi=x/\sqrt{a}$, $\eta=y/\sqrt{b}$ leads to an equation of the form 9.4.1.1:

$$\frac{\partial^2 w}{\partial \xi^2}+\frac{\partial^2 w}{\partial \eta^2}=f(w).$$

2. $ax\dfrac{\partial w}{\partial x}+ay\dfrac{\partial w}{\partial y}=\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}-f(w).$

Traveling-wave solution:

$$w=w(z),\qquad z=k_1x+k_2y,$$

where k_1 and k_2 are arbitrary constants and the function $w(z)$ is determined by the ordinary differential equation

$$azw'_z=(k_1^2+k_2^2)w''_{zz}-f(w).$$

Remark. The above solution is a nonclassical (noninvariant) traveling-wave solution, since the equation in question is not invariant under translation transformations.

⊙ *Reference*: A. D. Polyanin (2004 *a*).

3. $(a_1x + b_1y + c_1)\dfrac{\partial w}{\partial x} + (a_2x + b_2y + c_2)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This equation describes steady-state mass transfer with a volume chemical reaction in a translational-shear fluid flow.

Traveling-wave solution:

$$w = w(z), \qquad z = a_2x + (k - a_1)y,$$

where k is a root of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$\big[kz + a_2c_1 + (k - a_1)c_2\big]w'_z = \big[a_2^2 + (k - a_1)^2\big]w''_{zz} - f(w).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 = 0$.

⊙ *Reference for equation* 9.4.2.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 387).

4. $a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} + cx\dfrac{\partial w}{\partial x} - cy\dfrac{\partial w}{\partial y} = (bx^2 + ay^2)f(w).$

Solution:

$$w = w(z), \qquad z = xy,$$

where the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} = f(w).$$

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - \dfrac{1}{y}\dfrac{\partial w}{\partial y} = y^2 f(w) + g(w).$

Grad–Shafranov equation. This equation describes some classes of steady-state rotationally symmetric (swirling) flows of an ideal incompressible fluid, where y plays the role of the radial coordinate. Also it is the equilibrium equation in ideal magnetohydrodynamics for a two-dimensional plasma, for example the axisymmetric toroidal plasma in a tokamak.

1°. Let $f(w) = 0$ and $g(w)$ is an arbitrary function.

Functional separable solution:

$$w = w(\xi), \qquad \xi = \sqrt{(x + C)^2 + y^2},$$

where the function $w = w(\xi)$ is determined by the autonomous ordinary differential equation

$$w''_{\xi\xi} = g(w).$$

2°. Let $f(w) = ag(w)$, where $a > 0$ and $g(w)$ is an arbitrary function.

New functional separable solutions:

$$w = w(z), \qquad z = x \pm \tfrac{1}{2}\sqrt{a}\,y^2,$$

where the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} = g(w).$$

3°. Let $f(w) = aw^{2k-1}$ and $g(w) = bw^k$, where a, b, and k are arbitrary constants.
Self-similar solution:

$$w = x^{\frac{2}{1-k}}\, u(\eta), \qquad \eta = \frac{y}{x},$$

where the function $u = u(\eta)$ is determined by the ordinary differential equation

$$(\eta^2 + 1)u''_{\eta\eta} - \left[\frac{2(1+k)}{1-k}\eta + \frac{1}{\eta}\right]u'_\eta + \frac{2(1+k)}{(1-k)^2}u = a\eta^2 u^{2k-1} + bu^k.$$

4°. Let $f(w) = ae^{2\lambda w}$ and $g(w) = be^{\lambda w}$.
Solution:

$$w = U(\eta) - \frac{2}{\lambda}\ln|x|, \qquad \eta = \frac{y}{x},$$

where the function $U = U(\eta)$ is determined by the ordinary differential equation

$$(\eta^2 + 1)U''_{\eta\eta} + \left(2\eta - \frac{1}{\eta}\right)U'_\eta + \frac{2}{\lambda} = a\eta^2 e^{2\lambda U} + be^{\lambda U}.$$

5°. Let $f(w) = aw$ and $g(w) = bw\ln w + cw$.
Multiplicative separable solution:

$$w = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the second-order ordinary differential equations

$$\varphi''_{xx} - (b\ln\varphi + K)\varphi = 0,$$
$$\psi''_{yy} - \frac{1}{y}\psi'_y - (b\ln\psi + ay^2 + c - K)\psi = 0,$$

and K is an arbitrary constant.

⊙ *References for equation* 9.4.2.5: H. Grad and H. Rubin (1958), V. D. Shafranov (1966), G. K. Batchelor (1970), O. V. Kaptsov (1990, 1992, 1998), V. F. Zaitsev and A. D. Polyanin (1996, pp. 486–487), E. Yu. Meshcheryakova (2003), Yu. E. Litvinenko (2010).

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{x}\dfrac{\partial w}{\partial x} + \dfrac{b}{y}\dfrac{\partial w}{\partial y} = f(x^2 + y^2, w).$

1°. Solution with central symmetry:

$$w = w(\xi), \qquad \xi = \left(x^2 + y^2\right)^{1/2},$$

where the function $w = w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{a+b+1}{\xi}w'_\xi = f(\xi^2, w).$$

2°. See also equation 9.4.5.5 with $n = m = 0$ and $f(u, w) = f(w)$.

⊙ *Reference for equation* 9.4.2.6: V. F. Zaitsev and A. D. Polyanin (1996, pp. 486–487), A. D. Polyanin and V. F. Zaitsev (2004, p. 388).

7. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{f(w)}{x}\dfrac{\partial w}{\partial x}+\dfrac{g(w)}{y}\dfrac{\partial w}{\partial y}=h(w).$

Solution with central symmetry:

$$w=w(\xi),\qquad \xi=\left(x^2+y^2\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{f(w)+g(w)+1}{\xi}w'_\xi=h(w).$$

Remark. The functions $f(w)$, $g(w)$, and $h(w)$ in the equation and solution can be replaced by functions of two arguments $f(w,\xi)$, $g(w,\xi)$, and $h(w,\xi)$, where $\xi=\left(x^2+y^2\right)^{1/2}$.

8. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f_1(x)\dfrac{\partial w}{\partial x}+f_2(y)\dfrac{\partial w}{\partial y}+kw\ln w+\left[g_1(x)+g_2(y)\right]w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}=f_1(x)\varphi'_x+k\varphi\ln\varphi+\left[g_1(x)+C\right]\varphi,$$
$$b\psi''_{yy}=f_2(y)\psi'_y+k\psi\ln\psi+\left[g_2(y)-C\right]\psi,$$

where C is an arbitrary constant.

9. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=f(x)\left(\dfrac{\partial w}{\partial y}\right)^2+g(x)w+h(x).$

Generalized separable solution quadratic in y:

$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x), \tag{1}$$

where the functions $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by the following system of ordinary differential equations (the arguments of f, g, and h are omitted):

$$\varphi''_{xx}=4f\varphi^2+g\varphi, \tag{2}$$
$$\psi''_{xx}=(4f\varphi+g)\psi, \tag{3}$$
$$\chi''_{xx}=g\chi+f\psi^2+h-2a\varphi. \tag{4}$$

Whenever a solution $\varphi=\varphi(x)$ of the nonlinear equation (2) is found, the functions $\psi=\psi(x)$ and $\chi=\chi(x)$ can be determined successively from equations (3) and (4), which are linear in ψ and χ, respectively.

It is apparent from the comparison of equations (2) and (3) that (3) has the particular solution $\psi=\varphi(x)$. Hence, the general solution to (3) is given by (see Polyanin and Zaitsev, 2003)

$$\psi(x)=C_1\varphi(x)+C_2\varphi(x)\int\frac{dx}{\varphi^2(x)},\qquad \varphi\not\equiv 0.$$

Note that equation (2) has the trivial particular solution $\varphi(x)\equiv 0$, to which there corresponds solution (1) linear in y. If the functions f and g are proportional, then a particular solution to equation (2) is given by $\varphi=-\frac{1}{4}g/f=\text{const}$.

10. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} = f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + bf(x)w^2 + g(x)w + h(x).$

1°. Generalized separable solutions:

$$w(x,y) = \varphi(x) + \psi(x)\exp\bigl(\pm y\sqrt{-b}\,\bigr), \qquad b < 0, \tag{1}$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the following system of ordinary differential equations (the arguments of f, g, and h are omitted):

$$\varphi''_{xx} = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi''_{xx} = (2bf\varphi + g + ab)\psi. \tag{3}$$

Whenever a solution $\varphi = \varphi(x)$ of equation (2) is found, the functions $\psi = \psi(x)$ can be determined by solving equation (3), which is linear in ψ.

If the functions f, g, and h are proportional, i.e.,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

then particular solutions of equation (2) are given by

$$\varphi = k_1, \quad \varphi = k_2, \tag{4}$$

where k_1 and k_2 are roots of the quadratic equation $bk^2 + \alpha k + \beta = 0$. In this case, equation (3) can be rewritten as

$$\psi''_{xx} = \bigl[(2bk_n + \alpha)f + ab\bigr]\psi, \qquad n = 1,\ 2. \tag{5}$$

The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of exact solutions to the linear equation (5) for various $f = f(x)$. In the special case $f = \text{const}$, the general solution of equation (5) is the sum of exponentials (or sine and cosine).

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,y) = \varphi(x) + \psi(x)\bigl[A\exp\bigl(y\sqrt{-b}\,\bigr) + B\exp\bigl(-y\sqrt{-b}\,\bigr)\bigr], \qquad b < 0, \tag{6}$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} = bf\bigl(\varphi^2 + 4AB\psi^2\bigr) + g\varphi + h,$$
$$\psi''_{xx} = 2bf\varphi\psi + g\psi + ab\psi.$$

Note two special cases of solution (6) that involve hyperbolic functions. These are:

$$w(x,y) = \varphi(x) + \psi(x)\cosh\bigl(y\sqrt{-b}\,\bigr), \quad A = \tfrac{1}{2},\ B = \tfrac{1}{2},$$
$$w(x,y) = \varphi(x) + \psi(x)\sinh\bigl(y\sqrt{-b}\,\bigr), \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,y) = \varphi(x) + \psi(x)\cos\bigl(y\sqrt{b} + c\bigr), \qquad b > 0,$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} = bf\bigl(\varphi^2 + \psi^2\bigr) + g\varphi + h,$$
$$\psi''_{xx} = 2bf\varphi\psi + g\psi + ab\psi.$$

⊙ *References for equation* 9.4.2.10: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996, p. 488).

11. $$a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}=f(x)\left(\frac{\partial w}{\partial y}\right)^2+g(x)\frac{\partial w}{\partial x}+\big[h_1(x)y+h_0(x)\big]\frac{\partial w}{\partial y}+p(x)w+q_2(x)y^2+q_1(x)y+q_0(x).$$

There is a generalized separable solution quadratic in y:

$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x).$$

12. $$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(w)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right].$$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad\text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the two-dimensional Laplace equation for $U=U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

13. $$a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}=f(x)\left(\frac{\partial w}{\partial x}\right)^n+g(y)\left(\frac{\partial w}{\partial y}\right)^m+kw.$$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}-f(x)\big(\varphi'_x\big)^n-k\varphi=C,$$
$$b\psi''_{yy}-g(y)\big(\psi'_y\big)^m-k\psi=-C,$$

where C is an arbitrary constant.

14. $$a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}=f_1(x)\left(\frac{\partial w}{\partial x}\right)^n+f_2(y)\left(\frac{\partial w}{\partial y}\right)^m+g_1(x)\frac{\partial w}{\partial x}+g_2(y)\frac{\partial w}{\partial y}+h_1(x)+h_2(y)+kw.$$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}-f_1(x)\big(\varphi'_x\big)^n-g_1(x)\varphi'_x-k\varphi-h_1(x)=C,$$
$$b\psi''_{yy}-f_2(y)\big(\psi'_y\big)^m-g_2(y)\psi'_y-k\psi-h_2(y)=-C,$$

where C is an arbitrary constant.

15. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(a_1x+b_1y+c_1)\left(\dfrac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\dfrac{\partial w}{\partial y}\right)^k+f\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\right).$

Solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By+C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^k+a_2B^k=A, \tag{1}$$
$$b_1A^k+b_2B^k=B, \tag{2}$$
$$c_1A^k+c_2B^k=C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C. The unknown function $w(z)$ is determined by the ordinary differential equation

$$(A^2+B^2)w''_{zz}=z(w'_z)^k+f\big(w,\ Aw'_z,\ Bw'_z\big).$$

16. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f_1\left(x,\dfrac{\partial w}{\partial x}\right)+f_2\left(y,\dfrac{\partial w}{\partial y}\right)+kw.$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}-f_1\big(x,\varphi'_x\big)-k\varphi=C,$$
$$b\psi''_{yy}-f_2\big(y,\psi'_y\big)-k\psi=-C,$$

where C is an arbitrary constant.

17. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f_1\left(x,\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right)w+f_2\left(y,\dfrac{1}{w}\dfrac{\partial w}{\partial y}\right)w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$a\frac{\varphi''_{xx}}{\varphi}-f_1\left(x,\frac{\varphi'_x}{\varphi}\right)=C,$$
$$b\frac{\psi''_{yy}}{\psi}-f_2\left(y,\frac{\psi'_y}{\psi}\right)=-C,$$

where C is an arbitrary constant.

9.4.3. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]=h(w)$

▷ *Equations of this form describe steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f=f(x)$ and $g=g(y)$ are the principal thermal diffusivities (diffusion coefficients) dependent on coordinates; $h=h(w)$ is the kinetic function (source function), which defines the law of heat (substance) release of absorption. The simple solutions dependent on a single coordinate, $w=w(x)$ or $w=w(y)$, are not considered in this subsection.*

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$ and $m\neq 2$:

$$w=w(\xi),\qquad \xi=\left[b(2-m)^2x^{2-n}+a(2-n)^2y^{2-m}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{A}{\xi}w'_\xi=Bf(w), \tag{1}$$

where

$$A=\frac{4-nm}{(2-n)(2-m)},\quad B=\frac{4}{ab(2-n)^2(2-m)^2}.$$

For $m=4/n$, a family of exact solutions to the original equation with arbitrary $f=f(w)$ follows from (1). It is given by

$$\int\left[C_1+\frac{2n^2}{ab(2-n)^4}F(w)\right]^{-1/2}dw=C_2\pm\xi,\qquad F(w)=\int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $\zeta=\xi^{1-A}$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\zeta\zeta}=\frac{B}{(1-A)^2}\zeta^{\frac{2A}{1-A}}f(w). \tag{2}$$

A large number of exact solutions to equation (2) for various $f=f(w)$ can be found in Polyanin and Zaitsev (2003).

⊙ *Reference for equation* 9.4.3.1: V. F. Zaitsev and A. D. Polyanin (1996, p. 485).

2. $\dfrac{\partial}{\partial x}\left[a(x+c)^n\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[b(y+s)^m\dfrac{\partial w}{\partial y}\right]=f(w).$

The transformation $\zeta=x+c$, $\eta=y+s$ leads to an equation of the form 9.4.3.1:

$$\frac{\partial}{\partial\zeta}\left(a\zeta^n\frac{\partial w}{\partial\zeta}\right)+\frac{\partial}{\partial\eta}\left(b\eta^m\frac{\partial w}{\partial\eta}\right)=f(w).$$

3. $\dfrac{\partial}{\partial x}\left[a\big(|x|+c\big)^n\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[b\big(|y|+s\big)^m\dfrac{\partial w}{\partial y}\right]=f(w).$

The transformation $\zeta = |x| + c$, $\eta = |y| + s$ leads to an equation of the form 9.4.3.1:

$$\frac{\partial}{\partial\zeta}\left(a\zeta^n\frac{\partial w}{\partial\zeta}\right)+\frac{\partial}{\partial\eta}\left(b\eta^m\frac{\partial w}{\partial\eta}\right)=f(w).$$

4. $a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \left[b\mu^2(x+C_1)^2+4ae^{-\mu y}\right]^{1/2},$$

where C_1 is an arbitrary constant and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$w''_{\xi\xi}=\frac{1}{ab\mu^2}f(w).$$

The general solution of this equation with arbitrary kinetic function $f = f(w)$ is defined implicitly by

$$\int\left[C_2+\frac{2}{ab\mu^2}F(w)\right]^{-1/2}dw=C_3\pm\xi, \qquad F(w)=\int f(w)\,dw,$$

where C_2 and C_3 are arbitrary constants.

5. $a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(be^{\mu|y|}\dfrac{\partial w}{\partial y}\right)=f(w).$

The substitution $\zeta = |y|$ leads to an equation of the form 9.4.3.4.

6. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \left(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}-\frac{1}{\xi}w'_\xi=Af(w), \qquad A=\frac{4}{ab\beta^2\mu^2}. \tag{1}$$

The substitution $\zeta = \xi^2$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\zeta\zeta}=\tfrac{1}{4}A\zeta^{-1}f(w),$$

whose solutions with $f(w) = (kw + s)^{-1}$ and $f(w) = (kw + s)^{-2}$ (k, s = const) can be found in Polyanin and Zaitsev (2003).

⊙ *Reference for equation* 9.4.3.6: V. F. Zaitsev and A. D. Polyanin (1996, p. 487).

7. $\dfrac{\partial}{\partial x}\left(ae^{\beta|x|}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu|y|}\dfrac{\partial w}{\partial y}\right)=f(w).$

The transformation $\zeta=|x|$, $\eta=|y|$ leads to an equation of the form 9.4.3.6.

8. $\dfrac{\partial}{\partial x}\left(ax^{n}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $n\neq 2$ and $\mu\neq 0$:

$$w=w(\xi),\qquad \xi=\left[b\mu^2x^{2-n}+a(2-n)^2e^{-\mu y}\right]^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{n}{2-n}\frac{1}{\xi}w'_{\xi}=\frac{4}{ab\mu^2(2-n)^2}f(w).$$

9. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]=kw\ln w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$[f(x)\varphi'_x]'_x=k\varphi\ln\varphi+C\varphi,$$
$$[g(y)\psi'_y]'_y=k\psi\ln\psi-C\psi,$$

and C is an arbitrary constant.

⊙ *Reference for equation* 9.4.3.9: A. D. Polyanin and V. F. Zaitsev (2002, p. 221).

10. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]=aw\ln w+bw.$

This is a special case of equation 9.4.4.6 with $k=a$, $h_1(x)=b$, and $h_2(y)=0$.

9.4.4. Equations of the Form $\dfrac{\partial}{\partial x}\left[f(x,y,w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(x,y,w)\dfrac{\partial w}{\partial y}\right]=h(x,y,w)$

1. $(ay+c)\dfrac{\partial^2 w}{\partial x^2}+(bx+s)\dfrac{\partial^2 w}{\partial y^2}=f(w).$

This equation can be rewritten in the divergence form

$$\frac{\partial}{\partial x}\left[(ay+c)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[(bx+s)\frac{\partial w}{\partial y}\right]=f(w).$$

For $ab\neq 0$, there is an exact solution of the form

$$w=w(\xi),\qquad \xi=(a^2b)^{-1/3}x+(ab^2)^{-1/3}y+(a^2b)^{-2/3}c+(ab^2)^{-2/3}s,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi}=f(w).$$

2. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right]=f(w).$

Solutions are sought in the traveling wave form

$$w=w(z),\qquad z=Ax+By+C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^2+a_2B^2=A, \tag{1}$$
$$b_1A^2+b_2B^2=B, \tag{2}$$
$$c_1A^2+c_2B^2=C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C. The unknown function $w(z)$ is determined by the ordinary differential equation

$$zw''_{zz}+(Aa_1+Bb_2)w'_z=f(w).$$

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left\{\left[f(x)w+g(x)\right]\dfrac{\partial w}{\partial y}\right\}=0.$

1°. Generalized separable solution linear in y:

$$w(x,y)=(Ax+B)y-\int_{x_0}^{x}(x-t)(At+B)^2f(t)\,dt+C_1x+C_2,$$

where A, B, C_1, C_2, and x_0 are arbitrary constants. This solution is degenerate.

2°. Generalized separable solution quadratic in y:

$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x),$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(x)$, and $\chi=\chi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx}+6f\varphi^2=0, \tag{1}$$
$$\psi''_{xx}+6f\varphi\psi=0, \tag{2}$$
$$\chi''_{xx}+2f\varphi\chi+2\varphi g+f\psi^2=0. \tag{3}$$

The nonlinear equation (1) is treated independently from the others. For $f\equiv\text{const}$, its solution can be expressed in terms of elliptic integrals. For $f=ae^{\lambda x}$, a particular solution to (1) is given by $\varphi=-\frac{\lambda^2}{6a}e^{-\lambda x}$. Equations (2) and (3) are solved successively (these are linear in their respective unknowns). Since $\psi=\varphi(x)$ is a particular solution to equation (2), the general solution is expressed as (see Polyanin and Zaitsev, 2003)

$$\psi(x)=C_1\varphi(x)+C_2\varphi(x)\int\frac{dx}{\varphi^2(x)},$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[\dfrac{f(y)}{\sqrt{w+a}}\dfrac{\partial w}{\partial y}\right]=0.$

The substitution $U=\sqrt{w+a}$ leads to the equation

$$\frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left[f(y)\frac{\partial U}{\partial y}\right]=0,$$

which has a generalized separable solution of the form

$$U(x,y)=\varphi(y)x+\psi(y),$$
$$U(x,y)=\varphi(y)x^2+\psi(y)x+\chi(y).$$

5. $\dfrac{\partial^2 w}{\partial x^2}+f(w)\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=w(C_1x+C_2,\pm C_1y+C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Degenerate solution: $w=C_1xy+C_2x+C_3y+C_4$.

3°. Self-similar solution:

$$w=w(z),\qquad z=y/x,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[z^2+f(w)]w''_{zz}+2zw'_z=0.$$

6. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]=kw\ln w+\big[h_1(x)+h_2(y)\big]w.$

Multiplicative separable solution:

$$w(x,y)=\exp\big[\varphi(x)+\psi(y)\big].$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$e^{-\varphi}[fe^{\varphi}\varphi'_x]'_x-k\varphi-h_1(x)=C,$$
$$e^{-\psi}[ge^{\psi}\psi'_y]'_y-k\psi-h_2(y)=-C,$$

where C is an arbitrary constant.

7. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left\{\big[g(x)w+h(x)\big]\dfrac{\partial w}{\partial y}\right\}=0.$

There are generalized separable solutions linear and quadratic in y:

$$w(x,y)=\varphi(x)y+\psi(x),$$
$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x).$$

8. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]=0.$

This is a *stationary anisotropic heat (diffusion) equation*; $f(w)$ and $g(w)$ are the principal thermal diffusivities (diffusion coefficients).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, \pm C_1y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$\int \left[A^2 f(w) + B^2 g(w)\right] dw = C_1(Ax + By) + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants.

3°. Self-similar solution (α and β are arbitrary constants):

$$w = w(\zeta), \quad \zeta = \frac{x+\alpha}{y+\beta},$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$[f(w)w'_\zeta]'_\zeta + [\zeta^2 g(w)w'_\zeta]'_\zeta = 0. \tag{1}$$

Integrating (1) and taking w to be the independent variable, one obtains a Riccati equation for $\zeta = \zeta(w)$:

$$C\zeta'_w = g(w)\zeta^2 + f(w), \tag{2}$$

where C is an arbitrary constant. A large number of exact solutions to equation (2) for various $f = f(w)$ and $g = g(w)$ can be found in Polyanin and Zaitsev (2003).

4°. Solution in parametric form:

$$x = C_1v^2 + C_2v - \int f(w)[2C_1G(w) + C_3]\,dw + C_4,$$
$$y = -[2C_1G(w) + C_3]v - C_2G(w) + C_5, \quad G(w) = \int g(w)\,dw,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. Solution in parametric form:

$$x = [C_1F(w) + C_2]v + C_3F(w) + C_4, \quad F(w) = \int f(w)\,dw,$$
$$y = \frac{1}{2}C_1v^2 + C_3v - \int g(w)[C_1F(w) + C_2]\,dw + C_5.$$

6°. Solution in parametric form:

$$x = [C_1F(w) + C_2]v^2 + C_3F(w) + C_4 - 2\int\left\{f(w)\int g(w)[C_1F(w) + C_2]\,dw\right\}dw,$$
$$y = \frac{1}{3}C_1v^3 + C_3v - 2v\int g(w)[C_1F(w) + C_2]\,dw + C_5.$$

7°. Solution in parametric form:

$$x = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_3,$$
$$y = \frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})\frac{1}{f(w)}H'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[H] + \lambda^2 g(w)H = 0$, and the differential operator $\mathbf{L}_f$ is defined as

$$\mathbf{L}_f[\varphi] \equiv \frac{d}{dw}\left[\frac{1}{f(w)}\frac{d\varphi}{dw}\right]. \tag{3}$$

8°. Solution in parametric form:

$$x = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_3,$$
$$y = \frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]\frac{1}{f(w)}Z'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[Z] - \lambda^2 g(w)Z = 0$.

9°. Solution in parametric form:

$$x = [2C_1 F(w) + C_3]v + C_2 F(w) + C_5, \quad F(w) = \int f(w)\,dw,$$
$$y = C_1 v^2 + C_2 v - \int g(w)[2C_1 F(w) + C_3]\,dw + C_4.$$

10°. Solution in parametric form:

$$x = \frac{1}{2}C_1 v^2 + C_3 v - \int f(w)[C_1 G(w) + C_2]\,dw + C_5,$$
$$y = -[C_1 G(w) + C_2]v - C_3 G(w) + C_4, \quad G(w) = \int g(w)\,dw.$$

11°. Solution in parametric form:

$$x = \frac{1}{3}C_1 v^3 + C_3 v - 2v\int f(w)[C_1 G(w) + C_2]\,dw + C_5,$$
$$y = -[C_1 G(w) + C_2]v^2 - C_3 G(w) + C_4 + 2\int\left\{g(w)\int f(w)[C_1 G(w) + C_2]\,dw\right\}dw.$$

12°. Solution in parametric form:

$$x = -\frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})\frac{1}{g(w)}H'_w(w) + C_3,$$
$$y = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_4,$$

where C_1, C_2, C_3, and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_g[H] + \lambda^2 f(w)H = 0$, and the differential operator $\mathbf{L}_g$ is defined by (3) with $f(w) = g(w)$.

13°. Solution in parametric form:

$$x = -\frac{1}{\lambda}[C_2\sin(\lambda v) - C_1\cos(\lambda v)]\frac{1}{g(w)}Z'_w(w) + C_3,$$
$$y = [C_1\sin(\lambda v) + C_2\cos(\lambda v)]Z(w) + C_4,$$

where C_1, C_2, C_3, and λ are arbitrary constants, the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_g[Z] - \lambda^2 f(w)Z = 0$, and the differential operator $\mathbf{L}_g$ is defined by (3) with $f(w) = g(w)$.

14°. The original equation can be represented as the system of the equations

$$f(w)\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \quad -g(w)\frac{\partial w}{\partial y} = \frac{\partial v}{\partial x}. \tag{4}$$

The hodograph transformation

$$x = x(w, v), \quad y = y(w, v), \tag{5}$$

where w, v are treated as the independent variables and x, y as the dependent ones, brings (4) to the linear system

$$f(w)\frac{\partial y}{\partial v} = \frac{\partial x}{\partial w}, \quad -g(w)\frac{\partial x}{\partial v} = \frac{\partial y}{\partial w}. \tag{6}$$

Eliminating y yields the following linear equation for $x = x(w, v)$:

$$\frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial x}{\partial w}\right] + g(w)\frac{\partial^2 x}{\partial v^2} = 0. \tag{7}$$

Likewise, we can obtain another linear equation for $y = y(w, v)$ from system (6). We have

$$\frac{\partial}{\partial w}\left[\frac{1}{g(w)}\frac{\partial y}{\partial w}\right] + f(w)\frac{\partial^2 y}{\partial v^2} = 0. \tag{8}$$

The procedure for constructing exact solutions to the original equation consists of the following two stages. First, one finds an exact solution to the linear equation (7) for $x = x(w, v)$. Then, this solution is substituted into the linear system (6), from which the function $y = y(w, v)$ is found in the form

$$y = \int_{v_0}^{v} \frac{1}{f(w)}\frac{\partial x}{\partial w}(w, \xi)\,d\xi - \int_{w_0}^{w} g(\eta)\frac{\partial x}{\partial v}(\eta, v_0)\,d\eta, \tag{9}$$

where w_0 and v_0 are any numbers. The thus obtained expressions of (5) define a solution to the original equation in parametric form.

Likewise, one can first construct an exact solution to the linear equation (8) for $y = y(w, v)$ and then find $x = x(w, v)$ from (6) in the form

$$x = -\int_{v_0}^{v} \frac{1}{g(w)}\frac{\partial y}{\partial w}(w, \xi)\,d\xi + \int_{w_0}^{w} f(\eta)\frac{\partial y}{\partial v}(\eta, v_0)\,d\eta,$$

where w_0 and v_0 are any numbers.

Remark 1. Let $x = \Phi(w, v; f, g)$ be a solution to equation (7). Then $y = \Phi(w, v; g, f)$ solves equation (8).

Remark 2. Let $x = \Phi(w, v; f, g)$, $y = \Psi(w, v; f, g)$ be a solution to system of equations (6). Then the functions $x = \Psi(w, v; -g, -f)$ and $y = \Phi(w, v; -g, -f)$ also solve this system.

15°. Solutions to equation (7) with even powers of v:

$$x = \sum_{k=0}^{n} \varphi_k(w) v^{2k}, \tag{10}$$

where the functions $\varphi_k = \varphi_k(w)$ are determined by the recurrence relations

$$\varphi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\varphi_{k-1}(w) = A_k F(w) + B_k - 2k(2k-1) \int f(w) \left\{ \int g(w)\varphi_k(w)\, dw \right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

The function $y = y(w, v)$ is defined by (9) and, in conjunction with relation (10), gives a solution to the original nonlinear equation in parametric form.

16°. Solutions to equation (7) with odd powers of v:

$$x = \sum_{k=0}^{n} \psi_k(w) v^{2k+1}, \tag{11}$$

where the functions $\psi_k = \psi_k(w)$ are determined by the recurrence relations

$$\psi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\psi_{k-1}(w) = A_k F(w) + B_k - 2k(2k+1) \int f(w) \left\{ \int g(w)\psi_k(w)\, dw \right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \ldots, 1$).

The function $y = y(w, v)$ is defined by (9) and, in conjunction with relation (11), gives a solution to the original nonlinear equation in parametric form.

17°. In the special case $g(w) = k^2 f(w)$, the transformation

$$\bar{x} = kx, \quad u = \int f(w)\, dw$$

leads to the Laplace equation

$$\frac{\partial^2 u}{\partial \bar{x}^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *References for equation* 9.4.4.8: V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002, pp. 223–226).

9. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]=ax\dfrac{\partial w}{\partial x}+ay\dfrac{\partial w}{\partial y}-h(w).$

Traveling-wave solution:

$$w=w(z),\qquad z=k_1x+k_2y,$$

where k_1 and k_2 are arbitrary constants, and the function $w(z)$ is determined by the ordinary differential equation

$$[\varphi(w)w_z']_z'=azw_z'-h(w),\qquad \varphi(w)=k_1^2f(w)+k_2^2g(w).$$

Remark. The above solution is a nonclassical (noninvariant) traveling-wave solution, since the equation in question is not invariant under translation transformations in the independent variables.

⊙ *Reference*: A. D. Polyanin (2004 *a*).

10. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]=(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}+(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}.$

This equation describes steady-state anisotropic heat/mass transfer in a translational-shear fluid flow.

Traveling-wave solution:

$$w=w(z),\qquad z=a_2x+(k-a_1)y,$$

where k is a root of the quadratic equation

$$k^2-(a_1+b_2)k+a_1b_2-a_2b_1=0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$[\varphi(w)w_z']_z'=[kz+a_2c_1+(k-a_1)c_2]w_z',\qquad \varphi(w)=a_2^2f(w)+(k-a_1)^2g(w).$$

Remark 1. The above remains the same if an arbitrary function, $h(w)$, is added to the right-hand side of the original equation.

Remark 2. In the case of an incompressible fluid, equation coefficients must satisfy the condition $a_1+b_2=0$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 399).

11. $\dfrac{\partial}{\partial x}\left\{\left[a_1x+b_1y+f(w)\right]\dfrac{\partial w}{\partial x}\right\}+\dfrac{\partial}{\partial y}\left\{\left[a_2x+b_2y+g(w)\right]\dfrac{\partial w}{\partial y}\right\}=0.$

Solutions are sought in the traveling-wave form

$$w=w(\xi),\qquad \xi=Ax+By,$$

where the constants A and B are determined by solving the algebraic system of equations

$$a_1A^2+a_2B^2=A,\qquad b_1A^2+b_2B^2=B.$$

The desired function $w(\xi)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant):

$$\left[\xi+A^2f(w)+B^2g(w)\right]w_\xi'=C.$$

Taking w to be the independent variable, we obtain a first-order linear equation for $\xi=\xi(w)$:

$$C\xi_w'=\xi+A^2f(w)+B^2g(w).$$

9.4.5. Other Equations

1. $\dfrac{\partial^2 w}{\partial x^2} + a w^4 \dfrac{\partial^2 w}{\partial y^2} = f(y) w^5.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\big(\pm C_1^2 x + C_2,\ y\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Let $u = u(y)$ be any nontrivial particular solution of the second-order linear ordinary differential equation

$$a u''_{yy} - f(y) u = 0. \tag{1}$$

The transformation

$$\zeta = \int \frac{dy}{u^2}, \quad \xi = \frac{w}{u}$$

simplifies the original equation considerably, bringing it to the form

$$\frac{\partial^2 \xi}{\partial x^2} + a\xi^4 \frac{\partial^2 \xi}{\partial \zeta^2} = 0. \tag{2}$$

This equation is independent of f explicitly and has a degenerate solution

$$\xi(x, \zeta) = Ax\zeta + B\zeta + Cx + D,$$

where A, B, C, and D are arbitrary constants. Furthermore, equation (2) has exact solutions with the following structures, for example:

$$\begin{aligned}
&\xi(x, \zeta) = \xi(kx + \lambda\zeta) && \text{(traveling-wave solution)},\\
&\xi(x, \zeta) = g(x)h(\zeta) && \text{(multiplicative separable solution)},\\
&\xi(x, \zeta) = x^{\beta}\varphi(\eta), \quad \eta = \zeta x^{-2\beta-1} && \text{(self-similar solution)},
\end{aligned}$$

where k, λ, and β are arbitrary constants.

⊙ *Reference for equation* 9.4.5.1: V. F. Zaitsev and A. D. Polyanin (1996, pp. 489–490).

2. $\dfrac{\partial^2 w}{\partial x^2} + a \dfrac{\partial}{\partial y}\left(w^n \dfrac{\partial w}{\partial y} \right) = f(y) w^{n+1} + g(x) w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(y),$$

where the functions $\psi = \psi(y)$ and $\varphi = \varphi(x)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$\begin{aligned}
\varphi''_{xx} - g(x)\varphi + C\varphi^{n+1} &= 0,\\
a(\psi^n \psi'_y)'_y - f(y)\psi^{n+1} - C\psi &= 0.
\end{aligned}$$

3. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right) = f(y)e^{\lambda w} + g(x).$

Additive separable solution:

$$w = \varphi(x) + \psi(y),$$

where the functions $\psi = \psi(y)$ and $\varphi = \varphi(x)$ are determined by the ordinary differential equations

$$\varphi''_{xx} - g(x) + Ce^{\lambda\varphi} = 0,$$
$$a(e^{\lambda\psi}\psi'_y)'_y - f(y)e^{\lambda\psi} - C = 0,$$

and C is an arbitrary constant. The second equation can be reduced, with the change of variable $U = e^{\lambda\psi}$, to the linear equation $aU''_{yy} - \lambda f(y)U - \lambda C = 0$.

4. $\dfrac{\partial^2 w}{\partial x^2} + \big[f_3(x)w + f_2(x)y^2 + f_1(x)y + f_0(x)\big]\dfrac{\partial^2 w}{\partial y^2} = g_2(x)\left(\dfrac{\partial w}{\partial y}\right)^2$
$+ g_1(x)\dfrac{\partial w}{\partial x} + \big[h_1(x)y + h_0(x)\big]\dfrac{\partial w}{\partial y} + s_3(x)w + s_2(x)y^2 + s_1(x)y + s_0(x).$

There is a generalized separable solution quadratic in y:

$$w(x, y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

5. $ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + kx^{n-1}\dfrac{\partial w}{\partial x} + sy^{m-1}\dfrac{\partial w}{\partial y} = f(w).$

Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\big]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$Aw''_{\xi\xi} + \frac{B}{\xi}w'_\xi = f(w), \tag{1}$$

where

$$A = \tfrac{1}{4}ab(2-n)^2(2-m)^2,$$
$$B = \tfrac{1}{4}(2-n)(2-m)\big[ab(3nm - 4n - 4m + 4) + 2bk(2-m) + 2as(2-n)\big].$$

Solution of equation (1) with $B = 0$ and arbitrary $f = f(w)$ in implicit form:

$$\int\left[C_1 + \frac{2}{A}F(w)\right]^{-1/2} dw = C_2 \pm \xi, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference for equation* 9.4.5.5: V. F. Zaitsev and A. D. Polyanin (1996, p. 486–487).

6. $ax^n\dfrac{\partial^2 w}{\partial x^2}+by^m\dfrac{\partial^2 w}{\partial y^2}+kx^{n-1}f(w)\dfrac{\partial w}{\partial x}+sy^{m-1}f(w)\dfrac{\partial w}{\partial y}=g(w).$

For $n\neq 2$ and $m\neq 2$, there is a functional separable solution of the form

$$w=w(\xi),\qquad \xi=\left[b(2-m)^2x^{2-n}+a(2-n)^2y^{2-m}\right]^{1/2}.$$

7. $ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2}+be^{\mu y}\dfrac{\partial^2 w}{\partial y^2}+ke^{\beta x}\dfrac{\partial w}{\partial x}+se^{\mu y}\dfrac{\partial w}{\partial y}=f(w).$

Functional separable solution for $\beta\mu\neq 0$:

$$w=w(\xi),\qquad \xi=\left(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\right)^{1/2}.$$

Here, the function $w=w(\xi)$ is determined by the ordinary differential equation

$$Aw''_{\xi\xi}+\frac{B}{\xi}w'_\xi=f(w), \tag{1}$$

where

$$A=\tfrac{1}{4}ab\beta^2\mu^2,\quad B=\tfrac{1}{4}\beta\mu(3ab\beta\mu-2bk\mu-2as\beta).$$

Solution of equation (1) with $B=0$ and arbitrary $f=f(w)$ in implicit form:

$$\int\left[C_1+\frac{2}{A}F(w)\right]^{-1/2}dw=C_2\pm\xi,\qquad F(w)=\int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

8. $ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2}+be^{\mu y}\dfrac{\partial^2 w}{\partial y^2}+ke^{\beta x}f(w)\dfrac{\partial w}{\partial x}+se^{\mu y}f(w)\dfrac{\partial w}{\partial y}=g(w).$

For $\beta\mu\neq 0$, there is a functional separable solution of the form

$$w=w(\xi),\qquad \xi=\left(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\right)^{1/2}.$$

9. $ax^n\dfrac{\partial^2 w}{\partial x^2}+be^{\beta y}\dfrac{\partial^2 w}{\partial y^2}+kx^{n-1}\dfrac{\partial w}{\partial x}+se^{\beta y}\dfrac{\partial w}{\partial y}=f(w).$

Functional separable solution $\beta\neq 0$ and $n\neq 2$:

$$w=w(\xi),\qquad \xi=\left[b\beta^2x^{2-n}+a(2-n)^2e^{-\beta y}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$Aw''_{\xi\xi}+\frac{B}{\xi}w'_\xi=f(w), \tag{1}$$

where

$$A=\tfrac{1}{4}ab\beta^2(2-n)^2,\quad B=\tfrac{1}{4}\beta(2-n)\left[ab\beta(4-3n)+2bk\beta-2as(2-n)\right].$$

Solution of equation (1) with $B=0$ and arbitrary $f=f(w)$ in implicit form:

$$\int\left[C_1+\frac{2}{A}F(w)\right]^{-1/2}dw=C_2\pm\xi,\qquad F(w)=\int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference for equation* 9.4.5.9: V. F. Zaitsev and A. D. Polyanin (1996, p. 488).

10. $ax^n\dfrac{\partial^2 w}{\partial x^2}+be^{\beta y}\dfrac{\partial^2 w}{\partial y^2}+kx^{n-1}f(w)\dfrac{\partial w}{\partial x}+se^{\beta y}f(w)\dfrac{\partial w}{\partial y}=g(w).$

For $\beta \neq 0$ and $n \neq 2$, there is a functional separable solution of the form

$$w = w(\xi), \qquad \xi = \left[b\beta^2 x^{2-n} + a(2-n)^2 e^{-\beta y}\right]^{1/2}.$$

11. $(ay+c)\dfrac{\partial^2 w}{\partial x^2}+(bx+s)\dfrac{\partial^2 w}{\partial y^2}=f\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\right).$

Functional separable solution for $ab \neq 0$:

$$w = w(\xi), \qquad \xi = (a^2b)^{-1/3}x + (ab^2)^{-1/3}y + (a^2b)^{-2/3}c + (ab^2)^{-2/3}s.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} = f\left(w,\ \beta w'_\xi,\ \mu w'_\xi\right),$$

where $\beta = (a^2b)^{-1/3}$, $\mu = (ab^2)^{-1/3}$.

12. $(a_1x+b_1y+c_1)\dfrac{\partial^2 w}{\partial x^2}+(a_2x+b_2y+c_2)\dfrac{\partial^2 w}{\partial y^2}=f\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\right).$

Solutions are sought in the traveling-wave form

$$w = w(\xi), \qquad \xi = Ax + By + C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^2 + a_2B^2 = A, \tag{1}$$
$$b_1A^2 + b_2B^2 = B, \tag{2}$$
$$c_1A^2 + c_2B^2 = C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C.

The desired function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} = f\left(w,\ Aw'_\xi,\ Bw'_\xi\right).$$

13. $f_1(x)\dfrac{\partial^2 w}{\partial x^2}+f_2(y)\dfrac{\partial^2 w}{\partial y^2}=g_1(x)\dfrac{\partial w}{\partial x}+g_2(y)\dfrac{\partial w}{\partial y}+kw\ln w+\left[h_1(x)+h_2(y)\right]w.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$f_1(x)\varphi''_{xx} = g_1(x)\varphi'_x + k\varphi\ln\varphi + \left[h_1(x) + C\right]\varphi,$$
$$f_2(y)\psi''_{yy} = g_2(y)\psi'_y + k\psi\ln\psi + \left[h_2(y) - C\right]\psi,$$

where C is an arbitrary constant.

14. $\big[a_1x + b_1y + f(w)\big]\dfrac{\partial^2 w}{\partial x^2} + \big[a_2x + b_2y + g(w)\big]\dfrac{\partial^2 w}{\partial y^2} = h\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\right).$

Traveling-wave solution:

$$w = w(\xi), \qquad \xi = Ax + By,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^2 + a_2B^2 = A, \quad b_1A^2 + b_2B^2 = B,$$

and the function $w(\xi)$ is determined by the ordinary differential equation

$$\big[\xi + A^2 f(w) + B^2 g(w)\big]w''_{\xi\xi} = h\big(w,\ Aw'_\xi,\ Bw'_\xi\big).$$

15. $\dfrac{\partial}{\partial x}\left\{\big[a_1x+b_1y+f(w)\big]\dfrac{\partial w}{\partial x}\right\}+\dfrac{\partial}{\partial y}\left\{\big[a_2x+b_2y+g(w)\big]\dfrac{\partial w}{\partial y}\right\}=h\left(w,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\right).$

Traveling-wave solution:

$$w = w(\xi), \qquad \xi = Ax + By,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$A^2a_1 + B^2a_2 = A, \quad A^2b_1 + B^2b_2 = B,$$

and the function $w(\xi)$ is determined by the ordinary differential equation

$$\big[\varphi(\xi, w)w'_\xi\big]'_\xi = h\big(w,\ Aw'_\xi,\ Bw'_\xi\big),$$
$$\varphi(\xi, w) = \xi + A^2 f(w) + B^2 g(w).$$

16. $\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2} + h(x)w = 0.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-3}w\big(x,\ C_1y + C_2\big) + \phi(x),$$

where C_1 and C_2 are arbitrary constants, and $\phi(x)$ is determined by the second-order linear ordinary differential equation $\phi''_{xx} + f(x)\phi'_x + h(x)\phi = 0$, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x, y) = \varphi_1(x) + \varphi_2(x)y^{3/2} + \varphi_3(x)y^3,$$

where the functions $\varphi_k = \varphi_k(x)$ are determined by the system of ordinary differential equations

$$\varphi''_1 + f(x)\varphi'_1 + \tfrac{9}{8}g(x)\varphi_2^2 + h(x)\varphi_1 = 0,$$
$$\varphi''_2 + f(x)\varphi'_2 + \tfrac{45}{4}g(x)\varphi_2\varphi_3 + h(x)\varphi_2 = 0,$$
$$\varphi''_3 + f(x)\varphi'_3 + 18g(x)\varphi_3^2 + h(x)\varphi_3 = 0,$$

where the prime stands for the differentiation with respect to x.

3°. Generalized separable solution cubic in y:

$$w(x,y)=\psi_1(x)+\psi_2(x)y+\psi_3(x)y^2+\psi_4(x)y^3,$$

where the functions $\psi_k=\psi_k(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\psi_1''+f(x)\psi_1'+2g(x)\psi_2\psi_3+h(x)\varphi_1=0,\\
&\psi_2''+f(x)\psi_2'+2g(x)(2\psi_3^2+3\psi_2\psi_4)+h(x)\varphi_2=0,\\
&\psi_3''+f(x)\psi_3'+18g(x)\psi_3\psi_4+h(x)\varphi_3=0,\\
&\psi_4''+f(x)\psi_4'+18g(x)\psi_4^2+h(x)\varphi_4=0.
\end{aligned}$$

4°. Generalized separable solution:

$$w(x,y)=\xi(x)+\eta(x)\theta(y).$$

Here, the functions $\xi=\xi(x)$ and $\eta=\eta(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\eta_{xx}''+f(x)\eta_x'+ag(x)\eta^2+h(x)\eta=0,\\
&\xi_{xx}''+f(x)\xi_x'+bg(x)\eta^2+h(x)\xi=0,
\end{aligned}$$

where a and b are arbitrary constants, and the function $\theta=\theta(y)$ is determined by the autonomous ordinary differential equation

$$\theta_y'\theta_{yy}''=a\theta+b,$$

whose solution can be written out in implicit form.

17. $\dfrac{\partial^2 w}{\partial x^2}+f\left(\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y}\right)\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1=C_1^{-1}w\big(C_1x+C_2,\ C_1y+C_3\big)+C_4,$$

where C_1, C_2, C_4, and C_4 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t)=Axy+Bx+Cy+D,$$

where A, B, C, and D are arbitrary constants.

3°. Self-similar solution:

$$w(x,y)=xu(z),\qquad z=y/x,$$

where the function $u(z)$ is determined by the ordinary differential equation

$$z(zu_z'-u)_z'+f(u-zu_z',\ u_z')u_{zz}''=0.$$

4°. The Legendre transformation

$$u(\xi,\eta)=x\xi+y\eta-w(x,y),\quad \xi=\frac{\partial w}{\partial x},\quad \eta=\frac{\partial w}{\partial y},$$

where u is the new dependent variable, and ξ and η are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial\eta^2}+f(\xi,\eta)\frac{\partial^2 u}{\partial\xi^2}=0.$$

Exact solutions of this equation for some $f(\xi,\eta)$ can be found in Polyanin (2002).

Chapter 10
Second-Order Elliptic Equations with Three or More Space Variables

10.1. Equations with Three Space Variables Involving Power-Law Nonlinearities

10.1.1. Equations of the Form $\frac{\partial}{\partial x}\Big[f(x)\frac{\partial w}{\partial x}\Big]+\frac{\partial}{\partial y}\Big[g(y)\frac{\partial w}{\partial y}\Big]+\frac{\partial}{\partial z}\Big[h(z)\frac{\partial w}{\partial z}\Big]=aw^p$

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right)=sw^p.$

This is a special case of equation 10.3.1.3 with $f(w)=sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ C_1^{\frac{p-1}{2-k}}z\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, $k\neq 2$, and $p\neq 1$:

$$w=\left[\frac{1}{s(1-p)}\left(\frac{p}{1-p}+\frac{1}{2-n}+\frac{1}{2-m}+\frac{1}{2-k}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=sw^p,\qquad A=\frac{2}{2-n}+\frac{2}{2-m}+\frac{2}{2-k}-1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z)=U(\xi,z),\quad \xi^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2},$$

$$w(x,y,z)=V(x,\eta),\quad \eta^2=\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2},$$

$$w(x,y,z)=W(y,\zeta),\quad \zeta^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{z^{2-k}}{c(2-k)^2},$$

$$w(x,y,z)=x^{\frac{n-2}{p-1}}F(\rho_1,\rho_2),\quad \rho_1=yx^{\frac{n-2}{2-m}},\quad \rho_2=zx^{\frac{n-2}{2-k}}.$$

2. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=sw^p.$

This is a special case of equation 10.3.1.5 with $f(w)=sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ z+\frac{1-p}{\lambda}\ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, $\lambda\neq 0$, and $p\neq 1$:

$$w=\left[\frac{1}{s(p-1)}\left(\frac{p}{1-p}+\frac{1}{2-n}+\frac{1}{2-m}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w_{rr}''+\frac{A}{r}w_r'=sw^p,\qquad A=\frac{2}{2-n}+\frac{2}{2-m}-1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z)=U(\xi,z),\quad \xi^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2},$$
$$w(x,y,z)=V(x,\eta),\quad \eta^2=\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=W(y,\zeta),\quad \zeta^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=x^{\frac{n-2}{p-1}}F(\rho_1,\rho_2),\quad \rho_1=yx^{\frac{n-2}{2-m}},\quad \rho_2=z+\frac{2-n}{\lambda}\ln x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=sw^p.$

This is a special case of equation 10.3.1.6 with $f(w)=sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=C_1 w\Big(C_1^{\frac{p-1}{2-n}}x,\ y+\frac{1-p}{\beta}\ln C_1,\ z+\frac{1-p}{\lambda}\ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n \neq 2$, $\beta \neq 0$, $\lambda \neq 0$, and $p \neq 1$:

$$w = \left[\frac{1}{s(p-1)}\left(\frac{p}{1-p}+\frac{1}{2-n}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n \neq 2$, $\beta \neq 0$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r = sw^p, \qquad A = \frac{n}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi,z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2},$$

$$w(x,y,z) = V(x,\eta), \quad \eta^2 = \frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = W(y,\zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = x^{\frac{n-2}{p-1}}F(\rho_1,\rho_2), \quad \rho_1 = y+\frac{2-n}{\beta}\ln x, \quad \rho_2 = z+\frac{2-n}{\lambda}\ln x.$$

4. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right) = sw^p.$

This is a special case of equation 10.3.1.4 with $f(w) = sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x+\frac{1-p}{\beta}\ln C_1,\ y+\frac{1-p}{\gamma}\ln C_1,\ z+\frac{1-p}{\lambda}\ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $p \neq 1$, $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$:

$$w = \left[\frac{p}{b(1-p)^2}\right]^{\frac{1}{p-1}}\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right)^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} - \frac{1}{r}w'_r = sw^p.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi,z), \quad \xi^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2},$$

$$w(x,y,z) = V(x,\eta), \quad \eta^2 = \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = W(y,\zeta), \quad \zeta^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = \exp\Big(\frac{\beta x}{p-1}\Big)F(\rho_1,\rho_2), \quad \rho_1 = y - \frac{\beta}{\gamma}x, \quad \rho_2 = z - \frac{\beta}{\lambda}x.$$

10.1.2. Equations of the Form $\dfrac{\partial}{\partial x}\Big[f(w)\dfrac{\partial w}{\partial x}\Big] + \dfrac{\partial}{\partial y}\Big[g(w)\dfrac{\partial w}{\partial y}\Big] + \dfrac{\partial}{\partial z}\Big[g(w)\dfrac{\partial w}{\partial z}\Big] = 0$

1. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial}{\partial z}\Big[(bw+c)\dfrac{\partial w}{\partial z}\Big] = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2 x + C_3, \pm C_2 y + C_4, \pm C_1 C_2 z + C_5) + \frac{c(1-C_1^2)}{bC_1^2},$$

$$w_2 = w(x\cos\beta + ya^{-1/2}\sin\beta, -xa^{1/2}\sin\beta + y\cos\beta,\ z),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z) = A\sqrt{C_1x + C_2y + C_3z + C_4} - \frac{C_1^2 + aC_2^2}{bC_3^2} - \frac{c}{b},$$

$$w(x,y,z) = (C_1x + C_2)z + C_3x + C_4y + C_5 - \tfrac{1}{12}bC_1^{-2}(C_1x + C_2)^4,$$

$$w(x,y,z) = (C_1x + C_2)z + C_3(ax^2 - y^2) - \tfrac{1}{12}bC_1^{-2}(C_1x + C_2)^4,$$

$$w(x,y,z) = |z|^{1/2}\big[C_1(ax^2 - y^2) + C_2x + C_3 + C_4)\big] - \frac{c}{b},$$

$$w(x,y,z) = C_1|z|^{1/2}\exp\big(\sqrt{a}C_2x\big)\sin(C_2y + C_3) - \frac{c}{b},$$

$$w(x,y,z) = C_1|z|^{1/2}\sin(\sqrt{a}C_2x + C_3)\exp(C_2y) - \frac{c}{b},$$

where $A, C_1, \ldots, C_5$ are arbitrary constants (the first solution is of the traveling-wave type).

3°. Solution:

$$w = u(\xi) - 4bC_1^2x^2, \quad \xi = z + bC_1x^2 + C_2y,$$

where C_1 and C_2 are arbitrary constants and the function $u(\xi)$ is determined by the first-order ordinary differential equation

$$(bu + c + aC_2^2)u'_\xi + 2bC_1u = 8bC_1^2\xi + C_3.$$

With appropriate translations in both variables, one can make the equation homogeneous, which means it is integrable by quadrature.

4°. Solution:

$$w = v(r) - 4bC_2^2x^2 - 4abC_1^2y^2, \qquad r = z + bC_1x^2 + bC_2y^2,$$

where C_1 and C_2 are arbitrary constants and the function $v(r)$ is determined by the first-order ordinary differential equation

$$(bv + c)v_r' + 2b(aC_2 + C_1)v = 8b(a^2C_1^2 + C_2^2)r + C_3.$$

With appropriate translations in both variables, one can make the equation homogeneous.

5°. Solution (generalizes the solution of Items 3° and 4°):

$$w = U(\zeta) + A_1x^2 + A_2y^2 + A_3xy + A_4x + A_5y, \qquad \zeta = z + b(B_1x^2 + B_2y^2 + B_3xy + B_4x + B_5y),$$

where B_1, B_2, B_3, B_4, and B_5 are arbitrary constants, the coefficients A_n are expressed in terms of B_n by

$$\begin{aligned} A_1 &= -b(4B_1^2 + aB_3^2), \\ A_2 &= -b(B_3^2 + 4aB_2^2), \\ A_3 &= -4bB_3(B_1 + aB_2), \\ A_4 &= -2b(2B_1B_4 + aB_3B_5), \\ A_5 &= -2b(B_3B_4 + 2aB_2B_5), \end{aligned}$$

and the function $U(\zeta)$ is determined by the first-order ordinary differential equation

$$(bU + c + ab^2B_5^2 + b^2B_4^2)U_\zeta' + 2b(aB_2 + B_1)U + 2(aA_2 + A_1)\zeta = C_1.$$

With appropriate translations in both variables, one can make the equation homogeneous, which means it is integrable by quadrature.

6°. "Two-dimensional" generalized separable solution linear in z (generalizes the second and third solutions of Item 2°):

$$w = f(x, \eta)z + g(x, \eta), \qquad \eta = a^{-1/2}y,$$

where the functions f and g are determined by the system of differential equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \eta^2} = 0, \tag{1}$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial \eta^2} = -bf^2. \tag{2}$$

Equation (1) is the Laplace equation. Given $f = f(x, \eta)$, (2) represents a Helmholtz equation. For solutions of these linear equations, see Tikhonov and Samarskii (1990) and Polyanin (2002).

7°. "Two-dimensional" generalized separable solution quadratic in z:

$$w = f(x, y)z^2 + g(x, y)z + h(x, y),$$

where the functions $f = f(x, y)$, $g = g(x, y)$, and $h = h(x, y)$ are determined by the system of differential equations

$$\begin{aligned} &f_{xx} + af_{yy} + 6bf^2 = 0, \\ &g_{xx} + ag_{yy} + 6bfg = 0, \\ &h_{xx} + ah_{yy} + bg^2 + 2bfh + 2cf = 0. \end{aligned}$$

Here, the subscripts denote the corresponding partial derivatives.

8°. "Two-dimensional" solution (generalizes the last three solutions of Item 2°):

$$w(x,y,z) = |z|^{1/2}U(x,\eta) - \frac{c}{b}, \qquad \eta = a^{-1/2}y,$$

where the function $U = U(x,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

9°. There are solutions of the following forms:

$$w(x,y,z) = F(z,r), \quad r = ax^2 + y^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = x^{2\lambda}G(\xi,\eta) - \frac{c}{b}, \quad \xi = \frac{y}{x}, \quad \eta = \frac{z}{x^{\lambda+1}} \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = H(\zeta), \quad \zeta = (ax^2 + y^2)z^{-2} \qquad \text{"one-dimensional" solution,}$$

where λ is an arbitrary constant.

10°. The substitution $u = w + (c/b)$ leads to a special case of equation 10.1.2.3 with $n = 1$.

Remark. In the special case $a = 1$, $b < 0$, and $c > 0$, the equation in question describes transonic flows of ideal polytropic gases (Pokhozhaev, 1989).

⊙ *Reference for equation* 10.1.2.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 408–409).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(a_1 w + b_1)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[(a_2 w + b_2)\dfrac{\partial w}{\partial z}\right] = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = (\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x,y,z) = A\sqrt{k_1 x + k_2 y + k_3 z + B} - \frac{k_1^2 + b_1 k_2^2 + b_2 k_3^2}{a_1 k_2^2 + a_2 k_3^2},$$

where A, B, k_1, k_2, and k_3 are arbitrary constants.

3°. Solution linear in y and z:

$$w(x,y,z) = (A_1 x + B_1)y + (A_2 x + B_2)z - \tfrac{1}{12}(a_1 A_1^2 + a_2 A_2^2)x^4 - \tfrac{1}{3}(a_1 A_1 B_1 + a_2 A_2 B_2)x^3 - \tfrac{1}{2}(a_1 B_1^2 + a_2 B_2^2)x^2 + Cx + D,$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. There is a generalized separable solution of the form

$$w(x,y,z) = f(x)y^2 + g(x)yz + h(x)z^2 + \varphi(x)y + \psi(x)z + \chi(x).$$

5°. For other solutions, see equation 10.3.2.3 with $f(w) = 1$, $g(w) = a_1 w + b_1$, and $h(w) = a_2 w + b_2$.

3. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}+b\dfrac{\partial}{\partial z}\left(w^n\dfrac{\partial w}{\partial z}\right)=0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2x+C_3,\pm C_2y+C_4,\pm C_1^nC_2z+C_5),$$
$$w_2 = w(x\cos\beta+ya^{-1/2}\sin\beta,-xa^{1/2}\sin\beta+y\cos\beta,\ z),$$

where $C_1,\ldots,C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z)=z^{\frac{1}{n+1}}[C_1(ax^2-y^2)+C_2x+C_3y+C_4],$$
$$w(x,y,z)=z^{\frac{1}{n+1}}[C_1\ln(ax^2+y^2)+C_2],$$
$$w(x,y,z)=C_1z^{\frac{1}{n+1}}\exp(\sqrt{a}C_2x)\cos(C_2y+C_3),$$

where $C_1,\ldots,C_4$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x,y,z)=z^{\frac{1}{n+1}}U(x,\eta),\qquad \eta=a^{-1/2}y,$$

where the function $U=U(x,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial\eta^2}=0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z)=u(x,\eta)z^{2/n},\qquad \eta=a^{-1/2}y,$$

where the function $u=u(x,\eta)$ is determined by a differential equation of the form 9.1.1.1:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial\eta^2}+\frac{2b(n+2)}{n^2}u^{n+1}=0.$$

For $n=-1$ and $n=-2$, this equation is linear.

Remark. The solutions of Items 2° to 4° are special cases of a multiplicative separable solution $w=u(x,y)\theta(z)$, where $\theta=\theta(z)$ is determined by the autonomous ordinary differential equation $(\theta^n\theta'_z)'_z=C\theta$.

5°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z)=F(z,r),\quad r=ax^2+y^2;$$
$$w(x,y,z)=x^{2\lambda}G(\xi,\eta),\quad \xi=\frac{y}{x},\quad \eta=\frac{z}{x^{n\lambda+1}};$$
$$w(x,y,z)=|x|^{-2/n}H(z,\zeta),\quad \zeta=y/x;$$
$$w(x,y,z)=|z|^{2/n}U(t_1,t_2),\quad t_1=x+k_1\ln|z|,\quad t_2=y+k_2\ln|z|;$$
$$w(x,y,z)=\exp\left(-\frac{2z}{n+1}\right)V(\rho_1,\rho_2),\quad \rho_1=x\exp\left(-\frac{nz}{n+1}\right),\quad \rho_2=y\exp\left(-\frac{nz}{n+1}\right),$$

where k_1, k_2, and λ are arbitrary constants.

6°. There are solutions of the following forms:

$$w(x,y,z) = W(\zeta), \quad \zeta = (ax^2 + y^2)z^{-2};$$
$$w(x,y,z) = S(r)z^{2/n}, \quad r = ax^2 + y^2.$$

7°. For other solutions, see equation 10.1.2.5, where n should be set equal to zero and then k should be renamed n.

⊙ *References for equation* 10.1.2.3: N. Ibragimov (1994, p. 224), A. D. Polyanin and V. F. Zaitsev (2004, pp. 408–409).

4. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right) + b\dfrac{\partial}{\partial z}\left(w^n\dfrac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2 x + C_3, \pm C_1^n C_2 y + C_4, \pm C_1^n C_2 z + C_5),$$
$$w_2 = w\big(x,\ y\cos\beta + z\sqrt{a/b}\,\sin\beta, -y\sqrt{b/a}\,\sin\beta + z\cos\beta\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Degenerate solutions:

$$w(x,y,z) = x\big[C_1(by^2 - ax^2) + C_2 x + C_3 y + C_4\big]^{\frac{1}{n+1}},$$
$$w(x,y,z) = x\big[C_1\ln\big(by^2 + az^2\big) + C_2\big]^{\frac{1}{n+1}},$$
$$w(x,y,z) = x\big[C_1\exp\big(\lambda\sqrt{b}\,y\big)\sin\big(\lambda\sqrt{a}\,z + C_2\big) + C_3\big]^{\frac{1}{n+1}},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x,y,z) = (C_1 x + C_2)\big[U(\xi,\eta)\big]^{\frac{1}{n+1}}, \quad \xi = \sqrt{b}\,y, \quad \eta = \sqrt{a}\,z,$$

where the function $U = U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z) = x^{-2/n}\Theta(y,z),$$

where the function $\Theta = \Theta(y,z)$ is determined by the differential equation

$$a\frac{\partial}{\partial y}\left(\Theta^n\frac{\partial\Theta}{\partial y}\right) + b\frac{\partial}{\partial z}\left(\Theta^n\frac{\partial\Theta}{\partial z}\right) + \frac{2(n+2)}{n^2}\Theta = 0.$$

For $n = -2$, the equation obtained can be reduced, with the transformation $u = 1/\Theta$, $\xi = \sqrt{b}\,y$, $\eta = \sqrt{a}\,z$, to the Laplace equation.

Remark. The solutions of Items 2° to 4° are special cases of a multiplicative separable solution $w = \varphi(x)u(y,z)$, where $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation $\varphi''_{zz} = C\varphi^{n+1}$.

5°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z) = F(x,r), \quad r = by^2 + az^2;$$
$$w(x,y,z) = x^{2\lambda}G(\xi,\eta), \quad \xi = \frac{y}{x^{n\lambda+1}}, \quad \eta = \frac{z}{x^{n\lambda+1}};$$
$$w(x,y,z) = z^{2/n}H(x,\zeta), \quad \zeta = z/y;$$
$$w(x,y,z) = |x|^{-2/n}U(z_1,z_2), \quad z_1 = y + k_1\ln|x|, \quad z_2 = z + k_2\ln|x|;$$
$$w(x,y,z) = e^{-2x}V(\rho_1,\rho_2), \quad \rho_1 = ye^{nx}, \quad \rho_2 = ze^{nx};$$
$$w(x,y,z) = W(\theta), \quad \theta = (by^2 + az^2)x^{-2},$$

where k_1, k_2, and λ are arbitrary constants.

6°. For other solutions, see equation 10.1.2.5 with $k = n$.

⊙ *References for equation* 10.1.2.4: N. Ibragimov (1994, p. 223), A. D. Polyanin and V. F. Zaitsev (2004, pp. 411–412).

5. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial}{\partial y}\left(w^n\dfrac{\partial w}{\partial y}\right) + b\dfrac{\partial}{\partial z}\left(w^k\dfrac{\partial w}{\partial z}\right) = 0.$

This is a special case of equation 10.3.2.3 with $f(w) = 1$, $g(w) = aw^n$, and $h(w) = bw^k$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2x + C_3, \pm C_1^nC_2y + C_4, \pm C_1^kC_2z + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\beta_1^2 w + \frac{a\beta_2^2}{n+1}w^{n+1} + \frac{b\beta_3^2}{k+1}w^{k+1} = C_1(\beta_1x + \beta_2y + \beta_3z) + C_2,$$

where C_1, C_2, β_1, β_2, and β_3 are arbitrary constants.

3°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,z) = u(x,\xi), \quad \xi = c_1y + c_2z,$$

where the function $u = u(x,\xi)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial\xi}\left[\varphi(u)\frac{\partial u}{\partial\xi}\right] = 0, \quad \varphi(u) = ac_1^2u^n + bc_2^2u^k,$$

which can be reduced to a linear equation.

4°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,z) = v(y,\eta), \quad \eta = s_1x + s_2z,$$

where the function $v = v(y,\eta)$ is determined by a differential equation of the form 9.4.4.8:

$$a\frac{\partial}{\partial y}\left(v^n\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial\eta}\left[\psi(v)\frac{\partial v}{\partial\eta}\right] = 0, \quad \psi(v) = s_1^2 + bs_2^2v^k,$$

which can be reduced to a linear equation.

5°. There is a "two-dimensional" solution of the form (generalize the solutions of Items 3° and 4°):

$$w(x,y,z)=U(z_1,z_2),\quad z_1=a_1y+b_1z+c_1x,\quad z_2=a_2y+b_2z+c_2x.$$

6°. There are exact solutions of the following forms:

$w(x,y,z)=x^{2\lambda}F(\xi,\eta),\quad \xi=\dfrac{y}{x^{n\lambda+1}},\quad \eta=\dfrac{z}{x^{k\lambda+1}}$ "two-dimensional" solution;

$w(x,y,z)=y^{2/n}G(\zeta,x),\quad \zeta=y^{-k/n}z$ "two-dimensional" solution;

$w(x,y,z)=e^{-2x}H(z_1,z_2),\quad z_1=ye^{nx},\quad z_2=ze^{kx}$ "two-dimensional" solution;

$w(x,y,z)=(y/x)^{2/n}U(\theta),\quad \theta=x^{k/n-1}y^{-k/n}z$ "one-dimensional" solution,

where λ is an arbitrary constant.

⊙ *References for equation* 10.1.2.5: N. Ibragimov (1994, p. 224), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2004, pp. 412–413).

6. $\dfrac{\partial}{\partial y}\left(w^{n_1}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(w^{n_2}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(w^{n_3}\dfrac{\partial w}{\partial z}\right)=0.$

This is a special case of equation 10.3.2.3 with $f(w)=a_1w^{n_1}$, $g(w)=a_2w^{n_2}$, and $h(w)=a_3w^{n_3}$.

10.2. Equations with Three Space Variables Involving Exponential Nonlinearities

10.2.1. Equations of the Form $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[h(z)\dfrac{\partial w}{\partial z}\right]=ae^{\lambda w}$

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right)=se^{\lambda w}.$

This is a special case of equation 10.3.1.3 with $f(w)=se^{\lambda w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(C_1^{\frac{2}{2-n}}x,\,C_1^{\frac{2}{2-m}}y,\,C_1^{\frac{2}{2-k}}y\Big)+\frac{2}{\lambda}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$:

$$w=-\frac{1}{\lambda}\ln\left\{\frac{s\lambda}{A}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right]\right\},\qquad A=1-\frac{1}{2-n}-\frac{1}{2-m}-\frac{1}{2-k}.$$

3°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{B}{r}w'_r = se^{\lambda w}, \qquad B = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k} - 1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi,z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2},$$
$$w(x,y,z) = V(x,\eta), \quad \eta^2 = \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2},$$
$$w(x,y,z) = W(y,\zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{z^{2-k}}{c(2-k)^2},$$
$$w(x,y,z) = F(\rho_1,\rho_2) + \frac{n-2}{\lambda}\ln x, \quad \rho_1 = yx^{\frac{n-2}{2-m}}, \quad \rho_2 = zx^{\frac{n-2}{2-k}}.$$

2. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right) = se^{\sigma w}.$

This is a special case of equation 10.3.1.5 with $f(w) = se^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ z - \frac{2}{\lambda}\ln C_1\Big) + \frac{2}{\sigma}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\sigma}\ln\left\{\frac{s\sigma}{A}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right]\right\}, \qquad A = 1 - \frac{1}{2-n} - \frac{1}{2-m}.$$

3°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{B}{r}w'_r = se^{\sigma w}, \qquad B = \frac{2}{2-n} + \frac{2}{2-m} - 1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi,z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2},$$
$$w(x,y,z) = V(x,\eta), \quad \eta^2 = \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = W(y,\zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = F(\rho_1,\rho_2) + \frac{n-2}{\sigma}\ln x, \quad \rho_1 = yx^{\frac{n-2}{2-m}}, \quad \rho_2 = z + \frac{2-n}{\lambda}\ln x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=se^{\sigma w}.$

This is a special case of equation 10.3.1.6 with $f(w)=se^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(C_1^{\frac{2}{2-n}}x,\ y-\frac{2}{\beta}\ln C_1,\ z-\frac{2}{\lambda}\ln C_1\Big)+\frac{2}{\sigma}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$:

$$w=-\frac{1}{\sigma}\ln\left\{\frac{s\sigma(2-n)}{1-n}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]\right\}.$$

3°. Functional separable solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=se^{\sigma w},\qquad A=\frac{n}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z)=U(\xi,z),\quad \xi^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2},$$
$$w(x,y,z)=V(x,\eta),\quad \eta^2=\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=W(y,\zeta),\quad \zeta^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=F(\rho_1,\rho_2)+\frac{n-2}{\sigma}\ln x,\quad \rho_1=y+\frac{2-n}{\beta}\ln x,\quad \rho_2=z+\frac{2-n}{\lambda}\ln x.$$

4. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=se^{\sigma w}.$

This is a special case of equation 10.3.1.4 with $f(w)=se^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(x-\frac{2}{\beta}\ln C_1,\ y-\frac{2}{\gamma}\ln C_1,\ z-\frac{2}{\lambda}\ln C_1\Big)+\frac{2}{\sigma}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$:

$$w = -\frac{1}{\sigma}\ln\left[s\sigma\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right)\right].$$

3°. Functional separable solution for $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} - \frac{1}{r}w'_r = se^{\sigma w}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi,z), \quad \xi^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2},$$

$$w(x,y,z) = V(x,\eta), \quad \eta^2 = \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = W(y,\zeta), \quad \zeta^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = F(\rho_1,\rho_2) + \frac{\beta}{\sigma}x, \quad \rho_1 = y - \frac{\beta}{\gamma}x, \quad \rho_2 = z - \frac{\beta}{\lambda}x.$$

10.2.2. Equations of the Form $a_1\frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right) + a_2\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) + a_3\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) = be^{\beta w}$

1. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} + b\dfrac{\partial}{\partial z}\left(e^{w}\dfrac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1x + C_3, \pm C_1y + C_4, C_2z + C_5) + \ln\frac{C_1^2}{C_2^2},$$

$$w_2 = w(x\cos\beta + ya^{-1/2}\sin\beta,\ -xa^{1/2}\sin\beta + y\cos\beta,\ z),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y,z) = C_1(ax^2 - y^2) + C_2xy + C_3x + C_4y + C_5 + \ln(C_6z + C_7),$$

$$w(x,y,z) = C_1\exp(\sqrt{a}\,C_2x)\sin(C_2y + C_3) + \ln(C_4z + C_5),$$

$$w(x,y,z) = C_1\exp(C_2y)\sin(\sqrt{a}\,C_2x + C_3) + \ln(C_4z + C_5),$$

$$w(x,y,z) = \ln\left[\frac{(C_1^2 + aC_2^2)(z + C_4)^2}{b\cosh^2(C_1x + C_2y + C_3)}\right],$$

$$w(x,y,z) = \ln\left(\frac{4aC_3}{b}\right) - 2\ln\left|(y + C_1)^2 + a(x + C_2)^2 + C_3\right| + 2\ln|z + C_4|,$$

where $C_1, \ldots, C_7$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the first three solutions of Item 2°):

$$w(x,y,z) = U(x,\eta) + \ln(C_1 z + C_2), \qquad \eta = a^{-1/2}y,$$

where C_1 and C_2 are arbitrary constants and the function $U(x,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z) = V(x,\eta) + 2\ln|z + C|, \qquad \eta = a^{-1/2}y,$$

where the function $V = V(x,\eta)$ is determined by a solvable differential equation of the form 9.2.1.1:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial \eta^2} = -2be^V.$$

5°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z) = F(r,z), \quad r = ax^2 + y^2;$$
$$w(x,y,z) = G(\xi_1,\xi_2) - 2k\ln|x|, \quad \xi_1 = yx^{-1}, \quad \xi_2 = z|x|^{k-1};$$
$$w(x,y,z) = H(\eta_1,\eta_2) + 2k\ln|z|, \quad \eta_1 = x|z|^{k-1}, \quad \eta_2 = y|z|^{k-1};$$
$$w(x,y,z) = U(\zeta_1,\zeta_2) + 2\ln|z|, \quad \zeta_1 = x + k_1\ln|z|, \quad \zeta_2 = y + k_2\ln|z|;$$
$$w(x,y,z) = V(\rho_1,\rho_2) + 2z, \quad \rho_1 = xe^z, \quad \rho_2 = ye^z;$$
$$w(x,y,z) = W(\chi), \quad \chi = (ax^2 + y^2)z^{-2},$$

where k, k_1, and k_2 are arbitrary constants.

6°. For other exact solutions, see equation 10.3.2.3 with $f(w) = 1$, $g(w) = a$, and $h(w) = be^w$.

2. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right) + b\dfrac{\partial}{\partial z}\left(e^{\lambda w}\dfrac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 x + C_3, C_2 y + C_4, C_2 z + C_5) + \frac{1}{\lambda}\ln\frac{C_1^2}{C_2^2},$$
$$w_2 = w\big(x,\ y\cos\beta + z\sqrt{a/b}\,\sin\beta,\ -y\sqrt{b/a}\,\sin\beta + z\cos\beta\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants.

2°. Solutions:

$$w(x,y,z) = C_1 x + C_2 + \frac{1}{\lambda}\ln(C_3 y + C_4 z + C_5);$$
$$w(x,y,z) = C_1 x + C_2 + \frac{1}{\lambda}\ln\big[C_3(by^2 - az^2) + C_4 yz + C_5\big];$$

$$w(x,y,z) = C_1x + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\ln(by^2 + az^2) + C_4\bigr];$$

$$w(x,y,z) = C_1x + C_2 + \sqrt{b}\,C_3y + \frac{1}{\lambda}\ln\cos\bigl(\sqrt{a}\,C_3\lambda z + C_4\bigr);$$

$$w(x,y,z) = C_1x + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\exp\bigl(\sqrt{b}\,C_4y\bigr)\cos\bigl(\sqrt{a}\,C_4z + C_5\bigr) + C_6\bigr];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{-aC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cos^2(aC_1x + C_5)}\right];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{-bC_1^2z^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cos^2(bC_1x + C_5)}\right];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{-aC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\sinh^2(aC_1x + C_5)}\right];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{-bC_1^2z^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\sinh^2(bC_1x + C_5)}\right];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{aC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cosh^2(aC_1x + C_5)}\right];$$

$$w(x,y,z) = \frac{1}{\lambda}\ln\left[\frac{bC_1^2z^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cosh^2(bC_1x + C_5)}\right],$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the first five solutions of Item 2°):

$$w(x,y,z) = C_1x + C_2 + \frac{1}{\lambda}\ln U(\xi,\eta), \qquad \xi = \frac{y}{\sqrt{a}}, \qquad \eta = \frac{z}{\sqrt{b}},$$

where C_1 and C_2 are arbitrary constants and the function $U = U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2U}{\partial\xi^2} + \frac{\partial^2U}{\partial\eta^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z) = f(x) + \frac{1}{\lambda}\ln V(\xi,\eta), \qquad \xi = \frac{y}{\sqrt{a}}, \qquad \eta = \frac{z}{\sqrt{b}},$$

where the function $f = f(x)$ is determined by the autonomous ordinary differential equation (k is an arbitrary constant)

$$f''_{xx} + ke^{\lambda f} = 0, \tag{1}$$

and the function $V = V(\xi,\eta)$ is a solution of the Poisson equation

$$\Delta V - k\lambda = 0, \qquad \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}. \tag{2}$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

The general solution of equation (1) is expressed as

$$f(x)=\begin{cases}-\dfrac{1}{\lambda}\ln\left[-\frac{1}{2}k\lambda(x+C_1)^2\right] & \text{if } k\lambda<0,\\ -\dfrac{1}{\lambda}\ln\left[-\dfrac{k\lambda}{2C_1^2}\cos^2(C_1x+C_2)\right] & \text{if } k\lambda<0,\\ -\dfrac{1}{\lambda}\ln\left[-\dfrac{k\lambda}{2C_1^2}\sinh^2(C_1x+C_2)\right] & \text{if } k\lambda<0,\\ -\dfrac{1}{\lambda}\ln\left[\dfrac{k\lambda}{2C_1^2}\cosh^2(C_1x+C_2)\right] & \text{if } k\lambda>0.\end{cases}$$

5°. There are "two-dimensional" and "one-dimensional" solutions of the following forms:

$$w(x,y,z)=F(x,\tau)+\frac{2}{\lambda}\ln|y|,\quad \tau=\frac{z}{y};$$
$$w(x,y,z)=G(x,r),\quad r=by^2+az^2;$$
$$w(x,y,z)=H(z_1,z_2)-\frac{2k}{\lambda}\ln|x|,\quad z_1=y|x|^{k-1},\quad z_2=z|x|^{k-1};$$
$$w(x,y,z)=U(\xi,\eta)-\frac{2}{\lambda}\ln|x|,\quad \xi=y+k_1\ln|x|,\quad \eta=z+k_2\ln|x|;$$
$$w(x,y,z)=V(\rho_1,\rho_2)-\frac{2}{\lambda}x,\quad \rho_1=ye^x,\quad \rho_2=ze^x;$$
$$w(x,y,z)=W(\zeta),\quad \zeta=\frac{by^2+az^2}{x^2}.$$

6°. For other exact solutions, see equation 10.3.2.3 with $f(w)=1$, $g(w)=ae^{\lambda w}$, and $h(w)=be^{\lambda w}$.

3. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial}{\partial y}\left(e^w\dfrac{\partial w}{\partial y}\right)+b\dfrac{\partial}{\partial z}\left(e^{\lambda w}\dfrac{\partial w}{\partial z}\right)=0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1x+C_3,\pm C_1C_2y+C_4,\pm C_1C_2^{\lambda}z+C_5)-2\ln|C_2|,$$

where $C_1,\ldots,C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$k_1^2w+ak_2^2e^w+bk_3^2\lambda^{-1}e^{\lambda w}=C_1(k_1x+k_2y+k_3z)+C_2,$$

where C_1, C_2, k_1, k_2, and k_3 are arbitrary constants.

3°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,z)=u(x,\xi),\quad \xi=c_1y+c_2z,$$

where the function $u=u(x,\xi)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial}{\partial\xi}\left[\varphi(u)\frac{\partial u}{\partial\xi}\right]=0,\quad \varphi(u)=ac_1^2e^u+bc_2^2e^{\lambda u},$$

which can be reduced to a linear equation.

4°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,z) = v(y,\eta), \quad \eta = s_1x + s_2z,$$

where the function $v = v(y,\eta)$ is determined by a differential equation of the form 9.4.4.8:

$$a\frac{\partial}{\partial y}\left(e^v\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right] = 0, \quad \psi(v) = bs_2^2e^{\lambda v} + s_1^2,$$

which can be reduced to a linear equation.

5°. There is a "two-dimensional" solution of the form (generalizes the solutions of Items 3° and 4°):

$$w(x,y,z) = U(z_1,z_2), \quad z_1 = a_1x + b_1y + c_1z, \quad z_2 = a_2x + b_2y + c_2z.$$

6°. There are exact solutions of the following forms:

$$w(x,y,z) = F(\xi_1,\xi_2) - 2k\ln|x|, \quad \xi_1 = y|x|^{k-1}, \quad \xi_2 = z|x|^{k\lambda-1} \quad \text{"two-dimensional"};$$
$$w(x,y,z) = G(x,\eta) + 2\ln|y|, \quad \eta = |y|^{-\lambda}z \quad \text{"two-dimensional"};$$
$$w(x,y,z) = H(\zeta_1,\zeta_2) - 2kx, \quad \zeta_1 = ye^{kx}, \quad \zeta_2 = ze^{k\lambda x} \quad \text{"two-dimensional"};$$
$$w(x,y,z) = V(\rho) + 2\ln|y/x|, \quad \rho = |x|^{\lambda-1}|y|^{-\lambda}z \quad \text{"one-dimensional"},$$

where k is an arbitrary constant.

7°. For other exact solutions, see equation 10.3.2.3 with $f(w) = 1$, $g(w) = ae^w$, and $h(w) = be^{\lambda w}$.

⊙ *References for equation* 10.2.2.3: N. Ibragimov (1994, pp. 222–223), A. D. Polyanin and V. F. Zaitsev (2004, p. 419).

4. $a_1\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + a_2\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + a_3\dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1C_2^{\lambda_1}x + C_3, \pm C_1C_2^{\lambda_2}y + C_4, \pm C_1C_2^{\lambda_3}z + C_5) - 2\ln|C_2|,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There is an exact solution of the form

$$w(x,y,z) = U(\xi) - \frac{2}{\lambda_1-\lambda_2}\ln\left|\frac{y}{x}\right|, \quad \xi = |x|^{\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_2}}|y|^{\frac{\lambda_3-\lambda_1}{\lambda_1-\lambda_2}}z.$$

3°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\eta_1,\eta_2) - 2k\ln|x|, \quad \eta_1 = y|x|^{k(\lambda_2-\lambda_1)-1}, \quad \eta_2 = z|x|^{k(\lambda_3-\lambda_1)-1},$$
$$w(x,y,z) = V(\zeta_1,\zeta_2) - 2kx, \quad \zeta_1 = y\exp\big[k(\lambda_2-\lambda_1)x\big], \quad \zeta_2 = z\exp\big[k(\lambda_3-\lambda_1)x\big],$$

where k is an arbitrary constant.

4°. For other exact solutions, see equation 10.3.2.3 with $f(w) = a_1e^{\lambda_1 w}$, $g(w) = a_2e^{\lambda_2 w}$, and $h(w) = a_3e^{\lambda_3 w}$.

5. $a_1\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right)+a_2\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right)+a_3\dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right)=be^{\beta w}.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1}x + C_2, \pm C_1^{\beta-\lambda_2}y + C_3, \pm C_1^{\beta-\lambda_3}z + C_4) + 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There is a "two-dimensional" solution of the form

$$w(x,y,z) = U(\xi,\eta) + \frac{2}{\lambda_1-\beta}\ln|x|, \quad \xi = y|x|^{\frac{\beta-\lambda_2}{\lambda_1-\beta}}, \quad \eta = z|x|^{\frac{\beta-\lambda_3}{\lambda_1-\beta}}.$$

10.3. Three-Dimensional Equations Involving Arbitrary Functions

10.3.1. Heat and Mass Transfer Equations of the Form

$$\frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[f_3(z)\frac{\partial w}{\partial z}\right]=g(w)$$

▷ *Equations of this type describe steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f_1(x)$, $f_2(y)$, and $f_3(z)$ are the principal thermal diffusivities (diffusion coefficients) dependent on coordinates, and $g = g(w)$ is the kinetic function, which defines the law of heat (substance) release or absorption.*

1. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}+c\dfrac{\partial^2 w}{\partial z^2}=f(w).$

1°. Traveling-wave solution:

$$w = w(\theta), \qquad \theta = Ax + By + Cz.$$

The function $w(\theta)$ is defined implicitly by

$$\int\left[C_1+\frac{2}{aA^2+bB^2+cC^2}F(w)\right]^{-1/2}dw = C_2 \pm \theta, \qquad F(w)=\int f(w)\,dw,$$

where A, B, C, C_1, and C_2 are arbitrary constants.

2°. Solution:

$$w = w(r), \qquad r^2 = \frac{(x+C_1)^2}{a}+\frac{(y+C_2)^2}{b}+\frac{(z+C_3)^2}{c},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{2}{r}w'_r = f(w).$$

3°. "Two-dimensional" solution:

$$w = U(\xi,\eta), \quad \xi = \frac{y}{\sqrt{b}}+\frac{x}{\sqrt{a}C}, \quad \eta = (C^2-1)\frac{x^2}{a}-2C\frac{xy}{\sqrt{ab}}+C^2\frac{z^2}{c},$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi\frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} = f(U).$$

Remark. The solution specified in Item 3° can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a) & \\ \nearrow & & \searrow \\ (z, c) & \longleftarrow & (y, b) \end{array}$$

4°. "Two-dimensional" solution:

$$w = V(\zeta, \rho), \quad \zeta = \frac{Ax}{\sqrt{a}} + \frac{By}{\sqrt{b}} + \frac{Cz}{\sqrt{c}}, \quad \rho^2 = \left(\frac{Bx}{\sqrt{a}} - \frac{Ay}{\sqrt{b}}\right)^2 + \left(\frac{Cy}{\sqrt{b}} - \frac{Bz}{\sqrt{c}}\right)^2 + \left(\frac{Az}{\sqrt{c}} - \frac{Cx}{\sqrt{a}}\right)^2,$$

where A, B, and C are arbitrary constants and the function $V = V(\zeta, \rho)$ is determined by the equation

$$\frac{\partial^2 V}{\partial \zeta^2} + \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{1}{A^2 + B^2 + C^2} f(V).$$

5°. The transformation $x = \sqrt{a}\,\bar{x}$, $y = \sqrt{b}\,\bar{y}$, $z = \sqrt{c}\,\bar{z}$ brings the original equation to the form $\Delta w = f(w)$.

2. $a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left(by^n\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^m\dfrac{\partial w}{\partial z}\right) = f(w).$

1°. For $n = m = 0$, see equation 10.3.1.1.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^2}{4a} + \frac{y^{2-n}}{b(2-n)^2} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r = f(w), \qquad A = \frac{2(4-n-m)}{(2-n)(2-m)}.$$

3°. "Two-dimensional" solution for $n \neq 2$ and $m \neq 2$:

$$w = U(x, \xi), \qquad \xi^2 = 4\left[\frac{y^{2-n}}{b(2-n)^2} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

where the function $U(x, \xi)$ is determined by the differential equation

$$a\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \xi^2} + \frac{B}{\xi}\frac{\partial U}{\partial \xi} = f(U), \qquad B = \frac{4 - nm}{(2-n)(2-m)}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w = V(y, \eta), \quad \eta^2 = 4\left[\frac{x^2}{4a} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

$$w = W(z, \zeta), \quad \zeta^2 = 4\left[\frac{x^2}{4a} + \frac{y^{2-n}}{b(2-n)^2}\right].$$

⊙ *Reference for equation* 10.3.1.2: A. D. Polyanin and A. I. Zhurov (1998).

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=f(w),\qquad A=2\left(\frac{1}{2-n}+\frac{1}{2-m}+\frac{1}{2-k}\right)-1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left[\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$
$$w=V(y,\eta),\quad \eta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$
$$w=W(z,\zeta),\quad \zeta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}\right].$$

⊙ *Reference for equation* 10.3.1.3: A. D. Polyanin and A. I. Zhurov (1998).

4. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $\beta\neq 0$, $\gamma\neq 0$, and $\lambda\neq 0$:

$$w=w(r),\qquad r^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}-\frac{1}{r}w'_r=f(w).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left(\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$
$$w=V(y,\eta),\quad \eta^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$
$$w=W(z,\zeta),\quad \zeta^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}\right).$$

⊙ *Reference for equation* 10.3.1.4: A. D. Polyanin and A. I. Zhurov (1998).

5. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $\lambda\neq 0$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=f(w),\qquad A=2\left(\frac{1}{2-n}+\frac{1}{2-m}\right)-1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left[\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w=V(y,\eta),\quad \eta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w=W(z,\zeta),\quad \zeta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}\right].$$

6. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{n}{2-n}\frac{1}{r}w'_r=f(w). \tag{1}$$

Special case 1. For $n=0$ and any $f=f(w)$, equation (1) can be solved by quadrature to obtain

$$\int\left[C_1+2\int f(w)\,dw\right]^{-1/2}dw=C_2\pm r,$$

where C_1 and C_2 are arbitrary constants.

Special case 2. For $n=1$ and $f(w)=Ae^{\beta w}$, equation (1) has the one-parameter solution

$$w(r)=\frac{1}{\beta}\ln\left(-\frac{8C}{\beta A}\right)-\frac{2}{\beta}\ln(r^2+C),$$

where C is an arbitrary constant.

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left[\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w=V(y,\eta),\quad \eta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w=W(z,\zeta),\quad \zeta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}\right].$$

⊙ *Reference for equation* 10.3.1.6: A. D. Polyanin and A. I. Zhurov (1998).

7. $\dfrac{\partial}{\partial x}\left[f_1(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(y)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[f_3(z)\dfrac{\partial w}{\partial z}\right]=aw\ln w+bw.$

This is a special case of equation 10.3.3.6 with $g_1(x)=b$ and $g_2(y)=g_3(z)=0$.

10.3.2. Heat and Mass Transfer Equations with Complicating Factors

1. $(a_1x+b_1y+c_1z+d_1)\dfrac{\partial w}{\partial x}+(a_2x+b_2y+c_2z+d_2)\dfrac{\partial w}{\partial y}$

$$+(a_3x+b_3y+c_3z+d_3)\frac{\partial w}{\partial z}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}-f(w).$$

This equation describes steady-state mass transfer with a volume chemical reaction in a three-dimensional translational-shear fluid flow.

Let k be a root of the cubic equation

$$\begin{vmatrix} a_1-k & a_2 & a_3 \\ b_1 & b_2-k & b_3 \\ c_1 & c_2 & c_3-k \end{vmatrix}=0,$$

and the constants A, B, and C solve the degenerate system of linear algebraic equations

$$\begin{aligned}(a_1-k)A+a_2B+a_3C&=0,\\ b_1A+(b_2-k)B+b_3C&=0,\\ c_1A+c_2B+(c_3-k)C&=0.\end{aligned}$$

One of the equations follows from the other two and, hence, can be omitted.

Solution:

$$w=w(\zeta),\qquad \zeta=Ax+By+Cz,$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$(k\zeta+Ad_1+Bd_2+Cd_3)w'_\zeta=(A^2+B^2+C^2)w''_{\zeta\zeta}-f(w).$$

Remark. In the case of an incompressible fluid, some of the equation coefficients must satisfy the condition $a_1+b_2+c_3=0$.

2. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1z+d_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2z+d_2)\dfrac{\partial w}{\partial y}\right]$

$$+\frac{\partial}{\partial z}\left[(a_3x+b_3y+c_3z+d_3)\frac{\partial w}{\partial z}\right]=f(w).$$

Solutions are sought in the form

$$w=w(\zeta),\qquad \zeta=Ax+By+Cz+D,$$

where the constants A, B, C, and D are determined by solving the algebraic system of equations

$$\begin{aligned}a_1A^2+a_2B^2+a_3C^2&=A,\\ b_1A^2+b_2B^2+b_3C^2&=B,\\ c_1A^2+c_2B^2+c_3C^2&=C,\\ d_1A^2+d_2B^2+d_3C^2&=D.\end{aligned}$$

The first three equations are first solved for A, B, and C. The resulting expressions are then substituted into the last equation to evaluate D. The desired function $w(\zeta)$ is determined by the ordinary differential equation

$$\zeta w''_{\zeta\zeta}+(a_1A+b_2B+c_3C)w'_\zeta=f(w).$$

3. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right]=0.$

This equation describes steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f(w)$, $g(w)$, and $h(w)$ are the principal thermal diffusivities (diffusion coefficients) dependent on the temperature w.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs in front of C_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[k_1^2 f(w) + k_2^2 g(w) + k_3^2 h(w)\right] dw = C_1(k_1 x + k_2 y + k_3 z) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

3°. Solution:

$$w = w(\theta), \quad \theta = \frac{C_1 y + C_2 z + C_3}{x + C_4}, \tag{1}$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $w(\theta)$ is determined by the ordinary differential equation

$$[\theta^2 f(w) w'_\theta]'_\theta + C_1^2 [g(w) w'_\theta]'_\theta + C_2^2 [h(w) w'_\theta]'_\theta = 0,$$

which admits the first integral

$$[\theta^2 f(w) + C_1^2 g(w) + C_2^2 h(w)] w'_\theta = C_5.$$

For $C_5 \neq 0$, treating w as the independent variable, one obtains a Riccati equation for $\theta = \theta(w)$:

$$C_5 \theta'_w = \theta^2 f(w) + C_1^2 g(w) + C_2^2 h(w). \tag{2}$$

For exact solutions of this equation, which can be reduced to a second-order linear equation; see Polyanin and Zaitsev (2003).

Relations (1) and equation (2) can be used to obtain two other "one-dimensional" solutions by means of the following cyclic permutations of variables and determining functions:

$$\begin{array}{ccc} & (x, f) & \\ \nearrow & & \searrow \\ (z, h) & \longleftarrow & (y, g) \end{array} \tag{3}$$

4°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x,y,z) = U(x,\zeta), \quad \zeta = ay + bz, \tag{4}$$

where the function $U = U(x,\zeta)$ is determined by a differential equation of the form 9.4.4.8:

$$\frac{\partial}{\partial x}\left[f(U)\frac{\partial U}{\partial x}\right]+\frac{\partial}{\partial \zeta}\left[\psi(U)\frac{\partial U}{\partial \zeta}\right]=0, \quad \psi(U) = a^2 g(U) + b^2 h(U), \tag{5}$$

which can be reduced to a linear equation.

Relations (4) and equation (5) can be used to obtain two other "two-dimensional" solutions by means of the cyclic permutations of variables and determining functions; see (3).

5°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = V(z_1, z_2), \quad z_1 = a_1x + a_2y + a_3z, \quad z_2 = b_1x + b_2y + b_3z;$$
$$w(x,y,z) = W(\xi, \eta), \quad \xi = y/x, \quad \eta = z/x,$$

where the a_n and b_n are arbitrary constants (the first solution generalizes the Items 3° and 4°).

6°. Let $g(w) = af(w)$. Then, there is a "two-dimensional" solution of the form

$$w(x,y,z) = u(r,z), \quad r = ax^2 + y^2.$$

7°. Let $g(w) = af(w)$ and $h(w) = bf(w)$. Then, the transformation

$$v = \int f(w)\,dw, \quad y = \sqrt{a}\,\overline{y}, \quad z = \sqrt{b}\,\overline{z}$$

leads to the Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \overline{y}^2} + \frac{\partial^2 v}{\partial \overline{z}^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4. $$\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(w)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[f_3(w)\frac{\partial w}{\partial z}\right]$$
$$= (a_1x+b_1y+c_1z+d_1)\frac{\partial w}{\partial x} + (a_2x+b_2y+c_2z+d_2)\frac{\partial w}{\partial y} + (a_3x+b_3y+c_3z+d_3)\frac{\partial w}{\partial z}.$$

This equation describes steady-state anisotropic heat/mass transfer with a volume chemical reaction in a three-dimensional translational-shear fluid flow.

Let k be a root of the cubic equation

$$\begin{vmatrix} a_1 - k & a_2 & a_3 \\ b_1 & b_2 - k & b_3 \\ c_1 & c_2 & c_3 - k \end{vmatrix} = 0,$$

and the constants A, B, and C solve the degenerate system of linear algebraic equations

$$(a_1 - k)A + a_2B + a_3C = 0,$$
$$b_1A + (b_2 - k)B + b_3C = 0,$$
$$c_1A + c_2B + (c_3 - k)C = 0.$$

One of the equations follows from the other two and, hence, can be omitted.

Solution:

$$w = w(\zeta), \quad \zeta = Ax + By + Cz, \tag{1}$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$[\varphi(w)w'_\zeta]'_\zeta = (k\zeta + Ad_1 + Bd_2 + Cd_3)w'_\zeta,$$
$$\varphi(w) = A^2f_1(w) + B^2f_2(w) + C^2f_3(w).$$

Remark 1. A more general equation, with an additional term $g(w)$ on the right-hand side, where g is an arbitrary function, also has a solution of the form (1).

Remark 2. In the case of an incompressible fluid, some of the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

10.3.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}=f(w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial z}\right)^2\right].$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad \text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the three-dimensional Laplace equation for $U=U(x,y,z)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. For a more complicated equation of the form $(\mathbf{v}\cdot\nabla)w=\Delta w-f(w)|\nabla w|^2$, with an additional convective term, see 10.4.1.1.

2. $ax^n\dfrac{\partial^2 w}{\partial x^2}+by^m\dfrac{\partial^2 w}{\partial y^2}+cz^k\dfrac{\partial^2 w}{\partial z^2}=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=f(w),\qquad A=5-2\left(\frac{1}{2-n}+\frac{1}{2-m}+\frac{1}{2-k}\right).$$

2°. There is a "two-dimensional" solution of the form

$$w(x,y,z)=U(x,\rho),\qquad \rho^2=4\left[\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right].$$

This solution can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a,n) & \\ \nearrow & & \searrow \\ (z,c,k) & \longleftarrow & (y,b,m) \end{array}$$

3. $ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2}+be^{\mu y}\dfrac{\partial^2 w}{\partial y^2}+ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2}=f(w).$

1°. Functional separable solution for $\lambda\neq 0$, $\mu\neq 0$, and $\nu\neq 0$:

$$w=w(r),\qquad r^2=4\left(\frac{e^{-\lambda x}}{a\lambda^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{5}{r}w'_r = f(w).$$

2°. There is a "two-dimensional" solution of the form

$$w(x,y,z) = U(x,\xi), \qquad \xi^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

This solution can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of the variables and determining parameters:

$$\begin{array}{ccc} & (x,a,\lambda) & \\ \nearrow & & \searrow \\ (z,c,\nu) & \longleftarrow & (y,b,\mu) \end{array}$$

4. $$ax^n\frac{\partial^2 w}{\partial x^2} + by^m\frac{\partial^2 w}{\partial y^2} + ce^{\nu z}\frac{\partial^2 w}{\partial z^2} = f(w).$$

1°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r = f(w), \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m}\right) + 1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(x,\xi), \quad \xi^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = V(y,\eta), \quad \eta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = W(z,\zeta), \quad \zeta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

5. $$ax^n\frac{\partial^2 w}{\partial x^2} + be^{\mu y}\frac{\partial^2 w}{\partial y^2} + ce^{\nu z}\frac{\partial^2 w}{\partial z^2} = f(w).$$

1°. Functional separable solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{8-5n}{2-n}\frac{1}{r}w'_r = f(w).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(x,\xi), \quad \xi^2 = 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = V(y,\eta), \quad \eta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = W(z,\zeta), \quad \zeta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2}\right].$$

6. $\dfrac{\partial}{\partial x}\left[f_1(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(y)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[f_3(z)\dfrac{\partial w}{\partial z}\right]$
$= aw\ln w+\big[g_1(x)+g_2(y)+g_3(z)\big]w.$

Multiplicative separable solution:

$$w(x,y,z)=\varphi(x)\psi(y)\chi(z),$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(y)$, and $\chi=\chi(z)$ are determined by the ordinary differential equations (C_1 and C_2 are arbitrary constants)

$$[f_1(x)\varphi'_x]'_x-a\varphi\ln\varphi-[g_1(x)+C_1]\varphi=0,$$
$$[f_2(y)\psi'_y]'_y-a\psi\ln\psi-[g_2(y)+C_2]\psi=0,$$
$$[f_3(z)\chi'_z]'_z-a\chi\ln\chi-[g_3(z)-C_1-C_2]\chi=0.$$

10.4. Equations with n Independent Variables

10.4.1. Equations of the Form $\dfrac{\partial}{\partial x_1}\left[f_1(x_1)\dfrac{\partial w}{\partial x_1}\right]+\cdots+\dfrac{\partial}{\partial x_n}\left[f_n(x_n)\dfrac{\partial w}{\partial x_n}\right]=g(x_1,\ldots,x_n,w)$

1. $\displaystyle\sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}=f(w)\sum_{k=1}^{n}\left(\frac{\partial w}{\partial x_k}\right)^2+\sum_{k=1}^{n}g_k(x_1,\ldots,x_n)\frac{\partial w}{\partial x_k}.$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad\text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the linear equation

$$\sum_{k=1}^{n}\frac{\partial^2 U}{\partial x_k^2}=\sum_{k=1}^{n}g_k(x_1,\ldots,x_n)\frac{\partial U}{\partial x_k}.$$

2. $\displaystyle\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(a_k x_k^{m_k}\frac{\partial w}{\partial x_k}\right)=f(w).$

Functional separable solution:

$$w=w(r),\qquad r^2=A\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2}+\frac{B}{r}\frac{dw}{dr}=\frac{4}{A}f(w),\qquad B=\sum_{k=1}^{n}\frac{2}{2-m_k}-1.$$

Special case 1. For $f(w) = bw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{b(1-p)}\left(\frac{p}{1-p} + \sum_{k=1}^{n}\frac{1}{2-m_k}\right)\right]^{\frac{1}{p-1}}\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right]^{\frac{1}{1-p}}.$$

Special case 2. For $f(w) = be^{\lambda w}$, there is an exact solution of the form

$$w = -\frac{1}{\lambda}\ln\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right] + \frac{1}{\lambda}\ln\frac{1-B}{2b\lambda}, \qquad B = \sum_{k=1}^{n}\frac{2}{2-m_k} - 1.$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

3. $\displaystyle\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(a_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k}\right) = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr} = \frac{4}{A}f(w).$$

Special case 1. For $f(w) = bw^p$, there is an exact solution of the form

$$w = \left[\frac{p}{b(1-p)^2}\right]^{\frac{1}{p-1}}\left(\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2}\right)^{\frac{1}{1-p}}.$$

Special case 2. For $f(w) = be^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta}\ln\left(b\beta\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2}\right).$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

4. $\displaystyle\sum_{k=1}^{s}\frac{\partial}{\partial x_k}\left(a_k x_k^{m_k}\frac{\partial w}{\partial x_k}\right) + \sum_{k=s+1}^{n}\frac{\partial}{\partial x_k}\left(b_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k}\right) = f(w).$

1°. Functional separable solution:

$$w = w(r), \qquad r^2 = A\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2} + A\sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A}f(w), \qquad B = \sum_{k=1}^{s}\frac{2}{2-m_k} - 1.$$

Special case 1. For $f(w) = cw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{c(1-p)}\left(\frac{p}{1-p} + \sum_{k=1}^{s}\frac{1}{2-m_k}\right)\right]^{\frac{1}{p-1}}\left[\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2} + \sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}\right]^{\frac{1}{1-p}}.$$

Special case 2. For $f(w) = ce^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta}\ln\left[\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2} + \sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}\right] + \frac{1}{\beta}\ln\frac{1-B}{2c\beta}, \qquad B = \sum_{k=1}^{s}\frac{2}{2-m_k} - 1.$$

2°. We divide the equation variables into two groups (responsible for both power-law and exponential terms) and look for exact solutions in the form

$$w = w(y, z),$$

where

$$y^2 = A_1\sum_{k=1}^{q}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2} + A_1\sum_{k=s+1}^{p}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}, \qquad 0 \le q \le s \le p \le n;$$
$$z^2 = A_2\sum_{k=q+1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2} + A_2\sum_{k=p+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}.$$

Then we obtain the following equation for w:

$$A_1\left(\frac{\partial^2 w}{\partial y^2} + \frac{B_1}{y}\frac{\partial w}{\partial y}\right) + A_2\left(\frac{\partial^2 w}{\partial z^2} + \frac{B_2}{z}\frac{\partial w}{\partial z}\right) = 4f(w),$$
$$B_1 = \sum_{k=1}^{q}\frac{2}{2-m_k} - 1, \quad B_2 = \sum_{k=q+1}^{s}\frac{2}{2-m_k} - 1.$$

For $B_1 = B_2 = 0$ and $A_1 = A_2 = 1$, this equation arises in plane problems of heat and mass transfer (see equations 9.1.1.1, 9.2.1.1, 9.3.1.1, 9.3.2.1, 9.3.3.1, and 9.4.1.1).

⊙ *Reference for equation* 10.4.1.4: A. D. Polyanin and A. I. Zhurov (1998).

5. $\displaystyle\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left[f_k(x_k)\frac{\partial w}{\partial x_k}\right] = aw\ln w + w\sum_{k=1}^{n}g_k(x_k).$

Multiplicative separable solution:

$$w(x_1, x_2, \ldots, x_n) = \varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_n(x_n),$$

where the functions $\varphi_1 = \varphi_1(x_1)$, $\varphi_2 = \varphi_2(x_2)$, $\ldots$, $\varphi_n = \varphi_n(x_n)$ are determined by the ordinary differential equations

$$\frac{d}{dx_k}\left[f_k(x_k)\frac{d\varphi_k}{dx_k}\right] - a\varphi_k\ln\varphi_k - \left[g_k(x_k) + C_k\right]\varphi_k = 0; \qquad k = 1, 2, \ldots, n.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by a single constraint, $C_1 + \cdots + C_n = 0$.

10.4.2. Other Equations

1. $\displaystyle\sum_{k=1}^{n} f_k(x_k)\frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} g_k(x_k)\frac{\partial w}{\partial x_k} = aw\ln w + w\sum_{k=1}^{n} h_k(x_k).$

Multiplicative separable solution:

$$w(x_1, x_2, \ldots, x_n) = \varphi_1(x_1)\varphi_2(x_2)\ldots\varphi_n(x_n),$$

where the functions $\varphi_1 = \varphi_1(x_1)$, $\varphi_2 = \varphi_2(x_2)$, $\ldots$, $\varphi_n = \varphi_n(x_n)$ are determined by the ordinary differential equations

$$f_k(x_k)\frac{d^2\varphi_k}{dx_k^2} + g_k(x_k)\frac{d\varphi_k}{dx_k} - a\varphi_k\ln\varphi_k - \big[h_k(x_k) + C_k\big]\varphi_k = 0; \qquad k = 1, 2, \ldots, n.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by a single constraint, $C_1 + \cdots + C_n = 0$.

2. $\displaystyle\sum_{k=1}^{n} a_k x_k^{m_k}\frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} b_k x_k^{m_k-1}\frac{\partial w}{\partial x_k} = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A}f(w), \qquad B = 2\sum_{k=1}^{n}\frac{a_k(1-m_k)+b_k}{a_k(2-m_k)} - 1.$$

Special case 1. For $f(w) = cw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{2c(1-p)}\left(\frac{1+p}{1-p} + B\right)\right]^{\frac{1}{p-1}}\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right]^{\frac{1}{1-p}}.$$

Special case 2. For $f(w) = ce^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta}\ln\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right] + \frac{1}{\beta}\ln\frac{1-B}{2c\beta}.$$

3. $\displaystyle\sum_{k=1}^{n} a_k e^{\lambda_k x_k}\frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} b_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k} = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A}f(w), \qquad B = 2n - 1 - 2\sum_{k=1}^{n}\frac{b_k}{a_k\lambda_k}.$$

Special case 1. For $f(w) = cw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{2c(1-p)}\left(\frac{1+p}{1-p} + B\right)\right]^{\frac{1}{p-1}} \left(\sum_{k=1}^{n} \frac{e^{-\lambda_k x_k}}{a_k \lambda_k^2}\right)^{\frac{1}{1-p}}.$$

Special case 2. For $f(w) = ce^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta} \ln\left(\sum_{k=1}^{n} \frac{e^{-\lambda_k x_k}}{a_k \lambda_k^2}\right) + \frac{1}{\beta} \ln \frac{1-B}{2c\beta}.$$

4. $$\sum_{k=1}^{m_1}\left(a_k x_k^{n_k} \frac{\partial^2 w}{\partial x_k^2} + a_k p_k x_k^{n_k-1} \frac{\partial w}{\partial x_k}\right) + \sum_{k=1}^{m_2}\left(b_k e^{\lambda_k x_k} \frac{\partial^2 w}{\partial x_k^2} + b_k q_k e^{\lambda_k x_k} \frac{\partial w}{\partial x_k}\right) = f(w).$$

Functional separable solution:

$$w = w(r), \qquad r^2 = A\sum_{k=1}^{m_1} \frac{x_k^{2-n_k}}{a_k(2-n_k)^2} + A\sum_{k=1}^{m_2} \frac{e^{-\lambda_k x_k}}{b_k \lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2 w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A} f(w), \qquad B = 2\sum_{k=1}^{m_1} \frac{1-n_k+p_k}{2-n_k} - 2\sum_{k=1}^{m_2} \frac{q_k}{\lambda_k} + 2m_2 - 1.$$

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Chapter 11
Second-Order Equations Involving Mixed Derivatives and Some Other Equations

Preliminary remarks. Semilinear equations with mixed derivatives, which can be reduced to the canonical form by standard transformations (see Section 25.1), are not considered in this chapter. For hyperbolic equations of the form $\frac{\partial^2 w}{\partial x \partial y} = F\big(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\big)$ that contain mixed derivatives, see Section 7.5.

11.1. Equations Linear in the Mixed Derivative

11.1.1. Calogero Equation and Related Equations

1. $\dfrac{\partial^2 w}{\partial x \partial t} = w\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 11.1.1.5 with $f(u) = 0$.

General solution:

$$w(x,t) = \int_a^{\Phi} F(\xi)e^{-t/\xi}d\xi + \frac{dG(t)}{dt},$$

where $\Phi = \Phi(x,t)$ is defined implicitly by

$$x - \int_a^{\Phi} \xi F(\xi)e^{-t/\xi}d\xi + G(t) = 0,$$

and $F(\xi)$ and $G(t)$ are arbitrary functions, and a is an arbitrary constant.

⊙ Yu. N. Kosovtsov (2007, from *EqWorld*: http://eqworld.ipmnet.ru/eqarchive/view.php?id=86).

2. $\dfrac{\partial^2 w}{\partial x \partial t} = w\dfrac{\partial^2 w}{\partial x^2} + a.$

This is a special case of equation 11.1.1.5 with $f(u) = a$.

3. $\dfrac{\partial^2 w}{\partial x \partial t} + w\dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 = 0.$

Hunter–Saxton equation. It is an integrable equation that arises in the theoretical study of nematic liquid crystals. If the molecules in the liquid crystal are initially all aligned, and some of them are then wiggled slightly, this disturbance in orientation will propagate through the crystal, and this equation describes certain aspects of such orientation waves.

The Hunter–Saxton equation is often written in the form $(w_t + ww_x)_x = \frac{1}{2}w_x^2$.
The substitution $w = -u$ leads to an equation of the form 11.1.1.5 with $a = \frac{1}{2}$:

$$\frac{\partial^2 u}{\partial x \partial t} = u\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2.$$

⊙ *References*: J. K. Hunter and R. Saxton (1991), J. K. Hunter and Y. Zheng (1994, 1995 *a, b*), P. G. Gennes and J. Prost (1994), R. Beals, D. H. Sattinger and J. Szmigielski (2001), A. Bressan and A. Constantin (2005), J. Lenells (2007).

4. $$\frac{\partial^2 w}{\partial x \partial t} = w\frac{\partial^2 w}{\partial x^2} + a\left(\frac{\partial w}{\partial x}\right)^2.$$

This is a special case of equation 11.1.1.5 with $f(u) = au^2$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_2\varphi(t), C_1 C_2 t + C_3) + \varphi_t'(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. General solution in parametric form:

$$w = f_t'(t) + \int \big[g(z) - at\big]^{\frac{1-a}{a}}\, dz,$$
$$x = -f(t) + \int \big[g(z) - at\big]^{\frac{1}{a}}\, dz,$$

where $f(t)$ and $g(z)$ are arbitrary functions and z is the parameter.

3°. Conservation laws:

$$D_t\big[(w_x)^{1/a}\big] + D_x\big[-w(w_x)^{1/a}\big] = 0,$$
$$D_t\big[(w_{xx})^{\frac{1}{2a+1}}\big] + D_x\big[-w(w_{xx})^{\frac{1}{2a+1}}\big] = 0.$$

⊙ *References for equation* 11.1.1.4: F. Calogero (1984), J. K. Hunter and R. Saxton (1991), M. V. Pavlov (2001).

5. $$\frac{\partial^2 w}{\partial x \partial t} = w\frac{\partial^2 w}{\partial x^2} + f\left(\frac{\partial w}{\partial x}\right).$$

Calogero equation. This is special case of equations 11.1.1.6 and 11.1.1.8.

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_1\varphi(t), t + C_2) + \varphi_t'(t),$$

where C_1 and C_2 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function and the function $\varphi(t)$ is defined implicitly by (C is an arbitrary constant)

$$\int \frac{d\varphi}{f(\varphi)} = t + C.$$

3°. Introduce the notation

$$u = \frac{\partial w}{\partial x}, \quad v = \Phi(u) = \exp\left[\int \frac{u\,du}{f(u)}\right]. \tag{1}$$

The transformation

$$dz = v\,dx + vw\,dt, \quad dy = dt \qquad \left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt\right) \tag{2}$$

defines the passage from x, t to the new independent variables z, y in accordance with the rule

$$\frac{\partial}{\partial x} = v\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial y} + vw\frac{\partial}{\partial z}. \tag{3}$$

This results in the first-order equation

$$\frac{\partial u}{\partial y} = f(u),$$

which is independent of z and can be treated as an ordinary differential equation. Integrating yields its solution in implicit form:

$$\int \frac{du}{f(u)} = y + \varphi(z), \tag{4}$$

where $\varphi(z)$ is an arbitrary function. With the first relations of (1) and (3), we obtain the equation

$$\frac{\partial w}{\partial z} = \frac{u}{v} \quad \Longrightarrow \quad \frac{\partial w}{\partial z} = \frac{u}{\Phi(u)},$$

whose general solution is given by

$$w = \int \frac{u\,dz}{\Phi(u)} + \psi(y), \tag{5}$$

where $\psi(y)$ is an arbitrary function, and $u = u(z,y)$ is defined implicitly by (4). The inverse of transformation (2) has the form

$$dx = \frac{1}{\Phi(u)}\,dz - w\,dy, \quad dt = dy. \tag{6}$$

Integrating the first relation in (6) yields

$$x = \int_{z_0}^{z} \frac{d\xi}{\Phi(u(\xi,y))} - \int_{y_0}^{y} w(z_0,\tau)\,d\tau, \tag{7}$$

where $w = w(z,y)$ is defined by (5), and x_0 and y_0 are any numbers.

Formulas (4), (5), and (7) with $y = t$ define the general solution of the equation in question in parametric form (z is the parameter).

4°. Conservation law:

$$D_t[\Phi(w_x)] + D_x[-w\Phi(w_x)] = 0,$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$, and the function $\Phi(u)$ is defined in (1).

⊙ *References for equation* 11.1.1.5: F. Calogero (1984), M. V. Pavlov (2001), H. C. Lei (2002), X. Yonga, Z. Zhang, and Y. Chen (2008).

6. $\dfrac{\partial^2 w}{\partial x \partial t} + [a(t)w + b(t)x]\dfrac{\partial^2 w}{\partial x^2} = F\left(t, \dfrac{\partial w}{\partial x}\right).$

By changing from the original variables to the Crocco variables

$$t,\ x,\ w = w(t,x) \quad \Longrightarrow \quad t,\ \eta,\ \Phi = \Phi(t,\eta), \quad \text{where} \quad \eta = \frac{\partial w}{\partial x}, \quad \Phi = \frac{\partial^2 w}{\partial x^2},$$

one arrives at the first-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi}F(t,\eta)\right],$$

which becomes linear after substituting $\Phi = 1/\Psi$.

For details, see Example 26.40 in Subsection 26.6.1.

⊙ *References*: A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov and A. D. Polyanin (2010).

7. $\dfrac{\partial^2 w}{\partial t \partial x} = \left[F\left(t, \dfrac{\partial w}{\partial x}\right) - a(t)w - b(t)x\right]\dfrac{\partial^2 w}{\partial x^2} + G\left(t, \dfrac{\partial w}{\partial x}\right).$

Generalized Calogero equation.

By changing from the original variables to the Crocco variables

$$t,\ x,\ w = w(t,x) \quad \Longrightarrow \quad t,\ \eta,\ \Phi = \Phi(t,\eta), \quad \text{where} \quad \eta = \frac{\partial w}{\partial x}, \quad \Phi = \frac{\partial^2 w}{\partial x^2},$$

one arrives at the first-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[F(t,\eta) + \frac{G(t,\eta)}{\Phi}\right],$$

which becomes linear after substituting $\Phi = 1/\Psi$.

For details, see Example 26.40 in Subsection 26.6.1.

⊙ *References for equation* 11.1.1.7: A. D. Polyanin and S. N. Aristov (2009), A. D. Polyanin and A. I. Zhurov (2009 *b*).

8. $\dfrac{\partial^2 w}{\partial x \partial t} = wF\left(t, \dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2} + G\left(t, \dfrac{\partial w}{\partial x}\right).$

By setting $v = w_x$, let us rewrite the nonlinear equation as a system of equations:

$$v = \frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial t} = wF(t,v)\frac{\partial v}{\partial x} + G(t,v).$$

Using the hodograph transformation

$$x = x(v,t), \qquad w = w(v,t),$$

with v and t treated as independent variables and x and w treated as the dependent variables, and expressing the "old" derivatives through the "new" ones,

$$v_x = \frac{1}{x_v}, \quad v_t = -\frac{x_t}{x_v}, \quad w_x = \frac{w_v}{x_v}, \quad w_t = w_t - \frac{x_t}{x_v}w_v,$$

we obtain the linear system of equations

$$v\frac{\partial x}{\partial v} = \frac{\partial w}{\partial v}, \qquad \frac{\partial x}{\partial t} + G(t,v)\frac{\partial x}{\partial v} + F(t,v)w = 0.$$

9. $\dfrac{\partial^2 w}{\partial x \partial t} + [a(t)w + b(t)x]\dfrac{\partial^2 w}{\partial x^2} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

By changing from the old variables t, x, and $w = w(x,t)$ to the new variable t, η, and $\Phi = \Phi(t,\eta)$ with η and Φ defined as $\eta = \frac{\partial w}{\partial x}$ and $\Phi = \frac{\partial^2 w}{\partial x^2}$, one arrives at the first-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi}F(t,\eta,\Phi)\right].$$

⊙ *References*: A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov and A. D. Polyanin (2010).

11.1.2. Khokhlov–Zabolotskaya and Related Equations

1. $\dfrac{\partial^2 w}{\partial x \partial t} - w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 - \dfrac{\partial^2 w}{\partial y^2} = 0.$

Two-dimensional Khokhlov–Zabolotskaya equation. It describes the propagation of a sound beam in a nonlinear medium; t and y play the role of the space coordinates and x is a linear combination of time and a coordinate.

The equation of unsteady transonic gas flows (see 11.1.3.1 with $a = b = 1/2$)

$$2u_{x\tau} + u_x u_{xx} - u_{yy} = 0$$

can be reduced to the Khokhlov–Zabolotskaya equation; see Lin, Reissner, and Tsien (1948). To this end, one should pass to the new variable $\tau = 2t$, differentiate the equation with respect to x, and then substitute $w = -\partial u/\partial x$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}C_2^2 w(C_1 x + C_3, C_2 y + C_4, C_1^{-1}C_2^2 t + C_5),$$
$$w_2 = w(x + \lambda y + \varphi(t), y + 2\lambda t, t) + \varphi_t'(t) - \lambda^2,$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = -\frac{x}{t + C_1} + \varphi y + \psi,$$
$$w(x,y,t) = 2\varphi x + (\varphi_t' - 2\varphi^2)y^2 + \psi y + \chi,$$
$$w(x,y,t) = (\varphi y + \psi)x - \frac{1}{12\varphi^2}(\varphi y + \psi)^4 + \frac{1}{6}\varphi_t' y^3 + \frac{1}{2}\psi_t' y^2 + \chi y + \theta,$$
$$w(x,y,t) = C_1\sqrt{x + C_2 y + \varphi} + \varphi_t' - C_2^2,$$
$$w(x,y,t) = \frac{C_1}{t}\sqrt{4t(x + \varphi) - (y + C_2)^2} + \varphi_t',$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, the prime stands for the differentiation, and C_1 and C_2 are arbitrary constants.

3°. Solution in implicit form:

$$tz + x + \lambda y + \lambda^2 t + \varphi(t) = F(z), \qquad z = w - \varphi_t'(t),$$

where $\varphi(t)$ and $F(z)$ are arbitrary functions. With $\lambda = 0$, this relation determines the general y-independent solution of the original equation.

4°. "Two-dimensional" generalized separable solution quadratic in x:

$$w = f(y,t)x^2 + g(y,t)x + h(y,t),$$

where the functions $f = f(y,t)$, $g = g(y,t)$, and $h = h(y,t)$ are determined by the system of differential equations

$$\begin{aligned} f_{yy} &= -6f^2, \\ g_{yy} &= -6fg + 2f_t, \\ h_{yy} &= -2fh + g_t - g^2. \end{aligned}$$

The subscripts y and t denote the corresponding partial derivatives. A particular solution of this system is given by

$$f = -\frac{1}{R^2}, \quad g = \frac{C_1(t)}{R^2} + C_2(t)R^3 - \frac{\varphi_t'(t)}{2R},$$
$$h = \frac{C_3(t)}{R} + C_4(t)R^2 + \frac{R^2}{3}\int \frac{1}{R}(g_t - g^2)\,dy - \frac{1}{3R}\int R^2(g_t - g^2)\,dy, \quad R = y + \varphi(t),$$

where $\varphi(t)$, $C_1(t)$, …, $C_4(t)$ are arbitrary functions.

5°. "Two-dimensional" solution:

$$w = xu(\xi,t), \quad \xi = yx^{-1/2},$$

where the function $u = u(\xi,t)$ is determined by the differential equation

$$2\xi\frac{\partial^2 u}{\partial\xi\partial t} + (\xi^2 u + 4)\frac{\partial^2 u}{\partial\xi^2} + \xi^2\left(\frac{\partial u}{\partial\xi}\right)^2 - 5\xi u\frac{\partial u}{\partial\xi} - 4\frac{\partial u}{\partial t} + 4u^2 = 0.$$

6°. "Two-dimensional" solution:

$$w = v(\zeta,t) + \frac{\alpha_t' + 4}{\alpha}x, \quad \zeta = y^2 + \alpha x,$$

where $\alpha = \alpha(t)$ is an arbitrary function and the function $v = v(\zeta,t)$ is determined by the differential equation

$$\alpha\frac{\partial^2 v}{\partial\zeta\partial t} - (\alpha^2 v + 4\zeta)\frac{\partial^2 v}{\partial\zeta^2} - \alpha^2\left(\frac{\partial v}{\partial\zeta}\right)^2 - (\alpha_t' + 10)\frac{\partial v}{\partial\zeta} + \beta_t' - \beta^2 = 0, \quad \beta = \frac{\alpha_t' + 4}{\alpha}.$$

The last equation has a particular solution of the form $v = \zeta\varphi(t)$, where the function $\varphi = \varphi(t)$ is determined by the Riccati equation $\alpha\varphi_t' - \alpha^2\varphi^2 - (\alpha_t' + 10)\varphi + \beta_t' - \beta^2 = 0$.

7°. "Two-dimensional" solution:

$$w = U(r,z), \quad z = x + \beta y + \lambda t, \quad r = y + \mu t,$$

where β, λ, and μ are arbitrary constants, and the function $U = U(r,z)$ is determined by the differential equation

$$(\lambda - \beta^2)\frac{\partial^2 U}{\partial z^2} + (\mu - 2\beta)\frac{\partial^2 U}{\partial r\partial z} - \frac{\partial^2 U}{\partial r^2} - U\frac{\partial^2 U}{\partial z^2} - \left(\frac{\partial U}{\partial z}\right)^2 = 0.$$

With $\lambda = \beta^2$ and $\mu = 2\beta$, we obtain an equation of the form 9.1.5.1.

8°. "Two-dimensional" solution:

$$w = x^{-2}V(p,q), \quad p = tx^{-3}, \quad q = yx^{-2},$$

where the function $V = V(p,q)$ is determined by the differential equation

$$3p(3Vp+1)\frac{\partial^2 V}{\partial p^2} + (4q^2V+1)\frac{\partial^2 V}{\partial q^2} + 2q(6pV+1)\frac{\partial^2 V}{\partial p\partial q} + \left(3p\frac{\partial V}{\partial p} + 2q\frac{\partial V}{\partial q}\right)^2 + (36pV+5)\frac{\partial V}{\partial p} + 22qV\frac{\partial V}{\partial q} + 10V^2 = 0.$$

9°. Solution:

$$w = u(r)x^2y^{-2}, \quad r = (At+B)^{-1}x^{-1}y^2,$$

where A and B are arbitrary constants, and the function $u = u(r)$ is determined by the ordinary differential equation

$$r^2(u - Ar + 4)u''_{rr} + r^2(u'_r)^2 - r(6u - Ar + 6)u'_r + 6(u+1)u = 0.$$

⊙ *References for equation* 11.1.2.1: Y. Kodama (1988), Y. Kodama and J. Gibbons (1989), N. H. Ibragimov (1994, pp. 299–300; 1995, pp. 447–450), A. M. Vinogradov and I. S. Krasil'shchik (1997), A. D. Polyanin and V. F. Zaitsev (2002, pp. 238–239).

2. $\dfrac{\partial^2 w}{\partial x\partial t} + a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial^2 w}{\partial y^2} = 0.$

The transformation

$$w(x,y,t) = \frac{b}{a}u(x,y,\tau), \quad \tau = -bt$$

leads to an equation of the form 11.1.2.1:

$$\frac{\partial^2 u}{\partial x\partial\tau} - \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial y^2} = 0.$$

3. $\dfrac{\partial^2 w}{\partial x\partial t} - f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) - g(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

Generalized Khokhlov–Zabolotskaya equation.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(C_1^2x + C_2, C_1y + C_3, t),$$
$$w_2 = w(\xi,\eta,t) + \varphi(t), \quad \xi = x + \lambda y + \int [f(t)\varphi(t) + \lambda^2 g(t)]\,dt, \quad \eta = y + 2\lambda\int g(t)\,dt,$$

where C_1, C_2, C_3, and λ are arbitrary constants and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = -x\left(\int f\,dt + C\right)^{-1} + \varphi y + \psi,$$

$$w(x,y,t) = 2\varphi x + \frac{\varphi_t' - 2f\varphi^2}{g}y^2 + \psi y + \chi,$$

$$w(x,y,t) = (\varphi y + \psi)x - \frac{f}{12g\varphi^2}(\varphi y + \psi)^4 + \frac{\varphi_t'}{6g}y^3 + \frac{\psi_t'}{2g}y^2 + \chi y + \theta,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions; C is an arbitrary constant; $f = f(t)$ and $g = g(t)$; the prime denotes a derivative with respect to t.

3°. "Two-dimensional" solution:

$$w(x,y,t) = U(z,t) + \varphi(t), \quad z = x + \lambda y,$$

where the function $\varphi(t)$ is an arbitrary function, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation [$\psi(t)$ is an arbitrary function]

$$\frac{\partial U}{\partial t} - f(t)U\frac{\partial U}{\partial z} - [f(t)\varphi(t) + \lambda^2 g(t)]\frac{\partial U}{\partial z} = \psi(t).$$

A complete integral of this equation is sought in the form $U = A(t)z + B(t)$, which allows obtaining the general solution (see Polyanin, Zaitsev, and Moussiaux, 2002).

4°. "Two-dimensional" generalized separable solution quadratic in x:

$$w = \varphi(y,t)x^2 + \psi(y,t)x + \chi(y,t),$$

where the function $\varphi = \varphi(y,t)$, $\psi = \psi(y,t)$, and $\chi = \chi(y,t)$ are determined by the system of differential equations

$$g\varphi_{yy} = -6f\varphi^2,$$
$$g\psi_{yy} = -6f\varphi\psi + 2\varphi_t,$$
$$g\chi_{yy} = -f(2\varphi\chi + \psi^2) + \psi_t.$$

The subscripts y and t denote the corresponding partial derivatives, $f = f(t)$ and $g = g(t)$.

⊙ *Reference for equation* 11.1.2.3: A. D. Polyanin and V. F. Zaitsev (2002, p. 240).

4. $\dfrac{\partial^2 w}{\partial x\partial t} - \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial y^2} - \dfrac{\partial^2 w}{\partial z^2} = 0.$

Three-dimensional Khokhlov–Zabolotskaya equation.

1°. Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}C_2^2 w(C_1x + C_3, C_2y + C_4, C_2z + C_5, C_1^{-1}C_2^2 t + C_6),$$
$$w_2 = w(x + \lambda y + \mu z + \varphi(t), y + 2\lambda t, z + 2\mu t, t) + \varphi_t'(t) - \lambda^2 - \mu^2,$$
$$w_3 = w(x, y\cos\beta + z\sin\beta, -y\sin\beta + z\cos\beta, t),$$

where $C_1, \ldots, C_6$, λ, μ, and β are arbitrary constants, and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y,z,t) = 2\alpha_1 x + (\alpha_1' - 2\alpha_1^2 - \alpha_2)y^2 + \alpha_3 y + \alpha_2 z^2 + \beta z + \gamma,$$
$$w(x,y,z,t) = \frac{C\sqrt{4tx - y^2 - z^2}}{t^{3/2}},$$

where α_1, α_2, α_3, β, and γ are arbitrary functions of t, and C is an arbitrary constant.

3°. "Three-dimensional" solution:

$$w = u(x,\xi,t), \quad \xi = y\sin\beta + z\cos\beta,$$

where β is an arbitrary constant and the function $u = u(x,\xi,t)$ is determined by the Khokhlov–Zabolotskaya equation of the form 11.1.2.1:

$$\frac{\partial^2 u}{\partial x \partial t} - \left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. "Three-dimensional" generalized separable solution linear in x:

$$w = f(y,z,t)x + g(y,z,t),$$

where the functions $f = f(y,z,t)$ and $g = g(y,z,t)$ are determined by the differential equations

$$f_{yy} + f_{zz} = 0,$$
$$g_{yy} + g_{zz} = f_t - f^2.$$

The subscripts y, z, and t denote the corresponding partial derivatives. The first equation represents the Laplace equation and the second one is a Poisson equation (for g). For solutions of these linear equations, see, for example, Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Three-dimensional" generalized separable solution quadratic in x:

$$w = f(y,z,t)x^2 + g(y,z,t)x + h(y,z,t),$$

where the functions $f = f(y,z,t)$, $g = g(y,z,t)$, and $h = h(y,z,t)$ are determined by the system of differential equations

$$f_{yy} + f_{zz} = -6f^2,$$
$$g_{yy} + g_{zz} = -6fg + 2f_t,$$
$$h_{yy} + h_{zz} = -2fh + g_t - g^2.$$

6°. Solution:

$$w(x,y,z,t) = u(\xi)t^{-\lambda}, \quad \xi = t^{\lambda-2}(4xt - y^2 - z^2),$$

where λ is an arbitrary constant, and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$[4u + (1-\lambda)\xi]u''_{\xi\xi} + 4(u'_\xi)^2 = 0.$$

For $\lambda \neq 1$, the passage to the inverse $\xi = \xi(u)$, the change of variable $\xi(u) = p(u) - \frac{4}{1-\lambda}u$, and the reduction of order with $p'_u = \frac{4}{1-\lambda}\eta(p)$ result in the first-order equation $p\eta\eta'_p - \eta + 1 = 0$. Integrating yields $(\eta - 1)e^\eta = C_1 p$.

For $\lambda = 1$, we have $u(\xi) = \pm\sqrt{C_1\xi + C_2}$.

7°. Solution:

$$w(x,y,z,t)=\frac{y^2+z^2}{t^2}U(\zeta),\quad \zeta=\frac{y^2+z^2}{xt},$$

where the function $U=U(\zeta)$ is determined by the ordinary differential equation

$$\zeta^2(\zeta^2U-\zeta+4)U''_{\zeta\zeta}+\zeta^4(U'_\zeta)^2+\zeta(2\zeta^2U-3\zeta+12)U'_\zeta+4U=0.$$

8°. Solution:

$$w(x,y,z,t)=\frac{z^2}{t^2}V(q),\quad q=\frac{4tx-y^2}{z^2},$$

where the function $V=V(q)$ is determined by the ordinary differential equation

$$2(4V+q^2-q)V''_{qq}+8(V'_q)^2+(1-q)V'_q+V=0.$$

⊙ *References for equation* 11.1.2.4: A. M. Vinogradov, I. S. Krasil'shchik, and V. V. Lychagin (1986), N. H. Ibragimov (1994, pp. 300–301; 1995, pp. 448–450), A. D. Polyanin and V. F. Zaitsev (2002, pp. 240–241).

5. $\dfrac{\partial^2 w}{\partial t\partial x}+a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial^2 w}{\partial y^2}+c\dfrac{\partial^2 w}{\partial z^2}=0.$

1°. For $a<0$, $b<0$, and $c<0$, the passage to the new independent variables according to

$$x=\bar{x}\sqrt{-a},\quad y=\bar{y}\sqrt{-b},\quad z=\bar{z}\sqrt{-a},\quad t=\bar{t}/\sqrt{-a}$$

leads to the three-dimensional Khokhlov–Zabolotskaya equation 11.1.2.4.

2°. Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the functions

$$w_1=C_1^{-2}C_2^2w(C_1x+C_3,C_2y+C_4,C_2z+C_5,C_1^{-1}C_2^2t+C_6),$$
$$w_2=w(x+\lambda y+\mu z+\varphi(t),y-2b\lambda t,z-2c\mu t,t)-\frac{1}{a}\varphi'_t(t)-\frac{b\lambda^2+c\mu^2}{a},$$

where $C_1,\ldots,C_6$, λ, μ, and β are arbitrary constants and $\varphi=\varphi(t)$ is an arbitrary function, are also solutions of the equation.

3°. Solutions:

$$w(x,y,z,t)=\alpha y+\beta z+\frac{1}{at+C}x+\gamma,$$
$$w(x,y,z,t)=\alpha\ln(cy^2+bz^2)-(\beta'_t+4abc\beta^2)(cy^2+bz^2)+4bc\beta x+\gamma,$$

where $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are arbitrary functions and C is an arbitrary constant.

4°. "Three-dimensional" generalized separable solution linear in x:

$$w=f(y,z,t)x+g(y,z,t),$$

where the functions $f=f(y,z,t)$ and $g=g(y,z,t)$ are determined by the differential equations

$$bf_{yy}+cf_{zz}=0, \tag{1}$$
$$bg_{yy}+cg_{zz}=-f_t-af^2. \tag{2}$$

The subscripts y, z, t denote the corresponding partial derivatives. If $bc>0$, then by the scaling $y=\bar{y}\sqrt{|b|}$, $z=\bar{z}\sqrt{|c|}$, equation (1) can be reduced to the Laplace equation, and if $bc<0$, to the wave equation. Likewise, equation (2) can be reduced to a Poisson equation and a nonhomogeneous wave equation, respectively. For solutions of these linear equations, see, for example, Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The above remains true if the coefficients a, b, and c are functions of y, z, and t.

5°. "Three-dimensional" generalized separable solution quadratic in x:

$$w = f(y,z,t)x^2 + g(y,z,t)x + h(y,z,t),$$

where the functions $f = f(y,z,t)$, $g = g(y,z,t)$, and $h = h(y,z,t)$ are determined by the system of differential equations

$$\begin{aligned} bf_{yy} + cf_{zz} &= -6af^2, \\ bg_{yy} + cg_{zz} &= -6afg - 2f_t, \\ bh_{yy} + ch_{zz} &= -2afh - g_t - ag^2. \end{aligned}$$

Remark. This remains true if the coefficients a, b, and c are functions of y, z, and t.

6°. There are "three-dimensional" solutions of the following forms:

$$\begin{aligned} &w(x,y,z,t) = u(x,t,\xi), \quad \xi = cy^2 + bz^2; \\ &w(x,y,z,t) = v(p,q,r)x^{k+2}, \quad p = tx^{k+1}, \quad q = yx^{k/2}, \quad r = zx^{k/2}, \end{aligned}$$

where k is an arbitrary constant.

7°. "Two-dimensional" solution:

$$w(x,y,z,t) = xU(\eta,t), \quad \eta = (cy^2 + bz^2)x^{-1},$$

where the function $U = U(\eta,t)$ is determined by the differential equation

$$\eta(a\eta U + 4bc)\frac{\partial^2 U}{\partial \eta^2} - \eta\frac{\partial^2 U}{\partial t\partial \eta} + a\eta^2\left(\frac{\partial U}{\partial \eta}\right)^2 - 2(a\eta U - 2bc)\frac{\partial U}{\partial \eta} + \frac{\partial U}{\partial t} + aU^2 = 0.$$

8°. "Two-dimensional" solution:

$$w(x,y,z,t) = V(\zeta,t) - \frac{\varphi_t' - 4bc}{a\varphi}x, \quad \zeta = cy^2 + bz^2 + \varphi x,$$

where $\varphi = \varphi(t)$ is an arbitrary function, and $V = V(\zeta,t)$ is determined by the differential equation

$$a\varphi^2(a\varphi^2 V + 4bc\zeta)\frac{\partial^2 V}{\partial \zeta^2} + a\varphi^3\frac{\partial^2 V}{\partial t\partial \zeta} + a\varphi^2\left(a\varphi^2\frac{\partial V}{\partial \zeta} - \varphi_t' + 12bc\right)\frac{\partial V}{\partial \zeta} - \varphi_{tt}''\varphi + 2(\varphi_t')^2 - 12bc\varphi_t' + 16b^2c^2 = 0.$$

⊙ *References for equation* 11.1.2.5: P. Kucharczyk (1967), S. V. Sukhinin (1978), N. H. Ibragimov (1994, pp. 300–301).

11.1.3. Equation of Unsteady Transonic Gas Flows

1. $\dfrac{\partial^2 w}{\partial x \partial t} + a\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} - b\dfrac{\partial^2 w}{\partial y^2} = 0.$

This is a special case of equation 11.1.3.2 with $f(t) = a$ and $g(t) = -b$. For $a = b = \frac{1}{2}$, this is the *Lin–Reissner–Tsien equation* for an unsteady transonic gas flow (a special case).

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-3}C_2^2 w(C_1 x + C_3, C_2 y + C_4, C_1^{-1}C_2^2 t + C_5) + C_6 yt + C_7 y + C_8 t + C_9,$$
$$w_2 = w(\xi, \eta, t) + \varphi_{tt}''(t)y^2 + 2b\varphi_t'(t)x + \psi(t)y + \chi(t),$$
$$\xi = x + \lambda y + b\lambda^2 t - 2ab\varphi(t), \quad \eta = y + 2b\lambda t,$$

where the C_n and λ are arbitrary constants and $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Generalized separable solution in the form of a polynomial of degree 4 in y:

$$w(x, y, t) = \frac{1}{12b^2}(\gamma_{tt}'' + 6a\gamma\gamma_t' + 4a^2\gamma^3)y^4 + \frac{1}{6b}(\alpha_t' + 2a\alpha\gamma)y^3$$
$$+ \frac{1}{2b}[2(\gamma_t' + 2a\gamma^2)x + \beta_t' + 2a\beta\gamma]y^2 + (\alpha x + \delta)y + \gamma x^2 + \beta x + \mu,$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\delta = \delta(t)$ are arbitrary functions.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(z, t) + \varphi(t)y + \psi(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z, t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t} + \frac{a}{2}\left(\frac{\partial U}{\partial z}\right)^2 - b\lambda^2\frac{\partial U}{\partial z} = 0. \tag{1}$$

A complete integral of this equation is given by

$$U = C_1 z + \left(b\lambda^2 C_1 - \tfrac{1}{2}aC_1^2\right)t + C_2,$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (1) can be written out in parametric form (Polyanin, Zaitsev, and Moussiaux, 2002):

$$U = sz + \left(b\lambda^2 s - \tfrac{1}{2}as^2\right)t + f(s),$$
$$z + \left(b\lambda^2 - as\right)t + f_s'(s) = 0,$$

where $f = f(s)$ is an arbitrary function and s is the parameter.

4°. "Two-dimensional" solution of a more general form:

$$w(x, y, t) = U(z, t) + \varphi(t)y^2 + \psi(t)y + \chi(t)x + \theta(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z, t)$ is determined by the first-order partial differential equation [$\sigma(t)$ is an arbitrary function]:

$$\frac{\partial U}{\partial t} + \frac{a}{2}\left(\frac{\partial U}{\partial z}\right)^2 + \left[a\chi(t) - b\lambda^2\right]\frac{\partial U}{\partial z} = \left[2b\varphi(t) - \chi_t'(t)\right]z + \sigma(t).$$

This equation can be fully integrated — a complete integral is sought in the form $U = f(t)z + g(t)$.

5°. "Two-dimensional" generalized separable solution cubic in x:

$$w(x,y,t) = f(y,t)x^3 + g(y,t)x^2 + h(y,t)x + r(y,t),$$

where the functions $f = f(y,t)$, $g = g(y,t)$, $h = h(y,t)$, and $r = r(y,t)$ are determined by the differential equations

$$\begin{aligned} bf_{yy} &= 18af^2, \\ bg_{yy} &= 18afg + 3f_t, \\ bh_{yy} &= 6afh + 4ag^2 + 2g_t, \\ br_{yy} &= 2agh + h_t. \end{aligned}$$

The subscripts y and t denote the corresponding partial derivatives. Setting $f = 0$ and $g = \varphi(t)y + \psi(t)$, where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, one can integrate the system with respect to y to obtain a solution dependent on six arbitrary functions.

6°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,r)t^{-1}, \quad r = yt^{-1/2},$$

where the function $v = v(x,r)$ is determined by the differential equation

$$r\frac{\partial^2 v}{\partial x \partial r} - 2a\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + 2b\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r} = 0.$$

7°. "Two-dimensional" solution:

$$w(x,y,t) = v(p,t) + \frac{\gamma\gamma''_{tt} - 2(\gamma'_t)^2 - 18b\gamma'_t - 40b^2}{12ab\gamma^3}y^4 + \left(\frac{4b+\gamma'_t}{a\gamma^3}p - \delta\right)y^2 + \mu y + \lambda, \quad p = y^2 + \gamma x,$$

where $\gamma = \gamma(t)$, $\mu = \mu(t)$, $\lambda = \lambda(t)$, and $\delta = \delta(t)$ are arbitrary functions, and the function $v = v(p,t)$ is determined by the differential equation

$$\left(\gamma'_t p + a\gamma^3\frac{\partial v}{\partial p}\right)\frac{\partial^2 v}{\partial p^2} + \gamma\frac{\partial^2 v}{\partial t \partial p} + (\gamma'_t - 2b)\frac{\partial v}{\partial p} - \frac{2b[p(\gamma'_t + 4b) - a\gamma^3\delta]}{a\gamma^3} = 0.$$

⊙ *References for equation* 11.1.3.1: C. C. Lin, E. Reissner, and H. S. Tsien (1948), O. S. Ryzhov and S. A. Khristianovich (1958), E. V. Mamontov (1969), J. D. Cole and L. P. Cook (1986), E. M. Vorob'ev, N. V. Ignatovich, and E. O. Semenova (1989), A. D. Polyanin and V. F. Zaitsev (2002, pp. 242–243).

2. $\dfrac{\partial^2 w}{\partial x \partial t} + a\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} - c\dfrac{\partial^2 w}{\partial x^2} - b\dfrac{\partial^2 w}{\partial y^2} = 0.$

For $a = \frac{1}{2}(\gamma + 1)$, $b = \frac{1}{2}$, $c = \frac{1}{2}(1 - M_\infty^2)$, it is the *complete Lin–Reissner–Tsien equation*, where γ is the adiabatic exponent of the gas and M_∞ is the *Mach number*.

The transformation

$$w = u(z,y,t), \quad z = x + ct$$

leads to an equation of the form 11.1.3.1:

$$\frac{\partial^2 u}{\partial x \partial t} + a\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - b\frac{\partial^2 u}{\partial y^2} = 0.$$

3. $\dfrac{\partial^2 w}{\partial x \partial t} + f(t)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + g(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-4} w(C_1^2 x + C_2, C_1 y + C_3, t) + C_4 yt + C_5 y + C_6 t + C_7,$$

$$w_2 = w(\xi, \eta, t) - \frac{\varphi_t'(t)}{2g(t)} y^2 + \psi(t) y + \varphi(t) x + \chi(t),$$

$$\xi = x + \lambda y - \int \left[\lambda^2 g(t) + f(t)\varphi(t)\right] dt, \quad \eta = y - 2\lambda \int g(t)\, dt,$$

where $C_1, \ldots, C_7$ and λ are arbitrary constants and $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Generalized separable solution in the form of a polynomial of degree 4 in y:

$$w(x, y, t) = a(t)y^4 + b(t)y^3 + [c(t)x + d(t)]y^2 + [\alpha(t)x + \beta(t)]y + \gamma(t)x^2 + \mu(t)x + \delta(t),$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\delta = \delta(t)$ are arbitrary functions, and the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, and $d = d(t)$ are given by

$$a = -\frac{c_t' + 2f\gamma c}{12g}, \quad b = -\frac{\alpha_t' + 2f\alpha\gamma}{6g}, \quad c = -\frac{\gamma_t' + 2f\gamma^2}{g}, \quad d = -\frac{\mu_t' + 2f\gamma\mu}{2g}.$$

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(z, t) + \varphi(t)y^2 + \psi(t)y + \chi(t)x + \theta(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z, t)$ is determined by the first-order partial differential equation [$\sigma(t)$ is an arbitrary function]:

$$\frac{\partial U}{\partial t} + \frac{1}{2} f(t) \left(\frac{\partial U}{\partial z}\right)^2 + [f(t)\chi(t) + \lambda^2 g(t)] \frac{\partial U}{\partial z} = -[2g(t)\varphi(t) + \chi_t'(t)]z + \sigma(t).$$

This equation can be fully integrated; a complete integral is sought in the form $U = f(t)z + g(t)$.

4°. "Two-dimensional" generalized separable solution cubic in x:

$$w(x, y, t) = \varphi(y, t)x^3 + \psi(y, t)x^2 + \chi(y, t)x + \theta(y, t),$$

where the functions $\varphi = \varphi(y, t)$, $\psi = \psi(y, t)$, $\chi = \chi(y, t)$, and $\theta = \theta(y, t)$ are determined by the differential equations

$$\begin{aligned}
&g\varphi_{yy} + 18f\varphi^2 = 0,\\
&g\psi_{yy} + 18f\varphi\psi + 3\varphi_t = 0,\\
&g\chi_{yy} + 6f\varphi\chi + 4f\psi^2 + 2\psi_t = 0,\\
&g\theta_{yy} + 2f\psi\chi + \chi_t = 0.
\end{aligned}$$

The subscripts y and t denote the corresponding partial derivatives, $f = f(t)$ and $g = g(t)$. These equations can be treated as ordinary differential equations for y with parameter t; the constants of integration will be functions of t. The first equation has the following particular solutions: $\varphi = 0$ and $\varphi = -\dfrac{g}{3f(y + h)^2}$, where $h = h(t)$ is an arbitrary function.

5°. "Two-dimensional" solution:

$$w(x,y,t) = u(p,t) + a(t)y^4 + [b(t)p + c(t)]y^2 + \mu(t)y + \lambda(t), \quad p = y^2 + \gamma(t)x.$$

Here, $c = c(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\lambda = \lambda(t)$ are arbitrary functions; and the function $u = u(p,t)$ is determined by the differential equation

$$\gamma\frac{\partial^2 u}{\partial p\partial t} + \left(\gamma_t' p + f\gamma^3\frac{\partial u}{\partial p}\right)\frac{\partial^2 u}{\partial p^2} + (\gamma_t' + 2g)\frac{\partial u}{\partial p} + 2g(bp + c) = 0,$$

where the functions $a = a(t)$ and $b = b(t)$ are given by

$$a = -\frac{(b\gamma)_t' + 10gb}{12g}, \quad b = \frac{\gamma_t' - 4g}{f\gamma^3}.$$

⊙ *Reference for equation* 11.1.3.3: A. D. Polyanin and V. F. Zaitsev (2002, pp. 243–244).

4. $$\frac{\partial^2 w}{\partial x\partial t} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0.$$

Lin–Reissner–Tsien equation. For $w = w(t,x,y)$ see equation 11.1.3.1.

In cylindrical coordinates $y = r\cos\theta$ and $z = r\sin\theta$, the original equation becomes

$$\frac{\partial^2 w}{\partial x\partial t} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right) = 0.$$

1°. *Ryzhov–Shefter solution*:

$$w = \lambda(t)x + \tfrac{1}{2}a(t)[x - b(t)]^2 + f(\theta,t)[x - b(t)]r^2 + g(\theta,t)r^4, \tag{1}$$

where the three functions $\lambda = \lambda(t)$, $a = a(t)$, and $b = b(t)$ are linked by a single relation (two of the functions can be chosen arbitrarily)

$$\lambda_t' + \tfrac{1}{2}a\lambda = ab_t',$$

and the functions $f = f(\theta,t)$ and $g = g(\theta,t)$ are described by the linear system of equations

$$f_{\theta\theta} + 4f = 2a_t' + a^2,$$
$$g_{\theta\theta} + 16g = 2f_t + af.$$

These equations are easy to integrate, since they are ordinary differential equations in θ with parameter t.

2°. Generalized separable solution (generalizes the solution from Item 1°):

$$w = F_1 + F_2x + F_3x^2 + F_4r^2 + F_5xr^2 + F_6r^4,$$

where the functions $F_k = F_k(\theta,t)$ are determined by the system of equations

$$\begin{aligned}
&(F_1)_{\theta\theta} = (F_2)_{\theta\theta} = (F_3)_{\theta\theta} = 0,\\
&(F_4)_{\theta\theta} + 4F_4 = 2(F_2)_t + 2F_2F_3,\\
&(F_5)_{\theta\theta} + 4F_5 = 4(F_3)_t + 4F_3^2,\\
&(F_6)_{\theta\theta} + 16F_6 = 2(F_5)_t + 2F_3F_5.
\end{aligned} \tag{2}$$

Solutions (1) then correspond to

$$F_1 = \tfrac{1}{2}ab^2, \quad F_2 = \lambda - ab, \quad F_3 = \tfrac{1}{2}a, \quad F_4 = -bf, \quad F_5 = f, \quad F_6 = g.$$

The general solution of system (2) is as follows:

$$\begin{aligned}
&F_1 = a_1(t)\theta + b_1(t), \quad F_2 = a_2(t)\theta + b_2(t), \quad F_3 = a_3(t)\theta + b_3(t),\\
&F_4 = c_1(t)\cos 2\theta + c_2(t)\sin 2\theta + \alpha_1(t)\theta^2 + \beta_1(t)\theta + \gamma_1(t),\\
&F_5 = c_3(t)\cos 2\theta + c_4(t)\sin 2\theta + \alpha_2(t)\theta^2 + \beta_2(t)\theta + \gamma_2(t),\\
&F_6 = c_5(t)\cos 4\theta + c_6(t)\sin 4\theta + [\mu_1(t) + \nu_1(t)\theta]\cos 2\theta + [\mu_2(t) + \nu_2(t)\theta]\sin 2\theta\\
&\qquad + \sigma(t)\theta^3 + \alpha_3(t)\theta^2 + \beta_3(t)\theta + \gamma_3(t),
\end{aligned}$$

where $a_k(t)$, $b_k(t)$, and $c_k(t)$ are arbitrary functions, and the functions $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(t)$, $\mu_n(t)$, $\nu_n(t)$, and $\sigma(t)$ are determined by substituting the above solutions into the last three equations in system (2).

⊙ *References for equation* 11.1.3.4: C. C. Lin, E. Reissner, and H. S. Tsien (1948), O. S. Ryzhov and G. M. Shefter (1959), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 4–5).

11.1.4. Equations of the Form $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = F\left(x, y, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = 0.$

General solution:

$$w(x, y) = F\big(y + G(x)\big),$$

where $F(z)$ and $G(x)$ are arbitrary functions.

⊙ *Reference*: D. Zwillinger (1989, p. 397).

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-1} w(x, C_1 y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solutions linear and quadratic in y:

$$\begin{aligned}
w(x, y) &= \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),\\
w(x, y) &= C_1 y^2 + \varphi(x)y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,
\end{aligned}$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants.

3°. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y), \tag{1}$$

brings the original equation to the first-order nonlinear equation

$$U\frac{\partial U}{\partial \xi} = f(\xi), \tag{2}$$

which is independent of η. On integrating (2) and taking into account the relations of (1), we obtain the first-order equation

$$\left(\frac{\partial w}{\partial y}\right)^2 = 2\int f(x)\,dx + \psi(w), \tag{3}$$

where $\psi(w)$ is an arbitrary function.

Integrating (3) yields the general solution in implicit form:

$$\int \frac{dw}{\sqrt{2F(x)+\psi(w)}} = \pm y + \varphi(x),$$

where $\varphi(x)$ and $\psi(w)$ are arbitrary functions, $F(x) = \int f(x)\,dx$.

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial w}{\partial y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(x, C_1 y + C_2) + C_3,$$
$$w_2 = w\big(x, y + \varphi(x)\big),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x,y) = y\left[\int f(x)\,dx + C\right] + \varphi(x),$$
$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{1}{\lambda}\int f(x)\,dx + C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

3°. The equation can be rewritten as the relation that the Jacobian of the functions w and $v = w_y - \int f(x)\,dx$ is equal to zero. It follows that w and v are functionally dependent, which means that v is expressible in terms of w:

$$\frac{\partial w}{\partial y} - \int f(x)\,dx = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the first-order equation (1) for any $\varphi(w)$ is a solution of the original equation.

Equation (1) can be treated as an ordinary differential equation in the independent variable y with parameter x. Integrating yields its general solution in implicit form:

$$\int\left[\varphi(w) + \int f(x)\,dx\right]^{-1} dw = y + \psi(x),$$

where $\psi(x)$ and $\varphi(w)$ are arbitrary functions.

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial w}{\partial y} + g(y)\dfrac{\partial w}{\partial x}.$

First integral:

$$\frac{\partial w}{\partial y} = \varphi(w) - \int g(y)\,dy + \int f(x)\,dx,$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a first-order ordinary differential equation in the independent variable y with parameter x.

5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)g\left(\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(x, C_1 y + C_2) + C_3,$$
$$w_2 = w\big(x, y + \varphi(x)\big),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. First integral:

$$\int \frac{U\,dU}{g(U)} = \varphi(w) + \int f(x)\,dx, \qquad U = \frac{\partial w}{\partial y},$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a first-order ordinary differential equation in the independent variable y with parameter x.

11.1.5. Other Equations with Two Independent Variables

1. $A\dfrac{\partial^2 w}{\partial x^2} + B\dfrac{\partial^2 w}{\partial x\partial y} + f(w)\dfrac{\partial w}{\partial x}\left(A\dfrac{\partial w}{\partial x} + B\dfrac{\partial w}{\partial y}\right) + C\dfrac{\partial w}{\partial x} = 0.$

Here $A = A(x, y)$, $B = B(x, y)$, $C = C(x, y)$, and $f = f(w)$ are arbitrary functions.

This equation can be represented as the first-order factor system

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = 0, \tag{1}$$

$$u = \exp\left(\int f(w)\,dw\right)\frac{\partial w}{\partial x}. \tag{2}$$

First, one solves the linear first-order equation (1) by the method of characteristics and then equation (2) by straightforward integration with respect to x.

2. $w\dfrac{\partial^2 w}{\partial x^2} = a\left(w\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\right) + f(x, y)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This equation can be represented as the first-order factor system

$$a\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} + [f(x, y) - 1]u^2 = 0,$$
$$u = \frac{1}{w}\frac{\partial w}{\partial x},$$

which is integrated successively.

3. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}=\dfrac{\partial w}{\partial y}\left[\dfrac{\partial^2 w}{\partial x^2}+a\left(\dfrac{\partial w}{\partial x}\right)^2\right].$

General solution in implicit form:

$$e^{aw}f(y)=x+g(w),$$

where $f(y)$ and $g(w)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}=\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}+aw^m\left(\dfrac{\partial w}{\partial x}\right)^n.$

General solution in implicit form with $n\neq 2$:

$$\int_b^{w(x,y)}\frac{d\xi}{Z(y,\xi)}=x+f(y),$$

where $Z=Z(y,\xi)$ is any solution of the transcendental equation

$$Z^{2-n}+a(n-2)y\xi^m=g(\xi),$$

and $f(y)$ and $g(\xi)$ are arbitrary functions; b is an arbitrary constant.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

5. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x\partial y}=\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}+aw^n\dfrac{\partial^2 w}{\partial x^2}+bw^m\left(\dfrac{\partial w}{\partial x}\right)^2.$

General solution in implicit form:

$$\int_c^{w(x,y)}\exp\left[\frac{b\xi^{m-n+1}}{a(m-n+1)}\right]G\left(\frac{\xi^{1-n}}{a(n-1)}-y\right)d\xi=F(y)-x,$$

where $F=F(y)$ and $G(z)$ are arbitrary functions, c is an arbitrary constant ($ab\neq 0$, $n\neq 1$, and $m\neq n-1$).

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

6. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}=\left[a\dfrac{\partial w}{\partial x}+f(x)\right]\dfrac{\partial^2 w}{\partial y^2}.$

First integral:

$$(1-a)y\frac{\partial w}{\partial y}+aw+\int f(x)\,dx=\Phi\left(\frac{\partial w}{\partial y}\right),$$

where $\Phi(z)$ is an arbitrary function.

7. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}=\left[\dfrac{\partial w}{\partial x}+f(x)w\right]\dfrac{\partial^2 w}{\partial y^2}.$

First integral:

$$\frac{\partial w}{\partial y}=\Phi\left(w\exp\left(\int f(x)\,dx\right)\right),$$

where $\Phi(z)$ is an arbitrary function.

⊙ *Reference*: V. F. Zaitsev (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=173).

8. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} = \left[a\dfrac{\partial w}{\partial x} + f(x)\dfrac{\partial w}{\partial y}\right]\dfrac{\partial^2 w}{\partial y^2}.$

First integral:

$$(1-a)y\frac{\partial w}{\partial y} + aw + \frac{\partial w}{\partial y}\int f(x)\,dx = \Phi\left(\frac{\partial w}{\partial y}\right),$$

where $\Phi(z)$ is an arbitrary function.

9. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial x^2} + 1 = 0.$

Gibbons–Tsarev equation. This equation arises in the theory of Benney moment equations (a system of hydrodynamic type). This is a special case of equation 11.1.5.10 with $f(x) = -1$, $g(x) = 0$, and $h(x) = -1$.

⊙ *References*: J. Gibbons and S. P. Tsarev (1999), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 22, 31), O. V. Kaptsov (2009).

10. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)w + h(x).$

1°. Generalized separable solution:

$$w = \varphi_2(x)y^2 + \varphi_1(x)y + \varphi_0(x), \tag{1}$$

where the functions $\varphi_n = \varphi_n(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f\varphi_2'' &= 2\varphi_2\varphi_2' - g\varphi_2,\\ f\varphi_1'' &= 2\varphi_1\varphi_2' - g\varphi_1,\\ f\varphi_0'' &= -2\varphi_2\varphi_0' - g\varphi_0 + \varphi_1\varphi_1' - h. \end{aligned} \tag{2}$$

Remark 1. In the original equation, f can also depend on the second derivative w_{yy}: $f = f(x, w_{yy})$. In this case, one should set $f = f(x, 2\varphi_2)$ in equations (2).

2°. Generalized separable solution:

$$w = \psi_1(x) + \psi_2(x)\cos(ky) + \psi_3(x)\sin(ky),$$

where k is an arbitrary constant and the functions $\psi_n = \psi_n(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f\psi_3'' &= k^2\psi_3\psi_1' - g\psi_3,\\ f\psi_2'' &= k^2\psi_2\psi_1' - g\psi_2,\\ f\psi_1'' &= k^2\psi_2\psi_2' + k^2\psi_3\psi_3' - g\psi_1 - h. \end{aligned}$$

3°. Generalized separable solution:

$$w = \psi_1(x) + \psi_2(x)\cosh(ky) + \psi_3(x)\sinh(ky),$$

where k is an arbitrary constant and the functions $\psi_n = \psi_n(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f\psi_3'' &= -k^2\psi_3\psi_1' - g\psi_3,\\ f\psi_2'' &= -k^2\psi_2\psi_1' - g\psi_2,\\ f\psi_1'' &= -k^2\psi_2\psi_2' + k^2\psi_3\psi_3' - g\psi_1 - h. \end{aligned}$$

4°. For $f(x) = a = \text{const}$ and $g(x) = h(x) = 0$, the original equation is linearized with the Legendre transformation (see Subsection 26.3.3).

Remark 2. The *Chaplygin gas equation* (Chaplygin, 1904)

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{w_t+\frac{1}{2}w_x^2}}\right)+\frac{\partial}{\partial x}\left(\frac{w_x}{\sqrt{w_t+\frac{1}{2}w_x^2}}\right)=0,$$

where $w=w(x,t)$ is the velocity potential, is reduced, by differentiating, to the equation

$$\frac{1}{2}\frac{\partial^2 w}{\partial t^2}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}-\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}=0,$$

which, after renaming $x \Rightarrow y$ and $t \Rightarrow x$, can be reduced to a special case of the equation in question with $f(x)=-\frac{1}{2}$ and $g(x)=h(x)=0$.

11. $$\left[\frac{\partial w}{\partial y}+f(y)\right]\frac{\partial^2 w}{\partial x\partial y}=\left[\frac{\partial w}{\partial x}+g(x)\right]\frac{\partial^2 w}{\partial y^2}.$$

First integral:

$$\frac{\partial w}{\partial y}=\Phi\left(w+\int f(y)\,dy+\int g(x)\,dx\right),$$

where $\Phi(z)$ is an arbitrary function.

⊙ *Reference*: V. F. Zaitsev (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=60).

12. $$\left[f(w)\frac{\partial w}{\partial y}+g(y)\right]\frac{\partial^2 w}{\partial x\partial y}=\left[f(w)\frac{\partial w}{\partial x}+h(x)\right]\frac{\partial^2 w}{\partial y^2}.$$

First integral:

$$\frac{\partial w}{\partial y}=\Phi\left(\int f(w)\,dw+\int g(y)\,dy+\int h(x)\,dx\right),$$

where $\Phi(z)$ is an arbitrary function.

⊙ *Reference*: V. F. Zaitsev (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=65).

13. $$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+f(y)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=g(y)w+h(y)x+s(y).$$

Generalized separable solution linear in x:

$$w=\varphi(y)x+\psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$f\varphi\varphi''_{yy}+(\varphi'_y)^2=g\varphi+h,$$
$$f\varphi\psi''_{yy}+\varphi'_y\psi'_y=g\psi+s.$$

14. $$\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial x^2}-2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial y^2}=0.$$

1°. First integrals:

$$\frac{w_y}{w_x}+\varphi(w)=0,\qquad x+y\frac{w_y}{w_x}+\psi(w)=0,$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions.

2°. General solution in implicit form:

$$x-\varphi(w)y+\psi(w)=0,$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions.

⊙ *References for equation* 11.1.5.14: M. E. Goursat (1896), O. V. Kaptsov (2009).

15. $\left[1+\left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. First integrals:

$$\frac{w_x}{\sqrt{1+w_y^2}}-\varphi(x)=0, \qquad y+ww_y-\psi(w_y)=0,$$

where $\varphi(x)$ and $\psi(u)$ are arbitrary functions.

2°. General solution in parametric form:

$$w=\sqrt{1+\xi^2}\,\Phi(x)+\xi y+\psi(\xi), \qquad y+\Phi'(\xi)+\frac{\xi}{\sqrt{1+\xi^2}}\psi(x)=0,$$

where $\Phi'(\xi)=\varphi(\xi)$ and ξ is the parameter.

⊙ *References for equation* 11.1.5.15: M. E. Goursat (1896), O. V. Kaptsov (2009).

16. $\left[1-\left(\dfrac{\partial w}{\partial t}\right)^2\right]\dfrac{\partial^2 w}{\partial x^2}+2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x\partial t}-\left[1+\left(\dfrac{\partial w}{\partial x}\right)^2\right]\dfrac{\partial^2 w}{\partial t^2}=0.$

Born–Infeld equation (see Born and Infeld, 1934). It is used in nonlinear electrodynamics and field theory.

1°. Solutions:

$$w(x,t)=\varphi(x+t),$$
$$w(x,t)=\psi(x-t),$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

2°. Cauchy problem with initial conditions:

$$w=f(x) \quad \text{at} \quad t=0, \qquad \partial_t w=g(x) \quad \text{at} \quad t=0.$$

The hyperbolicity condition $1+[f_x'(x)]^2-g^2(x)>0$ is assumed to hold.

Solution in parametric form:

$$t=\frac{1}{2}\int_\alpha^\beta \frac{1+[f_\zeta'(\zeta)]^2}{\sqrt{1+[f_\zeta'(\zeta)]^2-g^2(\zeta)}}\,d\zeta,$$
$$x=\frac{\alpha+\beta}{2}-\frac{1}{2}\int_\alpha^\beta \frac{f_\zeta'(\zeta)g(\zeta)\,d\zeta}{\sqrt{1+[f_\zeta'(\zeta)]^2-g^2(\zeta)}},$$
$$w=\frac{f(\alpha)+f(\beta)}{2}+\frac{1}{2}\int_\alpha^\beta \frac{g(\zeta)\,d\zeta}{\sqrt{1+[f_\zeta'(\zeta)]^2-g^2(\zeta)}}.$$

3°. By the introduction of the new variables

$$\xi=x-t, \quad \eta=x+t, \quad u=\frac{\partial w}{\partial \xi}, \quad v=\frac{\partial w}{\partial \eta},$$

the Born–Infeld equation can be rewritten as the equivalent system of equations

$$\frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \xi} = 0,$$
$$v^2 \frac{\partial u}{\partial \xi} - (1 + 2uv)\frac{\partial u}{\partial \eta} + u^2 \frac{\partial v}{\partial \eta} = 0.$$

The hodograph transformation (where u, v are treated as the independent variables and ξ, η as the dependent ones) leads to the linear system

$$\begin{aligned} \frac{\partial \xi}{\partial v} - \frac{\partial \eta}{\partial u} &= 0, \\ v^2 \frac{\partial \eta}{\partial v} + (1 + 2uv)\frac{\partial \xi}{\partial v} + u^2 \frac{\partial \xi}{\partial u} &= 0. \end{aligned} \tag{1}$$

On eliminating η, we can reduce this system to the second-order linear equation

$$u^2 \frac{\partial^2 \xi}{\partial u^2} + (1 + 2uv)\frac{\partial^2 \xi}{\partial u \partial v} + v^2 \frac{\partial^2 \xi}{\partial v^2} + 2u\frac{\partial \xi}{\partial u} + 2v\frac{\partial \xi}{\partial v} = 0.$$

Looking for solutions in the hyperbolic domain, we write out the equation of characteristics

$$u^2\, dv^2 - (1 + 2uv)\, du\, dv + v^2\, du^2 = 0.$$

Integrals of this equation are given by $r = C_1$ and $s = C_2$, where

$$r = \frac{\sqrt{1 + 4uv} - 1}{2v}, \qquad s = \frac{\sqrt{1 + 4uv} - 1}{2u}. \tag{2}$$

Passing in (1) to the new variables of (2), we obtain

$$\begin{aligned} r^2 \frac{\partial \xi}{\partial r} + \frac{\partial \eta}{\partial r} &= 0, \\ \frac{\partial \xi}{\partial s} + s^2 \frac{\partial \eta}{\partial s} &= 0. \end{aligned} \tag{3}$$

Eliminating η yields the simple equation

$$\frac{\partial^2 \xi}{\partial r \partial s} = 0,$$

whose solution is the sum of two arbitrary functions with different arguments, $\xi = \varphi(r) + \psi(s)$. The function η is found from system (3).

4°. The Legendre transformation

$$w(x,t) + u(\zeta,\tau) = x\zeta + t\tau, \quad \zeta = \frac{\partial w}{\partial x}, \quad \tau = \frac{\partial w}{\partial t}, \quad x = \frac{\partial u}{\partial \zeta}, \quad t = \frac{\partial u}{\partial \tau}$$

leads to the linear equation

$$\left(1 - \tau^2\right)\frac{\partial^2 u}{\partial \tau^2} - 2\zeta\tau\frac{\partial^2 u}{\partial \zeta \partial \tau} - \left(1 + \zeta^2\right)\frac{\partial^2 u}{\partial \zeta^2} = 0.$$

⊙ *References for equation* 11.1.5.16: B. M. Barbashov and N. A. Chernikov (1966), G. B. Whitham (1974).

17. $\dfrac{\partial^2 w}{\partial t^2} + 2\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x \partial t} + \left[a + b\dfrac{\partial w}{\partial t} + c\left(\dfrac{\partial w}{\partial x}\right)^2\right]\dfrac{\partial^2 w}{\partial x^2} = 0.$

Potential equation of the one-dimensional flow of a compressible gas.

1°. The *von Mises solution*:

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2,$$

where the functions $\varphi_n = \varphi_n(t)$ $(n = 0, 1, 2)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0'' &= -2\varphi_1\varphi_1' - 2a\varphi_2 - 2b\varphi_2\varphi_0' - 2c\varphi_1^2\varphi_2,\\
\varphi_1'' &= -4(\varphi_1\varphi_2)' - 2b\varphi_2\varphi_1' - 8c\varphi_1\varphi_2^2,\\
\varphi_2'' &= -2(4+b)\varphi_2\varphi_2' - 8c\varphi_2^3.
\end{aligned}$$

The last equation for φ_2 can be solved independently in terms of Jacobi elliptic functions.

2°. The Legendre transformation (see Subsection 26.3.3)

$$w(x,t) + u(\zeta,\tau) = x\zeta + t\tau, \quad \zeta = \frac{\partial w}{\partial x}, \quad \tau = \frac{\partial w}{\partial t}, \quad x = \frac{\partial u}{\partial \zeta}, \quad t = \frac{\partial u}{\partial \tau}$$

leads to the linear equation

$$\frac{\partial^2 u}{\partial \zeta^2} - 2\zeta\frac{\partial^2 u}{\partial \zeta \partial \tau} + (a + b\tau + c\zeta^2)\frac{\partial^2 u}{\partial \tau^2} = 0.$$

⊙ *Reference for equation* 11.1.5.17: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 2–3).

18. $\left[a + \left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial^2 w}{\partial x^2} - 2b\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} + \left[c + \left(\dfrac{\partial w}{\partial x}\right)^2\right]\dfrac{\partial^2 w}{\partial y^2} = 0.$

Equation of minimal surfaces (with $a = b = c = 1$). It describes, for example, the shape of a foam film bounded by a given contour.

1°. The Legendre transformation

$$w(x,y) + u(\xi,\eta) = x\xi + y\eta, \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}, \quad x = \frac{\partial u}{\partial \xi}, \quad y = \frac{\partial u}{\partial \eta}$$

leads to the linear equation

$$\left(a + \eta^2\right)\frac{\partial^2 u}{\partial \eta^2} + 2b\xi\eta\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(c + \xi^2\right)\frac{\partial^2 u}{\partial \xi^2} = 0.$$

2°. General solution in parametric form for $a = b = c = 1$:

$$x = \operatorname{Re} f_1(z), \quad y = \operatorname{Re} f_2(z), \quad w = \operatorname{Re} f_3(z),$$

where the $f_k(z)$ are arbitrary analytic functions of the complex variable $z = \alpha + i\beta$ with derivatives constrained by

$$[f_1'(z)]^2 + [f_2'(z)]^2 + [f_3'(z)]^2 = 0,$$

and $\operatorname{Re} f(z)$ stands for the real part of the analytic function $f(z)$ of the complex variable z.

Note that z can be taken as one of the functions $f_k(z)$.

⊙ *Reference for equation* 11.1.5.18: R. Courant and D. Hilbert (1989).

19. $f(w)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}+\left[g(w)\dfrac{\partial w}{\partial y}-f(w)\dfrac{\partial w}{\partial x}\right]\dfrac{\partial^2 w}{\partial x\partial y}-g(w)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. First integral:

$$f(w)\frac{\partial w}{\partial x}+g(w)\frac{\partial w}{\partial y}=\varphi(w),$$

where $\varphi(w)$ is an arbitrary function.

2°. General solution in implicit form:

$$\Psi\left(x-\int\frac{f(w)}{\varphi(w)}\,dw,\ y-\int\frac{g(w)}{\varphi(w)}\,dw\right)=0,$$

where $\Psi(z_1,z_2)$ and $\varphi(w)$ are arbitrary functions.

20. $\dfrac{\partial^2 w}{\partial x\partial y}=F\left(y,\dfrac{\partial w}{\partial x}\right)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1=C_1^{-1}w(C_1x+C_2,y)+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The Euler transformation

$$w(x,y)+u(\xi,\eta)=x\xi,\quad x=\frac{\partial u}{\partial\xi},\quad y=\eta$$

leads to the linear equation (for details, see Subsection 26.3.4)

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=F(\eta,\xi)\frac{\partial u}{\partial\eta},$$

whose order can be reduced with the substitution $U=\frac{\partial u}{\partial\eta}$.

21. $\dfrac{\partial^2 w}{\partial y^2}+f(\theta)\dfrac{\partial^2 w}{\partial x\partial y}-[\theta f(\theta)+\theta^2]\dfrac{\partial^2 w}{\partial x^2}=0,\qquad \theta=\dfrac{\partial w}{\partial y}\Big/\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1=C_1w(C_2x+C_3,\ C_2y+C_4)+C_5,$$
$$w_2=\Phi\big(w(x,y)\big),$$

where $C_1,\dots,C_5$ are arbitrary constants, and $\Phi(u)$ is an arbitrary function, are also solutions of the equation.

2°. Traveling-wave solution:

$$w=\Phi(k_1x+k_2y),$$

where $\Phi(z)$ is an arbitrary function, and k_1 and k_2 are arbitrary constants.

3°. There are solutions of the forms

$$w(x,y) = e^{kx}U_1(y) \quad \text{(multiplicative separable solution)},$$
$$w(x,y) = e^{ky}U_2(x) \quad \text{(multiplicative separable solution)},$$
$$w(x,y) = x^kU_1(y/x) \quad \text{(self-similar solution)},$$

where k is an arbitrary constant.

4°. The Legendre transformation

$$x = \frac{\partial u}{\partial \xi}, \quad y = \frac{\partial u}{\partial \eta}, \quad w = x\xi + y\eta - u$$

leads to the linear equation

$$\frac{\partial^2 u}{\partial \xi^2} + f(\theta)\frac{\partial^2 u}{\partial \xi \partial \eta} - [\theta f(\theta) + \theta^2]\frac{\partial^2 u}{\partial \eta^2} = 0, \qquad \theta = \frac{\eta}{\xi}.$$

⊙ *References for equation* 11.1.5.21: V. I. Arnold (2000, p. 139), O. F. Menshikh (2004).

22. $$\left[f^2 - \left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \left[f^2 - \left(\frac{\partial w}{\partial y}\right)^2\right]\frac{\partial^2 w}{\partial y^2} = 0,$$
$$f = f(w_x^2 + w_y^2).$$

This equation describes a two-dimensional steady-state isentropic flow of a compressible gas; w is the velocity potential and f is the sound speed.

The Legendre transformation

$$w(x,y) + U(\xi,\eta) = x\xi + y\eta, \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}, \quad x = \frac{\partial U}{\partial \xi}, \quad y = \frac{\partial U}{\partial \eta}$$

leads to the linear equation

$$\left(f^2 - \xi^2\right)\frac{\partial^2 U}{\partial \eta^2} + 2\xi\eta\frac{\partial^2 U}{\partial \xi \partial \eta} + \left(f^2 - \eta^2\right)\frac{\partial^2 U}{\partial \xi^2} = 0, \qquad f = f(\xi^2 + \eta^2).$$

⊙ *Reference*: R. Courant and D. Hilbert (1989).

11.1.6. Other Equations with Three and More Independent Variables

1. $$\frac{\partial^2 w}{\partial x \partial t} + f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial y^2} = 0.$$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1}w(C_1x + C_2, C_1y + C_3, C_1t + C_4) + C_5yt + C_6y + C_7t + C_8,$$
$$w_2 = w(x + \lambda y - b\lambda^2 t, y - 2b\lambda t, t) + \varphi(t)y + \psi(t),$$

where $C_1, \ldots, C_8$ and λ are arbitrary constants and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(z,t) + \varphi(t)y^2 + \psi(t)y + \chi(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t} + F\left(\frac{\partial U}{\partial z}\right) + b\lambda^2\frac{\partial U}{\partial z} + 2b\varphi(t)z = \sigma(t), \quad F(u) = \int f(u)\,du,$$

where $\sigma(t)$ is an arbitrary function. A complete integral of this equation has the form

$$U = A(t)z + B(t),$$

where the functions $A(t)$ and $B(t)$ are given by

$$A(t) = -2b\int \varphi(t)\,dt + C_1, \quad B(t) = \int\left[\sigma(t) - F(A(t)) - b\lambda^2 A(t)\right]dt + C_2,$$

and C_1 and C_2 are arbitrary constants.

⊙ *References for equation* 11.1.6.1: A. D. Polyanin and V. F. Zaitsev (2002, p. 245).

2. $\dfrac{\partial^2 w}{\partial x\partial t} + f(t)\Phi\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2} + g(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the function

$$w_1 = w(\xi,\eta,t) + \varphi(t)y + \psi(t), \quad \xi = x + \lambda y - \lambda^2\int g(t)\,dt + C_1, \quad \eta = y - 2\lambda\int g(t)\,dt + C_2,$$

where C_1, C_2, and λ are arbitrary constants and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, is also a solution of the equation.

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(z,t) + \varphi(t)y^2 + \psi(t)y + \chi(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t} + f(t)\Psi\left(\frac{\partial U}{\partial z}\right) + \lambda^2 g(t)\frac{\partial U}{\partial z} + 2g(t)\varphi(t)z = \sigma(t), \quad \Psi(u) = \int \Phi(u)\,du,$$

where $\sigma(t)$ is an arbitrary function. This equation can be fully integrated—a complete integral is sought in the form $U = A(t)z + B(t)$.

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial t} = \dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x\partial y}.$

General solution:

$$w(x,y,t) = G(x, F(y,t)),$$

where $F(y,t)$ and $G(x,z)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2007, http://eqworld.ipmnet.ru/eqarchive/view.php?id=67).

4. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}\dfrac{\partial^2 w}{\partial y\partial t} = \dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial z}.$

Solutions:

$$w(x,y,z,t) = F(G(x,t),H(y,z)),$$
$$w(x,y,z,t) = S(G(x,t)+F(z,t)+K(x,y)+H(y,z)),$$

where F, G, H, and K are arbitrary functions of two arguments, and S is an arbitrary function of one argument.

⊙ *Reference*: Yu. N. Kosovtsov (2007, http://eqworld.ipmnet.ru/eqarchive/view.php?id=90).

11.2. Equations Quadratic in the Highest Derivatives

11.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = F(x,y)$

▷ *Suppose $w(x,y)$ is a solution of the equation in question. Then the function*

$$w_1 = w(x,y) + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

1. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-k-2}w(x,C_1^2y) + C_2xy + C_3x + C_4y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solutions:

$$w(x,y) = (C_1x+C_2)y^{k+1} + \frac{y}{k(k+1)}\int_0^x \frac{(x-t)f(t)}{(C_1t+C_2)}\,dt + C_3xy + C_4x + C_5y + C_6,$$
$$w(x,y) = (C_1x+C_2)y^{k+2} + \frac{1}{(k+1)(k+2)}\int_0^x \frac{(x-t)f(t)}{(C_1t+C_2)}\,dt + C_3xy + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. Generalized separable solution:

$$w(x,y) = \varphi(x)y^{\frac{k+2}{2}} + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)\varphi\varphi''_{xx} = 4f(x).$$

2. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)g(y).$

1°. Additive separable solution:

$$w(x,y) = C_1\int_0^x (x-t)f(t)\,dt + C_2x + \frac{1}{C_1}\int_0^y (y-\tau)g(\tau)\,d\tau + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

2°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations (C_1 is an arbitrary constant)

$$\varphi\varphi''_{xx} = C_1 f(x),$$
$$\psi\psi''_{yy} = C_1^{-1} g(y).$$

3. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax+by).$

Solutions:

$$w(x,y) = \pm\frac{1}{ab}\int_0^z (z-t)\sqrt{f(t)}\,dt + C_1xy + C_2x + C_3y + C_4, \qquad z = ax+by,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^{2k} + g(x)y^k + h(x)y^{k-1}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)y^{k+1} + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$k(k+1)\varphi\varphi''_{xx} = f(x),$$
$$k(k+1)\varphi\psi''_{xx} = g(x),$$
$$k(k+1)\varphi\chi''_{xx} = h(x).$$

5. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{\lambda y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\Big(x, y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2xy + C_3x + C_4y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,y) = (C_1x + C_2)e^{\lambda y} + \frac{1}{\lambda^2}\int_{x_0}^{x} \frac{(x-t)f(t)}{C_1t + C_2}\,dt + C_3xy + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, and x_0 is any number such that the integrand does not have a singularity at $x = x_0$.

3°. Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y/2} + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $\lambda^2\varphi\varphi''_{xx} = 4f(x)$.

6. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{2\lambda y} + g(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} + \frac{1}{\lambda^2}\int_{x_0}^{x}(x-t)\frac{g(t)}{\varphi(t)}\,dt + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $\lambda^2\varphi\varphi''_{xx} = f(x)$.

11.2.2. Monge–Ampère Equation $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = F(x,y)$

Preliminary remarks. The Monge–Ampère equation is encountered in differential geometry, gas dynamics, and meteorology.

1°. Suppose $w(x,y)$ is a solution of the Monge–Ampère equation. Then the functions

$$w_1 = \pm w(x,y) + C_1x + C_2y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$\bar{x} = a_1x + b_1y + c_1, \quad \bar{y} = a_2x + b_2y + c_2, \quad \bar{w} = kw + a_3x + b_3y + c_3, \quad \bar{F} = k^2(a_1b_2 - a_2b_1)^{-2}F,$$

where the a_n, b_n, c_n, and k are arbitrary constants, takes the Monge–Ampère equation to an equation of the same form.

3°. The transformation

$$\bar{x} = x(1+\alpha x+\beta y)^{-1}, \quad \bar{y} = y(1+\alpha x+\beta y)^{-1}, \quad \bar{w} = w(1+\alpha x+\beta y)^{-1}, \quad \bar{F} = F(1+\alpha x+\beta y)^4,$$

where α and β are arbitrary constants, takes the Monge–Ampère equation to an equation of the same form.

4°. In the cylindrical coordinates r, φ, where $x = r\cos\varphi$, $y = r\sin\varphi$, the Monge–Ampère equation has the form

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right)\right]^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = F(r\cos\varphi, r\sin\varphi).$$

5°. In Lagrangian coordinates, the system of equations of the one-dimensional gas dynamics with plane waves is as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial \xi} = 0, \qquad \frac{\partial V}{\partial t} - \frac{\partial u}{\partial \xi} = 0,$$

where t is time, u the velocity, p the pressure, ξ the Lagrangian coordinate, and V the specific volume. The equation of state is assumed to have the form $V = V\big(p, S(\xi)\big)$, where $S = S(\xi)$ is a prescribed entropy profile.

The *Martin transformation*

$$u(\xi, t) = \frac{\partial w}{\partial x}(x, y), \quad t = \frac{\partial w}{\partial y}(x, y), \quad x = \xi, \quad y = p(\xi, t)$$

reduces the equations of the one-dimensional gas dynamics to the nonhomogeneous Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = F(x, y),$$

where $F(x, y) = -\frac{\partial V}{\partial p}\big(p, S(\xi)\big)$.

⊙ *References for Preliminary remarks*: M. N. Martin (1953), B. L. Rozhdestvenskii and N. N. Yanenko (1983), S. V. Khabirov (1990 *b*), N. H. Ibragimov (1994, pp. 94–101).

1. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = 0.$

Homogeneous Monge–Ampère equation.

1°. Suppose $w(x, y)$ is a solution of the homogeneous Monge–Ampère equation. Then the functions

$$w_1 = C_1 w(C_2 x + C_3 y + C_4, C_5 x + C_6 y + C_7) + C_8 x + C_9 y + C_{10},$$
$$w_2 = (1 + C_1 x + C_2 y) w\left(\frac{x}{1 + C_1 x + C_2 y}, \frac{y}{1 + C_1 x + C_2 y}\right),$$

where $C_1, \ldots, C_{10}$ are arbitrary constants, are also solutions of the equation.

2°. First integrals:

$$\Phi_1\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0,$$
$$\Phi_2\left(\frac{\partial w}{\partial x}, w - x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y}\right) = 0,$$

where $\Phi_1(u, v)$ and $\Phi_2(u, z)$ are arbitrary functions of two arguments.

3°. General solution in parametric form:

$$w = tx + \varphi(t)y + \psi(t),$$
$$x + \varphi'(t)y + \psi'(t) = 0,$$

where t is the parameter, and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions.

4°. Solutions involving one arbitrary function:

$$w(x,y) = \varphi(C_1x + C_2y) + C_3x + C_4y + C_5,$$
$$w(x,y) = (C_1x + C_2y)\varphi\left(\frac{y}{x}\right) + C_3x + C_4y + C_5,$$
$$w(x,y) = \sqrt{x^2+y^2}\,\varphi\left(\frac{y}{x}\right) + C_1x + C_2y + C_3,$$
$$w(x,y) = (C_1x + C_2y + C_3)\varphi\left(\frac{C_4x + C_5y + C_6}{C_1x + C_2y + C_3}\right) + C_7x + C_8y + C_9,$$

where $C_1, \ldots, C_9$ are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

5°. Solutions involving arbitrary constants:

$$w(x,y) = C_1y^2 + C_2xy + \frac{C_2^2}{4C_1}x^2 + C_3y + C_4x + C_5,$$
$$w(x,y) = \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) + C_4y + C_5x + C_6,$$
$$w(x,y) = \pm(C_1x + C_2y + C_3)^k + C_4x + C_5y + C_6,$$
$$w(x,y) = \pm\frac{(C_1x + C_2y + C_3)^{k+1}}{(C_4x + C_5y + C_6)^k} + C_7x + C_8y + C_9,$$
$$w(x,y) = \pm\sqrt{C_1(x+a)^2 + C_2(x+a)(y+b) + C_3(y+b)^2} + C_5x + C_6y + C_7,$$

where the a, b, and the C_n are arbitrary constants.

⊙ *References for equation* 11.2.2.1: E. Goursat (1933, p. 62), S. V. Khabirov (1990 *b*), N. H. Ibragimov (1994, pp. 94–101).

2. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = A.$

1°. First integrals for $A = a^2 > 0$:

$$\Phi_1\left(\frac{\partial w}{\partial x} + ay, \frac{\partial w}{\partial y} - ax\right) = 0,$$
$$\Phi_2\left(\frac{\partial w}{\partial x} - ay, \frac{\partial w}{\partial y} + ax\right) = 0,$$

where the $\Phi_n(u,v)$ are arbitrary functions of two arguments ($n = 1, 2$).

2°. General solution in parametric form for $A = a^2 > 0$:

$$x = \frac{\beta - \lambda}{2a}, \quad y = \frac{\psi'(\lambda) - \varphi'(\beta)}{2a}, \quad w = \frac{(\beta + \lambda)[\psi'(\lambda) - \varphi'(\beta)] + 2\varphi(\beta) - 2\psi(\lambda)}{4a},$$

where β and λ are the parameters and $\varphi = \varphi(\beta)$ and $\psi = \psi(\lambda)$ are arbitrary functions.

3°. Solutions:

$$w(x,y) = \pm\frac{\sqrt{A}}{C_2}x(C_1x + C_2y) + \varphi(C_1x + C_2y) + C_3x + C_4y,$$

$$w(x,y) = C_1y^2 + C_2xy + \frac{1}{4C_1}(C_2^2 - A)x^2 + C_3y + C_4x + C_5,$$

$$w(x,y) = \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) - \frac{A}{12C_2}(x^3 + 3C_1x^2) + C_4y + C_5x + C_6,$$

$$w(x,y) = \pm\frac{2\sqrt{A}}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2} + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

Another five solutions can be obtained:

(a) from the solution of equation 11.2.2.18 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant;
(b) from the solution of equation 11.2.2.22 with $f(u) = A$, where a, b, and c are arbitrary constants;
(c) from the solution of equation 11.2.2.23 with $f(u) = A$, where a, b, c, k, and s are arbitrary constants;
(d) from the solution of equation 11.2.2.24 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant;
(e) from the solution of equation 11.2.2.26 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant.

4°. The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi, \eta)$ is the new independent variable, and ξ and η are the new dependent variables, leads to an equation of the similar form

$$\left(\frac{\partial^2 u}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 u}{\partial\xi^2}\frac{\partial^2 u}{\partial\eta^2} = \frac{1}{A}.$$

⊙ *References for equation* 11.2.2.2: E. Goursat (1933, pp. 63–64), A. D. Polyanin and V. F. Zaitsev (2004, pp. 452–453).

3. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1}w(x, C_2x \pm C_1y + C_3) + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions quadratic in y:

$$w(x,y)=C_1y^2+C_2xy+\frac{C_2^2}{4C_1}x^2-\frac{1}{2C_1}\int_0^x(x-t)f(t)\,dt+C_3y+C_4x+C_5,$$

$$w(x,y)=\frac{1}{x+C_1}\left(C_2y^2+C_3y+\frac{C_3^2}{4C_2}\right)-\frac{1}{2C_2}\int_0^x(x-t)(t+C_1)f(t)\,dt+C_4y+C_5x+C_6,$$

where $C_1,\ldots,C_6$ are arbitrary constants.

3°. Generalized separable solutions for $f(x)>0$:

$$w(x,y)=\pm y\int\sqrt{f(x)}\,dx+\varphi(x)+C_1y,$$

where $\varphi(x)$ is an arbitrary function.

4°. Conservation law:

$$D_x\big[y(w_xw_{yy}-w_yw_{xy}+g_y)-g-w_xw_y\big]+D_y\big[y(w_yw_{xx}-w_xw_{xy}+g_x)+(w_x)^2\big]=0,$$

where $D_x=\dfrac{\partial}{\partial x}$, $D_y=\dfrac{\partial}{\partial y}$, $g=y\displaystyle\int f(x)\,dx+\varphi(x)+\psi(y)$, and $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

5°. Let us consider some specific functions $f=f(x)$. Solutions that can be obtained by the formulas of Items 1° and 2° are omitted.

5.1. Solutions with $f(x)=Ax^k$ can be obtained:

(a) from the solution of equation 11.2.2.18 with $f(u)=A$ and $\alpha=k/2$, where β is an arbitrary constant;
(b) from the solution of equation 11.2.2.26 with $f(u)=A$ and $\alpha=k/2$, where β is an arbitrary constant.

5.2. Solutions for $f(x)=Ae^{\lambda x}$:

$$w(x,y)=\pm\frac{2\sqrt{A}}{C_2\lambda}e^{\lambda x/2}\sin(C_1x+C_2y+C_3)+C_4x+C_5y+C_6,$$

$$w(x,y)=\pm\frac{2\sqrt{A}}{C_2\lambda}e^{\lambda x/2}\sinh(C_1x+C_2y+C_3)+C_4x+C_5y+C_6,$$

$$w(x,y)=\pm\frac{2\sqrt{-A}}{C_2\lambda}e^{\lambda x/2}\cosh(C_1x+C_2y+C_3)+C_4x+C_5y+C_6.$$

Another solution can be obtained from the solution of equation 11.2.2.24 with $\alpha=\lambda$ and $f(u)=A$, where β is an arbitrary constant.

⊙ *References for equation* 11.2.2.3: M. N. Martin (1953), C. S. Ludford (1955), B. L. Rozhdestvenskii and N. N. Yanenko (1983), S. V. Khabirov (1990 *b*), A. D. Polyanin and V. F. Zaitsev (2002, p. 250).

4. $\left(\dfrac{\partial^2w}{\partial x\partial y}\right)^2-\dfrac{\partial^2w}{\partial x^2}\dfrac{\partial^2w}{\partial y^2}=f(x)y.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=\pm C_1^{-3}w(x,C_1^2y)+C_2x+C_3y+C_4,$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in y:

$$w(x,y) = C_1y^2 - y\int F(x)\,dx + \frac{1}{2C_1}\int_a^x (x-t)F^2(t)\,dt + C_2x + C_3y + C_4,$$
$$F(x) = \frac{1}{2C_1}\int f(x)\,dx + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where

$$\varphi(x) = \frac{1}{C_1x + C_2}, \quad \psi(x) = C_3\varphi(x) + C_4 + \frac{\varphi(x)}{2C_1}\int \frac{f(x)\,dx}{[\varphi(x)]^3} - \frac{1}{2C_1}\int \frac{f(x)\,dx}{[\varphi(x)]^2},$$
$$\chi(x) = \frac{1}{2}\int_a^x (x-t)\frac{[\psi_t'(t)]^2}{\varphi(t)}\,dt + C_5x + C_6.$$

4°. Generalized separable solutions cubic in y:

$$w(x,y) = C_1y^3 - \frac{1}{6C_1}\int_a^x (x-t)f(t)\,dt + C_2x + C_3y + C_4,$$
$$w(x,y) = \frac{y^3}{(C_1x + C_2)^2} - \frac{1}{6}\int_a^x (x-t)(C_1t + C_2)^2 f(t)\,dt + C_3x + C_4y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. See solution of equation 11.2.2.7 in Item 3° with $k = 1$.

⊙ *Reference for equation* 11.2.2.4: A. D. Polyanin and V. F. Zaitsev (2002, p. 250).

5. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^2.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = \pm C_1^{-2}w(x, C_1y) + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \left[C_1\int \varphi^2(x)\,dx + C_2\right]y + \frac{1}{2}C_1^2\int_a^x (x-t)\varphi^3(t)\,dt + C_3x + C_4,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi_{xx}'' = 2(\varphi_x')^2 - \tfrac{1}{2}f(x).$$

3°. Generalized separable solutions in the form of polynomials of degree 4 in y:

$$w(x,y) = C_1y^4 - \frac{1}{12C_1}\int_a^x (x-t)f(t)\,dt + C_2x + C_3y + C_4,$$
$$w(x,y) = \frac{y^4}{(C_1x + C_2)^3} - \frac{1}{12}\int_a^x (x-t)(C_1t + C_2)^3 f(t)\,dt + C_3x + C_4y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

4°. See solution of equation 11.2.2.7 in Item 3° with $k = 2$.

⊙ *Reference for equation* 11.2.2.5: A. D. Polyanin and V. F. Zaitsev (2002, p. 251).

6. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^2 + g(x)y + h(x).$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi\varphi''_{xx} &= 2(\varphi'_x)^2 - \tfrac{1}{2}f(x),\\
\varphi\psi''_{xx} &= 2\varphi'_x\psi'_x - \tfrac{1}{2}g(x),\\
\varphi\chi''_{xx} &= \tfrac{1}{2}(\psi'_x)^2 - \tfrac{1}{2}h(x).
\end{aligned}$$

7. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-k-2} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x,y) = \frac{C_1 y^{k+2}}{(k+1)(k+2)} - \frac{1}{C_1}\int_a^x (x-t)f(t)\,dt + C_2 x + C_3 y + C_4,$$

$$w(x,y) = \frac{y^{k+2}}{(C_1 x + C_2)^{k+1}} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)(C_1 t + C_2)^{k+1} f(t)\,dt + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x) y^{\frac{k+2}{2}},$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + 4f(x) = 0.$$

4°. Let us consider the case where f is a power-law function of x, $f(x) = Ax^n$, in more detail.

Solutions:

$$w(x,y) = \frac{C_1 x^{n+2}}{(n+1)(n+2)} - \frac{A y^{k+2}}{C_1(k+1)(k+2)} + C_2 y + C_3 x + C_4,$$

$$w(x,y) = \frac{C_1 x^{n+2}}{(n+1)(n+2)y^{n+1}} - \frac{A y^{k+n+3}}{C_1(k+n+2)(k+n+3)} + C_2 y + C_3 x + C_4,$$

$$w(x,y) = \frac{C_1 y^{k+2}}{(k+1)(k+2)x^{k+1}} - \frac{A x^{k+n+3}}{C_1(k+n+2)(k+n+3)} + C_2 y + C_3 x + C_4,$$

$$w(x,y) = (C_1 x + C_2)^{-k-1} y^{k+2} - \frac{A}{(k+1)(k+2)}\int_a^x (x-t)t^n (C_1 t + C_2)^{k+1}\,dt + C_3 y + C_4 x,$$

$$w(x,y) = (C_1 y + C_2)^{-n-1} x^{n+2} - \frac{A}{(n+1)(n+2)}\int_a^y (y-t)t^k (C_1 t + C_2)^{n+1}\,dt + C_3 y + C_4 x,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

There is also a multiplicative separable solution, see Item 3° with $f(x) = Ax^n$, and a solution of the same type:

$$w(x,y) = \psi(y)x^{\frac{n+2}{2}},$$

where the function $\psi = \psi(y)$ is determined by the ordinary differential equation

$$n(n+2)\psi\psi''_{yy} - (n+2)^2(\psi'_y)^2 + 4Ay^k = 0.$$

The substitution $\psi = U^{-n/2}$ brings it to the Emden–Fowler equation

$$U''_{yy} = \frac{8A}{n^2(n+2)}y^kU^{n+1},$$

whose exact solutions for various values of k and n can be found in the books by Polyanin and Zaitsev (1995, 2003).

Another exact solution for $f(x) = Ax^n$ can be obtained from the solution of equation 11.2.2.18 with $f(u) = Au^k$ and $n = 2\alpha + k\beta$, where α and β are arbitrary constants.

⊙ *Reference for equation* 11.2.2.7: A. D. Polyanin and V. F. Zaitsev (2002, pp. 251–252).

8. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^{2k+2} + g(x)y^k.$

Generalized separable solution:

$$w(x,y) = \varphi(x)y^{k+2} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)\frac{g(t)}{\varphi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$(k+1)(k+2)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + f(x) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 252).

9. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{\lambda y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1 w\Big(x,\ y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x,y) = C_1\int_a^x (x-t)f(t)\,dt + C_2x - \frac{1}{C_1\lambda^2}e^{\lambda y} + C_3y + C_4,$$

$$w(x,y) = C_1e^{\beta x+\lambda y} - \frac{1}{C_1\lambda^2}\int_a^x (x-t)e^{-\beta t}f(t)\,dt + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ and β are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\exp\big(\tfrac{1}{2}\lambda y\big),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi''_{xx} - (\varphi'_x)^2 + 4\lambda^{-2}f(x) = 0.$$

⊙ *Reference for equation* 11.2.2.9: A. D. Polyanin and V. F. Zaitsev (2002, p. 252).

10. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{2\lambda y} + g(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{1}{\lambda^2}\int_a^x (x-t)\frac{g(t)}{\varphi(t)}\,dt + C_1 x + C_2 y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi''_{xx} - (\varphi'_x)^2 + \lambda^{-2} f(x) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 252).

11. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)g(y) + A^2.$

Generalized separable solutions:

$$w(x,y) = C_1\int_a^x (x-t)f(t)\,dt - \frac{1}{C_1}\int_b^y (y-\xi)g(\xi)\,d\xi \pm Axy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants; a and b are any numbers.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 457).

12. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax + by).$

1°. Solutions:

$$w(x,y) = \pm\frac{x}{b}\int \sqrt{f(z)}\,dz + \varphi(z) + C_1 x + C_2 y, \qquad z = ax + by,$$

where C_1 and C_2 are arbitrary constants and $\varphi(z)$ is an arbitrary function.

2°. The transformation

$$w = U(x,z), \quad z = ax + by$$

leads to an equation of the form 11.2.2.3:

$$\left(\frac{\partial^2 U}{\partial x\partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2} f(z).$$

Here, x and z play the role of y and x in 11.2.2.3, respectively.

⊙ *References for equation* 11.2.2.12: M. N. Martin (1953), C. S. Lundford (1955), B. L. Rozhdestvenskii and N. N. Yanenko (1983).

13. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^k f(ax + by).$

The transformation

$$w = U(x,z), \quad z = ax + by$$

leads to an equation of the form 11.2.2.7:

$$\left(\frac{\partial^2 U}{\partial x\partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2} x^k f(z).$$

Here, x and z play the role of y and x in 11.2.2.7, respectively.

14. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^{2k+2} f(ax+by) + x^k g(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax + by$$

leads to an equation of the form 11.2.2.8:

$$\left(\frac{\partial^2 U}{\partial x\partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}x^{k+2}f(z) + b^{-2}x^k g(z).$$

Here, x and z play the role of y and x in 11.2.2.8, respectively.

15. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{\lambda x} f(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax + by$$

leads to an equation of the form 11.2.2.9:

$$\left(\frac{\partial^2 U}{\partial x\partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}e^{\lambda x}f(z).$$

Here, x and z play the role of y and x in 11.2.2.9, respectively.

16. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{2\lambda x} f(ax+by) + e^{\lambda x} g(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax + by$$

leads to an equation of the form 11.2.2.10:

$$\left(\frac{\partial^2 U}{\partial x\partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}e^{2\lambda x}f(z) + b^{-2}e^{\lambda x}g(z).$$

Here, x and z play the role of y and x in 11.2.2.10, respectively.

17. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = \dfrac{1}{x^4} f\left(\dfrac{y}{x}\right).$

This is a special case of equation 11.2.2.18 with $\alpha = -2$ and $\beta = -1$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm w(C_1 x,\, C_1 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. First integral:

$$w - x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} \pm \int \sqrt{f(z)}\,dz = C, \qquad z = \frac{y}{x},$$

where C is an arbitrary constant.

3°. Solutions:

$$w = x\varphi\left(\frac{y}{x}\right) \pm \int \sqrt{f(z)}\,dz + C, \qquad z = \frac{y}{x},$$

where $\varphi(z)$ is an arbitrary function.

4°. Conservation law:

$$D_x\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) + D_y\left[-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + x^{-3}F\left(\frac{y}{x}\right)\right] = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(z) = \displaystyle\int f(z)\,dz$.

⊙ *References for equation* 11.2.2.17: M. N. Martin (1953), C. S. Lundford (1955), B. L. Rozhdestvenskii and N. N. Yanenko (1983), S. V. Khabirov (1990 *b*).

18. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^{2\alpha} f(x^\beta y).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{\beta-\alpha-1} w(C_1 x,\, C_1^{-\beta} y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x, y) = x^{\alpha-\beta+1} u(z) \quad z = x^\beta y,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$[\beta(\beta+1)zu'_z + (\alpha-\beta)(\beta-\alpha-1)u]u''_{zz} + (\alpha+1)^2(u'_z)^2 - f(z) = 0.$$

⊙ *Reference for equation* 11.2.2.18: S. V. Khabirov (1990 *b*).

19. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax - by^2).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = w(x + 2bC_1 y + abC_1,\, y + aC_1) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x, y) = \pm \int \sqrt{F(z) + C_1}\,dz + C_2 x + C_3 y + C_4, \qquad F(z) = \frac{1}{a^2 b}\int f(z)\,dz, \quad z = ax - by^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

⊙ *Reference for equation* 11.2.2.19: A. D. Polyanin and V. F. Zaitsev (2002, p. 254).

20. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x^2+y^2).$

Solutions in polar coordinates:

$$w = C_1\varphi \pm \int \sqrt{C_2 - 2F(r) - C_1^2 r^{-2}}\,dr + C_3,$$
$$r = \sqrt{x^2+y^2}, \quad \tan\varphi = \frac{y}{x}, \quad F(r) = \int r f(r)\,dr,$$

where C_1, C_2, and C_3 are arbitrary constants.

21. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = (x^2+y^2)^k f\Big(\dfrac{y}{x}\Big).$

Solutions in polar coordinates:

$$w = r^{k+2}u(\varphi) \qquad (x = r\cos\varphi, \quad y = r\sin\varphi),$$

where the function $u = u(\varphi)$ is determined by the ordinary differential equation

$$(k+1)^2(u'_\varphi)^2 - (k+1)(k+2)u[u''_{\varphi\varphi} + (k+2)u] = f(\tan\varphi).$$

22. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax^2+bxy+cy^2).$

Solution for $b^2 \neq 4ac$:

$$w(x,y) = u(z) \quad z = ax^2 + bxy + cy^2,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2(4ac-b^2)zu'_z u''_{zz} + (4ac-b^2)(u'_z)^2 + f(z) = 0.$$

Integrating yields

$$u(z) = \pm\int \sqrt{\frac{F(z)}{z}}\,dz + C_1, \quad F(z) = \frac{1}{b^2-4ac}\int f(z)\,dz + C_2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 254).

23. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax^2+bxy+cy^2+kx+sy).$

Solution:

$$w(x,y) = u(z), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2\big[(4ac-b^2)z + as^2 + ck^2 - bks\big]u'_z u''_{zz} + (4ac-b^2)(u'_z)^2 + f(z) = 0.$$

The substitution $V(z) = (u'_z)^2$ leads to a first-order linear equation.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, p. 254).

24. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{\alpha x} f(e^{\beta x} y).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{\alpha-2\beta} w(x - 2\ln C_1,\ C_1^{2\beta} y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized self-similar solution:

$$w(x, y) = e^{\mu x} U(z), \quad z = e^{\beta x} y, \quad \mu = \tfrac{1}{2}\alpha - \beta,$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$\beta^2 z U_z' U_{zz}'' - \mu^2 U U_{zz}'' + (\beta + \mu)^2 (U_z')^2 - f(z) = 0.$$

⊙ *Reference for equation* 11.2.2.24: S. V. Khabirov (1990 *b*).

25. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{ky/x} f(x).$

Solution:

$$w(x, y) = \exp\left(\frac{ky}{2x}\right)\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$x^2\varphi\varphi_{xx}'' - x^2(\varphi_x')^2 + 2x\varphi\varphi_x' - \varphi^2 + 4k^{-2}x^4 f(x) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990 *b*).

26. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^{2\alpha} f(x^\beta e^{y/x}).$

Solution:

$$w(x, y) = x^{\alpha+2} u(z), \quad z = x^\beta e^{y/x},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$z^2\left[\beta z u_z' + (\alpha+2)(\alpha+1)u\right]u_{zz}'' + z\left\{\left[\beta - (\alpha+1)^2\right]z u_z' + (\alpha+2)(\alpha+1)u\right\}u_z' + f(z) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990 *b*).

27. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = y^{-4}\exp(2\alpha y^{-1}) f(xy^{-1} + \beta y^{-2}).$

Solution:

$$w = y\exp(\alpha y^{-1})\varphi(z) + C_1 y + C_2 x + C_3, \quad z = xy^{-1} + \beta y^{-2},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$(2\beta\varphi_z' + \alpha^2\varphi)\varphi_{zz}'' - \alpha^2\varphi_z'^2 + f(z) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990 *b*).

▷ *For exact solutions of the nonhomogeneous Monge–Ampère equation for some specific $F = F(x, y)$ (without functional arbitrariness), see Khabirov (1990 b,c) and Ibragimov (1994). The Cauchy problem for the Monge–Ampère equation is discussed in Courant and Hilbert (1989).*

11.2.3. Equations of the Form

$$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)$$

1. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)w + g(x).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x, y + C_1 x + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in y:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by the system of ordinary differential equations

$$2\varphi\varphi''_{xx} + f(x)\varphi - 4(\varphi'_x)^2 = 0,$$
$$2\varphi\psi''_{xx} + f(x)\psi - 4\varphi'_x\psi'_x = 0,$$
$$2\varphi\chi''_{xx} + f(x)\chi + g(x) - (\psi'_x)^2 = 0.$$

Note that the second equation is linear in ψ and has a particular solution $\psi = \varphi$ (hence, its general solution can be expressed via the particular solution of the first equation).

2. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)w^2.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, y + C_2 x + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x, y) = e^{\lambda y}u(x),$$

where λ is an arbitrary constant and the function $u = u(x)$ is determined by the ordinary differential equation

$$uu''_{xx} - (u'_x)^2 + \lambda^{-2}f(x)u^2 = 0.$$

3. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^n w^k.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C^{n+2}w(x, C^{k-2}y),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution with $n \neq -2$ and $k \neq 2$:

$$w(x, y) = y^{\frac{n+2}{2-k}}U(x),$$

where the function $U(x)$ is determined by the ordinary differential equation

$$(n + 2)(n + k)UU''_{xx} - (n + 2)^2(U'_x)^2 + (k - 2)^2 f(x)U^k = 0.$$

4. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{\lambda y}w^k.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = Cw\Big(x, y + \frac{k-2}{\lambda}\ln C\Big),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution with $k \neq 2$ and $\lambda \neq 0$:

$$w(x, y) = \exp\left(\frac{\lambda y}{2-k}\right)U(x),$$

where the function $U(x)$ is determined by the ordinary differential equation

$$UU''_{xx} - (U'_x)^2 + (k-2)^2\lambda^{-2}f(x)U^k = 0.$$

5. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(w).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(A_1x + B_1y + C_1, A_2x + B_2y + C_2), \qquad |A_2B_1 - A_1B_2| = 1,$$

where C_1, C_2, and any three of the four constants A_1, A_2, B_1, and B_2 are arbitrary, is also a solution of the equation.

2°. Functional separable solution:

$$w(x, y) = u(z), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z)$ is determined by the ordinary differential equation

$$2\big[(4ac - b^2)z + as^2 + ck^2 - bks\big]u'_zu''_{zz} + (4ac - b^2)(u'_z)^2 + f(u) = 0.$$

⊙ *Reference for equation* 11.2.3.5: A. D. Polyanin and V. F. Zaitsev (2004, p. 462).

6. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + f(x)\exp\left(\dfrac{ay}{x}\right)w^k.$

Solution:

$$w(x, y) = \exp\left(\frac{\lambda y}{x}\right)u(x), \qquad \lambda = \frac{a}{2-k},$$

where the function $u = u(x)$ is determined by the ordinary differential equation

$$x^2uu''_{xx} - (xu'_x - u)^2 + \lambda^{-2}x^4f(x)u^k = 0.$$

7. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + a\dfrac{\partial w}{\partial y}.$

This equation is used in meteorology for describing wind fields in near-equatorial regions.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-2}C_2^{-1}w(C_1x + C_3, C_2y + C_4x + C_5) + C_6x + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w = \varphi(x),$$
$$w = \tfrac{1}{4}(\sqrt{a}\,x + C)^2y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant.

3°. Solutions:

$$w = C_1e^{\lambda y} - \tfrac{1}{2}a\lambda^{-1}x^2 + C_2x + C_3,$$
$$w = \tfrac{1}{4}a(x + C_1)^2(y + C_2),$$
$$w = \tfrac{1}{4}aC_2^{-1}(x + C_1)^2\tanh(C_2y + C_3),$$
$$w = \tfrac{1}{4}aC_2^{-1}(x + C_1)^2\coth(C_2y + C_3),$$
$$w = \tfrac{1}{4}aC_2^{-1}(x + C_1)^2\tan(C_2y + C_3),$$

where C_1, C_2, C_3, and λ are arbitrary constants. The first solution is a solution in additive separable form and the other four are multiplicative separable solutions.

4°. Generalized separable solution quadratic in y:

$$w = F(x)y^2 + G(x)y + H(x),$$

where

$$F(x) = \frac{1}{C_1x + C_2}, \quad G(x) = -\frac{a}{6C_1^2}(C_1x + C_2)^2 + \frac{C_3}{C_1x + C_2} + C_4,$$
$$H(x) = \frac{1}{2}\int_0^x (x - t)\frac{[G_t'(t)]^2 - aG(t)}{F(t)}\,dt + C_5x + C_6,$$

and $C_1, \ldots, C_6$ are arbitrary constants.

5°. Generalized separable solution:

$$w = C_1\exp(C_2x + C_3y) - \frac{a}{2C_3}x^2 + C_4x + C_5.$$

6°. There are exact solutions of the following forms:

$$w(x, y) = |x|^{k+2}U(z), \quad z = y|x|^{-k};$$
$$w(x, y) = e^{kx}V(\xi), \quad \xi = ye^{-kx};$$
$$w(x, y) = x^2W(\eta), \quad \eta = y + k\ln|x|,$$

where k is an arbitrary constant.

7°. The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi,\eta)$ is the new dependent variable and ξ and η are the new independent variables, leads to the simpler equation

$$\left(\frac{\partial^2 u}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 u}{\partial\xi^2}\frac{\partial^2 u}{\partial\eta^2} = \frac{1}{a\eta}.$$

For exact solutions of this equation, see 11.2.2.3.

⊙ *References for equation* 11.2.3.7: E. R. Rozendorn (1984), A. D. Polyanin and V. F. Zaitsev (2004, p. 463).

8. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f(x)\frac{\partial w}{\partial y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(x, C_1 y + C_2 x + C_3) + C_4 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w = \varphi(x),$$
$$w = \frac{1}{4}\left[\int f(x)\,dx + C\right]^2 y + \varphi(x),$$
$$w = C_1 \exp(C_2 x + C_3 y) - \frac{1}{C_3}\int_0^x (x-t) f(t)\,dt + C_4 x + C_5,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant. For $C_2 = 0$, the last solution is an additive separable solution.

3°. Generalized separable solution quadratic in y:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where

$$\varphi(x) = \frac{1}{C_1 x + C_2}, \quad \psi(x) = -\int\left[\varphi^2(x)\int\frac{f(x)}{\varphi^2(x)}\,dx\right]dx + \frac{C_3}{C_1 x + C_2} + C_4,$$
$$\chi(x) = \frac{1}{2}\int_0^x (x-t)\frac{[\psi_t'(t)]^2 - f(t)\psi(t)}{\varphi(t)}\,dt + C_5 x + C_6,$$

and $C_1, \ldots, C_6$ are arbitrary constants.

⊙ *Reference for equation* 11.2.3.8: A. D. Polyanin and V. F. Zaitsev (2004, p. 464).

9. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(w)\left(\dfrac{\partial w}{\partial x}\right)^4.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$x - y\int \sqrt{f(w)}\,dw = \varphi(w),$$
$$x + y\int \sqrt{f(w)}\,dw = \psi(w),$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions.

10. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(w)\left(\dfrac{\partial w}{\partial x}\right)^2\left(\dfrac{\partial w}{\partial y}\right)^2.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_3 y + C_4),$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$C_1 x + C_2 y \exp\left(\int \sqrt{f(w)}\,dw\right) = \varphi(w),$$
$$C_3 x + C_4 y \exp\left(-\int \sqrt{f(w)}\,dw\right) = \psi(w),$$

where $\varphi(w)$ and $\psi(w)$ are arbitrary functions.

11. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = -f(w)\left(\dfrac{\partial w}{\partial x}\right)^3\dfrac{\partial w}{\partial y}.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1 y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$x - y\left(\frac{1}{2}\int \sqrt{f(w)}\,dw + C\right)^2 = \varphi(w),$$

where $\varphi(w)$ is an arbitrary function and C is an arbitrary constant.

12. $$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f_1(w)\left(\frac{\partial w}{\partial x}\right)^4 + f_2(w)\left(\frac{\partial w}{\partial x}\right)^3\frac{\partial w}{\partial y} + f_3(w)\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + f_4(w)\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y}\right)^3 + f_5(w)\left(\frac{\partial w}{\partial y}\right)^4.$$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$x - \varphi(w)y = \psi(w),$$

where $\psi(w)$ is an arbitrary function and the function $\varphi = \varphi(w)$ is determined by the first-order ordinary differential equation

$$(\varphi'_w)^2 = f_1(w) - f_2(w)\varphi + f_3(w)\varphi^2 - f_4(w)\varphi^3 + f_5(w)\varphi^4.$$

13. $$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f\left(\frac{\partial w}{\partial x}\right).$$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1}w(C_1x + C_2y + C_3, \pm y + C_4) + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x, y) = \varphi(y)x + \psi(y),$$

where $\psi(y)$ is an arbitrary function and the function $\varphi(y)$ is defined implicitly by

$$\int \frac{d\varphi}{\sqrt{f(\varphi)}} = \pm y + C,$$

where C is an arbitrary constant.

3°. Additive separable solution:

$$w(x, y) = C_1y^2 + C_2y + C_3 + z(x),$$

where the function $z(x)$ is determined by the autonomous ordinary differential equation $2C_1z''_{xx} + f(z'_x) = 0$. Its general solution can be written out in parametric form as

$$x = -2C_1\int \frac{dt}{f(t)} + C_3, \qquad z = -2C_1\int \frac{t\,dt}{f(t)} + C_4.$$

4°. Solution (generalizes the solution of Item 3°):

$$w(x, y) = C_1y^2 + C_2y + C_3 + z(\zeta), \quad \zeta = k_1x + k_2y,$$

where the function $z(\zeta)$ is determined by the autonomous ordinary differential equation $2C_1k_1^2z''_{\zeta\zeta} + f(k_1z'_\zeta) = 0$, whose general solution can be written out in parametric form in a similar manner as in Item 3°.

5°. The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi,\eta)$ is the new dependent variable and ξ and η are the new independent variables, leads to an equation of the form 11.2.2.3:

$$\left(\frac{\partial^2 u}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 u}{\partial\xi^2}\frac{\partial^2 u}{\partial\eta^2} = \frac{1}{f(\xi)}.$$

14. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(w)\left(\frac{\partial w}{\partial x}\right)^4 g\left(\frac{\partial w}{\partial y}\Big/\frac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$x + y\varphi_{1,2}(w) = \psi(w).$$

Here $\psi(w)$ is an arbitrary function and the functions $\varphi_{1,2}(w)$ are defined parametrically by

$$\int \frac{d\varphi}{\sqrt{|g(\varphi)|}} = \pm\int \sqrt{|f(w)|}\,dw + C,$$

where C is an arbitrary constant ($fg > 0$).

15. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + F\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right).$

The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi,\eta)$ is the new dependent variable and ξ and η are the new independent variables, leads to the simpler equation

$$\left(\frac{\partial^2 u}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 u}{\partial\xi^2}\frac{\partial^2 u}{\partial\eta^2} = \frac{1}{F(\xi,\eta)}.$$

For exact solutions of this equation, see Subsection 11.2.2.

11.2.4. Equations of the Form $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x,y)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x,y)$

1. $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_2 y + C_3) + C_4 x + C_5 y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Degenerate solutions involving arbitrary functions:

$$w(x,y) = \varphi(x) + C_1 y + C_2,$$
$$w(x,y) = \varphi(y) + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + [C_1\varphi(x) + C_2]y + \frac{C_1^2}{2}\int_0^x (x-t)\frac{[\varphi_t'(t)]^2}{f(t)\varphi(t)}\,dt + C_3 x + C_4,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - 2(\varphi_x')^2 = 0.$$

4°. Generalized separable solution involving an arbitrary power of y:

$$w(x,y) = \varphi(x)y^k + C_1 x + C_2 y + C_3$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$(k-1)f(x)\varphi\varphi_{xx}'' - k(\varphi_x')^2 = 0.$$

5°. Generalized separable solution involving an exponential of y:

$$w(x,y) = \varphi(x)e^{\lambda y} + C_1 x + C_2 y + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - (\varphi_x')^2 = 0.$$

2. $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-1} w(x, C_1 y + C_2) + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in y:

$$w(x,y) = \pm y \int \sqrt{g(x)}\,dx + \varphi(x) + C_1 y,$$

where $\varphi(x)$ is an arbitrary function.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + [C_1\varphi(x) + C_2]y + \frac{1}{2}\int_0^x (x-t)\frac{C_1^2[\varphi_t'(t)]^2 - g(t)}{f(t)\varphi(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - 2(\varphi_x')^2 = 0,$$

which has a particular solution $\varphi = C_6$.

3. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x)y.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-3} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution cubic in y:

$$w(x,y) = C_1 y^3 + C_2 y - \frac{1}{6C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

A more general solution is given by

$$w(x,y) = \varphi(x)y^3 + C_1 y - \frac{1}{6}\int_a^x (x-t)\frac{g(t)\,dt}{f(t)\varphi(t)} + C_2 x + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$2f(x)\varphi\varphi_{xx}'' - 3(\varphi_x')^2 = 0.$$

3°. For an exact solution quadratic in y, see equation 11.2.4.5 with $g_2 = g_0 = 0$.

4°. See the solution of equation 11.2.4.6 in Item 3° with $k = 1$.

4. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x)y^2.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-2} w(x, C_1 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution involving y to the power of four:

$$w(x,y) = C_1 y^4 + C_2 y - \frac{1}{12C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

A more general solution is given by

$$w(x,y) = \varphi(x)y^4 + C_1 y - \frac{1}{12}\int_a^x (x-t)\frac{g(t)\,dt}{f(t)\varphi(t)} + C_2 x + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$3f(x)\varphi\varphi''_{xx} - 4(\varphi'_x)^2 = 0.$$

3°. For an exact solution quadratic in y, see equation 11.2.4.5 with $g_1 = g_0 = 0$.

4°. See the solution of equation 11.2.4.6 in Item 3° with $k = 2$.

5. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g_2(x)y^2 + g_1(x)y + g_0(x).$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$f(x)\varphi\varphi''_{xx} = 2(\varphi'_x)^2 - \tfrac{1}{2}g_2(x),$$
$$f(x)\varphi\psi''_{xx} = 2\varphi'_x\psi'_x - \tfrac{1}{2}g_1(x),$$
$$f(x)\varphi\chi''_{xx} = \tfrac{1}{2}(\psi'_x)^2 - \tfrac{1}{2}g_0(x).$$

6. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)y^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-k-2} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y) = \frac{C_1 y^{k+2}}{(k+1)(k+2)} + C_2 y - \frac{1}{C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)y^{\frac{k+2}{2}},$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)f(x)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + 4g(x) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = \psi(x)y^{k+2} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)\frac{g(t)}{f(t)\psi(t)}\,dt + C_1 x + C_2 y + C_3,$$

where the function $\psi = \psi(x)$ is determined by the ordinary differential equation

$$(k+1)f(x)\psi\psi''_{xx} - (k+2)(\psi'_x)^2 = 0.$$

7. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)y^{2k+2} + h(x)y^k.$

Generalized separable solution:

$$w(x,y) = \varphi(x)y^{k+2} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)\frac{h(t)}{f(t)\varphi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$(k+1)(k+2)f(x)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + g(x) = 0.$$

8. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)e^{\lambda y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(x, y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y) = C_1e^{\lambda y} + C_2y - \frac{1}{C_1\lambda^2}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\exp\big(\tfrac{1}{2}\lambda y\big),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi''_{xx} - (\varphi'_x)^2 + 4\lambda^{-2}g(x) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = \psi(x)e^{\lambda y} - \frac{1}{\lambda^2}\int_a^x (x-t)\frac{g(t)}{f(t)\psi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\psi = \psi(x)$ is determined by the ordinary differential equation

$$f(x)\psi\psi''_{xx} - (\psi'_x)^2 = 0.$$

9. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)e^{2\lambda y} + h(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{1}{\lambda^2}\int_a^x (x-t)\frac{h(t)}{f(t)\varphi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi''_{xx} - (\varphi'_x)^2 + \lambda^{-2}g(x) = 0.$$

10. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f_1(x)g_1(y)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + f_2(x)g_2(y).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x, y) + C_1x + C_2y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution for $f_1g_1 \neq 0$:

$$w(x, y) = C_1\int_a^x (x-t)\frac{f_2(t)}{f_1(t)}\,dt - \frac{1}{C_1}\int_b^y (y-\xi)\frac{g_2(\xi)}{g_1(\xi)}\,d\xi + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Degenerate solutions for $f_2g_2 = 0$:

$$w(x, y) = \varphi(x) + C_1y + C_2,$$
$$w(x, y) = \varphi(y) + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

4°. Generalized separable solution for $f_2g_2 = 0$:

$$w(x, y) = \varphi(x)\psi(y) + C_1x + C_2y + C_3,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations

$$f_1(x)\varphi\varphi''_{xx} - C_4(\varphi'_x)^2 = 0,$$
$$C_4g_1(y)\psi\psi''_{yy} - (\psi'_y)^2 = 0.$$

11. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = f(ax + by)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(ax + by).$

Solution:

$$w(x, y) = \varphi(z) + C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y, \quad z = ax + by,$$

where $C_1, \ldots, C_5$ are arbitrary constants and the function $\varphi(z)$ is determined by the ordinary differential equation

$$(ab\varphi''_{zz} + C_2)^2 = f(z)(a^2\varphi''_{zz} + 2C_1)(b^2\varphi''_{zz} + 2C_3) + g(z),$$

which is easy to integrate; to this end, the equation should first be solved for φ''_{zz}.

11.2.5. Other Equations with Two Independent Variables

1. $aw\left(\dfrac{\partial w}{\partial y}\right)^2\dfrac{\partial^2 w}{\partial x\partial y} + \dfrac{\partial w}{\partial x}\left(\dfrac{\partial^2 w}{\partial y^2}\right)^2 = 0.$

Solution:

$$w = \frac{2C}{a}\tanh\big[Cy + \Phi(x)\big],$$

where $\Phi(x)$ is an arbitrary function and C is an arbitrary constant.

⊙ *Reference*: V. F. Zaitsev (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=211).

2. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial w}{\partial y}.$

General solution:

$$w = -\frac{4F'(y)}{F(y)+G(x)} + \int \left(\frac{F''(y)}{F'(y)}\right)^2 dy,$$

where $F(y)$ and $G(x)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2006, http://eqworld.ipmnet.ru/eqarchive/view.php?id=41).

3. $a\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 \dfrac{\partial^2 w}{\partial x^2} = 0.$

Solution:

$$w = a\ln\big[x+\Phi(y)\big] + C,$$

where $\Phi(y)$ is an arbitrary function.

⊙ *Reference*: V. F. Zaitsev (2008, http://eqworld.ipmnet.ru/eqarchive/view.php?id=210).

4. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + f(x)\dfrac{\partial^2 w}{\partial y^2}.$

The substitution

$$w = U(x,y) - \int_a^x (x-t)f(t)\,dt$$

leads to an equation of the form 11.2.2.1:

$$\left(\frac{\partial^2 U}{\partial x\partial y}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2}.$$

5. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + f(x)\dfrac{\partial^2 w}{\partial x\partial y}.$

First integral:

$$\frac{\partial w}{\partial y} = \Phi\left(\frac{\partial w}{\partial x}\right) + \int f(x)\,dx,$$

where $\Phi(u)$ is an arbitrary function.

6. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + b.$

The substitution

$$w = U(x,y) - \tfrac{1}{2}a_2x^2 - \tfrac{1}{2}a_1y^2$$

leads to an equation of the form 11.2.2.2:

$$\left(\frac{\partial^2 U}{\partial x\partial y}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2} + b - a_1a_2.$$

7. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + b_1\dfrac{\partial w}{\partial x} + b_2\dfrac{\partial w}{\partial y}.$

This equation is used in meteorology for describing horizontal air flows; w is the stream function for the wind velocity, and x and y are coordinates on the earth surface.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, y + C_2) + C_3(b_2x - b_1y) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = C_3 \exp\left[-\frac{b_1C_1 + b_2C_2}{a_1C_1^2 + a_2C_2^2}(C_1x + C_2y)\right] + C_4.$$

3°. Generalized separable solution linear in y:

$$w = y(C_1e^{-\lambda x} + C_2) + \frac{C_1^2}{2a_1}e^{-2\lambda x} + \left(\frac{b_2C_1}{b_1}x + C_3\right)e^{-\lambda x} - \frac{b_2C_2}{b_1}x + C_4, \qquad \lambda = \frac{b_1}{a_1}$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. Generalized separable solution quadratic in y:

$$w = f(x)y^2 + g(x)y + h(x),$$

where the functions $f(x)$, $g(x)$, and $h(x)$ are determined by the system of ordinary differential equations

$$2ff''_{xx} + a_1f''_{xx} + b_1f'_x - 4(f'_x)^2 = 0, \tag{1}$$
$$2fg''_{xx} + a_1g''_{xx} + b_1g'_x - 4f'_xg'_x + 2b_2f = 0, \tag{2}$$
$$2fh''_{xx} + a_1h''_{xx} + b_1h'_x + 2a_2f + b_2g - (g'_x)^2 = 0. \tag{3}$$

This system can be fully integrated. To this end, equation (1) is first reduced, with the change of variable $U(f) = f'_x$, to a first-order linear equation. Equation (2) is linear in g and a fundamental system of solutions of the corresponding homogeneous equation has the form $g_1 = 1$, $g_2 = f(x)$. Equation (3) is finally reduced, with the substitution $V(x) = h'_x$, to first-order linear equation.

Remark. The solutions of Items 3° and 4° can be used to obtain two other solutions by means of the following renaming: $(x, a_1, b_1) \rightleftarrows (y, a_2, b_2)$.

⊙ *References for equation* 11.2.5.7: E. R. Rozendorn (1984), A. D. Polyanin and V. F. Zaitsev (2004, p. 470).

8. $\left(\dfrac{\partial^2 w}{\partial x\partial y}\right)^2 = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + f_1(x)\dfrac{\partial^2 w}{\partial x^2} + f_2(x)\dfrac{\partial^2 w}{\partial y^2} + g_1(x)\dfrac{\partial w}{\partial x} + g_2(x)\dfrac{\partial w}{\partial y}.$

There are generalized separable solutions linear and quadratic in y:

$$w(x, y) = \varphi_1(x)y + \varphi_0(x),$$
$$w(x, y) = \psi_2(x)y^2 + \psi_1(x)y + \psi_0(x).$$

9. $$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = -a\left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial x^2}.$$

General solution:

$$w(x,y) = \frac{1}{a}\big\{aF(Z) + axZ - \ln[G(Z) + ay]\big\},$$

where $Z = Z(x,t)$ is any solution of the transcendental equation

$$-G'_Z(Z) + axG(Z) + aF'_Z(Z)[G(Z) + ay] + a^2xy = 0,$$

and $F(Z)$ and $G(Z)$ are arbitrary functions.

⊙ *Reference*: Yu. N. Kosovtsov (2008 *a*).

10. $$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x)w + h_2(x)y^2 + h_1(x)y + h_0(x).$$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&2f(x)\varphi\varphi''_{xx} - 4(\varphi'_x)^2 + g(x)\varphi + h_2(x) = 0,\\
&2f(x)\varphi\psi''_{xx} - 4\varphi'_x\psi'_x + g(x)\psi + h_1(x) = 0,\\
&2f(x)\varphi\chi''_{xx} - (\psi'_x)^2 + g(x)\chi + h_0(x) = 0.
\end{aligned}$$

11. $$\begin{aligned}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 &= f_1(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \big[f_2(x)w + f_3(x)y^2 + f_4(x)y + f_5(x)\big]\left(\frac{\partial^2 w}{\partial y^2}\right)^2\\
&\quad + g_1(x)\frac{\partial^2 w}{\partial x^2} + \big[g_2(x)w + g_3(x)y^2 + g_4(x)y + g_5(x)\big]\frac{\partial^2 w}{\partial y^2}\\
&\quad + \big[g_6(x)y + g_7(x)\big]\frac{\partial^2 w}{\partial x\partial y} + h_1(x)\left(\frac{\partial w}{\partial y}\right)^2 + h_2(x)\frac{\partial w}{\partial x}\\
&\quad + \big[h_3(x)y + h_4(x)\big]\frac{\partial w}{\partial y} + s_1(x)w + s_2(x)y^2 + s_3(x)y + s_4(x).\end{aligned}$$

There is a generalized separable solution of the form

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

12. $$\frac{\partial^2 w}{\partial x^2}\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 w}{\partial y^2}.$$

This equation occurs in plane problems of plasticity; w is the generating function.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm C_1^{-2}w(C_1x + C_2, C_3y + C_4) + C_5x + C_6y + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants, are also solutions of the equation.

2°. Introduce the new variable

$$U(x,y)=\frac{\partial w}{\partial x}$$

and apply the Legendre transformation (for details, see Subsection 26.3.3)

$$X=\frac{\partial U}{\partial x},\quad Y=\frac{\partial U}{\partial y},\quad Z=x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}-U$$

to obtain a second-order linear equation:

$$(1+X^2)^2\frac{\partial^2 Z}{\partial X^2}+2XY(1+X^2)\frac{\partial^2 Z}{\partial X\partial Y}+Y^2(X^2-1)\frac{\partial^2 Z}{\partial Y^2}=0. \tag{1}$$

This equation is hyperbolic. The transformation

$$t=\arctan X,\quad \xi=\tfrac{1}{2}\ln(1+X^2)-\ln Y,\quad F=\frac{Z}{\sqrt{1+X^2}}$$

brings (1) to a constant-coefficient linear equation:

$$\frac{\partial^2 F}{\partial t^2}=\frac{\partial^2 F}{\partial \xi^2}-F. \tag{2}$$

For solutions of equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The original equation is invariant under the Legendre transformation

$$\overline{x}=\frac{\partial w}{\partial x},\quad \overline{y}=\frac{\partial w}{\partial y},\quad \overline{w}=x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}-w.$$

⊙ *References for equation* 11.2.5.12: Yu. N. Radayev (1988), V. I. Astafiev, Yu. N. Radayev, and L. V. Stepanova (2001).

11.2.6. Plebański Heavenly Equations

1. $\dfrac{\partial^2 w}{\partial t\partial z}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial^2 w}{\partial t\partial x}\dfrac{\partial^2 w}{\partial y\partial z}=1.$

Plebański first heavenly equation.

1°. Suppose $w(t,x,y,z)$ is a solution of the equation in question.

1.1. The functions

$$w_1=\pm w(t,x,y,z)+F(t,y)+G(x,z),$$

where $F(t,y)$ and $G(x,z)$ are arbitrary functions, c is an arbitrary constant, are also solutions of the equation.

1.2. The functions $w_2=\pm w(\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{z})$ with

$$\widetilde{t}=a_1t+a_2y+a_3,\quad \widetilde{x}=b_1x+b_2z+b_3,$$
$$\widetilde{y}=c_1t+\frac{1}{a_1}\left(a_2c_1+\frac{1}{b_1k_2-b_2k_1}\right)y+c_2,\quad \widetilde{z}=k_1x+k_2z+k_3,$$

where a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, k_1, k_2, and k_3 are arbitrary constants, are also solutions of the equation.

1.3. The function $w_3 = aw(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ with

$$\hat{t} = \varphi(t), \quad \hat{x} = \frac{R(x,z)}{a^2} + G(K(x,z)), \quad \hat{y} = \frac{y}{\varphi_t'(t)} + \psi(t), \quad \hat{z} = K(x,z),$$

where $\varphi(t)$, $\psi(t)$, $K(x,z)$, and $G(K)$ are arbitrary functions, a is an arbitrary constant, is also a solution of the equation. The function $R(x,z)$ solves the first-order linear equation $K_z R_x - K_x R_z = 1$. By changing from x and z to the new independent variables x and $\zeta = K(x,z)$, we obtain the equation $R_x = 1/K_z$ for $R = R(x,\zeta)$; this equation is solved by direct integration with respect to x (the derivative K_z must first be expressed in terms of x and ζ).

2°. Solutions:

$$w(t,x,y,z) = \sqrt{z}\left[\frac{xy}{\varphi_t'(t)} + 2\varphi(t)\right],$$

$$w(t,x,y,z) = 2[\varphi(t) + \psi(x)]\sqrt{\frac{yz}{\varphi_t'(t)\psi_x'(x)}},$$

$$w(t,x,y,z) = F(x,y) + \frac{t\varphi(x)}{\psi_y'(y)} - \frac{z\psi(y)}{\varphi_x'(x)},$$

$$w(t,x,y,z) = F(x,y) + tG(x,y) - z\int \frac{dy}{G_x(x,y)}, \qquad G_x(x,y) = \frac{\partial G}{\partial x},$$

$$w(t,x,y,z) = [y\theta(t) + \eta(t)]\int \frac{dx}{\zeta(x)} + [z\zeta(x) + \varphi(x)]\int \frac{dt}{\theta(t)} + F\big(y\theta(t) + \eta(t), z\zeta(x) + \varphi(x)\big),$$

$$w(t,x,y,z) = \sqrt{2[t\varphi(y) + \psi(y)}\left(\zeta(y) + \eta(x) + \theta(x)\int \frac{dy}{\varphi(y)} + \frac{z}{\theta_x'(x)}\right) + \int G\left(y, \zeta(y) + \eta(x) + \theta(x)\int \frac{dy}{\varphi(y)} + \frac{z}{\theta_x'(x)}\right)dy,$$

where F and G are arbitrary functions of two arguments and φ, ψ, η, θ, and ζ are arbitrary functions of a single argument.

3°. See also exact solutions with arbitrary constants in Dryuma (2006).

⊙ *References for equation* 11.2.6.1: J. F. Plebański (1975), A. Macias, J. L. Cervantes-Cota, and C. Lammerzahl (2001), E. V. Ferapontov and M. V. Pavlov (2003), Yu. Kosovtsov (2008 *b*).

2. $\dfrac{\partial^2 w}{\partial t \partial x} + \dfrac{\partial^2 w}{\partial y \partial z} + \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - \left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = 0.$

Plebański second heavenly equation is descriptive of self-dual Einstein spaces with Ricci-flat metrics. It is a completely integrable equation.

1°. Suppose $w(t,x,y,z)$ is a solution of the equation in question.

1.1. The function

$$w_1 = w(z,y,x,t)$$

is also a solution of the equation.

1.2. The function

$$w_2 = w(t,x,y,z) + (y-cx)F(t+cz) + G(t,z),$$

where $F(\zeta)$ and $G(t,z)$ are arbitrary functions, c is an arbitrary constant, is also a solution of the equation.

1.3. The function

$$w_3 = w(t,x,y,z) + y\frac{\partial F(t,z)}{\partial t} - x\frac{\partial F(t,z)}{\partial z} + G(t,z),$$

where $F(t,z)$ and $H(t,z)$ are arbitrary functions, is also a solution of the equation.

1.4. The function

$$w_4 = w(t,x,y,z) + a_t'(t)yz + b(t)y + c(t)z - a(t)x + d(t) + A_z'(z)xt + B(z)x + C(z)t - A(z)y + D(z),$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are arbitrary functions, is also a solution of the equation.

1.5. The function

$$\begin{aligned}
w_5 &= w(\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{z}),\\
\widetilde{t} &= ct + \frac{b^2 - a_1a_2^2c}{a_2^3}z + k_1,\\
\widetilde{x} &= a_1x + a_2y + a_3z - \frac{a_2a_3}{a_1}t + k_2,\\
\widetilde{y} &= \frac{b^2 - a_1a_2^2c}{a_2^2b}x - \frac{a_2c}{b}y + \frac{a_3(b^2 - a_1a_2^2c)}{a_1a_2^2b}z + \frac{a_2a_3c}{a_1b}t + k_3,\\
\widetilde{z} &= bt - \frac{a_1b}{a_2}z + k_4,
\end{aligned}$$

where a_1, a_2, a_3, b, c, k_1, k_2, k_3, and k_4 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$\begin{aligned}
w(t,x,y,z) &= F(y+ax,\ t-az) + G(t,z),\\
w(t,x,y,z) &= \tfrac{1}{2}ay^2 + by + F(x-at,\ z) + G(t,z),\\
w(t,x,y,z) &= y\frac{\partial F(t,z)}{\partial t} - x\frac{\partial F(t,z)}{\partial z} + G(t,z),\\
w(t,x,y,z) &= y\frac{\partial F(x,z)}{\partial x} + t\left[\int\left(\frac{\partial^2 F(x,z)}{\partial x^2}\right)^2 dx - \frac{\partial F(x,z)}{\partial z}\right] + G(t,z),
\end{aligned}$$

where the functions F and G are arbitrary functions of their arguments and a and b are arbitrary constants.

3°. There are generalized separable solutions of the forms

$$\begin{aligned}
w &= \varphi_0(y,z,t) + \varphi_1(y,z,t)x + \varphi_2(y,z,t)x^2,\\
w &= \psi_0(x,z,t) + \psi_1(x,z,t)y + \psi_2(x,z,t)y^2.
\end{aligned}$$

⊙ *References for equation* 11.2.6.2: J. F. Plebański (1975), A. Macias, J. L. Cervantes-Cota, and C. Lammerzahl (2001), F. Neyzi, Y. Nutku, and M. B. Sheftel (2005), V. S. Dryuma (2006), V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 271), Yu. Kosovtsov (2008 *c*).

11.3. Bellman Type Equations and Related Equations

11.3.1. Equations with Quadratic Nonlinearities

1. $\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} - f(t)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} - g(t)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko (1971) and Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, y + C_3, t) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" solutions:

$$w = U(z, \tau), \quad z = y \pm 2\left[x \int g(t)\,dt + C_1 x\right]^{1/2} + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

3°. "Two-dimensional" solution:

$$w = u(\xi, \tau), \quad \xi = y + C_1 x + \frac{1}{C_1}\int g(t)\,dt + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(\xi, \eta)$ is determined by the linear heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. The solutions of Items 2° and 3° are special cases of a more general solution with the form

$$w = U(z, \tau), \quad z = y + \varphi(x, t), \quad \tau = \int f(t)\,dt,$$

where the function $\varphi = \varphi(x, t)$ satisfies the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x} = g(t), \tag{1}$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

A complete integral of equation (1) is given by

$$\varphi = C_1 x + \frac{1}{C_1}\int g(t)\,dt + C_2, \tag{2}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (1) can be represented in parametric form with the complete integral (2) and the two relations (see Subsection 24.1.1)

$$C_2 = \psi(C_1),$$
$$x - \frac{1}{C_1^2}\int g(t)\,dt + \psi'(C_1) = 0,$$

where $\psi = \psi(C_1)$ is an arbitrary function and the prime denotes a derivative with respect to the argument; C_1 and C_2 play the role of parameters.

Remark. To the solution of Item 2° there corresponds $\psi(C_1) = \text{const}$.

5°. "Two-dimensional" solutions:

$$w = \pm\exp\big[\lambda y + \zeta(x,t)\big],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x,t)$ is determined by the first-order partial differential equation

$$\frac{\partial\zeta}{\partial t}\frac{\partial\zeta}{\partial x} - \lambda^2 f(t)\frac{\partial\zeta}{\partial x} - \lambda^2 g(t) = 0. \tag{3}$$

A complete integral of equation (3) is given by

$$\zeta = C_1 x + \lambda^2\int\left[f(t) + \frac{1}{C_1}g(t)\right]dt + C_2, \tag{4}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (3) can be represented in parametric form with the complete integral (4) and the two relations

$$C_2 = \varphi(C_1),$$
$$x - \frac{\lambda^2}{C_1^2}\int g(t)\,dt + \varphi'(C_1) = 0,$$

where $\varphi = \varphi(C_1)$ is an arbitrary function; C_1 and C_2 play the role of parameters.

6°. "Two-dimensional" solution:

$$w = e^{\lambda x}\theta(y,t),$$

where λ is an arbitrary constant, and the function $\theta = \theta(y,t)$ is determined by the "two-dimensional" equation

$$\lambda\theta\frac{\partial\theta}{\partial t} - \lambda f(t)\theta\frac{\partial^2\theta}{\partial y^2} - g(t)\left(\frac{\partial\theta}{\partial y}\right)^2 = 0.$$

7°. Cauchy problems and self-similar solutions of the equation for power-law $f(t)$ and $g(t)$ are discussed in Chernousko (1971) and Chernousko and Kolmanovskii (1978).

⊙ *References for equation* 11.3.1.1: A. S. Bratus' and K. A. Volosov (2002), A. D. Polyanin and V. F. Zaitsev (2002, pp. 259–260).

2. $\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} - f(t)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} - g(t)h(x)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

The substitution $z = \int h(x)\,dx$ leads to an equation of the form 11.3.1.1 for $w = w(z, y, t)$.

3. $\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} - f(t)\dfrac{\partial w}{\partial x}\left(\dfrac{\partial^2 w}{\partial y^2} + \dfrac{n}{y}\dfrac{\partial w}{\partial y}\right) - g(t)h(x)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time; $n + 1$ is the dimensionality of the equations of motion of the controllable system (n is a nonnegative integer).

"Two-dimensional" solution:

$$w(x, y, t) = \exp\left[\lambda \int h(x)\,dx\right] U(y, t),$$

where the function $U(y, t)$ is determined by the differential equation (λ is an arbitrary constant)

$$\lambda U\frac{\partial U}{\partial t} - \lambda f(t)U\left(\frac{\partial^2 U}{\partial y^2} + \frac{n}{y}\frac{\partial U}{\partial y}\right) - g(t)\left(\frac{\partial U}{\partial y}\right)^2 = 0.$$

4. $\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} - f(t)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} - g(x, t)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

1°. "Two-dimensional" solution:

$$w = U(z, \tau), \quad z = y + \varphi(x, t), \quad \tau = \int f(t)\,dt.$$

Here, the function $\varphi = \varphi(x, t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x} = g(x, t), \tag{1}$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0. \tag{2}$$

Complete integrals and the general solutions (integrals) of equation (1) for various $g(x, t)$ can be found in Subsection 2.2.1. For solutions of equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. "Two-dimensional" solutions:

$$w = \pm\exp\big[\lambda y + \zeta(x, t)\big],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x, t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x} - \lambda^2 f(t)\frac{\partial \zeta}{\partial x} - \lambda^2 g(x, t) = 0.$$

⊙ *Reference for equation* 11.3.1.4: A. D. Polyanin and V. F. Zaitsev (2002, p. 261).

5. $\dfrac{\partial w}{\partial t}\dfrac{\partial w}{\partial x} - f(x,t)\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} - g(x,t)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

"Two-dimensional" solutions:

$$w = \pm\exp\big[\lambda y + \zeta(x,t)\big],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x} - \lambda^2 f(x,t)\frac{\partial \zeta}{\partial x} - \lambda^2 g(x,t) = 0.$$

11.3.2. Equations with Power-Law Nonlinearities

1. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^k - f(t)\left(\dfrac{\partial w}{\partial x}\right)^k\dfrac{\partial^2 w}{\partial y^2} - g(t)\left(\dfrac{\partial w}{\partial y}\right)^{k+1} = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko (1971) and Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, y + C_3, t) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" solution:

$$w = U(z,\tau), \quad \tau = \int f(t)\,dt + C_1,$$

$$z = y + (x + C_2)^{\frac{k}{k+1}}\left[\frac{(k+1)^{k+1}}{k^k}\int g(t)\,dt + C_3\right]^{\frac{1}{k+1}} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $U = U(z,\tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

3°. "Two-dimensional" solution:

$$w = u(\xi,\tau), \quad \xi = y + C_1 x + \frac{1}{C_1^k}\int g(t)\,dt + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants and the function $u = u(\xi,\eta)$ is determined by the linear heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. The solutions of Items 2° and 3° are special cases of the more general solution

$$w = U(z,\tau), \quad z = y + \varphi(x,t), \quad \tau = \int f(t)\,dt,$$

where the function $\varphi = \varphi(x, t)$ satisfies the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)^k = g(t), \tag{1}$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

A complete integral of equation (1) is given by

$$\varphi = C_1 x + \frac{1}{C_1^k}\int g(t)\,dt + C_2, \tag{2}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (1) can be expressed in parametric form with the complete integral (2) and the two relations (see Subsection 24.1.1)

$$C_2 = \psi(C_1),$$
$$x - \frac{k}{C_1^{k+1}}\int g(t)\,dt + \psi'(C_1) = 0,$$

where $\psi = \psi(C_1)$ is an arbitrary function and the prime denotes a derivative; C_1 and C_2 play the role of parameters.

5°. "Two-dimensional" solution:

$$w = \exp\bigl[\lambda y + \zeta(x, t)\bigr],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x, t)$ is determined by the first-order partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(t) = 0. \tag{3}$$

A complete integral of this equation has the form (see Subsection 24.1.1)

$$\zeta = C_1 x + \int\left[\lambda^2 f(t) + \frac{\lambda^{k+1}}{C_1^k} g(t)\right] dt + C_2, \tag{4}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (3) can be expressed in parametric form with the complete integral (4) and the two relations

$$C_2 = \varphi(C_1),$$
$$x - k\frac{\lambda^{k+1}}{C_1^{k+1}}\int g(t)\,dt + \varphi'(C_1) = 0,$$

where $\varphi = \varphi(C_1)$ is an arbitrary function; C_1 and C_2 play the role of parameters.

6°. "Two-dimensional" solution:

$$w = e^{\lambda x}\theta(y, t),$$

where λ is an arbitrary constant and the function $\theta = \theta(y, t)$ is determined by the "two-dimensional" equation

$$\frac{\partial \theta}{\partial t} - f(t)\frac{\partial^2 \theta}{\partial y^2} - \frac{g(t)}{(\lambda\theta)^k}\left(\frac{\partial \theta}{\partial y}\right)^{k+1} = 0.$$

7°. Cauchy problems and self-similar solutions of the equation for power-law $f(t)$ and $g(t)$ are discussed in Chernousko (1971) and Chernousko and Kolmanovskii (1978).

⊙ *References for equation* 11.3.2.1: A. S. Bratus' and K. A. Volosov (2002), A. D. Polyanin and V. F. Zaitsev (2002, pp. 261–263).

2. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^k - f(t)\left(\dfrac{\partial w}{\partial x}\right)^k \dfrac{\partial^2 w}{\partial y^2} - g(t)h(x)\left(\dfrac{\partial w}{\partial y}\right)^{k+1} = 0.$

The substitution $z = \int [h(x)]^{1/k}\,dx$ leads to an equation of the form 11.3.2.1 for $w = w(z, y, t)$.

3. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^k - f(t)\left(\dfrac{\partial w}{\partial x}\right)^k \dfrac{\partial^2 w}{\partial y^2} - g(x,t)\left(\dfrac{\partial w}{\partial y}\right)^{k+1} = 0.$

1°. "Two-dimensional" solution:

$$w = U(z,\tau), \quad z = y + \varphi(x,t), \quad \tau = \int f(t)\,dt.$$

Here, the function $\varphi = \varphi(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)^k = g(x,t), \tag{1}$$

and the function $U = U(z,\tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0. \tag{2}$$

Complete integrals and the general solutions (integrals) of equation (1) for various $g(x,t)$ can be found in Polyanin, Zaitsev, and Moussiaux (2002). For solutions of equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. "Two-dimensional" solution:

$$w = \exp\big[\lambda y + \zeta(x,t)\big],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(x,t) = 0.$$

⊙ *Reference for equation* 11.3.2.3: A. D. Polyanin and V. F. Zaitsev (2002, p. 263).

4. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^k - f(x,t)\left(\dfrac{\partial w}{\partial x}\right)^k \dfrac{\partial^2 w}{\partial y^2} - g(x,t)\left(\dfrac{\partial w}{\partial y}\right)^{k+1} = 0.$

"Two-dimensional" solution:

$$w = \exp\big[\lambda y + \zeta(x,t)\big],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(x,t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(x,t) = 0.$$

Chapter 12
Second-Order Equations of General Form

12.1. Equations Involving the First Derivative in t

12.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right). \tag{1}$$

1°. Suppose $w(x,t)$ is a solution of equation (1). Then the function $w(x + C_1, t + C_2)$, where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. In the general case, equation (1) admits traveling-wave solution

$$w = w(\xi), \qquad \xi = kx + \lambda t, \tag{2}$$

where k and λ are arbitrary constants and the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(w, kw'_\xi, k^2 w''_{\xi\xi}\right) - \lambda w'_\xi = 0.$$

This subsection presents special cases where equation (1) admits exact solutions other than traveling wave (2).

1. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w(C_1 x + C_2, C_1^2 t + C_3) + C_4 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = F(A)t + \tfrac{1}{2}Ax^2 + Bx + C,$$

where A, B, and C are arbitrary constants.

3°. Generalized separable solution:

$$w(x,t) = (Ax + B)t + C + \varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\left(\varphi''_{xx}\right) = Ax + B.$$

4°. Solution:

$$w(x,t) = At + B + \psi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k^2\psi''_{\xi\xi}\right) = \lambda\psi'_\xi + A.$$

5°. Solution:

$$w(x,t) = \tfrac{1}{2}Ax^2 + Bx + C + U(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants and the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k^2U''_{\xi\xi} + A\right) = \lambda U'_\xi.$$

6°. Self-similar solution:

$$w(x,t) = t\,\Theta(\zeta), \qquad \zeta = \frac{x}{\sqrt{t}},$$

where the function $\Theta(\zeta)$ is determined by the ordinary differential equation

$$F\left(\Theta''_{\zeta\zeta}\right) + \tfrac{1}{2}\zeta\Theta'_\zeta - \Theta = 0.$$

7°. The substitution $u(x,t) = \dfrac{\partial w}{\partial x}$ brings the original equation to an equation of the form 5.6.18.4:

$$\frac{\partial u}{\partial t} = f\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}, \qquad f(z) = F'_z(z).$$

8°. The transformation

$$\begin{aligned}
&\bar{t} = \alpha t + \gamma_1, \quad \bar{x} = \beta_1 x + \beta_2 w + \gamma_2,\\
&\bar{w} = \beta_1\left(\beta_4 w + \tfrac{1}{2}\beta_3 x^2 + \gamma_3 x\right) + \gamma_4 t + \gamma_5 + \beta_2\left[\beta_3(xw_x - w) + \gamma_3 w_x + \tfrac{1}{2}\beta_4 w_x^2\right],\\
&\bar{w}_{\bar{x}} = \beta_3 x + \beta_4 w_x + \gamma_3,
\end{aligned}$$

where α, the β_i, and the γ_i are arbitrary constants ($\alpha \neq 0$, $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$) and the subscripts x and $\bar{x}$ denote the corresponding partial derivatives, takes the equation in question to an equation of the same form. The right-hand side of the equation becomes

$$\bar{F}(\bar{w}_{\bar{x}\bar{x}}) = \frac{\beta_1\beta_4 - \beta_2\beta_3}{\alpha}F(w_{xx}) + \frac{\gamma_4}{\alpha}.$$

Special case 1. Equation with a quadratic nonlinearity in the highest derivative:

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + b.$$

1°. Additive separable solution:

$$w(x,t) = Ax^2 + Bx + C + (4aA^2 + b)t,$$

where A, B, and C are arbitrary constants.

2°. Generalized separable solution:

$$w = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi_4' &= 144a\varphi_4^2,\\
\varphi_3' &= 144a\varphi_3\varphi_4,\\
\varphi_2' &= 36a\varphi_3^2 + 48a\varphi_2\varphi_4,\\
\varphi_1' &= 24a\varphi_2\varphi_3,\\
\varphi_0' &= 4a\varphi_2^2 + b.
\end{aligned}$$

This system is easy to solve by successive integration.

Special case 2. Equation:

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2}\right)^k, \qquad k > 0, \quad k \neq 1.$$

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}C_1x^2 + C_2x + aC_1^k t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solution:

$$w(x,t) = \left[a(1-k)t + C_1\right]^{\frac{1}{1-k}} u(x) + C_2,$$

where the function $u(x)$ is determined by the autonomous ordinary differential equation $(u_{xx}'')^k - u = 0$, whose general solution can be written out in implicit form:

$$\int \left(\frac{2k}{1+k}u^{\frac{1+k}{k}} + C_3\right)^{-1/2} du = \pm x + C_4.$$

Special case 3. Equation:

$$\frac{\partial w}{\partial t} = a\exp\left(\lambda\frac{\partial^2 w}{\partial x^2}\right).$$

Generalized separable solution:

$$w(x,t) = U(x) - \frac{1}{2\lambda}(x^2 + A_1x + A_2)\ln(B_1t + B_2) + C_1x + C_2,$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants, and the function $U(x)$ is determined by the ordinary differential equation

$$2a\lambda\exp(\lambda U_{xx}'') + B_1(x^2 + A_1x + A_2) = 0,$$

which is easy to integrate; to this end, the equation should first be solved for U_{xx}''.

Special case 4. Equation:

$$\frac{\partial w}{\partial t} = a\ln\left|\frac{\partial^2 w}{\partial x^2}\right|.$$

Generalized separable solution:

$$w(x,t) = (at + C)\left[\ln\frac{2A^2(at + C)}{\cos^2(Ax + B)} - 1\right] + D,$$

where A, B, C, and D are arbitrary constants.

⊙ *References for equation* 12.1.1.1: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994, pp. 115–118, 131–132), A. D. Polyanin and V. F. Zaitsev (2002, pp. 264–265; 2004, pp. 479–481).

2. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$.

Apart from a traveling-wave solution, this equation has a more complicated exact solution of the form

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k\varphi'_\xi,\ k^2\varphi''_{\xi\xi}\right) - \lambda\varphi'_\xi - A = 0.$$

Special case. Equation:

$$\frac{\partial w}{\partial t} = a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}.$$

1°. Generalized separable solution:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)x^{3/2} + \varphi_3(t)x^3,$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_1 = \tfrac{9}{8}a\varphi_2^2,$$
$$\varphi'_2 = \tfrac{45}{4}a\varphi_2\varphi_3,$$
$$\varphi'_3 = 18a\varphi_3^2.$$

The prime denotes a derivative with respect to t.

2°. Generalized separable solution cubic in x:

$$w(x,t) = \psi_1(t) + \psi_2(t)x + \psi_3(t)x^2 + \psi_4(t)x^3,$$

where the functions $\psi_k = \psi_k(t)$ are determined by the autonomous system of ordinary differential equations

$$\psi'_1 = 2a\psi_2\psi_3,$$
$$\psi'_2 = 2a(2\psi_3^2 + 3\psi_2\psi_4),$$
$$\psi'_3 = 18a\psi_3\psi_4,$$
$$\psi'_4 = 18a\psi_4^2.$$

3°. Generalized separable solution:

$$w(x,t) = \frac{\theta(x) + C_3}{C_1 t + C_2} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\theta = \theta(x)$ is determined by the autonomous ordinary differential equation

$$a\theta'_x\theta''_{xx} + C_1\theta + C_1C_3 = 0,$$

whose solution can be written out in implicit form.

3. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + aw$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, t + C_2) + C_3e^{at},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = (C_1x + C_2)e^{at} + e^{at}\int e^{-at}F(C_1e^{at}, 0)\,dt.$$

3°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\left(w'_z, w''_{zz}\right) - \lambda w'_z + aw = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = -\frac{x + C_1}{a\tau} + \frac{1}{\tau}\int \tau F\Big(-\frac{1}{a\tau}, 0\Big)\, d\tau, \qquad \tau = t + C_2.$$

3°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$F\big(U'_\zeta, U''_{\zeta\zeta}\big) + aUU'_\zeta = C_2 U'_\zeta + 2C_1.$$

In the special case $C_1 = 0$, the above solution converts to a traveling-wave solution.

⊙ *Reference for equation* 12.1.1.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 482).

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + bw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1 e^{bt} + C_2, t + C_3) + C_1 b e^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = a\varphi^2 + b\varphi,$$
$$\psi'_t = a\varphi\psi + b\psi + F(\varphi, 0),$$

which is easy to integrate (the first equation is a Bernoulli equation and the second one is linear in ψ).

3°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\big(w'_z, w''_{zz}\big) + aww'_z - \lambda w'_z + bw = 0.$$

⊙ *Reference for equation* 12.1.1.5: A. D. Polyanin and V. F. Zaitsev (2004, p. 482).

6. $\dfrac{\partial w}{\partial t} = e^{\beta w} F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, C_2 t + C_3) + \frac{1}{\beta}\ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\varphi} F\left(\varphi'_x, \varphi''_{xx}\right) + A = 0.$$

3°. Solution:

$$w(x,t) = -\frac{1}{\beta}\ln(t + C) + \Theta(\xi), \quad \xi = kx + \lambda\ln(t + C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\Theta} F\left(k\Theta'_\xi, k^2\Theta''_{\xi\xi}\right) = \lambda\Theta'_\xi - \frac{1}{\beta}.$$

7. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = t\varphi(z), \quad z = kx + \lambda\ln|t|,$$

where k and λ are arbitrary constants and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$F\left(k\frac{\varphi'_z}{\varphi}, k^2\frac{\varphi''_{zz}}{\varphi}\right) = \lambda\varphi'_z + \varphi.$$

8. $\dfrac{\partial w}{\partial t} = wF\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = Ce^{\lambda t}\varphi(x),$$

where C and λ are arbitrary constants and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{\varphi'_x}{\varphi}, \frac{\varphi''_{xx}}{\varphi}\right) = \lambda.$$

This equation has particular solutions of the form $\varphi(x) = e^{\alpha x}$, where α is a root of the algebraic (or transcendental) equation $F(\alpha, \alpha^2) - \lambda = 0$.

3°. Solution:

$$w(x,t) = Ce^{\lambda t}\psi(\xi), \quad \xi = kx + \beta t,$$

where C, k, λ, and β are arbitrary constants, and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$\psi F\left(k\frac{\psi'_\xi}{\psi}, k^2\frac{\psi''_{\xi\xi}}{\psi}\right) = \beta\psi'_\xi + \lambda\psi.$$

This equation has particular solutions of the form $\psi(\xi) = e^{\mu\xi}$.

9. $\dfrac{\partial w}{\partial t} = w^n F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

For the cases $n = 0$ and $n = 1$, see equations 12.1.1.7 and 12.1.1.8, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{n-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \big[(1-n)At + B\big]^{\frac{1}{1-n}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\varphi^{n-1}F\left(\frac{\varphi'_x}{\varphi}, \frac{\varphi''_{xx}}{\varphi}\right) = A.$$

3°. Solution:

$$w(z,t) = (t + C)^{\frac{1}{1-n}}\Theta(z), \quad z = kx + \lambda\ln(t + C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$\Theta^n F\left(k\frac{\Theta'_z}{\Theta}, k^2\frac{\Theta''_{zz}}{\Theta}\right) = \lambda\Theta'_z + \frac{1}{1-n}\Theta.$$

10. $\dfrac{\partial w}{\partial t} = w^n F\left(w^k \dfrac{\partial w}{\partial x},\ w^{2k+1}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1^{-k-1}x + C_2, C_1^{n-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution with $n \neq 1$ and $k \neq -1$:

$$w(x,t) = t^{\frac{1}{1-n}}U(z), \quad z = xt^{\frac{k+1}{n-1}},$$

where the function $U(z)$ is determined by the ordinary differential equation

$$\frac{1}{1-n}U + \frac{k+1}{n-1}zU_z' = U^n F\big(U^k U_z',\ U^{2k+1}U_{zz}''\big).$$

3°. Generalized self-similar solution with $n = 1$ and $k \neq -1$:

$$w(x,t) = \exp\left(-\frac{1}{k+1}t\right)u(\xi), \quad \xi = xe^t,$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$-\frac{1}{k+1}u + \xi u_\xi' = uF\big(u^k u_\xi',\ u^{2k+1}u_{\xi\xi}''\big).$$

4°. For $k = -1$, see equations 8.1.1.7–8.1.1.9.

11. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 12.1.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1}w(x + C_2, C_1 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(\xi), \quad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(k\varphi_{\xi\xi}''/\varphi_\xi'\big) = \lambda\varphi_\xi' + A.$$

If $A = 0$, the equation has a traveling-wave solution.

3°. Solution:

$$w(x,t) = t\Theta(z) + C, \quad z = kx + \lambda\ln|t|$$

where C, k, β, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$F\big(k\Theta_{zz}''/\Theta_z'\big) = \lambda\Theta_z' + \Theta.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(\dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 12.1.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$k\varphi'_z F\left(k\varphi''_{zz}/\varphi'_z\right) = \lambda\varphi'_z + A.$$

3°. Solution:

$$w(x,t) = Ae^{\beta t}\Theta(\xi) + B, \qquad \xi = kx + \lambda t,$$

where A, B, k, β, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$k\Theta'_\xi F\left(k\Theta''_{\xi\xi}/\Theta'_\xi\right) = \lambda\Theta'_\xi + \beta\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(\dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 12.1.1.2. For the cases $\beta = 0$ and $\beta = 1$, see equations 12.1.1.11 and 12.1.1.12, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{\beta-1} t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,t) = \left[A(1-\beta)t + B\right]^{\frac{1}{1-\beta}} \varphi(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\left(\varphi'_x\right)^{\beta} F\left(\varphi''_{xx}/\varphi'_x\right) = A\varphi.$$

3°. Solution:

$$w(x,t) = (t + A)^{\frac{1}{1-\beta}} \Theta(z) + B, \qquad z = kx + \lambda \ln(t + A),$$

where A, B, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$k^{\beta}\left(\Theta'_z\right)^{\beta} F\left(k\Theta''_{zz}/\Theta'_z\right) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

14. $\dfrac{\partial w}{\partial t} = wF\left(\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

This is a special case of equation 12.1.2.11.

15. $\dfrac{\partial w}{\partial t} = wF\left(\dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 12.1.2.12.

16. $\dfrac{\partial w}{\partial t} = wF\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 12.1.2.13.

17. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\big[F\big(w, w_x\big)\big], \qquad w_x = \dfrac{\partial w}{\partial x}.$

1°. The transformation

$$\bar{t} = t - t_0, \quad \bar{x} = -\int_{x_0}^{x} w(y,t)\,dy - \int_{t_0}^{t} F\big(w(x_0,\tau), w_x(x_0,\tau)\big)\,d\tau, \quad \bar{w}(\bar{x},\bar{t}) = \frac{1}{w(x,t)}$$

converts a (nonzero) solution $w(x,t)$ of the original equation to a solution $\bar{w}(\bar{x},\bar{t})$ of a similar equation:

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left[\bar{F}\big(\bar{w},\ \bar{w}_{\bar{x}}\big)\right],$$

where

$$\bar{F}\big(w, w_x\big) = wF\big(w^{-1},\ w^{-3}w_x\big). \tag{1}$$

2°. In the special case

$$F\big(w, w_x\big) = g(w)(w_x)^k,$$

it follows from (1) that

$$\bar{F}\big(w, w_x\big) = \bar{g}(w)(w_x)^k, \qquad \bar{g}(w) = w^{1-3k}g(w^{-1}).$$

⊙ *References for equation* 12.1.1.17: W. Strampp (1982), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), N. H. Ibragimov (1994, pp. 115–118, pp. 129–130).

12.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\dfrac{\partial w}{\partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C are arbitrary constants, is also a solution of the equation.

2. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + \psi(t),\, t\big) + \varphi(t), \quad \varphi(t) = C\exp\left[\int g(t)\,dt\right], \quad \psi(t) = -\int f(t)\varphi(t)\,dt,$$

where C is an arbitrary constant, is also a solution of the equation.

3. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solutions:

$$w(x,t) = A\exp\left[\lambda x + \int F(t,\lambda^2)\,dt\right],$$
$$w(x,t) = \big[A\cosh(\lambda x) + B\sinh(\lambda x)\big]\exp\left[\int F(t,\lambda^2)\,dt\right],$$
$$w(x,t) = \big[A\cos(\lambda x) + B\sin(\lambda x)\big]\exp\left[\int F(t,-\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + \int F(t,\lambda,\lambda^2)\,dt\right],$$

where A and λ are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = wF\left(t, w^k\dfrac{\partial w}{\partial x}, w^{2k+1}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $k \neq -1$:

$$w(x,t) = \big[C_1(k+1)x + C_2\big]^{\frac{1}{k+1}}\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the first-order ordinary differential equation

$$\varphi_t' = \varphi F\big(t, C_1\varphi^{k+1}, -kC_1^2\varphi^{2k+2}\big).$$

2°. For $k = -1$, see equation 12.1.2.4.

6. $\dfrac{\partial w}{\partial t} = f(t)w^{\beta}\Phi\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 12.1.1.9:

$$\frac{\partial u}{\partial \tau} = u^{\beta}\Phi\left(\frac{1}{u}\frac{\partial u}{\partial x}, \frac{1}{u}\frac{\partial^2 u}{\partial x^2}\right),$$

which has a traveling-wave solution $u = u(Ax + B\tau)$ and a solution in the multiplicative form $u = \varphi(x)\psi(\tau)$.

7. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^{k}\Phi\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi'_t = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

C is an arbitrary constant, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\left(\Theta'_x\right)^k\Phi\left(\Theta''_{xx}/\Theta'_x\right) = A\Theta + B. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[C - kA\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int\left[Bf(t)\varphi^k(t) + h(t)\right]\frac{dt}{G(t)},$$

where A, B, C, and D are arbitrary constants.

For $k = 1$, a solution to equation (3) is given by

$$\Theta(x) = \alpha e^{\lambda x} - B/A,$$

where α is an arbitrary constant and λ is determined from the algebraic (or transcendental) equation $\lambda\Phi(\lambda) = A$.

8. $\dfrac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\big[\beta G(t)\big]\,dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 12.1.1.6:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right),$$

which has a traveling-wave solution $u = u(Ax + B\tau)$ and an additive separable solution $u = \varphi(x) + \psi(\tau)$.

9. $\dfrac{\partial w}{\partial t} = f(t)\Phi\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)\dfrac{\partial w}{\partial x}.$

With the transformation

$$w = U(z,\tau), \quad z = x + \int g(t)\,dt, \quad \tau = \int f(t)\,dt$$

one arrives at the simpler equation

$$\frac{\partial U}{\partial \tau} = \Phi\left(U, \frac{\partial U}{\partial z}, \frac{\partial^2 U}{\partial z^2}\right),$$

which has a traveling-wave solution $U = U(kz + \lambda\tau)$.

10. $\dfrac{\partial w}{\partial t} = wF\left(t, w\dfrac{\partial^2 w}{\partial x^2} + aw^2\right).$

Multiplicative separable solutions:

$$w(x,t) = \big[C_1\sin\big(x\sqrt{a}\,\big) + C_2\cos\big(x\sqrt{a}\,\big)\big]\exp\left[\int F(t,0)\,dt\right] \qquad \text{if } a > 0,$$

$$w(x,t) = \big[C_1\sinh\big(x\sqrt{|a|}\,\big) + C_2\cosh\big(x\sqrt{|a|}\,\big)\big]\exp\left[\int F(t,0)\,dt\right] \quad \text{if } a < 0,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 488).

11. $\dfrac{\partial w}{\partial t} = wF\left(t, \left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \big[C_1\sin\big(x\sqrt{a}\,\big) + C_2\cos\big(x\sqrt{a}\,\big)\big]\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi_t' = \varphi F\big(t, a(C_1^2 + C_2^2)\varphi^2, -a\big)$.

2°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \big(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\big)\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi_t' = \varphi F\big(t,\, 4C_1C_2a\varphi^2,\, -a\big)$.

Special case. For $C_1C_2 = 0$, a solution is given by

$$w(x,t) = C\exp\left[\pm\sqrt{|a|}\,x + \int F(t,0,-a)\,dt\right],$$

where C is an arbitrary constant.

⊙ *Reference for equation* 12.1.2.11: A. D. Polyanin and V. F. Zaitsev (2004, p. 489).

12. $\dfrac{\partial w}{\partial t} = wF\left(t,\, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\, w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right)$.

1°. Multiplicative separable solution:

$$w(x,t) = C\exp\left[\lambda x + \int F(t,\lambda^2,0)\,dt\right],$$

where C and λ are arbitrary constants.

2°. Multiplicative separable solution:

$$w(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi_t' = \varphi F\big(t, \lambda^2, 4AB\lambda^2\varphi^2\big)$.

3°. Multiplicative separable solution:

$$w(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi_t' = \varphi F\big(t, -\lambda^2, -\lambda^2(A^2+B^2)\varphi^2\big)$.

⊙ *References for equation* 12.1.2.12: Ph. W. Doyle (1996, the case $\partial_t F \equiv 0$ was treated), A. D. Polyanin and V. F. Zaitsev (2004, p. 489).

13. $\dfrac{\partial w}{\partial t} = wF\left(t,\, \dfrac{\partial^2 w}{\partial x^2},\, 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(t,\, \dfrac{\partial^2 w}{\partial x^2},\, 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right)$.

Generalized separable solution quadratic in x:

$$w = \varphi_1(t)x^2 + \varphi_2(t)x + \varphi_3(t),$$

where the functions φ_1, φ_2, and φ_3 are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi_1' &= \varphi_1 F\big(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\big),\\
\varphi_2' &= \varphi_2 F\big(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\big),\\
\varphi_3' &= \varphi_3 F\big(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\big) + G\big(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\big).
\end{aligned}$$

It follows from the first two equations that $\varphi_2 = C\varphi_1$, where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 489).

12.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(x,\, w,\, \frac{\partial w}{\partial x},\, \frac{\partial^2 w}{\partial x^2}\right). \tag{1}$$

Suppose that the auxiliary ordinary differential equation

$$w = F(x,\, w,\, w'_x,\, w''_{xx})$$

is reduced, by a linear transformation

$$x = \varphi(z), \quad w = \psi(z)u + \chi(z)$$

and the subsequent division of the resulting equation by $\psi(z)$, to the autonomous form

$$u = \mathcal{F}(u,\, u'_z,\, u''_{zz}),$$

where $\mathcal{F} = F/\psi$. Then, the original equation (1) can be reduced, by the same transformation

$$x = \varphi(z), \quad w(x,t) = \psi(z)u(z,t) + \chi(z),$$

to the equation

$$\frac{\partial u}{\partial t} = \mathcal{F}\left(u,\, \frac{\partial u}{\partial z},\, \frac{\partial^2 u}{\partial z^2}\right),$$

which has a traveling-wave solution $u = u(kz + \lambda t)$.

The above allows using various known transformations of ordinary differential equations (see Kamke, 1977, and Polyanin and Zaitsev, 2003) for constructing exact solutions to partial differential equations. If the original equation is linear, then such transformations will result in linear constant-coefficient equations.

1. $\dfrac{\partial w}{\partial t} = F\left(x,\, \dfrac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution:

$$w(x,t) = Axt + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\left(x,\, \varphi''_{xx}\right) = Ax + B.$$

Special case. Equation

$$\frac{\partial w}{\partial t} = f(x)\exp\left(a\frac{\partial^2 w}{\partial x^2}\right).$$

Generalized separable solution:

$$w = \varphi(x) + \psi(x)\theta(t),$$

where

$$\varphi(x) = \frac{1}{2a}A_1x^2 + C_3x + C_4 + \frac{1}{a}\int_{x_0}^{x}(x-\xi)\ln\frac{\psi(\xi)}{f(\xi)}\,d\xi,$$

$$\psi(x) = \frac{1}{2}A_2x^2 + C_1x + C_2,$$

$$\theta(t) = -\frac{1}{A_2a}\ln\left(C_5 - A_2ae^{A_1}t\right).$$

2. $\dfrac{\partial w}{\partial t} = F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x, \varphi'_x, \varphi''_{xx}\big) = A.$$

3. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial w}{\partial x} + F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$F\big(w, w'_z, w''_{zz}\big) + azw'_z = 0.$$

⊙ *Reference for equation* 12.1.3.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 350).

4. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, x\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x, C_1 t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(\varphi'_x, x\varphi''_{xx}\big) = A.$$

3°. Solution:

$$w(x,t) = t\Theta(\xi) + C, \quad \xi = x/t,$$

where C is an arbitrary constant, and the function $\Theta(\xi)$ is determined by the ordinary differential equation

$$F\big(\Theta'_\xi, \xi\Theta''_{\xi\xi}\big) + \xi\Theta'_\xi - \Theta = 0.$$

5. $\dfrac{\partial w}{\partial t} = F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2}\right).$

The substitution $x = \pm e^z$ leads to the equation

$$\frac{\partial w}{\partial t} = F\left(w,\ \frac{\partial w}{\partial z},\ \frac{\partial^2 w}{\partial z^2} - \frac{\partial w}{\partial z}\right),$$

which has a traveling-wave solution $w = w(kz + \lambda t)$.

6. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x, C_1^{-k} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x, t) = w(z), \quad z = xt^{1/k},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$kz^{k-1} F\big(w,\ zw'_z,\ z^2 w''_{zz}\big) - w'_z = 0.$$

7. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2}\right) + ax\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xe^{at}, \quad \tau = \frac{1}{ak}\big(1 - e^{-akt}\big),$$

we obtain an equation of the form 12.1.3.6:

$$\frac{\partial w}{\partial \tau} = z^k F\left(w,\ z\frac{\partial w}{\partial z},\ z^2\frac{\partial^2 w}{\partial z^2}\right).$$

8. $\dfrac{\partial w}{\partial t} = e^{\lambda x} F\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1,\ e^{-\lambda C_1} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x, t) = w(z), \quad z = \lambda x + \ln t,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$e^z F\big(w,\ \lambda w'_z,\ \lambda^2 w''_{zz}\big) - w'_z = 0.$$

9. $\dfrac{\partial w}{\partial t} = wF\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x,\ t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda t}\varphi(x),$$

where λ is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\big) = \lambda.$$

10. $\dfrac{\partial w}{\partial t} = w^{\beta}F\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

For $\beta = 1$, see equation 12.1.3.9.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x,\ C_1^{\beta-1}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \big[(1-\beta)At + B\big]^{\frac{1}{1-\beta}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi^{\beta-1}F\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\big) = A.$$

11. $\dfrac{\partial w}{\partial t} = e^{\beta w}F\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,\ C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$e^{\beta\varphi}F\big(x,\ \varphi'_x,\ \varphi''_{xx}\big) + A = 0.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(x,\ \dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x,\ t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x F\big(x,\ \varphi''_{xx}/\varphi'_x\big) = A.$$

3°. Generalized separable solution:

$$w(x,t) = Ae^{\mu t}\Theta(x) + B$$

where A, B, and μ are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\Theta'_x F\big(x,\ \Theta''_{xx}/\Theta'_x\big) = \mu\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(x,\ \dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

For $\beta = 1$, see equation 12.1.3.12.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x,\ C_1^{\beta-1} t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\big(\varphi'_x\big)^{\beta} F\big(x,\ \varphi''_{xx}/\varphi'_x\big) = A.$$

3°. Generalized separable solution:

$$w(x,t) = \big[A(1-\beta)t + C_1\big]^{\frac{1}{1-\beta}}\big[\Theta(x) + B\big] + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\big(\Theta'_x\big)^{\beta} F\big(x,\ \Theta''_{xx}/\Theta'_x\big) = A\Theta + AB.$$

12.1.4. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial w}{\partial x}\right)^m \left(\dfrac{\partial^2 w}{\partial x^2}\right)^n + \left[f(t)x + g(t)\right]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{m+2n}(t)H^{m+n-1}(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial u}{\partial z}\right)^m \left(\frac{\partial^2 u}{\partial z^2}\right)^n.$$

The last equation admits a traveling-wave solution, a self-similar solution, and a multiplicative separable solution.

2. $\dfrac{\partial w}{\partial t} = f(w)\left(\dfrac{\partial^2 w}{\partial x^2}\right)^k + \left[xg(t) + h(t)\right]\dfrac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{2k}(t)\,dt, \quad \text{with} \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial w}{\partial \tau} = f(w)\left(\frac{\partial^2 w}{\partial z^2}\right)^k.$$

The last equation admits a traveling-wave solution and a self-similar solution.

3. $\dfrac{\partial w}{\partial t} = F\left(x, t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

4. $\dfrac{\partial w}{\partial t} = F\left(ax + bt,\ w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right).$

Solution:

$$w = w(\xi), \quad \xi = ax + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi,\ a^2 w''_{\xi\xi}\right) - bw'_\xi = 0.$$

5. $\dfrac{\partial w}{\partial t} = f(t)x^k\Phi\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2}\right) + xg(t)\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xG(t), \quad \tau = \int f(t)G^{-k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

we obtain a simpler equation of the form 12.1.3.6:

$$\frac{\partial w}{\partial \tau} = z^k\Phi\left(w,\ z\frac{\partial w}{\partial z},\ z^2\frac{\partial^2 w}{\partial z^2}\right).$$

6. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{f(x)}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\exp\left[\int F(t,\lambda)\,dt\right],$$

where the function $\varphi = \varphi(x)$ satisfies the linear ordinary differential equation $f(x)\varphi''_{xx} = \lambda\varphi$.

7. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + Be^{-\lambda x}E(t), \quad E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + e^{-\lambda x}E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = wF_1\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{\lambda x}F_2\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{-\lambda x}F_3\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

10. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cosh(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sinh(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cos(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + B\sin(\lambda x)E(t),$$
$$E(t) = \exp\left[\int \Phi(t,-\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sin(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t,-\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = wF_1\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \cos(\lambda x)F_2\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \sin(\lambda x)F_3\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

14. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x}.$

Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right], \quad E(t) = \exp\left[\int \Phi(t,\lambda,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

15. $\dfrac{\partial w}{\partial t} = f(t)w^{\beta}\Phi\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \quad \text{with} \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 12.1.3.10:

$$\frac{\partial u}{\partial \tau} = u^{\beta}\Phi\left(x,\ \frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2}\right),$$

which has a multiplicative separable solution $u = \varphi(x)\psi(\tau)$.

16. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^{k}\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi_t' = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

A and B are arbitrary constants, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\left(\Theta_x'\right)^k\Phi\left(x,\ \Theta_{xx}''/\Theta_x'\right) = A\Theta + B. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[C - kA\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int\left[Bf(t)\varphi^k(t) + h(t)\right]\frac{dt}{G(t)},$$

where C and D are arbitrary constants.

For $k = 1$ and $\Phi(x,y) = \Phi(y)$, a solution of equation (3) is given by

$$\Theta(x) = \alpha e^{\lambda x} - B/A,$$

where α is an arbitrary constant, and λ is found from the algebraic (or transcendental) equation $\lambda\Phi(\lambda) = A$.

17. $\dfrac{\partial w}{\partial t} = \big[f_1(t)w + f_0(t)\big]\left(\dfrac{\partial w}{\partial x}\right)^k \Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right) + g_1(t)w + g_0(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant):

$$\varphi_t' = Cf_1(t)\varphi^{k+1} + g_1(t)\varphi, \tag{1}$$

$$\psi_t' = \big[Cf_1(t)\varphi^k + g_1(t)\big]\psi + Cf_0(t)\varphi^k + g_0(t), \tag{2}$$

and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\big(\Theta_x'\big)^k \Phi\big(x,\ \Theta_{xx}''/\Theta_x'\big) = C. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[A - kC\int f_1(t)G^k(t)\,dt\right]^{-1/k}, \quad G(t) = \exp\left[\int g_1(t)\,dt\right],$$

$$\psi(t) = B\varphi(t) + \varphi(t)\int \big[Cf_0(t)\varphi^k(t) + g_0(t)\big]\frac{dt}{\varphi(t)},$$

where A, B, and C are arbitrary constants.

Further, we assume that Φ is independent of x explicitly, i.e., $\Phi(x,y) = \Phi(y)$. For $\Phi(0) \neq 0$ and $\Phi(0) \neq \infty$, particular solution to equation (3) has the form $\Theta(x) = \alpha x + \beta$, where $\alpha^k\Phi(0) = C$ and β is an arbitrary constant.

For $k = 0$, the general solution of equation (3) is expressed as

$$\Theta(x) = \alpha e^{\lambda x} + \beta,$$

where α and β are arbitrary constants, and λ is determined from the algebraic (transcendental) equation $\Phi(\lambda) = C$.

18. $\dfrac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\big[\beta G(t)\big]\,dt, \qquad G(t) = \int g(t)\,dt$$

leads to a simpler equation of the form 12.1.3.11:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(x,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2}\right),$$

which has an additive separable solution $u = \varphi(x) + \psi(\tau)$.

19. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial w}{\partial x} - x\dfrac{\partial^2 w}{\partial x^2},\ 2w - 2x\dfrac{\partial w}{\partial x} + x^2\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = (C_2x^2 + C_1x + C_0)\varphi(t),$$

where C_0, C_1, and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\big(t, 2C_2\varphi, C_1\varphi, 2C_0\varphi\big)$.

⊙ *References for equation* 12.1.4.19: Ph. W. Doyle (1996, the case $\partial_t F \equiv 0$ was treated), A. D. Polyanin and V. F. Zaitsev (2004, p. 498).

20. $\dfrac{\partial w}{\partial t} = F\left(x,\ t,\ w,\ \dfrac{\partial w}{\partial x}\right)G\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \int h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$G\big(x, \varphi'_x, \varphi''_{xx}\big) = 0.$$

21. $\dfrac{\partial w}{\partial t} = F\left(x,\ t,\ w,\ \dfrac{\partial w}{\partial x}\right)G\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = C\exp\left[\int h(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$G\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\big) = 0.$$

22. $\dfrac{\partial w}{\partial t} = g_0(t)F_0\left(\dfrac{\partial^2 w}{\partial x^2}\right) + xg_1(t)F_1\left(\dfrac{\partial^2 w}{\partial x^2}\right) + x^2g_2(t)F_2\left(\dfrac{\partial^2 w}{\partial x^2}\right)$
$+ h(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[p_0(t) + xp_1(t)\big]\dfrac{\partial w}{\partial x} + q(t)w + s_0(t) + xs_1(t) + x^2s_2(t).$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

23. $\dfrac{\partial w}{\partial t} = x^2f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution quadratic in x:

$$w(x,t) = x^2\varphi(t) + x\int f_1(t, 2\varphi)\,dt + \int f_0(t, 2\varphi)\,dt + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the first-order ordinary differential equation

$$\varphi'_t = f_2(t, 2\varphi).$$

24. $\dfrac{\partial w}{\partial t} = x^2 f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + x f_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

12.1.5. Equations of the Form $F\left(x, t, w, \dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) = 0$

1. $F\left(at + bx,\ w,\ \dfrac{\partial w}{\partial t},\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

Solution:

$$w = w(\xi), \quad \xi = at + bx,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi,\ bw'_\xi,\ b^2 w''_{\xi\xi}\right) = 0.$$

2. $F\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

1°. Multiplicative separable solution:

$$w(x,t) = \left[A\cosh(\lambda x) + B\sinh(\lambda x)\right]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, \lambda^2\right) = 0.$$

2°. Multiplicative separable solution:

$$w(x,t) = \left[A\cos(\lambda x) + B\sin(\lambda x)\right]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, -\lambda^2\right) = 0.$$

3. $F\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

Multiplicative separable solution:

$$w(x,t) = Ae^{\lambda x}\varphi(t),$$

where A and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, \lambda, \lambda^2\right) = 0.$$

4. $\boldsymbol{F\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.}$

Multiplicative separable solution:

$$w(x,t) = Ae^{\lambda t}\varphi(x),$$

where A and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$F\big(x, \lambda, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\big) = 0.$$

5. $\boldsymbol{F_1\left(t, \dfrac{\partial w}{\partial t}\right) + F_2\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) = kw.}$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$F_1\big(t, \varphi'_t\big) - k\varphi = C,$$
$$F_2\big(x, \psi'_x, \psi''_{xx}\big) - k\psi = -C,$$

and C is an arbitrary constant.

6. $\boldsymbol{F_1\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}\right) + w^k F_2\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.}$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$\varphi^{-k} F_1\big(t, \varphi'_t/\varphi\big) = C,$$
$$\psi^k F_2\big(x, \psi'_x/\psi, \psi''_{xx}/\psi\big) = -C,$$

and C is an arbitrary constant.

7. $\boldsymbol{F_1\left(t, \dfrac{\partial w}{\partial t}\right) + e^{\lambda w} F_2\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) = 0.}$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$e^{-\lambda\varphi} F_1\big(t, \varphi'_t\big) = C,$$
$$e^{\lambda\psi} F_2\big(x, \psi'_x, \psi''_{xx}\big) = -C,$$

and C is an arbitrary constant.

8. $\boldsymbol{F_1\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial t}\right) + F_2\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = k\ln w.}$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$F_1\big(t, \varphi_t'/\varphi\big) - k\ln\varphi = C,$$
$$F_2\big(x, \psi_x'/\psi, \psi_{xx}''/\psi\big) - k\ln\psi = -C,$$

and C is an arbitrary constant.

12.1.6. Equations with Three Independent Variables

1. $\boldsymbol{\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial^2 w}{\partial y^2}\right).}$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1t + C_2, y + C_3, t + C_4) + C_1,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \quad z = C_1x + C_2y + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\big(C_1w_z', C_2w_z', C_1^2w_{zz}'', C_2^2w_{zz}''\big) + aC_1ww_z' = \lambda w_z'.$$

3°. Solution:

$$w = u(\xi) + 2C_1t, \quad \xi = x + C_2y + aC_1t^2 + C_3t,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(u_\xi', C_2u_\xi', u_{\xi\xi}'', C_2^2u_{\xi\xi}''\big) + auu_\xi' = C_3u_\xi' + 2C_1.$$

4°. "Two-dimensional" solution:

$$w(x,y,t) = U(y,\eta) + 2C_1t, \quad \eta = x + aC_1t^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(y,\eta)$ is determined by the differential equation

$$2C_1 + C_2\frac{\partial U}{\partial \eta} = aU\frac{\partial U}{\partial \eta} + F\left(\frac{\partial U}{\partial \eta},\ \frac{\partial U}{\partial y},\ \frac{\partial^2 U}{\partial \eta^2},\ \frac{\partial^2 U}{\partial y^2}\right).$$

5°. There is a "two-dimensional" solution of the form

$$w(x,y,t) = V(\zeta_1,\zeta_2), \quad \zeta_1 = a_1x + b_1y + c_1t, \quad \zeta_2 = a_2x + b_2y + c_2t.$$

2. $\dfrac{\partial w}{\partial t}+(a_1x+b_1y)\dfrac{\partial w}{\partial x}+(a_2x+b_2y)\dfrac{\partial w}{\partial y}=F\left(w,\dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y},\dfrac{\partial^2 w}{\partial x^2},\dfrac{\partial^2 w}{\partial x\partial y},\dfrac{\partial^2 w}{\partial y^2}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(x + Cb_1e^{\lambda t}, y + C(\lambda - a_1)e^{\lambda t}, t\big),$$

where C is an arbitrary constant, and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1b_2 - a_2b_1 = 0, \tag{1}$$

are also solutions of the equation.

2°. Solutions:

$$w = w(z), \qquad z = a_2x + (\lambda - a_1)y + Ce^{\lambda t},$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\big[\lambda z + a_2c_1 + (\lambda - a_1)c_2\big]w'_z = F\big(w,\ a_2w'_z,\ (\lambda - a_1)w'_z,\ a_2^2w''_{zz},\ a_2(\lambda - a_1)w''_{zz},\ (\lambda - a_1)^2w''_{zz}\big).$$

3°. "Two-dimensional" solutions:

$$w = u(\zeta, t), \qquad \zeta = a_2x + (\lambda - a_1)y,$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $u(\zeta,t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t} + \big[\lambda\zeta + a_2c_1 + (\lambda - a_1)c_2\big]\frac{\partial u}{\partial \zeta} = F\left(u,\ a_2\frac{\partial u}{\partial \zeta},\ (\lambda - a_1)\frac{\partial u}{\partial \zeta},\ a_2^2\frac{\partial^2 u}{\partial \zeta^2},\ a_2(\lambda - a_1)\frac{\partial^2 u}{\partial \zeta^2},\ (\lambda - a_1)^2\frac{\partial^2 u}{\partial \zeta^2}\right).$$

12.2. Equations Involving Two or More Second Derivatives

12.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(C_1x + C_2, \pm C_1t + C_3) + C_4xt + C_5x + C_6t + C_7,$$

where the C_n are arbitrary constants, are also solutions of the equation.

2°. Solution quadratic in x and t:

$$w(x,t) = \tfrac{1}{2}Ax^2 + Bxt + \tfrac{1}{2}F(A)t^2 + C_1x + C_2t + C_3,$$

where A, B, C_1, C_2, and C_3 are arbitrary constants.

3°. Generalized separable solution quadratic in x:

$$w(x,t)=\frac{1}{2}(C_1t+C_2)x^2+(C_3t+C_4)x+\int_0^t(t-\xi)F(C_1\xi+C_2)\,d\xi+C_5t+C_6,$$

where $C_1,\ldots,C_6$ are arbitrary constants.

4°. Generalized separable solution quadratic in t:

$$w(x,t)=\frac{1}{2}(C_1x+C_2)t^2+(C_3x+C_4)t+\int_0^x(x-\xi)\Phi(C_1\xi+C_2)\,d\xi+C_5x+C_6,$$

where $C_1,\ldots,C_6$ are arbitrary constants, and the function $\Phi(u)$ is the inverse of $F(u)$.

5°. Self-similar solution:

$$w=t^2U(z),\qquad z=x/t,$$

where the function $U=U(z)$ is determined by the ordinary differential equation

$$2U-2zU'_z+z^2U''_{zz}=F(U''_{zz}).$$

6°. The substitution $u(x,t)=\dfrac{\partial w}{\partial x}$ leads to an equation of the form 7.4.7.10:

$$\frac{\partial^2u}{\partial t^2}=f\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2u}{\partial x^2},\qquad f(\xi)=F'_\xi(\xi).$$

Special case 1. Let $F(\xi)=a\xi^n$.

1°. Multiplicative separable solution:

$$w(x,t)=\varphi(x)\psi(t),$$

where the functions $\varphi=\varphi(x)$ and $\psi=\psi(t)$ are determined by the second-order autonomous ordinary differential equations (C is an arbitrary constant)

$$\varphi''_{xx}=C\varphi^{1/n},$$
$$\psi''_{tt}=aC^n\psi^n,$$

whose general solutions can be written out in implicit form.

2°. Self-similar solution:

$$w(x,t)=t^\sigma U(z),\qquad z=t^\beta x,\qquad \sigma=\frac{2(1+n\beta)}{1-n},$$

where β is an arbitrary constant, and the function $U=U(z)$ is determined by the ordinary differential equation

$$\sigma(\sigma-1)U+\beta(2\sigma+\beta-1)zU'_z+\beta^2z^2U''_{zz}=a(U''_{zz})^n.$$

3°. Generalized separable solution quadratic in x:

$$w(x,t)=\frac{1}{2}(C_1t+C_2)x^2+(C_3t+C_4)x+\frac{a}{C_1^2(n+1)(n+2)}(C_1t+C_2)^{n+2}+C_5t+C_6,$$

where $C_1,\ldots,C_6$ are arbitrary constants.

4°. Generalized separable solution quadratic in t:

$$w(x,t)=\frac{1}{2}(C_1x+C_2)t^2+(C_3x+C_4)t+\frac{4a^{1/n}}{C_1^2(4n^2-1)}(C_1x+C_2)^{(2n+1)/2}+C_5x+C_6,$$

where $C_1,\ldots,C_6$ are arbitrary constants.

Special case 2. Let $F(\xi) = a\exp(\lambda\xi)$. Generalized separable solution:

$$w = (A_2x^2 + A_1x + A_0)\varphi(t) + \psi(x),$$

where A_2, A_1, and A_0 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations (B_2 is an arbitrary constant)

$$\varphi''_{tt} = aB_2\exp(2A_2\lambda\varphi), \tag{1}$$

$$\exp(\lambda\psi''_{xx}) = B_2(A_2x^2 + A_1x + A_0). \tag{2}$$

The general solution of equation (1) is expressed as

$$\varphi(t) = -\frac{1}{2A_2\lambda}\ln\left[\frac{A_2B_2a\lambda}{C_1^2}\cos^2(C_1t + C_2)\right] \quad \text{if} \quad A_2B_2a\lambda > 0,$$

$$\varphi(t) = -\frac{1}{2A_2\lambda}\ln\left[\frac{A_2B_2a\lambda}{C_1^2}\sinh^2(C_1t + C_2)\right] \quad \text{if} \quad A_2B_2a\lambda > 0,$$

$$\varphi(t) = -\frac{1}{2A_2\lambda}\ln\left[-\frac{A_2B_2a\lambda}{C_1^2}\cosh^2(C_1t + C_2)\right] \quad \text{if} \quad A_2B_2a\lambda < 0,$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (2) is given by

$$\psi(x) = \frac{1}{\lambda}\int_{t_0}^{t}(t-\xi)\ln(A_2B_2\xi^2 + A_1B_2\xi + A_0B_2)\,d\xi + B_1t + B_0,$$

where B_1 and B_0 are arbitrary constants.

Special case 3. Let $F(\xi) = a\ln\xi + b$. Generalized separable solution:

$$w = (A_2t^2 + A_1t + A_0)\varphi(x) + \psi(t),$$

where A_2, A_1, and A_0 are arbitrary constants, and the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (B_2 is an arbitrary constant)

$$a\ln\varphi''_{xx} - 2A_2\varphi = B_2, \tag{3}$$

$$\psi''_{tt} - a\ln(A_2t^2 + A_1t + A_0) - b = B_2. \tag{4}$$

The general solution of equation (3) is given by

$$\varphi(x) = -\frac{a}{2A_2}\ln\left[\frac{A_2}{aC_1^2}\cos^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} \quad \text{if} \quad A_2a > 0,$$

$$\varphi(x) = -\frac{a}{2A_2}\ln\left[\frac{A_2}{aC_1^2}\sinh^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} \quad \text{if} \quad A_2a > 0,$$

$$\varphi(x) = -\frac{a}{2A_2}\ln\left[-\frac{A_2}{aC_1^2}\cosh^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} \quad \text{if} \quad A_2a < 0,$$

$$\varphi(x) = \tfrac{1}{2}e^{B_2/a}x^2 + C_1x + C_2 \quad \text{if} \quad A_2 = 0,$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (4) is expressed as

$$\psi(t) = a\int_{t_0}^{t}(t-\xi)\ln(A_2\xi^2 + A_1\xi + A_0)\,d\xi + \frac{1}{2}(B_2 + b)t^2 + B_1t + B_0,$$

where B_1 and B_0 are arbitrary constants.

⊙ *References for equation* 12.2.1.1: N. H. Ibragimov (1994, pp. 214–218), A. D. Polyanin and V. F. Zaitsev (2004, pp. 501–503).

2. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x + C_1, \pm t + C_2) + C_3t + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F(kw'_z, k^2 w''_{zz}) - \lambda^2 w''_{zz} = 0.$$

3°. Additive separable solution:

$$w(x,t) = At^2 + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation

$$F(\varphi'_x, \varphi''_{xx}) - 2A = 0.$$

4°. Solution (generalizes the solutions of Items 2° and 3°):

$$w(x,t) = At^2 + Bt + C + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, C, k, and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$F(k\varphi'_z, k^2\varphi''_{zz}) - \lambda^2\varphi''_{zz} - 2A = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$.

1°. Degenerate solution linear in x:

$$w = (C_1 t + C_2)x + C_3 t + C_4 + \int_0^t (t-\tau)F(C_1\tau + C_2, 0)\,d\tau.$$

2°. Traveling-wave solution:

$$w(x,t) = w(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $w = w(\xi)$ is determined by the autonomous ordinary differential equation

$$(a\beta^2 w - \lambda^2)w''_{\xi\xi} + F(\beta w'_\xi, \beta^2 w''_{\xi\xi}) = 0.$$

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U''_{zz} - 2aC_1U'_z + F(U'_z, U''_{zz}) = 8aC_1^2.$$

Special case 1. Let $F(w_x, w_{xx}) = F(w_x)$. Self-similar solution:

$$w(x,t) = t^2 u(\zeta), \quad \zeta = xt^{-2},$$

where the function $u = u(\zeta)$ is determined by the ordinary differential equation

$$2u - 2\zeta u'_\zeta + 4\zeta^2 u''_{\zeta\zeta} = auu''_{\zeta\zeta} + F(u'_\zeta).$$

Special case 2. Let $F(w_x, w_{xx}) = F(w_{xx})$. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6a\varphi^2,$$
$$\psi''_{tt} = 6a\varphi\psi,$$
$$\chi''_{tt} = 2a\varphi\chi + F(2\varphi).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

This is a special case of equation 12.2.2.7.

5. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} + aw^2\right).$

This is a special case of equation 12.2.2.8.

6. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 12.2.2.9.

7. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 12.2.2.11.

12.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial t}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = F\left(t, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 xt + C_3 x + C_4 t + C_5,$$

where the C_n are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{1}{2}(C_1 t + C_2)x^2 + (C_3 t + C_4)x + \int_0^t (t-\xi)F(\xi, C_1\xi + C_2)\,d\xi + C_5 t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. The substitution $u(x,t) = \frac{\partial w}{\partial x}$ leads to a simpler equation which is linear in the highest derivatives:

$$\frac{\partial^2 u}{\partial t^2} = f\left(t, \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}, \qquad f(t,\xi) = \frac{\partial}{\partial \xi}F(t,\xi).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x, \pm t + C_1) + C_2 xt + C_3 x + C_4 t + C_5,$$

where C_n are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}(C_1 x + C_2)t^2 + (C_3 x + C_4)t + \varphi(x) + C_5 x + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$C_1 x + C_2 = F(x, \varphi''_{xx}).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, w,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right).$

Suppose the auxiliary ordinary differential equation

$$w = F(x,\, w,\, w'_x,\, w''_{xx})$$

is reduced, with the linear transformation

$$x = \varphi(z), \quad w = \psi(z)u + \chi(z)$$

followed by the division of the resulting equation by $\psi(z)$, to the autonomous form

$$u = \mathcal{F}(u,\, u'_z,\, u''_{zz}),$$

where $\mathcal{F} = F/\psi$. Then the original partial differential equation can be reduced, with the same transformation,

$$x = \varphi(z), \quad w(x,t) = \psi(z)u(z,t) + \chi(z),$$

to

$$\frac{\partial^2 u}{\partial t^2} = \mathcal{F}\left(u,\, \frac{\partial u}{\partial z},\, \frac{\partial^2 u}{\partial z^2}\right),$$

which has a traveling-wave solution $u = u(z + \lambda t)$.

The above allows using various known transformations of ordinary differential equations (see Kamke, 1977; Polyanin and Zaitsev, 2003) for constructing exact solutions to partial differential equations. If the original equation is linear, then such transformations will result in linear constant-coefficient equations.

4. $\dfrac{\partial^2 w}{\partial t^2} = (aw + bx)\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

The substitution $w = u - (b/a)x$ leads to an equation of the form 12.2.1.3:

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + F\left(\frac{\partial u}{\partial x} - \frac{b}{a}, \frac{\partial^2 u}{\partial x^2}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right) + G\left(t,\, \dfrac{\partial w}{\partial t}\right) + bw.$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$F\big(x, \varphi'_x, \varphi''_{xx}\big) + b\varphi = C,$$
$$\psi''_{tt} - G\big(t, \psi'_t\big) - b\psi = C.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, t,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x,t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } a = k^2 > 0,$$
$$w_2 = w(x,t) + C_1\cos(kt) + C_2\sin(kt) \quad \text{if } a = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

7. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(t,\, \dfrac{1}{w}\dfrac{\partial w}{\partial t},\, \left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2,\, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \left[C_1\sin\left(x\sqrt{a}\,\right) + C_2\cos\left(x\sqrt{a}\,\right)\right]\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t,\, \varphi'_t/\varphi,\, a(C_1^2 + C_2^2)\varphi^2,\, -a\right).$$

2°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \left(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\right)\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t,\, \varphi'_t/\varphi,\, 4aC_1C_2\varphi^2,\, -a\right).$$

⊙ *Reference for equation* 12.2.2.7: A. D. Polyanin and V. F. Zaitsev (2004, p. 507).

8. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(t,\, \dfrac{1}{w}\dfrac{\partial w}{\partial t},\, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\, w\dfrac{\partial^2 w}{\partial x^2} + aw^2\right).$

Multiplicative separable solutions:

$$w(x,t) = \left[C_1\sin\left(x\sqrt{a}\,\right) + C_2\cos\left(x\sqrt{a}\,\right)\right]\varphi(t) \quad \text{if } a > 0,$$
$$w(x,t) = \left(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\right)\varphi(t) \quad \text{if } a < 0,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t, \varphi'_t/\varphi, -a, 0\right).$$

⊙ *Reference for equation* 12.2.2.8: A. D. Polyanin and V. F. Zaitsev (2004, p. 507).

9. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial t},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

1°. Multiplicative separable solution:

$$w(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\big(t,\ \varphi'_t/\varphi,\ \lambda^2, 4AB\lambda^2\varphi^2\big)$.

2°. Multiplicative separable solution:

$$w(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\big(t,\ \varphi'_t/\varphi, -\lambda^2, -\lambda^2(A^2 + B^2)\varphi^2\big)$.

⊙ *Reference for equation* 12.2.2.9: A. D. Polyanin and V. F. Zaitsev (2004, p. 507).

10. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial w}{\partial x} - x\dfrac{\partial^2 w}{\partial x^2},\ 2w - 2x\dfrac{\partial w}{\partial x} + x^2\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = (C_2x^2 + C_1x + C_0)\varphi(t),$$

where C_0, C_1, and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\big(t, 2C_2\varphi, C_1\varphi, 2C_0\varphi\big)$.

11. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

Generalized separable solution quadratic in x:

$$w = \varphi_1(t)x^2 + \varphi_2(t)x + \varphi_3(t),$$

where the functions φ_1, φ_2, and φ_3 are determined by the solution of the ordinary differential equations

$$\begin{aligned}
\varphi''_1 &= \varphi_1 F\big(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\big),\\
\varphi''_2 &= \varphi_2 F\big(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\big),\\
\varphi''_3 &= \varphi_3 F\big(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\big) + G\big(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\big).
\end{aligned}$$

It follows from the first two equations that

$$\varphi_2 = C_1\varphi_1 + C_2\varphi_1\int\frac{dt}{\varphi_1^2},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference for equation* 12.2.2.11: A. D. Polyanin and V. F. Zaitsev (2004, p. 508).

12. $\dfrac{\partial^2 w}{\partial t^2} = wF_1\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{\lambda x}F_2\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{-\lambda x}F_3\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

13. $\dfrac{\partial^2 w}{\partial t^2} = wF_1\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \cos(\lambda x)F_2\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \sin(\lambda x)F_3\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

14. $\dfrac{\partial^2 w}{\partial t^2} = x^2 f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution quadratic in x:

$$w(x,t) = x^2\varphi(t) + x\int_0^t (t-\xi)f_1\big(\xi, 2\varphi(\xi)\big)\,d\xi + \int_0^t (t-\xi)f_0\big(\xi, 2\varphi(\xi)\big)\,d\xi + C_1xt + C_2x + C_3t + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = f_2(t, 2\varphi).$$

15. $\dfrac{\partial^2 w}{\partial t^2} = x^2 f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

12.2.3. Equations Linear in the Mixed Derivative

1. $\dfrac{\partial^2 w}{\partial x \partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + \varphi(t), t) + \frac{\varphi'_t(t)}{g(t)},$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function and $\varphi(t)$ is determined by the first-order ordinary differential equation $\varphi'_t = F(t, \varphi, 0)$.

3°. For $g(t) = a$ and $F = F(w_x, w_{xx})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$k\lambda U''_{zz} = ak^2 UU''_{zz} + F(kU'_z, k^2U''_{zz}).$$

2. $\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial t\partial x} - \dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x^2} = F\left(t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Let $\widetilde{w}(t,x)$ be a solution of the equation in question. Then the function

$$w(t,x) = \widetilde{w}\big(t, x+\varphi(t)\big),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. In the original equation, let us change from the old variables to the von Mises variables:

$$t,\ x,\ w = w(t,x) \quad \Longrightarrow \quad t,\ w,\ \eta = \eta(t,w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial x}. \tag{1}$$

The derivatives are transformed as follows:

$$\frac{\partial^2 w}{\partial x^2} = \eta\frac{\partial \eta}{\partial w}, \quad \frac{\partial^2 w}{\partial t\partial x} = \frac{\partial \eta}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \eta}{\partial w}.$$

The von Mises transformation reduces the original second-order equation to a first-order equation:

$$\eta\frac{\partial \eta}{\partial t} = F\left(t, w, \eta, \eta\frac{\partial \eta}{\partial w}\right). \tag{2}$$

Special case 1. The original equation with $F = f(t,w)w_{xx} + g(t,w)w_x^2 + h(t,w)w_x$ reduces to equation (2) of a special form, which is further reduced, by canceling by η, to a linear first-order equation:

$$\eta_t - f(t,w)\eta_w = g(t,w)\eta + h(t,w).$$

Special case 2. With $F = f(t,w)w_{xx} + g(t,w)w_x^2 + h(t,w)$, equation (2) is reduced, with the change of variable $Z = \eta^2$, to a linear first-order equation:

$$Z_t - f(t,w)Z_w = 2g(t,w)Z + h(t,w).$$

3. $f\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)\dfrac{\partial^2 w}{\partial x^2} + g\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)\dfrac{\partial^2 w}{\partial x\partial y} + h\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)\dfrac{\partial^2 w}{\partial y^2} = 0.$

The Legendre transformation

$$w(x,y) + u(\xi,\eta) = x\xi + y\eta, \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}$$

leads to the linear equation (for details, see Subsection 26.3.3)

$$f(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} - g(\xi,\eta)\frac{\partial^2 u}{\partial \xi\partial \eta} + h(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} = 0.$$

4. $\dfrac{\partial^2 w}{\partial x\partial t} = F\left(t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)\dfrac{\partial^2 w}{\partial y^2}.$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t), \qquad z = x + C_1 y + C_1^2\int g(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial z \partial t} = F\left(t, u, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}\right).$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi,t), \qquad \xi = x + \varphi(t)(y+C_1)^2, \qquad \varphi(t) = -\left[4\int g(t)\,dt + C_2\right]^{-1},$$

where the function $U(\xi,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial \xi \partial t} = F\left(t, U, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2}\right) + 2g(t)\varphi(t)\frac{\partial U}{\partial \xi}.$$

12.2.4. Equations with Two Independent Variables, Nonlinear in Two or More Highest Derivatives

1. $\boldsymbol{f_1\left(\dfrac{\partial^2 w}{\partial x^2}\right) f_2\left(\dfrac{\partial^2 w}{\partial y^2}\right) = g_1(x)g_2(y).}$

Generalized separable solution:

$$w(x,y) = \varphi(x) + \psi(y) + C_1 xy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations (a is any)

$$f_1(\varphi''_{xx}) = a g_1(x),$$
$$a f_2(\psi''_{yy}) = g_2(y).$$

2. $\boldsymbol{F\left(x, y, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}\right) = 0.}$

The substitution $u = \frac{\partial w}{\partial x}$ leads to the first-order partial differential equation

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$$

For details about integration methods and exact solutions for such equations (with various F), see Section 24.1 and Sections 3.2–3.3.

3. $\boldsymbol{\dfrac{\partial^2 w}{\partial y^2} = F\left(\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}\right).}$

1°. Solution quadratic in both variables:

$$w(x,y) = \tfrac{1}{2}C_1 x^2 + C_2 xy + \tfrac{1}{2}F(C_1, C_2)y^2 + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

2°. We differentiate the equation with respect to x, introduce the new variable

$$U(x,y)=\frac{\partial w}{\partial x},$$

and then apply the Legendre transformation (for details, see Subsection S.2.3)

$$X=\frac{\partial U}{\partial x},\quad Y=\frac{\partial U}{\partial y},\quad Z=x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}-U$$

to obtain the second-order linear equation

$$\frac{\partial^2 Z}{\partial X^2}=F_X(X,Y)\frac{\partial^2 Z}{\partial Y^2}-F_Y(X,Y)\frac{\partial^2 Z}{\partial X\partial Y},$$

where the subscripts X and Y denote the corresponding partial derivatives.

Special case. Let $F(X,Y)=aX+f(Y)$, or

$$\frac{\partial^2 w}{\partial y^2}=a\frac{\partial^2 w}{\partial x^2}+f\left(\frac{\partial^2 w}{\partial x\partial y}\right).$$

Solution:

$$w=\varphi(z)+\tfrac{1}{6}(A_2A_3-A_1A_4)x^3+\tfrac{1}{2}aA_1A_3x^2y+\tfrac{1}{2}aA_2A_3xy^2+\tfrac{1}{6}(a^2A_1A_3+aA_2A_4)y^3$$
$$+\tfrac{1}{2}B_1x^2+B_2xy+\tfrac{1}{2}B_3y^2+B_4x+B_5y+B_6,\qquad z=A_1x+A_2y,$$

where the A_n and B_m are arbitrary constants and the function $\varphi(z)$ is determined by the ordinary differential equation

$$(A_2^2-aA_1^2)\varphi''_{zz}+aA_4z+B_3-aB_1=f(A_1A_2\varphi''_{zz}+aA_3z+B_2).$$

4. $F\left(\dfrac{\partial^2 w}{\partial x^2},\dfrac{\partial^2 w}{\partial x\partial y},\dfrac{\partial^2 w}{\partial y^2}\right)=0.$

1°. Solution quadratic in both variables:

$$w(x,y)=A_{11}x^2+A_{12}xy+A_{22}y^2+B_1x+B_2y+C,$$

where A_{11}, A_{12}, A_{22}, B_1, B_2, and C are arbitrary constants constrained by

$$F(2A_{11},A_{12},2A_{22})=0.$$

2°. Solving the equation for w_{yy} (or w_{xx}), one arrives at an equation of the form 12.2.4.3.

5. $F_1\left(x,\dfrac{\partial w}{\partial x},\dfrac{\partial^2 w}{\partial x^2},\dfrac{\partial^2 w}{\partial x\partial y}\right)+F_2\left(y,\dfrac{\partial w}{\partial y},\dfrac{\partial^2 w}{\partial x\partial y},\dfrac{\partial^2 w}{\partial y^2}\right)=kw.$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\big(x,\varphi'_x,\varphi''_{xx},0\big)-k\varphi=C,$$
$$F_2\big(y,\psi'_y,0,\psi''_{yy}\big)-k\psi=-C,$$

where C is an arbitrary constant.

6. $F_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + F_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}, \frac{1}{w}\frac{\partial^2 w}{\partial y^2}\right) = \ln w.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\big(x, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\big) - \ln\varphi = C,$$
$$F_2\big(y, \psi'_y/\psi, \psi''_{yy}/\psi\big) - \ln\psi = -C,$$

where C is an arbitrary constant.

7. $F_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + w^k F_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}, \frac{1}{w}\frac{\partial^2 w}{\partial y^2}\right) = 0.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi^{-k} F_1\big(x, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\big) = C,$$
$$\psi^k F_2\big(y, \psi'_y/\psi, \psi''_{yy}/\psi\big) = -C,$$

where C is an arbitrary constant.

8. $F\left(ax + by,\ w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial y^2},\ \frac{\partial^2 w}{\partial x \partial y}\right) = 0.$

Solution:

$$w = w(\xi), \quad \xi = ax + by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\big(\xi,\ w,\ aw'_\xi,\ bw'_\xi,\ a^2 w''_{\xi\xi},\ b^2 w''_{\xi\xi},\ abw''_{\xi\xi}\big) = 0.$$

9. $F\left(ax + by,\ w + kx + sy,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial y^2},\ \frac{\partial^2 w}{\partial x \partial y}\right) = 0.$

The substitution $u(x, y) = w(x, y) + kx + sy$ leads to an equation of the form 12.2.4.8:

$$F\left(ax + by,\ u,\ \frac{\partial u}{\partial x} - k,\ \frac{\partial u}{\partial y} - s,\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial y^2},\ \frac{\partial^2 u}{\partial x \partial y}\right) = 0.$$

10. $(a_1 x + b_1 y)\frac{\partial w}{\partial x} + (a_2 x + b_2 y)\frac{\partial w}{\partial y} = F\left(w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial x \partial y},\ \frac{\partial^2 w}{\partial y^2}\right).$

Traveling-wave type solutions:

$$w = w(z), \qquad z = a_2 x + (k - a_1)y,$$

where k is a root of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1 b_2 - a_2 b_1 = 0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$kzw'_z = F\big(w,\ a_2 w'_z,\ (k - a_1)w'_z,\ a_2^2 w''_{zz},\ a_2(k - a_1)w''_{zz},\ (k - a_1)^2 w''_{zz}\big).$$

11. $$(a_1x+b_1y+c_1)\left(\frac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\frac{\partial w}{\partial y}\right)^k = F\left(w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial x\partial y},\ \frac{\partial^2 w}{\partial y^2}\right).$$

Exact solutions are sought in the traveling-wave form

$$w = w(z), \qquad z = Ax + By + C,$$

where the constants A, B, and C are determined by solving the algebraic system

$$a_1A^k + a_2B^k = A, \tag{1}$$
$$b_1A^k + b_2B^k = B, \tag{2}$$
$$c_1A^k + c_2B^k = C. \tag{3}$$

Equations (1) and (2) are first solved for A and B, and then C is evaluated from (3).

The desired function $w(z)$ is determined by the ordinary differential equation

$$z(w_z')^k = F\big(w,\ Aw_z',\ Bw_z',\ A^2w_{zz}'',\ ABw_{zz}'',\ B^2w_{zz}''\big).$$

12. $$(a_1x+b_1y)\frac{\partial^2 w}{\partial x^2}+(a_2x+b_2y)\frac{\partial^2 w}{\partial x\partial y}+(a_3x+b_3y)\frac{\partial^2 w}{\partial y^2} = F\left(w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial x\partial y},\ \frac{\partial^2 w}{\partial y^2}\right).$$

Traveling-wave type solutions:

$$w = w(z), \qquad z = Ax + By,$$

where the constants A and B are determined by solving the algebraic system of equation

$$a_1A^2 + a_2AB + a_3B^2 = A,$$
$$b_1A^2 + b_2AB + b_3B^2 = B,$$

and the desired function $w(z)$ is determined by the ordinary differential equation

$$zw_{zz}'' = F\big(w,\ Aw_z',\ Bw_z',\ A^2w_{zz}'',\ ABw_{zz}'',\ B^2w_{zz}''\big).$$

13. $$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x\partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0.$$

Traveling-wave type solutions:

$$w = w(z), \quad z = C_1x + C_2y,$$

where C_1 and C_2 are arbitrary constants, and the function $w(z)$ is determined by the ordinary differential equation

$$F(w, C_1w_z', C_2w_z', zw_z', C_1^2w_{zz}'', C_1C_2w_{zz}'', C_2^2w_{zz}'') = 0.$$

Remark. The above solution is a nonclassical (noninvariant) traveling-wave solution, since the equation is not invariant under translation transformations.

⊙ *Reference*: A. D. Polyanin (2004 *a*).

14. $$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, x\frac{\partial^2 w}{\partial x^2} + y\frac{\partial^2 w}{\partial x \partial y}, x\frac{\partial^2 w}{\partial x \partial y} + y\frac{\partial^2 w}{\partial y^2}\right) = 0.$$

Traveling-wave type solutions:

$$w = w(z), \quad z = C_1 x + C_2 y,$$

where C_1 and C_2 are arbitrary constants, and the function $w(z)$ is determined by the ordinary differential equation

$$F(w, C_1 w'_z, C_2 w'_z, z w'_z, C_1^2 w''_{zz}, C_1 C_2 w''_{zz}, C_2^2 w''_{zz}, C_1 z w''_{zz}, C_2 z w''_{zz}) = 0.$$

15. $$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2}\right) = 0.$$

Traveling-wave type solutions:

$$w = w(z), \quad z = C_1 x + C_2 y,$$

where C_1 and C_2 are arbitrary constants, and the function $w(z)$ is determined by the ordinary differential equation

$$F(w, C_1 w'_z, C_2 w'_z, z w'_z, C_1^2 w''_{zz}, C_1 C_2 w''_{zz}, C_2^2 w''_{zz}, z^2 w''_{zz}) = 0.$$

12.2.5. Equations with n Independent Variables

1. $$\sum_{k=1}^{n} f_k\left(x_k, \frac{\partial w}{\partial x_k}, \frac{\partial^2 w}{\partial x_k^2}\right) = aw.$$

Additive separable solution:

$$w(x_1, \ldots, x_n) = \sum_{k=1}^{n} \varphi_k(x_k),$$

where the functions $\varphi_k = \varphi_k(x_k)$ are determined by the second-order ordinary differential equations

$$\sum_{k=1}^{n} f_k\left(x_k, \frac{d\varphi_k}{dx_k}, \frac{d^2\varphi_k}{dx_k^2}\right) - a\varphi_k = C_k, \qquad k = 1, 2, \ldots, n.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by the constraint $C_1 + \cdots + C_n = 0$.

Remark. The functions f_k in the original equation can also depend on any number of mixed derivatives $\partial_{x_i x_j} w$. In this case, the arguments corresponding to $\partial_{x_i x_j} w$ in the second-order ordinary differential equations obtained will be replaced by zeros.

2. $\displaystyle\sum_{k=1}^{n} f_k\left(x_k, \frac{1}{w}\frac{\partial w}{\partial x_k}, \frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right) = a\ln w.$

Multiplicative separable solution:

$$w(x_1,\ldots,x_n) = \prod_{k=1}^{n}\varphi_k(x_k),$$

where the functions $\varphi_k = \varphi_k(x_k)$ are determined by the second-order ordinary differential equations

$$f_k\left(x_k, \frac{1}{\varphi_k}\frac{d\varphi_k}{dx_k}, \frac{1}{\varphi_k}\frac{d^2\varphi_k}{dx_k^2}\right) - a\ln\varphi_k = C_k; \qquad k = 1,\ldots,n.$$

The arbitrary constants $C_1,\ldots,C_n$ are related by a single constraint, $C_1+\cdots+C_n = 0$.

3. $\displaystyle F\left(x_1,\ldots,x_k; \frac{\partial w}{\partial x_1},\ldots,\frac{\partial w}{\partial x_k}; \frac{\partial^2 w}{\partial x_1^2},\ldots,\frac{\partial^2 w}{\partial x_k^2}\right)$
$\displaystyle + G\left(x_{k+1},\ldots,x_n; \frac{\partial w}{\partial x_{k+1}},\ldots,\frac{\partial w}{\partial x_n}; \frac{\partial^2 w}{\partial x_{k+1}^2},\ldots,\frac{\partial^2 w}{\partial x_n^2}\right) = aw.$

Additive separable solution:

$$w(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = \varphi(x_1,\ldots,x_k) + \psi(x_{k+1},\ldots,x_n).$$

Here, the functions $\varphi = \varphi(x_1,\ldots,x_k)$ and $\psi = \psi(x_{k+1},\ldots,x_n)$ are determined by solving the two simpler partial differential equations

$$F\left(x_1,\ldots,x_k; \frac{\partial\varphi}{\partial x_1},\ldots,\frac{\partial\varphi}{\partial x_k}; \frac{\partial^2\varphi}{\partial x_1^2},\ldots,\frac{\partial^2\varphi}{\partial x_k^2}\right) = a\varphi + C,$$
$$G\left(x_{k+1},\ldots,x_n; \frac{\partial\psi}{\partial x_{k+1}},\ldots,\frac{\partial\psi}{\partial x_n}; \frac{\partial^2\psi}{\partial x_{k+1}^2},\ldots,\frac{\partial^2\psi}{\partial x_n^2}\right) = a\psi - C,$$

where C is an arbitrary constant.

4. $\displaystyle F\left(x_1,\ldots,x_k; \frac{1}{w}\frac{\partial w}{\partial x_1},\ldots,\frac{1}{w}\frac{\partial w}{\partial x_k}; \frac{1}{w}\frac{\partial^2 w}{\partial x_1^2},\ldots,\frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right)$
$\displaystyle + G\left(x_{k+1},\ldots,x_n; \frac{1}{w}\frac{\partial w}{\partial x_{k+1}},\ldots,\frac{1}{w}\frac{\partial w}{\partial x_n}; \frac{1}{w}\frac{\partial^2 w}{\partial x_{k+1}^2},\ldots,\frac{1}{w}\frac{\partial^2 w}{\partial x_n^2}\right) = a\ln w.$

Multiplicative separable solution:

$$w(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = \varphi(x_1,\ldots,x_k)\psi(x_{k+1},\ldots,x_n).$$

Here, the functions $\varphi = \varphi(x_1,\ldots,x_k)$ and $\psi = \psi(x_{k+1},\ldots,x_n)$ are determined by solving the two simpler partial differential equations

$$F\left(x_1,\ldots,x_k; \frac{1}{\varphi}\frac{\partial\varphi}{\partial x_1},\ldots,\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_k}; \frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_1^2},\ldots,\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_k^2}\right) = a\ln\varphi + C,$$
$$G\left(x_{k+1},\ldots,x_n; \frac{1}{\psi}\frac{\partial\psi}{\partial x_{k+1}},\ldots,\frac{1}{\psi}\frac{\partial\psi}{\partial x_n}; \frac{1}{\psi}\frac{\partial^2\psi}{\partial x_{k+1}^2},\ldots,\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_n^2}\right) = a\ln\psi - C,$$

where C is an arbitrary constant.

5. $$F\left(x_1,\dots,x_k;\frac{1}{w}\frac{\partial w}{\partial x_1},\dots,\frac{1}{w}\frac{\partial w}{\partial x_k};\frac{1}{w}\frac{\partial^2 w}{\partial x_1^2},\dots,\frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right)$$
$$+\,w^{\beta}G\left(x_{k+1},\dots,x_n;\frac{1}{w}\frac{\partial w}{\partial x_{k+1}},\dots,\frac{1}{w}\frac{\partial w}{\partial x_n};\frac{1}{w}\frac{\partial^2 w}{\partial x_{k+1}^2},\dots,\frac{1}{w}\frac{\partial^2 w}{\partial x_n^2}\right)=0.$$

Multiplicative separable solution:

$$w(x_1,\dots,x_k,x_{k+1},\dots,x_n)=\varphi(x_1,\dots,x_k)\psi(x_{k+1},\dots,x_n).$$

Here, the functions $\varphi=\varphi(x_1,\dots,x_k)$ and $\psi=\psi(x_{k+1},\dots,x_n)$ are determined by solving the two simpler partial differential equations

$$\varphi^{-\beta}F\left(x_1,\dots,x_k;\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_1},\dots,\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_k};\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_1^2},\dots,\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_k^2}\right)=C,$$
$$\psi^{\beta}G\left(x_{k+1},\dots,x_n;\frac{1}{\psi}\frac{\partial\psi}{\partial x_{k+1}},\dots,\frac{1}{\psi}\frac{\partial\psi}{\partial x_n};\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_{k+1}^2},\dots,\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_n^2}\right)=-C,$$

where C is an arbitrary constant.

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Chapter 13
Third-Order Equations

13.1. Equations Involving the First Derivative in t

13.1.1. Korteweg–de Vries Equation $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} = 0$

1. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w\dfrac{\partial w}{\partial x} = 0.$

Korteweg–de Vries equation in canonical form (or *KdV equation* for short). It is used in many sections of nonlinear mechanics and theoretical physics for describing one-dimensional *nonlinear dispersive nondissipative waves* (in which the dissipation law for linear waves has the form $\omega = a_1 k + a_3 k^3$, where k is the wavenumber). In particular, the mathematical modeling of moderate-amplitude shallow-water surface waves is based on this equation (see Korteweg and de Vries, 1895).

The Korteweg–de Vries equation is solved by the inverse scattering method; see Items 9 and 10 and the references at the end of this equation.

1. The similarity formula.

Suppose $w(x,t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 6C_1 C_2 t + C_3, C_1^3 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

⊙ *Reference*: P. J. Olver (1986).

2. Traveling-wave solutions. Soliton. Cnoidal waves. Periodic solutions.

2.1. Traveling-wave solution:

$$w = w(z), \quad z = x - \lambda t,$$

where the function $w(z)$ defined in implicit form as

$$\int \frac{dw}{\sqrt{2w^3 + \lambda w^2 + C_1 w + C_2}} = \pm z + C_3. \tag{1}$$

Here, λ, C_1, C_2, and C_3 are arbitrary constants; to $\lambda = 0$ there corresponds a stationary solution.

Below are important cases where solution (1) can be written out in explicit form.

2.2. *Soliton*. The unique solution regular for all real values of $z = x - \lambda t$ and vanishing as $z \to \pm\infty$ is expressed as

$$w(z) = -\frac{\lambda}{2\cosh^2\left[\frac{1}{2}\sqrt{\lambda}\,(z + B)\right]}, \qquad \lambda > 0, \tag{2}$$

where B is an arbitrary constant.

2.3. *Cnoidal waves*. There are periodic solutions that are real and regular for any real $z = x - \lambda t$:

$$w(z) = A\,\mathrm{cn}^2\big[p(z+B),k\big], \quad A = -2p^2k^2, \quad \lambda = 4p^2(2k^2-1). \tag{3}$$

They depend on an arbitrary positive constant $k^2 < 1$. Here, $\mathrm{cn}(y,k)$ is the Jacobian elliptic cosine. Solution (2) can be obtained from (3) by letting $k^2 \to 1$. The periods of solution (3) are $\omega_1 = 4K$ and $\omega_2 = 2K + 2iK_*$, where K and K_* are complete elliptic integrals of the first kind:

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K_* = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k_*^2t^2)}}, \quad k^2 + k_*^2 = 1.$$

2.4. *Unlimited periodic solutions*:

$$w(z) = \tfrac{1}{6}\lambda\Big\{1 + 3\tan^2\big[\tfrac{1}{2}\sqrt{\lambda}\,(z+B)\big]\Big\} \quad \text{if } \lambda > 0,$$
$$w(z) = -\frac{\lambda}{2\cos^2\big[\frac{1}{2}\sqrt{-\lambda}\,(z+B)\big]} \quad \text{if } \lambda < 0,$$

where B and λ are arbitrary constants.

2.5. *Rational solution*. It has the form

$$w(z) = \frac{2}{(z+B)^2} - \frac{\lambda}{6}, \qquad z = x - \lambda t,$$

where B is an arbitrary constant.

2.6. Other traveling-wave solutions:

$$w(z) = \frac{\lambda}{2\sinh^2\big[\frac{1}{2}\sqrt{\lambda}\,(z+B)\big]} \quad \text{if } \lambda > 0,$$
$$w(z) = \tfrac{1}{6}\lambda\Big\{1 - 3\tanh^2\big[\tfrac{1}{2}\sqrt{-\lambda}\,(z+B)\big]\Big\} \quad \text{if } \lambda < 0,$$
$$w(z) = \tfrac{1}{6}\lambda\Big\{1 - 3\coth^2\big[\tfrac{1}{2}\sqrt{-\lambda}\,(z+B)\big]\Big\} \quad \text{if } \lambda < 0,$$

where B and λ are arbitrary constants.

3. Two-, Three-, and N-soliton solutions.

3.1. Two-soliton solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\big(1 + B_1e^{\theta_1} + B_2e^{\theta_2} + AB_1B_2e^{\theta_1+\theta_2}\big),$$
$$\theta_1 = a_1x - a_1^3t, \quad \theta_2 = a_2x - a_2^3t, \quad A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2,$$

where B_1, B_2, a_1, and a_2 are arbitrary constants.

⊙ *References*: R. Hirota (1971, 1972).

3.2. Three-soliton solution:

$$w = -2\frac{\partial^2}{\partial x^2}F(x,t), \qquad F(x,t) = 1 + \exp(\theta_1) + \exp(\theta_2) + \exp(\theta_3)$$
$$+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\theta_1+\theta_2) + \frac{(k_1-k_3)^2}{(k_1+k_3)^2}\exp(\theta_1+\theta_3) + \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\exp(\theta_2+\theta_3)$$
$$+ \frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_3+k_3)^2}\exp(\theta_1+\theta_2+\theta_3),$$
$$\theta_1 = k_1x - k_1^3t + C_1, \quad \theta_2 = k_2x - k_2^3t + C_2, \quad \theta_3 = k_3x - k_3^3t + C_3,$$

where C_1, C_2, C_3, k_1, k_2, and k_3 are arbitrary constants.

⊙ *Reference*: A. M. Wazwaz (2008).

3.3. N-soliton solutions:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\Big\{\ln\det\big[\mathbf{I} + \mathbf{C}(\mathbf{x},\mathbf{t})\big]\Big\}.$$

Here, $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{C}(x,t)$ the $N\times N$ symmetric matrix with entries

$$C_{mn}(x,t) = \frac{\sqrt{\rho_m(t)\rho_n(t)}}{p_m+p_n}\exp\big[-(p_m+p_n)x\big],$$

where the normalizing factors $\rho_n(t)$ are given by

$$\rho_n(t) = \rho_n(0)\exp\big(8p_n^3t\big), \quad n = 1,2,\ldots,N.$$

The solution involves $2N$ arbitrary constants p_n and $\rho_n(0)$.

The following asymptotic formula holds:

$$w(x,t) \approx -2\sum_{n=1}^{N}\frac{p_n^2}{\cosh^2\big[p_n\big(x-\xi_n^{\pm}-v_nt\big)\big]} \quad \text{as} \quad t\to\pm\infty,$$

where $v_n = 4p_n^2$ is the speed of the nth soliton and the real constants $\xi_n^{\pm}$ are constrained by

$$\xi_n^+ - \xi_n^- = \sum_{m=1}^{n-1}p_n^{-1}\ln\frac{p_n+p_m}{p_n-p_m} - \sum_{m=n+1}^{N}p_n^{-1}\ln\frac{p_n+p_m}{p_n-p_m}.$$

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

4. "Soliton + pole" solutions.

4.1. "One soliton + one pole" solution:

$$w(x,t) = -2p^2\big[\cosh^{-2}(pz) - (1+px)^{-2}\tanh^2(pz)\big]\big[1-(1+px)^{-1}\tanh(pz)\big]^{-2}, \qquad z = x-4p^2t-c,$$

where p and c are arbitrary constants.

4.2. "N solitons + one pole" solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\Big\{x\ln\det\big[\mathbf{I} + \mathbf{D}(\mathbf{x},\mathbf{t})\big]\Big\}.$$

Here, $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{D}(x,t)$ the $N \times N$ symmetric matrix with entries

$$D_{mn}(x,t) = c_m(t)c_n(t)\big[(p_m + p_n)^{-1} + (p_m p_n x)^{-1}\big] \exp\big[-(p_m + p_n)x\big],$$

where the normalizing factors $c_n(t)$ are given by

$$c_n(t) = c_n(0)\exp\big(4p_n^3 t\big), \quad n = 1, 2, \ldots, N.$$

The solution involves $2N$ arbitrary constants p_n and $c_n(0)$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

5. Rational solutions.

5.1. The simplest rational solution is as follows:

$$w(x,t) = 2(x-\xi)^{-2},$$

where ξ is an arbitrary constant that can be complex (if it is real, the solution is singular for real values of x).

5.2. General form of a rational solution:

$$w(x,t) = 2\sum_{j=1}^{N}\big[x - \xi_j(t)\big]^{-2}. \tag{4}$$

The functions $\xi_j(t)$ must meet the conditions

$$\begin{aligned} &\sum_{\substack{k=1\\ j\neq k}}^{N}\big[\xi_j(t) - \xi_k(t)\big]^{-3} = 0, \qquad j = 1, 2, \ldots, N, \\ &\xi_j(t) = -12\sum_{\substack{k=1\\ j\neq k}}^{N}\big[\xi_j(t) - \xi_k(t)\big]^{-2}, \quad j = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

A solution exists if $N = \frac{1}{2}m(m+1)$, $m = 1,\ 2,\ 3,\ \ldots$; if $m > 1$, there are no real solutions. In particular, if $m = 2$, there are three poles $\xi_j(t) = -e^{2\pi i j/3}(12t)^{1/3}$ ($j = 1,\ 2,\ 3$) and solution (4) can be written out as follows:

$$w(x,t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2} \qquad (\text{for } N = 3).$$

A solution for $m = 3$ (corresponds to $N = 6$) is given by

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\big(x^6 + 60x^3 t - 720t^2\big).$$

Note that (4) can be rewritten in the equivalent form

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\big[\ln P_N(x,t)\big], \quad \text{where} \quad P_N(x,t) = \prod_{j=1}^{N}\big[x - \xi_j(t)\big].$$

⊙ *References*: M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982).

6. Self-similar solutions.

6.1. Simplest self-similar solution (degenerate solution):

$$w(x,t) = -\frac{1}{6}\frac{x-x_0}{t-t_0},$$

where x_0 and t_0 are arbitrary constants.

6.2. Self-similar solution:

$$w(x,t) = \big[3(t-t_0)\big]^{-2/3} f(y), \quad y = \big[3(t-t_0)\big]^{-1/3}(x-x_0),$$

where the function $f(y)$ is determined by the third-order ordinary differential equation

$$f'''_{yyy} - yf'_y - 2f - 6ff'_y = 0. \tag{6}$$

Equation (6) has a first integral

$$(y+2f)[f''_{yy} - (y+2f)f] - (1+f'_y)f'_y = C,$$

where C is the constant of integration. A solution of equation (6) can be represented as

$$f(y) = g'_y(y) + g^2(y),$$

where the function $g(y)$ is any solution of the *second Painlevé equation* (see Conte and Musette, 2008)

$$g''_{yy} - 2g^3 - yg = A, \qquad A \text{ is an arbitrary constant.} \tag{7}$$

For $A = 2^{-2/3}$, equation (7) has a solution

$$g(y) = \frac{d}{dy}\Big[\ln F\big(-2^{-1/3}y\big)\Big],$$

where the function $F = F(z)$ satisfies the *Airy equation* $F''_{zz} = zF$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

7. General similarity solutions.

7.1. Solution:

$$w(x,t) = 2\varphi(z) + 2C_1 t, \quad z = x + 6C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(z)$ is determined by the second-order ordinary differential equation

$$\varphi''_{zz} = 6\varphi^2 - C_2\varphi - C_1 z + C_3,$$

where C_3 is an arbitrary constant. To the case $C_1 = -1$, $C_2 = C_3 = 0$ there corresponds the *first Painlevé equation* (if all C_n are nonzero, the equation for φ can also be reduced to the first Painlevé equation).

7.2. Solution:

$$w = \varphi^2 F(z) + \frac{1}{6\varphi}(\varphi'_t x + \psi'_t), \quad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are given by

$$\varphi(t) = (3At + C_1)^{-1/3}, \qquad \psi(t) = C_2(3At + C_1)^{2/3} + C_3(3At + C_1)^{-1/3},$$

where A, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$F'''_{zzz} - 6FF'_z - AF + \tfrac{2}{3}A^2 z = 0.$$

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

8. Miura transformation and Bäcklund transformations.

8.1. The Korteweg–de Vries equation can be reduced, with the differential change of variable (*Miura transformation*)

$$w = \frac{\partial u}{\partial x} + u^2, \tag{8}$$

to the form

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = \left(\frac{\partial}{\partial x} + 2u\right)\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x}\right) = 0.$$

It follows that any solution $u = u(x, t)$ of the modified Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x} = 0 \tag{9}$$

generates a solution (8) of the Korteweg–de Vries equation.

⊙ *References*: R. M. Miura (1968), F. Calogero and A. Degasperis (1982).

8.2. The Bäcklund transformations

$$\begin{aligned} &\frac{\partial u}{\partial x} = \varepsilon(u^2 - w), \qquad \varepsilon = \pm 1, \\ &\frac{\partial u}{\partial t} = -\varepsilon\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial}{\partial x}(uw) \end{aligned} \tag{10}$$

link solutions of the Korteweg–de Vries equation with those of the modified Korteweg–de Vries equation (9). With $\varepsilon = 1$, the first relation in (10) turns into the Miura transformation (8).

⊙ *References*: G. L. Lamb (1974), N. H. Ibragimov (1985).

8.3. The auto-Bäcklund transformation expressed via the potential functions $\frac{\partial \varphi}{\partial x} = -\frac{1}{2}w$ and $\frac{\partial \widetilde{\varphi}}{\partial x} = -\frac{1}{2}\widetilde{w}$ has the form

$$\begin{aligned} &\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) = k^2 - (\widetilde{\varphi} - \varphi)^2, \\ &\frac{\partial}{\partial t}(\widetilde{\varphi} - \varphi) = 6(\widetilde{\varphi} - \varphi)^2\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) - 6k^2\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) - \frac{\partial^3}{\partial x^3}(\widetilde{\varphi} - \varphi), \end{aligned}$$

where k is an arbitrary constant.

⊙ *Reference*: R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982).

9. Gelfand–Levitan–Marchenko integral equation.

Any rapidly decreasing function $F = F(x, y; t)$ as $x \to +\infty$ that satisfies simultaneously the two linear equations

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0,$$
$$\frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F = 0$$

generates a solution of the Korteweg–de Vries equation in the form

$$w = -2\frac{d}{dx}K(x, x; t), \tag{11}$$

where $K(x, y; t)$ is a solution of the linear *Gelfand–Levitan–Marchenko integral equation*

$$K(x, y; t) + F(x, y; t) + \int_x^\infty K(x, z; t)F(z, y; t)\,dz = 0. \tag{12}$$

Time t appears in this equation as a parameter.

10. Cauchy problem.

Consider the Cauchy problem for the Korteweg–de Vries equation subject to the initial conditions

$$w = f(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty), \tag{13}$$

where the function $f(x)$ is quite rapidly vanishing as $|x| \to \infty$. The solution of the Cauchy problem falls into several stages.

First stage. Initially, a linear eigenvalue problem is solved for the auxiliary ordinary differential equation

$$\psi''_{xx} - \big[f(x) - \lambda\big]\psi = 0. \tag{14}$$

The eigenvalues fall into two types:

$$\begin{aligned} &\lambda_n = -\varkappa_n^2, \quad n = 1, 2, \ldots, N \quad \text{(discrete spectrum)},\\ &\lambda = k^2, \qquad -\infty < k < \infty \qquad \text{(continuous spectrum)}. \end{aligned} \tag{15}$$

Let the $\lambda_n = -\varkappa_n^2$ be discrete eigenvalues and let the $\psi_n = \psi_n(x)$ be the corresponding normalized eigenfunctions, which vanish at infinity and are square summable, so that

$$\int_{-\infty}^{\infty} \psi_n^2(x)\,dx = 1.$$

The leading asymptotic term in the expansion of ψ_n for large x is given by

$$\psi_n \to c_n \exp(-\varkappa_n x) \quad \text{as} \quad x \to \infty. \tag{16}$$

For continuous spectrum, $\lambda = k^2$, the wave function ψ at infinity is determined by a linear combination of the exponentials $\exp(\pm ikx)$ (since $f \to 0$ as $|x| \to \infty$). The conditions

$$\begin{aligned} &\psi \to e^{-ikx} && \text{as} \quad x \to -\infty,\\ &\psi \to a(k)e^{-ikx} + b(k)e^{ikx} && \text{as} \quad x \to \infty, \end{aligned} \tag{17}$$

and equation (14) enable us to uniquely determine the transmission and reflection coefficients $a(k)$ and $b(k)$; note that $|a|^2 - |b|^2 = 1$.

Second stage. At the next stage, one considers the linear Gelfand–Levitan–Marchenko integral equation (12), where

$$F(x, y; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{i[8k^3 t + k(x+y)]}\, dk + \sum_{n=1}^{N} c_n^2 e^{8\varkappa_n^3 t - \varkappa_n (x+y)}. \tag{18}$$

It involves the constants $\varkappa_n$ and c_n and the function $b(k)$ determined at the first state; see (15)–(17). It is apparent from (18) that $F(x, y; t) = F(x + y; t)$.

Third stage. The solution of the integral equation (12), (18) is finally substituted into (11) to give a solution of the Cauchy problem for the Korteweg–de Vries equation with the initial condition (13).

Remark. Solving the Cauchy problem for the given nonlinear equation is reduced to solving two linear problems sequentially. The procedure for solving the Cauchy problem for the Korteweg–de Vries equation is described Section 36.4 in more detail.

⊙ *References*: R. M. Miura (1977), F. Calogero and A. Degasperis (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984).

11. Conservation laws and motion integrals.

11.1. The Korteweg–de Vries equation has infinitely many conservation laws. The simplest conservation laws are expressed as

$$\begin{aligned}
&D_t(w) + D_x\left(w_{xx} - 3w^2\right) = 0,\\
&D_t\left(\tfrac{1}{2}w^2\right) + D_x\left(ww_{xx} - \tfrac{1}{2}w_x^2 - 2w^3\right) = 0,\\
&D_t\left(w_x^2 + 2w^3\right) + D_x\left(2w_x w_{xxx} - w_{xx}^2 + 6w^2 w_{xx} - 12ww_x^2 - 9w^4\right) = 0,\\
&D_t\left(3tw^2 + xw\right) + D_x\left[t\left(6ww_{xx} - 3w_x^2 - 12w^3\right) - w_x + xw_{xx} - 3xw^2\right] = 0,
\end{aligned}$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$.

⊙ *References*: G. B. Whitham (1965), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), N. H. Ibragimov (1994).

11.2. The Korteweg–de Vries equation has infinitely many motion integrals:

$$I_n = \int_{-\infty}^{\infty} P_n(w, w_x, \dots)\, dx = \text{const}, \qquad n = 0,\ 1,\ 2,\ \dots,$$

where P_n is a polynomial in w and its derivatives; w is assumed to decay rapidly as $|x| \to \infty$. In particular, the first four polynomials are as follows:

$$P_0 = w, \quad P_1 = w^2, \quad P_2 = w_x^2 + 2w^3, \quad P_3 = \tfrac{1}{2}\left(w_{xx}^2 - 5w^2 w_{xx} + 5w^4\right).$$

⊙ *References for equation* 13.1.1.1: G. B. Whitham (1965), C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura (1967, 1974), R. M. Miura (1968, 1977), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), P. D. Lax (1968, 1975), R. Hirota (1971, 1972, 2004), V. E. Zakharov and L. D. Faddeev (1971), G. L. Lamb (1974, 1980), C. P. Novikov (1974), K. Sawada and T. Kotera (1974), R. K. Bullough and P. J. Caudrey (1980), M. Wadati and K. Sawada (1980), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), N. H. Ibragimov (1985, 1994), A. C.Newell (1985), P. J. Olver (1986), J. Hietarinta (1987 *a*), B. G. Konopelchenko (1987), G. W. Bluman and S. Kumei (1989), P. A. Clarkson and M. D. Kruskal (1989), M. J. Ablowitz and P. A. Clarkson (1991), R. S. Palais (1997), M. Remoissenet (1999), W.-X. Ma and Y. You (2005).

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} = 0.$

Unnormalized Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x - bC_1C_2 t + C_3, C_1^3 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w = \frac{3ac}{b\cosh^2\left[\frac{1}{2}\sqrt{c}\,(x - act + B)\right]} \quad \text{if } c > 0,$$

$$w = -\frac{3ac}{b\sinh^2\left[\frac{1}{2}\sqrt{c}\,(x - act + B)\right]} \quad \text{if } c > 0,$$

$$w = -\frac{ac}{b}\left\{1 - 3\tanh^2\left[\tfrac{1}{2}\sqrt{-c}\,(x - act + B)\right]\right\} \quad \text{if } c < 0,$$

$$w = -\frac{ac}{b}\left\{1 - 3\coth^2\left[\tfrac{1}{2}\sqrt{-c}\,(x - act + B)\right]\right\} \quad \text{if } c < 0,$$

$$w = -\frac{ac}{b}\left\{1 + 3\tan^2\left[\tfrac{1}{2}\sqrt{c}\,(x - act + B)\right]\right\} \quad \text{if } c > 0,$$

$$w = \frac{3ac}{b\cos^2\left[\frac{1}{2}\sqrt{-c}\,(x - act + B)\right]} \quad \text{if } c < 0,$$

where c and B are arbitrary constants.

3°. Consider the Cauchy problem with the initial condition

$$w = f(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty),$$

where the function $f(x)$ decays quite rapidly as $|x| \to \infty$.

The asymptotic solution as $t \to \infty$ (for sufficiently large x) is the sum of solitons

$$w(x,t) = 2\sum_{n=1}^{N} |\lambda_n| \cosh^{-2}\left[\sqrt{\frac{b|\lambda_n|}{6a}}\left(x - \tfrac{2}{3}b|\lambda_n|t + c_n\right)\right],$$

where the λ_n are discrete eigenvalues of the linear Schrödinger equation

$$\Psi''_{xx} + \frac{b}{6a}\left[\lambda + f(x)\right]\Psi = 0, \qquad \Psi(\pm\infty) = 0.$$

4°. The transformation $w(x,t) = -\dfrac{6a}{b}u(x,\tau)$, $\tau = at$ leads to the Korteweg–de Vries equation in canonical form 13.1.1.1:

$$\frac{\partial u}{\partial \tau} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0.$$

⊙ *References for equation* 13.1.1.2: C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura (1974), G. I. Barenblatt (1989).

13.1.2. Cylindrical, Spherical, and Modified Korteweg–de Vries Equations

1. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-6w\dfrac{\partial w}{\partial x}+\dfrac{1}{2t}w=0.$

Cylindrical Korteweg–de Vries equation. This is a special case of equation 13.1.2.3 for $a=1$, $b=-6$, and $k=\frac{1}{2}$.

The transformation

$$w(x,t)=-\frac{x}{12t}-\frac{1}{2t}u(z,\tau),\quad x=\frac{z}{\tau},\quad t=-\frac{1}{2\tau^2}$$

leads to the Korteweg–de Vries equation in canonical form 13.1.1.1:

$$\frac{\partial u}{\partial \tau}+\frac{\partial^3 u}{\partial z^3}-6u\frac{\partial u}{\partial z}=0.$$

⊙ *References*: R. S. Johnson (1979), F. Calogero and A. Degasperis (1982), G. C. Zhu and H. H. Chen (1986), G. W. Bluman and S. Kumei (1989, p. 313).

2. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-6w\dfrac{\partial w}{\partial x}+\dfrac{1}{t}w=0.$

Spherical Korteweg–de Vries equation. This is a special case of equation 13.1.2.3 for $a=1$, $b=-6$, and $k=1$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1^2 w\big(C_1x+6C_1C_2\ln|t|+C_3,\ C_1^3t\big)+C_2t^{-1},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t)=\frac{C_1-x}{t(C_2+6\ln|t|)}.$$

3°. Self-similar solution:

$$w(x,t)=t^{-2/3}u(z),\qquad z=xt^{-1/3},$$

where the function $u=u(z)$ is determined by the ordinary differential equation

$$au'''_{zzz}+buu'_z-\tfrac{1}{3}zu'_z+\tfrac{1}{3}u=0.$$

3. $\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}+bw\dfrac{\partial w}{\partial x}+\dfrac{k}{t}w=0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=C_1^2 w\left(C_1x-\frac{bC_1C_2}{1-k}t^{1-k}+C_3,\ C_1^3t\right)+C_2t^{-k},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t)=\frac{(1-k)x+C_1}{C_2t^k+bt}.$$

3°. Self-similar solution:

$$w(x,t)=t^{-2/3}u(z),\quad z=xt^{-1/3},$$

where the function $u=u(z)$ is determined by the ordinary differential equation

$$au'''_{zzz}+buu'_z-\tfrac{1}{3}zu'_z+\left(k-\tfrac{2}{3}\right)u=0.$$

⊙ *References for equation* 13.1.2.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 522).

4. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-6w^2\dfrac{\partial w}{\partial x}=0.$

Modified Korteweg–de Vries equation in canonical form (or *mKdV equation* for short). This is a special case of equation 13.1.3.2 with $a=-6$ and $k=2$ and is a special case of equation 13.1.2.6 with $a=-6$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1w(C_1x+C_2,C_1^3t+C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

$$w=\frac{2kC\exp(kx-k^3t)}{1-C^2\exp[2(kx-k^3t)]},$$

where C and k are arbitrary constants.

3°. Traveling-wave solutions:

$$w=\pm\frac{\sqrt{c}}{\sinh\left[\sqrt{c}\,(x-ct+B)\right]} \quad \text{if } c>0,$$

$$w=\pm\sqrt{-c}\,\tanh\left[\sqrt{-c}\,(x-2ct+B)\right] \quad \text{if } c<0,$$

$$w=\pm\sqrt{-c}\,\coth\left[\sqrt{-c}\,(x-2ct+B)\right] \quad \text{if } c<0,$$

$$w=\pm\frac{\sqrt{-c}}{\cos\left[\sqrt{-c}\,(x-ct+B)\right]} \quad \text{if } c<0,$$

$$w=\pm\tfrac{1}{2}\sqrt{-c}\left\{2+\tan^2\left[\tfrac{1}{2}\sqrt{-c}\,(x-ct+B)\right]+\cot^2\left[\tfrac{1}{2}\sqrt{-c}\,(x-ct+B)\right]\right\}^{1/2} \quad \text{if } c<0,$$

where B and c are arbitrary constants.

4°. Self-similar solution (x_0 and t_0 are arbitrary constants):

$$w(x,t)=\left[3(t-t_0)\right]^{-1/3}f(y),\quad y=\left[3(t-t_0)\right]^{-1/3}(x-x_0),$$

where the function $f(y)$ is determined by the third-order ordinary differential equation

$$f'''_{yyy}-yf'_y-f-6f^2f'_y=0.$$

Integrating yields the second Painlevé equation (see Conte and Musette, 2008)

$$f''_{yy}-2f^3-yf=a,$$

where a is an arbitrary constant.

5°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function $u(x,t)$ obtained with the Miura transformation

$$u(x,t) = \frac{\partial w}{\partial x} + w^2 \tag{1}$$

satisfies the Korteweg–de Vries equation 13.1.1.1:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0. \tag{2}$$

In general, the converse is not true: if $u(x,t)$ is a solution of the Korteweg–de Vries equation (2), the function $w(x,t)$ linked to it with the Miura transformation (1) satisfies the nonlinear integro-differential equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w^2\frac{\partial w}{\partial x} = c(t)\exp\left[-2\int w(x,t)\,dx\right].$$

6°. The modified Korteweg–de Vries equation is an integrable equation and can be solved by inverse scattering method.

Solutions of the modified Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6\sigma w^2\frac{\partial w}{\partial x} = 0, \qquad \sigma = \pm 1 \tag{3}$$

may be obtained from solutions of the linear Gelfand–Levitan–Marchenko integral equation. Any function $F = F(x,y;t)$ rapidly decaying as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\begin{aligned} \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} &= 0, \\ \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F &= 0, \end{aligned} \tag{4}$$

generates a solution of equation (3) in the form

$$w = K(x,x;t),$$

where $K(x,y;t)$ is a solution of the linear Gelfand–Levitan–Marchenko integral equation,

$$K(x,y;t) = F(x,y;t) + \frac{\sigma}{4}\int_x^\infty \int_x^\infty K(x,z;t)F(z,u;t)F(u,y;t)\,dz\,du. \tag{5}$$

Time t appears in (5) as a parameter. It follows from the first equation in (4) that $F(x,y;t) = F(x+y;t)$.

7°. Conservation laws:

$$\begin{aligned} &D_t(w) + D_x\left(w_{xx} - 2w^3\right) = 0, \\ &D_t\left(\tfrac{1}{2}w^2\right) + D_x\left(ww_{xx} - \tfrac{1}{2}w_x^2 - \tfrac{3}{2}w^4\right) = 0, \end{aligned}$$

where $D_t = \frac{\partial}{\partial t}$ and $D_x = \frac{\partial}{\partial x}$.

⊙ *References for equation* 13.1.2.4: G. B. Whitham (1965), R. M. Miura (1968), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), M. Wadati (1972, 1973), M. Wadati and K. Sawada (1980), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), J. Hietarinta (1987 *b*), P. G. Drazin and R. S. Johnson (1996), A. M. Wazwaz (2008).

5. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + 6w^2\dfrac{\partial w}{\partial x} = 0.$

Modified Korteweg–de Vries equation (another form). This is a special case of equation 13.1.3.2 with $a = 6$ and $k = 2$ and is a special case of equation 13.1.2.6 with $a = 6$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm C_1 w(C_1 x + C_2, C_1^3 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. One-soliton solution:

$$w(x,t) = a + \frac{k^2}{\sqrt{4a^2 + k^2}\,\cosh z + 2a}, \qquad z = kx - (6a^2k + k^3)t + b,$$

where a, b, and k are arbitrary constants.

3°. Two-soliton solution:

$$w(x,t) = 2\frac{a_1 e^{\theta_1} + a_2 e^{\theta_2} + A a_2 e^{2\theta_1+\theta_2} + A a_1 e^{\theta_1+2\theta_2}}{1 + e^{2\theta_1} + e^{2\theta_2} + 2(1-A)e^{\theta_1+\theta_2} + A^2 e^{2(\theta_1+\theta_2)}},$$

$$\theta_1 = a_1 x - a_1^3 t + b_1, \quad \theta_2 = a_2 x - a_2^3 t + b_2, \quad A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

4°. Rational solutions (algebraic solitons):

$$w(x,t) = a - \frac{4a}{4a^2z^2 + 1}, \qquad z = x - 6a^2 t,$$

$$w(x,t) = a - \frac{12a\left(z^4 + \frac{3}{2}a^{-2}z^2 - \frac{3}{16}a^{-4} - 24tz\right)}{4a^2\left(z^3 + 12t - \frac{3}{4}a^{-2}z\right)^2 + 3\left(z^2 + \frac{1}{4}a^{-2}\right)^2},$$

where a is an arbitrary constant.

5°. Traveling-wave solutions involving hyperbolic functions:

$$w = \pm\tfrac{1}{2}\sqrt{c}\left\{2 - \tanh^2\left[\tfrac{1}{2}\sqrt{c}\,(x - ct + B)\right] - \coth^2\left[\tfrac{1}{2}\sqrt{c}\,(x - ct + B)\right]\right\}^{1/2},$$

where B and $c > 0$ are arbitrary constants.

6°. Traveling-wave solutions involving trigonometric functions:

$$w = \pm\frac{2\sqrt{\lambda/3}\,\cos^2\left[\frac{1}{2}\sqrt{\lambda}\,(x - \lambda t)\right]}{3 - 2\cos^2\left[\frac{1}{2}\sqrt{\lambda}\,(x - \lambda t)\right]}.$$

Setting $w = 0$ for $\frac{1}{2}\sqrt{\lambda}\,|x - \lambda t| \geq \frac{\pi}{2}$, one obtains (formal) compacton-like solutions, which are located in the interval of length $2\pi/\sqrt{\lambda}$.

7°. The modified Korteweg–de Vries equation is an integrable equation and can be solved by inverse scattering method; see Item 6° in equation 13.1.2.4 with $\sigma = -1$.

⊙ *References for equation* 13.1.2.5: H. Ono (1976), M. J. Ablowitz and H. Segur (1981), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), J. Hietarinta (1987 *b*), P. G. Drazin and R. S. Johnson (1996), S. N. Kumar and P. K. Panigrahi (1999), P. Rosenau (2000).

6. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}+aw^2\dfrac{\partial w}{\partial x}=0.$

Unnormalized modified Korteweg–de Vries equation. This is a special case of equation 13.1.3.2 with $k=2$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1 w(C_1x+C_2,C_1^3t+C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w=\pm\left\{\frac{6c}{a\cosh^2\left[\sqrt{c}\,(x-ct+B)\right]}\right\}^{1/2} \quad \text{if } a>0 \text{ and } c>0,$$

$$w=\pm\left\{-\frac{6c}{a\sinh^2\left[\sqrt{c}\,(x-ct+B)\right]}\right\}^{1/2} \quad \text{if } a<0 \text{ and } c>0,$$

$$w=\pm\left\{\frac{6c}{a\cos^2\left[\sqrt{-c}\,(x-ct+B)\right]}\right\}^{1/2} \quad \text{if } a<0 \text{ and } c<0,$$

$$w=\pm\sqrt{\frac{6c}{a}}\,\tanh\left[\sqrt{-c}\,(x-2ct+B)\right] \quad \text{if } a<0 \text{ and } c<0,$$

$$w=\pm\sqrt{\frac{6c}{a}}\,\coth\left[\sqrt{-c}\,(x-2ct+B)\right] \quad \text{if } a<0 \text{ and } c<0,$$

where B and c are arbitrary constants.

3°. Traveling-wave solutions:

$$w=\pm\left\{\frac{3c}{2a}\left(2-\tanh^2\left[\tfrac{1}{2}\sqrt{c}\,(x-ct+B)\right]-\coth^2\left[\tfrac{1}{2}\sqrt{c}\,(x-ct+B)\right]\right)\right\}^{1/2}$$
$$\text{if } a>0 \text{ and } c>0,$$

$$w=\pm\left\{\frac{3c}{2a}\left(2+\tan^2\left[\tfrac{1}{2}\sqrt{-c}\,(x-ct+B)\right]+\cot^2\left[\tfrac{1}{2}\sqrt{-c}\,(x-ct+B)\right]\right)\right\}^{1/2}$$
$$\text{if } a<0 \text{ and } c<0,$$

where B and c are arbitrary constants.

4°. The substitution $w=\pm\sqrt{6/|a|}\,u$ leads to an equation of the form 13.1.2.4 or 13.1.2.5:

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6\operatorname{sign} a\,u^2\frac{\partial u}{\partial x}=0.$$

5°. See Item 6° in equation 13.1.2.4 with $\sigma=-\frac{1}{6}a$.

⊙ *References for equation* 13.1.2.6: J. Hietarinta (1987 *b*), A. M. Wazwaz (2008).

7. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-\dfrac{3}{2}\dfrac{w_{xx}^2}{w_x}=0.$

Schwarzian Korteweg–de Vries equation. It is an integrable equation.

⊙ *References*: I. M. Krichever and S. P. Novikov (1980), J. Weiss (1983), G. Wilson (1988), I. Ya. Dorfman (1989, 1993), F. Guil and M. Mañas (1991).

13.1.3. Generalized Korteweg–de Vries Equation

$$\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + f(w)\frac{\partial w}{\partial x} = 0$$

Preliminary remarks. For $f(w) = bw$, see equations 13.1.1.1 and 13.1.1.2; for $f(w) = bw^2$, see equations 13.1.2.4 and 13.1.2.5.

1°. Equations of this form admit traveling-wave solutions

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the second-order autonomous ordinary differential equation (C is an arbitrary constant)

$$\alpha k^3 w''_{zz} + k\int f(w)\,dw - \lambda w = C.$$

2°. Conservation laws:

$$D_t\big(w\big) + D_x\big[\alpha w_{xx} + F_0(w)\big] = 0,$$
$$D_t\big(\tfrac{1}{2}w^2\big) + D_x\big[\alpha w w_{xx} - \tfrac{1}{2}\alpha w_x^2 + F_1(w)\big] = 0,$$

where

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}, \quad F_0(w) = \int f(w)\,dw, \quad F_1(w) = \int w f(w)\,dw.$$

1. $\dfrac{\partial w}{\partial t} + \big(2aw - 3bw^2\big)\dfrac{\partial w}{\partial x} + \dfrac{\partial^3 w}{\partial x^3} = 0.$

Gardner equation or the *combined KdV-mKdV equation*. It is used to model nonlinear phenomena in plasma and solid-state physics and in quantum field theory.

1°. Traveling-wave solutions:

$$w = \frac{a}{3b}\left\{1 \pm \tanh\left[\frac{a}{3\sqrt{2b}}\left(x - \frac{2a^2}{9b}t + B\right)\right]\right\} \quad \text{if } b > 0;$$
$$w = \frac{a}{3b}\left\{1 \pm \coth\left[\frac{a}{3\sqrt{2b}}\left(x - \frac{2a^2}{9b}t + B\right)\right]\right\} \quad \text{if } b > 0;$$
$$w = \frac{6c}{2a \pm \sqrt{4a^2 - 18bc}\,\cosh[\sqrt{c}\,(x - ct + B)]} \quad \text{if } c > 0 \text{ and } 2a^2 > 9bc;$$
$$w = \frac{6c}{2a \pm \sqrt{18bc - 4a^2}\,\sinh[\sqrt{c}\,(x - ct + B)]} \quad \text{if } c > 0 \text{ and } 2a^2 < 9bc,$$

where B and c are arbitrary constants.

2°. Traveling-wave solution with $b > 0$ and $k = a^2 - 3bc > 0$:

$$w = \frac{a}{3b} + \frac{\sqrt{3k}}{3b}\operatorname{csch}\left[\sqrt{\frac{2k}{3b}}\,(x - ct + B)\right] + \frac{\sqrt{3k}}{3b}\coth\left[\sqrt{\frac{2k}{3b}}\,(x - ct + B)\right].$$

3°. Traveling-wave solution with $k = 9b^2 - 16a^2 > 0$:

$$w = \operatorname{csch}\xi\left(1 + \frac{3b - \sqrt{k}}{4a}\operatorname{csch}\xi\right)^{-1}, \qquad \xi = \frac{\sqrt{9b + 3\sqrt{k}}}{6}\left(x - \frac{3b + \sqrt{k}}{12}\,t + B\right),$$

where B and c are arbitrary constants.

4°. Traveling-wave solutions:

$$w = \operatorname{sech}\xi\left(1 + \frac{3b \pm \sqrt{k}}{4a}\operatorname{sech}\xi\right)^{-1}, \qquad \xi = \pm\frac{\sqrt{-9b \pm 3\sqrt{k}}}{6}\left(x - \frac{-3b \pm \sqrt{k}}{12}\,t + B\right),$$

where $k = 9b^2 + 16a^2$, and B and c are arbitrary constants.

5°. The transformation

$$w = u(z,t) + \frac{a}{3b}, \qquad z = x - \frac{a^2}{3b}t$$

leads to an integrable equation of the form 13.1.2.6:

$$\frac{\partial u}{\partial t} - 3bu^2\frac{\partial u}{\partial z} + \frac{\partial^3 u}{\partial z^3} = 0.$$

⊙ *References for equation* 13.1.3.1: Z. Fu, S. Liu, and Sh. Liu (2004), A. M. Wazwaz (2007, 2008).

2. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + aw^k\dfrac{\partial w}{\partial x} = 0.$

Korteweg–de Vries type equation with a power-law nonlinearity. For $k = 1$ and $k = 2$ see equations 13.1.1.1–13.1.1.2 and 13.1.2.4–13.1.2.5. For $k = 1/2$, the equation describes ion-acoustic waves in a cold-ion plasma with nonisothermal electrons.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{2/k}w(C_1x + C_2, C_1^3t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cosh^2\left[\frac{1}{2}k\sqrt{c}\,(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } c > 0,$$

$$w = \left\{-\frac{c(k+1)(k+2)}{2a\sinh^2\left[\frac{1}{2}k\sqrt{c}\,(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } c > 0,$$

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cos^2\left[\frac{1}{2}k\sqrt{-c}\,(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } c < 0,$$

where B and c are arbitrary constants. The first solution is the soliton solution.

3°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{8a}\left(2 - \tanh^2\left[\tfrac{1}{4}k\sqrt{c}\,(x - ct + B)\right] - \coth^2\left[\tfrac{1}{4}k\sqrt{c}\,(x - ct + B)\right]\right)\right\}^{1/k}$$
$$\text{if } c > 0,$$

$$w = \left\{\frac{c(k+1)(k+2)}{8a}\left(2 + \tan^2\left[\tfrac{1}{4}k\sqrt{-c}\,(x - ct + B)\right] + \cot^2\left[\tfrac{1}{4}k\sqrt{-c}\,(x - ct + B)\right]\right)\right\}^{1/k}$$
$$\text{if } c < 0,$$

where B and c are arbitrary constants.

4°. Self-similar solution:

$$w(x,t) = t^{-\frac{2}{3k}} U(z), \qquad z = xt^{-\frac{1}{3}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$-\frac{2}{3k}U - \frac{1}{3}zU'_z + U'''_{zzz} + aU^k U'_z = 0.$$

5°. Conservation laws:

$$D_t w + D_x\left(w_{xx} + \frac{a}{k+1}w^{k+1}\right) = 0,$$
$$D_t\left(w^2\right) + D_x\left(2ww_{xx} - w_x^2 + \frac{2a}{k+2}w^{k+2}\right) = 0.$$

⊙ *References for equation* 13.1.3.2: H. Schamel (1973), M. J. Ablowitz and H. Segur (1981), P. G. Drazin and R. S. Johnson (1996), A. M. Wazwaz (2008).

3. $\dfrac{\partial w}{\partial t} + \left(aw^n - bw^{2n}\right)\dfrac{\partial w}{\partial x} + \dfrac{\partial^3 w}{\partial x^3} = 0.$

This equation describes the propagation of nonlinear long acoustic-type waves.

1°. Traveling-wave solutions:

$$w = \left\{\frac{a(2n+1)}{2b(n+2)}\left[1 \pm \tanh\left(knx - 4k^3nt + C\right)\right]\right\}^{1/n}; \quad k = \frac{a}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}},$$
$$w = \left\{\frac{a(2n+1)}{2b(n+2)}\left[1 \pm \coth\left(knx - 4k^3nt + C\right)\right]\right\}^{1/n}, \quad k = \frac{a}{2(n+2)}\sqrt{\frac{2n+1}{b(n+1)}},$$

where C is an arbitrary constant (the two first solutions with tanh are the kink solutions).

2°. Traveling-wave solutions:

$$w = \left\{\frac{(n+1)(n+2)c}{a \pm a\sqrt{k}\,\cosh[n\sqrt{c}\,(x-ct)+B]}\right\}^{1/n}, \quad k = 1 - \frac{n^3+5n^2+8n+4}{2n+1}\frac{bc}{a^2} \quad \text{if } c>0;$$
$$w = \left\{\frac{(n+1)(n+2)c}{a \pm a\sqrt{s}\,\sinh[n\sqrt{c}\,(x-ct)+B]}\right\}^{1/n}, \quad s = \frac{n^3+5n^2+8n+4}{2n+1}\frac{bc}{a^2} - 1 \quad \text{if } c>0;$$
$$w = \left\{\frac{(n+1)(n+2)c}{a \pm a\sqrt{k}\,\cos[n\sqrt{-c}\,(x-ct)+B]}\right\}^{1/n}, \quad k = 1 - \frac{n^3+5n^2+8n+4}{2n+1}\frac{bc}{a^2} \quad \text{if } c<0,$$

where B and c are arbitrary constants.

⊙ *References for equation* 13.1.3.3: A. M. Wazwaz (2006, 2008; some other traveling-wave solutions can also be found there).

4. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + ae^w \dfrac{\partial w}{\partial x} = 0.$

Korteweg–de Vries type equation with an exponential nonlinearity.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x + C_2, C_1^3 t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w(z)$ is determined by the second-order autonomous ordinary differential equation

$$w''_{zz} + \lambda w + ae^w = C,$$

and λ and C are arbitrary constants.

3°. Solution:

$$w(x,t) = U(\xi) - \tfrac{2}{3}\ln t, \qquad \xi = xt^{-\frac{1}{3}},$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$U'''_{\xi\xi\xi} + \left(ae^U - \tfrac{1}{3}\xi\right)U'_\xi - \tfrac{2}{3} = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

Korteweg–de Vries type equation with a logarithmic nonlinearity.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = e^{C_1} w(x + bC_1 t + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{C_2 - x}{bt + C_1} + \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w(x,t) = e^{\lambda t}u(z), \qquad z = x + \tfrac{1}{2}b\lambda t^2 + kt,$$

where k and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$au'''_{zzz} + (b\ln u + c - k)u'_z - \lambda u = 0.$$

To $\lambda = 0$ there corresponds a traveling-wave solution.

⊙ *References for equation* 13.1.3.5: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991), V. A. Galaktionov (1999), A. D. Polyanin and V. F. Zaitsev (2004, p. 525).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b\operatorname{arcsinh} w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \sinh\left[\frac{C_2 - x}{bt + C_1} + \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b\operatorname{arccosh} w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \cosh\left[\frac{C_2 - x}{bt + C_1} + \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b\arcsin w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \sin\left[\frac{C_2 - x}{bt + C_1} - \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b\arccos w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \cos\left[\frac{C_2 - x}{bt + C_1} - \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

13.1.4. Equations Reducible to the Korteweg–de Vries Equation

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} = f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x - b\int_{t_0}^{t} (t-\tau) f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 13.1.1.2:

$$\frac{\partial u}{\partial t} + a\frac{\partial^3 u}{\partial x^3} + bu\frac{\partial u}{\partial x} = 0.$$

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + \left[bw^2 + f(t)\right]\dfrac{\partial w}{\partial x} = 0.$

The transformation

$$w = ku(z,t), \qquad z = a^{-1/3}x - a^{-1/3}\int f(t)\,dt, \quad \text{with} \quad k = \sqrt{|6a^{1/3}b^{-1}|},$$

leads to an equation of the form 13.1.2.4 or 13.1.2.5:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial z^3} + 6\,\mathrm{sign}(ab)\,u^2\frac{\partial u}{\partial z} = 0.$$

3. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + a\left(\dfrac{\partial w}{\partial x}\right)^2 = 0.$

Potential Korteweg–de Vries equation (or *potential KdV equation* for short).

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1 x - 2aC_1C_2 t + C_3, C_1^3 t + C_4) + C_2 x - aC_2^2 t + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

$$w = \frac{2kC\exp\left(\frac{1}{3}akx - \frac{1}{27}a^3k^3t\right)}{1 - C\exp\left(\frac{1}{3}akx - \frac{1}{27}a^3k^3t\right)},$$

where C and k are arbitrary constants.

3°. Traveling-wave solutions:

$$w = A + \frac{3\sqrt{c}}{a}\tanh\left[\frac{\sqrt{c}}{2}(x - ct + B)\right] \qquad \text{if } c > 0,$$

$$w = A + \frac{3\sqrt{c}}{a}\coth\left[\frac{\sqrt{c}}{2}(x - ct + B)\right] \qquad \text{if } c > 0,$$

$$w = A + \left\{\sqrt{\frac{3cB^2 - aB\sqrt{c}}{3c}} + B\tanh\left[\frac{\sqrt{c}}{2}(x - ct + C)\right]\right\}^{-1} \qquad \text{if } c > 0,$$

$$w = A + \left\{\sqrt{\frac{3cB^2 - aB\sqrt{c}}{3c}} + B\coth\left[\frac{\sqrt{c}}{2}(x - ct + C)\right]\right\}^{-1} \qquad \text{if } c > 0,$$

$$w = A - \frac{3\sqrt{-c}}{a}\tan\left[\sqrt{-c}\,(x - ct + B)\right] - \frac{3\sqrt{-c}}{a}\sec\left[\sqrt{-c}\,(x - ct + B)\right] \qquad \text{if } c < 0,$$

$$w = A + \frac{3\sqrt{-c}}{a}\cot\left[\sqrt{-c}\,(x - ct + B)\right] + \frac{3\sqrt{-c}}{a}\csc\left[\sqrt{-c}\,(x - ct + B)\right] \qquad \text{if } c < 0,$$

where A, B, C, and c are arbitrary constants.

4°. The Bäcklund transformation

$$\frac{\partial w}{\partial x} = -\frac{3}{a}u, \qquad \frac{\partial w}{\partial t} = \frac{3}{a}\frac{\partial^2 u}{\partial x^2} - \frac{9}{a}u^2 \tag{1}$$

links the equation in question with the Korteweg–de Vries equation 13.1.1.1:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0. \tag{2}$$

Let $u = u(x,t)$ be a solution of equation (2). Then the linear system of first-order equations (1) enables us to find the corresponding solution $w = w(x,t)$ of the original equation.

⊙ *References for equation* 13.1.4.3: N. H. Ibragimov (1985), A. M. Wazwaz (2008).

4. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - a\left(\dfrac{\partial w}{\partial x}\right)^2 = f(t).$

The substitution $w = u(x,t) + \int f(t)\,dt$ leads to an equation of the form 13.1.4.3:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - a\left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

5. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 3\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{2t}w = 0.$

The Bäcklund transformation

$$w_t = -\xi_{xx} + 3\xi^2 - \frac{1}{2t}w,$$
$$w_x = \xi$$

links the equation in question with the cylindrical Korteweg–de Vries equation 13.1.2.1:

$$\frac{\partial \xi}{\partial t} + \frac{\partial^3 \xi}{\partial x^3} - 6\xi\frac{\partial \xi}{\partial x} + \frac{1}{2t}\xi = 0.$$

6. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - a\left(\dfrac{\partial w}{\partial x}\right)^3 = 0.$

Potential modified Korteweg–de Vries equation (or *potential mKdV equation* for short).

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm w(C_1 x + C_2, C_1^3 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. The Bäcklund transformation

$$\frac{\partial w}{\partial x} = bu, \quad \frac{\partial w}{\partial t} = -b\frac{\partial^2 u}{\partial x^2} + 2bu^3, \qquad \text{where} \quad b = \pm\sqrt{2/a}, \tag{1}$$

links the equation in question with the modified Korteweg–de Vries equation 13.1.2.4:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x} = 0. \tag{2}$$

Let $u = u(x,t)$ be a solution of equation (2). Then the linear system of first-order equations (1) enables us to find the corresponding solution $w = w(x,t)$ of the original equation.

⊙ *Reference for equation* 13.1.4.6: N. H. Ibragimov (1985).

7. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-a\left(\dfrac{\partial w}{\partial x}\right)^3=f(t).$

The substitution $w=u(x,t)+\int f(t)\,dt$ leads to an equation of the form 13.1.4.6:

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}-a\left(\frac{\partial u}{\partial x}\right)^3=0.$$

8. $\dfrac{\partial w}{\partial t}=\dfrac{\partial^3 w}{\partial x^3}-\dfrac{1}{8}\left(\dfrac{\partial w}{\partial x}\right)^3-\left(ae^{w}+be^{-w}\right)\dfrac{\partial w}{\partial x}.$

Solutions can be found from the first-order equation

$$\frac{\partial w}{\partial x}-\frac{4}{\sqrt{6}}\left(\sqrt{a}\,e^{w/2}+\sqrt{b}\,e^{-w/2}\right)=4u, \tag{1}$$

where the function $u=u(x,t)$ satisfies

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial x^3}+\left(\lambda-6u^2\right)\frac{\partial u}{\partial x},\quad \lambda=-2\sqrt{ab}. \tag{2}$$

Equation (1) can be treated as an ordinary differential equation for x with parameter t. In the special cases $a=0$ or $b=0$, equation (2) coincides with the modified Korteweg–de Vries equation 13.1.2.4.

Remark. By changing in (2) from t and x to the new independent variables t and $z=x+\lambda t$, one arrives at the modified Korteweg–de Vries equation 13.1.2.4:

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial z^3}-6u^2\frac{\partial u}{\partial z}.$$

⊙ *Reference*: N. H. Ibragimov (1985).

9. $\dfrac{\partial w}{\partial t}=w^3\dfrac{\partial^3 w}{\partial x^3}.$

Harry Dym equation (brief *Dym equation*).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1 w(C_2x+C_3,\,C_1^3C_2^3t+C_4),$$

where $C_1,\ldots,C_4$ are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the following forms:

$$\begin{aligned}
&w=U(z), && z=x+\lambda t && \Longrightarrow && U''_{zz}+\tfrac{1}{2}\lambda U^{-2}=C;\\
&w=t^{-\lambda-1/3}U(z), && z=xt^{\lambda} && \Longrightarrow && U^3U'''_{zzz}-\lambda zU'_z+\left(\lambda+\tfrac{1}{3}\right)U=0;\\
&w=e^{-\lambda t}U(z) && z=xe^{\lambda t} && \Longrightarrow && U^3U'''_{zzz}-\lambda zU'_z+\lambda U=0;\\
&w=t^{-1/3}U(z), && z=x+\lambda\ln|t| && \Longrightarrow && U^3U'''_{zzz}-\lambda U'_z+\tfrac{1}{3}U=0,
\end{aligned}$$

where λ and C are arbitrary constants. The first solution represents a traveling wave and the second one is a self-similar solution.

3°. We now show that the equation in question is connected with the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial y^3} + u\frac{\partial u}{\partial y}. \tag{1}$$

The substitution

$$u = 3\left(\frac{\partial v}{\partial y}\right)^{-1}\frac{\partial^3 v}{\partial y^3} - \frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-2}\left(\frac{\partial^2 v}{\partial y^2}\right)^2$$

brings (1) to the form

$$\frac{\partial v}{\partial t} = \frac{\partial^3 v}{\partial y^3} - \frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-1}\left(\frac{\partial^2 v}{\partial y^2}\right)^2. \tag{2}$$

Differentiating (2) with respect to y yields

$$\frac{\partial^2 v}{\partial y\partial t} = \frac{\partial^4 v}{\partial y^4} - 3\left(\frac{\partial v}{\partial y}\right)^{-1}\frac{\partial^2 v}{\partial y^2}\frac{\partial^3 v}{\partial y^3} + \frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-2}\left(\frac{\partial^2 v}{\partial y^2}\right)^3.$$

The transformation $x = v$, $w = \frac{\partial v}{\partial y}$ leads to the original equation.

⊙ *References for equation* 13.1.4.9: M. Kruskal (1975), F. Calogero and A. Degasperis (1982, p. 53), M. Leo, R. A. Leo, G. Soliani, L. Solombrino, and L. Martina (1983), N. H. Ibragimov (1985), P. A. Clarkson, A. S. Fokas, and M. J. Ablowitz (1989).

13.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + F\left(w, \dfrac{\partial w}{\partial x}\right) = 0$

▷ *For $F(w,u) = f(w)u$, see Subsections 13.1.1 and 13.1.3. For $F(w,u) = bu^2$ and $F(w,u) = bu^3$, see equations 13.1.4.3 and 13.1.4.5, respectively. Equations of this form admit traveling-wave solutions, $w = w(kx + \lambda t)$.*

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} + cw = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + bC_1e^{-ct} + C_2,\ t + C_3\big) + cC_1e^{-ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = U(z) + C_1e^{-ct}, \qquad z = x + bC_1e^{-ct} + C_2t,$$

where C_1 and C_2 are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + (bU + C_2)U'_z + cU = 0.$$

To the special case $C_1 = 0$ there corresponds a traveling-wave solution.

3°. There is a degenerate solution of the form $w = (x + C)\varphi(t)$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^3 w}{\partial x^3} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw^2.$

1°. Generalized separable solutions for $ab < 0$:

$$w(x,t) = \frac{C_2}{(t+C_1)^2}\exp(\lambda x + \lambda^3 t) - \frac{1}{b(t+C_1)}, \qquad \lambda = \pm\sqrt{-\frac{b}{a}},$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution for $ab < 0$:

$$w(x,t) = \frac{1}{2}\left(\frac{1}{bt+C_1} - \frac{1}{bt+C_2}\right)\cosh(\lambda x + \lambda^3 t + C_3) - \frac{1}{2}\left(\frac{1}{bt+C_1} + \frac{1}{bt+C_2}\right), \qquad \lambda = \sqrt{-\frac{b}{a}},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Generalized separable solution for $ab > 0$:

$$w(x,t) = \frac{1}{2}\left(\frac{1}{bt+C_1} - \frac{1}{bt+C_2}\right)\sin(\lambda x - \lambda^3 t + C_3) - \frac{1}{2}\left(\frac{1}{bt+C_1} + \frac{1}{bt+C_2}\right), \qquad \lambda = \sqrt{\frac{b}{a}},$$

where C_1, C_2, and C_3 are arbitrary constants.

Remark. In Items 2° and 3°, one can set $C_1 = \infty$ (or $C_2 = \infty$). In particular, using the formula given in Item 3° with $a = b = 1$, one can obtain the solution

$$w = \frac{1}{t_0 - t}\cos^2[\tfrac{1}{2}(x-t)],$$

where $t_0 > 0$ is the finite blow-up time.

⊙ *References for equation* 13.1.5.2: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin and V. F. Zaitsev (2002, pp. 289–290; 2004, p. 529), V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 164).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Traveling-wave solutions with $bc < 0$:

$$w = C\exp\Big\{|c/b|^{1/2}x + \big(s + a|c/b|^{3/2}\big)t\Big\},$$
$$w = C\exp\Big\{-|c/b|^{1/2}x + \big(s - a|c/b|^{3/2}\big)t\Big\},$$

where C is an arbitrary constant.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\big(x\sqrt{|c/b|}\,\big) + \psi(t)\exp\big(-x\sqrt{|c/b|}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = \big(s + a|c/b|^{3/2}\big)\varphi + 4c\varphi^2\psi,$$
$$\psi'_t = \big(s - a|c/b|^{3/2}\big)\psi + 4c\varphi\psi^2.$$

3°. Generalized separable solution with $bc > 0$:

$$w = \varphi(t)\cos\big(x\sqrt{c/b}\,\big) + \psi(t)\sin\big(x\sqrt{c/b}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = \big[s + a(c/b)^{3/2}\big]\varphi + c(\varphi^2 + \psi^2)\varphi,$$
$$\psi'_t = \big[s - a(c/b)^{3/2}\big]\psi + c(\varphi^2 + \psi^2)\psi.$$

4. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\left(\dfrac{\partial w}{\partial x}\right)^k = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{3-k} w(C_1^k x + C_2, C_1^{3k} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w = x(kbt)^{-1/k} + Ct^{-1/k}.$$

3°. Self-similar solution:

$$w = t^{\frac{k-3}{3k}} U(z), \qquad z = xt^{-\frac{1}{3}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$\frac{k-3}{3k}U - \frac{1}{3}zU'_z + bU(U'_z)^k + aU'''_{zzz} = 0.$$

5. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + \left(b_1 w^{\frac{2-k}{2}} + b_2 w^{\frac{1-k}{2}}\right)\left(\dfrac{\partial w}{\partial x}\right)^k = 0.$

Degenerate solution quadratic in x:

$$w(x,t) = \left[\frac{x}{2}\left(\frac{kb_1 t}{2}\right)^{-1/k} + Ct^{-1/k} - \frac{b_2}{b_1}\right]^2.$$

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

6. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + (b_1 \ln w + b_2) w^{1-k}\left(\dfrac{\partial w}{\partial x}\right)^k = 0.$

Generalized traveling-wave solutions:

$$w = \begin{cases} \exp\left[-a\dfrac{k(kb_1)^{-3/k}}{k-2}t^{(k-3)/k} - \dfrac{b_2}{b_1} + Ct^{-1/k} + (kb_1 t)^{-1/k}x\right] & \text{if } k \neq 2, \\ \exp\left[-a(2b_1)^{-3/2}t^{-1/2}\ln t - \dfrac{b_2}{b_1} + Ct^{-1/2} + (2b_1 t)^{-1/2}x\right] & \text{if } k = 2, \end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

7. $\dfrac{\partial w}{\partial t} - \dfrac{\partial^3 w}{\partial x^3} + (b_1 \arcsin w + b_2)(1-w^2)^{\frac{1-k}{2}}\left(\dfrac{\partial w}{\partial x}\right)^k = 0.$

Generalized traveling-wave solutions:

$$w = \begin{cases} \sin\left[-\dfrac{k(kb_1)^{-3/k}}{k-2}t^{(k-3)/k} - \dfrac{b_2}{b_1} + Ct^{-1/k} + (kb_1 t)^{-1/k}x\right] & \text{if } k \neq 2, \\ \sin\left[-(2b_1)^{-3/2}t^{-1/2}\ln t - \dfrac{b_2}{b_1} + Ct^{-1/2} + (2b_1 t)^{-1/2}x\right] & \text{if } k = 2, \end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

8. $\dfrac{\partial w}{\partial t} - \dfrac{\partial^3 w}{\partial x^3} + (b_1 \operatorname{arcsinh} w + b_2)(1 + w^2)^{\frac{1-k}{2}} \left(\dfrac{\partial w}{\partial x}\right)^k = 0.$

Generalized traveling-wave solutions:

$$w = \begin{cases} \sinh\left[\dfrac{k(kb_1)^{-3/k}}{k-2} t^{(k-3)/k} - \dfrac{b_2}{b_1} + Ct^{-1/k} + (kb_1 t)^{-1/k} x\right] & \text{if } k \neq 2, \\ \sinh\left[(2b_1)^{-3/2} t^{-1/2} \ln t - \dfrac{b_2}{b_1} + Ct^{-1/2} + (2b_1 t)^{-1/2} x\right] & \text{if } k = 2, \end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

13.1.6. Equations of the Form $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + F\left(x, t, w, \dfrac{\partial w}{\partial x}\right) = 0$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (bx + c)\dfrac{\partial w}{\partial x} + f(w).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-bt},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$aw'''_{zzz} + (bz + c)w'_z + f(w) = 0.$$

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + f(t)w\dfrac{\partial w}{\partial x} + g(t)w = 0.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + C_1\psi(t) + C_2,\ t\big) - C_1\varphi(t),$$

where

$$\varphi(t) = \exp\left[-\int g(t)\,dt\right], \quad \psi(t) = \int f(t)\varphi(t)\,dt,$$

is also a solution of the equation (C_1 and C_2 are arbitrary constants).

2°. There is a degenerate solution of the form $w = \beta(t)(x + C)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\ln w + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.*

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\,\mathrm{arcsinh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sinh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\,\mathrm{arccosh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cosh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\,\mathrm{arcsin}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) - a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

* The constant a in equations 13.1.6.3 to 13.1.6.7 and their solutions can be replaced by an arbitrary function of time, $a = a(t)$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\arccos(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) - a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi_k = \varphi_k(t)$ satisfy an appropriate system of ordinary differential equations.

2°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct}\int e^{-ct} f(t)\,dt + \theta(z), \qquad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta'''_{zzz} + b\big(\theta'_z\big)^2 - \lambda\theta'_z + c\theta = 0.$$

3°. The substitution

$$w = U(x,t) + e^{ct}\int e^{-ct} f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^3 U}{\partial x^3} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

13.1.7. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^3 w}{\partial x^3}\right)$

1. $\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} + a\dfrac{\partial^3 w}{\partial x^3} = b\dfrac{\partial^2 w}{\partial x^2}.$

Burgers–Korteweg–de Vries equation (also known as *Korteweg–de Vries–Burgers equation*). It describes nonlinear waves in dispersive-dissipative media with instabilities, waves arising in thin films flowing down an inclined surface, changes of the concentration of substances in chemical reactions, etc.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - C_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \frac{x + C_1}{t + C_2}.$$

3°. Traveling-wave solutions:

$$w(x,t) = C_1 - \frac{12b^2}{25a(1 + C_2 e^y)^2}, \qquad y = -\frac{b}{5a}x + \left(\frac{b}{5a}C_1 - \frac{6b^3}{125a^2}\right)t;$$

$$w(x,t) = C_1 - \frac{12b^2}{25a(1 - C_2 e^y)^2}, \qquad y = -\frac{b}{5a}x + \left(\frac{b}{5a}C_1 - \frac{6b^3}{125a^2}\right)t;$$

$$w(x,t) = C_1 + \frac{12b^2}{25a}\frac{1 + 2C_2 e^z}{(1 + C_2 e^z)^2}, \qquad z = \frac{b}{5a}x - \left(\frac{b}{5a}C_1 + \frac{6b^3}{125a^2}\right)t,$$

where C_1 and C_2 are arbitrary constants.

4°. Traveling-wave solutions:

$$w(x,t) = C_1 \mp \frac{12b^2}{25a}\xi^2\varphi(\xi), \qquad \xi = C_2 \exp\left[\frac{b}{5a}x + \left(\frac{6b^3}{125a^2} - \frac{b}{5a}C_1\right)t\right],$$

where the function $\varphi(\xi)$ is defined implicitly by

$$\xi = \int \frac{d\varphi}{\sqrt{\pm(4\varphi^3 - 1)}} - C_3,$$

and C_1, C_2, and C_3 are arbitrary constants. For the upper sign, the inversion of this relation leads to the classical Weierstrass elliptic function, $\varphi(\xi) = \wp(\xi + C_3,\, 0,\, 1)$.

5°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x - C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the second-order ordinary differential equation (C_3 is an arbitrary constant)

$$aU''_{\zeta\zeta} - bU'_\zeta + \tfrac{1}{2}U^2 + C_2 U = -2C_1\zeta + C_3.$$

To the special case $C_1 = 0$ there corresponds a traveling-wave solution.

⊙ *References for equation* 13.1.7.1: Y. Kuramoto and T. Tsuzuki (1976), B. J. Cohen, J. A. Krommes, W. M. Tang, and M. N. Rosenbluth (1976), V. Ya. Shkadov (1977), J. Topper and T. Kawahara (1978), G. I. Sivashinsky (1983), J. L. Bona and M. E. Schonbek (1985), N. A. Kudryashov (1990 *a*), R. F. Bikbaev (1995), Y. N. Zayko and I. S. Nefedov (2001), Z. Feng (2002, 20005, 2007), A. D. Polyanin and V. F. Zaitsev (2004, pp. 532–533), L. Zhang (2011).

2. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-aw^2\dfrac{\partial w}{\partial x}=b\dfrac{\partial^2 w}{\partial x^2}.$

Modified Burgers–Korteweg–de Vries equation (also known as *modified Korteweg–de Vries–Burgers equation*).

Traveling-wave solutions with $a>0$:

$$w=\pm\frac{b}{\sqrt{6a}}\left[1+\tanh\left(\frac{b}{6}x+\frac{b^3}{27}t+C\right)\right],$$

where C is an arbitrary constant.

⊙ *References*: A. Jeffrey and S. Xu (1989), D. Jacobs, B. McKinney, and M. Shearer (1995), Z. Feng (2003), L. Zhang (2011).

3. $\dfrac{\partial w}{\partial t}-\dfrac{\partial^3 w}{\partial x^3}+aw^2\dfrac{\partial w}{\partial x}=b\dfrac{\partial^2 w}{\partial x^2}.$

A variant of the modified Burgers–Korteweg–de Vries equation.

Traveling-wave solutions with $a>0$:

$$w=\pm\frac{b}{\sqrt{6a}}\left[1-\tanh\left(\frac{b}{6}x-\frac{b^3}{27}t+C\right)\right],$$

where C is an arbitrary constant.

⊙ *Reference*: L. Zhang (2011).

4. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^3 w}{\partial x^3}+w\dfrac{\partial^2 w}{\partial x^2}-\left(\dfrac{\partial w}{\partial x}\right)^2+bw+c.$

1°. Generalized separable solution:

$$w=\varphi_1(t)+\varphi_2(t)x+\varphi_3(t)\exp[\lambda(t)x],$$

where the functions $\varphi_n=\varphi_n(t)$ and $\lambda=\lambda(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= -\varphi_2^2+b\varphi_1+c,\\
\varphi_2' &= b\varphi_2,\\
\varphi_3' &= (\lambda^2\varphi_1-2\lambda\varphi_2+a\lambda^3+b)\varphi_3,\\
\lambda' &= \varphi_2\lambda^2.
\end{aligned}$$

Integrating the second, fourth, and first equation successively, we obtain

$$\text{for } b\neq 0:\quad \varphi_2=C_1e^{bt},\quad \lambda=\frac{b}{C_2-C_1e^{bt}},\quad \varphi_1=C_3e^{bt}-\frac{C_1^2}{b}e^{2bt}-\frac{c}{b};$$

$$\text{for } b=0:\quad \varphi_2=C_1,\qquad \lambda=\frac{1}{C_2-C_1t},\qquad \varphi_1=C_3+(c-C_1^2)t,$$

where C_1, C_2, and C_3 are arbitrary constants. Substituting these expressions into the third equation yields a separable equation, which is to integrate.

2°. Generalized separable solution:

$$w = \varphi(t) + \psi_1(t)e^{\lambda x} + \psi_2(t)e^{-\lambda x},$$

where λ is an arbitrary constant and the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi' &= 4\lambda^2\psi_1\psi_2 + b\varphi + c,\\
\psi_1' &= (\lambda^2\varphi + a\lambda^3 + b)\psi_1,\\
\psi_2' &= (\lambda^2\varphi - a\lambda^3 + b)\psi_2.
\end{aligned}$$

3°. Generalized separable solution:

$$w = \varphi(t) + \psi_1(t)\cos(\lambda x) + \psi_2(t)\sin(\lambda x),$$

where λ is an arbitrary constant and the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi' &= -\lambda^2(\psi_1^2 + \psi_2^2) + b\varphi + c,\\
\psi_1' &= -a\lambda^3\psi_2 - \lambda^2\varphi\psi_1 + b\psi_1,\\
\psi_2' &= a\lambda^3\psi_1 - \lambda^2\varphi\psi_2 + b\psi_2.
\end{aligned}$$

5. $\dfrac{\partial w}{\partial t} + 3w^2\dfrac{\partial w}{\partial x} + 3\left(\dfrac{\partial w}{\partial x}\right)^2 + 3w\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^3 w}{\partial x^3} = 0.$

Sharma–Tasso–Olver equation (an integrable equation of the Burgers hierarchy). This is a special case of equation 15.1.11.1 with $a = 1$ and $n = 2$.

1°. Traveling-wave solution:

$$w(x,t) = \tfrac{1}{2}\lambda\big[1 + \tanh\big(\tfrac{1}{2}\lambda x - \tfrac{1}{2}\lambda^3 t + C\big)\big],$$

where C and λ are arbitrary constants.

2°. Solution:

$$w(x,t) = \frac{C_1k_1\exp(k_1x - k_1^3t) + C_2k_2\exp(k_2x - k_2^3t)}{C_1\exp(k_1x - k_1^3t) + C_2\exp(k_2x - k_2^3t)},$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

3°. Rational solutions:

$$\begin{aligned}
w(x,t) &= \frac{3x^2}{x^3 - 6t},\\
w(x,t) &= \frac{4(x^3 - 6t)}{x(x^3 - 24t)},\\
w(x,t) &= \frac{5(x^3 - 24t)}{x(x^3 - 60t)},\\
w(x,t) &= \frac{6x^2(x^3 - 60t)}{x^6 - 120x^3t + 360t^2},\\
w(x,t) &= \frac{7(x^6 - 120x^3t + 360t^2)}{x(x^6 - 210x^3t + 2520t^2)}.
\end{aligned}$$

4°. Solutions of the original equation are expressed in terms of solutions $u = u(x,t)$ of the third-order linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$$

by the formula (the *Hopf–Cole transformation*)

$$w(x,t) = \frac{1}{u}\frac{\partial u}{\partial x}.$$

⊙ *References for equation* 13.1.7.5: B. Erbaş and E. Yusufoğlu (2009), N. A. Kudryashov (2010 *d*), L. Zhang (2011).

6. $\dfrac{\partial w}{\partial t} = w\dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + aw + b.$

1°. Generalized separable solution:

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)\cosh[\lambda(t)x] + \varphi_3(t)\sinh[\lambda(t)x],$$

where the functions $\varphi_n = \varphi_n(t)$ and $\lambda = \lambda(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= a\varphi_1 + b, \quad \varphi_2' = a\varphi_2,\\
\varphi_3' &= \lambda^3\varphi_1\varphi_4 - \lambda^2\varphi_2\varphi_3 + a\varphi_3,\\
\varphi_4' &= \lambda^3\varphi_1\varphi_3 - \lambda^2\varphi_2\varphi_4 + a\varphi_4,\\
\lambda' &= \lambda^3\varphi_2.
\end{aligned}$$

2°. Generalized separable solution:

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)\cos[\lambda(t)x] + \varphi_3(t)\sin[\lambda(t)x],$$

where the functions $\varphi_n = \varphi_n(t)$ and $\lambda = \lambda(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= a\varphi_1 + b, \quad \varphi_2' = a\varphi_2,\\
\varphi_3' &= -\lambda^3\varphi_1\varphi_4 + \lambda^2\varphi_2\varphi_3 + a\varphi_3,\\
\varphi_4' &= \lambda^3\varphi_1\varphi_3 + \lambda^2\varphi_2\varphi_4 + a\varphi_4,\\
\lambda' &= -\lambda^3\varphi_2.
\end{aligned}$$

⊙ *References for equation* 13.1.7.6: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 220).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + c.$

1°. Generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi_4' &= 144b\varphi_4^2,\\
\varphi_3' &= 144b\varphi_3\varphi_4,\\
\varphi_2' &= 12b(3\varphi_3^2 + 4\varphi_2\varphi_4),\\
\varphi_1' &= 24(a\varphi_4 + b\varphi_2\varphi_3),\\
\varphi_0' &= 6a\varphi_3 + 4b\varphi_2^2 + c.
\end{aligned}$$

2°. Solution:

$$w(x,t) = C_1x^2 + C_2x + C_3t + \int u(z)\,dz, \quad z = kx + \lambda t,$$

where C_1, C_2, C_3, k, and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$\lambda u + C_3 = ak^3u''_{zz} + b(k^2u'_z + 2C_1)^2 + c,$$

whose order can be reduced with the standard substitution $F(u) = u'_z$.

To the special case $C_1 = C_2 = C_3 = 0$ there corresponds a traveling-wave solution. For $\lambda = 0$, we get an additive separable solution.

3°. The substitution

$$w(x,t) = V(x,t) + ct$$

leads to the simpler equation

$$\frac{\partial V}{\partial t} = a\frac{\partial^3 V}{\partial x^3} + b\left(\frac{\partial^2 V}{\partial x^2}\right)^2,$$

which admits a self-similar solution of the form $V = t^{1/3}\theta(\xi)$, where $\xi = xt^{-1/3}$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3}{\partial x^3}(w^2) + b\dfrac{\partial}{\partial x}(w^2)$.

This equation describes the effect of nonlinear dispersion in the pattern formation in liquid drops.

1°. Traveling-wave solutions:

$$w = -\frac{2\lambda}{3b} \pm \frac{2\lambda}{3b}\sin\left[\frac{1}{2}\sqrt{\frac{b}{a}}\,(x - \lambda t) + C\right],$$

where C and λ are arbitrary constants.

Special case. Traveling-wave solution with $a = b = 1$:

$$w = -\tfrac{4}{3}\lambda\cos^2[\tfrac{1}{4}(x - \lambda t) + C],$$

where C and λ are arbitrary constants. Using this solution, one can construct a moving compactly supported soliton-type solution (called a *compacton*):

$$w = \begin{cases} -\frac{4}{3}\lambda\cos^2[\frac{1}{4}(x-\lambda t)] & \text{if } |x-\lambda t| \le 2\pi, \\ 0 & \text{if } |x-\lambda t| > 2\pi. \end{cases}$$

⊙ *References*: P. Rosenau and J. M. Hyman (1993), L. Zhang (2011).

2°. Multiplicative separable solution:

$$w(x,t) = \frac{\varphi(x)}{t + C},$$

where C is an arbitrary constant and the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation

$$a(\varphi^2)'''_{xxx} + b(\varphi^2)'_x + \varphi = 0.$$

3°. See also solutions of equation 13.1.7.16 with $n = 2$.

9. $\dfrac{\partial w}{\partial t} = aw^{3/2}\dfrac{\partial^3 w}{\partial x^3}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = B^2 w(Ax + C_1, A^3B^3t + C_2),$$

where A, B, C_1, and C_2 are arbitrary constants, is also a solution of the equation.

2°. The substitution $w = U^{-2}$ leads to an equation of the form 13.1.7.10:

$$\frac{\partial U}{\partial t} = a\frac{\partial^2}{\partial x^2}\left(U^{-3}\frac{\partial U}{\partial x}\right).$$

⊙ *References for equation* 13.1.7.9: N. Euler, M. L. Gandarias, M. Euler, and O. Lindblom (2001), A. D. Polyanin and V. F. Zaitsev (2004, pp. 534–535).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2}{\partial x^2}\left(w^{-3}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = Bw(Ax + C_1, A^3B^{-3}t + C_2),$$

where A, B, C_1, and C_2 are arbitrary constants, is also a solution of the equation.

2°. Consider the third-order linear equation

$$x_t = ax_{zzz}.$$

Let us divide the equation by x_z, then differentiate with respect to z, and change to the von Mises variables:

$$t,\ z,\ x = x(t,z) \quad \Longrightarrow \quad t,\ x,\ \eta = \eta(t,x), \quad \text{where} \quad \eta = \frac{\partial x}{\partial z}.$$

The derivatives are transformed as follows:

$$\frac{\partial}{\partial z} = \eta\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + x_t\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial z}\left(\frac{x_t}{x_z}\right) \equiv \frac{x_z x_{tz} - x_t x_{zz}}{x_z^2} = \frac{\eta_t}{\eta}.$$

As a result, we obtain the equation

$$\eta_t = a\eta^2(\eta\eta_x)_{xx},$$

which is reduced to the original equation with the substitution $\eta = 1/w$.

3°. See also Table 13.1 (first equation in the third row) and Examples 26.38 and 26.46.

⊙ *References for equation* 13.1.7.10: N. Euler, M. L. Gandarias, M. Euler, and O. Lindblom (2001), A. D. Polyanin and V. F. Zaitsev (2004, pp. 534–535), A. D. Polyanin (2010).

11. $\dfrac{\partial w}{\partial t} + c\dfrac{\partial}{\partial x}(xw) + sw = a\dfrac{\partial^2}{\partial x^2}\left(w^{-3}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right).$

Here $a = a(t)$, $b = b(t)$, $c = c(t)$, and $s = s(t)$ are arbitrary functions.

The equation can be reduced to the linear third-order equation

$$x_t = ax_{zzz} + bx_{zz} + szx_z + cx.$$

For details, see Example 26.37.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2}{\partial x^2}\left(w^{-3/2}\dfrac{\partial w}{\partial x}\right).$

Modified Harry Dym equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^2 C_2^{-2} w(C_1 x + C_3, C_2^3 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. The transformation $u = w^{-1/2}$, $\tau = at$ leads to an equation of the form 13.1.4.9:

$$\frac{\partial u}{\partial \tau} = u^3 \frac{\partial^3 u}{\partial x^3}.$$

3°. The equation is invariant under the transformation

$$d\bar{x} = w\,dx + [a(w^{-3/2}w_x)_x]\,dt, \quad d\bar{t} = dt, \quad \bar{w} = 1/w.$$

⊙ *References for equation* 13.1.7.12: M. Kruskal (1975), F. Calogero and A. Degasperis (1982), N. H. Ibragimov (1985).

13. $\dfrac{\partial w}{\partial t} = aw^{3/2}\dfrac{\partial}{\partial x}\left(x^3\dfrac{\partial^2 w}{\partial x^2}\right).$

The equation is reduced to the Harry Dym equation (for details, see Table 26.1, row 13).

14. $\dfrac{\partial w}{\partial t} = aw^{3/2}\dfrac{\partial}{\partial x}\left(x^{3/2}\dfrac{\partial^2 w}{\partial x^2}\right).$

The equation is reduced to the linear equation (for details, see Table 26.1, row 14).

15. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2}{\partial x^2}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + bw + c.$

Multiplicative separable solution:

$$w = C_1 \exp\left(-\frac{c}{6a}x^3 + C_2 x^2 + C_3 x + bt\right),$$

where C_1, C_2, and C_3 are arbitrary constants.

16. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial}{\partial x}(w^n) + \dfrac{\partial^3}{\partial x^3}(w^n) = 0.$

It is the $K(n,n)$ *equation*. This is a special case of equation 13.1.7.20 with $f(w) = -nw^{n-1}$ and $g(w) = -anw^{n-1}$.

1°. Traveling-wave solutions with $n \neq 0,\ 1$:

$$w = \left\{\frac{2cn}{a(n+1)}\sin^2\left[\frac{n-1}{2n}\sqrt{a}\,(x-ct+B)\right]\right\}^{\frac{1}{n-1}} \quad \text{if } a > 0;$$

$$w = \left\{-\frac{2cn}{a(n+1)}\sinh^2\left[\frac{n-1}{2n}\sqrt{-a}\,(x-ct+B)\right]\right\}^{\frac{1}{n-1}} \quad \text{if } a < 0;$$

$$w = \left\{\frac{2cn}{a(n+1)}\cosh^2\left[\frac{n-1}{2n}\sqrt{-a}\,(x-ct+B)\right]\right\}^{\frac{1}{n-1}} \quad \text{if } a < 0,$$

where B and c are arbitrary constants.

2°. Multiplicative separable solution:

$$w(x,t) = (t+C)^{\frac{1}{1-n}}\varphi(x),$$

where C is an arbitrary constant and the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{1-n}\varphi + a(\varphi^n)'_x + (\varphi^n)'''_{xxx} = 0.$$

⊙ *References for equation* 13.1.7.16: P. Rosenau and J. M. Hyman (1993), A. M. Wazwaz (2002 *a,b*; 2008).

17. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial}{\partial x}(w^{n+1}) + \dfrac{\partial}{\partial x}\left[w\dfrac{\partial^2}{\partial x^2}(w^n)\right] = 0.$

It is the *modified* $K(n,n)$ *equation*.

1°. Traveling-wave solutions with $n \neq 0,\ 1$:

$$w = \left\{\frac{2c}{a}\sin^2\left[\frac{\sqrt{a}}{2}(x-ct+B)\right]\right\}^{1/n} \quad \text{if } a>0;$$

$$w = \left\{\frac{2c}{a}\sinh^2\left[\frac{\sqrt{-a}}{2}(x-ct+B)\right]\right\}^{1/n} \quad \text{if } a<0;$$

$$w = -\left\{\frac{2c}{a}\cosh^2\left[\frac{\sqrt{-a}}{2}(x-ct+B)\right]\right\}^{1/n} \quad \text{if } a<0,$$

where B and c are arbitrary constants.

2°. Multiplicative separable solution:

$$w(x,t) = (t+C)^{-1/n}\varphi(x),$$

where C is an arbitrary constant and the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation

$$-\frac{1}{n}\varphi + a(\varphi^{n+1})'_x + [\varphi(\varphi^n)''_{xx}]'_x = 0.$$

⊙ *References for equation* 13.1.7.17: P. Rosenau (1994, 2000), A. M. Wazwaz (2008).

18. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2}{\partial x^2}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{a}{f(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{1}{6}bx^3 + C_1x^2 + C_2x + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 534).

19. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^3}{\partial x^3}\big[f(w)\big] + \dfrac{af(w)+b}{f'(w)} + c[af(w)+b].$

1°. Functional separable solution in implicit form with $a \neq 0$:

$$f(w) = \frac{1}{a}[\rho(x)e^{at} - b],$$

where

$$\rho(x) = \begin{cases} C_1 + C_2x + C_3x^2 & \text{if } c = 0, \\ C_1e^{-kx} + e^{kx/2}\left(C_2\cos\dfrac{kx\sqrt{3}}{2} + C_3\sin\dfrac{kx\sqrt{3}}{2}\right), \quad k = (ac)^{1/3} & \text{if } c \neq 0, \end{cases}$$

and C_1, C_2, and C_3 are arbitrary constants.

2°. Functional separable solution in implicit form with $a = 0$:

$$f(w) = bt + C_1 + C_2x + C_3x^2 - \tfrac{1}{6}bcx^3.$$

20. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2}{\partial x^2}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x}.$

1°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$k^3[f(w)w_z']_z' + kG(w) - \lambda w + C = 0, \qquad G(w) = \int g(w)\,dw.$$

The substitution $U(w) = f(w)w_z'$ leads to a first-order separable equation.

2°. The transformation

$$dz = w\,dx + \big\{[f(w)w_x]_x + G(w)\big\}\,dt, \quad d\tau = dt, \quad u = 1/w \qquad \left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt\right) \quad (1)$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2}{\partial z^2}\left[\Phi(u)\frac{\partial u}{\partial z}\right] + \Psi(u)\frac{\partial u}{\partial z}, \qquad (2)$$

where

$$\Phi(u) = \frac{1}{u^3}f\left(\frac{1}{u}\right), \quad \Psi(u) = \frac{1}{u}g\left(\frac{1}{u}\right) - G\left(\frac{1}{u}\right), \quad G(w) = \int g(w)\,dw.$$

The inverse of transformation (1) is written out as

$$dx = \frac{1}{w}\,dz - \frac{1}{w}\big\{w[wf(w)w_z]_z + G(w)\big\}\,d\tau, \quad dt = d\tau, \quad w = 1/u.$$

TABLE 13.1
Some integrable nonlinear third-order equations of the form 13.1.7.20

Type of generated equation	Form of generated equation	Solvable equation of the form (3)
Linear equation	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3}$	$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right)$
Korteweg–de Vries equation 13.1.1.2	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw\frac{\partial w}{\partial x}$	$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right) - \frac{b}{2u^2}\frac{\partial u}{\partial z}$
Modified Korteweg–de Vries equation 13.1.2.4	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw^2\frac{\partial w}{\partial x}$	$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right) - \frac{2b}{3u^3}\frac{\partial u}{\partial z}$

Table 13.1 lists some solvable equations of the form (2) generated by known solvable third-order equations.

Equation (2) can be reduced to the form (see equation 13.1.7.21)

$$\frac{\partial v}{\partial \tau} = \varphi(v)\frac{\partial^3 v}{\partial z^3} + \psi(v)\frac{\partial v}{\partial z}, \tag{3}$$

where

$$v = \int w f(w)\,dw, \quad \varphi(v) = w^3 f(w), \quad \psi(v) = wg(w) - G(w).$$

3°. Conservation laws:

$$D_t(w) + D_x\left\{-[f(w)w_x]_x - G(w)\right\} = 0,$$
$$D_t\left[\Phi(w)\right] + D_x\left\{-F(w)[f(w)w_x]_x + \tfrac{1}{2}[f(w)w_x]^2 - \Psi(w)\right\} = 0,$$

where

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}, \quad G(w) = \int g(w)\,dw, \quad F(w) = \int f(w)\,dw,$$
$$\Phi(w) = \int F(w)\,dw, \quad \Psi(w) = \int F(w)g(w)\,dw.$$

⊙ *Reference for equation* 13.1.7.20: A. D. Polyanin and V. F. Zaitsev (2004, pp. 534–535).

21. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^3 w}{\partial x^3} + g(w)\dfrac{\partial w}{\partial x}$.

1°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$w''_{zz} = \int \frac{\lambda - g(w)}{f(w)}\,dw + C,$$

which is easy to integrate.

TABLE 13.2
Some integrable nonlinear third-order equations of the form 13.1.7.21; $k = (8/a)^{1/2}$

Type of generated equation	Form of generated equation	Solvable equation of the form (3)
Linear equation	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3}$
Korteweg–de Vries equation 13.1.1.2	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw\frac{\partial w}{\partial x}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3} - bU\frac{\partial U}{\partial z}$
Modified Korteweg–de Vries equation 13.1.2.4	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw^2\frac{\partial w}{\partial x}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3} - \frac{2bk}{3a}U^{3/2}\frac{\partial U}{\partial z}$

2°. Conservation law:

$$D_t\big[\varphi(w)\big] + D_x\big[-w_{xx} - \psi(w)\big] = 0,$$

where

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}, \quad \varphi(w) = \int \frac{dw}{f(w)}, \quad \psi(w) = \int \frac{g(w)}{f(w)}\,dw. \tag{1}$$

3°. The transformation

$$dz = \varphi(w)\,dx + \big[w_{xx} + \psi(w)\big]\,dt, \quad d\tau = dt, \quad U = \int \varphi(w)\,dw \quad \left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt\right) \tag{2}$$

leads to an equation of the similar form

$$\frac{\partial U}{\partial \tau} = F(U)\frac{\partial^3 U}{\partial z^3} + G(U)\frac{\partial U}{\partial z}. \tag{3}$$

The functions $F(U)$ and $G(U)$ in (3) are defined parametrically by

$$F(U) = f(w)\varphi^3(w), \quad G(U) = g(w)\varphi(w) - \psi(w), \quad U = \int \varphi(w)\,dw,$$

where $\varphi(w)$ and $\psi(w)$ are defined in (1).

Table 13.2 presents some solvable equations of the form (3) generated by known solvable third-order equations.

4°. The substitution $\varphi = \int \frac{dw}{f(w)}$ leads to an equation of the form 13.1.7.20:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\mathcal{F}(\varphi)\frac{\partial \varphi}{\partial z}\right] + \mathcal{G}(\varphi)\frac{\partial \varphi}{\partial z},$$

where the functions $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F}(\varphi) = f(w), \quad \mathcal{G}(\varphi) = g(w), \quad \varphi = \int \frac{dw}{f(w)}.$$

⊙ *Reference for equation* 13.1.7.21: A. D. Polyanin and V. F. Zaitsev (2004, pp. 535–536).

22. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^3 w}{\partial x^3} + \big[g(w) + ax\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$f(w)w'''_{zzz} + [g(w) + az]w'_z + h(w) = 0.$$

⊙ *Reference for equation* 13.1.7.22: A. D. Polyanin and V. F. Zaitsev (2004, p. 536).

23. $w\dfrac{\partial w}{\partial t} + a\dfrac{\partial w}{\partial x} + bw\dfrac{\partial^3 w}{\partial x^3} = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w(C_1 x + C_2, C_1^3 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = U(\xi), \qquad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $U = U(\xi)$ is determined by the second-order autonomous ordinary differential equation

$$bU''_{\xi\xi} + a\ln|U| + \lambda U = C_1.$$

3°. Self-similar solution:

$$w = t^{2/3}u(z), \qquad z = xt^{-1/3},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$buu'''_{zzz} - \tfrac{1}{3}zuu'_z + au'_z + \tfrac{2}{3}u^2 = 0.$$

4°. The substitution $x = b^{1/3}\hat{z}$ leads to an equation of the form 13.1.3.2 with $k = -1$ for $w = w(z,t)$.

13.2. Equations Involving the Second Derivative in t

13.2.1. Equations with Quadratic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

This is a special case of equation 15.3.2.1 with $n = 3$.

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)f(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi'''_{xxx} + b(\varphi'_x)^2 - A = 0,$$

whose order can be reduced with the change of variable $U(x) = \varphi'_x$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)f(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^3 u}{\partial x^3} + b\left(\frac{\partial u}{\partial x}\right)^2.$$

This equation admits a traveling-wave solution, $u = u(kx + \lambda t)$, and a self-similar solution, $u = t^{-2/3}\phi(z)$, where $z = xt^{-2/3}$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^3 w}{\partial x^3} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + f(t).$

This is a special case of equation 15.3.2.1 with $n = 3$.

1°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - f(t) = 0,$$
$$\psi'''_{zzz} - \lambda^2\psi''_{zz} + a(\psi'_z)^2 + b\psi = 0.$$

The solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + a\left(\frac{\partial u}{\partial x}\right)^2 + bu.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial^2 w}{\partial x^2} + c.$

This is a special case of equation 15.3.3.1 with $n = 3$.

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^3 u'''_{\xi\xi\xi} + (b\beta^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + bUU''_{zz} - b^2C_2^2U''_{zz} - 2bC_1U'_z + c - 8bC_1^2 = 0.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + b\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + c.$

This is a special case of equation 15.3.3.2 with $n = 3$.

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^3 u'''_{\xi\xi\xi} + b\beta^2(uu'_\xi)'_\xi - \lambda^2 u''_{\xi\xi} + c = 0. \tag{1}$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + b(UU'_z)'_z - b^2C_2^2U''_{zz} - 2bC_1U'_z + c - 8bC_1^2 = 0. \tag{2}$$

Remark. Equations (1) and (2) can each be integrated once with respect to the independent variable.

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + c.$

1°. Generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi_4'' &= 144b\varphi_4^2,\\
\varphi_3'' &= 144b\varphi_3\varphi_4,\\
\varphi_2'' &= 12b(3\varphi_3^2 + 4\varphi_2\varphi_4),\\
\varphi_1'' &= 24(a\varphi_4 + b\varphi_2\varphi_3),\\
\varphi_0'' &= 6a\varphi_3 + 4b\varphi_2^2 + c.
\end{aligned}$$

2°. Generalized separable solution:

$$w(x,t) = C_1xt^2 + C_2t^2 + C_3t + f(x),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $f(x)$ is determined by the ordinary differential equation

$$2C_1x + 2C_2 = af_{xxx}''' + b(f_{xx}'')^2 + c.$$

3°. Solution (generalizes the solution of Item 2°):

$$w(x,t) = C_1x^2 + C_2xt + C_3t^2 + C_4x + C_5t + u(z), \quad z = kx + \lambda t,$$

where $C_1, \ldots, C_5$, k, and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$\lambda^2 u_{zz}'' + 2C_3 = ak^3 u_{zzz}''' + b(k^2u_{zz}'' + 2C_1)^2 + c,$$

which can be integrated using the substitution $U(z) = u_{zz}''$. To the special case $C_1 = \cdots = C_5 = 0$ there corresponds a traveling-wave solution.

4°. The substitution

$$w(x,t) = V(x,t) + \tfrac{1}{2}ct^2$$

leads to the simpler equation

$$\frac{\partial^2 V}{\partial t^2} = a\frac{\partial^3 V}{\partial x^3} + b\left(\frac{\partial^2 V}{\partial x^2}\right)^2,$$

which admits a self-similar solution of the form $V = t^{2/3}\theta(\xi)$, where $\xi = xt^{-2/3}$.

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^3 w}{\partial x^3} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)(A_3x^3 + A_2x^2 + A_1x) + \psi(t),$$

where A_1, A_2, and A_3 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_{tt}'' &= 6A_3a\varphi^2 + f(t)\varphi,\\
\psi_{tt}'' &= 6A_3a\varphi\psi + f(t)\psi + g(t).
\end{aligned}$$

7. $\dfrac{\partial^2 w}{\partial t^2} + k(t)\dfrac{\partial w}{\partial t} = f(t)w\dfrac{\partial^3 w}{\partial x^3} + g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial w}{\partial x} + p(t)w + q(t).$

Generalized separable solution cubic in x:

$$w(x,t) = \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_3'' + k(t)\varphi_3' &= [6f(t)\varphi_3 + p(t)]\varphi_3,\\
\varphi_2'' + k(t)\varphi_2' &= [6f(t)\varphi_3 + p(t)]\varphi_2 + 3h(t)\varphi_3,\\
\varphi_1'' + k(t)\varphi_1' &= [6f(t)\varphi_3 + p(t)]\varphi_1 + 6g(t)\varphi_3 + 2h(t)\varphi_2,\\
\varphi_0'' + k(t)\varphi_0' &= [6f(t)\varphi_3 + p(t)]\varphi_0 + 2g(t)\varphi_2 + h(t)\varphi_1 + q(t).
\end{aligned}$$

8. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^3 w}{\partial x^3} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of second-order ordinary differential equations (C is an arbitrary constant)

$$\begin{aligned}
\varphi_{tt}'' &= C\varphi^2 + b\varphi\psi + f(t)\varphi,\\
\psi_{tt}'' &= C\varphi\psi + b\psi^2 + f(t)\psi + g(t),
\end{aligned}$$

and the function $\Theta(x)$ satisfies the third-order constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\Theta_{xxx}''' + b\Theta = C.$$

13.2.2. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + bw\ln w + \big[f(x) + g(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
\varphi_{tt}'' - \big[b\ln\varphi + g(t) + C\big]\varphi &= 0,\\
a\psi_{xxx}''' + \big[b\ln\psi + f(x) - C\big]\psi &= 0,
\end{aligned}$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_{xxx}''' + f(x)\varphi\ln\varphi + \big[g(x) - b^2\big]\varphi = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Multiplicative separable solutions with $bc < 0$:

$$w = \begin{cases} \exp\big(|c/b|^{1/2}x\big)\big[C_1\exp(kt) + C_2\exp(-kt)\big] & \text{if } s + a|c/b|^{3/2} = k^2 > 0, \\ \exp\big(|c/b|^{1/2}x\big)\big[C_1\cos(kt) + C_2\sin(kt)\big] & \text{if } s + a|c/b|^{3/2} = -k^2 < 0, \end{cases}$$

$$w = \begin{cases} \exp\big(-|c/b|^{1/2}x\big)\big[C_1\exp(kt) + C_2\exp(-kt)\big] & \text{if } s - a|c/b|^{3/2} = k^2 > 0, \\ \exp\big(-|c/b|^{1/2}x\big)\big[C_1\cos(kt) + C_2\sin(kt)\big] & \text{if } s - a|c/b|^{3/2} = -k^2 < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

Remark. The more general equation

$$w_{tt} = aw_{xxx} + f(w)\big(bw_x^2 + cw^2\big) + sw,$$

where $f(w)$ is an arbitrary function, has the same solutions.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\big(x\sqrt{|c/b|}\,\big) + \psi(t)\exp\big(-x\sqrt{|c/b|}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_{tt}'' = \big(s + a|c/b|^{3/2}\big)\varphi + 4c\varphi^2\psi,$$
$$\psi_{tt}'' = \big(s - a|c/b|^{3/2}\big)\psi + 4c\varphi\psi^2.$$

3°. Generalized separable solution with $bc > 0$:

$$w = \varphi(t)\cos\big(x\sqrt{c/b}\,\big) + \psi(t)\sin\big(x\sqrt{c/b}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_{tt}'' = \big[s + a(c/b)^{3/2}\big]\varphi + c(\varphi^2 + \psi^2)\varphi,$$
$$\psi_{tt}'' = \big[s - a(c/b)^{3/2}\big]\psi + c(\varphi^2 + \psi^2)\psi.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + F\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = C_1t^2 + C_2t + \int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau + \varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_{xxx}''' + F\big(x, \varphi_x'\big) - 2C_1 = 0,$$

whose order can be reduced with the change of variable $u(x) = \varphi_x'$.

2°. The substitution

$$w = U(x,t) + \int_0^t (t-\tau)g(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^3 U}{\partial x^3} + F\left(x, \frac{\partial U}{\partial x}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + F\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi'''_{xxx} + F\left(x, \psi'_x\right) + b\psi = 0.$$

The solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^3 U}{\partial x^3} + F\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2}{\partial x^2}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + bw + c.$

Multiplicative separable solution:

$$w = \varphi(t)\exp\left(-\frac{c}{6a}x^3 + C_1x^2 + C_2x\right), \qquad \varphi(t) = \begin{cases} C_3\cosh(\sqrt{b}\,t) + C_4\sinh(\sqrt{b}\,t) & \text{if } b > 0, \\ C_3t + C_4 & \text{if } b = 0, \\ C_3\cos(\sqrt{|b|}\,t) + C_4\sin(\sqrt{|b|}\,t) & \text{if } b < 0, \end{cases}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2}{\partial x^2}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{1}{6}bx^3 + C_1x^2 + C_2x + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

8. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^3 w}{\partial x^3}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, y + C_2) + C_3 xt + C_4 x + C_5 t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = u(z) + C_3 x^2 + C_4 t^2, \qquad z = C_1 x + C_2 t,$$

where the function $u(z)$ is determined by the autonomous ordinary differential equation

$$C_2^2 u''_{zz} + 2C_4 = F(C_1^2 u''_{zz} + 2C_3,\ C_1^3 u'''_{zzz}),$$

whose order can be reduced by two with the change of variable $\theta(z) = u''_{zz}$.

13.3. Hydrodynamic Boundary Layer Equations

13.3.1. Steady Hydrodynamic Boundary Layer Equations for a Newtonian Fluid

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu \dfrac{\partial^3 w}{\partial y^3}.$

This is an equation of a *steady laminar hydrodynamic boundary layer* on a flat plate; w is the *stream function*, x and y are the longitudinal and normal coordinates, respectively, and ν is the kinematic viscosity of the fluid. A similar equation governs the steady-state flow of a plane laminar jet out of a slot.

Preliminary remarks. The system of *steady-state hydrodynamic boundary layer equations*

$$u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu \frac{\partial^2 u_1}{\partial y^2},$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal components of the fluid velocity, respectively, is reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

⊙ *References*: H. Schlichting (1981), L. G. Loitsyanskiy (1996).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x)),$$
$$w_2 = C_1 w(C_2 x + C_3, C_1 C_2 y + C_4) + C_5,$$

where $\varphi(x)$ is an arbitrary function and $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation.

⊙ *References*: Yu. N. Pavlovskii (1961), L. V. Ovsiannikov (1982).

2°. Degenerate solutions (linear and quadratic in y):

$$w(x,y) = C_1 y + \varphi(x),$$
$$w(x,y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\varphi^2(x) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function. These solutions are independent of ν and correspond to inviscid fluid flows.

TABLE 13.3
Invariant solutions to the hydrodynamic boundary layer equation (the additive constant is omitted)

No.	Solution structure	Function F or equation for F	Remarks
1	$w = F(y) + \nu\lambda x$	$F(y) = \begin{cases} C_1\exp(-\lambda y) + C_2 y & \text{if } \lambda \neq 0, \\ C_1 y^2 + C_2 y & \text{if } \lambda = 0 \end{cases}$	λ is any
2	$w = F(x)y^{-1}$	$F(x) = 6\nu x + C_1$	—
3	$w = x^{\lambda+1}F(z),\ z = x^{\lambda}y$	$(2\lambda+1)(F_z')^2 - (\lambda+1)FF_{zz}'' = \nu F_{zzz}'''$	λ is any
4	$w = e^{\lambda x}F(z),\ z = e^{\lambda x}y$	$2\lambda(F_z')^2 - \lambda FF_{zz}'' = \nu F_{zzz}'''$	λ is any
5	$w = F(z) + a\ln\lvert x\rvert,\ z = y/x$	$-(F_z')^2 - aF_{zz}'' = \nu F_{zzz}'''$	a is any

3°. Solutions involving arbitrary functions:

$$w(x,y) = \frac{6\nu x + C_1}{y+\varphi(x)} + \frac{C_2}{[y+\varphi(x)]^2} + C_3,$$
$$w(x,y) = \varphi(x)\exp(-C_1 y) + \nu C_1 x + C_2,$$
$$w(x,y) = C_1\exp\bigl[-C_2 y - C_2\varphi(x)\bigr] + C_3 y + C_3\varphi(x) + \nu C_2 x + C_4,$$
$$w(x,y) = 6\nu C_1 x^{1/3}\tanh\xi + C_2, \qquad \xi = C_1\frac{y}{x^{2/3}} + \varphi(x),$$
$$w(x,y) = -6\nu C_1 x^{1/3}\tan\xi + C_2, \qquad \xi = C_1\frac{y}{x^{2/3}} + \varphi(x),$$

where $C_1, \ldots, C_4$ are arbitrary constants and $\varphi(x)$ is an arbitrary function.

Special case 1. For $C_1 = \sqrt{k/\nu}$ and $\varphi(x) = -\sqrt{k\nu}\,x$, the second solution becomes

$$w = \sqrt{k\nu}\,x\bigl[1 - \exp\bigl(-\sqrt{k/\nu}\,y\bigr)\bigr] + \text{const}.$$

It describes a fluid flow induced by the motion of surface particles at $y = 0$ with a velocity of $u_1|_{y=0} = kx$. The fluid velocity components in this case meet the boundary conditions

$$u_1 = 0 \quad \text{at} \quad x = 0, \qquad u_1 = kx \quad \text{at} \quad y = 0, \qquad u_2 = 0 \quad \text{at} \quad y = 0, \qquad u_1 \to 0 \quad \text{as} \quad y \to \infty.$$

⊙ *References*: N. V. Ignatovich (1993), A. D. Polyanin (2001 *a*).

4°. Table 13.3 lists invariant solutions to the hydrodynamic boundary layer equation that are obtained with the classical group-theoretic methods. Solution 1 is expressed in additive separable form, solution 2 is in multiplicative separable form, solution 3 is self-similar, and solution 4 is generalized self-similar. Solution 5 degenerates at $a = 0$ into a self-similar solution (see solution 3 with $\lambda = -1$). Equations 3–5 for F are autonomous and generalized homogeneous; hence, their order can be reduced by two.

⊙ *References*: Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. G. Loitsyanskiy (1996), G. I. Burde (1996).

Special case 2. The *Blasius problem* on a translational fluid flow with a velocity U_i past a flat plate is characterized by the boundary conditions

$$\partial_x w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to U_\mathrm{i} \quad \text{as} \quad y \to \infty, \qquad \partial_y w = U_\mathrm{i} \quad \text{at} \quad x = 0.$$

The form of the solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is given in the third row of Table 13.3 with $\lambda = -1/2$. The boundary conditions for $F(z)$ are as follows:

$$F = F_z' = 0 \quad \text{at} \quad z = 0, \qquad F_z' \to U_i \quad \text{as} \quad z \to \infty.$$

For details, see Blasius (1908), Schlichting (1981), and Loitsyanskiy (1996).

Special case 3. The *Schlichting problem* on the axisymmetric flow of a plane laminar jet out of a thin slit is characterized by the boundary conditions

$$\partial_x w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to 0 \quad \text{as} \quad y \to \infty,$$

which are supplemented with the integral condition of conservation of momentum

$$\int_0^\infty (\partial_y w)^2 \, dy = A \qquad (A = \text{const}).$$

The form of the solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is given in the third row of Table 13.3 with $\lambda = -2/3$. On integrating the ordinary differential equation for F with appropriate boundary conditions,

$$F = F''_{zz} = 0 \quad \text{at} \quad z = 0, \qquad F'_z \to 0 \quad \text{as} \quad z \to \infty,$$

and the integral condition

$$\int_0^\infty (F'_z)^2 = A,$$

we finally obtain

$$w(x, y) = k(A\nu x)^{1/3} \tanh \xi, \quad \xi = \tfrac{1}{6} k (A/\nu^2)^{1/3} y x^{-2/3}, \quad k = 3^{2/3}.$$

For details, see the book by Schlichting (1981) and Loitsyanskiy (1996).

Special case 4. Note two cases where the equation specified in row 3 of Table 13.3 can be integrated.

For $\lambda = -1$, the solution can be obtained in parametric form:

$$F = -\frac{\nu}{2C_1} \int \frac{\tau \, d\tau}{\sqrt{1 + \tau^3}} + C_2, \quad z = 3C_1 \int \frac{d\tau}{\sqrt{1 + \tau^3}} + C_3.$$

There is a solution $F = 6\nu z^{-1}$.

For $\lambda = -\frac{2}{3}$, the twofold integration yields the Riccati equation

$$\nu F'_z + \tfrac{1}{6} F^2 + C_1 z + C_2 = 0.$$

If $C_1 = 0$, it can be readily integrated; if $C_1 \neq 0$, the solution can be expressed in terms of the Bessel functions of order $1/3$.

5°. Generalized separable solution linear in x:

$$w(x, y) = x f(y) + g(y), \tag{1}$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the autonomous system of ordinary differential equations

$$(f'_y)^2 - f f''_{yy} = \nu f'''_{yyy}, \tag{2}$$

$$f'_y g'_y - f g''_{yy} = \nu g'''_{yyy}. \tag{3}$$

The order of equation (2) can be reduced by two. Suppose a solution of equation (2) is known. Equation (3) is linear in g and has two linearly independent solutions:

$$g_1 = 1, \qquad g_2 = f(y).$$

The second particular solution follows from the comparison of (2) and (3). The general solution of equation (3) can be written out in the form

$$\begin{gathered} g(y) = C_1 + C_2 f + C_3 \left(f \int \psi \, dy - \int f \psi \, dy \right), \\ f = f(y), \quad \psi = \frac{1}{(f'_y)^2} \exp\left(-\frac{1}{\nu} \int f \, dy \right); \end{gathered} \tag{4}$$

see Polyanin and Zaitsev (2003).

It is not difficult to verify that equation (2) has the following particular solutions:

$$f(y) = 6\nu(y + C)^{-1},$$
$$f(y) = Ce^{\lambda y} - \lambda\nu, \tag{5}$$

where C and λ are arbitrary constants. The first solution in (5) leads, taking into account (1) and (4), to the first solution of Item 3° with $\varphi(x) = \text{const}$. Substituting the second expression of (5) into (1) and (4), one may obtain another solution.

⊙ *Reference*: K. Hiemenz (1911), C. Y. Wang (1991), A. D. Polyanin (2001 *a*).

6°. Generalized separable solution (special case of solution 3 in Item 3°):

$$w(x, y) = (a + be^{-\lambda y})z(x) + cy,$$

where a, b, c, and λ are arbitrary constants, and the function $z = z(x)$ is defined implicitly by

$$c \ln|z| + a\lambda z = \nu\lambda^2 x.$$

⊙ *Reference*: N. V. Ignatovich (1993), B. I. Burde (1996).

7°. Below are two transformations that reduce the order of the boundary layer equation.

7.1. The *von Mises transformation*

$$\xi = x, \quad \eta = w, \quad U(\xi, \eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x, y),$$

leads to a nonlinear heat equation of the form 5.10.1.1:

$$\frac{\partial U}{\partial \xi} = \nu \frac{\partial}{\partial \eta}\left(U \frac{\partial U}{\partial \eta}\right).$$

7.2. The *Crocco transformation*

$$\xi = x, \quad \zeta = \frac{\partial w}{\partial y}, \quad \Psi(\xi, \zeta) = \frac{\partial^2 w}{\partial y^2}, \qquad \text{where} \quad w = w(x, y),$$

leads to the second-order nonlinear equation

$$\zeta \frac{\partial}{\partial \xi}\left(\frac{1}{\Psi}\right) + \nu \frac{\partial^2 \Psi}{\partial \zeta^2} = 0.$$

⊙ *Reference*: L. G. Loitsyanskiy (1996).

8°. Conservation law:

$$D_x\left(w_y^2\right) + D_y\left(-w_x w_y - \nu w_{yy}\right) = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

⊙ *References for equation* 13.3.1.1: H. Blasius (1908), Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. V. Ovsiannikov (1982), D. Zwillinger (1989, pp. 396–397), N. V. Ignatovich (1993), B. I. Burde (1996), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *a*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 291–294).

TABLE 13.4
Invariant solutions to the hydrodynamic boundary layer equation with pressure gradient (a, k, m, and β are arbitrary constants)

No.	Function $f(x)$	Form of solution $w = w(x, y)$	Function u or equation for u
1	$f(x) = 0$	See equation 13.3.1.1	See equation 13.3.1.1
2	$f(x) = ax^m$	$w = x^{\frac{m+3}{4}} u(z),\ z = x^{\frac{m-1}{4}} y$	$\frac{m+1}{2}(u'_z)^2 - \frac{m+3}{4} u u''_{zz} = \nu u'''_{zzz} + a$
3	$f(x) = ae^{\beta x}$	$w = e^{\frac{1}{4}\beta x} u(z),\ z = e^{\frac{1}{4}\beta x} y$	$\frac{1}{2}\beta(u'_z)^2 - \frac{1}{4}\beta u u''_{zz} = \nu u'''_{zzz} + a$
4	$f(x) = a$	$w = kx + u(y)$	$u(y) = \begin{cases} C_1 \exp\left(-\frac{k}{\nu} y\right) - \frac{a}{2k} y^2 + C_2 y & \text{if } k \neq 0, \\ -\frac{a}{6\nu} y^3 + C_2 y^2 + C_1 y & \text{if } k = 0 \end{cases}$
5	$f(x) = ax^{-3}$	$w = k \ln\lvert x\rvert + u(z),\ z = y/x$	$-(u'_z)^2 - k u''_{zz} = \nu u'''_{zzz} + a$

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu \dfrac{\partial^3 w}{\partial y^3} + f(x).$

This is a *hydrodynamic boundary layer equation with pressure gradient*. The formula $f(x) = UU'_x$ holds true; $U = U(x)$ is the fluid velocity in the stream core* at the interface between the core and the boundary layer.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x)) + C,$$
$$w_2 = -w(x, -y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, are also solutions of the equation.

⊙ *References*: Yu. N. Pavlovskii (1961), L. V. Ovsiannikov (1982).

2°. Degenerate solutions (linear and quadratic in y) for arbitrary $f(x)$:

$$w(x, y) = \pm y \left[2 \int f(x)\, dx + C_1\right]^{1/2} + \varphi(x),$$
$$w(x, y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\left[\varphi^2(x) - 2 \int f(x)\, dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function, and C_1 and C_2 are arbitrary constants. These solutions are independent of ν and correspond to inviscid fluid flows.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

3°. Table 13.4 lists invariant solutions to the hydrodynamic boundary layer equation with pressure gradient that are obtained with the classical group-theoretic methods.

Note that the *Falkner–Skan problem* (see Falkner and Skan, 1931) on a symmetric fluid flow past a wedge is described by the equation specified in the second row of Table 13.4. The case $m = 1$ corresponds to a fluid flow near a stagnation point, and the case $m = 0$ corresponds to a symmetric flow past a wedge with an angle of $\alpha = \frac{2}{3}\pi$.

⊙ *References*: Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. G. Loitsyanskiy (1996), G. I. Burde (1996).

* The hydrodynamic problem on the flow of an ideal (inviscid) fluid about the body is solved in the stream core.

4°. Generalized separable solution (linear in x) for $f(x) = ax + b$:

$$w(x, y) = xF(y) + G(y),$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the system of ordinary differential equations

$$(F'_y)^2 - FF''_{yy} = \nu F'''_{yyy} + a, \tag{1}$$
$$F'_y G'_y - FG''_{yy} = \nu G'''_{yyy} + b. \tag{2}$$

The order of the autonomous equation (1) can be reduced by one. Given a particular solution of equation (1), the corresponding equation (2) can be reduced with the substitution $H(y) = G'_y$ to a second-order equation. For $F(y) = \pm\sqrt{a}\, y + C$, equation (2) is integrable by quadrature (since, for $b = 0$, we know two of its particular solutions: $G_1 = 1$ and $G_2 = \pm\frac{1}{2}\sqrt{a}\, y^2 + Cy$).

⊙ *Reference*: A. D. Polyanin (2001 *a*).

5°. Solutions for $f(x) = -ax^{-5/3}$:

$$w(x, y) = \frac{6\nu x}{y + \varphi(x)} \pm \frac{\sqrt{3a}}{x^{1/3}}[y + \varphi(x)],$$

where $\varphi(x)$ is an arbitrary function.

⊙ *Reference*: B. I. Burde (1996).

6°. Solutions for $f(x) = ax^{-1/3} - bx^{-5/3}$:

$$w(x, y) = \pm\sqrt{3b}\, z + x^{2/3}\theta(z), \qquad z = yx^{-1/3},$$

where the function $\theta = \theta(z)$ is determined by the ordinary differential equation

$$\tfrac{1}{3}(\theta'_z)^2 - \tfrac{2}{3}\theta\theta''_{zz} = \nu\theta'''_{zzz} + a.$$

⊙ *Reference*: B. I. Burde (1996).

7°. Generalized separable solution for $f(x) = ae^{\beta x}$:

$$w(x, y) = \varphi(x)e^{\lambda y} - \frac{a}{2\beta\lambda^2\varphi(x)}e^{\beta x - \lambda y} - \nu\lambda x + \frac{2\nu\lambda^2}{\beta}y + \frac{2\nu\lambda}{\beta}\ln|\varphi(x)|,$$

where $\varphi(x)$ is an arbitrary function and λ is an arbitrary constant.

⊙ *References*: A. D. Polyanin (2001 *a*, 2002).

8°. For

$$f(x) = a^2\nu^2 x^{-3}(xgg'_x - g^2), \qquad g = -\tfrac{1}{4}a \pm \left(\tfrac{1}{16}a^2 + bx^{2/3}\right)^{1/2},$$

there are exact solutions of the form

$$w(x, y) = a\nu z + 6\nu g \tanh z, \qquad z = \frac{yg}{x}.$$

⊙ *Reference*: B. I. Burde (1996).

9°. Below are two transformations that reduce the order of the boundary layer equation.

9.1. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y),$$

leads to the nonlinear heat equation

$$U\frac{\partial U}{\partial \xi} = \nu U\frac{\partial}{\partial \eta}\left(U\frac{\partial U}{\partial \eta}\right) + f(\xi).$$

9.2. The Crocco transformation

$$\xi = x, \quad \zeta = \frac{\partial w}{\partial y}, \quad \Psi(\xi,\zeta) = \frac{\partial^2 w}{\partial y^2}, \qquad \text{where} \quad w = w(x,y),$$

leads to the second-order nonlinear equation

$$\zeta\frac{\partial}{\partial \xi}\left(\frac{1}{\Psi}\right) + \nu\frac{\partial^2 \Psi}{\partial \zeta^2} - f(\xi)\frac{\partial}{\partial \zeta}\left(\frac{1}{\Psi}\right) = 0.$$

⊙ *Reference*: L. G. Loitsyanskiy (1996).

10°. Conservation law:

$$D_x\left[w_y^2 - F(x)\right] + D_y\left(-w_x w_y - \nu w_{yy}\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(x) = \displaystyle\int f(x)\,dx$.

⊙ *References for equation* 13.3.1.2: V. M. Falkner and S. W. Skan (1931), Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. V. Ovsiannikov (1982), B. I. Burde (1996), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *a, b*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 294–295).

3. $\dfrac{\partial w}{\partial z}\dfrac{\partial^2 w}{\partial x \partial z} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial z^2} = \nu\dfrac{\partial}{\partial z}\left(z\dfrac{\partial^2 w}{\partial z^2}\right) + f(x).$

Preliminary remarks. The system of *axisymmetric steady laminar hydrodynamic boundary layer equations*

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + f(x), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{2}$$

where u and v are the axial and radial fluid velocity components, respectively, and x and r are cylindrical coordinates, can be reduced to the equation in question by the introduction of a stream function w and a new variable z such that

$$u = \frac{2}{r}\frac{\partial w}{\partial r}, \quad v = -\frac{2}{r}\frac{\partial w}{\partial x}, \quad z = \frac{1}{4}r^2.$$

System (1), (2) is used for describing an axisymmetric jet and a boundary layer on an extensive body of revolution. The function $f(x)$ is expressed via the longitudinal fluid velocity $U = U(x)$ in the inviscid flow core as $f = UU'_x$.

⊙ *References*: F. L. Crabtree, D. Küchemann, and L. Sowerby (1963), H. Schlichting (1981), L. G. Loitsyanskiy (1996).

1°. Self-similar solution for $f(x) = Ax^k$:

$$w(x, z) = xU(\zeta), \qquad \zeta = zx^{\frac{k-1}{2}},$$

where the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$-\tfrac{1}{2}(k+1)(U'_\zeta)^2 + UU''_{\zeta\zeta} + A + \nu(\zeta U''_{\zeta\zeta})'_\zeta = 0.$$

Special case. An *axisymmetric jet* is characterized by the values $A = 0$ and $k = -3$. In this case, the solution of the equation just obtained with appropriate boundary conditions is given by

$$U(\zeta) = \frac{2\nu\zeta}{\zeta + C},$$

where the constant of integration C can be expressed via the jet momentum.

⊙ *References*: H. Schlichting (1981), L. G. Loitsyanskiy (1996).

2°. Generalized separable solutions (linear and quadratic in z) for arbitrary $f(x)$:

$$w(x, z) = \pm z\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$

$$w(x, z) = C_1 z^2 + \varphi(x)z + \frac{1}{4C_1}\varphi^2(x) - \frac{1}{2C_1}\int f(x)\,dx - \nu x + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. The first solution is "inviscid" (independent of ν).

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3°. Functional separable solution for arbitrary $f(x)$:

$$w(x, z) = 2\nu x + \nu F(x)\left(\frac{2C_1}{\xi} + C_2\xi\right), \qquad \xi = \frac{z}{F^2(x)} - C_1 F'_x(x),$$

$$F(x) = \pm\nu C_2\left[2\int f(x)\,dx + C_3\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: G. I. Burde (1994).

4°. Functional separable solution for $f(x) = ax + b$:

$$w(x, z) = \nu\lambda\varphi(x) + \frac{\nu}{a}(ax + b)\left(Ce^{-\lambda\xi} + \lambda\xi - 3\right), \qquad \xi = z - \varphi'_x(x), \qquad \lambda = \pm\frac{\sqrt{a}}{\nu},$$

where C and λ are arbitrary constants and $\varphi(x)$ is an arbitrary function.

⊙ *Reference*: G. I. Burde (1994).

5°. Generalized separable solution (linear in x) for $f(x) = ax + b$:

$$w(x, z) = x\varphi(z) + \psi(z),$$

where the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are determined by the system of ordinary differential equations

$$(\varphi'_z)^2 - \varphi\varphi''_{zz} = \nu(z\varphi''_{zz})'_z + a,$$
$$\varphi'_z\psi'_z - \varphi\psi''_{zz} = \nu(z\psi''_{zz})'_z + b.$$

The first equation has particular solutions $\varphi = \pm\sqrt{a}\,z + C$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6°. Additive separable solutions for $f(x) = a$:

$$w(x,z) = \nu(1-k)x + C_1 z^k + \frac{a}{2\nu(k-2)}z^2 + C_2 z + C_3,$$
$$w(x,z) = -\nu x - \frac{a}{2\nu}z^2 \ln z + C_1 z^2 + C_2 z + C_3,$$

where $C_1, \ldots, k$ are arbitrary constants.

7°. Conservation law:

$$D_x\left[w_z^2 - F(x)\right] + D_z\left(-w_x w_z - \nu z w_{zz}\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_z = \dfrac{\partial}{\partial z}$, and $F(x) = \displaystyle\int f(x)\,dx$.

⊙ *References for equation* 13.3.1.3: F. L. Crabtree, D. Küchemann, and L. Sowerby (1963), H. Schlichting (1981), G. I. Burde (1994), L. G. Loitsyanskiy (1996), A. D. Polyanin and V. F. Zaitsev (2002, pp. 295–296).

13.3.2. Steady Boundary Layer Equations for Non-Newtonian Fluids

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = k\left(\dfrac{\partial^2 w}{\partial y^2}\right)^{n-1}\dfrac{\partial^3 w}{\partial y^3}.$

This equation describes a boundary layer on a flat plane in the flow of a power-law non-Newtonian fluid; w is the stream function, x and y are the longitudinal and normal coordinates, and n and k are rheological parameters ($n > 0$, $k > 0$).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(C_1^{2-n} C_2^{2n-1} x + C_3, C_2 y + C_4) + C_5,$$
$$w_2 = w(x, y + \varphi(x)),$$

where $C_1, \ldots, C_5$ are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Additive separable solutions:

$$w(x,y) = \frac{1}{C_1^2 n(2n-1)}\left[C_1(n-1)y + C_2\right]^{\frac{2n-1}{n-1}} + C_3 y + C_4 - kC_1 x \quad \text{if} \quad n \neq 1/2,$$
$$w(x,y) = -\frac{1}{C_1^2}\ln(C_1 y + C_2) + C_3 y + C_4 + 2kC_1 x \quad \text{if} \quad n = 1/2.$$

3°. Multiplicative separable solutions:

$$w(x,y) = \left[\lambda(2-n)x + C_1\right]^{\frac{1}{2-n}} F(y) \quad \text{if} \quad n \neq 2,$$
$$w(x,y) = C_1 e^{\lambda x} F(y) \quad \text{if} \quad n = 2,$$

where $F = F(y)$ is determined by the autonomous ordinary differential equation

$$\lambda(F_y')^2 - \lambda F F_{yy}'' = k(F_{yy}'')^{n-1} F_{yyy}''',$$

whose order can be reduced by two. The equation for F has a particular solution in the form of a power-law function, $F = A_n(y + C)^{\beta_n}$, where $\beta_n = \frac{2n-1}{n-2}$.

4°. Self-similar solution ($n \neq 2$ and λ is any):

$$w(x,y) = x^{\frac{2\lambda n-\lambda+1}{2-n}}\psi(z), \qquad z = x^{\lambda}y, \tag{1}$$

where the function $\psi = \psi(z)$ is determined by the autonomous ordinary differential equation

$$\frac{\lambda n+\lambda+1}{2-n}(\psi_z')^2 - \frac{2\lambda n-\lambda+1}{2-n}\psi\psi_{zz}'' = k(\psi_{zz}'')^{n-1}\psi_{zzz}''', \tag{2}$$

whose order can be reduced by two.

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

Special case 1. The *generalized Blasius problem* on a translational flow with an incident velocity $U_{\rm i}$ past a flat plate is characterized by the boundary conditions

$$\partial_x w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to U_{\rm i} \quad \text{as} \quad y \to \infty, \qquad \partial_y w = U_{\rm i} \quad \text{at} \quad x = 0.$$

A solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is sought in the form (1) with $\lambda = -\frac{1}{n+1}$. The boundary conditions for $\psi(z)$ are the following:

$$\psi = \psi_z' = 0 \quad \text{at} \quad z = 0, \qquad \psi_z' \to U_i \quad \text{as} \quad z \to \infty. \tag{3}$$

In Zaitsev and Polyanin (1989, 1994), exact solutions to problem (2)–(3) are specified for $\lambda = -\frac{1}{n+1}$ with $n = \frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{5}{7}$, 2.

Special case 2. The *generalized Schlichting problem* on the symmetric flow of a plane laminar power-law fluid jet out of a thin slit is characterized by the boundary conditions

$$\partial_x w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to 0 \quad \text{as} \quad y \to \infty,$$

which are supplemented with the integral condition of conservation of momentum

$$\int_0^\infty (\partial_y w)^2\, dy = A \qquad (A = \text{const}).$$

A solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is sought in the form (1) with $\lambda = -\frac{2}{3n}$. A solution to equation (2) for $\psi(z)$ with appropriate boundary conditions and integral condition (see the conditions in Special case 3, Subsection 13.3.1, where F should be replaced by ψ) can be found in the books by Shulman and Berkovskii (1966) and Polyanin, Kutepov, et al. (2002).

5°. Self-similar solution for $n = 2$ (λ is any):

$$w(x,y) = x^{\lambda}U(z), \qquad z = yx^{-1/3},$$

where the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$(\lambda - \tfrac{1}{3})(U_z')^2 + \lambda U U_{zz}'' = kU_{zz}''U_{zzz}''',$$

whose order can be reduced by two.

6°. Generalized self-similar solution (λ is any):

$$w(x,y) = e^{\lambda(2n-1)x}\Phi(\tau), \qquad \tau = e^{\lambda(2-n)x}y,$$

where the function $\Phi = \Phi(\tau)$ is determined by the autonomous ordinary differential equation

$$\lambda(n+1)(\Phi_\tau')^2 - \lambda(2n-1)\Phi\Phi_{\tau\tau}'' = k(\Phi_{\tau\tau}'')^{n-1}\Phi_{\tau\tau\tau}''',$$

whose order can be reduced by two.

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

7°. Solution for $n \neq 1/2$:

$$w(x,y) = C_1 \ln|x| + C_2 + g(\xi), \qquad \xi = x^{\frac{1}{1-2n}} y,$$

where the function $g = g(\xi)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{1-2n}(g'_\xi)^2 - C_1 g''_{\xi\xi} = k(g''_{\xi\xi})^{n-1} g'''_{\xi\xi\xi},$$

whose order can be reduced by two.

8°. Solution for $n = 1/2$:

$$w(x,y) = C_1 x + C_2 + h(\zeta), \qquad \zeta = e^{\lambda x} y,$$

where the function $h = h(\zeta)$ is determined by the autonomous ordinary differential equation

$$\lambda(h'_\zeta)^2 - C_1 h''_{\zeta\zeta} = k(h''_{\zeta\zeta})^{-1/2} h'''_{\zeta\zeta\zeta},$$

whose order can be reduced by two.

9°. Conservation law:

$$D_x\left(nw_y^2\right) + D_y\left(-nw_x w_y - kw_{yy}^n\right) = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

⊙ *References for equation* 13.3.2.1: Z. P. Shulman and B. M. Berkovskii (1966), A. D. Polyanin and V. F. Zaitsev (2002, pp. 297–298).

2. $$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = k\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3} + f(x).$$

This is a steady *boundary layer equation for a power-law fluid with pressure gradient*.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, is also a solution of the equation.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

2°. Degenerate solutions (linear and quadratic in y) for any $f(x)$:

$$w(x,y) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$

$$w(x,y) = C_1 y^2 + \varphi(x)y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. These solutions are independent of k and correspond to inviscid fluid flows.

3°. Self-similar solution for $f(x) = ax^m$:

$$w(x,y) = x^{\frac{2nm+2n-m+1}{2(n+1)}}\psi(z), \qquad z = x^{\frac{2m-n-nm}{2(n+1)}}y,$$

where the function $\psi = \psi(z)$ is determined by the autonomous ordinary differential equation

$$\frac{nm+n+m+1}{2(n+1)}(\psi_z')^2 - \frac{2nm+2n-m+1}{2(n+1)}\psi\psi_{zz}'' = k(\psi_{zz}'')^{n-1}\psi_{zzz}''' + a.$$

Note that solving the generalized Falkner–Skan problem on a symmetric power-law fluid flow past a wedge is reduced to solving the equation just obtained.

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

4°. Generalized self-similar solution for $f(x) = ae^{\beta x}$:

$$w(x,y) = \exp\left(\beta\frac{2n-1}{2n+2}x\right)\Phi(\tau), \qquad \tau = \exp\left(\beta\frac{2-n}{2n+2}x\right)y,$$

where the function $\Phi = \Phi(\tau)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{2}\beta(\Phi_\tau')^2 - \beta\frac{2n-1}{2n+2}\Phi\Phi_{\tau\tau}'' = k(\Phi_{\tau\tau}'')^{n-1}\Phi_{\tau\tau\tau}''' + a.$$

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

5°. Additive separable solution for $f(x) = a$:

$$w(x,y) = C_1x + h(y),$$

where the function $h = h(y)$ is determined by the autonomous ordinary differential equation

$$k(h_{yy}'')^{n-1}h_{yyy}''' + C_1h_{yy}'' + a = 0.$$

Its general solution can be written out in parametric form:

$$y = -k\int_{C_2}^{t}\frac{u^{n-1}\,du}{C_1u+a}, \quad h = k^2\int_{C_3}^{t}\frac{u^{n-1}\varphi(u)\,du}{C_1u+a}, \quad \text{where} \quad \varphi(u) = \int_{C_4}^{u}\frac{v^n\,dv}{C_1v+a}.$$

6°. Multiplicative separable solution for $f(x) = ax^{\frac{n}{2-n}}$, $n \neq 2$:

$$w(x,y) = x^{\frac{1}{2-n}}F(y),$$

where the function $F = F(y)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{2-n}(F_y')^2 - \frac{1}{2-n}FF_{yy}'' = k(F_{yy}'')^{n-1}F_{yyy}''' + a.$$

7°. Self-similar solution for $f(x) = ax^m$, $n = 2$:

$$w(x,y) = x^{\frac{1}{2}m+\frac{5}{6}}U(z), \qquad z = yx^{-1/3},$$

where the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{1}{2}(m+1)(U_z')^2 + \tfrac{1}{6}(3m+5)UU_{zz}'' = kU_{zz}''U_{zzz}''' + a.$$

8°. Multiplicative separable solution for $f(x) = ae^{\lambda x}$, $n = 2$:

$$w(x,y) = e^{\frac{1}{2}\lambda x}G(y),$$

where the function $G = G(y)$ is determined by the autonomous ordinary differential equation

$$\tfrac{1}{2}\lambda(G_y')^2 - \tfrac{1}{2}\lambda GG_{yy}'' = k(G_{yy}'')^{n-1}G_{yyy}''' + a.$$

9°. Solution $f(x) = ax^{\frac{2n+1}{1-2n}}$, $n \neq 1/2$:

$$w(x,y) = C_1 \ln|x| + C_2 + g(\xi), \qquad \xi = x^{\frac{1}{1-2n}}y,$$

where the function $g = g(\xi)$ is determined by the autonomous ordinary differential equation

$$k(g_{\xi\xi}'')^{n-1}g_{\xi\xi\xi}''' + C_1 g_{\xi\xi}'' - \frac{1}{1-2n}(g_\xi')^2 + a = 0,$$

10°. Solution $f(x) = ae^{\lambda x}$, $n = 1/2$:

$$w(x,y) = C_1 x + C_2 + h(\zeta), \qquad \zeta = e^{\frac{1}{2}\lambda x}y,$$

where the function $h = h(\zeta)$ is determined by the autonomous ordinary differential equation

$$k(h_{\zeta\zeta}'')^{-1/2}h_{\zeta\zeta\zeta}''' + C_1 h_{\zeta\zeta}'' - \tfrac{1}{2}\lambda(h_\zeta')^2 + a = 0.$$

11°. Conservation law:

$$D_x\left[nw_y^2 - nF(x)\right] + D_y\left(-nw_xw_y - kw_{yy}^n\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(x) = \displaystyle\int f(x)\,dx$.

⊙ *References for equation* 13.3.2.2: Z. P. Shulman and B. M. Berkovskii (1966), A. D. Polyanin (2001 *a*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 298–300).

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left[f\left(\dfrac{\partial^2 w}{\partial y^2}\right)\right]$.

This is an equation of a steady boundary layer on a flat plate in the flow of a non-Newtonian fluid of general form; w is the steam function, and x and y are the coordinates along and normal to the plate.

Preliminary remarks. The system of *non-Newtonian fluid boundary layer equations*

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial u_1}{\partial y}\right)\right],$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal fluid velocity components, can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(C_1^3 x + C_2, C_1 y + C_3) + C_4,$$
$$w_2 = w(x, y + \varphi(x)),$$

where $C_1, \ldots, C_4$ are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

2°. Solutions involving arbitrary functions:

$$w(x, y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\varphi^2(x) + C_2,$$
$$w(x, y) = g(z) + C_1 x + C_2, \quad z = y + \varphi(x),$$

where C_1 and C_2 are arbitrary constants, and $\varphi(x)$ is an arbitrary function. The function $g = g(z)$ in the second formula is determined by the autonomous ordinary differential equation

$$f(g''_{zz}) + C_1 g'_z = C_3,$$

whose general solution can be written out in parametric form:

$$g = \frac{1}{C_1^2}\int \frac{f'_t(t)}{t}\big[f(t) - C_3\big]\,dt + C_4, \quad z = C_5 - \frac{1}{C_1}\int \frac{f'_t(t)}{t}\,dt.$$

3°. Self-similar solution:

$$w(x, y) = x^{2/3}\psi(\xi), \quad \xi = yx^{-1/3},$$

where the function $\psi = \psi(\xi)$ is determined by the autonomous ordinary differential equation

$$(\psi'_\xi)^2 - 2\psi\psi''_{\xi\xi} = 3[f(\psi''_{\xi\xi})]'_\xi.$$

4°. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi, \eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x, y),$$

leads to the second-order nonlinear equation

$$\frac{\partial U}{\partial \xi} = \frac{\partial}{\partial \eta}\left[f\left(U\frac{\partial U}{\partial \eta}\right)\right].$$

It admits, for example, a traveling-wave solution $U = U(a\xi + b\eta)$ and a self-similar solution $U = \xi^{1/3}\Phi(\eta\xi^{-2/3})$.

5°. Conservation law:

$$D_x\big(w_y^2\big) + D_y\big[-w_x w_y - f(w_{yy})\big] = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

⊙ *References for equation* 13.3.2.3: A. D. Polyanin (2001 *a*), A. D. Polyanin and V. F. Zaitsev (2002, p. 300).

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left[f\left(\dfrac{\partial^2 w}{\partial y^2}\right)\right] + g(x).$

This is a steady *boundary layer equation for a non-Newtonian fluid of general form* with pressure gradient.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, is also a solution of the equation.

2°. There are degenerate solutions; see Item 2° in 13.3.2.2, where $f(x)$ should be replaced by $g(x)$.

3°. Solution for $g(x) = a$:

$$w(x, y) = \zeta(z) + C_1 x + C_2, \quad z = y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. The function $\zeta = \zeta(z)$ is determined by the ordinary differential equation

$$f(\zeta_{zz}'') + C_1\zeta_z' + a\zeta = C_3.$$

4°. Self-similar solution for $g(x) = a(x + b)^{-1/3}$:

$$w(x, y) = (x + b)^{2/3}\psi(\xi), \quad \xi = y(x + b)^{-1/3},$$

where the function $\psi = \psi(\xi)$ is determined by the autonomous ordinary differential equation

$$(\psi_\xi')^2 - 2\psi\psi_{\xi\xi}'' = 3[f(\psi_{\xi\xi}'')]_\xi' + 3a.$$

5°. Conservation law:

$$D_x\left[w_y^2 - G(x)\right] + D_y\left[-w_x w_y - f(w_{yy})\right] = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $G(x) = \displaystyle\int g(x)\,dx$.

⊙ *Reference for equation* 13.3.2.4: A. D. Polyanin and V. F. Zaitsev (2002, p. 301).

13.3.3. Unsteady Boundary Layer Equations for a Newtonian Fluid

1. $\dfrac{\partial^2 w}{\partial t\partial y} + \dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu\dfrac{\partial^3 w}{\partial y^3}.$

This equation describes an *unsteady hydrodynamic boundary layer* on a flat plate; w is the stream function, x and y are the coordinates along and normal to the plate, respectively, and ν is the kinematic viscosity of the fluid. A similar equation describes an unsteady flow of a plane laminar jet out of a thin slit.

Preliminary remarks. The system of unsteady hydrodynamic boundary layer equations

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = \nu\frac{\partial^2 u_1}{\partial y^2}, \quad \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal fluid velocity components, can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x, t), t) + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx + \chi(t),$$
$$w_2 = C_1 w(C_2 x + C_2 C_3 t + C_4,\ C_1 C_2 y + C_1 C_2 C_5 t + C_6,\ C_1^2 C_2^2 t + C_7) + C_5 x - C_3 y + C_8,$$

where $\varphi(x, t)$ and $\chi(t)$ are arbitrary functions and the C_n are arbitrary constants, are also solutions of the equation.

⊙ *References*: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982).

2°. Degenerate solutions linear and quadratic in y:

$$w = C_1 y + \varphi(x, t),$$
$$w = C_1 y^2 + \varphi(x, t)y + \frac{1}{4C_1}\varphi^2(x, t) + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx,$$

where $\varphi(x, t)$ is an arbitrary function of two variables and C_1 is an arbitrary constant. Here and henceforth, the additive arbitrary function of time, $\chi = \chi(t)$, in exact solutions for the stream function is omitted. These solutions are independent of ν and correspond to inviscid fluid flows.

⊙ *Reference*: A. D. Polyanin (2001 *b*).

3°. Solutions involving arbitrary functions:

$$w = \frac{6\nu x + C_1}{y + \varphi(x, t)} + \frac{C_2}{[y + \varphi(x, t)]^2} + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx,$$
$$w = C_1 \exp\big[-C_2 y - C_2\varphi(x, t)\big] + C_3 y + C_3\varphi(x, t) + \nu C_2 x + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx,$$
$$w = 6\nu C_1 x^{1/3}\tanh\xi + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx, \qquad \xi = C_1\frac{y + \varphi(x, t)}{x^{2/3}},$$
$$w = -6\nu C_1 x^{1/3}\tan\xi + \frac{\partial}{\partial t}\int \varphi(x, t)\,dx, \qquad \xi = C_1\frac{y + \varphi(x, t)}{x^{2/3}},$$

where $\varphi(x, t)$ is an arbitrary function of two variables, and C_1, C_2, and C_3 are arbitrary constants. The construction of these solutions was based on the simpler, stationary solutions specified in 13.3.1.1.

Note also the solution

$$w = f(x)\exp\big[-\lambda y - \lambda g(t)\big] + \big[\nu\lambda + g_t'(t)\big]x,$$

where $f(x)$ and $g(t)$ are arbitrary functions and λ is an arbitrary constant. It can be obtained from the second of the solutions specified above with $\varphi(x, t) = -\frac{1}{\lambda}\ln f(x) + g(t)$, $C_2 = \lambda$, and $C_3 = 0$.

⊙ *References*: G. I. Burde (1995), A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001).

4°. Generalized separable solution linear in x:

$$w(x, y, t) = xF(y, t) + G(y, t), \tag{1}$$

TABLE 13.5
Exact solutions of equation (2) in 13.3.3.1

No.	Function $F = F(y,t)$ (or general form of solution)	Remarks (or determining equations)
1	$F = \psi(t)$	$\psi(t)$ is an arbitrary function
2	$F = \frac{y}{t+C_1} + \psi(t)$	$\psi(t)$ is an arbitrary function, C_1 is any
3	$F = \frac{6\nu}{y+\psi(t)} + \psi_t'(t)$	$\psi(t)$ is an arbitrary function
4	$F = C_1 \exp\big[-\lambda y + \lambda\psi(t)\big] - \psi_t'(t) + \nu\lambda$	$\psi(t)$ is an arbitrary function, C_1, λ are any
5	$F = F(\xi),\ \xi = y + \lambda t$	$\lambda F_{\xi\xi}'' + (F_\xi')^2 - FF_{\xi\xi}'' = \nu F_{\xi\xi\xi}'''$
6	$F = t^{-1/2}\big[H(\xi) - \frac{1}{2}\xi\big],\ \xi = yt^{-1/2}$	$\frac{3}{4} - 2H_\xi' + (H_\xi')^2 - HH_{\xi\xi}'' = \nu H_{\xi\xi\xi}'''$

where the functions $F = F(y,t)$ and $G = G(y,t)$ are determined from the simpler equations in two variables

$$\frac{\partial^2 F}{\partial t\partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu\frac{\partial^3 F}{\partial y^3}, \tag{2}$$

$$\frac{\partial^2 G}{\partial t\partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu\frac{\partial^3 G}{\partial y^3}. \tag{3}$$

Equation (2) is solved independently of (3). If $F = F(y,t)$ is a solution of equation (2), then the functions

$$F_1 = F(y + \psi(t), t) + \psi_t'(t),$$
$$F_2 = C_1 F(C_1 y + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

Given a particular solution $F = F(y,t)$ of equation (2), the corresponding equation (3) can be reduced, with the substitution $U = \frac{\partial G}{\partial y}$, to the second-order linear equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial F}{\partial y}U. \tag{4}$$

Table 13.5 lists exact solutions of equation (2). The ordinary differential equations in the last two rows, determining a traveling-wave solution and a self-similar one, are both autonomous and, hence, their order can be reduced.

Table 13.6 presents transformations that simplify equation (4) corresponding to respective solutions of equation (2) in Table 13.5. It is apparent that in the first three cases, solutions of equation (4) are expressed via solutions of a linear constant-coefficient heat equation. In the other three cases, equation (4) is reduced to linear equations, which can be solved by the method of separation of variables.

The fourth equation in Table 13.6 has the following particular solutions (A and B are any):

$$Z(\eta) = A + B\int \Phi(\eta)\,d\eta, \quad \Phi(\eta) = \exp\left(\frac{C_1}{\nu\lambda}e^{\eta} - \eta\right);$$

$$Z(\eta,t) = A\nu\lambda^2 t + A\int \Phi(\eta)\left[\int \frac{d\eta}{\Phi(\eta)}\right]d\eta.$$

TABLE 13.6
Transformations of equation (4) for the respective exact solutions of equation (2); the number in the first column corresponds to the number of the exact solution $F = F(y,t)$ in Table 13.5

No.	Transformation of equation (4)	Resulting equation
1	$U = u(\zeta,t),\ \zeta = y + \int \psi(t)\,dt$	$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial \zeta^2}$
2	$U = \frac{1}{t+C_1}u(z,\tau),\ \tau = \frac{1}{3}(t+C_1)^3 + C_2,$ $z = (t+C_1)y + \int \psi(t)(t+C_1)\,dt + C_3$	$\frac{\partial u}{\partial \tau} = \nu\frac{\partial^2 u}{\partial z^2}$
3	$U = \zeta^{-3}u(\zeta,t),\ \zeta = y + \psi(t)$	$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial \zeta^2}$
4	$U = e^{\eta}Z(\eta,t),\ \eta = -\lambda y + \lambda\psi(t)$	$\frac{\partial Z}{\partial t} = \nu\lambda^2\frac{\partial^2 Z}{\partial \eta^2} + (\nu\lambda^2 - C_1\lambda e^{\eta})\frac{\partial Z}{\partial \eta}$
5	$U = u(\xi,t),\ \xi = y + \lambda t$	$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial \xi^2} + \big[F(\xi) - \lambda\big]\frac{\partial u}{\partial \xi} - F'_{\xi}(\xi)u$
6	$U = t^{-1/2}u(\xi,\tau),\ \xi = yt^{-1/2},\ \tau = \ln t$	$\frac{\partial u}{\partial \tau} = \nu\frac{\partial^2 u}{\partial \xi^2} + H(\xi)\frac{\partial u}{\partial \xi} + \big[1 - H'_{\xi}(\xi)\big]u$

For other exact solutions of this equation, see the book by Polyanin (2002), where a more general equation, $\partial_t w = f(x)\partial_{xx} w + g(x)\partial_x w$, was considered.

Equation 5 in Table 13.6 has a stationary particular solution $u_0 = F'_{\xi}(\xi)$ (cf. equation 5 in Table 13.5). Consequently, other particular solutions of this equation are given by

$$u(\xi) = C_1 F'_{\xi}(\xi) + C_2 F'_{\xi}(\xi)\int \frac{\Psi(\xi)\,d\xi}{[F'_{\xi}(\xi)]^2}, \quad \Psi(\xi) = \exp\left[\frac{\lambda}{\nu}\xi - \frac{1}{\nu}\int F(\xi)\,d\xi\right];$$

$$u(\xi,t) = C_1\nu t F'_{\xi}(\xi) + C_1 F'_{\xi}(\xi)\int \frac{\Psi(\xi)\Phi(\xi)}{[F'_{\xi}(\xi)]^2}\,d\xi, \quad \Phi(\xi) = \int \frac{[F'_{\xi}(\xi)]^2}{\Psi(\xi)}\,d\xi;$$

see Polyanin (2002).

⊙ *References*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001*b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001, 2002).

Special case 1. Solution exponentially dependent on time:

$$w(x,y,t) = f(y)x + e^{-\lambda t}\int g(y)\,dy,$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - f f''_{yy} = \nu f'''_{yyy},$$
$$-\lambda g + g f'_y - f g'_y = \nu g''_{yy}.$$

Special case 2. Periodic solution:

$$w(x,y,t) = f(y)x + \sin(\lambda t)\int g(y)\,dy + \cos(\lambda t)\int h(y)\,dy,$$

where the functions $f = f(y)$, $g = g(y)$, and $h = h(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - f f''_{yy} = \nu f'''_{yyy},$$
$$-\lambda h + f'_y g - f g'_y = \nu g''_{yy},$$
$$\lambda g + f'_y h - f h'_y = \nu h''_{yy}.$$

5°. Generalized separable solution:

$$w(x,y,t) = \big[A(t)e^{k_1 x} + B(t)e^{k_2 x}\big]e^{\lambda y} + \varphi(t)x + ay,$$
$$A(t) = C_1 \exp\left[(\nu\lambda^2 - ak_1)t + \lambda\int \varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[(\nu\lambda^2 - ak_2)t + \lambda\int \varphi(t)\,dt\right],$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, a, k_1, k_2, and λ are arbitrary constants.

6°. Generalized separable solution:

$$w(x,y,t) = A(t)\exp(kx + \lambda y) + B(t)\exp(\beta kx + \beta\lambda y) + \varphi(t)x + ay,$$
$$A(t) = C_1 \exp\left[(\nu\lambda^2 - ak)t + \lambda\int \varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[(\nu\beta^2\lambda^2 - ak\beta)t + \beta\lambda\int \varphi(t)\,dt\right],$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, a, k, β, and λ are arbitrary constants.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001).

7°. "Two-dimensional" solution:

$$w(x,y,t) = \int u(z,t)\,dz + \varphi(t)y + \psi(t)x, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the second-order linear differential equation

$$\frac{\partial u}{\partial t} + \big[k\varphi(t) - \lambda\psi(t)\big]\frac{\partial u}{\partial z} = \nu\lambda^2\frac{\partial^2 u}{\partial z^2} - \frac{1}{\lambda}\varphi_t'(t).$$

The transformation

$$u = U(\xi,t) - \frac{1}{\lambda}\varphi(t), \qquad \xi = z - \int \big[k\varphi(t) - \lambda\psi(t)\big]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu\lambda^2\frac{\partial^2 U}{\partial \xi^2}.$$

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001).

8°. Solutions:

$$w = e^{\nu\lambda^2 t}(C_1 e^{\lambda z} + C_2 e^{-\lambda z}) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$
$$w = e^{-\nu\lambda^2 t}\big[C_1\sin(\lambda z) + C_2\cos(\lambda z)\big] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$
$$w = C_1 e^{-\nu\lambda^2 z}\sin(\lambda z - 2\nu\lambda^2 t + C_2) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and λ are arbitrary constants. For periodic function $\varphi(x,t) = \varphi(x,t+T)$, the last solution is also periodic, $w(x,y,t) = w(x,y,t+T)$, if $\lambda = \sqrt{\pi/(\nu T)}$.

9°. "Two-dimensional" solution:

$$w = W(\xi,\eta) + a_1 x + a_2 y, \quad \xi = k_1 x + \lambda_1 t, \quad \eta = k_2 y + \lambda_2 t,$$

where the function W is determined by the differential equation

$$(\lambda_1 + a_2 k_1)\frac{\partial^2 W}{\partial\xi\partial\eta} + (\lambda_2 - a_1 k_2)\frac{\partial^2 W}{\partial\eta^2} + k_1 k_2\left(\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta} - \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\eta^2}\right) = \nu k_2^2\frac{\partial^3 W}{\partial\eta^3}.$$

In the special case

$$a_1 = \lambda_2/k_2, \quad a_2 = -\lambda_1/k_1,$$

we have the steady boundary layer equation 13.3.1.1:

$$\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta} - \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\eta^2} = \beta\frac{\partial^3 W}{\partial\eta^3}, \qquad \beta = \nu\frac{k_2}{k_1}.$$

10°. "Two-dimensional" solution:

$$w = V(\xi,\eta), \quad \xi = \frac{x}{\sqrt{t}}, \quad \eta = \frac{y}{\sqrt{t}},$$

where the function V is determined by the differential equation

$$-\frac{1}{2}\frac{\partial V}{\partial\eta} - \frac{1}{2}\xi\frac{\partial^2 V}{\partial\xi\partial\eta} - \frac{1}{2}\eta\frac{\partial^2 V}{\partial\eta^2} + \frac{\partial V}{\partial\eta}\frac{\partial^2 V}{\partial\xi\partial\eta} - \frac{\partial V}{\partial\xi}\frac{\partial^2 V}{\partial\eta^2} = \nu\frac{\partial^3 V}{\partial\eta^3}.$$

For example, this equation has solutions of the form $V = F(\eta)\xi + G(\eta)$.

⊙ *References for equation* 13.3.3.1: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982), G. I. Burde (1995), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001; 2002, pp. 301–305; 2004, pp. 553–557).

2. $\dfrac{\partial^2 w}{\partial t\partial y} + \dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu\dfrac{\partial^3 w}{\partial y^3} + f(x,t).$

This equation describes an unsteady hydrodynamic boundary layer with pressure gradient.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x,t), t) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$

$$w_2 = -w(x,-y,t) + \psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, are also solutions of the equation.

⊙ *References*: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982).

2°. For $f(x,t) = g(t)$, the transformation

$$w = u(\xi,y,t) - h_t'(t)y, \quad \xi = x + h(t), \qquad \text{where} \quad h(t) = -\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau, \quad (1)$$

leads to a simpler equation of the form 13.3.3.1:

$$\frac{\partial^2 u}{\partial t\partial y} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial\xi\partial y} - \frac{\partial u}{\partial\xi}\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3}.$$

Note that $f = g(t)$ and $h = h(t)$ are related by the simple constraint $h_{tt}'' = -g$.

In the general case, transformation (1) brings the equation in question to a similar equation with the function $f(x,t)$ modified according to

$$f(x,t) \quad \xrightarrow{\text{transformation (1)}} \quad f(x,t) - g(t).$$

⊙ *Reference*: L. V. Ovsiannikov (1982).

3°. Degenerate solution (quadratic in y) for any $f(x,t)$:

$$w(x,y,t) = Cy^2 + \varphi(x,t)y + \frac{1}{4C}\varphi^2(x,t) + \frac{1}{2C}\int\left[\frac{\partial\varphi}{\partial t} - f(x,t)\right]dx,$$

where $\varphi(x,t)$ is an arbitrary function of two arguments and C is an arbitrary constant. From now on, the arbitrary additive function of time $\psi = \psi(t)$ is omitted in exact solutions for the stream function. These solutions are independent of ν and correspond to inviscid fluid flows.

Degenerate solution (linear in y) for any $f(x,t)$:

$$w(x,y,t) = \psi(x,t)y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function, and $\psi = \psi(x,t)$ is determined by the first-order partial differential equation

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial x} = f(x,t).$$

For information about the methods of integration of such equations, see Section 23.1.

Degenerate solutions for $f(x,t) = f(x)$:

$$w(x,y,t) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function.

4°. Generalized separable solution (linear in x) for $f(x,t) = f_1(t)x + f_2(t)$:

$$w(x,y,t) = xF(y,t) + G(y,t), \tag{2}$$

where the functions $F = F(y,t)$ and $G = G(y,t)$ are determined by the simpler equations in two variables

$$\frac{\partial^2 F}{\partial t\partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu\frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{3}$$

$$\frac{\partial^2 G}{\partial t\partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu\frac{\partial^3 G}{\partial y^3} + f_2(t). \tag{4}$$

Equation (3) is solved independently of equation (4).

If $F = F(y,t)$ is a solution to equation (3), then the function

$$F_1 = F(y + \psi(t), t) + \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

Table 13.7 lists exact solutions of equation (3) for various $f_1 = f_1(t)$; two more complicated solutions of this equation are given at the end of Item 4°. Note that, for $G \equiv 0$, solutions (2) specified in the first and the last rows of Table 13.7 were treated in the book by Ovsiannikov (1982).

The substitution $U = \frac{\partial G}{\partial y}$ brings equation (4) to the second-order linear equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial F}{\partial y}U + f_2(t). \tag{5}$$

TABLE 13.7
Exact solutions of equation (3) in 13.3.3.2 for various $f_1(t)$; $\psi(t)$ is an arbitrary function

Function $f_1 = f_1(t)$	Function $F = F(y,t)$ (or general form of solution)	Determining equation (or determining coefficients)
Any	$F = a(t)y + \psi(t)$	$a'_t + a^2 = f_1(t)$
$f_1(t) = Ae^{-\beta t}$, $A > 0$, $\beta > 0$	$F = Be^{-\frac{1}{2}\beta t}\sin[\lambda y + \lambda\psi(t)] + \psi'_t(t)$, $F = Be^{-\frac{1}{2}\beta t}\cos[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2A\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, $A > 0$, $\beta > 0$	$F = Be^{\frac{1}{2}\beta t}\sinh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2A\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, $A < 0$, $\beta > 0$	$F = Be^{\frac{1}{2}\beta t}\cosh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2\lvert A\rvert\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, A is any, $\beta > 0$	$F = \psi(t)e^{\lambda y} - \frac{Ae^{\beta t-\lambda y}}{4\lambda^2\psi(t)} + \frac{\psi'_t(t)}{\lambda\psi(t)} - \nu\lambda$	$\lambda = \pm\sqrt{\dfrac{\beta}{2\nu}}$
$f_1(t) = At^{-2}$	$F = t^{-1/2}\big[H(\xi) - \frac{1}{2}\xi\big]$, $\xi = yt^{-1/2}$	$\frac{3}{4} - A - 2H'_\xi + (H'_\xi)^2 - HH''_{\xi\xi} = \nu H'''_{\xi\xi\xi}$
$f_1(t) = A$	$F = F(\xi)$, $\xi = y + \lambda t$	$-A + \lambda F''_{\xi\xi} + (F'_\xi)^2 - FF''_{\xi\xi} = \nu F'''_{\xi\xi\xi}$

Let us dwell on the first solution to (3) specified in Table 13.7:

$$F(y,t) = a(t)y + \psi(t), \qquad \text{where} \qquad a'_t + a^2 = f_1(t). \tag{6}$$

The Riccati equation for $a = a(t)$ is reduced by the substitution $a = h'_t/h$ to the second-order linear equation $h''_{tt} - f_1(t)h = 0$. Exact solutions of this equation for various $f_1(t)$ can be found in Kamke (1977) and Polyanin and Zaitsev (2003). In particular, for $f_1(t) = \text{const}$ we have

$$a(t) = k\frac{C_1\cos(kt) - C_2\sin(kt)}{C_1\sin(kt) + C_2\cos(kt)} \quad \text{if} \quad f_1 = -k^2 < 0,$$
$$a(t) = k\frac{C_1\cosh(kt) + C_2\sinh(kt)}{C_1\sinh(kt) + C_2\cosh(kt)} \quad \text{if} \quad f_1 = k^2 > 0.$$

On substituting solution (6), with arbitrary $f_1(t)$, into equation (5), one obtains

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial y^2} + \big[a(t)y + \psi(t)\big]\frac{\partial U}{\partial y} - a(t)U + f_2(t).$$

The transformation (Polyanin, 2002)

$$U = \frac{1}{\Phi(t)}\left[u(z,\tau) + \int f_2(t)\Phi(t)\,dt\right], \quad \tau = \int \Phi^2(t)\,dt + C_1,$$
$$z = y\Phi(t) + \int \psi(t)\Phi(t)\,dt + C_2, \quad \Phi(t) = \exp\left[\int a(t)\,dt\right],$$

leads to the linear heat equation

$$\frac{\partial u}{\partial \tau} = \nu\frac{\partial^2 u}{\partial z^2}.$$

⊙ *References*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001*b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001, 2002).

Remark 1. The ordinary differential equations in the last two rows of Table 13.7 (see the last column), which determine a self-similar and a traveling-wave solution, are both autonomous and, hence, their order can be reduced.

Remark 2. Suppose $w(x, y, t)$ is a solution of the unsteady hydrodynamic boundary layer equation with $f(x, t) = f_1(t)x + f_2(t)$. Then the function

$$w_1 = w(x + h(t), y, t) - h_t'(t)y, \qquad \text{where} \quad h_{tt}'' - f_1(t)h = 0,$$

is also a solution of the equation.

⊙ *Reference*: L. V. Ovsiannikov (1982).

Remark 3. In the special case $f_2(t) = 0$, equation (4) admits a particular solution $G = G(t)$, where $G(t)$ is an arbitrary function.

Special case 1. Solution with $f(x, t) = Ax + Be^{-\lambda t}$:

$$w(x, y, t) = xg(y) + e^{-\lambda t} \int h(y)\, dy,$$

where the functions $g = g(y)$ and $h = h(y)$ are determined by the system of ordinary differential equations

$$(g')^2 - gg'' = \nu g''' + A,$$
$$-\lambda h + hg' - gh' = \nu h'' + B.$$

The prime denotes a derivative with respect to y.

Special case 2. Periodic solution with $f(x, t) = Ax + B_1 \sin(\lambda t) + B_2 \cos(\lambda t)$:

$$w(x, y, t) = xg(y) + \sin(\lambda t) \int h_1(y)\, dy + \cos(\lambda t) \int h_2(y)\, dy,$$

where the functions $g = g(y)$, $h_1 = h_1(y)$, and $h_2 = h_2(y)$ are determined by the system of ordinary differential equations

$$(g')^2 - gg'' = \nu g''' + A,$$
$$-\lambda h_2 + g'h_1 - gh_1' = \nu h_1'' + B_1,$$
$$\lambda h_1 + g'h_2 - gh_2' = \nu h_2'' + B_2.$$

Below are two more complex solutions of equation (3).

The solution

$$F(y, t) = -\frac{\gamma_t'}{\gamma} y + \gamma^3 \exp\left(\nu \int \frac{dt}{\gamma^2}\right)\left(A \cosh \frac{y}{\gamma} + B \sinh \frac{y}{\gamma}\right),$$

where A and B are arbitrary constants and $\gamma = \gamma(t)$ is an arbitrary function, corresponds to the right-hand side of equation (3) in the form

$$f_1(t) = -\frac{\gamma_{tt}''}{\gamma} + 2\left(\frac{\gamma_t'}{\gamma}\right)^2 + (B^2 - A^2)\gamma^4 \exp\left(2\nu \int \frac{dt}{\gamma^2}\right).$$

The solution

$$F(y, t) = -\frac{\gamma_t'}{\gamma} y + \gamma^3 \exp\left(-\nu \int \frac{dt}{\gamma^2}\right)\left(A \cos \frac{y}{\gamma} + B \sin \frac{y}{\gamma}\right),$$

where A and B are arbitrary constants and $\gamma = \gamma(t)$ is an arbitrary function, corresponds to the right-hand side of equation (3) in the form

$$f_1(t) = -\frac{\gamma_{tt}''}{\gamma} + 2\left(\frac{\gamma_t'}{\gamma}\right)^2 + (A^2 + B^2)\gamma^4 \exp\left(-2\nu \int \frac{dt}{\gamma^2}\right).$$

This solution was obtained in Burde (1995) for the case $A = 0$.

5°. Generalized separable solution for $f(x,t) = g(x)e^{\beta t}$, $\beta > 0$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} + \psi(x,t)e^{-\lambda y} + \frac{1}{\lambda}\frac{\partial}{\partial t}\int \ln|\varphi(x,t)|\,dx - \nu\lambda x,$$
$$\psi(x,t) = -\frac{e^{\beta t}}{2\lambda^2\varphi(x,t)}\int g(x)\,dx, \qquad \lambda = \pm\sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

6°. Generalized separable solutions for $f(x,t) = g(x)e^{\beta t}$, $\beta > 0$:

$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\left(\tfrac{1}{2}\beta t\right)\sqrt{\psi(x)}\,\sinh\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\left(\tfrac{1}{2}\beta t\right)\sqrt{\psi(x)}\,\cosh\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$\psi(x) = 2\int g(x)\,dx + C_1, \qquad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

7°. Generalized separable solutions for $f(x,t) = g(x)e^{-\beta t}$, $\beta > 0$:

$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\left(-\tfrac{1}{2}\beta t\right)\sqrt{\psi(x)}\,\sin\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\left(-\tfrac{1}{2}\beta t\right)\sqrt{\psi(x)}\,\cos\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$\psi(x) = 2\int g(x)\,dx + C_1, \qquad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

8°. Solution for $f(x,t) = xg(t)$:

$$w(x,y,t) = \frac{\psi_t'}{\psi}xy + \left(\frac{2\psi_t'}{\psi^2} - \nu\psi\right)x + \varphi(z)\exp(\psi y), \qquad z = \frac{x}{\psi}, \qquad \psi = \psi(t),$$

where $\varphi(z)$ is an arbitrary function and the function $\psi = \psi(t)$ is determined by the second-order linear ordinary differential equation

$$\psi_{tt}'' = g(t)\psi.$$

⊙ *Reference*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000).

9°. Generalized separable solution for $f(x,t) = ae^{\beta x-\gamma t}$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} - \frac{a}{2\beta\lambda^2\varphi(x,t)}e^{\beta x-\lambda y-\gamma t} + \frac{1}{\lambda}\frac{\partial}{\partial t}\int \ln|\varphi(x,t)|\,dx - \nu\lambda x + \frac{2\nu\lambda^2+\gamma}{\beta}\left(y + \frac{1}{\lambda}\ln|\varphi(x,t)|\right),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments and λ is an arbitrary constant.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

10°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = \int u(z,t)\,dz + \varphi(t)y + \psi(t)x, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the second-order linear equation

$$\frac{\partial u}{\partial t} + \big[k\varphi(t) - \lambda\psi(t)\big]\frac{\partial u}{\partial z} = \nu\lambda^2\frac{\partial^2 u}{\partial z^2} - \frac{1}{\lambda}\varphi_t'(t) + \frac{1}{\lambda}f(t).$$

The transformation

$$u = U(\xi,t) - \frac{1}{\lambda}\varphi(t) + \frac{1}{\lambda}\int f(t)\,dt, \qquad \xi = z - \int\big[k\varphi(t) - \lambda\psi(t)\big]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu\lambda^2\frac{\partial^2 U}{\partial \xi^2}.$$

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

11°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = Ce^{-\lambda y+\lambda\varphi(x,t)} - a(t)\varphi(x,t) - \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + a(t)y + \nu\lambda x, \qquad a(t) = \int f(t)\,dt,$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C and λ are arbitrary constants.

12°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} + \psi(x,t)e^{-\lambda y} + \chi(x,t) + a(t)y,$$

where λ is any, $\varphi(x,t)$ is an arbitrary function of two arguments, and the remaining functions are given by

$$\psi(x,t) = \frac{C\nu e^{2\nu\lambda^2 t}}{\varphi(x,t)}\left[x - \int a(t)\,dt\right], \qquad a(t) = \int f(t)\,dt + Ce^{2\nu\lambda^2 t},$$
$$\chi(x,t) = \frac{1}{\lambda}a(t)\ln|\varphi(x,t)| + \frac{1}{\lambda}\frac{\partial}{\partial t}\int \ln|\varphi(x,t)|\,dx - \nu\lambda x.$$

TABLE 13.8
Solutions of the unsteady hydrodynamic boundary layer equation that depends on two generalized variables. Notation: $\mathcal{R}[z] = \nu z_{\eta\eta\eta} + z_\xi z_{\eta\eta} - z_\eta z_{\xi\eta}$ and $g = g(u)$ is an arbitrary function.

Function $f = f(x,t)$	General form of solution	Equation for $z = z(\xi,\eta)$
$f = f(x+\lambda t)$	$w = z(\xi,y) - \lambda y,\ \xi = x + \lambda t$	$\nu z_{yyy} + z_\xi z_{yy} - z_y z_{\xi y} + f(\xi) = 0$
$f = g(x)t^{-2}$	$w = z(x,\eta)t^{-1/2},\ \eta = yt^{-1/2}$	$\nu z_{\eta\eta\eta} + z_x z_{\eta\eta} - z_\eta z_{x\eta} + \frac{1}{2}\eta z_{\eta\eta} + z_\eta + g(x) = 0$
$f = e^{\lambda t} g(xe^{-\lambda t})$	$w = e^{\lambda t} z(\xi,y),\ \xi = xe^{-\lambda t}$	$\nu z_{yyy} + z_\xi z_{yy} - z_y z_{\xi y} + \lambda\xi z_{\xi y} - \lambda z_y + g(\xi) = 0$
$f = t^{-n-2} g(xt^n)$	$w = z(\xi,\eta)t^{-(2n+1)/2},$ $\xi = xt^n,\ \eta = yt^{-1/2}$	$\mathcal{R}[z] + \frac{1}{2}\eta z_{\eta\eta} - n\xi z_{\xi\eta} + (1+n)z_\eta + g(\xi) = 0$
$f = ax^n$	$w = z(\xi,\eta)t^{-(n+3)/(2n-2)},$ $\xi = xt^{2/(n-1)},\ \eta = yt^{-1/2}$	$\mathcal{R}[z] + \frac{1}{2}\eta z_{\eta\eta} - \frac{2\xi}{n-1} z_{\xi\eta} + \frac{n+1}{n-1} z_\eta + a\xi^n = 0$
$f = ae^{\lambda x}$	$w = z(\xi,\eta)t^{-1/2},$ $\xi = x + \frac{2}{\lambda}\ln t,\ \eta = yt^{-1/2}$	$\mathcal{R}[z] + \frac{1}{2}\eta z_{\eta\eta} - \frac{2}{\lambda} z_{\xi\eta} + z_\eta + ae^{\lambda\xi} = 0$

13°. Solutions for $f(x,t) = f(t)$:

$$w = e^{\nu\lambda^2 t}(C_1 e^{\lambda z} + C_2 e^{-\lambda z}) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \qquad z = y + \varphi(x,t);$$

$$w = e^{-\nu\lambda^2 t}\big[C_1\sin(\lambda z) + C_2\cos(\lambda z)\big] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \quad z = y + \varphi(x,t);$$

$$w = C_1 e^{-\lambda z}\sin(\lambda z - 2\nu\lambda^2 t + C_2) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and λ are arbitrary constants. For periodic function $f(t) = f(t+T)$ satisfying the condition $\int_0^T f(t)\,dt = 0$, the last solution is also periodic, $w(x,y,t) = w(x,y,t+T)$, if $\varphi(x,t) = \varphi(x)$ and $\lambda = \sqrt{\pi/(\nu T)}$.

14°. Solutions for $f(x,t) = A$:

$$w = -\frac{A}{6\nu}z^3 + C_2 z^2 + C_1 z + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$

$$w = kx + C_1\exp\left(-\frac{k}{\nu}z\right) - \frac{A}{2k}z^2 + C_2 z + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and k are arbitrary constants.

15°. Table 13.8 presents solutions of the unsteady hydrodynamic boundary layer equation with pressure gradient that depends on two generalized variables (used results of group-theoretic analyses in Ovsiannikov, 1982).

For $f(x,t) = f(k_1 x + \lambda_1 t)$, there is also a wide class of "two-dimensional" solutions with the form

$$w = z(\xi,\eta) + a_1 x + a_2 y, \qquad \xi = k_1 x + \lambda_1 t, \qquad \eta = k_2 y + \lambda_2 t,$$

where the function z is determined by the differential equation

$$(\lambda_1 + a_2 k_1)\frac{\partial^2 z}{\partial\xi\partial\eta} + (\lambda_2 - a_1 k_2)\frac{\partial^2 z}{\partial\eta^2} + k_1 k_2\left(\frac{\partial z}{\partial\eta}\frac{\partial^2 z}{\partial\xi\partial\eta} - \frac{\partial z}{\partial\xi}\frac{\partial^2 z}{\partial\eta^2}\right) = \nu k_2^2\frac{\partial^3 z}{\partial\eta^3} + f(\xi).$$

16°. For

$$f(x,t) = a'(t)X^{-1/3} - \tfrac{1}{3}a^2(t)X^{-5/3} - b''(t), \qquad X = x + b(t),$$

where $a(t)$ and $b(t)$ are some functions, a solution is given by (Burde, 1995)

$$w = [a(t)X^{-1/3} - b'(t)]y + 6\nu X y^{-1}.$$

⊙ *References for equation* 13.3.3.2: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982), G. I. Burde (1995), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001; 2002, pp. 305–310; 2004, pp. 557–563).

3. $\dfrac{\partial^2 w}{\partial z\partial t} + \dfrac{\partial w}{\partial z}\dfrac{\partial^2 w}{\partial x\partial z} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial z^2} = \nu\dfrac{\partial}{\partial z}\left(z\dfrac{\partial^2 w}{\partial z^2}\right) + f(x,t).$

Preliminary remarks. The system of *axisymmetric unsteady laminar boundary layer equations*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + f(x,t), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{2}$$

where u and v are the axial and radial components of the fluid velocity, respectively, and x and r the axial and radial coordinates, is reduced to the equation in question by the introduction of a stream function w and a new variable z such that

$$u = \frac{2}{r}\frac{\partial w}{\partial r}, \quad v = -\frac{2}{r}\frac{\partial w}{\partial x}, \quad z = \frac{1}{4}r^2.$$

System (1), (2) describes an axisymmetric jet ($f \equiv 0$) and a boundary layer on an extensive body of revolution ($f \not\equiv 0$).

1°. The equation remains the same under the replacement of w by $w + \varphi(t)$, where $\varphi(t)$ is an arbitrary function.

2°. Generalized separable solution (quadratic in z) for arbitrary $f(x,t)$:

$$w(x,z,t) = Cz^2 + \varphi(x,t)z + \frac{1}{4C}\varphi^2(x,t) + \frac{1}{2C}\frac{\partial}{\partial t}\int \varphi(x,t)\,dx - \frac{1}{2C}\int f(x,t)\,dx - \nu x + \psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions and C is an arbitrary constant.

The equation also has an "inviscid" solution of the form $w = \varphi(x,t)z + \psi(x,t)$, where $\psi(x,t)$ is an arbitrary function, and the function $\varphi = \varphi(x,t)$ is described by the first-order partial differential equation $\partial_t\varphi + \varphi\partial_x\varphi = f(x,t)$.

3°. Generalized separable solution (linear in x) for $f(x,t) = a(t)x + b(t)$:

$$w(x,z,t) = x\varphi(z,t) + \psi(z,t),$$

where the functions $\varphi = \varphi(z,t)$ and $\psi = \psi(z,t)$ are described by the system of partial differential equations

$$\frac{\partial^2\varphi}{\partial z\partial t} + \left(\frac{\partial\varphi}{\partial z}\right)^2 - \varphi\frac{\partial^2\varphi}{\partial z^2} = \nu\frac{\partial}{\partial z}\left(z\frac{\partial^2\varphi}{\partial z^2}\right) + a(t),$$

$$\frac{\partial^2\psi}{\partial z\partial t} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z} - \varphi\frac{\partial^2\psi}{\partial z^2} = \nu\frac{\partial}{\partial z}\left(z\frac{\partial^2\psi}{\partial z^2}\right) + b(t).$$

The first equation has an exact solution $\varphi = C(t)z$, where the function $C = C(t)$ is determined by the Riccati equation $C_t' + C^2 = a(t)$. The second equation is reduced by the change of variable $V = \frac{\partial\psi}{\partial z}$ to a second-order linear equation.

4°. "Two-dimensional" solution for $f(x,t) = f(x+\lambda t)$:

$$w(x,z,t) = U(\xi,z) - \lambda z, \quad \xi = x + \lambda t,$$

where the function $U = U(\xi, z)$ is determined by the differential equation

$$\frac{\partial U}{\partial z}\frac{\partial^2 U}{\partial \xi \partial z} - \frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial z^2} = \nu \frac{\partial}{\partial z}\left(z\frac{\partial^2 U}{\partial z^2}\right) + f(\xi),$$

which coincides, up to renaming, with the stationary equation (see equation 13.3.1.3 and its solutions).

5°. Generalized separable solution (linear in x) for $f(x,t) = f(t)$:

$$w(x,z,t) = A(t)x + B(t) + z\int f(t)\,dt + u(z,t),$$

where $A(t)$ and $B(t)$ are arbitrary functions, and the function $u = u(z,t)$ is determined by the second-order linear parabolic differential equation

$$\frac{\partial u}{\partial t} - A(t)\frac{\partial u}{\partial z} = \nu z \frac{\partial^2 u}{\partial z^2}.$$

6°. Suppose $w(x,z,t)$ is a solution of the unsteady axisymmetric boundary layer equation with $f(x,t) = a(t)x + b(t)$. Then the function

$$w_1 = w(\xi, z, t) - \varphi_t'(t)z + \psi(t), \quad \xi = x + \varphi(t),$$

where $\psi(t)$ is an arbitrary function and $\varphi = \varphi(t)$ is a solution of the linear ordinary differential equation $\varphi_{tt}'' - a(t)\varphi = 0$, is also a solution of the equation.

⊙ *Reference for equation* 13.3.3.3: A. D. Polyanin and V. F. Zaitsev (2002, pp. 310–311).

13.3.4. Unsteady Boundary Layer Equations for Non-Newtonian Fluids

1. $$\frac{\partial^2 w}{\partial t \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = k\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3}.$$

This equation describes an unsteady boundary layer on a flat plate in a power-law fluid flow; w is the steam function, and x and y are coordinates along and normal to the plate.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(C_1^{n-2}C_2^{2n-1}x + C_1^{n-2}C_2^{2n-1}C_3 t,\ C_2 y + C_2 C_5 t,\ C_1^{n-1}C_2^{2n}t) + C_5 x - C_3 y,$$
$$w_2 = w(x + C_6, y + C_7, t + C_8) + C_9,$$
$$w_3 = w\big(x, y + \varphi(x,t), t\big) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + \psi(t),$$

where the C_n are arbitrary constants and $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x,y,t)=\psi(t)x+\int U(z,t)\,dz,\quad z=y+\int \psi(t)\,dt,$$

where $\psi(t)$ is an arbitrary function, and the function $U(z,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t}=k\left(\frac{\partial U}{\partial z}\right)^{n-1}\frac{\partial^2 U}{\partial z^2}.$$

For details about this equation, see 5.6.18.2 with $f(x)=$ const and 5.6.18.4 with $f(U)=kU^{n-1}$.

3°. Generalized separable solution linear in x:

$$w(x,y,t)=\frac{xy}{t+C_1}+\psi(t)x+\int U(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function, C_1 is an arbitrary constant, and the function $U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t}=k\left(\frac{\partial U}{\partial y}\right)^{n-1}\frac{\partial^2 U}{\partial y^2}+\left[\frac{y}{t+C_1}+\psi(t)\right]\frac{\partial U}{\partial y}-\frac{1}{t+C}U.$$

With the transformation

$$U=\frac{1}{t+C_1}u(\zeta,\tau),\quad \tau=\tfrac{1}{3}(t+C_1)^3+C_2,\quad \zeta=(t+C_1)y+\int \psi(t)(t+C_1)\,dt+C_3$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau}=k\left(\frac{\partial u}{\partial \zeta}\right)^{n-1}\frac{\partial^2 u}{\partial \zeta^2}.$$

For details about this equation, see 5.6.18.2 with $f(x)=$ const and 5.6.18.4 with $f(U)=kU^{n-1}$.

4°. "Two-dimensional" solution:

$$w(x,y,t)=\int v(\eta,t)\,d\eta+\varphi(t)y+\psi(t)x,\qquad \eta=kx+\lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t}+\big[k\varphi(t)-\lambda\psi(t)\big]\frac{\partial v}{\partial \eta}=k\lambda^{2n}\left(\frac{\partial v}{\partial \eta}\right)^{n-1}\frac{\partial^2 v}{\partial \eta^2}-\frac{1}{\lambda}\varphi_t'(t).$$

With the transformation

$$v=R(\zeta,t)-\frac{1}{\lambda}\varphi(t),\quad \zeta=\eta-\int\big[k\varphi(t)-\lambda\psi(t)\big]\,dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t}=k\lambda^{2n}\left(\frac{\partial R}{\partial \zeta}\right)^{n-1}\frac{\partial^2 R}{\partial \zeta^2}.$$

⊙ *Reference for equation* 13.3.4.1: A. D. Polyanin and V. F. Zaitsev (2002, pp. 311–312).

2. $\dfrac{\partial^2 w}{\partial t\partial y}+\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=\dfrac{\partial}{\partial y}\left[f\left(\dfrac{\partial^2 w}{\partial y^2}\right)\right].$

This equation describes an unsteady boundary layer on a flat plate in a non-Newtonian fluid flow; w is the stream function, and x and y are coordinates along and normal to the plate.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y+\varphi(x,t), t)+\frac{\partial}{\partial t}\int \varphi(x,t)\,dx+\psi(t),$$

$$w_2 = C_1^{-2} w(C_1^3 x + C_1^3 C_2 t + C_3,\ C_1 y + C_1 C_4 t + C_5,\ C_1^2 t + C_6) + C_4 x - C_2 y + C_7,$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions and the C_n are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x,y,t)=\psi(t)x+\int U(z,t)\,dz,\quad z=y+\int \psi(t)\,dt,$$

where $\psi(t)$ is an arbitrary function and the function $U(z,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial z}\left[f\left(\frac{\partial U}{\partial z}\right)\right].$$

It admits, for any $f=f(v)$, exact solutions of the following forms:

$$U(z,t)=H(\zeta),\quad \zeta=kz+\lambda t \quad\Longrightarrow\quad \text{equation}\quad \lambda H=kf(kH'_\zeta)+C;$$

$$U(z,t)=az+H(\zeta),\quad \zeta=kz+\lambda t \quad\Longrightarrow\quad \text{equation}\quad \lambda H=kf(kH'_\zeta+a)+C;$$

$$U(z,t)=\sqrt{t}\,H(\zeta),\quad \zeta=z/\sqrt{t} \quad\Longrightarrow\quad \text{equation}\quad \tfrac{1}{2}H-\tfrac{1}{2}\zeta H'_\zeta=[f(H'_\zeta)]'_\zeta,$$

where a, k, C, and λ are arbitrary constants. Solutions of the first two equations with $H=H(\zeta)$ can be obtained in parametric form; see Kamke (1977) and Polyanin and Zaitsev (2003).

3°. Generalized separable solution linear in x:

$$w(x,y,t)=\frac{xy}{t+C}+\psi(t)x+\int U(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function, C is an arbitrary constant, and the function $U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial U}{\partial y}\right)\right]+\left[\frac{y}{t+C}+\psi(t)\right]\frac{\partial U}{\partial y}-\frac{1}{t+C}U.$$

With the transformation

$$U=\frac{1}{t+C_1}u(\zeta,\tau),\quad \tau=\tfrac{1}{3}(t+C_1)^3+C_2,\quad \zeta=(t+C_1)y+\int \psi(t)(t+C_1)\,dt+C_3$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau}=\frac{\partial}{\partial \zeta}\left[f\left(\frac{\partial u}{\partial \zeta}\right)\right].$$

For details about this equation, see Item 2°.

4°. "Two-dimensional" solution:

$$w(x,y,t)=\int v(\eta,t)\,d\eta+\varphi(t)y+\psi(t)x,\qquad \eta=kx+\lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t}+\big[k\varphi(t)-\lambda\psi(t)\big]\frac{\partial v}{\partial \eta}=\frac{\partial}{\partial \eta}\left[f\left(\lambda^2\frac{\partial v}{\partial \eta}\right)\right]-\frac{1}{\lambda}\varphi_t'(t).$$

With the transformation

$$v=R(\zeta,t)-\frac{1}{\lambda}\varphi(t),\qquad \zeta=\eta-\int\big[k\varphi(t)-\lambda\psi(t)\big]\,dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t}=\frac{\partial}{\partial \zeta}\left[f\left(\lambda^2\frac{\partial R}{\partial \zeta}\right)\right].$$

⊙ *References for equation* 13.3.4.2: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 312–313).

3. $\dfrac{\partial^2 w}{\partial t\partial y}+\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=\dfrac{\partial}{\partial y}\left[f\left(\dfrac{\partial^2 w}{\partial y^2}\right)\right]+g(x,t).$

This is an unsteady boundary layer equation for a non-Newtonian fluid with pressure gradient.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1=w(x,y+\varphi(x,t),t)+\frac{\partial}{\partial t}\int\varphi(x,t)\,dx+\psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, is also a solution of the equation.

2°. There are degenerate solutions; see Item 3° in 13.3.3.2, where $f(x,t)$ should be substituted by $g(x,t)$.

3°. For $g(x,t)=g(t)$, the transformation

$$w=u(\xi,y,t)-\varphi_t'(t)y,\qquad \xi=x+\varphi(t),\qquad \text{where}\quad \varphi(t)=-\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

leads to a simpler equation of the form 13.3.4.2:

$$\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial \xi\partial y}-\frac{\partial u}{\partial \xi}\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 u}{\partial y^2}\right)\right].$$

Note that $g=g(t)$ and $\varphi=\varphi(t)$ are related by the simple equation $\varphi_{tt}''=-g$.

4°. "Two-dimensional" solution (linear in x) for $g(x,t) = g(t)$:

$$w(x,y,t) = a(t)x + \int U(y,t)\,dy,$$

where the function $U = U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t} - a(t)\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial U}{\partial y}\right)\right] + g(t).$$

With the transformation

$$U = u(\xi,t) + \int g(t)\,dt, \qquad \xi = y + \int a(t)\,dt$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial u}{\partial y}\right)\right].$$

For details about this equation, see 13.3.4.2, Item 2°.

5°. "Two-dimensional" solution (linear in x) for $g(x,t) = s(t)x + h(t)$:

$$w(x,y,t) = \big[a(t)y + \psi(t)\big]x + \int Q(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function and $a = a(t)$ is determined by the Riccati equation

$$a'_t + a^2 = s(t),$$

and the function $Q = Q(y,t)$ satisfies the second-order equation

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial Q}{\partial y}\right)\right] + \big[a(t)y + \psi(t)\big]\frac{\partial Q}{\partial y} - a(t)Q + h(t).$$

With the transformation

$$Q = \frac{1}{\Phi(t)}\left[Z(\xi,\tau) + \int h(t)\Phi(t)\,dt\right], \quad \tau = \int \Phi^2(t)\,dt + A, \quad \xi = y\Phi(t) + \int \psi(t)\Phi(t)\,dt + B,$$

where $\Phi(t) = \exp\left[\int a(t)\,dt\right]$, one arrives at the simpler equation

$$\frac{\partial Z}{\partial \tau} = \frac{\partial}{\partial \xi}\left[f\left(\frac{\partial Z}{\partial \xi}\right)\right].$$

For details about this equation, see 13.3.4.2, Item 2°.

6°. "Two-dimensional" solution for $g(x,t) = g(t)$:

$$w(x,y,t) = \int v(\eta,t)\,d\eta + \varphi(t)y + \psi(t)x, \qquad \eta = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t} + \big[k\varphi(t) - \lambda\psi(t)\big]\frac{\partial v}{\partial \eta} = \frac{\partial}{\partial \eta}\left[f\left(\lambda^2\frac{\partial v}{\partial \eta}\right)\right] - \frac{1}{\lambda}\varphi_t'(t) + \frac{1}{\lambda}g(t).$$

With the transformation

$$v = R(\zeta,t) - \frac{1}{\lambda}\varphi(t) + \frac{1}{\lambda}\int g(t)\,dt, \qquad \zeta = \eta - \int \big[k\varphi(t) - \lambda\psi(t)\big]\,dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial \zeta}\left[f\left(\lambda^2\frac{\partial R}{\partial \zeta}\right)\right].$$

⊙ *References for equation* 13.3.4.3: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 313–314).

13.3.5. Related Equations

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1(x,y) = C_1 w(x, C_1 y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solutions linear and quadratic in y:

$$w(x,y) = C_1 y + \varphi(x),$$
$$w(x,y) = C_1 y^2 + \varphi(x)y + \frac{1}{4C_1}\varphi^2(x) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function.

3°. Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} - \lambda\int f(x)\,dx + C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

4°. Generalized separable solution:

$$w(x,y)=\varphi(y)\int f(x)\,dx+\psi(y),$$

where the functions $\varphi=\varphi(y)$ and $\psi=\psi(y)$ are determined by the autonomous system of ordinary differential equations

$$(\varphi'_y)^2-\varphi\varphi''_{yy}=\varphi'''_{yyy},$$
$$\varphi'_y\psi'_y-\varphi\psi''_{yy}=\psi'''_{yyy}.$$

For exact solutions of this system, see 13.3.1.1, Item 5° [equations (2)–(3) with $\nu=1$].

5°. Generalized self-similar solution:

$$w(x,y)=\varphi(x)U(z),\qquad z=\psi(x)y$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(x)$, and $U=U(z)$ are determined by the system of ordinary differential equations

$$(\varphi\psi)'_x=C_1f(x)\psi^2,$$
$$\varphi'_x=C_2f(x)\psi,$$
$$C_1(U'_z)^2-C_2UU''_{zz}=U'''_{zzz}.$$

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=f(y)\dfrac{\partial^3 w}{\partial y^3}+g(y)x+h(y).$

Generalized separable solution linear in x:

$$w=\varphi(y)x+\psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$f\varphi'''_{yyy}+\varphi\varphi''_{yy}-(\varphi'_y)^2+g=0,$$
$$f\psi'''_{yyy}+\varphi\psi''_{yy}-\varphi'_y\psi'_y+h=0.$$

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=\dfrac{\partial}{\partial y}\left[f(y)\dfrac{\partial^2 w}{\partial y^2}\right]+g(y)x+h(y).$

Generalized separable solution linear in x:

$$w=\varphi(y)x+\psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$(f\varphi''_{yy})'_y+\varphi\varphi''_{yy}-(\varphi'_y)^2+g=0,$$
$$(f\psi''_{yy})'_y+\varphi\psi''_{yy}-\varphi'_y\psi'_y+h=0.$$

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=f(x)\Big(\dfrac{\partial^2 w}{\partial y^2}\Big)^{n-1}\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1(x,y)=C_1^{2n-1}w(x,C_1^{2-n}y+\varphi(x))+C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,y)=\left[(2-n)\int f(x)\,dx+C\right]^{\frac{1}{2-n}}\theta(y),$$

where C is an arbitrary constant and the function $\theta=\theta(y)$ is determined by the autonomous ordinary differential equation

$$(\theta'_y)^2-\theta\theta''_{yy}=(\theta''_{yy})^{n-1}\theta'''_{yyy}.$$

3°. Generalized traveling-wave solution:

$$w=U(z),\qquad z=y\left[\int f(x)\,dx+C\right]^{\frac{1}{1-2n}}+\varphi(x),$$

where $\varphi(x)$ is an arbitrary function and the function $U=U(z)$ is determined by the autonomous ordinary differential equation

$$(U'_z)^2=(1-2n)(U''_{zz})^{n-1}U'''_{zzz}.$$

This equation can be fully integrated.

5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=F\left(x,w,\dfrac{\partial w}{\partial y},\dfrac{\partial^2 w}{\partial y^2},\dfrac{\partial^3 w}{\partial y^3}\right).$

This is a special case of equation 15.4.1.5 with $n=3$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y)=w\big(x,y+\varphi(x)\big),$$

where $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. The von Mises transformation

$$\xi=x,\quad \eta=w,\quad u(\xi,\eta)=\frac{\partial w}{\partial y},\qquad \text{where}\quad w=w(x,y),\tag{1}$$

reduces the order of the equation by one. Formulas for computing derivatives:

$$\frac{\partial w}{\partial y}=u,\quad \frac{\partial^2 w}{\partial y^2}=u\frac{\partial u}{\partial \eta},\quad \frac{\partial^3 w}{\partial y^3}=u\frac{\partial}{\partial \eta}\left(u\frac{\partial u}{\partial \eta}\right),\quad \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=u\frac{\partial u}{\partial \xi}.$$

As a result, we obtain the second-order equation

$$uu_\xi=F\big(\xi,\eta,u,uu_\eta,u(uu_\eta)_\eta\big).\tag{2}$$

Special case 1. For $F=f(x,w)w_{yyy}/w_y$, equation (2) is reduced, with the change of variable $Z=\eta^2$, to the linear second-order equation $Z_\xi=f(\xi,\eta)Z_{\eta\eta}$.

Special case 2. For $F=aw_y^3w_{yyy}$, equation (2) is reduced, with the change of variable $Z=\eta^2$, to the equation $Z_\xi=aZ^2Z_{\eta\eta}$, which can further be reduced to the linear heat equation (see equation 5.1.9.13).

13.4. Equations of Motion of Ideal Fluid (Euler Equations)

13.4.1. Stationary Equations

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w)-\dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w)=0, \qquad \Delta w=\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The stationary two-dimensional equations of motion of an ideal fluid (Euler equations)

$$u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0$$

are reduced to this equation by the introduction of a stream function, w, such that $u_1=\frac{\partial w}{\partial y}$ and $u_2=-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with the cross differentiation) from the first two equations; the third equation is then satisfied automatically.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1=C_1w(C_2x+C_3,C_2y+C_4)+C_5,$$
$$w_2=w(x\cos\alpha+y\sin\alpha,\ -x\sin\alpha+y\cos\alpha),$$

where $C_1,\dots,C_5$ and α are arbitrary constants, are also solutions of the equation.

2°. Solutions of general form:

$$w(x,y)=\varphi_1(\xi),\quad \xi=a_1x+b_1y;$$
$$w(x,y)=\varphi_2(r),\quad r=\sqrt{(x-a_2)^2+(y-b_2)^2};$$

where $\varphi_1(\xi)$ and $\varphi_2(r)$ are arbitrary functions; a_1, b_1, a_2, and b_2 are arbitrary constants.

3°. Any solutions of the linear equations

$\Delta w=0$ (Laplace equation),
$\Delta w=C$ (Poisson equation),
$\Delta w=\lambda w$ (Helmholtz equation),
$\Delta w=\lambda w+C$ (nonhomogeneous Helmholtz equation),

where C and λ are arbitrary constants, are also solutions of the original equation. For details about the Laplace, Poisson, and Helmholtz equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The solutions of the Laplace equation $\Delta w=0$ correspond to irrotational (potential) solutions of the Euler equation. Such solutions are discussed in detail in textbooks on hydrodynamics (e.g., see Sedov, 1980, and Loitsyanskiy, 1996), where the methods of the theory of functions of a complex variable are extensively used.

4°. The Jacobian of the functions w and $v=\Delta w$ appears on the left-hand side of the equation in question. The fact that the Jacobian of two functions is zero means that the two functions are functionally dependent. Hence, v must be a function of w, so that

$$\Delta w=f(w), \tag{1}$$

where $f(w)$ is an arbitrary function. Any solution of the second-order equation (1) for arbitrary $f(w)$ is a solution of the original equation.

The results of Item 3° correspond to special cases of the linear function $f(w)=\lambda w+C$. For solutions of equation (1) with some nonlinear $f=f(w)$, see 9.1.1.1, 9.2.1.1, 9.3.1.1, 9.3.2.1, 9.3.3.1, and 9.4.1.1.

5°. Additive separable solutions:

$$w(x,y) = A_1x^2 + A_2x + B_1y^2 + B_2y + C,$$
$$w(x,y) = A_1\exp(\lambda x) + A_2\exp(-\lambda x) + B_1\exp(\lambda y) + B_2\exp(-\lambda y) + C,$$
$$w(x,y) = A_1\sin(\lambda x) + A_2\cos(\lambda x) + B_1\sin(\lambda y) + B_2\cos(\lambda y) + C,$$

where A_1, A_2, B_1, B_2, C, and λ are arbitrary constants. These solutions are special cases of solutions presented in Item 3°.

6°. Generalized separable solutions:

$$w(x,y) = (Ax + B)e^{-\lambda y} + C,$$
$$w(x,y) = \big[A_1\sin(\beta x) + A_2\cos(\beta x)\big]\big[B_1\sin(\lambda y) + B_2\cos(\lambda y)\big] + C,$$
$$w(x,y) = \big[A_1\sin(\beta x) + A_2\cos(\beta x)\big]\big[B_1\sinh(\lambda y) + B_2\cosh(\lambda y)\big] + C,$$
$$w(x,y) = \big[A_1\sinh(\beta x) + A_2\cosh(\beta x)\big]\big[B_1\sin(\lambda y) + B_2\cos(\lambda y)\big] + C,$$
$$w(x,y) = \big[A_1\sinh(\beta x) + A_2\cosh(\beta x)\big]\big[B_1\sinh(\lambda y) + B_2\cosh(\lambda y)\big] + C,$$
$$w(x,y) = Ae^{\alpha x+\beta y} + Be^{\gamma x+\lambda y} + C, \quad \alpha^2 + \beta^2 = \gamma^2 + \lambda^2,$$

where A, B, C, D, k, β, and λ are arbitrary constants. These solutions are special cases of solutions presented in Item 3°.

7°. Solution:

$$w(x,y) = F(z)x + G(z), \quad z = y + kx,$$

where k is an arbitrary constant and the functions $F = F(z)$ and $G = G(z)$ are determined by the autonomous system of third-order ordinary differential equations:

$$F'_zF''_{zz} - FF'''_{zzz} = 0, \tag{2}$$
$$G'_zF''_{zz} - FG'''_{zzz} = \frac{2k}{(k^2+1)}FF''_{zz}. \tag{3}$$

On integrating the system once, we arrive at the following second-order equations:

$$(F'_z)^2 - FF''_{zz} = A_1, \tag{4}$$
$$G'_zF'_z - FG''_{zz} = \frac{2k}{k^2+1}\int FF''_{zz}\,dz + A_2, \tag{5}$$

where A_1 and A_2 are arbitrary constants.

The autonomous equation (4) can be reduced, with the change of variable $Z(F) = (F'_z)^2$, to a first-order linear equation.

The general solution of equation (2), or (4), is given by

$$F(z) = B_1z + B_2, \qquad A_1 = B_1^2;$$
$$F(z) = B_1\exp(\lambda z) + B_2\exp(-\lambda z), \qquad A_1 = -4\lambda^2B_1B_2;$$
$$F(z) = B_1\sin(\lambda z) + B_2\cos(\lambda z), \qquad A_1 = \lambda^2(B_1^2 + B_2^2),$$

where B_1, B_2, and λ are arbitrary constants.

The general solution of equation (3), or (5), is expressed as

$$G = C_1\int F\,dz - \int F\left(\int\frac{\psi\,dz}{F^2}\right)dz + C_2,$$
$$F = F(z), \quad \psi = \frac{2k}{k^2+1}\int FF''_{zz}\,dz + A_2,$$

where C_1 and C_2 are arbitrary constants.

8°. There are exact solutions of the following forms:

$$w(x,y) = x^a U(\zeta), \qquad \zeta = y/x;$$
$$w(x,y) = e^{ax} V(\rho), \qquad \rho = bx + cy;$$
$$w(x,y) = W(\zeta) + a\ln|x|, \quad \zeta = y/x,$$

where a, b, and c are arbitrary constants.

▷ *For other exact solutions, see equation 13.4.1.2.*

⊙ *References for equation* 13.4.1.1: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 314–316).

2. $$\frac{\partial w}{\partial \theta}\frac{\partial}{\partial r}(\Delta w) - \frac{\partial w}{\partial r}\frac{\partial}{\partial \theta}(\Delta w) = 0, \qquad \Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

Preliminary remarks. Equation 13.4.1.1 is reduced to this equation by passing to polar coordinates r, θ with origin at a point (x_0, y_0), where x_0 and y_0 are any, such that

$$x = r\cos\theta + x_0, \qquad y = r\sin\theta + y_0 \qquad \text{(direct transformation)},$$
$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad \tan\theta = \frac{y-y_0}{x-x_0} \qquad \text{(inverse transformation)}.$$

The radial and angular components of the fluid velocity are expressed in terms of the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$ and $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Multiplicative separable solution:

$$w(r,\theta) = r^\lambda U(\theta),$$

where the function $U = U(\theta)$ is determined by the second-order autonomous ordinary differential equation

$$U''_{\theta\theta} + \lambda^2 U = C U^{\frac{\lambda-2}{\lambda}} \qquad (\lambda \text{ and } C \text{ are any}).$$

Its general solution can be written out in implicit form. In particular, if $C = 0$, we have

$$U = A_1\sin(\lambda\theta) + A_2\cos(\lambda\theta) \quad \text{if} \quad \lambda \neq 0,$$
$$U = A_1\theta + A_2 \quad \text{if} \quad \lambda = 0.$$

To $\lambda = 0$ there corresponds a solution dependent on the angle θ only.

Special case. For $\lambda = 2$, we have the solution

$$w = r^2[A_1\sin(2\theta) + A_2\cos(2\theta) + \tfrac{1}{4}C].$$

2°. Multiplicative separable solution (generalizes the solution of Item 1°):

$$w(r,\theta) = f(r)g(\theta),$$

where the functions $f = f(r)$ and $g = g(\theta)$ are determined by the linear ordinary differential equations

$$\mathbf{L}(f) = (\beta - \lambda r^{-2})f,$$
$$g''_{\theta\theta} = \lambda g,$$

where β and λ are arbitrary constants; $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

3°. Solution:

$$w = b\theta + U(\xi), \qquad \xi = \theta + a\ln r, \tag{1}$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$abU'''_{\xi\xi\xi} = 2bU''_{\xi\xi} + 2U'_\xi U''_{\xi\xi}.$$

The onefold integration yields

$$abU''_{\xi\xi} = (U'_\xi)^2 + 2bU'_\xi + C_1, \tag{2}$$

where C_1 is an arbitrary constant. The further integration results in

$$\xi = ab\int \frac{dz}{z^2 + 2bz + C_1} + C_2, \qquad z = U'_\xi.$$

4°. Generalized separable solution linear in θ:

$$w(r,\theta) = f(r)\theta + g(r).$$

Here, the functions $f = f(r)$ and $g = g(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned} -f'_r\mathbf{L}(f) + f[\mathbf{L}(f)]'_r &= 0,\\ -g'_r\mathbf{L}(f) + f[\mathbf{L}(g)]'_r &= 0, \end{aligned} \tag{3}$$

where $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

System (3) admits first integrals, which allow us to obtain the following second-order linear ordinary differential equations for f and g:

$$\begin{aligned} \mathbf{L}(f) &= Af,\\ \mathbf{L}(g) &= Ag + B, \end{aligned} \tag{4}$$

where A and B are arbitrary constants. For $A = 0$, the solutions of equations (4) are given by

$$\begin{aligned} f(r) &= C_1\ln r + C_2,\\ g(r) &= \tfrac{1}{4}Br^2 + C_3\ln r + C_4. \end{aligned}$$

For $A \neq 0$, the solutions of equations (4) are expressed in terms of Bessel functions.

▷ *For other exact solutions, see equation 13.4.1.1.*

⊙ *References for equation* 13.4.1.2: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 316–317).

3. $\dfrac{\partial w}{\partial z}\dfrac{\partial \mathrm{E}w}{\partial r} - \dfrac{\partial w}{\partial r}\dfrac{\partial \mathrm{E}w}{\partial z} - \dfrac{2}{r}\dfrac{\partial w}{\partial z}\mathrm{E}w = 0, \qquad \mathrm{E}w = r\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial r}\right) + \dfrac{\partial^2 w}{\partial z^2}.$

Preliminary remarks. The stationary Euler equations written in cylindrical coordinates for the axisymmetric case are reduced to the equation in question by the introduction of a stream function w such that $u_r = \frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z = -\frac{1}{r}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2 + y^2}$, and u_r and u_z are the radial and axial fluid velocity components.

1°. Any function $w = w(r, z)$ that solves the second-order linear equation $\mathrm{E}w = 0$ will also be a solution of the given equation.

2°. Solutions:

$$w = \varphi(r),$$
$$w = (C_1 z^2 + C_2 z + C_3) r^2 + C_4 z + C_5,$$

where $\varphi(r)$ is an arbitrary function and $C_1, \ldots, C_5$ are arbitrary constants.

3°. Generalized separable solution linear in z:

$$w(r, z) = \varphi(r) z + \psi(r).$$

Here, $\varphi = \varphi(r)$ and $\psi = \psi(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &\varphi[\mathbf{L}(\varphi)]'_r - \varphi'_r \mathbf{L}(\varphi) - 2r^{-1}\varphi\,\mathbf{L}(\varphi) = 0, \\ &\varphi[\mathbf{L}(\psi)]'_r - \psi'_r \mathbf{L}(\varphi) - 2r^{-1}\varphi\,\mathbf{L}(\psi) = 0, \end{aligned} \tag{1}$$

where $\mathbf{L}(\varphi) = \varphi''_{rr} - r^{-1}\varphi'_r$.

System (1) admits first integrals, which allow us to obtain the following second-order linear ordinary differential equations for φ and ψ:

$$\begin{aligned} &\mathbf{L}(\varphi) = 4C_1 r^2 \varphi, \\ &\mathbf{L}(\psi) = 4C_1 r^2 \psi + 4C_2 r^2, \end{aligned} \tag{2}$$

where C_1 and C_2 are arbitrary constants. The substitution $\xi = r^2$ brings (2) to the linear constant-coefficient equations

$$\begin{aligned} &\varphi''_{\xi\xi} = C_1 \varphi, \\ &\psi''_{\xi\xi} = C_1 \psi + C_2. \end{aligned}$$

Integrating yields

$$\varphi = \begin{cases} A_1 \cosh(k\xi) + B_1 \sinh(k\xi) & \text{if } C_1 = k^2 > 0, \\ A_1 \cos(k\xi) + B_1 \sin(k\xi) & \text{if } C_1 = -k^2 < 0, \\ A_1 \xi + B_1 & \text{if } C_1 = 0, \end{cases}$$

$$\psi = \begin{cases} A_2 \cosh(k\xi) + B_2 \sinh(k\xi) - C_2/C_1 & \text{if } C_1 = k^2 > 0, \\ A_2 \cos(k\xi) + B_2 \sin(k\xi) - C_2/C_1 & \text{if } C_1 = -k^2 < 0, \\ \frac{1}{2} C_2 \xi^2 + A_2 \xi + B_2 & \text{if } C_1 = 0, \end{cases}$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

⊙ *References for equation* 13.4.1.3: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 317–318).

13.4.2. Nonstationary Equations

1. $\dfrac{\partial}{\partial t}(\Delta w) + \dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = 0, \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional nonstationary equations of an ideal incompressible fluid (Euler equations)

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\ &\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\ &\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \end{aligned}$$

are reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

For stationary solutions, see Subsection 13.4.1.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = -w(y,\ x,\ t),$$
$$w_2 = C_1 w(C_2 x + C_3,\ C_2 y + C_4,\ C_1 C_2^2 t + C_5) + C_6,$$
$$w_3 = w(x\cos\alpha + y\sin\alpha,\ -x\sin\alpha + y\cos\alpha,\ t),$$
$$w_4 = w(x\cos\beta t + y\sin\beta t,\ -x\sin\beta t + y\cos\beta t,\ t) - \tfrac{1}{2}\beta(x^2 + y^2),$$
$$w_5 = w(x + \varphi(t),\ y + \psi(t),\ t) + \psi_t'(t)x - \varphi_t'(t)y + \chi(t),$$

where $C_1, \ldots, C_6$, α, and β are arbitrary constants and $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation. Solutions of the Laplace equation $\Delta w = 0$ describe irrotational (potential) flows of an ideal incompressible fluid.

3°. Solutions of general form:

$$w(x, y, t) = Q(z) + \psi_t'(t)x - \varphi_t'(t)y, \quad z = C_1[x + \varphi(t)] + C_2[y + \psi(t)];$$
$$w(x, y, t) = Q(z) + \psi_t'(t)x - \varphi_t'(t)y, \quad z = [x + \varphi(t)]^2 + [y + \psi(t)]^2,$$

where $Q(z)$, $\varphi(t)$, and $\psi(t)$ are arbitrary functions; C_1 and C_2 are arbitrary constants.

Likewise, the formulas of Item 1° can be used to construct nonstationary solutions based on other, stationary solutions (see Subsection 13.4.1).

4°. Generalized separable solution linear in x:

$$w(x, y, t) = F(y, t)x + G(y, t), \tag{1}$$

where the functions $F(y, t)$ and $G = G(y, t)$ are determined by the system of one-dimensional third-order equations

$$\frac{\partial^3 F}{\partial t \partial y^2} + \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 F}{\partial y^3} = 0, \tag{2}$$
$$\frac{\partial^3 G}{\partial t \partial y^2} + \frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 G}{\partial y^3} = 0. \tag{3}$$

Equation (2) is solved independently of (3). If $F = F(y, t)$ is a solution of equation (2), then the functions

$$F_1 = F(y + \psi(t), t) + \psi_t'(t),$$
$$F_2 = C_1 F(C_1 y + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

Integrating (2) and (3) with respect to y yields the system of second-order equations

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = f_1(t), \tag{4}$$
$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = f_2(t), \tag{5}$$

TABLE 13.9
Solutions of equations (2) and (4)

No.	Function $F = F(y,t)$ (or general form of solutions)	Function $f_1(t)$ in equation (4)	Determining functions (of determining equation)
1	$F = \varphi(t)y + \psi(t)$	$f_1(t) = \varphi_t' + \varphi^2$	$\varphi(t)$ and $\psi(t)$ are arbitrary
2	$F = A\exp[-\lambda y - \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = 0$	$\psi(t)$ is arbitrary; A and λ are any
3	$F = A\sinh[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
4	$F = A\cosh[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = -A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
5	$F = A\sin[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
6	$F = A\cos[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
7	$F = t^{-1}H(z) + \psi_t'(t),\ z = y + \psi(t)$	$f_1(t) = At^{-2}$	$-A - H_z' + (H_z')^2 - HH_{zz}'' = 0$
8	$F = t^{-1/2}\big[H(z) - \frac{1}{2}z\big],\ z = yt^{-1/2}$	$f_1(t) = At^{-2}$	$\frac{3}{4} - A - 2H_z' + (H_z')^2 - HH_{zz}'' = 0$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions. Equation (5) is linear in G. Then the substitution

$$G = \int U\,dy - hF + h_t'y, \quad \text{where} \quad U = U(y,t), \quad F = F(y,t), \tag{6}$$

and the function $h = h(t)$ is determined by the second-order linear ordinary differential equation

$$h_{tt}'' - f_1(t)h = f_2(t), \tag{7}$$

brings (5) to the first-order linear homogeneous partial differential equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial y} = -\frac{\partial F}{\partial y}U. \tag{8}$$

Thus, whenever a particular solution of equation (2) or (4) is known, finding G is reduced to solving the linear equations (7) and (8) followed by integrating by formula (6).

Solutions of equation (2) are listed in Table 13.9. The ordinary differential equations in the last two rows can be reduced, with the substitution $H_z' = V(H)$, to first-order separable equations. Table 13.10 presents the general solutions of equation (5) that correspond to exact solutions of equation (2) in Table 13.9.

The general solution of the linear nonhomogeneous equation (7) can be obtained by the formula

$$h(t) = C_1h_1(t) + C_2h_2(t) + \frac{1}{W_0}\left[h_2(t)\int h_1(t)f_2(t)\,dt - h_1(t)\int h_2(t)f_2(t)\,dt\right], \tag{9}$$

where $h_1 = h_1(t)$ and $h_2 = h_2(t)$ are fundamental solutions of the corresponding homogeneous equation (with $f_2 \equiv 0$), and $W_0 = h_1(h_2)_t' - h_2(h_1)_t'$ is the Wronskian determinant ($W_0 = \text{const}$).

For exact solutions 2–8 in Table 13.9, one should set

$h_1 = 1$, $h_2 = t$, $W_0 = 1$ for solution 2;

$h_1 = e^{-A\lambda t}$, $h_2 = e^{A\lambda t}$, $W_0 = 2A\lambda$ for solutions 3, 5, 6;

$h_1 = \cos(A\lambda t)$, $h_2 = \sin(A\lambda t)$, $W_0 = A\lambda$ for solution 4;

$h_1 = |t|^{\frac{1}{2}-\mu}$, $h_2 = |t|^{\frac{1}{2}+\mu}$, $W_0 = 2\mu = (1+4A)^{\frac{1}{2}}$ for solutions 7, 8

in formula (9).

TABLE 13.10
Solutions of equation (5); $\Theta(\xi)$ is an arbitrary function everywhere; the number in the first column corresponds to the number of an exact solution in Table 13.9

No.	General solution of equation (5)	Notation
1	$G = \frac{1}{\Phi^2(t)}\Theta(\xi) + \frac{y}{\Phi(t)}\int f_2(t)\Phi(t)\,dt,\ \xi = y\Phi(t) + \int \psi(t)\Phi(t)\,dt$	$\Phi(t) = \exp\left[\int \varphi(t)\,dt\right]$
2	Formula (6), where $U = e^{-\lambda z}\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}e^{\lambda z}$	$z = y + \psi(t)$
3	Formula (6), where $U = \sinh(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\left\lvert\tanh\frac{\lambda z}{2}\right\rvert$	$z = y + \psi(t)$
4	Formula (6), where $U = \cosh(\lambda z)\Theta(\xi),\ \xi = t + \frac{2}{A\lambda}\arctan\left(e^{\lambda z}\right)$	$z = y + \psi(t)$
5	Formula (6), where $U = \sin(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\left\lvert\tan\frac{\lambda z}{2}\right\rvert$	$z = y + \psi(t)$
6	Formula (6), where $U = \cos(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\left\lvert\tan\left(\frac{\lambda z}{2} + \frac{\pi}{4}\right)\right\rvert$	$z = y + \psi(t)$
7	Formula (6), where $U = \Theta(\xi)H(z),\ \xi = t\exp\left[\int \frac{dz}{H(z)}\right]$	$z = y + \psi(t)$
8	Formula (6), where $U = \Theta(\xi)H(z)\exp\left[-\frac{1}{2}\int \frac{dz}{H(z)}\right],\ \xi = t\exp\left[\int \frac{dz}{H(z)}\right]$	$z = \frac{y}{\sqrt{t}}$

5°. Solution:

$$w(x, y, t) = F(\zeta, t)x + G(\zeta, t), \qquad \zeta = y + kx,$$

where the functions $F(\zeta, t)$ and $G = G(\zeta, t)$ are determined from the system of one-dimensional third-order equations

$$\frac{\partial^3 F}{\partial t\partial\zeta^2} + \frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\zeta^2} - F\frac{\partial^3 F}{\partial\zeta^3} = 0, \tag{10}$$

$$\frac{\partial^3 G}{\partial t\partial\zeta^2} + \frac{\partial G}{\partial\zeta}\frac{\partial^2 F}{\partial\zeta^2} - F\frac{\partial^3 G}{\partial\zeta^3} = \frac{2k}{k^2+1}\left(F\frac{\partial^2 F}{\partial\zeta^2} - \frac{\partial^2 F}{\partial t\partial\zeta}\right). \tag{11}$$

Integrating (10) and (11) with respect to ζ yields

$$\frac{\partial^2 F}{\partial t\partial\zeta} + \left(\frac{\partial F}{\partial\zeta}\right)^2 - F\frac{\partial^2 F}{\partial\zeta^2} = f_1(t), \tag{12}$$

$$\frac{\partial^2 G}{\partial t\partial\zeta} + \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\zeta} - F\frac{\partial^2 G}{\partial\zeta^2} = Q(\zeta, t), \tag{13}$$

where $f_1(t)$ is an arbitrary function, and the function $Q(\zeta, t)$ is given by

$$Q(\zeta, t) = -\frac{2k}{k^2+1}\frac{\partial F}{\partial t} + \frac{2k}{k^2+1}\int F\frac{\partial^2 F}{\partial\zeta^2}\,d\zeta + f_2(t), \qquad f_2(t) \text{ is any.}$$

Equation (13) is linear in G. Consequently, the substitution $U = \frac{\partial G}{\partial\zeta}$ brings it to the first-order linear equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial\zeta} = -\frac{\partial F}{\partial\zeta}U + Q(\zeta, t). \tag{14}$$

Equation (10) coincides, up to renaming, with equation (2), whose exact solutions are listed in Table 13.9. In these cases, solutions of the corresponding equation (14) can be found by quadrature.

6°. Solution [special case of a solution of the form (1)]:

$$w(x,y,t)=\exp\left[-\lambda y-\lambda\int\varphi(t)\,dt\right]\left[C_1x+C_2-C_1\int\psi(t)\,dt\right]+\varphi(t)x+\psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, and λ are arbitrary constants.

7°. Generalized separable solution:

$$w(x,y,t)=e^{-\lambda y}\left[A(t)e^{\beta x}+B(t)e^{-\beta x}\right]+\varphi(t)x+\psi(t)y,$$
$$A(t)=C_1\exp\left[-\beta\int\psi(t)\,dt-\lambda\int\varphi(t)\,dt\right],$$
$$B(t)=C_2\exp\left[\beta\int\psi(t)\,dt-\lambda\int\varphi(t)\,dt\right],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

8°. Generalized separable solution:

$$w(x,y,t)=e^{-\lambda y}\left[A(t)\sin(\beta x)+B(t)\cos(\beta x)\right]+\varphi(t)x+\psi(t)y,$$
$$A(t)=\exp\left(-\lambda\int\varphi\,dt\right)\left[C_1\sin\left(\beta\int\psi\,dt\right)+C_2\cos\left(\beta\int\psi\,dt\right)\right],$$
$$B(t)=\exp\left(-\lambda\int\varphi\,dt\right)\left[C_1\cos\left(\beta\int\psi\,dt\right)-C_2\sin\left(\beta\int\psi\,dt\right)\right],$$

where $\varphi=\varphi(t)$ and $\psi=\psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

9°. Generalized separable solutions:

$$w(x,y,t)=A(t)\exp(k_1x+\lambda_1y)+B(t)\exp(k_2x+\lambda_2y)+\varphi(t)x+\psi(t)y,$$
$$A(t)=C_1\exp\left[\lambda_1\int\varphi(t)\,dt-k_1\int\psi(t)\,dt\right],$$
$$B(t)=C_2\exp\left[\lambda_2\int\varphi(t)\,dt-k_2\int\psi(t)\,dt\right],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions; C_1 and C_2 are arbitrary constants; and k_1, λ_1, k_2, and λ_2 are arbitrary parameters related by one of the two constraints

$$k_1^2+\lambda_1^2=k_2^2+\lambda_2^2 \quad \text{(first family of solutions)},$$
$$k_1\lambda_2=k_2\lambda_1 \quad \text{(second family of solutions)}.$$

10°. Generalized separable solution:

$$w(x,y,t)=\left[C_1\sin(\lambda x)+C_2\cos(\lambda x)\right]\left[A(t)\sin(\beta y)+B(t)\cos(\beta y)\right]+\varphi(t)x,$$
$$A(t)=C_3\cos\left(\beta\int\varphi\,dt+C_4\right),\quad B(t)=C_3\sin\left(\beta\int\varphi\,dt+C_4\right),$$

where $\varphi=\varphi(t)$ is an arbitrary function and $C_1,\ldots,C_4$, λ, and β are arbitrary constants.

11°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sinh(\lambda x) + C_2\cosh(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$
$$A(t) = C_3\cos\left(\beta\int\varphi\,dt + C_4\right), \quad B(t) = C_3\sin\left(\beta\int\varphi\,dt + C_4\right),$$

where $\varphi = \varphi(t)$ is an arbitrary function and $C_1, \ldots, C_4$, λ, and β are arbitrary constants.

12°. Solution:

$$w(x,y,t) = f(z) + g(t)z + \varphi(t)x + \psi(t)y, \qquad z = kx + \lambda y + \int\big[\lambda\varphi(t) - k\psi(t)\big]\,dt,$$

where $f(z)$, $g(t)$, $\varphi(t)$, and $\psi(t)$ are arbitrary functions and k and λ are arbitrary constants.

13°. There is a "two-dimensional" solution of the form

$$w = W(\rho_1,\rho_2) + c_1x + c_2y, \qquad \rho_1 = a_1x + a_2y + a_3t, \qquad \rho_2 = b_1x + b_2y + b_3t.$$

14°. "Two-dimensional" solution:

$$w = t^{(2-k)/k}\Psi(\xi,\eta),$$
$$\xi = t^{-1/k}\big[x\cos(\lambda\ln t) - y\sin(\lambda\ln t)\big], \quad \eta = t^{-1/k}\big[x\sin(\lambda\ln t) + y\cos(\lambda\ln t)\big],$$

where k and λ are arbitrary constants and the function $\Psi(\xi,\eta)$ is determined by the differential equation

$$-\widetilde{\Delta}\Psi + \left(\frac{\partial\Psi}{\partial\eta} - \frac{1}{k}\xi - \lambda\eta\right)\frac{\partial}{\partial\xi}\widetilde{\Delta}\Psi - \left(\frac{\partial\Psi}{\partial\xi} + \frac{1}{k}\eta - \lambda\xi\right)\frac{\partial}{\partial\eta}\widetilde{\Delta}\Psi = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

15°. "Two-dimensional" solution:

$$w(x,y,t) = \frac{\varphi_t'(x^2 - y^2 + 2\varphi xy)}{2(1+\varphi^2)} + \frac{y - \varphi x}{1+\varphi^2}F(\zeta,t) - 2G(\zeta,t), \qquad \zeta = x + \varphi y,$$

where $\varphi = \varphi(t)$ is an arbitrary function and the functions $F = F(\zeta,t)$ and $G = G(\zeta,t)$ are determined by the differential equations

$$F\frac{\partial^3F}{\partial\zeta^3} - \frac{\partial F}{\partial\zeta}\frac{\partial^2F}{\partial\zeta^2} + \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial^2F}{\partial\zeta^2} + \frac{\partial^3F}{\partial\zeta^2\partial t} = 0, \tag{15}$$
$$F\frac{\partial^3G}{\partial\zeta^3} - \frac{\partial^2F}{\partial\zeta^2}\frac{\partial G}{\partial\zeta} + \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial^2G}{\partial\zeta^2} + \frac{\partial^3G}{\partial\zeta^2\partial t} = -\frac{\varphi_t'}{(1+\varphi^2)^2}\zeta\frac{\partial^2F}{\partial\zeta^2}. \tag{16}$$

Equation (15) is solved independently of equation (16). If $F = F(\zeta,t)$ is a solution to (15), then the function

$$F_1 = F(y + \sigma(t), t) - \sigma_t'(t),$$

where $\sigma(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (15) and (16) with respect to ζ yields

$$F\frac{\partial^2F}{\partial\zeta^2} - \left(\frac{\partial F}{\partial\zeta}\right)^2 + \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial F}{\partial\zeta} + \frac{\partial^2F}{\partial\zeta\partial t} = \psi_1(t),$$
$$F\frac{\partial^2G}{\partial\zeta^2} - \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\zeta} + \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial G}{\partial\zeta} + \frac{\partial^2G}{\partial\zeta\partial t} = \frac{\varphi_t'}{(1+\varphi^2)^2}\left(F - \zeta\frac{\partial F}{\partial\zeta}\right) + \psi_2(t),$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions. The change of variable $u = \frac{\partial G}{\partial \zeta}$ brings the last equation to a first-order linear equation (for known F).

Note that equation (15) admits particular solutions of the following forms:

$$F(\zeta, t) = a(t)\zeta + b(t),$$
$$F(\zeta, t) = a(t)e^{-\lambda\zeta} + \frac{a_t'(t)}{\lambda a(t)} + \frac{2\varphi\varphi_t'}{\lambda(1+\varphi^2)},$$

where $a(t)$ and $b(t)$ are arbitrary functions and λ is an arbitrary constant.

⊙ *References for equation* 13.4.2.1: A. A. Buchnev (1971), B. J. Cantwell (1978), P. J. Olver (1986), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 318–322; 2004, pp. 574–579).

2. $\dfrac{\partial Q}{\partial t} + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\dfrac{\partial Q}{\partial r} - \dfrac{1}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial Q}{\partial \theta} = 0, \qquad Q = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}.$

Preliminary remarks. Equation 13.4.4.1 is reduced to the equation in question by passing to the polar coordinate system with center at a point (x_0, y_0), where x_0 and y_0 are any, by the formulas

$$x = r\cos\theta + x_0, \qquad y = r\sin\theta + y_0 \qquad \text{(direct transformation)},$$
$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad \tan\theta = \frac{y-y_0}{x-x_0} \qquad \text{(inverse transformation)}.$$

The radial and angular components of the fluid velocity are expressed via the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$, $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Generalized separable solution linear in θ:

$$w(r, \theta, t) = f(r, t)\theta + g(r, t), \tag{1}$$

where the functions $f = f(r, t)$ and $g = g(r, t)$ satisfy the system of equations

$$\mathbf{L}(f_t) - r^{-1}f_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(f)]_r = 0, \tag{2}$$
$$\mathbf{L}(g_t) - r^{-1}g_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(g)]_r = 0. \tag{3}$$

Here, the subscripts r and t denote the corresponding partial derivatives, $\mathbf{L}(f) = r^{-1}(rf_r)_r$.

2°. For particular solutions to equation (2) of the form

$$f = \varphi(t)\ln r + \psi(t) \tag{4}$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, equation (3) can be reduced, with the change of variable $U = \mathbf{L}(g)$, to the first-order linear equation $U_t + r^{-1}fU_r = 0$. Two families of particular solutions to this equation are given by

$$U = \Theta(\zeta), \quad \zeta = r^2 - 2\int \psi(t)\,dt \qquad \text{(first family of solutions, } \varphi = 0),$$
$$U = \Theta(\zeta), \quad \zeta = \int \frac{r\,dr}{\ln r} - \int \varphi(t)\,dt \qquad \text{(second family of solutions, } \psi = 0),$$

where $\Theta(\zeta)$ is an arbitrary function. The second term in solution (1) is expressed via $U = U(r, t)$, provided the first term has the form (4), as follows:

$$g(r, t) = C_1(t)\ln r + C_2(t) + \int \Phi(r, t)\,dr, \qquad \Phi(r, t) = \frac{1}{r}\int rU(r, t)\,dr,$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions.

Remark. Equation (2) has also a solution $f = -\dfrac{r^2}{2(t+C)}$.

3°. "Two-dimensional" solution:

$$w(r,\theta,t) = Ar^2t + H(\xi,\eta), \qquad \xi = r\cos(\theta + At^2), \qquad \eta = r\sin(\theta + At^2),$$

where A is an arbitrary constant and the function $H(\xi,\eta)$ is determined by the differential equation

$$\frac{\partial H}{\partial \eta}\frac{\partial}{\partial \xi}\widetilde{\Delta}H - \frac{\partial H}{\partial \xi}\frac{\partial}{\partial \eta}\widetilde{\Delta}H + 4A = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

▷ *For other exact solutions, see equation 13.4.2.1.*

⊙ *References for equation* 13.4.2.2: A. A. Buchnev (1971), P. J. Olver (1986), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 322–323; 2004, pp. 579–580).

13.5. Other Third-Order Nonlinear Equations

13.5.1. Equations Involving Second-Order Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial x \partial t} = \alpha w \dfrac{\partial^2 w}{\partial x^2} + \beta \dfrac{\partial^3 w}{\partial x^3}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1 x + \alpha C_1 \varphi(t), C_1^2 t + C_2) + \varphi_t'(t),$$

where C_1 and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. There are exact solutions of the following forms:

$$w = U(z), \qquad z = kx + \lambda t \quad \text{traveling-wave solution;}$$
$$w = |t|^{-1/2} V(\xi), \qquad \xi = x|t|^{-1/2} \quad \text{self-similar solution.}$$

3°. The order of the equation can be reduce by one; see equation 13.5.1.8 with $a(t) = -\alpha$, $b(t) = 0$, and $F(t,u_1,u_2,u_3) = \beta u_3$.

⊙ *Reference for equation* 13.5.1.1: A. D. Polyanin and V. F. Zaitsev (2004, p. 580).

2. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = \nu \dfrac{\partial^3 w}{\partial x^3}.$

This equation occurs in fluid dynamics; see 13.3.3.1, equation (2) and 14.3.3.1, equation (4) with $f_1(t) = 0$.

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x + \psi(t), t) + \psi_t'(t),$$
$$w_2 = C_1 w(C_1 x + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = \frac{C_1 x}{C_1 t + C_2} + \psi(t),$$
$$w(x,t) = \frac{6\nu}{x + \psi(t)} + \psi_t'(t),$$
$$w(x,t) = C_1 \exp\bigl[-\lambda x + \lambda\psi(t)\bigr] - \psi_t'(t) + \nu\lambda,$$

where $\psi(t)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants. The first solution is "inviscid" (independent of ν).

3°. Traveling-wave solution (λ is an arbitrary constant):

$$w = F(z), \qquad z = x + \lambda t,$$

where the function $F(z)$ is determined by the autonomous ordinary differential equation

$$\lambda F_{zz}'' + (F_z')^2 - F F_{zz}'' = \nu F_{zzz}'''.$$

4°. Self-similar solution:

$$w = t^{-1/2}\bigl[G(\xi) - \tfrac{1}{2}\xi\bigr], \qquad \xi = xt^{-1/2},$$

where the function $G = G(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{3}{4} - 2G_\xi' + (G_\xi')^2 - GG_{\xi\xi}'' = \nu G_{\xi\xi\xi}'''.$$

The solutions of Items 3° and 4° can be generalized using the formulas of Item 1°.

5°. The equation admits order reduction; see Item 12° of equation 13.5.1.4.

⊙ *References for equation* 13.5.1.2: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 323–324).

3. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = \nu\dfrac{\partial^3 w}{\partial x^3} + f(t).$

This equation occurs in fluid dynamics; see 13.3.3.2, equation (3) and 14.3.3.1, equation (4).

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \psi(t), t) + \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution (linear in x) for any $f(t)$:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function, and the function $\varphi = \varphi(t)$ is described by the Riccati equation $\varphi_t' + \varphi^2 = f(t)$. For exact solutions of this equation, see Polyanin and Zaitsev (2003).

3°. Generalized separable solutions for $f(t) = Ae^{-\beta t}$, $A > 0$, $\beta > 0$:

$$w(x,t) = Be^{-\frac{1}{2}\beta t}\sin[\lambda x + \lambda\psi(t)] + \psi_t'(t),$$
$$w(x,t) = Be^{-\frac{1}{2}\beta t}\cos[\lambda x + \lambda\psi(t)] + \psi_t'(t), \qquad B = \pm\sqrt{\frac{2A\nu}{\beta}}, \quad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

4°. Generalized separable solution for $f(t) = Ae^{\beta t}$, $A > 0$, $\beta > 0$:

$$w(x,t) = Be^{\frac{1}{2}\beta t}\sinh[\lambda x + \lambda\psi(t)] + \psi_t'(t), \qquad B = \pm\sqrt{\frac{2A\nu}{\beta}}, \quad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

5°. Generalized separable solution for $f(t) = Ae^{\beta t}$, $A < 0$, $\beta > 0$:

$$w(x,t) = Be^{\frac{1}{2}\beta t}\cosh[\lambda x + \lambda\psi(t)] + \psi_t'(t), \qquad B = \pm\sqrt{\frac{2|A|\nu}{\beta}}, \quad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

6°. Generalized separable solution for $f(t) = Ae^{\beta t}$, A is any, $\beta > 0$:

$$w(x,t) = \psi(t)e^{\lambda x} - \frac{Ae^{\beta t-\lambda x}}{4\lambda^2\psi(t)} + \frac{\psi_t'(t)}{\lambda\psi(t)} - \nu\lambda, \qquad \lambda = \pm\sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

7°. Self-similar solution for $f(t) = At^{-2}$:

$$w(x,t) = t^{-1/2}\big[u(z) - \tfrac{1}{2}z\big], \quad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{3}{4} - A - 2u_z' + (u_z')^2 - uu_{zz}'' = \nu u_{zzz}''',$$

whose order can be reduced by one.

8°. Traveling-wave solution for $f(t) = A$:

$$w = w(\xi), \qquad \xi = x + \lambda t,$$

where the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$-A + \lambda w_{\xi\xi}'' + (w_\xi')^2 - ww_{\xi\xi}'' = \nu w_{\xi\xi\xi}''',$$

whose order can be reduced by one.

9°. The equation admits order reduction; see Item 12° of equation 13.5.1.4.

⊙ *References for equation* 13.5.1.3: V. A. Galaktionov (1995), A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 324–325).

4. $\dfrac{\partial^2 w}{\partial t\partial x}+w\dfrac{\partial^2 w}{\partial x^2}-\left(\dfrac{\partial w}{\partial x}\right)^2=\nu\dfrac{\partial^3 w}{\partial x^3}+q(t)\dfrac{\partial w}{\partial x}+p(t).$

This equation arises in fluid dynamics and describes some classes of exact solutions to the Navier–Stokes equations; see Subsection 21.5.2 (equations (12) and (42)) and Subsection 21.5.3. The functions $q = q(t)$ and $p = p(t)$ can be defined arbitrarily. If we set $q \equiv 0$ and $p(t) = -f(t)$ and substitute $-w$ for w, we obtain equation 13.5.1.3.

1°. Suppose $w = w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \psi(t), t) - \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w = a(t)x + b(t),$$

where $b(t)$ is an arbitrary function and the function $a = a(t)$ satisfies the Riccati $a'_t = a^2 + q(t)a + p(t)$.

3°. Functional separable solution:

$$w = -a'_t(t) + b(t)[x + a(t)] - \frac{6\nu}{x + a(t)}, \quad q = -4b, \quad p = b'_t + 3b^2,$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions.

4°. Generalized separable solution exponentially dependent on x:

$$w = a(t)e^{-\sigma x} + b(t), \quad p = 0, \quad q = \frac{a'_t}{a} - \sigma b - \sigma^2\nu,$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions. On choosing periodic $a(t)$ and $b(t)$, one arrives at solutions periodic in time t.

5°. Multiplicative separable solution:

$$w = a(t)(C_1 e^{\sigma x} + C_2 e^{-\sigma x}), \quad p = 4C_1C_2\sigma^2 a^2(t), \quad q = \frac{a'_t}{a} - \sigma^2\nu,$$

where $a = a(t)$ is an arbitrary function and C_1, C_2, and σ are arbitrary constants.

6°. Traveling-wave solution:

$$w = w(z), \quad z = x - \lambda t, \quad p = \text{const}, \quad q = \text{const},$$

where p, q, and λ are arbitrary constants, and the function $w = w(z)$ is determined by the ordinary differential equation

$$(w - \lambda)w''_{zz} - (w'_z)^2 = \nu w'''_{zzz} + qw'_z + p.$$

Special case 1. Monotonic traveling-wave solution:

$$w = -6\nu\sigma \tanh[\sigma(x - \lambda t) + B] + \lambda, \quad p = 0, \quad q = 8\nu\sigma^2,$$

where B, σ, and λ are arbitrary constants.

Special case 2. Unbounded periodic traveling-wave solution:

$$w = 6\nu\sigma \tan[\sigma(x - \lambda t) + B] + \lambda, \quad p = 0, \quad q = -8\nu\sigma^2,$$

where B, σ, and λ are arbitrary constants.

7°. Multiplicative separable solution periodic in x:

$$w = a(t)\sin(\sigma x + B), \quad a(t) = C\exp\Big[-\nu\sigma^2 t + \int q(t)\,dt\Big],$$
$$p = -\sigma^2 a^2(t), \quad q = q(t) \text{ is an arbitrary function,}$$

where B, C, and σ are arbitrary constants. For $q(t) = \nu\sigma^2 + \varphi_t'(t)$, where $\varphi(t)$ is a periodic function, we have a periodic solution in both arguments x and t.

8°. Functional separable solution:

$$w = \frac{a(t)}{\lambda(t)}\exp[-\lambda(t)x] + b(t) + c(t)x,$$

where the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, and $\lambda = \lambda(t)$ satisfy the system of ordinary differential equations

$$\lambda_t' = -c\lambda, \quad a_t' = (\nu\lambda^2 + q + 2c + b\lambda)a, \quad c_t' = c^2 + qc + p.$$

Three out of the six functions $a(t)$, $b(t)$, $c(t)$, $\lambda(t)$, $p(t)$, and $q(t)$ can be defined arbitrarily.

9°. Functional separable solution:

$$w = \omega(t)x + \frac{\xi(t)}{\theta(t)}\sin[\theta(t)x + a], \tag{1}$$

where a is an arbitrary constant and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, and $\theta = \theta(t)$ are determined by the system of ordinary differential equations

$$\theta_t' = -\omega\theta, \quad \omega_t' = \omega^2 + q(t)\omega + p(t) + \xi^2, \quad \xi_t' = [2\omega - \nu\theta^2 + q(t)]\xi. \tag{2}$$

The functions $\theta(t)$ and $\xi(t)$ in system (2) can be regarded as prescribed (in an arbitrary way). Then the functions $\omega(t)$, $p(t)$, and $q(t)$ are easily determinable (without integrals). To periodic $\theta(t)$ and $\xi(t)$, there corresponds a periodic solution (1).

10°. Functional separable solution:

$$w = \omega(t)x + \frac{\xi(t)}{\theta(t)}\Big[C_1 e^{\theta(t)x} + C_2 e^{-\theta(t)x}\Big],$$

where C_1 and C_2 are arbitrary constants, and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, and $\theta = \theta(t)$ are determined by the system of ordinary differential equations

$$\theta_t' = -\omega\theta, \quad \omega_t' = \omega^2 + q(t)\omega + p(t) - 4C_1C_2\xi^2, \quad \xi_t' = [2\omega + \nu\theta^2 + q(t)]\xi.$$

11°. Self-similar solution:

$$w = t^{-1/2}u(\eta), \quad \eta = xt^{-1/2}; \quad p = At^{-2}, \quad q = Bt^{-1},$$

where A and B are arbitrary constants, and the function $u = u(\eta)$ is determined by the ordinary differential equation

$$(u - \tfrac{1}{2}\eta)u''_{\eta\eta} - (u'_\eta)^2 = \nu u'''_{\eta\eta\eta} + (B+1)u'_\eta + A.$$

12°. Denote

$$\eta = \frac{\partial w}{\partial x}, \qquad \Phi = \frac{\partial^2 w}{\partial x^2}. \tag{3}$$

Moving the term $-w_x^2$ (from now on, we use short notation for the derivatives) to the right-hand side of the original equation, dividing the resulting equation by $w_{xx} = \Phi$, differentiating with respect to x, and taking into account (3), we obtain

$$\frac{\Phi_t}{\Phi} - \frac{w_{xt}\Phi_x}{\Phi^2} + \eta = \frac{\partial}{\partial x}\frac{\nu\Phi_x + \eta^2 + q(t)\eta + p(t)}{\Phi}. \tag{4}$$

Let us change in (4) from the old variables t, x, and $w = w(x,t)$ to the new variables t, η, and $\Phi = \Phi(t,\eta)$ with η and Φ defined by formulas (3). The derivatives are transformed as follows:

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = w_{xx}\frac{\partial}{\partial \eta} = \Phi\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + w_{xt}\frac{\partial}{\partial \eta}.$$

As a result, equation (4) is reduced to the second-order equation

$$\frac{\eta}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left(\frac{\eta^2 + q\eta + p}{\Phi}\right) + \nu\frac{\partial^2 \Phi}{\partial \eta^2},$$

or

$$\frac{\partial \Phi}{\partial t} + (\eta^2 + q\eta + p)\frac{\partial \Phi}{\partial \eta} = (\eta + q)\Phi + \nu\Phi^2\frac{\partial^2 \Phi}{\partial \eta^2}. \tag{5}$$

Note that in the degenerate case (inviscid fluid with $\nu = 0$), the original nonlinear second-order equation is reduced to the linear first-order equation (5), which can be integrated using the method of characteristics.

Whenever a solution to the original equation is known, formulas (3) give a solution to equation (5) in parametric form.

Suppose $\Phi = \Phi(\eta, t)$ is a solution to equation (5). Then the corresponding solution of the original equation can also be represented in parametric form:

$$x = \int \frac{ds}{\Phi(s,t)} + \psi(t), \qquad w = \int \frac{s\,ds}{\Phi(s,t)} - \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function (in the integration, t is treated as a parameter).

⊙ *References for equation* 13.5.1.4: S. N. Aristov and A. D. Polyanin (2009, 2010), A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

5. $\dfrac{\partial^2 w}{\partial t \partial x} + w\dfrac{\partial^2 w}{\partial x^2} - \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 = \nu\dfrac{\partial^3 w}{\partial x^3} + p(t).$

This equation arises in fluid dynamics and describes some classes of exact solutions to the Navier–Stokes equations; see Subsections 21.5.2 (Eq. (26)) and 21.5.3 (Eq. (26)).

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \psi(t), t) - \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w = a(t)x + b(t),$$

where $b(t)$ is an arbitrary function and the function $a = a(t)$ satisfies the Riccati equation $a'_t = \frac{1}{2}a^2 + p(t)$.

3°. Degenerate solution quadratic in x:

$$w = ax^2 + b(t)x + \frac{1}{4a}[b^2(t) - 2b'_t(t) + 2p(t)],$$

where a is an arbitrary constant ($a \neq 0$) and $b(t)$ is an arbitrary function.

4°. Solution:

$$w = -\frac{4\nu}{x + a} - a'_t, \quad p = 0,$$

where $a = a(t)$ is an arbitrary function.

5°. Traveling-wave solution:

$$w = w(z), \quad z = x - \lambda t, \quad p = \text{const},$$

where p and λ are arbitrary constants, and the function $w = w(z)$ is determined by the ordinary differential equation

$$(w - \lambda)w''_{zz} - \tfrac{1}{2}(w'_z)^2 = \nu w'''_{zzz} + p.$$

6°. Self-similar solution:

$$w = t^{-1/2}u(\eta), \quad \eta = xt^{-1/2}, \quad p = At^{-2},$$

where A is an arbitrary constant and the function $u = u(\eta)$ is determined by the ordinary differential equation

$$(u - \tfrac{1}{2}\eta)u''_{\eta\eta} - \tfrac{1}{2}(u'_\eta)^2 = \nu u'''_{\eta\eta\eta} + u'_\eta + A.$$

7°. Denote

$$\eta = \frac{\partial w}{\partial x}, \quad \Phi = \frac{\partial^2 w}{\partial x^2}. \tag{1}$$

Moving the term $-\frac{1}{2}w_x^2$ (from now on, we use short notation for the derivatives) to the right-hand side of the original equation, dividing the resulting equation by $w_{xx} = \Phi$, differentiating with respect to x, and taking into account (1), we obtain

$$\frac{\Phi_t}{\Phi} - \frac{w_{xt}\Phi_x}{\Phi^2} + \eta = \frac{\partial}{\partial x}\frac{\nu\Phi_x + \frac{1}{2}\eta^2 + q(t)\eta + p(t)}{\Phi}. \tag{2}$$

Let us change in (2) from the old variables t, x, and $w = w(x,t)$ to the new variables t, η, and $\Phi = \Phi(t,\eta)$ with η and Φ defined by formulas (1). The derivatives are transformed as follows:

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = w_{xx}\frac{\partial}{\partial \eta} = \Phi\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + w_{xt}\frac{\partial}{\partial \eta}.$$

As a result, after rearrangements, equation (2) is reduced to the second-order equation

$$\frac{\partial \Phi}{\partial t} + (\tfrac{1}{2}\eta^2 + p)\frac{\partial \Phi}{\partial \eta} = \nu\Phi^2\frac{\partial^2 \Phi}{\partial \eta^2}. \tag{3}$$

Note that in the degenerate case (inviscid fluid with $\nu = 0$), the original nonlinear second-order equation is reduced to the linear first-order equation (3), which can be integrated using the method of characteristics.

Whenever a solution to the original equation is known, formulas (1) give a solution to equation (3) in parametric form.

Suppose $\Phi = \Phi(\eta, t)$ is a solution to equation (3). Then the corresponding solution of the original equation can also be represented in parametric form:

$$x = \int \frac{ds}{\Phi(s,t)} + \psi(t), \qquad w = \int \frac{s\,ds}{\Phi(s,t)} - \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function (in the integration, t is treated as a parameter).

⊙ *References for equation* 13.5.1.5: S. N. Aristov and A. D. Polyanin (2009, 2010), A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

6. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = f(t)\dfrac{\partial^3 w}{\partial x^3}$.

1°. Suppose $w = w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + \psi(t), t) + \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized separable solutions:

$$w(x,t) = \frac{C_1 x}{C_1 t + C_2} + \varphi(t),$$
$$w(x,t) = \varphi(t)e^{-\lambda x} - \frac{\varphi_t'(t)}{\lambda\varphi(t)} + \lambda f(t),$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants. The first solution is degenerate.

3°. The order of the equation can be reduced by changing from the old variables t, x, and $w = w(x,t)$ to the new variables t, η, and $\Phi = \Phi(t,\eta)$ with η and Φ defined as $\eta = \frac{\partial w}{\partial x}$ and $\Phi = \frac{\partial^2 w}{\partial x^2}$.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^3 w}{\partial x^3} - 3\left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x^2} + 3\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x \partial t}.$

1°. Self-similar solution:

$$w = w(z), \qquad z = x + kt,$$

where the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} = w'_z(w'_z - k)^2 + C,$$

and k and C are arbitrary constants.

2°. The parametrization of this equation is given by

$$w_x = \frac{w_{yy}}{w_y^2}, \qquad w_t = -\frac{w_{yyy}}{w_y^3} + 3\frac{w_{yy}^2}{w_y^4}.$$

Any solution $w = w(x, t, y)$ of this system gives a one-parameter family of solutions of the initial partial differential equation.

3°. It is an integrable equation that possesses a degenerate Lax representation, implying the existence of infinitely many higher symmetries and corresponding invariant solutions.

⊙ *References*: A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov (1991), A. L. Martínez and A. B. Shabat (2003), R. H. Heredero, A. B. Shabat, and V. V. Sokolov (2003).

8. $\dfrac{\partial^2 w}{\partial x \partial t} + [a(t)w + b(t)x]\dfrac{\partial^2 w}{\partial x^2} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^3 w}{\partial x^3}\right).$

By changing from the old variables t, x, and $w = w(x, t)$ to the new variables t, η, and $\Phi = \Phi(t, \eta)$ with η and Φ defined as $\eta = \frac{\partial w}{\partial x}$, one arrives at the second-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi}F\left(t, \eta, \Phi, \Phi\frac{\partial \Phi}{\partial \eta}\right)\right].$$

⊙ *References*: A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov and A. D. Polyanin (2010).

9. $\dfrac{\partial^3 w}{\partial x^3} = a\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 \dfrac{\partial^2 w}{\partial x \partial y}.$

1°. The *Euler transformation* (see Subsection 26.3.4)

$$w(x, y) + u(\xi, \eta) = x\xi, \quad x = \frac{\partial u}{\partial \xi}, \quad y = \eta$$

leads to a linear constant-coefficient equation

$$\frac{\partial^3 u}{\partial \xi^3} = a\frac{\partial^2 u}{\partial \xi \partial \eta}.$$

Integrating with respect to ξ yields the nonhomogeneous heat equation

$$\frac{\partial u}{\partial \eta} = \frac{1}{a}\frac{\partial^2 u}{\partial \xi^2} + f(\eta),$$

where $f(\eta)$ is an arbitrary function.

2°. The substitution $u = w_x$ lead to a special case of a linearizable equation of the form 5.6.18.6 with $f(t, w) = 1/a$, $u_{xx} = au_x^2 u_y$.

▷ *For other equations with second-order mixed derivatives, see Sections 13.3 and 13.4.*

13.5.2. Equations Involving Third-Order Mixed Derivatives

1. $\dfrac{\partial^3 w}{\partial x^2 \partial y} = ae^{\lambda w}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w\big(C_1 x + C_2,\ C_3 y + C_4\big) + \frac{1}{\lambda}\ln(C_1^2 C_3),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x, y) = -\frac{3}{\lambda}\ln z, \qquad z = f(y)x - \frac{1}{6}a\lambda f(y)\int \frac{dy}{f^3(y)},$$

where $f(y)$ is an arbitrary function.

⊙ *Reference for equation* 13.5.2.1: A. D. Polyanin and V. F. Zaitsev (2004, pp. 582–583).

2. $\dfrac{\partial w}{\partial t} = w\dfrac{\partial w}{\partial x} + \beta\dfrac{\partial^3 w}{\partial x^2 \partial t}.$

BBM equation (Benjamin–Bona–Mahony equation). It describes long waves in dispersive systems.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm x + C_2, \pm C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solutions:

$$w = -\frac{3c}{\cosh^2\left[\frac{1}{2}\beta^{-1/2}(x - ct + B)\right]} \quad \text{if } \beta > 0,$$

$$w = \frac{3c}{\sinh^2\left[\frac{1}{2}\beta^{-1/2}(x - ct + B)\right]} \quad \text{if } \beta > 0,$$

$$w = -\frac{3c}{\cos^2\left[\frac{1}{2}(-\beta)^{-1/2}(x - ct + B)\right]} \quad \text{if } \beta < 0,$$

where B and c are arbitrary constants. The first solution is the soliton solution.

3°. Traveling-wave solutions:

$$w = -\frac{3c}{4}\left\{2 - \tanh^2\left[\frac{1}{4\sqrt{\beta}}(x - ct + B)\right] - \coth^2\left[\frac{1}{4\sqrt{\beta}}(x - ct + B)\right]\right\} \quad \text{if } \beta > 0,$$

$$w = -\frac{3c}{4}\left\{2 + \tan^2\left[\frac{1}{4\sqrt{-\beta}}(x - ct + B)\right] + \cot^2\left[\frac{1}{4\sqrt{-\beta}}(x - ct + B)\right]\right\} \quad \text{if } \beta < 0,$$

where B and c are arbitrary constants.

4°. Traveling-wave solution:

$$w(x,t) = -a + \wp\left(\frac{x-at}{2\sqrt{3a\beta}} + C_1, C_2, C_3\right),$$

where $\wp(z, C_2, C_3)$ is the Weierstrass elliptic function $\left(\wp'_z = \sqrt{4\wp^3 - C_2\wp - C_3}\right)$; a, C_1, C_2, and C_3 are arbitrary constants. See also equation 13.5.2.3, Item 2° with $a = -1$, $b = \beta$, and $k = 1$.

5°. Multiplicative separable solution:

$$w(x,t) = u(x)/t,$$

where the function $u = u(x)$ is determined by the autonomous ordinary differential equation $\beta u''_{xx} - uu'_x - u = 0$. Its solution can be written out in parametric form

$$u = \sqrt{2\beta}(\tau - \ln|\tau| + C_1)^{1/2}, \qquad x = \frac{1}{2}\sqrt{2\beta}\int \frac{d\tau}{\tau(\tau - \ln|\tau| + C_1)^{1/2}} + C_2.$$

6°. Solution:

$$w(x,t) = U(\xi)/t, \qquad \xi = x - a\ln|t|,$$

where the function $U = U(\xi)$ is determined by the autonomous ordinary differential equation $\beta(aU'''_{\xi\xi\xi} + U''_{\xi\xi}) - (U + a)U'_\xi - U = 0$.

7°. Conservation laws for $\beta = 1$:

$$\begin{aligned}
&D_t w + D_x\left(-w_{tx} - \tfrac{1}{2}w^2\right) = 0,\\
&D_t\left(\tfrac{1}{2}w^2 + \tfrac{1}{2}w_x^2\right) + D_x\left(-ww_{tx} - \tfrac{1}{3}w^3\right) = 0,\\
&D_t\left(\tfrac{1}{3}w^3\right) + D_x\left(w_t^2 - w_{tx}^2 - w^2w_{tx} - \tfrac{1}{4}w^4\right) = 0,
\end{aligned}$$

where $D_x = \frac{\partial}{\partial x}$ and $D_t = \frac{\partial}{\partial t}$.

⊙ *Reference for equation* 13.5.2.2: D. N. Peregrine (1966), T. B. Benjamin, J. L. Bona, and J. J. Mahony (1972), P. O. Olver (1979), N. H. Ibragimov (1994, pp. 194–196), A. D. Polyanin and V. F. Zaitsev (2004, p. 583).

3. $\dfrac{\partial w}{\partial t} + aw^k\dfrac{\partial w}{\partial x} - b\dfrac{\partial^3 w}{\partial x^2\partial t} = 0.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm x + C_2, \pm C_1^k t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cosh^2\left[\frac{1}{2}kb^{-1/2}(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } b > 0,$$

$$w = \left\{-\frac{c(k+1)(k+2)}{2a\sinh^2\left[\frac{1}{2}kb^{-1/2}(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } b > 0,$$

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cos^2\left[\frac{1}{2}k(-b)^{-1/2}(x - ct + B)\right]}\right\}^{1/k} \quad \text{if } b < 0,$$

where B and c are arbitrary constants. The first solution is the soliton solution.

3°. Traveling-wave solutions:

$$w=\left\{\frac{c(k+1)(k+2)}{8a}\left(2-\tanh^2\left[\frac{k}{4\sqrt{b}}\,(x-ct+B)\right]-\coth^2\left[\frac{k}{4\sqrt{b}}\,(x-ct+B)\right]\right)\right\}^{1/k}$$

if $b>0$,

$$w=\left\{\frac{c(k+1)(k+2)}{8a}\left(2+\tan^2\left[\frac{k}{4\sqrt{-b}}\,(x-ct+B)\right]+\cot^2\left[\frac{k}{4\sqrt{-b}}\,(x-ct+B)\right]\right)\right\}^{1/k}$$

if $b<0$,

where B and c are arbitrary constants.

4°. There is a multiplicative separable solution of the form $w(x,t)=t^{-1/k}\theta(x)$.

⊙ *References for equation* 13.5.2.3: W. E. Schiesser (2003, a private communication), S. Hamdi, W. H. Enright, W. E. Schiesser, and J. J. Gottlieb (2003), A. D. Polyanin and V. F. Zaitsev (2004, p. 583).

4. $\dfrac{\partial w}{\partial t}+f(w)\dfrac{\partial w}{\partial x}=\alpha\dfrac{\partial^3 w}{\partial x^2\partial t}$.

1°. Let $w=w(k_1x+\lambda_1t)$ be a traveling-wave solution of the generalized Korteweg–de Vries equation

$$\frac{\partial w}{\partial t}+\beta f(w)\frac{\partial w}{\partial x}=\gamma\frac{\partial^3 w}{\partial x^3}. \tag{1}$$

Then the original equation has the traveling-wave solutions

$$w=w(k_2x+\lambda_2t),\qquad k_2=\pm\left(\frac{k_1^3\gamma}{\alpha\lambda_1}\right)^{1/2},\qquad \lambda_2=\frac{k_1^2\gamma}{k_2\alpha\beta}. \tag{2}$$

Remark 1. It should be emphasized that the original equation and equation (1) are of principally different types (the former has a mixed derivative, w_{xxt}, and the latter has a third derivative in one variable, w_{xxx}). The above similarity between solutions to equations of different types can only occur for traveling-wave solutions.

Remark 2. In the original equation and equation (1), the parameters α and γ can be replaced by $\alpha g(w)$ and $\gamma g(w)$, respectively, where $g(w)$ is an arbitrary function.

Example. Consider the BBM equation 13.5.2.2,

$$\frac{\partial w}{\partial t}=w\frac{\partial w}{\partial x}+b\frac{\partial^3 w}{\partial x^2\partial t}, \tag{3}$$

which corresponds to $f(w)=-w$ and $\alpha=b$. As equation (1), we take the Korteweg–de Vries equation in canonical form,

$$\frac{\partial w}{\partial t}+\frac{\partial^3 w}{\partial x^3}-6w\frac{\partial w}{\partial x}=0, \tag{4}$$

which corresponds to $\beta=6$ and $\gamma=-1$. Consider the traveling-wave solution of equation (4) (see equation 13.1.1.1, Item 2.2)

$$w=-\frac{\lambda}{2\cosh^2\left[\frac{1}{2}\sqrt{\lambda}\,(x-\lambda t)+B\right]}=-\frac{\lambda}{2\cosh^2\left(k_1x+\lambda_1t+B\right)} \tag{5}$$

with the determining parameters $k_1=\frac{1}{2}\lambda^{1/2}$ and $\lambda_1=-\frac{1}{2}\lambda^{3/2}$, where $\lambda>0$ is an arbitrary constant. Substituting these values of k_1 and λ_1 into (2), we get

$$k_2=\pm\frac{1}{2\sqrt{b}},\qquad \lambda_2=\mp\frac{\lambda}{12\sqrt{b}}. \tag{6}$$

Using formulas (2) and taking into account (5) and (6), we obtain the corresponding solutions of equation (3):

$$w=-\frac{\lambda}{2\cosh^2\left(k_2x+\lambda_2t+B\right)}=-\frac{\lambda}{2}\cosh^{-2}\left(\pm\frac{x}{2\sqrt{b}}\mp\frac{\lambda t}{12\sqrt{b}}+B\right).$$

2°. For traveling-wave solutions of the equation in question with $f(w)=aw^k$, see equation 13.5.2.3.

3°. The technique outlined in Item 1° allows one to construct solutions of the equation in question with $f(w) = aw^n - bw^{2n}$ by using solutions of equation 13.1.3.3.

5. $\dfrac{\partial w}{\partial t} + aw^k\dfrac{\partial w}{\partial x} + b\dfrac{\partial^3 w}{\partial x\partial t^2} = 0.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{-k}x + C_2, \pm C_1^k t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cosh^2\left[\frac{1}{2}k(bc)^{-1/2}(x-ct+B)\right]}\right\}^{1/k} \quad \text{if } bc>0,$$

$$w = \left\{-\frac{c(k+1)(k+2)}{2a\sinh^2\left[\frac{1}{2}k(bc)^{-1/2}(x-ct+B)\right]}\right\}^{1/k} \quad \text{if } bc>0,$$

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cos^2\left[\frac{1}{2}k(-bc)^{-1/2}(x-ct+B)\right]}\right\}^{1/k} \quad \text{if } bc<0,$$

where B and c are arbitrary constants. The first solution is the soliton solution.

3°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{8a}\left(2-\tanh^2\left[\frac{k}{4\sqrt{bc}}(x-ct+B)\right]-\coth^2\left[\frac{k}{4\sqrt{bc}}(x-ct+B)\right]\right)\right\}^{1/k}$$
if $bc>0$,

$$w = \left\{\frac{c(k+1)(k+2)}{8a}\left(2+\tan^2\left[\frac{k}{4\sqrt{-bc}}(x-ct+B)\right]+\cot^2\left[\frac{k}{4\sqrt{-bc}}(x-ct+B)\right]\right)\right\}^{1/k}$$
if $bc<0$,

where B and c are arbitrary constants.

4°. There is a self-similar solution of the form $w(x,t) = x^{2/k}U(z)$, where $z = xt$.

5°. Generalized separable solution for $k = 1$:

$$w(x,t) = \frac{x+C_2}{at+C_1} + \frac{2ab}{(at+C_1)^2}.$$

⊙ *References for equation* 13.5.2.5: W. E. Schiesser (2003, a private communication), A. D. Polyanin and V. F. Zaitsev (2004, p. 584).

6. $\dfrac{\partial w}{\partial t} + f(w)\dfrac{\partial w}{\partial x} = \alpha\dfrac{\partial^3 w}{\partial x\partial t^2}.$

1°. Let $w = w(k_1x + \lambda_1 t)$ be the traveling-wave solution of the generalized Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \beta f(w)\frac{\partial w}{\partial x} = \gamma\frac{\partial^3 w}{\partial x^3}. \tag{1}$$

Then the original equation has the traveling-wave solutions

$$w = w(k_2x + \lambda_2 t), \qquad \lambda_2 = \pm\left(\frac{k_1^2\gamma}{\alpha\beta}\right)^{1/2}, \quad k_2 = \frac{k_1^3\gamma}{\alpha\lambda_1\lambda_2}. \tag{2}$$

Remark 1. It should be emphasized that the original equation and equation (1) are of principally different types (the former has a mixed derivative, w_{xtt}, and the latter has a third derivative in one variable, w_{xxx}). The above similarity between solutions to equations of different types can only occur for traveling-wave solutions.

Remark 2. In the original equation and equation (1), the parameters α and γ can be replaced by $\alpha g(w)$ and $\gamma g(w)$, respectively, where $g(w)$ is an arbitrary function.

Example. Consider the equation

$$\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x \partial t^2} + bw\frac{\partial w}{\partial x} = 0, \tag{3}$$

which corresponds to $f(w) = bw$ and $\alpha = -a$. As equation (1), we take the unnormalized Korteweg–de Vries equation 13.1.1.2,

$$\frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + bw\frac{\partial w}{\partial x} = 0, \tag{4}$$

which corresponds to $\beta = 1$ and $\gamma = -a$. Consider the traveling-wave solution of equation (4) (see equation 13.1.1.2, Item 2°)

$$w = \frac{3ac}{b\cosh^2\left[\frac{1}{2}\sqrt{c}\,(x - act) + B\right]} = \frac{3ac}{b\cosh^2\left[(k_1x + \lambda_1 t) + B\right]} \tag{5}$$

with the determining parameters $k_1 = \frac{1}{2}c^{1/2}$ and $\lambda_1 = -\frac{1}{2}ac^{3/2}$, where B and $c > 0$ are arbitrary constants. Substituting these values of k_1 and λ_1 into (2), we get

$$k_2 = \pm\frac{1}{2a\sqrt{c}}, \qquad \lambda_2 = \mp\frac{1}{2}\sqrt{c}. \tag{6}$$

Using formulas (2) and taking into account (5) and (6), we obtain the corresponding solutions of equation (3):

$$w = \frac{3ac}{b\cosh^2\left(k_2x + \lambda_2 t + B\right)} = \frac{3ac}{b}\cosh^{-2}\left(\pm\frac{x}{2a\sqrt{c}} \mp \frac{1}{2}\sqrt{c}\,t + B\right).$$

2°. For traveling-wave solutions of the equation in question with $f(w) = aw^k$, see equation 13.5.2.5.

3°. The technique outlined in Item 1° allows one to construct solutions of the equation in question with $f(w) = aw^n - bw^{2n}$ on the basis of solutions of equation 13.1.3.3.

7. $\dfrac{\partial^3 w}{\partial x^2 \partial t} = a\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial^3 w}{\partial x^3}$.

For $a = -2$ and $b = -1$, this equation describes the asymptotic behavior of nematic liquid crystals (see Hunter and Saxton, 1991).

1°. There are the traveling-wave and self-similar solutions:

$$w = w(z), \qquad z = x - \lambda t, \quad aw'_z w''_{zz} + (bw + \lambda)w'''_{zzz} = 0;$$
$$w = t^{k-1}f(\xi), \quad \xi = xt^{-k}, \qquad (af'_\xi + k + 1)f''_{\xi\xi} + (bf + k\xi)f'''_{\xi\xi\xi} = 0,$$

where λ and k are arbitrary constants.

2°. The first integral:

$$\frac{\partial^2 w}{\partial x \partial t} = \frac{a - b}{2}\left(\frac{\partial w}{\partial x}\right)^2 + bw\frac{\partial^2 w}{\partial x^2} + \psi(t), \tag{1}$$

where $\psi(t)$ is an arbitrary function.

Equation (1) is a special case of equation 11.1.1.5 and, hence, is reducible to a linear first-order equation.

3°. *Special case* $b = -3a$.

There is a generalized separable solution of the form

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3,$$

where $\varphi_2 = \varphi_2(t)$ and $\varphi_3 = \varphi_3(t)$ are arbitrary functions, and the functions $\varphi_0 = \varphi_0(t)$ and $\varphi_1 = \varphi_1(t)$ are determined by the system of linear algebraic equations

$$\varphi_2' = a\varphi_1\varphi_2 - 9a\varphi_0\varphi_3,$$
$$\varphi_3' = \tfrac{2}{3}a\varphi_2^2 - 2a\varphi_1\varphi_3.$$

4°. *Special case* $b = -a$.

There are generalized separable solutions of the form

$$w = C_1\cosh[\lambda x + \varphi(t)] + C_2\sinh[\lambda x + \varphi(t)] - \frac{1}{a\lambda}\varphi_t'(t),$$
$$w = C_1\cos[\lambda x + \varphi(t)] + C_2\sin[\lambda x + \varphi(t)] - \frac{1}{a\lambda}\varphi_t'(t),$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants.

There is a generalized separable solution of the form

$$w = -\frac{3\lambda_t'}{a\lambda^2} - \frac{\varphi_t'}{a\lambda\varphi} - \frac{\lambda_t'}{a\lambda}x + \varphi e^{\lambda x},$$

where $\varphi = \varphi(t)$ and $\lambda = \lambda(t)$ are arbitrary functions.

8. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^2 \partial t} = aw\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + cw\dfrac{\partial^3 w}{\partial x^3}.$

1°. Traveling-wave solutions for $a(b + c) < 0$:

$$w = A + B\exp\left(\pm\lambda x \pm A\frac{a\lambda + c\lambda^3}{1+\lambda^2}t\right), \quad \lambda = \sqrt{-\frac{a}{b+c}},$$
$$w = A + B\cosh\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1+\lambda^2}t + C\right),$$
$$w = A + B\sinh\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1+\lambda^2}t + C\right),$$

where A, B, and C are arbitrary constants.

2°. Traveling-wave solutions for $a(b + c) > 0$:

$$w = A + B\cos\left(\lambda x + A\frac{a\lambda - c\lambda^3}{1-\lambda^2}t\right), \quad \lambda = \sqrt{\frac{a}{b+c}},$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solution for $a = 0$, $b = -c$:

$$w = A + B\exp\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1+\lambda^2}t\right),$$

where A, B, and λ are arbitrary constants.

4°. Multiplicative separable solution:

$$w = \frac{f(x)}{t + C},$$

where the function $f = f(x)$ is determined by the ordinary differential equation

$$af f' + bf' f'' + cf f''' + f + f'' = 0.$$

5°. Generalized separable solution for $a = b + c$ (here b and c are any numbers):

$$w = \varphi_1(t)\cos x + \varphi_2(t)\sin x,$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are arbitrary functions.

6°. Generalized separable solution for $a = c$, $b = 0$:

$$w = \varphi_1(t)\cos x + \varphi_2(t)\sin x + C,$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are arbitrary functions, and C is an arbitrary constant.

7°. For $a = 0$ and b and c are any numbers, there is a generalized separable solution of the form

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= 2b\varphi_1\varphi_2 + 6c\varphi_0\varphi_3 - 12(3b + c)\varphi_2\varphi_3,\\
\varphi_1' &= 6(b + c)\varphi_1\varphi_3 + 4b\varphi_2^2 - 36(3b + c)\varphi_3^2,\\
\varphi_2' &= 6(3b + c)\varphi_2\varphi_3,\\
\varphi_3' &= 6(3b + c)\varphi_3^2.
\end{aligned}$$

This system is easy to integrate in the reverse order, starting from the last equation.

⊙ *References for equation* 13.5.2.8: P. Rosenau (1994), P. Rosenau and D. Levy (1999), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 210–211).

9. $\dfrac{\partial w}{\partial t} - \dfrac{\partial^3 w}{\partial x^2 \partial t} = aw\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + cw\dfrac{\partial^3 w}{\partial x^3}.$

1°. Traveling-wave solutions for $a(b + c) < 0$:

$$\begin{aligned}
w &= A + B\exp\left(\pm\lambda x \pm A\frac{a\lambda + c\lambda^3}{1 - \lambda^2}t\right), \quad \lambda = \sqrt{-\frac{a}{b + c}},\\
w &= A + B\cosh\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1 - \lambda^2}t + C\right),\\
w &= A + B\sinh\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1 - \lambda^2}t + C\right),
\end{aligned}$$

where A, B, and C are arbitrary constants.

2°. Traveling-wave solutions for $a(b+c)>0$:

$$w = A + B\cos\left(\lambda x + A\frac{a\lambda - c\lambda^3}{1+\lambda^2}t\right), \qquad \lambda = \sqrt{\frac{a}{b+c}},$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solution for $a=0$, $b=-c$:

$$w = A + B\exp\left(\lambda x + A\frac{a\lambda + c\lambda^3}{1-\lambda^2}t\right),$$

where A, B, and λ are arbitrary constants ($\lambda \neq \pm 1$).

4°. Multiplicative separable solution:

$$w = \frac{f(x)}{t+C},$$

where the function $f = f(x)$ is determined by the ordinary differential equation

$$af f' + bf' f'' + cf f''' + f - f'' = 0.$$

5°. Generalized separable solution for $a=-b-c$ (here b and c are any numbers):

$$w = \varphi_1(t)e^x + \varphi_2(t)e^{-x},$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are arbitrary functions.

6°. Generalized separable solution for $a=-c$, $b=0$:

$$w = \varphi_1(t)e^x + \varphi_2(t)e^{-x} + C,$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are arbitrary functions, and C is an arbitrary constant.

7°. For $a=-3$, $b=2$, and $c=1$, this equation is derived from the Euler equations for inviscid fluids and represents an asymptotic model that describes the wave dynamics at the free surface of fluids under the force of gravity.

This equation is a completely integrable equation. There are generalized separable solutions of the form (the *Camassa–Holm multipeakon solutions*)

$$w = \sum_{i=1}^{N} p_i(t)e^{-|x-q_i(t)|}, \tag{1}$$

where the functions $p_i = p_i(t)$ and $q_i = q_i(t)$ are governed by the Hamiltonian ODEs:

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i}. \tag{2}$$

The Hamiltonian in equations (2) is

$$H(p_i, q_i) = \frac{1}{2}\sum_{i,j=1}^{N} p_i(t)p_j(t)e^{-|q_i(t)-q_j(t)|}. \tag{3}$$

8°. For $a=-b-1$, $c=1$ ($b \neq 2,\ 3$ is any number), there exist the multipeakon solutions of the form (1), where the functions $p_i = p_i(t)$ and $q_i = q_i(t)$ are governed by the system

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -(b-1)\frac{\partial H}{\partial q_i}$$

with the same "Hamiltonian" (3).

9°. For $a = 0$ (b and c are any numbers), there is a generalized separable solution of the form

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= 2b\varphi_1\varphi_2 + 6c\varphi_0\varphi_3 + 12(3b + c)\varphi_2\varphi_3,\\
\varphi_1' &= 6(b + c)\varphi_1\varphi_3 + 4b\varphi_2^2 + 36(3b + c)\varphi_3^2,\\
\varphi_2' &= 6(3b + c)\varphi_2\varphi_3,\\
\varphi_3' &= 6(3b + c)\varphi_3^2.
\end{aligned}$$

This system is easy to integrate in the reverse order, starting from the last equation.

⊙ *References for equation* 13.5.2.9: R. Camassa and D. D. Holm (1993), B. Fuchssteiner (1996), R. Camassa and A. I. Zenchuk (2001), A. Degasperis, D. D. Holm, and A. N. W. Hone (2002), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 210, 211, 217).

10. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^2 \partial t} = aw\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + cw\dfrac{\partial^3 w}{\partial x^3} + d\dfrac{\partial w}{\partial x}.$

The transformation

$$w = u(z, t) + \frac{d}{c - a}, \qquad z = x + \frac{cd}{c - a} \qquad (a \neq c)$$

leads to an equation of the form 13.5.2.8:

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial z^2 \partial t} = au\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + cu\frac{\partial^3 u}{\partial z^3}.$$

11. $\dfrac{\partial w}{\partial t} - \dfrac{\partial^3 w}{\partial x^2 \partial t} = aw\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + cw\dfrac{\partial^3 w}{\partial x^3} + d\dfrac{\partial w}{\partial x}.$

1°. The transformation

$$w = u(z, t) - \frac{d}{a + c}, \qquad z = x + \frac{cd}{a + c} \qquad (a + c \neq 0)$$

leads to an equation of the form 13.5.2.9:

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial z^2 \partial t} = au\frac{\partial u}{\partial z} + b\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + cu\frac{\partial^3 u}{\partial z^3}.$$

2°. For $a = -1$, $b = 3$, $c = 1$, and $d = -1$, this equation describes qualitative behavior of wave-breaking (see Whitham, 1965).

12. $\dfrac{\partial^3 w}{\partial x^2 \partial t} = kw\dfrac{\partial^3 w}{\partial x^3}.$

This equation is encountered at the interface between projective geometry and gravitational theory.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\big(C_2 x + C_2 k\varphi(t), C_1 C_2 t + C_3\big) + \varphi_t'(t),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = Cx^2 + \varphi(t)x + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C is an arbitrary constant.

3°. Self-similar solution:

$$w(x,t) = t^{-\alpha-1}U(z), \qquad z = t^{\alpha}x,$$

where α is an arbitrary constant and the function $U(z)$ is determined by the ordinary differential equation $(\alpha-1)U_{zz}'' + \alpha z U_{zzz}''' = kUU_{zzz}'''$.

4°. Multiplicative separable solution:

$$w(x,t) = (Akt + B)^{-1}u(x),$$

where A and B are arbitrary constants, and the function $u(x)$ is determined by the autonomous ordinary differential equation $uu_{xxx}''' + Au_{xx}'' = 0$.

5°. There is a first integral:

$$\frac{\partial^2 w}{\partial x \partial t} = kw\frac{\partial^2 w}{\partial x^2} - \frac{k}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \psi(t), \tag{1}$$

where $\psi(t)$ is an arbitrary function. By changing in (1) from the old variables t, x, and $w = w(x,t)$ to the new variables t, η, and $\Phi = \Phi(t,\eta)$ with η and Φ defined as $\eta = \frac{\partial w}{\partial x}$ and $\Phi = \frac{\partial^2 w}{\partial x^2}$, we arrive at the first-order equation

$$\frac{\partial}{\partial t}\frac{1}{\Phi} + \frac{k\eta}{\Phi} + \frac{\partial}{\partial \eta}\left(\frac{\psi(t) - \frac{1}{2}k\eta^2}{\Phi}\right) = 0. \tag{2}$$

The substitution $\Phi = 1/U$ leads (2) to the first-order linear equation.

⊙ *References for equation* 13.5.2.12: V. S. Dryuma (2000), M. V. Pavlov (2001), A. D. Polyanin and V. F. Zaitsev (2004, p. 584), A. D. Polyanin (2010).

13. $\dfrac{\partial^3 w}{\partial x^2 \partial t} = f(t)w\dfrac{\partial^3 w}{\partial x^3} + g(x,t).$

There is a first integral:

$$\frac{\partial^2 w}{\partial x \partial t} = f(t)w\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}f(t)\left(\frac{\partial w}{\partial x}\right)^2 + \int g(x,t)\,dx + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function.

14. $$w^2\frac{\partial w}{\partial t}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial t}+w\frac{\partial^3 w}{\partial x^2\partial t}=0.$$

Ostrovsky equation. This equation is used to describe waves in the ocean.

Traveling-wave solutions in the form of solitary waves:

$$w=2\lambda-6\lambda\tanh^2[\sqrt{\lambda}(x-ct)+b],$$
$$w=6\lambda-6\lambda\tanh^2[\sqrt{\lambda}(x-ct)+b],$$

where b, c, and λ are arbitrary constants.

⊙ *References*: E. J. Parkes (1996, 2007), N. A. Kudryashov (2010 *c*).

15. $$\frac{\partial w}{\partial t}+2k\frac{\partial w}{\partial x}+3w\frac{\partial w}{\partial x}-2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}-w\frac{\partial^3 w}{\partial x^3}-\frac{\partial^3 w}{\partial x^2\partial t}=0.$$

Camassa–Holm equation. The Camassa–Holm equation models breaking waves: a smooth initial profile with sufficient decay at infinity develops into either a wave that exists for all times or into a breaking wave (wave breaking being characterized by the fact that the solution remains bounded but its slope becomes unbounded in finite time). It is the integrable equation.

⊙ *References for equation* 13.5.2.15: R. Camassa and D. D. Holm (1993), R. Camassa, D. D. Holm, and J. M. Hyman (1994), C.-R. Boldea (1995), A. Constantin and J. Escher (1998), R. S. Johnson (2002, 2003), E. G. Reyes (2002), H. P. McKean (2003), A. Parker (2004, 2005 *a, b*, 2006), J. Lenells (2005 *a*), E. Loubet (2005), A. Constantin, V. S. Gerdjikov, and R. I. Ivanov (2006), A. Bressan and A. Constantin (2007 *a,b*), V. Golovko, P. Kersten, I. Krasil'shchik, and A. Verbovetsky (2008).

16. $$\frac{\partial w}{\partial t}+2k\frac{\partial w}{\partial x}+4w\frac{\partial w}{\partial x}-3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}-w\frac{\partial^3 w}{\partial x^3}-\frac{\partial^3 w}{\partial x^2\partial t}=0.$$

The *Degasperis–Procesi equation* arises in the theory of water waves. It is integrable.

Special case. For $k=0$, the Degasperis–Procesi equation has *multipeakon solutions* of the form

$$w(x,t)=\sum_{i=1}^{n} m_i(t)e^{-|x-x_i(t)|},$$

where the functions $m_i=m_i(t)$ and $x_i=x_i(t)$ are determined by the system of ordinary differential equations

$$x_i'=\sum_{j=1}^{n} m_j e^{-|x_i-x_j|},$$
$$m_i'=2m_i\sum_{j=1}^{n} m_j\,\operatorname{sign}(x_i-x_j)e^{-|x_i-x_j|}.$$

These ordinary differential equations can be solved explicitly in terms of elementary functions, using inverse spectral methods.

When $k>0$ the soliton solutions of the Degasperis–Procesi equation are smooth; they converge to peakons in the limit as k tends to zero.

⊙ *References for equation* 13.5.2.16: A. Degasperis and M. Procesi (1999), A. Degasperis, D. D. Holm, and A. N. W. Hone (2002), R. S. Johnson (2003), H. Lundmark and J. Szmigielski (2003), Z. Yin (2003, 2004), V. O. Vakhnenko and E. J. Parkes (2004), R. Ivanov (2005, 2007), J. Lenells (2005 *b*), Y. Matsuno (2005), G. M. Coclite and K. H. Karlsen (2006), J. Escher, Y. Liu, and Z. Yin (2006, 2007), H. Lundmark (2007).

17. $\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x^2 \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x \partial y^2} = 0.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_4 y + C_5) + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x, y) = axy + f(x) + g(y);$$
$$w(x, y) = \frac{1}{\lambda}\big[f(x) + g(y)\big] - \frac{2}{\lambda}\ln\left|b\int \exp\big[f(x)\big]\,dx + \frac{a\lambda}{2b}\int \exp\big[g(y)\big]\,dy\right|,$$
$$w(x, y) = \varphi(z), \quad z = ax + by;$$
$$w(x, y) = \psi(\xi), \quad \xi = xy,$$

where $f = f(y)$, $g = g(y)$, $\varphi(z)$, and $\psi(\xi)$ are arbitrary functions; a, b, and λ are arbitrary constants.

3°. There are exact solutions of the following forms:

$$w(x, y) = |x|^a F(r), \quad r = y|x|^b;$$
$$w(x, y) = e^{ax} G(\eta), \quad \eta = bx + cy;$$
$$w(x, y) = e^{ax} H(\zeta), \quad \zeta = ye^{bx};$$
$$w(x, y) = |x|^a U(\rho), \quad \rho = y + b\ln|x|;$$
$$w(x, y) = V(r) + a\ln|x|, \quad r = y|x|^b;$$
$$w(x, y) = W(\rho) + a\ln|x|, \quad \rho = y + b\ln|x|,$$

where a, b, and c are arbitrary constants. Another set of solutions can be obtained by swapping x and y in the above formulas.

4°. The left-hand side of the original equation represents the Jacobian of w and $v = w_{xy}$. The fact that the Jacobian of two quantities is zero means that these are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x \partial y} = \Phi(w), \tag{1}$$

where $\Phi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\Phi(w)$ will be a solution of the original equation.

18. $\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x^2 \partial y} = 0.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(C_2 x + C_3, C_4 y + C_5) + C_6,$$
$$w_2 = w\big(x + \varphi(y), y\big),$$

where $C_1, \ldots, C_6$ are arbitrary constants and $\varphi(y)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y) = Ax^2 + fx + g;$$
$$w(x,y) = f\exp(Ax) + g\exp(-Ax);$$
$$w(x,y) = f\sin(Ax) + g\cos(Ax);$$
$$w(x,y) = A\ln\left[(x+f)^2\right] + B;$$
$$w(x,y) = A\ln\left[\sin^2(fx+g)\right] + B;$$
$$w(x,y) = A\ln\left[\sinh^2(fx+g)\right] + B;$$
$$w(x,y) = A\ln\left[\cosh^2(fx+g)\right] + B;$$
$$w(x,y) = \varphi(z), \qquad z = Ax + By,$$

where $f = f(y)$, $g = g(y)$, and $\varphi(z)$ are arbitrary functions; A and B are arbitrary constants.

3°. The left-hand side of the original equation represents the Jacobian of w and $v = w_{xx}$. The fact that the Jacobian of two quantities is zero means that these are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x^2} = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\varphi(w)$ will be a solution of the original equation.

Integrating (1) yields the general solution of the original equation in implicit form:

$$\int\left[f(y) + 2\int\varphi(w)\,dw\right]^{-1/2} dw = g(y) \pm x,$$

where $f = f(y)$, $g = g(y)$, and $\varphi(w)$ are arbitrary functions.

19. $\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x^2 \partial y} = f(y)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(C_1 x + C_2, y) + C_3,$$
$$w_2 = w\big(x + \varphi(y), y\big),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(y)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,y) = -\frac{1}{2}x^2\left[\int f(y)\,dy + C\right] + x\varphi(y) + \psi(y),$$

where $\varphi(y)$ and $\psi(y)$ are arbitrary functions and C is an arbitrary constant.

3°. Additive separable solutions:

$$w(x,y) = C_1 e^{kx} + C_2 e^{-kx} + C_3 + \frac{1}{k^2}\int f(y)\,dy,$$
$$w(x,y) = C_1\cos(kx) + C_2\sin(kx) + C_3 - \frac{1}{k^2}\int f(y)\,dy,$$

where C_1, C_2, C_3, and k are arbitrary constants.

4°. The original equation can be rewritten as the relation where the Jacobian of two functions, w and $v = w_{xx} + \int f(y)\,dy$, is equal to zero. It follows that w and v are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x^2} + \int f(y)\,dy = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\varphi(w)$ will be a solution of the original equation.

Equation (1) may be treated as an ordinary differential equation with independent variable x and parameter y. Integrating yields the general solution of (1) in implicit form:

$$\int \left[\psi_1(y) - 2w\int f(y)\,dy + 2\int \varphi(w)\,dw\right]^{-1/2} dw = \psi_2(y) \pm x,$$

where $\psi_1(y)$, $\psi_2(y)$, and $\varphi(w)$ are arbitrary functions.

20. $\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x\partial y^2} - \dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial y^3} = f(x,w)\dfrac{\partial w}{\partial y}.$

This is a special case of equation 15.4.1.6 with $G = w_{yy}$ and $F = f(x,w)w_y$.

The von Mises transformation

$$x,\ y,\ w = w(x,y) \quad \Longrightarrow \quad x,\ w,\ \eta = \eta(x,w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial y},$$

leads to the second-order equation $(\eta\eta_w)_x = f(x,w)$, whose general solution is expressed as

$$\eta^2 = 2\int_{x_0}^{x}\int_{w_0}^{w} f(\tau,\xi)\,d\xi\,d\tau + \varphi(x) + \psi(w),$$

where $\varphi(x)$ and $\psi(w)$ are arbitrary functions; x_0 and w_0 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2010).

21. $\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x^2\partial y} = f(y)\dfrac{\partial w}{\partial x} + g(x)\dfrac{\partial w}{\partial y}.$

First integral:

$$\frac{\partial^2 w}{\partial x^2} = \varphi(w) + \int g(x)\,dx - \int f(y)\,dy,$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a second-order ordinary differential equation with independent variable x and parameter y.

22. $\dfrac{\partial w}{\partial y}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial y^3} - \dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x\partial y^2}\right) = 2\dfrac{\partial^2 w}{\partial y^2}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} - \dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}\right).$

General solution:

$$w = f\big(\varphi(x)y + \psi(x)\big),$$

where $\varphi(x)$, $\psi(x)$, and $f(z)$ are arbitrary functions.

Remark. The equation in question can be represented as the equality of the Jacobian of two functions, w and v, to zero:

$$w_x v_y - w_y v_x = 0, \qquad \text{where} \quad v = w_{yy}/w_y^2.$$

23. $\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x\partial y^2}-\dfrac{\partial w}{\partial y}\dfrac{\partial^3 w}{\partial x^2\partial y}\right)=\dfrac{\partial^2 w}{\partial x\partial y}\left[\left(\dfrac{\partial w}{\partial x}\right)^2\dfrac{\partial^2 w}{\partial y^2}-\left(\dfrac{\partial w}{\partial y}\right)^2\dfrac{\partial^2 w}{\partial x^2}\right].$

Two forms of representation of the general solution:

$$w = f\big(\varphi(x)+\psi(y)\big),$$
$$w = \bar{f}\big(\bar{\varphi}(x)\bar{\psi}(y)\big),$$

where $\varphi(x)$, $\psi(y)$, $\bar{\varphi}(x)$, $\bar{\psi}(y)$, $f(z_1)$, and $\bar{f}(z_2)$ are arbitrary functions.

Remark. The equation in question can be represented as the equality of the Jacobian of two functions to zero:

$$w_x v_y - w_y v_x = 0, \qquad \text{where} \quad v = w_{xy}/(w_x w_y).$$

24. $\dfrac{\partial^3 w}{\partial x\partial y\partial z} = e^w.$

Third-order Liouville type equation.

Solution:

$$w = \ln\frac{-6\alpha'(x)\beta'(y)\gamma'(z)}{[\alpha(x)+\beta(y)+\gamma(z)]^3},$$

where $\alpha(x)$, $\beta(y)$, and $\gamma(z)$ are arbitrary functions; a prime denotes a derivative with respect to the argument of the function.

⊙ *Reference*: A. N. Mironov (2010).

25. $\dfrac{\partial^3 w}{\partial t^3} = \left(\dfrac{\partial^3 w}{\partial t\partial x^2}\right)^2 - \dfrac{\partial^3 w}{\partial t^2\partial x}\dfrac{\partial^3 w}{\partial x^3}.$

One of the *associativity equations* is the two-dimensional field theory.

1°. Generalized separable solution:

$$w = \varphi(x) + C_1x^2t + \tfrac{2}{3}C_1^2t^3 + C_2t^2 + C_3t,$$

where $\varphi(x)$ is an arbitrary function and C_1, C_2, and C_3 are arbitrary constants.

2°. Generalized separable solution:

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3 + \varphi_4(t)x^4,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi_0''' = -6\varphi_3\varphi_1'' + 4(\varphi_2')^2,$$
$$\varphi_1''' = -24\varphi_4\varphi_1'' - 12\varphi_3\varphi_2'' + 24\varphi_2'\varphi_3',$$
$$\varphi_2''' = -48\varphi_4\varphi_2'' - 18\varphi_3\varphi_3'' + 48\varphi_2'\varphi_4' + 36(\varphi_3')^2,$$
$$\varphi_3''' = -72\varphi_4\varphi_3'' - 24\varphi_3\varphi_4'' + 144\varphi_3'\varphi_4',$$
$$\varphi_4''' = -96\varphi_4\varphi_4'' + 144(\varphi_3')^2.$$

3°. Generalized separable solution:

$$w = \varphi(x) + \psi(x)e^{\lambda t},$$

where λ is an arbitrary constant and the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$\psi'\varphi''' = -\lambda\psi,$$
$$\psi'\psi''' = (\psi'')^2.$$

The general solution of this system is

$$\varphi = \frac{C_2\lambda}{C_1k^4}e^{-kx} - \frac{\lambda}{6k}x^3 + C_3x^2 + C_4x + C_5,$$
$$\psi = C_1e^{kx} + C_2,$$

where $C_1, \ldots, C_5$ and k are arbitrary constants ($C_1 \neq 0$ and $k \neq 0$); to $C_1 = 0$ and $k = 0$ there corresponds the degenerate solution from Item 1°).

4°. There is a self-similar solution of the form

$$w = t^{4k-1}f(z), \quad z = xt^{-k},$$

where k is an arbitrary constant.

⊙ *References for equation* 13.5.2.25: B. A. Dubrovin (1996), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 314–315).

13.5.3. Other Equations

1. $\dfrac{\partial w}{\partial t} + aw^k\dfrac{\partial w}{\partial x} - b\dfrac{\partial^3 w}{\partial t^3} = 0.$

1°. $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-1}w(\pm C_1^k x + C_2, \pm t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (both upper signs or both lower signs being taken at a time).

2°. Traveling-wave solutions:

$$w = \left\{\frac{c(k+1)(k+2)}{2a\cosh^2\big[m(x-ct+B)\big]}\right\}^{1/k}, \quad m = \frac{k}{2c\sqrt{b}} \quad \text{if } b > 0,$$
$$w = \left\{-\frac{c(k+1)(k+2)}{2a\sinh^2\big[m(x-ct+B)\big]}\right\}^{1/k}, \quad m = \frac{k}{2c\sqrt{b}} \quad \text{if } b > 0,$$
$$w = \left\{\frac{c(k+1)(k+2)}{2a\cos^2\big[m(x-ct+B)\big]}\right\}^{1/k}, \quad m = \frac{k}{2c\sqrt{-b}} \quad \text{if } b < 0,$$

where B and c are arbitrary constants.

Remark. The first solution is the soliton solution. This solution was obtained by W. E. Schiesser (2003, private communication).

3°. Traveling-wave solutions:

$$w = \left\{ \frac{c(k+1)(k+2)}{8a} \left(2 - \tanh^2\left[\frac{k}{4c\sqrt{b}}(x - ct + B) \right] - \coth^2\left[\frac{k}{4c\sqrt{b}}(x - ct + B) \right] \right) \right\}^{1/k}$$

if $b > 0$,

$$w = \left\{ \frac{c(k+1)(k+2)}{8a} \left(2 + \tan^2\left[\frac{k}{4c\sqrt{-b}}(x - ct + B) \right] + \cot^2\left[\frac{k}{4c\sqrt{-b}}(x - ct + B) \right] \right) \right\}^{1/k}$$

if $b < 0$,

where B and c are arbitrary constants.

4°. Multiplicative separable solution:

$$w(x,t) = x^{1/k} f(t),$$

where the function $f(t)$ is determined by the autonomous ordinary differential equation

$$b f'''_{ttt} - f'_t - \frac{a}{k} f^{k+1} = 0.$$

5°. Generalized separable solution with $k = 1$:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$b\varphi'''_{ttt} - \varphi'_t - a\varphi^2 = 0,$$
$$b\psi'''_{ttt} - \psi'_t - a\varphi\psi = 0.$$

2. $\dfrac{\partial w}{\partial t} + f(w)\dfrac{\partial w}{\partial x} = \alpha\dfrac{\partial^3 w}{\partial t^3}$.

1°. Let $w = w(k_1 x + \lambda_1 t)$ be the traveling-wave solution of the generalized Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \beta f(w)\frac{\partial w}{\partial x} = \gamma\frac{\partial^3 w}{\partial x^3}. \tag{1}$$

Then the original equation has the traveling-wave solutions

$$w = w(k_2 x + \lambda_2 t), \qquad \lambda_2 = \pm\left(\frac{k_1^3\gamma}{\alpha\lambda_1}\right)^{1/2}, \quad k_2 = \frac{\alpha\beta\lambda_2^3}{k_1^2\gamma}, \tag{2}$$

where either plus or minus signs are taken.

Remark 1. It should be emphasized that the original equation and equation (1) are of principally different types (these equations have different third derivatives, w_{ttt} and w_{xxx}). The above similarity between solutions to equations of different types can only occur for traveling-wave solutions.

Remark 2. In the original equation and equation (1), the parameters α and γ can be replaced by $\alpha g(w)$ and $\gamma g(w)$, respectively, where $g(w)$ is an arbitrary function.

Example. Consider the equation

$$\frac{\partial w}{\partial t} + aw^k \frac{\partial w}{\partial x} - b\frac{\partial^3 w}{\partial t^3} = 0, \tag{3}$$

which corresponds to $f(w) = aw^k$ and $\alpha = b$. As equation (1), we take equation 13.1.3.2,

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + aw^k \frac{\partial w}{\partial x} = 0, \tag{4}$$

which corresponds to $\beta = 1$ and $\gamma = -1$. Consider the traveling-wave solution of equation (4) (see equation 13.1.1.2, first solution in Item 2.2)

$$w = \left\{ \frac{c(k+1)(k+2)}{2a\cosh^2\left[\frac{1}{2}k\sqrt{c}\,(x-ct)+B\right]} \right\}^{1/k} = \left\{ \frac{c(k+1)(k+2)}{2a\cosh^2\left(k_1 x + \lambda_1 t + B\right)} \right\}^{1/k} \tag{5}$$

with the determining parameters $k_1 = \frac{1}{2}kc^{1/2}$ and $\lambda_1 = -\frac{1}{2}kc^{3/2}$, where $c > 0$ is an arbitrary constant. Substituting these values of k_1 and λ_1 into (2), we get

$$k_2 = \pm\frac{k}{2c\sqrt{b}}, \qquad \lambda_2 = \mp\frac{k}{2\sqrt{b}} \qquad (b > 0). \tag{6}$$

Using formulas (2) and taking into account (5) and (6), we obtain the corresponding solutions of equation (3):

$$w = \left\{ \frac{c(k+1)(k+2)}{2a\cosh^2\left(k_2 x + \lambda_2 t + B\right)} \right\}^{1/k} = \left\{ \frac{c(k+1)(k+2)}{2a} \cosh^{-2}\left(\pm\frac{kx}{2c\sqrt{b}} \mp \frac{kt}{2\sqrt{b}} + B \right) \right\}^{1/k}.$$

For other traveling-wave solutions of equation (3), see equation 13.3.5.1.

2°. The technique outlined in Item 1° allows one to construct solutions of the equation in question with $f(w) = aw^n - bw^{2n}$ on the basis of solutions of equation 13.1.3.3.

3. $a\dfrac{\partial^3 w}{\partial x^3} + b\dfrac{\partial^3 w}{\partial y^3} = (ay^3 + bx^3) f(w)$.

Solution:

$$w = w(z), \qquad z = xy,$$

where the function $w(w)$ is determined by the autonomous ordinary differential equation

$$w'''_{zzz} = f(w).$$

Remark. The above remains true if the constants a and b in the original equation are replaced by arbitrary functions $a = a(x, y, w, w_x, w_y, \dots)$ and $b = b(x, y, w, w_x, w_y, \dots)$.

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = b\dfrac{\partial^3 w}{\partial x^3} + c\dfrac{\partial^3 w}{\partial y^3}$.

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x + C_2, C_1 y + C_3) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solutions:

$$w(x, y) = c\lambda x + C_1 e^{a\lambda y} + C_2 y + C_3,$$
$$w(x, y) = C_1 e^{\lambda x} + C_2 x + b\lambda y + C_3,$$
$$w(x, y) = C_1 e^{-a\lambda x} + \frac{c\lambda}{a} x + C_2 e^{\lambda y} - ab\lambda y + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants.

3°. Solution:

$$w = u(z) + C_3 x, \qquad z = C_1 x + C_2 y,$$

where C_1, C_2 and C_3 are arbitrary constants. The function $u(z)$ is determined by the second-order autonomous ordinary differential equation (C_4 is an arbitrary constant)

$$C_1 C_2 (C_1 + aC_2)(u'_z)^2 + 2aC_2^2 C_3 u'_z = 2(bC_1^3 + cC_2^3)u''_{zz} + C_4.$$

To $C_3 = 0$ there corresponds a traveling-wave solution. In this case, the substitution $F(u) = (u'_z)^2$ leads to a first-order linear equation.

4°. There is a self-similar solution of the form $w = w(y/x)$.

5. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = a\dfrac{\partial^3 w}{\partial x^3} + b\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 y + C_3) + C_4 xy + C_5 x + C_6 y + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x, y) = -\frac{aC_1^3 + bC_2^3}{C_1^2 C_2^2} z(\ln|z| - 1), \qquad z = C_1 x + C_2 y + C_3.$$

3°. Additive separable solutions:

$$w(x, y) = \tfrac{1}{2} bC_1 x^2 + C_2 x + C_3 \exp(C_1 y) + C_4 y + C_5,$$
$$w(x, y) = \tfrac{1}{2} aC_1 y^2 + C_2 y + C_3 \exp(C_1 x) + C_4 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

4°. Solution:

$$w = U(\zeta) + C_3 x^2 + C_4 y^2, \qquad \zeta = C_1 x + C_2 y,$$

where the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$(C_1^2 U''_{\zeta\zeta} + 2C_3)(C_2^2 U''_{\zeta\zeta} + 2C_4) = (aC_1^3 + bC_2^3)U'''_{\zeta\zeta\zeta},$$

which can be integrated with the substitution $F(\zeta) = U''_{\zeta\zeta}$.

5°. There is a self-similar solution of the form $w = xu(y/x)$.

Chapter 14
Fourth-Order Equations

14.1. Equations Involving the First Derivative in t

14.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + F\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + bw\ln w + f(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{aA^4}{3b}e^{4bt} + e^{bt}\int e^{-bt}f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt}f(t)\,dt\right]\varphi(z), \quad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$a\varphi'''''_{zzzz} - \lambda\varphi'_z + b\varphi\ln\varphi = 0,$$

whose order can be reduced by one.

3°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}f(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^4 u}{\partial x^4} + bu\ln u.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + f(t)w\ln w + [g(t)x + h(t)]w.$

This is a special case of equation 15.1.3.5 with $n = 4$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + (bx + c)\dfrac{\partial w}{\partial x} + f(w).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''''_{zzzz} + (bz + c)w'_z + f(w) = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} - b\dfrac{\partial^4 w}{\partial x^4}.$

This equation describes the evolution of nonlinear waves in a dispersive medium.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^3 w(C_1 x + aC_1 C_2 t + C_3, C_1^4 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x, t) = -\frac{x + C_1}{at + C_2},$$
$$w(x, t) = -\frac{120b}{a(x + aC_1 t + C_2)^3} + C_1.$$

The first solution is degenerate and the second one is a traveling-wave solution.

3°. Traveling-wave solution in implicit form:

$$\Big(\frac{40b}{9a}\Big)^{1/3} \int_0^{C_1^3 w + C_2} \frac{d\eta}{(1 - \eta^2)^{2/3}} = C_1 x + aC_1 C_2 t + C_3.$$

With $C_1 = 1$ and $C_2 = C_3 = 0$, we have the stationary solution.

4°. Traveling-wave solution (generalizes the second solution of Item 2° and the solution of Item 3°):

$$w = w(\xi), \quad \xi = x - \lambda t,$$

where the function $w(\xi)$ is determined by the third-order autonomous ordinary differential equation

$$bw'''_{\xi\xi\xi} = \tfrac{1}{2}aw^2 + \lambda w + C.$$

Here, C and λ are arbitrary constants.

5°. Self-similar solution:

$$w(x,t) = t^{-3/4}u(\eta), \qquad \eta = xt^{-1/4},$$

where the function $u(\eta)$ is determined by the ordinary differential equation

$$bu''''_{\eta\eta\eta\eta} = auu'_\eta + \tfrac{1}{4}\eta u'_\eta + \tfrac{3}{4}u.$$

6°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the third-order ordinary differential equation

$$bU'''_{\zeta\zeta\zeta} - \tfrac{1}{2}aU^2 + C_2 U = -2C_1\zeta + C_3.$$

7°. Solution:

$$w = \varphi^3 F(z) + \frac{1}{a\varphi}(\varphi'_t x + \psi'_t), \qquad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are defined by

$$\varphi(t) = (4At + C_1)^{-1/4},$$
$$\psi(t) = C_2(4At + C_1)^{3/4} + C_3(4At + C_1)^{-1/4},$$

where A, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$bF''''_{zzzz} - aFF'_z - 2AF + 3\frac{A^2}{a}z = 0.$$

8°. Let us set $a = b = -1$ (the original equation can be reduced to this case by appropriate scaling of the independent variables).

The equation admits a formal series solution of the form

$$w(x,t) = \frac{1}{[x-\varphi(t)]^3}\sum_{n=0}^{\infty} w_n(t)[x-\varphi(t)]^n.$$

The series coefficients $w_n = w_n(t)$ are expressed as

$$w_0 = -120, \quad w_1 = w_2 = 0, \quad w_3 = -\varphi'(t), \quad w_4 = w_5 = 0, \quad w_6 = \psi(t),$$
$$(n+1)(n-6)(n^2-13n+60)w_n = \sum_{m=6}^{n-6}(m-3)w_{n-m}w_m + w'_{n-4},$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. This solution has a singularity at $x = \varphi(t)$.

9°. If $a = b = -1$, the equation also admits the formal series solution

$$w(x,t) = \frac{x}{t} + \frac{1}{x}\sum_{n=1}^{\infty}\left(\frac{t}{x^4}\right)^{n-1}\sum_{k=0}^{n-1} A_k^n x^{2k},$$

where A_0^1 is an arbitrary constant and the other coefficients can be expressed in terms of A_0^1 with recurrence relations. This solution can be generalized with the help of translations in the independent variables.

⊙ *References for equation* 14.1.1.4: O. V. Rudenko and V. A. Robsman (2002), A. D. Polyanin and V. F. Zaitsev (2004, pp. 590–591). The solutions of Items 8° and 9° were obtained by V. G. Baydulov (2002).

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} - b\dfrac{\partial^4 w}{\partial x^4} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x + a\int_{t_0}^{t} (t-\tau)f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 14.1.1.4:

$$\frac{\partial u}{\partial t} = au\frac{\partial u}{\partial x} - b\frac{\partial^4 u}{\partial x^4}.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + bw\dfrac{\partial w}{\partial x} + cw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + bC_1e^{ct} + C_2,\ t + C_3) + C_1ce^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = U(z) + C_1ce^{ct}, \qquad z = x + bC_1e^{ct} + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU''''_{zzzz} + bUU'_z - C_2U'_z + cU = 0.$$

If $C_1 = 0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + [f(t)\ln w + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^3(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 2bC_1C_2 t + C_3, C_1^4 t + C_4) + C_2 x + bC_2^2 t + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t + C_2 + \int \theta(z)\,dz, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the third-order autonomous ordinary differential equation

$$a\theta'''_{zzz} + b\theta^2 - \lambda\theta - C_1 = 0.$$

To $C_1 = 0$ there corresponds a traveling-wave solution.

3°. Self-similar solution:

$$w(x,t) = t^{-1/2}u(\zeta), \qquad \zeta = xt^{-1/4},$$

where the function $u(\zeta)$ is determined by the ordinary differential equation

$$au''''_{\zeta\zeta\zeta\zeta} + b(u'_\zeta)^2 + \tfrac{1}{4}\zeta u'_\zeta + \tfrac{1}{2}u = 0.$$

4°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

The substitution $w = U(x,t) + \int f(t)\,dt$ leads to a simpler equation of the form 14.1.1.8:

$$\frac{\partial U}{\partial t} = a\frac{\partial^4 U}{\partial x^4} + b\left(\frac{\partial U}{\partial x}\right)^2.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct}\int e^{-ct} f(t)\,dt + \theta(z), \qquad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta''''_{zzzz} + b\left(\theta'_z\right)^2 - \lambda\theta'_z + c\theta = 0.$$

2°. There is a degenerate solution of the form

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

3°. The substitution

$$w = U(x,t) + e^{ct}\int e^{-ct} f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^4 U}{\partial x^4} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

14.1.2. Other Equations

1. $\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} + \alpha\dfrac{\partial^2 w}{\partial x^2} + \beta\dfrac{\partial^3 w}{\partial x^3} + \gamma\dfrac{\partial^4 w}{\partial x^4} = 0.$

Kuramoto–Sivashinsky equation. It describes nonlinear waves in dispersive-dissipative media with an instability, waves arising in a fluid flowing down an inclined plane, the evolution of the concentration of a substance in chemical reactions, and others.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - C_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \frac{x + C_1}{t + C_2}.$$

3°. Traveling-wave solutions:

$$w(x,t) = C_1 + \left[\tfrac{15}{76}\left(16\alpha - \beta^2\gamma^{-1}\right) + 15\beta k + 60\gamma k^2\right]F - \left(15\beta + 180\gamma k\right)F^2 + 60\gamma F^3,$$
$$F = k\left[1 + C_2\exp(-kx - \lambda t)\right]^{-1},$$

where C_1 and C_2 are arbitrary constants, and the coefficients β, k, and λ are defined by

$\beta = 0$, $\quad k = \pm\sqrt{\frac{11}{19}\alpha\gamma^{-1}}$, $\quad \lambda = -C_1 k - \frac{30}{19}\alpha k^2$ (first set of solutions);

$\beta = \pm 4\sqrt{\alpha\gamma}$, $\quad k = \pm\sqrt{\alpha\gamma^{-1}}$, $\quad \lambda = -C_1 k - \frac{3}{2}\beta k^3$ (second set of solutions);

$\beta = \pm\frac{12}{\sqrt{47}}\sqrt{\alpha\gamma}$, $\quad k = \pm\sqrt{\frac{1}{47}\alpha\gamma^{-1}}$, $\quad \lambda = -C_1 k - \frac{60}{47}\alpha k^2$ (third set of solutions);

$\beta = \pm\frac{16}{\sqrt{73}}\sqrt{\alpha\gamma}$, $\quad k = \pm\sqrt{\frac{1}{73}\alpha\gamma^{-1}}$, $\quad \lambda = -C_1 k - \frac{90}{73}\alpha k^2$ (fourth set of solutions).

Special case. For $\beta = 0$, $\alpha = \gamma = 1$, $C_1 = 0$, and $C_2 = 1$, we have a solution

$$w(x,t) = \tfrac{15}{19}k\left(11H^3 - 9H + 2\right), \quad H = \tanh\left(\tfrac{1}{2}kx - \tfrac{15}{19}k^2 t\right), \quad k = \pm\sqrt{\tfrac{11}{19}},$$

which describes concentration waves in chemical reactions.

4°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x - C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the third-order ordinary differential equation (C_3 is an arbitrary constant)

$$\gamma U'''_{\zeta\zeta\zeta} + \beta U''_{\zeta\zeta} + \alpha U'_\zeta + \tfrac{1}{2}U^2 + C_2 U = -2C_1\zeta + C_3.$$

The special cases $C_1 = 0$ corresponds to a traveling-wave solution.

⊙ *References for equation* 14.1.2.1: Y. Kuramoto and T. Tsuzuki (1976), B. J. Cohen, J. A. Krommes, W. M. Tang, and M. N. Rosenbluth (1976), V. Ya. Shkadov (1977), J. Topper and T. Kawahara (1978), G. I. Sivashinsky (1983), A. V. Marchenko (1988), R. Conte and M. Musette (1989), N. A. Kudryashov (1989, 1990 *b*), A. D. Polyanin and V. F. Zaitsev (2004, p. 593).

2. $\dfrac{\partial w}{\partial t} + w^k\dfrac{\partial w}{\partial x} + \alpha\dfrac{\partial^2 w}{\partial x^2} + \beta\dfrac{\partial^3 w}{\partial x^3} + \gamma\dfrac{\partial^4 w}{\partial x^4} = 0.$

Generalized Kuramoto–Sivashinsky equation.

1°. Traveling-wave solution with $\alpha = \dfrac{k\beta^2}{(k+3)^2\gamma}$:

$$w = \left[\frac{3(k+1)(2k+3)\beta^3}{8(k+3)^2\gamma^2}\right]^{1/k} \exp\left[-\frac{\beta z}{2(k+3)\gamma}\right]\left[\cosh\frac{k\beta z}{2(k+3)\gamma}\right]^{-3/k},$$
$$z = x - \frac{2(k+2)\beta^3}{(k+3)^3\gamma^2}t + C,$$

where C is an arbitrary constant.

2°. Traveling-wave solution with $\alpha = \dfrac{(2k^2+18k+27)\beta^2}{9(k+3)^2\gamma}$:

$$w = \left[\frac{(k+1)(2k+3)\beta^3}{72(k+3)^2\gamma^2}\right]^{1/k}\left\{\tanh\left[\frac{k\beta z}{6(k+3)\gamma}\right] - 1\right\}^{3/k}, \qquad z = x + \frac{(2k+3)\beta^3}{9(k+3)^3\gamma^2}t + C,$$

where C is an arbitrary constant.

⊙ *Reference for equation* 14.1.2.2: N. A. Kudryashov (2010 *d*).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + bw + c.$

Generalized separable solution:

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)\exp[\lambda(t)x],$$

where the functions $\varphi_n = \varphi_n(t)$ and $\lambda = \lambda(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= -\varphi_2^2 + b\varphi_1 + c,\\
\varphi_2' &= b\varphi_2,\\
\varphi_3' &= (\lambda^2\varphi_1 - 2\lambda\varphi_2 + a\lambda^4 + b)\varphi_3,\\
\lambda' &= \varphi_2\lambda^2.
\end{aligned}$$

Integrating the second, fourth, and first equations successively, we get

$$\text{for } b \neq 0: \quad \varphi_2 = C_1e^{bt}, \quad \lambda = \frac{b}{C_2 - C_1e^{bt}}, \quad \varphi_1 = C_3e^{bt} - \frac{C_1^2}{b}e^{2bt} - \frac{c}{b};$$
$$\text{for } b = 0: \quad \varphi_2 = C_1, \qquad \lambda = \frac{1}{C_2 - C_1t}, \qquad \varphi_1 = C_3 + (c - C_1^2)t,$$

where C_1, C_2, and C_3 are arbitrary constants. Substituting these expressions into the third equation yields a separable equation, which is easy to integrate.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 38–39).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + c.$

There is a generalized separable solution in the form of a fourth-degree polynomial in x:

$$w = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

5. $\dfrac{\partial w}{\partial t} = -a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial^3 w}{\partial x^3}\right).$

This equation describes flows in a porous medium or in a Hele-Shaw cell which is formed by two immiscible fluids that are separated by a thin interface with thickness $2w$.

1°. Generalized separable solution:

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3 + \varphi_4(t)x^4,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -6a(\varphi_1\varphi_3 + 4\varphi_0\varphi_4),\\
\varphi_1' &= -12a(\varphi_2\varphi_3 + 4\varphi_1\varphi_4),\\
\varphi_2' &= -18a(\varphi_3^2 + 4\varphi_2\varphi_4),\\
\varphi_3' &= -120a\varphi_3\varphi_4,\\
\varphi_4' &= -120a\varphi_4^2.
\end{aligned}$$

This system is easy to integrate in the reverse order, starting from the last equation.

2°. For other solutions see equation 14.1.2.6 with $b = c = 0$ and 14.1.2.9 with $k = 1$.

⊙ *References for equation* 14.1.2.5: H. P. Greenspan (1978), N. F. Smyth and J. M. Hill (1988), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 102–103).

6. $\dfrac{\partial w}{\partial t} = -a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial^3 w}{\partial x^3}\right) + bw + c.$

1°. Generalized separable solution:

$$w = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 + \varphi_3(t)x^3 + \varphi_4(t)x^4,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -6a(\varphi_1\varphi_3 + 4\varphi_0\varphi_4) + b\varphi_0 + c,\\
\varphi_1' &= -12a(\varphi_2\varphi_3 + 4\varphi_1\varphi_4) + b\varphi_1,\\
\varphi_2' &= -18a(\varphi_3^2 + 4\varphi_2\varphi_4) + b\varphi_2,\\
\varphi_3' &= -120a\varphi_3\varphi_4 + b\varphi_3,\\
\varphi_4' &= -120a\varphi_4^2 + b\varphi_4.
\end{aligned}$$

2°. For $c = 0$, there is a generalized separable solution of the form

$$w = \psi_1(t)x^4 + \psi_2(t)x^{3/2}.$$

7. $\dfrac{\partial w}{\partial t} = -a\dfrac{\partial^4}{\partial x^4}(w^2).$

Generalized separable solution:

$$w = C_1 t^{-\frac{9}{128}} x^{\frac{1}{2}} + C_2 t^{-\frac{33}{128}} x^{\frac{3}{2}} + \frac{1}{1680at} x^4,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 138).

8. $\dfrac{\partial w}{\partial t} = -w\dfrac{\partial^4 w}{\partial x^4} + a\dfrac{\partial w}{\partial x}\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2.$

There are exact solutions, polynomial in x, of the following forms:

$$w = \sum_{k=0}^{n} \varphi_k(t) x^k,$$

where

$n = 4$ if a and b are arbitrary constants;
$n = 5$ if $15a + 20b = 6$;
$n = 6$ if $4a + 5b = 2$;
$n = 7$ if $a = \frac{16}{7},\ b = -\frac{10}{7}$;
$n = 8$ if $a = \frac{5}{2},\ b = -\frac{45}{28}$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 102–103).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^k \dfrac{\partial^3 w}{\partial x^3}\right).$

Thin film equation. With $k = 3$, this equation occurs in problems on the motion of long bubbles in tubes and on the spread of drops over a rigid surface.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^k C_2^4 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = A(x + C_1 t + C_2)^{3/k}, \quad A = \left[\frac{C_1 k^3}{3a(k-3)(2k-3)}\right]^{1/k};$$

$$w(x,t) = (Bt + C_1)^{-1/k}(x + C_2)^{4/k}, \quad B = 8ak^{-3}(k+4)(k-4)(2-k).$$

3°. Traveling-wave solution (generalizes the first solution of Item 2°):

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w = w(z)$ is determined by the third-order autonomous ordinary differential equation $aw^k w'''_{zzz} - \lambda w = C_1$. The substitution $U(z) = (w'_z)^2$ leads to the second-order equation

$$aU''_{ww} = \pm 2(\lambda w^{1-k} + C_1 w^{-k})U^{-1/2}.$$

For its solutions at some values of k and C_1, see Polyanin and Zaitsev (2003).

4°. Self-similar solution:

$$w(x,t) = t^{-\frac{4\beta+1}{k}} u(\xi), \qquad \xi = xt^{\beta},$$

where β is an arbitrary constant and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$-(4\beta+1)u + k\beta\xi u'_{\xi} = ak(u^k u'''_{\xi\xi\xi})'_{\xi}.$$

5°. Solution:

$$w(x,t) = (C_1 t + C_2)^{-1/k} V(\zeta), \qquad \zeta = x + C_3 \ln|C_1 t + C_2|,$$

where the function $V = V(\zeta)$ is determined by the autonomous ordinary differential equation

$$ak(V^k V'''_{\zeta\zeta\zeta})'_{\zeta} - kC_1C_3V'_{\zeta} + C_1V = 0.$$

Remark. For a special case $C_3 = 0$, we have a solution in multiplicative separable form.

6°. Generalized self-similar solution:

$$w(x,t) = e^{-4\beta t}\varphi(\eta), \qquad \eta = xe^{k\beta t},$$

where β is an arbitrary constant and the function $\varphi = \varphi(\eta)$ is determined by the ordinary differential equation

$$-4\beta\varphi + k\beta\eta\varphi'_{\eta} = a(\varphi^k\varphi'''_{\eta\eta\eta})'_{\eta}.$$

⊙ *References for equation* 14.1.2.9: F. P. Bretherton (1962), V. M. Starov (1983), A. D. Polyanin and V. F. Zaitsev (2004, p. 594).

10. $\dfrac{\partial w}{\partial t} = aw^{3/2}\dfrac{\partial^2}{\partial x^2}\left(w^{1/2}\dfrac{\partial^2 w}{\partial x^2}\right).$

The equation reduces to a linear equation (for details, see Table 26.1, row 15).

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^3}{\partial x^3}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{a}{f(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{b}{24}x^4 + C_1x^3 + C_2x^2 + C_3x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^4}{\partial x^4}\Big[f(w)\Big] + \dfrac{af(w)+b}{f'(w)} + c[af(w)+b].$

1°. Functional separable solution in implicit form with $a \neq 0$:

$$f(w) = \frac{1}{a}[\rho(x)e^{at} - b],$$

where

$$\rho(x) = \begin{cases} C_1 + C_2x + C_3x^2 + C_4x^3 & \text{if } c = 0, \\ \cosh kx(C_1\cos kx + C_2\sin kx) & \\ \quad + \sinh kx(C_3\cos kx + C_4\sin kx) & \text{if } ac = 4k^4 > 0, \\ C_1\cos kx + C_2\sin kx + C_3\cosh kx + C_4\sinh kx & \text{if } ac = -k^4 < 0, \end{cases}$$

and C_1, C_2, C_3, and C_4 are arbitrary constants.

2°. Functional separable solution in implicit form with $a = 0$:

$$f(w) = bt + C_1 + C_2x + C_3x^2 + C_4x^3 - \tfrac{1}{24}bcx^4.$$

14.2. Equations Involving the Second Derivative in t

14.2.1. Boussinesq Equation and Its Modifications

1. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial^4 w}{\partial x^4} = 0.$

Boussinesq equation in canonical form. This equation arises in several physical applications: propagation of long waves in shallow water, one-dimensional nonlinear lattice-waves, vibrations in a nonlinear string, and ion sound waves in a plasma.

⊙ *References*: J. Boussinesq (1872), M. Toda (1975), A. C. Scott (1975).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(C_1 x + C_2, \pm C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = 2C_1 x - 2C_1^2 t^2 + C_2 t + C_3,$$
$$w(x,t) = (C_1 t + C_2)x - \frac{1}{12C_1^2}(C_1 t + C_2)^4 + C_3 t + C_4,$$
$$w(x,t) = -\frac{(x + C_1)^2}{(t + C_2)^2} + \frac{C_3}{t + C_2} + C_4(t + C_2)^2,$$
$$w(x,t) = -\frac{x^2}{t^2} + C_1 t^3 x - \frac{C_1^2}{54}t^8 + C_2 t^2 + \frac{C_4}{t},$$
$$w(x,t) = -\frac{(x + C_1)^2}{(t + C_2)^2} - \frac{12}{(x + C_1)^2},$$
$$w(x,t) = -3\lambda^2 \cos^{-2}\left[\tfrac{1}{2}\lambda(x \pm \lambda t) + C_1\right],$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 2°):

$$w = w(\zeta), \qquad \zeta = x + \lambda t,$$

where the function $w(\zeta)$ is determined by the second-order ordinary differential equation (C_1 and C_2 are arbitrary constants)

$$w''_{\zeta\zeta} + w^2 + 2\lambda^2 w + C_1\zeta + C_2 = 0.$$

For $C_1 = 0$, this equation is integrable by quadrature.

⊙ *References*: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982).

4°. Self-similar solution:

$$w = \frac{1}{t}U(z), \qquad z = \frac{x}{\sqrt{t}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$U''''_{zzzz} + (UU'_z)'_z + \tfrac{1}{4}z^2 U''_{zz} + \tfrac{7}{4}zU'_z + 2U = 0.$$

⊙ *Reference*: T. Nishitani and M. Tajiri (1982).

5°. Degenerate solution (generalizes the first four solutions of Item 2°):

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi''_{tt} &= -6\varphi^2,\\
\psi''_{tt} &= -6\varphi\psi,\\
\chi''_{tt} &= -2\varphi\chi - \psi^2.
\end{aligned}$$

6°. Solution:

$$w = f(\xi) - 4C_1^2t^2 - 4C_1C_2t, \qquad \xi = x - C_1t^2 - C_2t,$$

where the function $f(\xi)$ is determined by the third-order ordinary differential equation

$$f'''_{\xi\xi\xi} + ff'_\xi + C_2^2f'_\xi - 2C_1f = 8C_1^2\xi + C_3, \tag{1}$$

and C_1, C_2, and C_3 are arbitrary constants. Equation (1) is reduced to the second Painlevé equation.

⊙ *References*: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982), P. A. Clarkson and M. D. Kruskal (1989).

7°. Generalized separable solution (generalizes the penultimate solution of Item 2°):

$$w = (x + C_1)^2u(t) - \frac{12}{(x + C_1)^2},$$

where the function $u = u(t)$ is determined by the second-order autonomous ordinary differential equation

$$u''_{tt} = -6u^2.$$

The function $u(t)$ is representable in terms of the Weierstrass elliptic function.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

8°. Solution:

$$w = \frac{1}{t}F(z) - \frac{1}{4}\left(\frac{x}{t} + Ct\right)^2, \qquad z = \frac{x}{\sqrt{t}} - \frac{1}{3}Ct^{3/2},$$

where C is an arbitrary constant and the function $F = F(z)$ is determined by the fourth-order ordinary differential equation

$$F''''_{zzzz} + (FF'_z)'_z + \tfrac{3}{4}zF'_z + \tfrac{3}{2}F - \tfrac{9}{8}z^2 = 0.$$

Its solutions are expressed via solutions of the fourth Painlevé equation.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

9°. Solution:

$$w(x,t) = (a_1t + a_0)^2U(z) - \left(\frac{a_1x + b_1}{a_1t + a_0}\right)^2, \qquad z = x(a_1t + a_0) + b_1t + b_0.$$

Here, a_1, a_0, b_1, and b_0 are arbitrary constants, and the function $U = U(z)$ is determined by the second-order ordinary differential equation

$$U''_{zz} + \tfrac{1}{2}U^2 = c_1z + c_2, \tag{2}$$

where c_1 and c_2 are arbitrary constants. For $c_1 = 0$, the general solution of equation (2) can be written out in implicit form. If $c_1 \neq 0$, the equation is reduced to the first Painlevé equation (see Conte and Musette, 2008).

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

10°. Solution:

$$w(x,t) = (a_1t + a_0)^2U(z) - \left[\frac{a_1^2x + \lambda(a_1t + a_0)^5 + a_1b_1}{a_1(a_1t + a_0)}\right]^2,$$
$$z = x(a_1t + a_0) + \frac{\lambda}{6a_1^2}(a_1t + a_0)^6 + b_1t + b_0.$$

Here, a_1, a_0, b_1, and b_0 are arbitrary constants, and the function $U = U(z)$ is determined by the third-order ordinary differential equation

$$U'''_{zzz} + UU'_z + 5\lambda U = 50\lambda^2z + c, \tag{3}$$

where c is an arbitrary constant. Equation (3) is reduced to the second Painlevé equation.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

11°. Solution:

$$w(x,t) = \varphi^2(t)U(z) - \frac{1}{\varphi^2(t)}\left[x\varphi'_t(t) + \psi'_t(t)\right]^2, \qquad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of second-order ordinary differential equations

$$\varphi''_{tt} = A\varphi^5, \tag{4}$$
$$\psi''_{tt} = A\varphi^4\psi, \tag{5}$$

where A is an arbitrary constant and the function $U = U(z)$ is determined by the fourth-order ordinary differential equation

$$U''''_{zzzz} + UU''_{zz} + (U'_z)^2 + AzU'_z + 2AU = 2A^2z^2.$$

A first integral of equation (4) is given by

$$(\varphi'_t)^2 = \tfrac{1}{3}A\varphi^6 + B,$$

where B is an arbitrary constant. The general solution of this equation can be expressed in terms of Jacobi elliptic functions. The general solution of equation (5) can be expressed in terms of $\varphi = \varphi(t)$ by

$$\psi = C_1\varphi(t) + C_2\varphi(t)\int \frac{dt}{\varphi^2(t)},$$

where C_1 and C_2 are arbitrary constants.

⊙ *References*: P. A. Clarkson and M. D. Kruskal (1989), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997).

12°. The Boussinesq equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x, y; t)$ as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\frac{1}{\sqrt{3}}\frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0,$$
$$\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial y^3} = 0$$

generates a solution of the Boussinesq equation in the form

$$w = 12\frac{d}{dx}K(x, x; t),$$

where $K(x, y; t)$ is a solution of the linear Gelfand–Levitan–Marchenko integral equation

$$K(x, y; t) + F(x, y; t) + \int_x^\infty K(x, s; t)F(s, y; t)\,ds = 0.$$

Time t appears here as a parameter.

⊙ *References for equation* 14.2.1.1: J. Boussinesq (1871, 1872), R. Hirota (1973), V. E. Zakharov (1973), M. Toda (1975), A. C. Scott (1975), M. J. Ablowitz and H. Segur (1981), T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982), J. Weiss (1985), P. A. Clarkson and M. D. Kruskal (1989), P. A. Clarkson (1995), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997), L. V. Bogdanov and V. E. Zakharov (2002), A. D. Polyanin and V. F. Zaitsev (2004, pp. 595–597).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial^4 w}{\partial x^4}$.

Unnormalized Boussinesq equation.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(C_1 x + C_2, \pm C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = 2C_1x + 2aC_1^2t^2 + C_2t + C_3,$$
$$w(x,t) = (C_1t + C_2)x + \frac{a}{12C_1^2}(C_1t + C_2)^4 + C_3t + C_4,$$
$$w(x,t) = \frac{(x + C_1)^2}{a(t + C_2)^2} + \frac{C_3}{t + C_2} + C_4(t + C_2)^2,$$
$$w(x,t) = \frac{x^2}{at^2} + C_1t^3x + \frac{aC_1^2}{54}t^8 + C_2t^2 + \frac{C_4}{t},$$
$$w(x,t) = \frac{(x + C_1)^2}{a(t + C_2)^2} - \frac{12b}{a(x + C_1)^2},$$
$$w(x,t) = \frac{3\lambda^2}{a}\cosh^{-2}\left[\frac{\lambda}{2\sqrt{b}}(x \pm \lambda t) + C_1\right],$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 2°):

$$w = u(\zeta), \quad \zeta = x + \lambda t,$$

where the function $u = u(\zeta)$ is determined by the second-order ordinary differential equation (C_1 and C_2 are arbitrary constants)

$$bu''_{\zeta\zeta} + au^2 - 2\lambda^2u + C_1\zeta + C_2 = 0.$$

For $C_1 = 0$, this equation is integrable by quadrature.

4°. Self-similar solution:

$$w = \frac{1}{t}U(z), \quad z = \frac{x}{\sqrt{t}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$2U + \tfrac{7}{4}zU'_z + \tfrac{1}{4}z^2U''_{zz} = a(UU'_z)'_z + bU''''_{zzzz}.$$

5°. Degenerate solution (generalizes the first four solutions of Item 2°):

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = 6a\varphi^2,$$
$$\psi''_{tt} = 6a\varphi\psi,$$
$$\chi''_{tt} = 2a\varphi\chi + a\psi^2.$$

6°. Solution:

$$w = f(\xi) + 4aC_1^2t^2 + 4aC_1C_2t, \quad \xi = x + aC_1t^2 + aC_2t,$$

where the function $f(\xi)$ is determined by the third-order ordinary differential equation

$$bf'''_{\xi\xi\xi} + aff'_\xi - a^2C_2^2f'_\xi - 2aC_1f = 8aC_1^2\xi + C_3,$$

and C_1, C_2, and C_3 are arbitrary constants.

7°. Solution (generalizes the penultimate solution of Item 2°):

$$w = (x + C_1)^2 u(t) - \frac{12b}{a(x + C_1)^2},$$

where the function $u = u(t)$ is determined by the second-order autonomous ordinary differential equation

$$u''_{tt} = 6au^2.$$

The function $u(t)$ is expressible in terms of the Weierstrass elliptic function.

8°. Solution:

$$w = \frac{1}{t}F(z) + \frac{1}{4a}\left(\frac{x}{t} + Ct\right)^2, \qquad z = \frac{x}{\sqrt{t}} - \frac{1}{3}Ct^{3/2},$$

where C is an arbitrary constant and the function $F = F(z)$ is determined by the ordinary differential equation

$$a(FF'_z)'_z + bF''''_{zzzz} = \frac{3}{4}zF'_z + \frac{3}{2}F + \frac{9}{8a}z^2.$$

9°. See also equation 14.2.1.3, Item 6°.

⊙ *References for equation* 14.2.1.2: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982), P. A. Clarkson and M. D. Kruskal (1989), A. D. Polyanin and V. F. Zaitsev (2004, pp. 597–599).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + 6\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial^4 w}{\partial x^4}$.

Solutions of this equation can be represented in the form

$$w(x,t) = 2\frac{\partial^2}{\partial x^2}(\ln u), \tag{1}$$

where the function $u = u(x,t)$ is determined by the bilinear equation

$$u\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial u}{\partial t}\right)^2 - u\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} - 3\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0. \tag{2}$$

1°. One- or two-soliton solutions of the original equation are generated by the following solutions of equation (2):

$$u = 1 + A\exp\left(kx \pm kt\sqrt{1+k^2}\right),$$

$$u = 1 + A_1\exp(k_1x + m_1t) + A_2\exp(k_2x + m_2t) + A_1A_2p_{12}\exp\left[(k_1+k_2)x + (m_1+m_2)t\right],$$

where A, A_1, A_2, k, k_1, and k_2 are arbitrary constants, and

$$m_i = \pm k_i\sqrt{1+k_i^2}, \quad p_{12} = \frac{3(k_1-k_2)^2 + (n_1-n_2)^2}{3(k_1+k_2)^2 + (n_1-n_2)^2}, \quad n_i = \frac{m_i}{k_i}.$$

2°. Rational solutions are generated by the following solutions of equation (2):

$$u = x \pm t,$$

$$u = x^2 - t^2 - 3,$$

$$u = (x \pm t)^3 + x \mp 5t.$$

3°. Solution of equation (2):

$$u = \exp(2kx - 2mt) + (Cx - At)\exp(kx - mt) - B,$$

$$A = \frac{C(2k^2+1)}{\sqrt{1+k^2}}, \quad B = \frac{C^2(4k^2+3)}{12k^2(1+k^2)}, \quad m = \sqrt{k^2+k^4},$$

where k and C are arbitrary constants.

4°. Solutions of equation (2):

$$u = \sin(kx - mt) + Ax + Bt,$$
$$u = \sin(kx) + C\sin(mt) + E\cos(mt),$$

where k and C are arbitrary constants, and

$$m = \sqrt{k^2 - k^4}, \quad A = \sqrt{\frac{3m^2}{3-4k^2}}, \quad B = \frac{A(2k^2-1)}{\sqrt{1-k^2}}, \quad E = \sqrt{\frac{1-C^2+k^2C^2-4k^2}{1-k^2}}.$$

5°. Solution (C is an arbitrary constant):

$$u = \sin(kx) + C\exp\bigl(t\sqrt{k^4-k^2}\,\bigr) + \frac{4k^2-1}{4C(k^2-1)}\exp\bigl(-t\sqrt{k^4-k^2}\,\bigr).$$

6°. The substitution $w = \frac{1}{6}(U-1)$ leads to an equation of the form 14.2.1.2:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + \frac{\partial^4 U}{\partial x^4}.$$

⊙ *References for equation* 14.2.1.3: R. Hirota (1973), M. J. Ablowitz and H. Segur (1981), O. V. Kaptsov (1998).

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + c\dfrac{\partial^4 w}{\partial x^4}.$

The substitution $w = U - (a/b)$ leads to an equation of the form 14.2.1.2:

$$\frac{\partial^2 U}{\partial t^2} = b\frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + c\frac{\partial^4 U}{\partial x^4}.$$

14.2.2. Other Equations with Quadratic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + bw\dfrac{\partial^2 w}{\partial x^2} + c.$

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^4 u''''_{\xi\xi\xi\xi} + (b\beta^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w(x,t) = U(z) + bC_1^2t^2 + 2bC_1C_2t, \qquad z = x - \tfrac{1}{2}bC_1t^2 - bC_2t,$$

where C_1 and C_2 are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$aU''''_{zzzz} + bUU''_{zz} - b^2C_2^2U''_{zz} + bC_1U'_z + c - 2bC_1^2 = 0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

This is a special case of equation 15.3.2.2 with $n = 4$.

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)f(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''''_{xxxx} + b(\varphi'_x)^2 - A = 0,$$

whose order can be reduced with the change of variable $U(x) = \varphi'_x$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)f(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial u}{\partial x}\right)^2.$$

This equation admits a traveling-wave solution $u = u(kx + \lambda t)$ and a self-similar solution $u = t^{-1}\phi(z)$, where $z = xt^{-1/2}$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^4 w}{\partial x^4} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + f(t).$

1°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - f(t) = 0,$$
$$\psi''''_{zzzz} - \lambda^2\psi''_{zz} + a(\psi'_z)^2 + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4} + a\left(\frac{\partial u}{\partial x}\right)^2 + bu.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^4 w}{\partial x^4} + aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + c\dfrac{\partial w}{\partial x} + d.$

This is a special case of equation 15.3.5.3 with $n = 4$.

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + c.$

1°. Generalized separable solution:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi_4'' &= 144b\varphi_4^2,\\
\varphi_3'' &= 144b\varphi_3\varphi_4,\\
\varphi_2'' &= 12b(3\varphi_3^2 + 4\varphi_2\varphi_4),\\
\varphi_1'' &= 24b\varphi_2\varphi_3,\\
\varphi_0'' &= 24a\varphi_4 + 4b\varphi_2^2 + c.
\end{aligned}$$

2°. Solution:

$$w(x,t) = C_1x^2 + C_2xt + C_3t^2 + C_4x + C_5t + u(z), \quad z = kx + \lambda t,$$

where $C_1, \ldots, C_5$, k, and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$\lambda^2 u_{zz}'' + 2C_3 = ak^4 u_{zzzz}'''' + b(k^2 u_{zz}'' + 2C_1)^2 + c,$$

which can be integrated with the substitution $U(z) = u_{zz}''$. To the special case $C_1 = \ldots = C_5 = 0$ there corresponds a traveling-wave solution.

3°. Generalized separable solution:

$$w(x,t) = C_1xt^2 + C_2t^2 + C_3t + f(x),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $f(x)$ is determined by the ordinary differential equation

$$2C_1x + 2C_2 = af_{xxxx}'''' + b(f_{xx}'')^2 + c,$$

which can be integrated with the substitution $F(x) = f_{xx}''$.

4°. The substitution

$$w(x,t) = V(x,t) + \tfrac{1}{2}ct^2$$

leads to the simpler equation

$$\frac{\partial^2 V}{\partial t^2} = a\frac{\partial^4 V}{\partial x^4} + b\left(\frac{\partial^2 V}{\partial x^2}\right)^2,$$

which admits a self-similar solution of the form $V = V(\xi)$, where $\xi = xt^{-1/2}$.

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^4 w}{\partial x^4} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)(A_4x^4 + A_3x^3 + A_2x^2 + A_1x) + \psi(t),$$

where A_1, A_2, A_3, and A_4 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 24A_4a\varphi^2 + f(t)\varphi,$$
$$\psi''_{tt} = 24A_4a\varphi\psi + f(t)\psi + g(t).$$

7. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^4 w}{\partial x^4} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations (C is an arbitrary constant):

$$\varphi''_{tt} = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi''_{tt} = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

and the function $\Theta(x)$ satisfies the fourth-order constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\Theta''''_{xxxx} + b\Theta = C.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 - f(t)\dfrac{\partial^2 w}{\partial x^2} - g(t)\dfrac{\partial^4 w}{\partial x^4} - h(t)w - p(t).$

1°. Generalized separable solution for $c/(a+b) = k^2 > 0$:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi'_1 = c\varphi_1^2 + bk^2(\varphi_2^2 + \varphi_3^2) - h(t)\varphi_1 - p(t),$$
$$\varphi'_2 = (2c - ak^2)\varphi_1\varphi_2 + [k^2 f(t) - k^4 g(t) - h(t)]\varphi_2,$$
$$\varphi'_3 = (2c - ak^2)\varphi_1\varphi_3 + [k^2 f(t) - k^4 g(t) - h(t)]\varphi_3.$$

The prime denotes a derivative with respect to t. From the last two equations we have $\varphi''_2/\varphi_2 = \varphi''_3/\varphi_3$. It follows that

$$\varphi_3 = C_1\varphi_2 + C_2\varphi_2\int\frac{dt}{\varphi_2^2}, \tag{1}$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution for $c/(a+b) = -k^2 < 0$:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)\cosh(kx) + \varphi_3(t)\sinh(kx),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1'' &= c\varphi_1^2 + bk^2(\varphi_3^2 - \varphi_2^2) - h(t)\varphi_1 - p(t),\\
\varphi_2'' &= (2c + ak^2)\varphi_1\varphi_2 - [k^2 f(t) + k^4 g(t) + h(t)]\varphi_2,\\
\varphi_3'' &= (2c + ak^2)\varphi_1\varphi_3 - [k^2 f(t) + k^4 g(t) + h(t)]\varphi_3.
\end{aligned}$$

The function φ_3 can be expressed in terms of φ_2 by formula (1).

3°. *Special case*: $a/b = -\frac{4}{3}$ and $bc < 0$.

Generalized separable solution:

$$w(x,t) = \psi_1(t) + \psi_2(t)\cos(kx) + \psi_3(t)\cos(\tfrac{1}{2}kx), \qquad k = \sqrt{-3c/b}.$$

Here, the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1'' &= c\psi_1^2 + bk^2\psi_2^2 + \left(A + \tfrac{1}{4}bk^2\right)\psi_3^2 - h(t)\psi_1 - p(t),\\
\psi_2'' &= (2c - ak^2)\psi_1\psi_2 + A\psi_3^2 + \left[k^2 f(t) - k^4 g(t) - h(t)\right]\psi_2,\\
\psi_3'' &= \left(2c - \tfrac{1}{4}ak^2\right)\psi_1\psi_3 + bk^2\psi_2\psi_3 + \left[\tfrac{1}{4}k^2 f(t) - \tfrac{1}{16}k^4 g(t) - h(t)\right]\psi_3,
\end{aligned}$$

where $A = \frac{1}{8}\left[4c - (a+b)k^2\right]$.

There is a more general solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\cos(kx) + \psi_3(t)\sin(kx) + \psi_4(t)\cos(\tfrac{1}{2}kx) + \psi_5(t)\sin(\tfrac{1}{2}kx),$$

where $k = \sqrt{-3c/b}$.

4°. *Special case*: $a/b = -\frac{4}{3}$ and $bc > 0$.

Generalized separable solution:

$$w(x,t) = \psi_1(t) + \psi_2(t)\cosh(kx) + \psi_3(t)\cosh(\tfrac{1}{2}kx), \qquad k = \sqrt{3c/b}.$$

Here, the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1'' &= c\psi_1^2 - bk^2\psi_2^2 + \left(A - \tfrac{1}{4}bk^2\right)\psi_3^2 - h(t)\psi_1 - p(t),\\
\psi_2'' &= (2c + ak^2)\psi_1\psi_2 + A\psi_3^2 - \left[k^2 f(t) + k^4 g(t) + h(t)\right]\psi_2,\\
\psi_3'' &= \left(2c + \tfrac{1}{4}ak^2\right)\psi_1\psi_3 - k^2\psi_2\psi_3 - \left[\tfrac{1}{4}k^2 f(t) + \tfrac{1}{16}k^4 g(t) + h(t)\right]\psi_3,
\end{aligned}$$

where $A = \frac{1}{8}\left[4c + (a+b)k^2\right]$.

There is a more general solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\cosh(kx) + \psi_3(t)\sinh(kx) + \psi_4(t)\cosh(\tfrac{1}{2}kx) + \psi_5(t)\sinh(\tfrac{1}{2}kx),$$

where $k = \sqrt{3c/b}$.

⊙ *Reference for equation* 14.2.2.8: V. A. Galaktionov (1995).

9. $\dfrac{\partial^2 w}{\partial t^2} = w\dfrac{\partial^2 w}{\partial x^2} - \dfrac{3}{4}\left(\dfrac{\partial w}{\partial x}\right)^2 - a(t)\dfrac{\partial^4 w}{\partial x^4} - b(t)\dfrac{\partial^3 w}{\partial x^3} - c(t)\dfrac{\partial^2 w}{\partial x^2} - d(t)\dfrac{\partial w}{\partial x} - e(t)w - f(t).$

There is a generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

⊙ *Reference*: V. A. Galaktionov (1995).

10. $\dfrac{\partial^2 w}{\partial t^2} = f(t)w\dfrac{\partial^4 w}{\partial x^4} + g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial w}{\partial x} + p(t)w + q(t).$

Generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_4'' &= (24f\varphi_4 + p)\varphi_4,\\
\varphi_3'' &= (24f\varphi_4 + p)\varphi_3 + 4h\varphi_4,\\
\varphi_2'' &= (24f\varphi_4 + p)\varphi_2 + 12g\varphi_4 + 3h\varphi_3,\\
\varphi_1'' &= (24f\varphi_4 + p)\varphi_1 + 6g\varphi_3 + 2h\varphi_2,\\
\varphi_0'' &= (24f\varphi_4 + p)\varphi_2 + 2g\varphi_2 + h\varphi_1 + q.
\end{aligned}$$

▷ *For other equations with quadratic nonlinearities, see Subsection 14.2.1.*

14.2.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + bw\ln w + \big[f(x) + g(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
\varphi_{tt}'' - \big[b\ln\varphi + g(t) + C\big]\varphi &= 0,\\
a\psi_{xxxx}'''' + \big[b\ln\psi + f(x) - C\big]\psi &= 0,
\end{aligned}$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_{xxxx}'''' + f(x)\varphi\ln\varphi + \big[g(x) - b^2\big]\varphi = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t} (t-\tau) g(\tau)\, d\tau + \varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi'''''_{xxxx} + f\left(x, \varphi'_x\right) - 2C_1 = 0,$$

whose order can be reduced with the change of variable $u(x) = \varphi'_x$.

2°. The substitution $w = U(x,t) + \displaystyle\int_0^t (t-\tau) g(\tau)\, d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^4 U}{\partial x^4} + f\left(x, \frac{\partial U}{\partial x}\right).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi'''''_{xxxx} + f\left(x, \psi'_x\right) + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t g(\tau) \sinh\left[k(t-\tau)\right] d\tau \quad \text{if} \quad b = k^2 > 0,$$

$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t g(\tau) \sin\left[k(t-\tau)\right] d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is given in Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^4 U}{\partial x^4} + f\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^3}{\partial x^3}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\, dw = at - \frac{1}{24}bx^4 + C_1 x^3 + C_2 x^2 + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

14.3. Equations Involving Mixed Derivatives

14.3.1. Kadomtsev–Petviashvili Equation and Related Equations

1. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-6w\dfrac{\partial w}{\partial x}\right)+a\dfrac{\partial^2 w}{\partial y^2}=0.$

Kadomtsev–Petviashvili equation in canonical form. It arises in the theory of long, weakly nonlinear surface waves propagating in the x-direction, with the variation in y being sufficiently slow.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(C_1 x + 6C_1\varphi(t),\ \pm C_1^2 y + C_2,\ C_1^3 t + C_3\big) + \varphi_t'(t),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. The time-invariant solutions satisfy the Boussinesq equation 14.2.1.2 (see also equation 14.2.1.1). The y-independent solutions satisfy the Korteweg–de Vries equation 13.1.1.1 (see also equation 13.1.4.1).

3°. One-soliton solution:

$$w(x,y,t) = -2\frac{\partial^2}{\partial x^2}\ln\big[1 + Ae^{kx+kpy-k(k^2+ap^2)t}\big],$$

where A, k, and p are arbitrary constants.

4°. Two-soliton solution:

$$w(x,y,t) = -2\frac{\partial^2}{\partial x^2}\ln\big(1 + A_1e^{\eta_1} + A_2e^{\eta_2} + A_1A_2Be^{\eta_1+\eta_2}\big),$$
$$\eta_i = k_ix + k_ip_iy - k_i(k_i^2 + ap_i^2)t, \quad B = \frac{3(k_1-k_2)^2 - a(p_1-p_2)^2}{3(k_1+k_2)^2 - a(p_1-p_2)^2},$$

where A_1, A_2, k_1, k_2, p_1, and p_2 are arbitrary constants.

5°. N-soliton solution:

$$w(x,y,t) = -2\frac{\partial^2}{\partial x^2}\ln\det\mathbf{A},$$

where $\mathbf{A}$ is an $N\times N$ matrix with entries

$$A_{nm} = \delta_{nm} + f_n(y,t)\frac{\exp[(p_n+q_m)x]}{p_n+q_m}, \quad \delta_{nm} = \begin{cases} 1 & \text{if } n=m,\\ 0 & \text{if } n\neq m,\end{cases}$$
$$f_n(y,t) = C_n\exp\big[\sqrt{3/a}\,(q_n^2-p_n^2)y + 4(p_n^3+q_n^3)t\big], \quad n,m = 1,\,2,\,\ldots,\,N,$$

and the p_n, q_m, and C_n are arbitrary constants ($C_n > 0$).

6°. Rational solutions:

$$w(x,y,t) = -2\frac{\partial^2}{\partial x^2}\ln(x + py - ap^2t),$$
$$w(x,y,t) = -2\frac{\partial^2}{\partial x^2}\ln\left[(x+p_1y-ap_1^2t)(x+p_2y-ap_2^2t) + \frac{12}{a(p_1-p_2)^2}\right],$$

where p, p_1, and p_2 are arbitrary constants.

7°. Two-dimensional power-law decaying solution ($a = -1$):

$$w(x,y,t) = 4\frac{(\widetilde{x}+\beta\widetilde{y})^2 - \gamma^2(\widetilde{y})^2 - 3/\gamma^2}{\left[(\widetilde{x}+\beta\widetilde{y})^2 + \gamma^2(\widetilde{y})^2 + 3/\gamma^2\right]^2}, \qquad \widetilde{x} = x - (\beta^2+\gamma^2)t, \quad \widetilde{y} = y + 2\beta t,$$

where β and γ are arbitrary constants.

⊙ *Reference*: M. J. Ablowitz and H. Segur (1981).

8°. "Two-dimensional" solution:

$$w = U(z,t) + \tfrac{1}{6}a\lambda^2, \qquad z = x + \lambda y,$$

where λ is an arbitrary constant and the function $U = U(z,t)$ is determined by a third-order differential equation of the form 13.1.4.1:

$$\frac{\partial U}{\partial t} + \frac{\partial^3 U}{\partial z^3} - 6U\frac{\partial U}{\partial z} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function. For $\varphi = 0$ we have the Korteweg–de Vries equation 13.1.1.1.

9°. "Two-dimensional" solution:

$$w = V(\xi,t), \qquad \xi = x + C_1 y - aC_1^2 t,$$

where C_1 and C_2 are arbitrary constants and the function $V = V(\xi,t)$ is determined by a third-order differential equation of the form 13.1.4.1:

$$\frac{\partial V}{\partial t} + \frac{\partial^3 V}{\partial \xi^3} - 6V\frac{\partial V}{\partial \xi} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function. For $\varphi = 0$ we have the Korteweg–de Vries equation 13.1.1.1.

10°. "Two-dimensional" solution:

$$w(x,y,t) = u(\eta,t), \qquad \eta = x + \frac{y^2}{4at},$$

where the function $u(\eta,t)$ is determined by the third-order differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial \eta^3} - 6u\frac{\partial u}{\partial \eta} + \frac{1}{2t}u = \psi(t).$$

with $\psi(t)$ being an arbitrary function. For $\psi = 0$ we have the cylindrical Korteweg–de Vries equation 13.1.2.1.

⊙ *References*: R. S. Johnson (1979), F. Calogero and A. Degasperis (1982).

11°. There is a degenerate solution quadratic in x:

$$w = x^2\varphi(y,t) + x\psi(y,t) + \chi(y,t).$$

12°. The Kadomtsev–Petviashvili equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x,z;y,t)$ as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\sqrt{\frac{a}{3}}\,\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0,$$
$$\frac{\partial F}{\partial t} + 4\left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3}\right) = 0$$

generates a solution of the Kadomtsev–Petviashvili equation in the form

$$w = -2\frac{d}{dx}K(x,x;y,t),$$

where $K = K(x,z;y,t)$ is a solution to the linear Gelfand–Levitan–Marchenko integral equation

$$K(x,z;y,t) + F(x,z;y,t) + \int_x^\infty K(x,s;y,t)F(s,z;y,t)\,ds = 0.$$

The quantities y and t appear here as parameters.

⊙ *References for equation* 14.3.1.1: B. B. Kadomtsev and V. I. Petviashvili (1970), V. S. Dryuma (1974), V. E. Zakharov and A. B. Shabat (1974), I. M. Krichever and S. P. Novikov (1978), R. S. Johnson (1979), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), V. E. Adler, A. B. Shabat, and R. I. Yamilov (2000), A. D. Polyanin and V. F. Zaitsev (2004, pp. 605–606).

2. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x}\right) + c\dfrac{\partial^2 w}{\partial y^2} = 0.$

Unnormalized Kadomtsev–Petviashvili equation. The transformation $w = -\dfrac{6a}{b}U(x,y,\tau)$, $\tau = at$ leads to an equation of the form 14.3.1.1:

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t} + \frac{\partial^3 U}{\partial x^3} - 6U\frac{\partial U}{\partial x}\right) + \frac{c}{a}\frac{\partial^2 U}{\partial y^2} = 0.$$

3. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + 12w^2\dfrac{\partial w}{\partial x}\right) + k\dfrac{\partial^2 w}{\partial y^2} = 0.$

Kadomtsev–Petviashvili type equation with a cubic nonlinearity. The y-independent solutions satisfy the modified Korteweg–de Vries equation 13.1.1.6.

Traveling-wave solutions:

$$w(x,y,t) = \pm\sqrt{-\frac{ac+b^2k}{2a^2}}\,\cosh^{-1}\left[\sqrt{-\frac{ac+b^2k}{a^4}}\,(ax+by+ct+d)\right],$$

where a, b, c, and d are arbitrary constants, and $ac + b^2k < 0$.

⊙ *Reference*: L. Zhang (2011).

14.3.2. Stationary Hydrodynamic Equations (Navier–Stokes Equations)

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w, \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional stationary *equations of a viscous incompressible fluid*

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$
$$u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

are reduced to the equation in question by the introduction of a *stream function* w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = -w(y,\ x),$$
$$w_2 = w(C_1x + C_2,\ C_1y + C_3) + C_4,$$
$$w_3 = w(x\cos\alpha + y\sin\alpha,\ -x\sin\alpha + y\cos\alpha),$$

where $C_1, \ldots, C_4$ and α are arbitrary constants, are also solutions of the equation.

⊙ *Reference*: V. V. Pukhnachov (1960).

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). On the utilization of these solutions in the hydrodynamics of ideal fluids, see Lamb (1945), Batchelor (1970), Lavrent'ev and Shabat (1973), Sedov (1980), and Loitsyanskiy (1996).

3°. Solutions in the form of a one-variable function or the sum of functions with different arguments:

$$w(y) = C_1y^3 + C_2y^2 + C_3y + C_4,$$
$$w(x, y) = C_1x^2 + C_2x + C_3y^2 + C_4y + C_5,$$
$$w(x, y) = C_1\exp(-\lambda y) + C_2y^2 + C_3y + C_4 + \nu\lambda x,$$
$$w(x, y) = C_1\exp(\lambda x) - \nu\lambda x + C_2\exp(\lambda y) + \nu\lambda y + C_3,$$
$$w(x, y) = C_1\exp(\lambda x) + \nu\lambda x + C_2\exp(-\lambda y) + \nu\lambda y + C_3,$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants.

⊙ *References*: J. L. M. Poiseuille (1840), M. Couette (1890), R. Berker (1963, Section 15), V. V. Pukhnachov (1960), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *c*).

4°. Generalized separable solutions:

$$w(x, y) = A(kx + \lambda y)^3 + B(kx + \lambda y)^2 + C(kx + \lambda y) + D,$$
$$w(x, y) = Ae^{-\lambda(y+kx)} + B(y + kx)^2 + C(y + kx) + \nu\lambda(k^2 + 1)x + D,$$

where A, B, C, D, k, β, and λ are arbitrary constants.

⊙ *Reference*: V. V. Pukhnachov (1960).

5°. Generalized separable solutions:

$$w(x,y) = 6\nu x(y+\lambda)^{-1} + A(y+\lambda)^3 + B(y+\lambda)^{-1} + C(y+\lambda)^{-2} + D \quad (\nu \neq 0),$$
$$w(x,y) = (Ax+B)e^{-\lambda y} + \nu\lambda x + C,$$
$$w(x,y) = \big[A\sinh(\beta x) + B\cosh(\beta x)\big]e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2+\lambda^2)x + C,$$
$$w(x,y) = \big[A\sin(\beta x) + B\cos(\beta x)\big]e^{-\lambda y} + \frac{\nu}{\lambda}(\lambda^2-\beta^2)x + C,$$
$$w(x,y) = Ae^{\lambda y+\beta x} + Be^{\gamma x} + \nu\gamma y + \frac{\nu}{\lambda}\gamma(\beta-\gamma)x + C, \quad \gamma = \pm\sqrt{\lambda^2+\beta^2},$$

where A, B, C, D, k, β, and λ are arbitrary constants.

Special case. Setting $A = -\nu\lambda$, $B = C = 0$, $\lambda = \sqrt{k/\nu}$ in the second solution, we obtain

$$w = \sqrt{k\nu}\,x\big[1 - \exp\big(-\sqrt{k/\nu}\,y\big)\big].$$

This solution describes the steady-state motion of a fluid due to the motion of the surface particles at $y = 0$ with a velocity $u_1|_{y=0} = kx$.

⊙ *References*: D. Riabouchinsky (1924), L. I. J. Kovásznay (1948), C. Y. Wang (1966), S. P. Lin and M. Tobak (1986), A. D. Polyanin (2001 *c*).

6°. Generalized separable solution linear in x (generalizes the first equation from Item 5°):

$$w(x,y) = F(y)x + G(y), \tag{1}$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_y F''_{yy} - FF'''_{yyy} = \nu F''''_{yyyy}, \tag{2}$$
$$G'_y F''_{yy} - FG'''_{yyy} = \nu G''''_{yyyy}. \tag{3}$$

On integrating the equations once, we obtain the system of third-order equations

$$(F'_y)^2 - FF''_{yy} = \nu F'''_{yyy} + A, \tag{4}$$
$$G'_y F'_y - FG''_{yy} = \nu G'''_{yyy} + B, \tag{5}$$

where A and B are arbitrary constants. The order of the autonomous equation (4) can be reduced by one.

Equation (2) has the following particular solutions:

$$F(y) = ay + b, \tag{6}$$
$$F(y) = 6\nu(y+a)^{-1}, \tag{7}$$
$$F(y) = ae^{-\lambda y} + \lambda\nu, \tag{8}$$

where a, b, and λ are arbitrary constants.

In the general case, equation (5) is reduced, with the substitution $U = G'_y$, to the second-order linear nonhomogeneous equation

$$\nu U''_{yy} + FU'_y - F'_y U + B = 0, \qquad \text{where} \quad U = G'_y. \tag{9}$$

The corresponding homogeneous equation (with $B = 0$) has two linearly independent particular solutions:

$$U_1 = \begin{cases} F''_{yy} & \text{if } F''_{yy} \not\equiv 0, \\ F & \text{if } F''_{yy} \equiv 0, \end{cases} \quad U_2 = U_1 \int \frac{\Phi\,dy}{U_1^2}, \qquad \text{where} \quad \Phi = \exp\left(-\frac{1}{\nu}\int F\,dy\right); \tag{10}$$

the first solution follows from the comparison of (2) and (9) with $B = 0$. Therefore the general solutions of equations (9) and (3) are given by

$$U = C_1U_1 + C_2U_2 + C_3\left(U_2\int \frac{U_1}{\Phi}\,dy - U_1\int \frac{U_2}{\Phi}\,dy\right), \quad G = \int U\,dy + C_4, \quad C_3 = -\frac{B}{\nu}; \tag{11}$$

see Polyanin and Zaitsev (2003).

The general solution of equation (3) corresponding to the particular solution (7) is expressed as

$$G(y) = \widetilde{C}_1(y+a)^3 + \widetilde{C}_2 + \widetilde{C}_3(y+a)^{-1} + \widetilde{C}_4(y+a)^{-2},$$

where $\widetilde{C}_1, \ldots, \widetilde{C}_4$ are arbitrary constants (these are expressed in terms of $C_1, \ldots, C_4$).

The general solutions of equation (3) corresponding to the particular solutions (6) and (8) are given by (10) and (11), respectively.

Special case 1. For $G(y) \equiv 0$, formula (1) and equation (4) with $A = k^2$ and the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = k$$

describe the flow of a viscous fluid at a stagnation point. In this case, the stream function far away from the rigid surface is determined by the stream function for inviscid fluid, $w|_{y\to\infty} \to kxy$. This problem was numerically solved by Hiemenz (1911) and Howarth (1934).

Special case 2. A solution of the form (1) with $G(y) = kF(y)$ describes a laminar fluid flow in a plane channel with porous walls. In this case, equation (3) is satisfied by virtue of (2).

⊙ *References*: K. Hiemenz (1911), A. S. Berman (1953), N. Rott (1956), J. T. Stuart (1959), R. Berker (1963), E. Becker (1976), K. Tamada (1979), M. J. Dorrepaal (1986), A. D. Polyanin (2001 *c*), P. G. Drazin and N. Riley (2006).

7°. Solution (generalizes the solution of Item 6°):

$$w(x,y) = F(z)x + G(z), \qquad z = y + kx,$$

where the functions $F = F(z)$ and $G = G(z)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_zF''_{zz} - FF'''_{zzz} = \nu(k^2+1)F''''_{zzzz}, \tag{12}$$

$$G'_zF''_{zz} - FG'''_{zzz} = \nu(k^2+1)G''''_{zzzz} + 4k\nu F'''_{zzz} + \frac{2k}{k^2+1}FF''_{zz}. \tag{13}$$

On integrating the equations once, we obtain the system of third-order equations

$$(F'_z)^2 - FF''_{zz} = \nu(k^2+1)F'''_{zzz} + A, \tag{14}$$

$$G'_zF'_z - FG''_{zz} = \nu(k^2+1)G'''_{zzz} + \psi(z) + B, \tag{15}$$

where A and B are arbitrary constants, and the function $\psi(z)$ is defined by

$$\psi(z) = 4k\nu F''_{zz} + \frac{2k}{k^2+1}\int FF''_{zz}\,dz.$$

The order of the autonomous equation (14) can be reduced by one.

Equation (12) has the following particular solutions:

$$F(z) = az + b, \quad z = y + kx,$$
$$F(z) = 6\nu(k^2 + 1)(z + a)^{-1},$$
$$F(z) = ae^{-\lambda z} + \lambda\nu(k^2 + 1),$$

where a, b, and λ are arbitrary constants.

In the general case, equation (15) is reduced, with the substitution $U = G'_z$, to a second-order linear nonhomogeneous equation, a particular solution of which, in the homogeneous case $\psi = B = 0$, is given by

$$U = \begin{cases} F''_{zz} & \text{if } F''_{zz} \not\equiv 0, \\ F & \text{if } F''_{zz} \equiv 0. \end{cases}$$

Consequently, the general solution to (15) can be expressed by quadrature; see Polyanin and Zaitsev (2003).

⊙ *Reference*: A. D. Polyanin (2001 *c*).

8°. Self-similar solution:

$$w = \int F(z)\,dz + C_1, \quad z = \arctan\left(\frac{x}{y}\right),$$

where the function F is determined by the first-order autonomous ordinary differential equation

$$3\nu(F'_z)^2 - 2F^3 + 12\nu F^2 + C_2F + C_3 = 0, \tag{16}$$

and C_1, C_2, and C_3 are arbitrary constants. The general solution of equation (16) can be written out in implicit form and also can be expressed in terms of the Weierstrass elliptic function.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

9°. There is an exact solution of the form

$$w = a\ln|x| + \int V(z)\,dz + C_1, \quad z = \arctan\left(\frac{x}{y}\right).$$

To $a = 0$ there corresponds a self-similar solution of (16).

⊙ *References for equation* 14.3.2.1: A. S. Berman (1953), V. V. Pukhnachov (1960), R. Berker (1963), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *c*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 333–336).

▷ *For other exact solutions of the Navier–Stokes equations, see equation 14.3.2.4 and Subsection 21.3.1 (steady-state solutions).*

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w + f(y), \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The system

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1 + F(y),$$
$$u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$. The above system of equations describes the plane flow of a viscous incompressible fluid under the action of a transverse force. Here, $f(y) = F_y'(y)$.

The case $F(y) = a\sin(\lambda y)$ corresponds to *Kolmogorov's model*, which is used for describing subcritical and transitional (laminar-to-turbulent) flow modes.

⊙ *Reference*: O. M. Belotserkovskii and A. M. Oparin (2000, pp. 106–110).

1°. Solution in the form of a one-argument function:

$$w(y) = -\frac{1}{6\nu}\int_0^y (y-z)^3 f(z)\,dz + C_1y^3 + C_2y^2 + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

2°. Additive separable solution for arbitrary $f(y)$:

$$w(x,y) = -\frac{1}{2\nu}\int_0^y (y-z)^2\Phi(z)\,dz + C_1e^{-\lambda y} + C_2y^2 + C_3y + C_4 + \nu\lambda x,$$
$$\Phi(z) = e^{-\lambda z}\int e^{\lambda z} f(z)\,dz,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

Special case. If $f(y) = a\beta\cos(\beta y)$, which corresponds to $F(y) = a\sin(\beta y)$, it follows from the preceding formula with $C_1 = C_2 = C_4 = 0$ and $B = -\nu\lambda$ that

$$w(x,y) = -\frac{a}{\beta^2(B^2+\nu^2\beta^2)}\left[B\sin(\beta y) + \nu\beta\cos(\beta y)\right] + Cy - Bx,$$

where B and C are arbitrary constants. This solution is specified in the book by Belotserkovskii and Oparin (2000); it describes a flow with a periodic structure.

3°. Additive separable solution for $f(y) = Ae^{\lambda y} + Be^{-\lambda y}$:

$$w(x,y) = C_1e^{-\lambda x} + C_2x - \frac{A}{\lambda^3(C_2+\nu\lambda)}e^{\lambda y} + \frac{B}{\lambda^3(C_2-\nu\lambda)}e^{-\lambda y} - \nu\lambda y,$$

where C_1 and C_2 are arbitrary constants.

4°. Generalized separable solution linear in x:

$$w(x,y) = \varphi(y)x + \psi(y),$$

where the functions $\varphi = \varphi(y)$ and $\psi = \psi(y)$ are determined by the system of fourth-order ordinary differential equations

$$\varphi_y'\varphi_{yy}'' - \varphi\varphi_{yyy}''' = \nu\varphi_{yyyy}'''', \tag{1}$$
$$\psi_y'\varphi_{yy}'' - \varphi\psi_{yyy}''' = \nu\psi_{yyyy}'''' + f(y). \tag{2}$$

On integrating once, we obtain the system of third-order equations

$$(\varphi_y')^2 - \varphi\varphi_{yy}'' = \nu\varphi_{yyy}''' + A, \tag{3}$$
$$\psi_y'\varphi_y' - \varphi\psi_{yy}'' = \nu\psi_{yyy}''' + \int f(y)\,dy + B, \tag{4}$$

where A and B are arbitrary constants. The order of the autonomous equation (3) can be reduced by one.

Equation (1) has the following particular solutions:

$$\varphi(y) = ay + b, \quad \varphi(y) = 6\nu(y+a)^{-1}, \quad \varphi(y) = ae^{-\lambda y} + \lambda\nu,$$

where a, b, and λ are arbitrary constants.

In the general case, equation (4) is reduced, with the substitution $U = \psi'_y$, to the second-order linear nonhomogeneous equation

$$\nu U''_{yy} + \varphi U'_y - \varphi'_y U + F = 0, \qquad \text{where} \quad U = \psi'_y, \quad F = \int f(y)\,dy + B. \tag{5}$$

The corresponding homogeneous equation (with $F = 0$) has two linearly independent particular solutions:

$$U_1 = \begin{cases} \varphi''_{yy} & \text{if } \varphi \neq ay + b, \\ \varphi & \text{if } \varphi = ay + b, \end{cases} \quad U_2 = U_1 \int \frac{\Phi\,dy}{U_1^2}, \qquad \text{where} \quad \Phi = \exp\left(-\frac{1}{\nu}\int \varphi\,dy\right);$$

the first solution follows from the comparison of (1) and (5) with $F = 0$. Consequently, the general solutions of equations (5) and (2) are given by

$$U = C_1U_1 + C_2U_2 + \frac{1}{\nu}U_1 \int U_2 \frac{F}{\Phi}\,dy - \frac{1}{\nu}U_2 \int U_1 \frac{F}{\Phi}\,dy, \quad \psi = \int U\,dy + C_4;$$

see Polyanin and Zaitsev (2003).

⊙ *Reference for equation* 14.3.2.2: A. D. Polyanin and V. F. Zaitsev (2002, pp. 336–337).

3. $\left(\dfrac{\partial w}{\partial y} + ax\right)\dfrac{\partial}{\partial x}(\Delta w) - \left(\dfrac{\partial w}{\partial x} - ay\right)\dfrac{\partial}{\partial y}(\Delta w) + 2a\Delta w = \nu\Delta\Delta w.$

This equation is used for describing the motion of a viscous incompressible fluid induced by two parallel disks, moving towards each other; see Craik (1989) and equation 14.3.3.2 in the unsteady case.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= -w(y,\ x), \\ w_2 &= w(x + C_1,\ y + C_2) - aC_2x + aC_1y + C_3, \\ w_3 &= w(x\cos\beta + y\sin\beta,\ -x\sin\beta + y\cos\beta), \end{aligned}$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson equation, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Solution dependent on a single coordinate x:

$$w(x) = \int_0^x (x - \xi)U(\xi)\,d\xi + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $U(x)$ is determined by the second-order linear ordinary differential equation

$$axU'_x + 2aU = \nu U''_{xx}.$$

The general solution to this equation can be found in Polyanin and Zaitsev (2003).

Likewise, we can obtain solutions of the form $w = w(y)$.

4°. Generalized separable solution linear in x:

$$w(x,y) = F(y)x + G(y), \tag{1}$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the fourth-order ordinary differential equations

$$F'_y F''_{yy} - FF'''_{yyy} + a(3F''_{yy} + yF'''_{yyy}) = \nu F''''_{yyyy}, \tag{2}$$
$$F''_{yy} G'_y - FG'''_{yyy} + a(2G''_{yy} + yG'''_{yyy}) = \nu G''''_{yyyy}. \tag{3}$$

Equation (2) is solved independently of equation (3). If $F = F(y)$ is a solution to (2), then the function

$$F_1 = F(y + C) - aC,$$

where C is an arbitrary constant, is also a solution of the equation.

Integrating (2) and (3) with respect to y yields

$$(F'_y)^2 - FF''_{yy} + a(2F'_y + yF''_{yy}) = \nu F'''_{yyy} + C_1, \tag{4}$$
$$F'_y G'_y - FG''_{yy} + a(G'_y + yG''_{yy}) = \nu G'''_{yyy} + C_2, \tag{5}$$

where C_1 and C_2 are arbitrary constants.

Equation (2) has a particular solution

$$F(y) = Ay + B, \tag{6}$$

where A and B are arbitrary constants. On substituting (6) into (5) and performing the change of variable $Q = G''_{yy}$, we obtain the second-order linear ordinary differential equation

$$-\big[(A-a)y + B\big]Q'_y + 2aQ = \nu Q''_{yy},$$

whose general solution can be found in Polyanin and Zaitsev (2003).

Solutions of the form $w(x,y) = f(x)y + g(x)$ can be obtained likewise.

5°. Note that equation (2) has the following particular solutions:

$$F = ay + C_1 \exp\big(-2\sqrt{a/\nu}\, y\big) + C_2 \exp\big(2\sqrt{a/\nu}\, y\big) \quad \text{if} \quad a > 0,$$
$$F = ay + C_1 \cos\big(2\sqrt{-a/\nu}\, y\big) + C_2 \sin\big(2\sqrt{-a/\nu}\, y\big) \quad \text{if} \quad a < 0,$$

where C_1 and C_2 are arbitrary constants.

⊙ *References for equation* 14.3.2.3: S. N. Aristov and I. M. Gitman (2002), A. D. Polyanin and V. F. Zaitsev (2004, pp. 611–612).

4. $\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\dfrac{\partial}{\partial r}(\Delta w) - \dfrac{1}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial}{\partial \theta}(\Delta w) = \nu \Delta\Delta w, \quad \Delta w = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}.$

Preliminary remarks. Equation 14.3.2.1 is reduced to the equation in question by passing to the polar coordinate system with origin at (x_0, y_0), where x_0 and y_0 are any numbers, according to

$$x = r\cos\theta + x_0, \qquad y = r\sin\theta + y_0 \qquad \text{(direct transformation)},$$
$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad \tan\theta = \frac{y-y_0}{x-x_0} \qquad \text{(inverse transformation)}.$$

The radial and angular fluid velocity components are expressed via the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$, $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions).

2°. Solutions in the form of a one-variable function and the sum of functions with different arguments:

$$w(r) = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4,$$
$$w(r,\theta) = A\nu\theta + C_1 r^{A+2} + C_2 r^2 + C_3 \ln r + C_4,$$

where $A, C_1, \ldots, C_4$ are arbitrary constants.

3°. Solution:

$$w = b\theta + U(\xi), \qquad \xi = \theta + a \ln r, \tag{1}$$

where the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$\nu(a^2+1)U^{(4)}_{\xi} - a(b+4\nu)U'''_{\xi\xi\xi} + 2(b+2\nu)U''_{\xi\xi} + 2U'_{\xi}U''_{\xi\xi} = 0.$$

The onefold integration yields

$$\nu(a^2+1)U'''_{\xi\xi\xi} - a(b+4\nu)U''_{\xi\xi} + 2(b+2\nu)U'_{\xi} + (U'_{\xi})^2 = C_1, \tag{2}$$

where C_1 is an arbitrary constant. Equation (2) is autonomous and independent of U explicitly. The transformation

$$z = U'_{\xi}, \qquad u(z) = U''_{\xi\xi}$$

brings it to the Abel equation of the second kind

$$\nu(a^2+1)uu'_z - a(b+4\nu)u + 2(b+2\nu)z + z^2 = C_1, \tag{3}$$

which is integrable by quadrature in some cases; for example, in the cases $a = 0$ and $b = -4\nu$, we have

$$\nu u^2 + \tfrac{2}{3}z^3 + 2(b+2\nu)z^2 = 2C_1 z + C_2 \quad \text{if} \quad a = 0,$$
$$\nu(a^2+1)u^2 + \tfrac{2}{3}z^3 - 4\nu z^2 = 2C_1 z + C_2 \quad \text{if} \quad b = -4\nu.$$

Four other solvable cases for equation (3) are presented in the book by Polyanin and Zaitsev (2003); (3) is first reduced to a canonical form with the change of variable $u = k\bar{u}$, where $k = \text{const}$.

Note that to $a = b = 0$ in (1)–(3) there corresponds a solution dependent on the angle θ alone; this solution can be written out in implicit form (see equation 14.3.2.1, Item 8°).

4°. Generalized separable solution linear in θ:

$$w(r,\theta) = f(r)\theta + g(r).$$

Here, $f = f(r)$ and $g = g(r)$ are determined by the system of ordinary differential equations

$$-f'_r\mathbf{L}(f) + f[\mathbf{L}(f)]'_r = \nu r\mathbf{L}^2(f), \tag{4}$$
$$-g'_r\mathbf{L}(f) + f[\mathbf{L}(g)]'_r = \nu r\mathbf{L}^2(g), \tag{5}$$

where $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

A particular solution to (4) is given by $f(r) = C_1 \ln r + C_2$. The corresponding equation (5) is reduced, with the substitution $Q = \mathbf{L}(g)$, to a second-order linear equation, which is easy

to integrate (since it has a particular solution $Q = 1$). Consequently, we obtain an exact solution of system (4)–(5) in the form

$$f(r) = C_1 \ln r + C_2, \quad g(r) = C_3 r^2 + C_4 \ln r + C_5 \int \left[\int rQ(r)\,dr \right] \frac{dr}{r} + C_6,$$
$$Q(r) = \int r^{(C_2/\nu)-1} \exp\left(\frac{C_1}{2\nu} \ln^2 r \right) dr,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

⊙ *References for equation* 14.3.2.4: G. B. Jeffery (1915), V. V. Pukhnachov (1960), R. Berker (1963), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *c*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 337–338).

▷ *For other exact solutions of the Navier–Stokes equations in polar (cylindrical) coordinates, see Subsection 21.3.2 (steady-state solutions).*

5. $$\frac{1}{r}\left(\frac{\partial w}{\partial z}\frac{\partial \mathrm{E}w}{\partial r} - \frac{\partial w}{\partial r}\frac{\partial \mathrm{E}w}{\partial z} \right) - \frac{2}{r^2}\frac{\partial w}{\partial z}\mathrm{E}w = \nu \mathrm{E}^2 w,$$
$$\text{where } \mathrm{E}w = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2}, \quad \mathrm{E}^2 w = \mathrm{E}(\mathrm{E}w).$$

Preliminary remarks. The stationary Navier–Stokes equations written in cylindrical coordinates for the axisymmetric case can be reduced to the equation in question by the introduction of a stream function w such that $u_r = \frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z = -\frac{1}{r}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2 + y^2}$, and u_r and u_z are the radial and axial fluid velocity components; see Happel and Brenner (1965).

1°. Any function $w = w(r, z)$ that solves the second-order linear equation $\mathrm{E}w = 0$ is also a solution of the original equation.

2°. Solutions in the form of a one-argument function and the sum of functions with different arguments:

$$w(r) = C_1 r^4 + C_2 r^2 \ln r + C_3 r^2 + C_4,$$
$$w(r, z) = A\nu z + C_1 r^{A+2} + C_2 r^4 + C_3 r^2 + C_4,$$

where $A, C_1, \ldots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(r, z) = r^2 f(z),$$

where the function $f = f(z)$ is determined by the ordinary differential equation (C is an arbitrary constant):

$$\nu f'''_{zzz} + 2f f''_{zz} - (f'_z)^2 = C. \tag{1}$$

This solution describes an axisymmetric fluid flow towards a plane (flow near a stagnation point).

4°. Generalized separable solution quadratic in r (generalizes the solution of Item 3°):

$$w(r, z) = r^2 f(z) + Az + B,$$

where A and B are arbitrary constants, and the function $f = f(z)$ is determined by the ordinary differential equation (1).

5°. Generalized separable solution linear in z:

$$w(r,z)=\varphi(r)z+\psi(r).$$

Here, $\varphi=\varphi(r)$ and $\psi=\psi(r)$ are determined by the system of ordinary differential equations

$$\varphi[\mathbf{L}(\varphi)]'_r-\varphi'_r\mathbf{L}(\varphi)-2r^{-1}\varphi\,\mathbf{L}(\varphi)=\nu r\mathbf{L}^2(\varphi), \tag{2}$$

$$\varphi[\mathbf{L}(\psi)]'_r-\psi'_r\mathbf{L}(\varphi)-2r^{-1}\varphi\,\mathbf{L}(\psi)=\nu r\mathbf{L}^2(\psi), \tag{3}$$

where $\mathbf{L}(\varphi)=\varphi''_{rr}-r^{-1}\varphi'_r$.

Particular solution of equation (2):

$$\varphi(r)=C_1r^2+C_2,$$

where C_1 and C_2 are arbitrary constants. In this case, the change of variable $U=\mathbf{L}(\psi)$ brings (3) to a second-order linear equation.

⊙ *References for equation* 14.3.2.5: H. Schlichting (1981), A. D. Polyanin and V. F. Zaitsev (2002, pp. 338–339).

▷ *For axisymmetric steady-state solutions of the Navier–Stokes equations, see also Subsection 21.3.3.*

6. $$\frac{1}{r^2\sin\theta}\left(\frac{\partial w}{\partial\theta}\frac{\partial \mathrm{E}w}{\partial r}-\frac{\partial w}{\partial r}\frac{\partial \mathrm{E}w}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\left(2\cot\theta\frac{\partial w}{\partial r}-\frac{2}{r}\frac{\partial w}{\partial\theta}\right)\mathrm{E}w=\nu\mathrm{E}^2w,$$

$$\text{where}\quad \mathrm{E}w=\frac{\partial^2 w}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\theta}\right),\qquad \mathrm{E}^2w=\mathrm{E}(\mathrm{E}w).$$

Preliminary remarks. The stationary Navier–Stokes equations written in spherical coordinates for the axisymmetric case are reduced to the given equation through the introduction of a stream function w such that $u_r=\frac{1}{r^2\sin\theta}\frac{\partial w}{\partial\theta}$ and $u_\theta=-\frac{1}{r\sin\theta}\frac{\partial w}{\partial r}$, where $r=\sqrt{x^2+y^2+z^2}$, and u_r and u_θ are the radial and angular fluid velocity components.

1°. Any function $w=w(r,\theta)$ that solves the second-order linear equation $\mathrm{E}w=0$ is also a solution of the equation in question.

Special case 1. Solution:

$$w(r,\theta)=(C_1r^2+C_2r^{-1})\sin^2\theta,$$

where C_1 and C_2 are arbitrary constants.

2°. Self-similar solution:

$$w(r,\theta)=\nu r f(\xi),\quad \xi=\cos\theta,$$

where the function $f=f(\xi)$ is determined by the first-order ordinary differential equation

$$2(1-\xi^2)f'_\xi-f^2+4\xi f+C_1\xi^2+C_2\xi+C_3=0, \tag{1}$$

and C_1, C_2, and C_3 are arbitrary constants.

The Riccati equation (1) is reduced, with the change of variable $f=-2(1-\xi^2)g'_\xi/g$, to the hypergeometric equation

$$(1-\xi^2)^2g''_{\xi\xi}+(C_1\xi^2+C_2\xi+C_3)g=0,$$

which, in the case $C_1\xi^2+C_2\xi+C_3=A(1-\xi^2)$, has power-law solutions:

$$g=(1+\xi)^k,\qquad k=\tfrac{1}{2}\big(1\pm\sqrt{1+A}\,\big).$$

Special case 2. In the Landau problem on the outflow of an axisymmetric submerged jet source, the solution of equation (1) is given by

$$f(\xi)=\frac{2(1-\xi^2)}{B-\xi}\qquad (C_1=C_2=C_3=0),$$

where the constant of integration B can be expressed via the jet momentum.

3°. The homogeneous translational fluid flow with a velocity U_0 about a rigid spherical particle of radius a is characterized by the boundary conditions

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = a, \qquad w \to \tfrac{1}{2}U_0 r^2 \sin^2\theta \quad \text{as} \quad r \to \infty. \tag{2}$$

The asymptotic solution of the equation in question subject to the boundary conditions (2) for low Reynolds numbers, $\mathrm{Re} = aU_0/\nu \to 0$, in the domain $r/a \le O(\mathrm{Re}^{-1})$ is given by

$$\frac{w}{U_0} = \frac{1}{4}(r-a)^2\left(2+\frac{a}{r}\right)\sin^2\theta + \frac{3}{32}\mathrm{Re}\,(r-a)^2\left[2+\frac{a}{r}-\left(2+\frac{a}{r}+\frac{a^2}{r^2}\right)\cos\theta\right]\sin^2\theta + O(\mathrm{Re}^2).$$

For the case $\mathrm{Re} = aU_0/\nu \to 0$ in the domain $r/a \ge O(\mathrm{Re}^{-1})$, Oseen asymptotic solution holds true; specifically,

$$\frac{w}{U_0} = \frac{1}{2}r^2\sin^2\theta - \frac{3}{2\,\mathrm{Re}}(1+\cos\theta)\left[1 - e^{-\frac{1}{2}\mathrm{Re}r(1-\cos\theta)}\right] + O(1).$$

⊙ *References for equation* 14.3.2.6: N. A. Slezkin (1934), I. Proudman and J. R. A. Pearson (1957), A. Nayfeh (1973), M. D. Van Dyke (1975), L. D. Landau and E. M. Lifshitz (1987), L. G. Loitsyanskiy (1996).

14.3.3. Nonstationary Hydrodynamic Equations (Navier–Stokes Equations)

1. $\dfrac{\partial}{\partial t}(\Delta w) + \dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w, \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional *nonstationary equations of motion of a viscous incompressible fluid*,

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$
$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

are reduced to the equation in question through the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

For stationary solutions, see equation 14.3.2.1.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = -w(y,\, x,\, t),$$
$$w_2 = w(C_1x + C_2,\, C_1y + C_3,\, C_1^2t + C_4) + C_5,$$
$$w_3 = w(x\cos\alpha + y\sin\alpha,\, -x\sin\alpha + y\cos\alpha,\, t),$$
$$w_4 = w(x\cos\beta t + y\sin\beta t,\, -x\sin\beta t + y\cos\beta t,\, t) - \tfrac{1}{2}\beta(x^2+y^2),$$
$$w_5 = w(x+\varphi(t),\, y+\psi(t),\, t) + \psi_t'(t)x - \varphi_t'(t)y + \chi(t),$$

where $C_1, \ldots, C_4$, α, and β are arbitrary constants and $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, are also solutions of the equation.

⊙ *References*: V. V. Pukhnachov (1960), B. J. Cantwell (1978), S. P. Lloyd (1981), L. V. Ovsiannikov (1982).

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson equation, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Example of an inviscid solution involving five arbitrary functions:

$$w = \varphi(t)x^2 + \psi(t)xy + [C - \varphi(t)]y^2 + a(t)x + b(t)y + c(t).$$

3°. Solution dependent on a single space variable:

$$w = W(x, t),$$

where the function W satisfies the linear nonhomogeneous heat equation

$$\frac{\partial W}{\partial t} - \nu\frac{\partial^2 W}{\partial x^2} = f_1(t)x + f_0(t),$$

and $f_1(t)$ and $f_0(t)$ are arbitrary functions. Solutions of the form $w = V(y, t)$ are determined by a similar equation.

4°. Generalized separable solution linear in x:

$$w(x, y, t) = F(y, t)x + G(y, t), \tag{1}$$

where the functions $F(y, t)$ and $G = G(y, t)$ are determined by the system of fourth-order one-dimensional equations

$$\frac{\partial^3 F}{\partial t\partial y^2} + \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 F}{\partial y^3} = \nu\frac{\partial^4 F}{\partial y^4}, \tag{2}$$

$$\frac{\partial^3 G}{\partial t\partial y^2} + \frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 G}{\partial y^3} = \nu\frac{\partial^4 G}{\partial y^4}. \tag{3}$$

Equation (2) is solved independently of (3). If $F = F(y, t)$ is a solution of equation (2), then the functions

$$F_1 = F(y + \psi(t), t) + \psi_t'(t),$$
$$F_2 = C_1 F(C_1 y + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

Integrating (2) and (3) with respect to y yields

$$\frac{\partial^2 F}{\partial t\partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu\frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{4}$$

$$\frac{\partial^2 G}{\partial t\partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu\frac{\partial^3 G}{\partial y^3} + f_2(t), \tag{5}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions. Equation (5) is linear in G. The substitution

$$G = \int U\,dy - hF + h_t' y, \qquad \text{where} \quad U = U(y, t), \quad F = F(y, t), \tag{6}$$

TABLE 14.1
Solutions of equations (2) and (4); $\varphi(t)$ and $\psi(t)$ are arbitrary functions, and A and λ are arbitrary constants

No.	Function $F = F(y,t)$ (or general form of solution)	Function $f_1(t)$ in equation (4)	Determining coefficients (or determining equation)
1	$F = \varphi(t)y + \psi(t)$	$f_1(t) = \varphi_t' + \varphi^2$	N/A
2	$F = \frac{6\nu}{y+\psi(t)} + \psi_t'(t)$	$f_1(t) = 0$	N/A
3	$F = A\exp\big[-\lambda y - \lambda\psi(t)\big] + \psi_t'(t) + \nu\lambda$	$f_1(t) = 0$	N/A
4	$F = Ae^{-\beta t}\sin[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
5	$F = Ae^{-\beta t}\cos[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
6	$F = Ae^{\beta t}\sinh[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
7	$F = Ae^{\beta t}\cosh[\lambda y + \lambda\psi(t)] + \psi_t'(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = -A^2\lambda^2 < 0$
8	$F = \psi(t)e^{\lambda y} - \frac{Ae^{\beta t-\lambda y}}{4\lambda^2\psi(t)} + \frac{\psi_t'(t)}{\lambda\psi(t)} - \nu\lambda$	$f_1(t) = Ae^{\beta t}$	$\beta = 2\nu\lambda^2$
9	$F = F(\xi)$, $\xi = y + \lambda t$	$f_1(t) = A$	$-A + \lambda F_{\xi\xi}'' + (F_\xi')^2 - FF_{\xi\xi}'' = \nu F_{\xi\xi\xi}'''$
10	$F = t^{-1/2}\big[U(\xi) - \frac{1}{2}\xi\big]$, $\xi = yt^{-1/2}$	$f_1(t) = At^{-2}$	$\frac{3}{4} - A - 2U_\xi' + (U_\xi')^2 - UU_{\xi\xi}'' = \nu U_{\xi\xi\xi}'''$

and the function $h = h(t)$ satisfies the linear ordinary differential equation

$$h_{tt}'' - f_1(t)h = f_2(t), \tag{7}$$

brings (5) to the linear homogeneous parabolic second-order equation

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial y^2} + F\frac{\partial U}{\partial y} - \frac{\partial F}{\partial y}U. \tag{8}$$

Thus, whenever a particular solution of equation (2) or (4) is known, determining the function G is reduced to solving the linear equations (7)–(8) followed by computing integrals by formula (6).

Exact solutions of equation (2) are listed in Table 14.1 (two more complicated solutions are specified at the end of Item 4°). The ordinary differential equations in the last two rows, which determine a traveling-wave solution and a self-similar solution, are autonomous and, therefore, their order can be reduced. Note that solutions of the form (1) with $F(y,t) = Cy/t$ were treated in Pukhnachov (1960); these solutions correspond to $\varphi(t) = C/t$ in the first row.

The general solution of the linear nonhomogeneous equation (7) is expressed as

$$h(t) = C_1h_1(t) + C_2h_2(t) + \frac{1}{W_0}\left[h_2(t)\int h_1(t)f_2(t)\,dt - h_1(t)\int h_2(t)f_2(t)\,dt\right], \tag{9}$$

where $h_1 = h_1(t)$ and $h_2 = h_2(t)$ are fundamental solutions of the corresponding homogeneous equation (with $f_2 \equiv 0$) and $W_0 = h_1(h_2)_t' - h_2(h_1)_t'$ is the Wronskian determinant (in this case, $W_0 = \text{const}$). Table 14.2 lists fundamental solutions of the homogeneous equation (7) corresponding to the exact solutions of (2) specified in Table 14.1.

TABLE 14.2
Fundamental system of solutions determining the general solution (9) of the nonhomogeneous equation (7); the number in the first column corresponds to the respective number of an exact solution in Table 14.1

No.	Fundamental system of solutions	Wronskian W_0	Notation and remarks
1	$h_1 = \Phi(t),\ h_2 = \Phi(t)\int \frac{dt}{\Phi^2(t)}$	$W_0 = 1$	$\Phi(t) = \exp\big[\int \varphi(t)\,dt\big]$
2	$h_1 = 1,\ h_2 = t$	$W_0 = 1$	N/A
3	$h_1 = 1,\ h_2 = t$	$W_0 = 1$	N/A
4	$h_1 = I_0\big(\frac{A\lambda}{\beta}e^{-\beta t}\big),\ h_2 = K_0\big(\frac{A\lambda}{\beta}e^{-\beta t}\big)$	$W_0 = \beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta = \nu\lambda^2$
5	$h_1 = I_0\big(\frac{A\lambda}{\beta}e^{-\beta t}\big),\ h_2 = K_0\big(\frac{A\lambda}{\beta}e^{-\beta t}\big)$	$W_0 = \beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta = \nu\lambda^2$
6	$h_1 = I_0\big(\frac{A\lambda}{\beta}e^{\beta t}\big),\ h_2 = K_0\big(\frac{A\lambda}{\beta}e^{\beta t}\big)$	$W_0 = -\beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta = \nu\lambda^2$
7	$h_1 = J_0\big(\frac{A\lambda}{\beta}e^{\beta t}\big),\ h_2 = Y_0\big(\frac{A\lambda}{\beta}e^{\beta t}\big)$	$W_0 = \frac{2\beta}{\pi}$	$J_0(z)$, $Y_0(z)$ are Bessel functions; $\beta = \nu\lambda^2$
8	$h_1 = I_0\big(\frac{2\sqrt{A}}{\beta}e^{\beta t/2}\big),\ h_2 = K_0\big(\frac{2\sqrt{A}}{\beta}e^{\beta t/2}\big)$	$W_0 = -\frac{\beta}{2}$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta = 2\nu\lambda^2$
9	$h_1 = \cosh(kt),\ h_2 = \sinh(kt)$ $h_1 = \cos(kt),\ h_2 = \sin(kt)$	$W_0 = k$ $W_0 = k$	if $A = k^2 > 0$ if $A = -k^2 < 0$
10	$h_1 = \lvert t\rvert^{\frac12-\mu},\ h_2 = \lvert t\rvert^{\frac12+\mu}$ $h_1 = \lvert t\rvert^{\frac12},\ h_2 = \lvert t\rvert^{\frac12}\ln\lvert t\rvert$ $h_1 = \lvert t\rvert^{\frac12}\cos(\mu\ln\lvert t\rvert),\ h_2 = \lvert t\rvert^{\frac12}\sin(\mu\ln\lvert t\rvert)$	$W_0 = 2\mu$ $W_0 = 1$ $W_0 = \mu$	if $A > -\frac14;\ \mu = \frac12\lvert 1 + 4A\rvert^{\frac12}$ if $A = -\frac14$ if $A < -\frac14;\ \mu = \frac12\lvert 1 + 4A\rvert^{\frac12}$

Equation (8) with any $F = F(y, t)$ has the trivial solution. The expressions in Tables 14.1 and 14.2 together with formulas (6) and (9) with $U = 0$ describe some exact solutions of the form (1). Nontrivial solutions of equation (8) generate a wider class of exact solutions.

Table 14.3 presents transformations that simplify equation (8) for some of the solutions to (2) or (4) listed in Table 14.1. It is apparent that solutions to (8) are expressed via solutions to the linear constant-coefficient heat equation in the first two cases. Equation (8) admits the application of the method of separation of variables in three other cases.

The third equation in Table 14.3 has the following particular solutions (B_1 and B_2 are arbitrary constants):

$$Z(\eta) = B_1 + B_2\int \Phi(\eta)\,d\eta, \quad \Phi(\eta) = \exp\left(\frac{A}{\nu\lambda}e^{\eta} - \eta\right),$$

$$Z(\eta, t) = B_1\nu\lambda^2 t + B_1\int \Phi(\eta)\left[\int \frac{d\eta}{\Phi(\eta)}\right] d\eta.$$

For other exact solutions of this equation, see the book by Polyanin (2002), where a more general solution of the form $\partial_t w = f(x)\partial_{xx} w + g(x)\partial_x w$ was considered.

⊙ *References*: R. Berker (1963), A. D. Polyanin (2001 *c*, 2002), A. D. Polyanin and V. F. Zaitsev (2002).

Special case 1. Solution exponentially dependent on time:

$$w(x, y, t) = f(y)x + e^{-\lambda t}\int g(y)\,dy,$$

TABLE 14.3
Transformations of equation (8) for the corresponding exact solutions of equation (4); the number in the first column corresponds to the respective number of an exact solution $F = F(y,t)$ in Table 14.1

No.	Transformations of equation (8)	Resulting equation
1	$U = \frac{1}{\Phi(t)}u(z,\tau),\ \tau = \int \Phi^2(t)\,dt + C_1,$ $z = y\Phi(t) + \int \psi(t)\Phi(t)\,dt + C_2,\ \Phi(t) = \exp\big[\int \varphi(t)\,dt\big]$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial z^2}$
2	$U = \zeta^{-3}u(\zeta,t),\ \zeta = y + \psi(t)$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \zeta^2}$
3	$U = e^{\eta}Z(\eta,t),\ \eta = -\lambda y - \lambda\psi(t)$	$\frac{\partial Z}{\partial t} = \nu\lambda^2 \frac{\partial^2 Z}{\partial \eta^2} + (\nu\lambda^2 - A\lambda e^{\eta})\frac{\partial Z}{\partial \eta}$
9	$U = u(\xi,t),\ \xi = y + \lambda t$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \xi^2} + \big[F(\xi) - \lambda\big]\frac{\partial u}{\partial \xi} - F'_{\xi}(\xi)u$
10	$U = t^{-1/2}u(\xi,\tau),\ \xi = yt^{-1/2},\ \tau = \ln t$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial \xi^2} + H(\xi)\frac{\partial u}{\partial \xi} + \big[1 - H'_{\xi}(\xi)\big]u$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy} + C_1,$$
$$-\lambda g + gf'_y - fg'_y = \nu g''_{yy} + C_2,$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: N. Rott (1956).

Special case 2. Periodic solution:

$$w(x,y,t) = f(y)x + \sin(\lambda t)\int g(y)\,dy + \cos(\lambda t)\int h(y)\,dy,$$

where the functions $f = f(y)$, $g = g(y)$, and $h = h(y)$ are determined by the solution of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy} + C_1,$$
$$-\lambda h + f'_y g - fg'_y = \nu g''_{yy} + C_2,$$
$$\lambda g + f'_y h - fh'_y = \nu h''_{yy} + C_3.$$

Below are another two exact solutions of equation (2):

$$F(y,t) = -\frac{\gamma'_t}{\gamma}y + \gamma^3 \exp\left(\nu\int \frac{dt}{\gamma^2}\right)\left(A\cosh\frac{y}{\gamma} + B\sinh\frac{y}{\gamma}\right),$$
$$F(y,t) = -\frac{\gamma'_t}{\gamma}y + \gamma^3 \exp\left(-\nu\int \frac{dt}{\gamma^2}\right)\left(A\cos\frac{y}{\gamma} + B\sin\frac{y}{\gamma}\right),$$

where A and B are arbitrary constants, and $\gamma = \gamma(t)$ is an arbitrary function. The first formula of the two displayed after (3) allows us to generalize the above expressions to obtain solutions involving two arbitrary functions.

Special case. Formula (1) with $G \equiv 0$ and equation (4) with the boundary conditions

$$F(0,t) = F_y(0,t) = 0, \qquad F_y(\infty,t) = \varphi(t),$$

where $\varphi(t)$ is a periodic function, describe the motion of an infinite plane performing periodic fluctuations about its equilibrium position along a normal direction. This problem models the local dynamics around a stagnation point on an oscillating body and was studied by Grosch and Salwen (1982), Riley and Vasantha (1988), Merchant and Davis (1989), and Blyth and Hall (2003).

Formula (1) with $G \equiv 0$ and equation (4) with the above boundary conditions and $\varphi(t) = \text{const}$ describe an unsteady flow at a rear stagnation point. For details, see the papers by Proudman and Johnson (1962), Robins and Howarth (1972), and van Dommelen and Shen (1980).

5°. Solution (generalizes the solution of Item 4°):

$$w(x,y,t) = F(\xi,t)x + G(\xi,t), \qquad \xi = y + kx,$$

where k is an arbitrary constant and the functions $F(\xi,t)$ and $G = G(\xi,t)$ are determined from the system of one-dimensional fourth-order equations

$$\frac{\partial^3 F}{\partial t\partial\xi^2} + \frac{\partial F}{\partial\xi}\frac{\partial^2 F}{\partial\xi^2} - F\frac{\partial^3 F}{\partial\xi^3} = \nu(k^2+1)\frac{\partial^4 F}{\partial\xi^4}, \tag{10}$$

$$\frac{\partial^3 G}{\partial t\partial\xi^2} + \frac{\partial G}{\partial\xi}\frac{\partial^2 F}{\partial\xi^2} - F\frac{\partial^3 G}{\partial\xi^3} = \nu(k^2+1)\frac{\partial^4 G}{\partial\xi^4} + 4\nu k\frac{\partial^3 F}{\partial\xi^3} + \frac{2k}{k^2+1}\left(F\frac{\partial^2 F}{\partial\xi^2} - \frac{\partial^2 F}{\partial t\partial\xi}\right). \tag{11}$$

Integrating (10) and (11) with respect to ξ yields

$$\frac{\partial^2 F}{\partial t\partial\xi} + \left(\frac{\partial F}{\partial\xi}\right)^2 - F\frac{\partial^2 F}{\partial\xi^2} = \nu(k^2+1)\frac{\partial^3 F}{\partial\xi^3} + f_1(t), \tag{12}$$

$$\frac{\partial^2 G}{\partial t\partial\xi} + \frac{\partial F}{\partial\xi}\frac{\partial G}{\partial\xi} - F\frac{\partial^2 G}{\partial\xi^2} = \nu(k^2+1)\frac{\partial^3 G}{\partial\xi^3} + Q(\xi,t), \tag{13}$$

where $f_1(t)$ is an arbitrary function, and the function $Q(\xi,t)$ is given by

$$Q(\xi,t) = 4\nu k\frac{\partial^2 F}{\partial\xi^2} - \frac{2k}{k^2+1}\frac{\partial F}{\partial t} + \frac{2k}{k^2+1}\int F\frac{\partial^2 F}{\partial\xi^2}\,d\xi + f_2(t), \qquad f_2(t) \text{ is any.}$$

Equation (13) is linear in G. The substitution $U = \frac{\partial G}{\partial\xi}$ brings (13) to the second-order linear equation

$$\frac{\partial U}{\partial t} = \nu(k^2+1)\frac{\partial^2 U}{\partial\xi^2} + F\frac{\partial U}{\partial\xi} - \frac{\partial F}{\partial\xi}U + Q(\xi,t). \tag{14}$$

Thus, whenever a particular solution of equation (10) or (12) is known, determining the function G is reduced to solving the second-order linear equation (14). Equation (10) is reduced, by scaling the independent variables so that $\xi = (k^2+1)\zeta$ and $t = (k^2+1)\tau$, to equation (2) in which y and t should be replaced by ζ and τ; exact solutions of equation (2) are listed in Table 14.1.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

6°. Solutions:

$$w(x,y,t) = Az^3 + Bz^2 + Cz + \psi_t'(t)x, \quad z = y + kx + \psi(t);$$
$$w(x,y,t) = Ae^{-\lambda z} + Bz^2 + Cz + \nu\lambda(k^2+1)x + \psi_t'(t)x,$$

where A, B, C, k, and λ are arbitrary constants and $\psi(t)$ is an arbitrary function.

7°. Generalized separable solution [special case of a solution of the form (1)]:

$$w(x,y,t) = e^{-\lambda y}\big[f(t)x + g(t)\big] + \varphi(t)x + \psi(t)y,$$
$$f(t) = C_1E(t), \quad E(t) = \exp\left[\nu\lambda^2 t - \lambda\int\varphi(t)\,dt\right],$$
$$g(t) = C_2E(t) - C_1E(t)\int\psi(t)\,dt,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, and λ are arbitrary constants.

8°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\big[A(t)e^{\beta x} + B(t)e^{-\beta x}\big] + \varphi(t)x + \psi(t)y,$$
$$A(t) = C_1 \exp\left[\nu(\lambda^2+\beta^2)t - \beta\int \psi(t)\,dt - \lambda\int \varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[\nu(\lambda^2+\beta^2)t + \beta\int \psi(t)\,dt - \lambda\int \varphi(t)\,dt\right],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

9°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\big[A(t)\sin(\beta x) + B(t)\cos(\beta x)\big] + \varphi(t)x + \psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, λ and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= \big[\nu(\lambda^2-\beta^2) - \lambda\varphi(t)\big]A + \beta\psi(t)B,\\ B'_t &= \big[\nu(\lambda^2-\beta^2) - \lambda\varphi(t)\big]B - \beta\psi(t)A. \end{aligned} \tag{15}$$

The general solution of system (15) is expressed as

$$A(t) = \exp\left[\nu(\lambda^2-\beta^2)t - \lambda\int \varphi\,dt\right]\left[C_1\sin\left(\beta\int\psi\,dt\right) + C_2\cos\left(\beta\int\psi\,dt\right)\right],$$
$$B(t) = \exp\left[\nu(\lambda^2-\beta^2)t - \lambda\int \varphi\,dt\right]\left[C_1\cos\left(\beta\int\psi\,dt\right) - C_2\sin\left(\beta\int\psi\,dt\right)\right],$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$; C_1 and C_2 are arbitrary constants. In particular, for $\varphi = \dfrac{\nu}{\lambda}(\lambda^2-\beta^2)$ and $\psi = a$, we obtain the periodic solution

$$A(t) = C_1\sin(a\beta t) + C_2\cos(a\beta t),$$
$$B(t) = C_1\cos(a\beta t) - C_2\sin(a\beta t).$$

⊙ *Reference*: A. D. Polyanin (2001 *c*).

10°. Generalized separable solution:

$$w(x,y,t) = A(t)\exp(k_1x + \lambda_1 y) + B(t)\exp(k_2x + \lambda_2 y) + \varphi(t)x + \psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k_1, λ_1, k_2, and λ_2 are arbitrary constants, constrained by one of the two relations

$$\begin{aligned} &k_1^2 + \lambda_1^2 = k_2^2 + \lambda_2^2 \quad &&\text{(first family of solutions)},\\ &k_1\lambda_2 = k_2\lambda_1 &&\text{(second family of solutions)}, \end{aligned}$$

and the functions $A(t)$ and $B(t)$ satisfy the linear ordinary differential equations

$$A'_t = \big[\nu(k_1^2+\lambda_1^2) + \lambda_1\varphi(t) - k_1\psi(t)\big]A,$$
$$B'_t = \big[\nu(k_2^2+\lambda_2^2) + \lambda_2\varphi(t) - k_2\psi(t)\big]B.$$

These equations can be readily integrated to obtain

$$A(t) = C_1\exp\left[\nu(k_1^2+\lambda_1^2)t + \lambda_1\int\varphi(t)\,dt - k_1\int\psi(t)\,dt\right],$$
$$B(t) = C_2\exp\left[\nu(k_2^2+\lambda_2^2)t + \lambda_2\int\varphi(t)\,dt - k_2\int\psi(t)\,dt\right].$$

⊙ *Reference*: A. D. Polyanin (2001 *c*).

11°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sin(\lambda x) + C_2\cos(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$

where $\varphi(t)$ is an arbitrary function, C_1, C_2, λ, and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A_t' &= -\nu(\lambda^2+\beta^2)A - \beta\varphi(t)B, \\ B_t' &= -\nu(\lambda^2+\beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{16}$$

The general solution of system (16) is expressed as

$$\begin{aligned} A(t) &= \exp\big[-\nu(\lambda^2+\beta^2)t\big]\left[C_3\sin\left(\beta\int\varphi\,dt\right) + C_4\cos\left(\beta\int\varphi\,dt\right)\right], \quad \varphi = \varphi(t), \\ B(t) &= \exp\big[-\nu(\lambda^2+\beta^2)t\big]\left[-C_3\cos\left(\beta\int\varphi\,dt\right) + C_4\sin\left(\beta\int\varphi\,dt\right)\right], \end{aligned}$$

where C_3 and C_4 are arbitrary constants.

Special case. With $\varphi(t) \equiv 0$, $C_1 = C_4 = 0$, $C_2 = 1$, $C_3 = -k$, we have a simple solution

$$w = k\cos(\lambda x)\cos(\beta y)\exp\big[-\nu(\lambda^2+\beta^2)t\big],$$

that for special case $\lambda = \beta$ was obtained by Taylor (1923). It may be interpreted as a double array of vortices which decay exponentially with time.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

12°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sinh(\lambda x) + C_2\cosh(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$

where $\varphi(t)$ is an arbitrary function, C_1, C_2, λ, and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A_t' &= \nu(\lambda^2-\beta^2)A - \beta\varphi(t)B, \\ B_t' &= \nu(\lambda^2-\beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{17}$$

The general solution of system (17) is expressed as

$$\begin{aligned} A(t) &= \exp\big[\nu(\lambda^2-\beta^2)t\big]\left[C_3\sin\left(\beta\int\varphi\,dt\right) + C_4\cos\left(\beta\int\varphi\,dt\right)\right], \quad \varphi = \varphi(t), \\ B(t) &= \exp\big[\nu(\lambda^2-\beta^2)t\big]\left[-C_3\cos\left(\beta\int\varphi\,dt\right) + C_4\sin\left(\beta\int\varphi\,dt\right)\right], \end{aligned}$$

where C_3 and C_4 are arbitrary constants.

Special case. With $\varphi(t) \equiv 0$, $C_1 = C_4 = 0$, $C_2 = 1$, and $C_3 = -k$, we have a simple solution

$$w = k\cosh(\lambda x)\cos(\beta y)\exp\big[\nu(\lambda^2-\beta^2)t\big],$$

which was obtained by Wang (1966).

⊙ *Reference*: A. D. Polyanin (2001 *c*).

13°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sinh(\lambda x) + C_2\cosh(\lambda x)\big]\big[A(t)\sinh(\beta y) + B(t)\cosh(\beta y)\big] + \varphi(t)x,$$

where $\varphi(t)$ is an arbitrary function, C_1, C_2, λ, and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$A'_t = \nu(\lambda^2+\beta^2)A + \beta\varphi(t)B,$$
$$B'_t = \nu(\lambda^2+\beta^2)B + \beta\varphi(t)A.$$

The general solution of this system is expressed as

$$A(t) = \exp\big[\nu(\lambda^2+\beta^2)t\big]\left[C_3\sinh\left(\beta\int\varphi\,dt\right) + C_4\cosh\left(\beta\int\varphi\,dt\right)\right], \qquad \varphi = \varphi(t),$$
$$B(t) = \exp\big[\nu(\lambda^2+\beta^2)t\big]\left[C_3\cosh\left(\beta\int\varphi\,dt\right) + C_4\sinh\left(\beta\int\varphi\,dt\right)\right],$$

where C_3 and C_4 are arbitrary constants.

Special case. With $\varphi(t)\equiv 0$, $C_1 = 1$, $C_2 = C_3 = 0$, $C_4 = k$, we have a simple solution

$$w = k\sinh(\lambda x)\sinh(\beta y)\exp\big[\nu(\lambda^2+\beta^2)t\big],$$

which was interpreted by Wang (1966) as the impingement of two rotational flows.

14°. Solution:

$$w = \frac{a\nu}{\sqrt{4\pi^2h^{-2}-a}\,(1-e^{a\nu t})}\left\{y + b\sin\left(\frac{2\pi y}{h}\right)\exp\left[\left(\mathrm{Re}-\sqrt{\mathrm{Re}^2+16\pi^2}\right)\frac{x}{2h}\right]\right\}, \quad \mathrm{Re} = \frac{Uh}{\nu},$$

where a, b, h, and U are arbitrary constants, and Re is the Reynolds number.

⊙ *Reference*: C. Y. Wang (1989).

15°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t) + \varphi(t)x + \psi(t)y, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the fourth-order linear equation

$$\frac{\partial^3 u}{\partial t\partial z^2} + \big[k\psi(t) - \lambda\varphi(t)\big]\frac{\partial^3 u}{\partial z^3} = \nu(k^2+\lambda^2)\frac{\partial^4 u}{\partial z^4}.$$

The transformation

$$U(\xi,t) = \frac{\partial^2 u}{\partial z^2}, \qquad \xi = z - \int\big[k\psi(t) - \lambda\varphi(t)\big]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu(k^2+\lambda^2)\frac{\partial^2 U}{\partial \xi^2}.$$

Using the solutions of the linear heat equation, one can obtain the corresponding exact solutions of the Navier–Stokes equations.

Special case. Solution periodic in the coordinates x and y:

$$w = \exp(-a\mu^2 t)[A\cos(\mu\xi) + B\sin(\mu\xi)] + \varphi(t)x + \psi(t)y + C,$$
$$\xi = kx + \lambda y - \int [k\psi(t) - \lambda\varphi(t)]\,dt, \quad a = \nu(k^2 + \lambda^2),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, and A, B, C, k, λ, and μ are arbitrary constants.

In the special case of $\varphi(t) \equiv 0$, $\psi(t) = \text{const}$, $B = 0$, $\mu = 1$, $k = m\cos\beta$, and $\lambda m \sin\beta$, the above formula give the solution obtained by Hui (1987).

⊙ *References*: A. Grauel and W.-H. Steeb (1985), W. H. Hui (1987), R. L. Moore (1991), A. D. Polyanin (2001 *c*).

16°. There are "two-dimensional" solutions of the form

$$w(x, y, t) = W(\rho_1, \rho_2) + c_1 x + c_2 y, \qquad \rho_1 = a_1 x + a_2 y + a_3 t, \qquad \rho_2 = b_1 x + b_2 y + b_3 t.$$

17°. "Two-dimensional" solution (a, b, and c are arbitrary constants):

$$w(x, y, t) = Z(X, Y), \quad X = \frac{x + a}{\sqrt{t + c}}, \quad Y = \frac{y + b}{\sqrt{t + c}},$$

where the function $Z = Z(X, Y)$ is determined by the differential equation

$$-\bar{\Delta}Z + \left(\frac{\partial Z}{\partial Y} - \frac{1}{2}X\right)\frac{\partial}{\partial X}(\bar{\Delta}Z) - \left(\frac{\partial Z}{\partial X} + \frac{1}{2}Y\right)\frac{\partial}{\partial Y}(\bar{\Delta}Z) = \nu\bar{\Delta}\bar{\Delta}Z, \quad \bar{\Delta} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}.$$

⊙ *Reference*: V. V. Pukhnachov (1960).

18°. "Two-dimensional" solution:

$$w(x, y, t) = \Psi(\xi, \eta),$$
$$\xi = t^{-1/2}\big[x\cos(k\ln t) - y\sin(k\ln t)\big], \quad \eta = t^{-1/2}\big[x\sin(k\ln t) + y\cos(k\ln t)\big],$$

where k is an arbitrary constant and the function $\Psi(\xi, \eta)$ is determined by the differential equation

$$-\widetilde{\Delta}\Psi + \left(\frac{\partial\Psi}{\partial\eta} - \frac{1}{2}\xi - k\eta\right)\frac{\partial}{\partial\xi}\widetilde{\Delta}\Psi - \left(\frac{\partial\Psi}{\partial\xi} + \frac{1}{2}\eta - k\xi\right)\frac{\partial}{\partial\eta}\widetilde{\Delta}\Psi = \nu\widetilde{\Delta}\widetilde{\Delta}\Psi, \quad \widetilde{\Delta} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

⊙ *Reference*: B. J. Cantwell (1978).

19°. "Two-dimensional" solution:

$$w(x, y, t) = \frac{\varphi_t'(x^2 - y^2 + 2\varphi xy)}{2(1 + \varphi^2)} + \frac{y - \varphi x}{1 + \varphi^2}F(\zeta, t) - 2G(\zeta, t), \qquad \zeta = x + \varphi y,$$

where $\varphi = \varphi(t)$ is an arbitrary function and the functions $F = F(\zeta, t)$ and $G = G(\zeta, t)$ are determined by the differential equations

$$\nu(1 + \varphi^2)\frac{\partial^4 F}{\partial\zeta^4} - F\frac{\partial^3 F}{\partial\zeta^3} + \frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\zeta^2} - \frac{2\varphi\varphi_t'}{1 + \varphi^2}\frac{\partial^2 F}{\partial\zeta^2} - \frac{\partial^3 F}{\partial\zeta^2\partial t} = 0, \tag{18}$$

$$\nu(1 + \varphi^2)\frac{\partial^4 G}{\partial\zeta^4} - F\frac{\partial^3 G}{\partial\zeta^3} + \frac{\partial^2 F}{\partial\zeta^2}\frac{\partial G}{\partial\zeta} - \frac{2\varphi\varphi_t'}{1 + \varphi^2}\frac{\partial^2 G}{\partial\zeta^2} - \frac{\partial^3 G}{\partial\zeta^2\partial t} = \frac{\varphi_t'}{(1 + \varphi^2)^2}\zeta\frac{\partial^2 F}{\partial\zeta^2}. \tag{19}$$

⊙ *Reference*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999).

Equation (18) is solved independently of equation (19). If $F = F(\zeta, t)$ is a solution to (18), the function

$$F_1 = F(y + \sigma(t), t) - \sigma_t'(t),$$

where $\sigma(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (18) and (19) with respect to ζ yields

$$\nu(1+\varphi^2)\frac{\partial^3 F}{\partial\zeta^3} - F\frac{\partial^2 F}{\partial\zeta^2} + \left(\frac{\partial F}{\partial\zeta}\right)^2 - \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial F}{\partial\zeta} - \frac{\partial^2 F}{\partial\zeta\partial t} = \psi_1(t),$$

$$\nu(1+\varphi^2)\frac{\partial^3 G}{\partial\zeta^3} - F\frac{\partial^2 G}{\partial\zeta^2} + \frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\zeta} - \frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial G}{\partial\zeta} - \frac{\partial^2 G}{\partial\zeta\partial t} = \frac{\varphi_t'}{(1+\varphi^2)^2}\left(\zeta\frac{\partial F}{\partial\zeta} - F\right) + \psi_2(t),$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions. The change of variable $u = \frac{\partial G}{\partial\zeta}$ brings the last equation to a second-order parabolic linear equation (with known F).

Note that equation (18) admits particular solutions of the forms

$$F(\zeta, t) = a(t)\zeta + b(t),$$

$$F(\zeta, t) = a(t)e^{-\lambda\zeta} + \frac{a_t'(t)}{\lambda a(t)} + \frac{2\varphi\varphi_t'}{\lambda(1+\varphi^2)} - \nu\lambda(1+\varphi^2),$$

where $a(t)$ and $b(t)$ are arbitrary functions and λ is an arbitrary constant.

⊙ *References for equation* 14.3.3.1: N. Rott (1956), V. V. Pukhnachov (1960, 2006), R. Berker (1963), B. J. Cantwell (1978), S. P. Lloyd (1981), L. V. Ovsiannikov (1982), L. G. Loitsyanskiy (1996), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999), A. D. Polyanin (2001 *c*), A. D. Polyanin and V. F. Zaitsev (2002, pp. 340–346; 2004, pp. 616–623).

▷ *For other exact solutions of the Navier–Stokes equations, see equation 14.3.3.3 and Subsection 21.3.1 (unsteady solutions).*

2. $\dfrac{\partial}{\partial t}(\Delta w) + \left(\dfrac{\partial w}{\partial y} + ax\right)\dfrac{\partial}{\partial x}(\Delta w) - \left(\dfrac{\partial w}{\partial x} - ay\right)\dfrac{\partial}{\partial y}(\Delta w) + 2a\Delta w = \nu\Delta\Delta w.$

Preliminary remarks. The system

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 2a,$$

describing the motion of a viscous incompressible fluid induced by two parallel disks moving towards each other, is reduced to the given equation. Here, a is the relative velocity of the disks, u_1 and u_2 are the horizontal velocity components, and $u_3 = -2az$ is the vertical velocity component. The introduction of a stream function w such that $u_1 = ax + \frac{\partial w}{\partial y}$ and $u_2 = ay - \frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with the help of cross differentiation) leads to the equation in question. For $a = 0$, see equation 14.3.3.1.

For stationary solutions, see equation 14.3.2.3.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = -w(y,\ x,\ t),$$

$$w_2 = w(x\cos\beta + y\sin\beta,\ -x\sin\beta + y\cos\beta,\ t),$$

$$w_3 = w\big(x + \varphi(t),\ y + \psi(t),\ t + C\big) + \big[\psi_t'(t) - a\psi(t)\big]x + \big[a\varphi(t) - \varphi_t'(t)\big]y + \chi(t),$$

where $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions and C and β are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson equation, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Solution dependent on a single coordinate x:

$$w(x,t) = \int_0^x (x-\xi)U(\xi,t)\,d\xi + f_1(t)x + f_0(t),$$

where $f_1(t)$ and $f_0(t)$ are arbitrary functions and the function $U(x,t)$ satisfies the linear nonhomogeneous parabolic equation

$$\frac{\partial U}{\partial t} + ax\frac{\partial U}{\partial x} + 2aU = \nu\frac{\partial^2 U}{\partial x^2},$$

which can be reduced to a linear constant-coefficient heat equation; see Polyanin (2002, page 93).

Solutions of the form $w = w(y,t)$ can be obtained likewise.

4°. Generalized separable solution linear in x:

$$w(x,y,t) = F(y,t)x + G(y,t), \tag{1}$$

where the functions $F(y,t)$ and $G = G(y,t)$ are determined by the system of one-dimensional fourth-order equations

$$\frac{\partial^3 F}{\partial t\partial y^2} + \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 F}{\partial y^3} + a\left(3\frac{\partial^2 F}{\partial y^2} + y\frac{\partial^3 F}{\partial y^3}\right) = \nu\frac{\partial^4 F}{\partial y^4}, \tag{2}$$

$$\frac{\partial^3 G}{\partial t\partial y^2} + \frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 G}{\partial y^3} + a\left(2\frac{\partial^2 G}{\partial y^2} + y\frac{\partial^3 G}{\partial y^3}\right) = \nu\frac{\partial^4 G}{\partial y^4}. \tag{3}$$

Equation (2) is solved independently of equation (3). If $F = F(y,t)$ is a solution to (2), then the function

$$F_1 = F\big(y + \psi(t), t\big) + \psi_t'(t) - a\psi(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (2) and (3) with respect to y yields

$$\frac{\partial^2 F}{\partial t\partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} + a\left(2\frac{\partial F}{\partial y} + y\frac{\partial^2 F}{\partial y^2}\right) = \nu\frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{4}$$

$$\frac{\partial^2 G}{\partial t\partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} + a\left(\frac{\partial G}{\partial y} + y\frac{\partial^2 G}{\partial y^2}\right) = \nu\frac{\partial^3 G}{\partial y^3} + f_2(t), \tag{5}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions.

Equation (2) has a particular solution

$$F(y,t) = f_1(t)y + f_0(t), \tag{6}$$

where $f_1 = f_1(t)$ and $f_0 = f_0(t)$ are arbitrary functions. On substituting (6) into (5), we arrive at a linear equation whose order can be reduced by two:

$$\frac{\partial Q}{\partial t} - \big[(f_1 - a)y + f_0\big]\frac{\partial Q}{\partial y} + 2aQ = \nu\frac{\partial^2 Q}{\partial y^2}, \qquad Q = \frac{\partial^2 G}{\partial y^2}.$$

The equation for Q can be reduced to a linear constant-coefficient heat equation; see Polyanin (2002, page 135).

Note that equation (2) has the following particular solutions:

$$\begin{aligned} F(y,t) &= ay + \big[C_1\exp(-\lambda y) + C_2\exp(\lambda y)\big]\exp\big[(\nu\lambda^2 - 4a)t\big], \\ F(y,t) &= ay + \big[C_1\cos(\lambda y) + C_2\sin(\lambda y)\big]\exp\big[-(\nu\lambda^2 + 4a)t\big], \end{aligned} \tag{7}$$

where C_1, C_2, and λ are arbitrary constants.

Solutions of the form $w(x,y,t) = f(x,t)y + g(x,t)$ can be obtained likewise.

Remark. The results of Items 1°–4° exclusive of formula (7) remain true if $a = a(t)$ is an arbitrary function in the original equation (in this case, one should set $C = 0$ in Item 1°).

⊙ *References for equation* 14.3.3.2: A. Craik (1989), A. D. Polyanin and V. F. Zaitsev (2004, pp. 624–625).

3. $\dfrac{\partial Q}{\partial t} + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\dfrac{\partial Q}{\partial r} - \dfrac{1}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial Q}{\partial \theta} = \nu\Delta Q, \quad Q = \Delta w = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}.$

Preliminary remarks. Equation 14.3.3.1 is reduced to the given equation by passing to polar coordinates with origin at a point (x_0, y_0), where x_0 and y_0 are any numbers, according to

$$\begin{aligned} &x = r\cos\theta + x_0, && y = r\sin\theta + y_0 && \text{(direct transformation)}, \\ &r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, && \tan\theta = \frac{y-y_0}{x-x_0} && \text{(inverse transformation)}. \end{aligned}$$

The radial and angular fluid velocity components are expressed in terms of the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$ and $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Solutions with axial symmetry

$$w = W(r,t)$$

are described by the linear nonhomogeneous heat equation

$$\frac{\partial W}{\partial t} - \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = \varphi(t)\ln r + \psi(t), \tag{1}$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions.

2°. Generalized separable solution linear in θ:

$$w(r,\theta,t) = f(r,t)\theta + g(r,t), \tag{2}$$

where the functions $f = f(r,t)$ and $g = g(r,t)$ are determined by the differential equations

$$\mathbf{L}(f_t) - r^{-1}f_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(f)]_r = \nu\mathbf{L}^2(f), \tag{3}$$

$$\mathbf{L}(g_t) - r^{-1}g_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(g)]_r = \nu\mathbf{L}^2(g). \tag{4}$$

Here, the subscripts r and t denote partial derivatives with respect to r and t, $\mathbf{L}(f) = r^{-1}(rf_r)_r$, and $\mathbf{L}^2(f) = \mathbf{L}\mathbf{L}(f)$.

Equation (3) has a particular solution of the form

$$f = \varphi(t)\ln r + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. In this case, equation (4) is reduced by the change of variable $U = \mathbf{L}(g)$ to a second-order linear equation.

Remark. Equation (3) has also a particular solution $f = -\dfrac{r^2}{2(t+C)}$.

3°. Let us consider the case $f = \psi(t)$ in Item 2° in more detail. This case corresponds to $w = \psi(t)\theta + g(r,t)$; the existence of such an exact solution was established by Pukhnachov (1960). For g, we have the equation

$$\frac{\partial U}{\partial t} + \frac{\psi(t)}{r}\frac{\partial U}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right), \qquad \text{where} \quad U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right). \tag{5}$$

Below are some exact solutions of equation (5):

$$U = \frac{a}{t}\exp\left[-\frac{r^2}{4\nu t} + \frac{1}{2\nu}\int\frac{\psi(t)}{t}\,dt\right] + b,$$

$$U = r^2 + 4\nu t - 2\int\psi(t)\,dt + a,$$

$$U = r^4 + p(t)r^2 + q(t), \quad p(t) = 16\nu t - 4\int\psi(t)\,dt + a, \quad q(t) = 2\int\big[2\nu - \psi(t)\big]p(t)\,dt + b,$$

where a and b are arbitrary constants. The second and the third solutions are special cases of solutions having the form

$$U = r^{2n} + A_{2n-2}(t)r^{2n-2} + \cdots + A_2(t)r^2 + A_0(t)$$

with n arbitrary constants.

The function $g(r,t)$ can be expressed in terms of $U(r,t)$ by

$$g(r,t) = C_1(t)\ln r + C_2(t) + \int\Phi(r,t)\,dr, \qquad \Phi(r,t) = \frac{1}{r}\int rU(r,t)\,dr,$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions.

4°. "Two-dimensional" solution:

$$w(r,\theta,t) = Ar^2t + \nu H(\xi,\eta), \qquad \xi = r\cos(\theta + At^2), \qquad \eta = r\sin(\theta + At^2),$$

where A is an arbitrary constant and the function $H(\xi,\eta)$ is determined by the differential equation

$$\widetilde{\Delta}\widetilde{\Delta}H - \frac{\partial H}{\partial\eta}\frac{\partial}{\partial\xi}\widetilde{\Delta}H + \frac{\partial H}{\partial\xi}\frac{\partial}{\partial\eta}\widetilde{\Delta}H - \frac{4A}{\nu^2} = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

⊙ *References for equation* 14.3.3.3: V. V. Pukhnachov (1960, 2006), L. G. Loitsyanskiy (1996), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999), A. D. Polyanin and V. F. Zaitsev (2002, pp. 346–347).

▷ *For other exact solutions of the Navier–Stokes equations in polar (cylindrical) coordinates, see Subsection 21.3.2 (unsteady solutions).*

4. $$\frac{\partial Q}{\partial t} + ar\frac{\partial Q}{\partial r} + 2aQ + \frac{1}{r}\frac{\partial w}{\partial\theta}\frac{\partial Q}{\partial r} - \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial Q}{\partial\theta} = \nu\Delta Q,$$

$$\text{where } Q = \Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}.$$

Equation 14.3.3.2 is reduced to the given equation by passing to polar coordinates r, θ: $x = r\cos\theta$, $y = r\sin\theta$.

1°. Solutions with axial symmetry,

$$w = W(r,t),$$

are described by the linear parabolic equation

$$\frac{\partial Q}{\partial t} + ar\frac{\partial Q}{\partial r} + 2aQ = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q}{\partial r}\right), \qquad Q = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right).$$

2°. Generalized separable solution linear in θ:

$$w(r,\theta,t) = f(r,t)\theta + g(r,t), \tag{1}$$

where the functions $f = f(r,t)$ and $g = g(r,t)$ are determined by the differential equations

$$\mathbf{L}(f_t) + ar[\mathbf{L}(f)]_r + 2a\mathbf{L}(f) - r^{-1}f_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(f)]_r = \nu\mathbf{L}^2(f), \tag{2}$$
$$\mathbf{L}(g_t) + ar[\mathbf{L}(g)]_r + 2a\mathbf{L}(g) - r^{-1}g_r\mathbf{L}(f) + r^{-1}f[\mathbf{L}(g)]_r = \nu\mathbf{L}^2(g). \tag{3}$$

Here, the subscripts r and t denote partial derivatives with respect to r and t, $\mathbf{L}(f) = r^{-1}(rf_r)_r$, and $\mathbf{L}^2(f) = \mathbf{L}\mathbf{L}(f)$.

Equation (2) has particular solutions of the form

$$f = \varphi(t)\ln r + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. In this case, equation (3) is reduced by the change of variable $U = \mathbf{L}(g)$ to a second-order linear equation.

5. $$\frac{\partial \mathbf{E}w}{\partial t} + \frac{1}{r}\left(\frac{\partial w}{\partial z}\frac{\partial \mathbf{E}w}{\partial r} - \frac{\partial w}{\partial r}\frac{\partial \mathbf{E}w}{\partial z}\right) - \frac{2}{r^2}\frac{\partial w}{\partial z}\mathbf{E}w = \nu\mathbf{E}^2w,$$

$$\text{where } \mathbf{E}w = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}, \quad \mathbf{E}^2w = \mathbf{E}(\mathbf{E}w).$$

Preliminary remarks. The nonstationary Navier–Stokes equations written in cylindrical coordinates for the axisymmetric case are reduced to the equation in question by the introduction of a stream function w such that $u_r = \frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z = -\frac{1}{r}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2+y^2}$, and u_r and u_z are the radial and axial fluid velocity components (see Happel and Brenner (1965)).

1°. Any function $w = w(r,z,t)$ that solves the second-order linear stationary equation $\mathrm{E}w = 0$ is also a solution of the original equation.

2°. Solution with axial symmetry:

$$w = U(r,t) + \varphi(t)r^2 + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and the function $U = U(r,t)$ is determined by the linear parabolic equation

$$\frac{\partial U}{\partial t} - \nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial r}\right) = 0.$$

3°. Generalized separable solution linear in z:

$$w(r,z,t) = f(r,t)z + g(r,t).$$

Here, $f = f(r,t)$ and $g = g(r,t)$ satisfy the system

$$\mathbf{L}(f_t) + r^{-1}f[\mathbf{L}(f)]_r - r^{-1}f_r\mathbf{L}(f) - 2r^{-2}f\,\mathbf{L}(f) = \nu\mathbf{L}^2(f), \tag{1}$$

$$\mathbf{L}(g_t) + r^{-1}f[\mathbf{L}(g)]_r - r^{-1}g_r\mathbf{L}(f) - 2r^{-2}f\,\mathbf{L}(g) = \nu\mathbf{L}^2(g), \tag{2}$$

where $\mathbf{L}(f) = f_{rr} - r^{-1}f_r$; the subscripts denote the corresponding partial derivatives.

Particular solution of equation (1):

$$f(r,t) = C_1(t)r^2 + C_2(t),$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions. In this case, the change of variable $U = \mathbf{L}(g)$ brings (2) to a second-order linear equation.

⊙ *References for equation* 14.3.3.5: A. D. Polyanin and V. F. Zaitsev (2002, p. 347), V. V. Pukhnachov (2006).

▷ *For axisymmetric unsteady solutions of the Navier–Stokes equations, see also Subsection 21.3.3 (unsteady solutions).*

14.3.4. Other Equations

1. $\dfrac{\partial^3 w}{\partial t\partial x^2} + \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} - w\dfrac{\partial^3 w}{\partial x^3} = f(t)\dfrac{\partial^4 w}{\partial x^4}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + \varphi(t), t\big) + \varphi_t'(t),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Multiplicative separable solutions:

$$w = (Ae^{\lambda x} + Be^{-\lambda x})\exp\left[\lambda^2\int f(t)\,dt\right],$$

$$w = A\sin(\lambda x + B)\exp\left[-\lambda^2\int f(t)\,dt\right],$$

where A, B, and C are arbitrary constants.

3°. On integrating once with respect to x, we obtain the third-order equation

$$\frac{\partial^2 w}{\partial t\partial x} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = f(t)\frac{\partial^3 w}{\partial x^3} + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + 2a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial^4 w}{\partial t^2\partial x^2}.$

Boussinesq-type equation.

1°. *Olver–Rosenau solution*:

$$w = -\frac{1}{2a} + \frac{3b}{2at^2} + \frac{x^2}{2at^2}.$$

2°. Generalized separable solution (generalizes the solution from Item 1°):

$$w = \frac{aC_1^2}{27}t^8 + aC_1C_2t^3 + C_3t^2 - \frac{1}{2a} + \frac{C_4}{t} + \left(\frac{3b}{2a} + \frac{aC_2^2}{2}\right)\frac{1}{t^2} + \left(C_1t^3 + \frac{C_2}{t^2}\right)x + \frac{1}{2at^2}x^2.$$

⊙ *References for equation* 14.3.4.2: P. J. Olver and P. Rosenau (1986), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 6–7).

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^4 w}{\partial y^4}.$

This is a special case of equation 15.4.1.2 with $n = 4$.

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^4 w}{\partial y^4} + g(x).$

This is a special case of equation 15.4.1.3 with $n = 2$.

5. $\dfrac{\partial^2 w}{\partial x\partial t} + f(t)\dfrac{\partial^4 w}{\partial x^4} + g(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + h(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

Generalized Kadomtsev–Petviashvili equation.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(x + \varphi(t), \pm y + C, t\big) - \frac{\varphi_t'(t)}{g(t)},$$

where C is an arbitrary constant and $\varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w(x, y, t) = u(z, t), \qquad z = x + C_1 y - C_1^2 \int h(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z, t)$ is determined by the third-order differential equation

$$\frac{\partial u}{\partial t} + f(t)\frac{\partial^3 u}{\partial z^3} + g(t)u\frac{\partial u}{\partial z} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, t), \qquad \xi = x + \theta(t)(y + C_1)^2, \qquad \theta(t) = \left[4\int h(t)\,dt + C_2\right]^{-1},$$

where the function $U(\xi, t)$ is determined by the third-order differential equation

$$\frac{\partial U}{\partial t} + f(t)\frac{\partial^3 U}{\partial \xi^3} + g(t)U\frac{\partial U}{\partial \xi} + 2h(t)\theta(t)U = \psi(t),$$

with $\psi(t)$ being an arbitrary function.

6. $\dfrac{\partial^2 w}{\partial x \partial t} = a\dfrac{\partial^4 w}{\partial x^3 \partial y} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x \partial y} + c\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}.$

1°. Generalized separable solution:

$$w = \varphi(x,t) + \psi(x,t)y,$$

where the functions $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ are determined by the system of differential equations

$$\varphi_{xt} = a\psi_{xxx} + b\varphi_x\psi_x + c\psi\varphi_{xx},$$
$$\psi_{xt} = b\psi_x^2 + c\psi\psi_{xx}.$$

2°. There is a generalized separable solution of the form

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)y + \varphi_4(t)x^2 + \varphi_5(t)xy + \varphi_6(t)y^2 + \varphi_7(t)x^2y.$$

3°. For $c = -b$, there is a generalized separable solution of the form

$$w = \varphi(y,t) + \psi(y,t)e^{\lambda x},$$

where λ is an arbitrary constant and the two functions $\varphi = \varphi(y,t)$ and $\psi = \psi(y,t)$ are linked by a single relation

$$\psi_t = a\lambda^2\psi_y - b\lambda\psi\varphi_y.$$

By assuming the function $\psi = \psi(y,t)$ to be arbitrary, one finds $\varphi = \varphi(y,t)$ by simple integration with respect to y:

$$\varphi = \frac{a\lambda}{b}\ln|\psi| - \frac{1}{b\lambda}\frac{\partial}{\partial t}\int \ln|\psi(y,t)|\,dy + C(t).$$

4°. For $a = -1$, $b = 4$, and $c = 2$, it is the *breaking soliton equation* which describes the interaction of a Riemann wave along the y-axis and a long wave along the x-axis. In this case, the equation can be solved by the inverse spectral transform.

In the case $a = -\frac{1}{4}$, $b = -1$, and $c = -\frac{1}{2}$, it is also an integrable equation.

⊙ *References for equation* 14.3.4.6: F. Calogero and A. Degasperis (1976), S.-J. Yu, K. Toda, and T. Fukuyama (1998), X. Geng and C. Cao (2004), Y. Yao (2005), V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 333).

7. $\dfrac{\partial^4 w}{\partial x_1 \partial x_2 \partial x_3 \partial x_4} = e^w.$

Fourth-order Liouville type equation.

Solution:

$$w = \ln\frac{24\alpha_1'(x_1)\alpha_2'(x_2)\alpha_3'(x_3)\alpha_4'(x_4)}{[\alpha_1(x_1) + \alpha_2(x_2) + \alpha_3(x_3) + \alpha_4(x_4)]^4},$$

where $\alpha_1(x_1)$, $\alpha_2(x_2)$, $\alpha_3(x_3)$, and $\alpha_4(x_4)$ are arbitrary functions.

⊙ *Reference*: A. N. Mironov (2010).

Chapter 15
Equations of Higher Orders

15.1. Equations Involving the First Derivative in t and Linear in the Highest Derivative

15.1.1. Fifth-Order Equations

1. $$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = a\frac{\partial^5 w}{\partial x^5}.$$

This is a special case of equation 15.1.4.1 with $n = 5$ and $b = -1$.

2. $$\frac{\partial w}{\partial t} - bw^k\frac{\partial w}{\partial x} = a\frac{\partial^5 w}{\partial x^5}.$$

This is a special case of equation 15.1.4.2 with $n = 5$.

3. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + be^{\lambda w}\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.3 with $n = 5$.

4. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + (b\ln w + c)\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.4 with $n = 5$.

5. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + (b\operatorname{arcsinh} w + c)\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.5 with $n = 2$ and $k = 1$.

6. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + (b\operatorname{arccosh} w + c)\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.6 with $n = 2$ and $k = 1$.

7. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + (b\arcsin w + c)\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.7 with $n = 2$ and $k = 1$.

8. $$\frac{\partial w}{\partial t} = a\frac{\partial^5 w}{\partial x^5} + (b\arccos w + c)\frac{\partial w}{\partial x}.$$

This is a special case of equation 15.1.4.8 with $n = 2$ and $k = 1$.

9. $\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} + a\dfrac{\partial^3 w}{\partial x^3} = b\dfrac{\partial^5 w}{\partial x^5}.$

Kawahara's equation. It describes magnetoacoustic waves in plasma and long water waves under ice cover.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm t + C_2),$$
$$w_2 = w(x - C_3 t, t) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken in the first formula).

2°. Degenerate solution:

$$w(x,t) = \frac{x + C_1}{t + C_2}.$$

3°. Traveling-wave solutions:

$$w(x,t) = \frac{105a^2}{169b\cosh^4 z} + 2C_1, \quad z = \tfrac{1}{2}kx - (18bk^5 + C_1 k)t + C_2, \quad k = \sqrt{\frac{a}{13b}} \quad \text{if } ab > 0;$$
$$w(x,t) = \frac{105a^2}{169b\sinh^4 z} + 2C_1, \quad z = \tfrac{1}{2}kx - (18bk^5 + C_1 k)t + C_2, \quad k = \sqrt{\frac{a}{13b}} \quad \text{if } ab > 0;$$
$$w(x,t) = \frac{105a^2}{169b\cos^4 z} + 2C_1, \quad z = \tfrac{1}{2}kx - (18bk^5 + C_1 k)t + C_2, \quad k = \sqrt{-\frac{a}{13b}} \quad \text{if } ab < 0,$$

where C_1 and C_2 are arbitrary constants.

4°. Traveling-wave solution for $a = 0$:

$$w(x,t) = \frac{1680b}{(x + C_1 t + C_2)^4} - C_1.$$

5°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x - C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the fourth-order ordinary differential equation (C_3 is an arbitrary constant)

$$bU'''''_{\zeta\zeta\zeta\zeta} - aU''_{\zeta\zeta} - \tfrac{1}{2}U^2 - C_2 U = 2C_1\zeta + C_3.$$

The special case $C_1 = 0$ corresponds to a traveling-wave solution.

⊙ *References for equation* 15.1.1.9: T. Kawahara (1972), A. V. Marchenko (1988), N. A. Kudryashov (1990 *a*), A. D. Polyanin and V. F. Zaitsev (2004, p. 632).

10. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} + b\dfrac{\partial^3 w}{\partial x^3} = c\dfrac{\partial^5 w}{\partial x^5} + kw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - aC_1 e^{kt} + C_2, t + C_3) + C_1 k e^{kt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = U(z) + C_1 k e^{kt}, \qquad z = x - aC_1 e^{kt} + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$cU_z^{(5)} - bU_{zzz}''' - aUU_z' - C_2 U_z' + kU = 0.$$

If $C_1 = 0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

11. $$\frac{\partial w}{\partial t} + a_1\frac{\partial w}{\partial x} + a_2 w\frac{\partial w}{\partial x} + a_3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + a_4\frac{\partial^3 w}{\partial x^3} + a_5 w\frac{\partial^3 w}{\partial x^3} + a_6\frac{\partial^5 w}{\partial x^5} = 0.$$

This equation describes long water waves with surface tension.

1°. Traveling-wave solutions:

$$w(x,t) = A + C_1\exp(kx + C_2 t), \qquad k = \pm\sqrt{-\frac{a_2}{a_3+a_5}}, \quad A = -\frac{a_6k^5 + a_4k^3 + a_1k + C_2}{a_5k^3 + a_2k};$$

$$w(x,t) = A + C_1\sinh(kx + C_2 t + C_3), \quad k = \pm\sqrt{-\frac{a_2}{a_3+a_5}}, \quad A = -\frac{a_6k^5 + a_4k^3 + a_1k + C_2}{a_5k^3 + a_2k};$$

$$w(x,t) = A + C_1\cosh(kx + C_2 t + C_3), \quad k = \pm\sqrt{-\frac{a_2}{a_3+a_5}}, \quad A = -\frac{a_6k^5 + a_4k^3 + a_1k + C_2}{a_5k^3 + a_2k};$$

$$w(x,t) = A + C_1\sin(kx + C_2 t + C_3), \qquad k = \pm\sqrt{\frac{a_2}{a_3+a_5}}, \quad A = \frac{a_6k^5 - a_4k^3 + a_1k + C_2}{a_5k^3 - a_2k},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. There are traveling-wave solutions of the following forms:

$$w(x,t) = A + \frac{B}{\cosh z} + \frac{C}{\cosh^2 z},$$

$$w(x,t) = A + \frac{B}{\sinh z} + \frac{C}{\sinh^2 z},$$

$$w(x,t) = A + B\frac{\sinh z}{\cosh z} + \frac{C}{\cosh^2 z},$$

$$w(x,t) = A + \frac{B + C\sinh z + D\cosh z}{(E + \cosh z)^2},$$

where $z = kx + \lambda t + \text{const}$, and the constants A, B, C, D, E, k, and λ are identified by substituting these solutions into the original equation.

⊙ *References for equation* 15.1.1.11: P. J. Olver (1984), N. A. Kudryashov and M. B. Sukharev (2001), P. Saucez, A. Vande Wouwer, W. E. Schiesser, and P. Zegeling (2003).

12. $\dfrac{\partial w}{\partial t} + \alpha w^2 \dfrac{\partial w}{\partial x} + \beta \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + \gamma w \dfrac{\partial^3 w}{\partial x^3} + \sigma \dfrac{\partial^5 w}{\partial x^5} = 0.$

General fifth-order KdV equation (or *fKdV equation* for short). This equation describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice and it is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics, and quantum field theory. A great deal of research work has been invested during the past decades for the study of the fKdV equation.

1°. Given below are exact solutions that are valid for any values of the parameters of the equation in question.

1.1. Traveling-wave solutions with $\alpha \neq 0$:

$$w = \frac{k(A+\beta+2\gamma)}{\alpha}\left[3\tanh^2\left(\frac{\sqrt{-k}}{\alpha}(\alpha x - 2Bk^2 t) + D\right) - 2\right] \quad \text{if} \quad k < 0,$$

$$w = \frac{k(A+\beta+2\gamma)}{\alpha}\left[3\coth^2\left(\frac{\sqrt{-k}}{\alpha}(\alpha x - 2Bk^2 t) + D\right) - 2\right] \quad \text{if} \quad k < 0,$$

$$w = -\frac{k(A+\beta+2\gamma)}{\alpha}\left[3\tan^2\left(\frac{\sqrt{k}}{\alpha}(\alpha x - 2Bk^2 t) + D\right) + 2\right] \quad \text{if} \quad k > 0,$$

$$w = -\frac{k(A-\beta-2\gamma)}{\alpha}\left[3\tanh^2\left(\frac{\sqrt{-k}}{\alpha}(\alpha x + 2Ck^2 t) + D\right) - 2\right] \quad \text{if} \quad k < 0,$$

$$w = -\frac{k(A-\beta-2\gamma)}{\alpha}\left[3\coth^2\left(\frac{\sqrt{-k}}{\alpha}(\alpha x + 2Ck^2 t) + D\right) - 2\right] \quad \text{if} \quad k < 0,$$

$$w = \frac{k(A-\beta-2\gamma)}{\alpha}\left[3\tan^2\left(\frac{\sqrt{k}}{\alpha}(\alpha x + 2Ck^2 t) + D\right) + 2\right] \quad \text{if} \quad k > 0,$$

where D and k are arbitrary constants, and other constants are given by

$$A = \sqrt{(\beta+2\gamma)^2 - 40\alpha\sigma}, \quad B = \beta^2 + A\beta + 2\beta\gamma - 12\alpha\sigma, \quad C = -\beta^2 + A\beta - 2\beta\gamma + 12\alpha\sigma.$$

1.2. Traveling-wave solutions with $\alpha \neq 0$:

$$w = -4\mu^2 \frac{BC_1^2 e^z + 4Be^{-z} - AC_1}{C_1^2 e^z + 4e^{-z} + 4C_1}, \quad z = \mu^5\Big(2\beta B - \frac{3}{2}\sigma\Big)t - \mu x + C_2,$$

where

$$A = \frac{10(4B\gamma - 5\sigma)}{\sqrt{(\beta+2\gamma)^2 - 40\alpha\sigma} - \beta}, \quad B = \frac{\beta + 2\gamma + \sqrt{(\beta+2\gamma)^2 - 40\alpha\sigma}}{16\alpha},$$

and C_1, C_2, and μ are arbitrary constants.

Special case 1. With $C_1 = \pm 2$ and $\mu = ki$ ($i^2 = -1$), we obtain periodic solutions:

$$w = 4Bk^2 - \frac{(A+4B)k^2}{1 \pm \cos[kx + \frac{1}{2}k^5(3\sigma - 4B\beta)t + C_2]}.$$

1.3. Traveling-wave solutions with $\alpha \neq 0$:

$$w = -4\mu^2 \frac{BC_1^2 e^z + 4Be^{-z} + AC_1}{C_1^2 e^z + 4e^{-z} + 4C_1}, \qquad z = \mu^5 \Big(2\beta B - \frac{3}{2}\sigma\Big)t - \mu x + C_2,$$

where

$$A = \frac{10(4B\gamma - 5\sigma)}{\sqrt{(\beta + 2\gamma)^2 - 40\alpha\sigma} + \beta}, \qquad B = \frac{\beta + 2\gamma - \sqrt{(\beta + 2\gamma)^2 - 40\alpha\sigma}}{16\alpha},$$

and C_1, C_2, and μ are arbitrary constants.

Special case 2. With $C_1 = \pm 2$ and $\mu = ki$ ($i^2 = -1$), we obtain periodic solutions:

$$w = 4Bk^2 + \frac{(A - 4B)k^2}{1 \pm \cos[kx + \frac{1}{2}k^5(3\sigma - 4B\beta)t + C_2]}.$$

⊙ *References*: A. H. Salas (2008 *b*), the cited paper gives also more cumbersome solutions; see also A. H. Salas: http://eqworld.ipmnet.ru/eqarchive/view.php?id=313.

In the literature, four special cases of the fifth-order KdV equation are most frequently encountered; these case are considered below. The solutions from Item 1° remain valid, but also some other solutions are given.

2°. For $\alpha = 30$, $\beta = 20$, $\gamma = 10$, and $\sigma = 1$, we have the *Lax equation*:

$$\frac{\partial w}{\partial t} + 30w^2 \frac{\partial w}{\partial x} + 20 \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} + 10w \frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0.$$

Traveling-wave solutions:

$$w = \frac{2ck^2 \exp(kx - k^5 t)}{[1 + c\exp(kx - k^5 t)]^2},$$
$$w = 4k^2 - 6k^2 \tanh^2(kx - 896k^5 t + c) - 6k^2 \coth^2(kx - 896k^5 t + c),$$
$$w = \tfrac{4}{3}k^2 - 2k^2 \tanh^2\big(kx - \tfrac{128}{3}k^5 t + c\big) - 2k^2 \coth^2\big(kx - \tfrac{128}{3}k^5 t + c\big),$$
$$w = a - 2k^2 \tanh^2[kx - 2k(48k^4 - 40ak^2 + 15a^2)t + c]$$
$$\quad - 2k^2 \coth^2[kx - 2k(48k^4 - 40ak^2 + 15a^2)t + c],$$

where a, c, and k are arbitrary constants.

Remark. The Lax equation is a particular case of the Korteweg–de Vries hierarchy 5.1.11.2 with $n = 2$.

⊙ *References*: P. D. Lax (1968), A. M. Wazwaz (2008); see also C. A. Gómes (2007), where other solutions are also given.

3°. For $\alpha = \beta = \gamma = 5$, $\sigma = 1$ we have *Sawada–Kotera equation* (or *SK equation* for short):

$$\frac{\partial w}{\partial t} + 5w^2 \frac{\partial w}{\partial x} + 5 \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} + 5w \frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0.$$

3.1. Traveling-wave solutions:

$$w = \frac{6ck^2 \exp(kx - k^5 t)}{[1 + c\exp(kx - k^5 t)]^2},$$
$$w = 4k^2 - 6k^2 \tanh^2(kx + 64k^5 t + c) - 6k^2 \coth^2(kx + 64k^5 t + c),$$
$$w = 8k^2 - 12k^2 \tanh^2(kx - 256k^5 t + c) - 12k^2 \coth^2(kx - 256k^5 t + c),$$
$$w = a - 6k^2 \tanh^2[kx - k(16k^4 - 40ak^2 + 5a^2)t + c]$$
$$\quad - 6k^2 \coth^2[kx - k(16k^4 - 40ak^2 + 5a^2)t + c],$$

where a, c, and k are arbitrary constants.

⊙ *References*: K. Sawada and T. Kotera (1974), A. M. Wazwaz (2008); see also C. A. Gómes (2007), where other solutions are also given.

3.2. Traveling-wave solutions:

$$w = a + \frac{3k}{1 \pm \cosh\left[\sqrt{k}\,x - \sqrt{k}\,(5a^2 + 5ak + k^2)t + c\right]},$$

$$w = -k\left\{1 - \frac{6}{1 + \cosh\left[\sqrt{k}\,(x - k^2t) + c\right]}\right\},$$

$$w = a + \frac{6k}{\cosh^2\left[\sqrt{k}\,x - \sqrt{k}\,(5a^2 + 20ak + 16k^2)t + c\right]},$$

$$w = -4k\left\{1 - \frac{3}{\cosh^2\left[\sqrt{k}\,(x - 16k^2t) + c\right]}\right\},$$

$$w = -\frac{k\left\{a^2 - 6 - 4a\cosh\left[\sqrt{k}\,(x - k^2t) + c\right] + \cosh^2\left[\sqrt{k}\,(x - k^2t) + c\right]\right\}}{\left\{a + \cosh\left[\sqrt{k}\,(x - k^2t) + c\right]\right\}^2},$$

$$w = a - \frac{3k}{1 \pm \cos\left[\sqrt{k}\,x - \sqrt{k}\,(5a^2 - 5ak + k^2)t + c\right]},$$

$$w = k\left\{1 - \frac{6}{1 \pm \cos\left[\sqrt{k}\,(x - k^2t) + c\right]}\right\},$$

$$w = a - \frac{6k}{\cos^2\left[\sqrt{k}\,x - \sqrt{k}\,(5a^2 - 20ak + 16k^2)t + c\right]},$$

$$w = 4k\left\{1 - \frac{3}{\cos^2\left[\sqrt{k}\,(x - 16k^2t) + c\right]}\right\},$$

$$w = \frac{k\left\{a^2 - 6 - 4a\sin\left[\sqrt{k}\,(x - k^2t) + c\right] + \sin^2\left[\sqrt{k}\,(x - k^2t) + c\right]\right\}}{\left\{a + \sin\left[\sqrt{k}\,(x - k^2t) + c\right]\right\}^2},$$

where a, c, and k are arbitrary constants.

⊙ *References*: A. H. Salas (2008 *a*), see also http://eqworld.ipmnet.ru/eqarchive/view.php?id=319.

4°. For $\alpha = 20$, $\beta = 25$, $\gamma = 10$, and $\sigma = 1$, we have the *Kaup–Kupershmidt equation* (or *KK equation* for short):

$$\frac{\partial w}{\partial t} + 20w^2\frac{\partial w}{\partial x} + 25\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + 10w\frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0.$$

Traveling-wave solutions:

$$w = k^2 - \tfrac{3}{2}k^2\tanh^2(kx - 16k^5t + c) - \tfrac{3}{2}k^2\coth^2(kx - 16k^5t + c),$$

$$w = 8k^2 - 12k^2\tanh^2(kx - 2816k^5t + c) - 12k^2\coth^2(kx - 2816k^5t + c),$$

where c and k are arbitrary constants.

⊙ *References*: D. Kaup (1975), B. A. Kupershmidt (1984), A. M. Wazwaz (2008).

5°. For $\alpha = 2$, $\beta = 6$, $\gamma = 3$, and $\sigma = 1$, we have the *Ito equation*:

$$\frac{\partial w}{\partial t} + 2w^2\frac{\partial w}{\partial x} + 6\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + 3w\frac{\partial^3 w}{\partial x^3} + \frac{\partial^5 w}{\partial x^5} = 0.$$

Traveling-wave solutions:

$$w = 20k^2 - 30k^2\tanh^2(kx - 1536k^5t + c) - 30k^2\coth^2(kx - 1536k^5t + c),$$
$$w = 4k^2 - 6k^2\tanh^2(kx + c) - 6k^2\coth^2(kx + c),$$

where c and k are arbitrary constants. The second solution is independent of time t.

⊙ *References*: M. Ito (1980), A. M. Wazwaz (2008).

Remark. The Lax, Sawada–Kotera, and Kaup–Kupershmidt equations belong to the completely integrable hierarchy of higher-order KdV equations. These three equations have infinite sets of conservation laws (Hereman and Zhaung (1980), Hereman and Nuseir (1997)). The Ito equation is not completely integrable and has a limited number of special conservation laws.

13. $\dfrac{\partial w}{\partial t} + 30w^2\dfrac{\partial w}{\partial x} + 20\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + 10w\dfrac{\partial^3 w}{\partial x^3} + \dfrac{\partial^5 w}{\partial x^5} = f(t).$

Lax equation with forcing term.

1°. Solutions:

$$w = a - 2\mu^2\lambda^2 k\tan^2\Big[\mu\sqrt{k}\Big(\lambda x + \int g(t)\,dt\Big) + b\Big] + \int f(t)\,dt \qquad \text{if} \quad k > 0,$$
$$w = a + 2\mu^2\lambda^2 k\tanh^2\Big[\mu\sqrt{-k}\Big(\lambda x + \int g(t)\,dt\Big) + b\Big] + \int f(t)\,dt \quad \text{if} \quad k < 0,$$
$$w = a + 2\mu^2\lambda^2 k\coth^2\Big[\mu\sqrt{-k}\Big(\lambda x + \int g(t)\,dt\Big) + b\Big] + \int f(t)\,dt \quad \text{if} \quad k < 0,$$

where a, b, k, μ, and λ are arbitrary constants, and

$$g(t) = -56k^2\lambda^5\mu^4 - 80ak\lambda^3\mu^2 - 30a^2\lambda - 80k\lambda^3\mu^2F - 60a\lambda F - 30\lambda F^2, \qquad F = \int f(t)\,dt.$$

2°. Solutions:

$$w = a - 2\mu^2\lambda^2 k\big(\tan^2 z + \cot^2 z\big) + \int f(t)\,dt, \qquad z = \mu\sqrt{k}\Big(\lambda x + \int g(t)\,dt\Big) + b;$$
$$w = a + 2\mu^2\lambda^2 k\big(\tanh^2 z + \coth^2 z\big) + \int f(t)\,dt, \quad z = \mu\sqrt{-k}\Big(\lambda x + \int g(t)\,dt\Big) + b,$$

where a, b, k, μ, and λ are arbitrary constants, and

$$g(t) = -96k^2\lambda^5\mu^4 - 80ak\lambda^3\mu^2 - 30a^2\lambda - 80k\lambda^3\mu^2F - 60a\lambda F - 30\lambda F^2, \qquad F = \int f(t)\,dt.$$

3°. Solution:

$$w = a - 2\lambda^2\Big[\lambda x + \int g(t)\,dt + b\Big]^{-2} + F(t),$$
$$F(t) = \int f(t)\,dt, \quad g(t) = -30a^2\lambda - 60a\lambda F(t) - 30\lambda F^2(t),$$

where a, b, and λ are arbitrary constants.

⊙ *Reference for equation* 15.1.1.13: A. H. Salas and C. A. Gómes (2008).

14. $\dfrac{\partial w}{\partial t} + 5w^2\dfrac{\partial w}{\partial x} + 5\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + 5w\dfrac{\partial^3 w}{\partial x^3} + \dfrac{\partial^5 w}{\partial x^5} = f(t).$

Sawada–Kotera equation with forcing term.

1°. Solutions:

$$w = a - 6\mu^2\lambda^2 k\tan^2\left[\mu\sqrt{k}\left(\lambda x + \int g(t)\,dt\right) + b\right] + \int f(t)\,dt \quad \text{if} \quad k > 0,$$
$$w = a + 6\mu^2\lambda^2 k\tanh^2\left[\mu\sqrt{-k}\left(\lambda x + \int g(t)\,dt\right) + b\right] + \int f(t)\,dt \quad \text{if} \quad k < 0,$$
$$w = a + 6\mu^2\lambda^2 k\coth^2\left[\mu\sqrt{-k}\left(\lambda x + \int g(t)\,dt\right) + b\right] + \int f(t)\,dt \quad \text{if} \quad k < 0,$$

where a, b, k, μ, and λ are arbitrary constants, and

$$g(t) = -76k^2\lambda^5\mu^4 - 40ak\lambda^3\mu^2 - 5a^2\lambda - 40k\lambda^3\mu^2 F - 10a\lambda F - 5\lambda F^2, \qquad F = \int f(t)\,dt.$$

2°. Solutions:

$$w = a - 6\mu^2\lambda^2 k\big(\tan^2 z + \cot^2 z\big) + \int f(t)\,dt, \qquad z = \mu\sqrt{k}\left(\lambda x + \int g(t)\,dt\right) + b;$$
$$w = a + 6\mu^2\lambda^2 k\big(\tanh^2 z + \coth^2 z\big) + \int f(t)\,dt, \quad z = \mu\sqrt{-k}\left(\lambda x + \int g(t)\,dt\right) + b,$$

where a, b, k, μ, and λ are arbitrary constants, and

$$g(t) = -16k^2\lambda^5\mu^4 - 40ak\lambda^3\mu^2 - 5a^2\lambda - 40k\lambda^3\mu^2 F - 10a\lambda F - 5\lambda F^2, \qquad F = \int f(t)\,dt.$$

3°. Solution:

$$w = a - 6\lambda^2\left[\lambda x + \int g(t)\,dt + b\right]^{-2} + F(t),$$
$$F(t) = \int f(t)\,dt, \quad g(t) = -5a^2\lambda - 10a\lambda F(t) - 5\lambda F^2(t),$$

where a, b, and λ are arbitrary constants.

⊙ *Reference for equation* 15.1.1.14: A. H. Salas and C. A. Gómes (2008).

15. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5}{\partial x^5}(w^2) + b\dfrac{\partial^3}{\partial x^3}(w^2) + c\dfrac{\partial}{\partial x}(w^2) + dw + e.$

1°. Generalized separable solution:

$$w = \varphi(t) + \psi(t)e^{\lambda x},$$

where λ is a root of the biquadratic equation $16a\lambda^4 + 4b\lambda^2 + c = 0$ and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi' = d\varphi + c,$$
$$\psi' = 2\lambda(a\lambda^4 + b\lambda + c)\varphi\psi + d\psi.$$

This system is easy to integrate.

2°. Generalized separable solutions with $c = 4b - 16a$:

$$w = Ae^{dt} - \frac{e}{d} + Be^{dt}\cos\Big[x + \mu\Big(\frac{A}{d}e^{dt} - \frac{e}{d}t\Big) + C\Big] \quad \text{if} \quad d \neq 0,$$
$$w = A + et + B\cos\big[x + \mu\big(\tfrac{1}{2}et^2 + At\big) + C\big] \quad \text{if} \quad d = 0,$$

where $\mu = 6(b - 5a)$ and A, B, and C are arbitrary constants.

3°. For $b = 25a$, $c = 144a$, there is a generalized separable solution of the form

$$w = \varphi_1(t) + \varphi_2(t)\cos x + \varphi_3(t)\sin x + \varphi_4(t)\cos 2x + \varphi_5(t)\sin 2x,$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi_1' = d\varphi_1 + e,$$
$$\varphi_2' = 120a(\varphi_2\varphi_5 - \varphi_3\varphi_4 + 2\varphi_1\varphi_3) + d\varphi_2,$$
$$\varphi_3' = -120a(\varphi_2\varphi_4 + \varphi_3\varphi_5 + 2\varphi_1\varphi_2) + d\varphi_3,$$
$$\varphi_4' = 120a(2\varphi_1\varphi_5 + \varphi_2\varphi_3) + d\varphi_4,$$
$$\varphi_5' = 60a(\varphi_3^2 - \varphi_2^2 - 4\varphi_1\varphi_4) + d\varphi_5.$$

Special case. With $a = 1$, $d = e = 0$ and

$$\varphi_1 = -\frac{\lambda}{280}, \quad \varphi_2 = -\frac{\lambda}{210}\cos\lambda t, \quad \varphi_3 = -\frac{\lambda}{210}\sin\lambda t, \quad \varphi_4 = -\frac{\lambda}{840}\cos 2\lambda t, \quad \varphi_3 = -\frac{\lambda}{840}\sin 2\lambda t,$$

we have the compacton-type traveling-wave solution

$$w = -\tfrac{1}{105}\lambda\cos^4\big[\tfrac{1}{2}(x - \lambda t)\big].$$

⊙ *References for equation* 15.1.1.15: B. Dey (1998), V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 189–190).

15.1.2. Some Equations with Sixth- to Ninth-Order

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^6 w}{\partial x^6} + bw\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.1 with $n = 6$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^6 w}{\partial x^6} + bw^k\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.2 with $n = 6$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^6 w}{\partial x^6} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.3 with $n = 6$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^6 w}{\partial x^6} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.4 with $n = 6$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + bw\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.1 with $n = 7$.

6. $\dfrac{\partial w}{\partial t} + 6w\dfrac{\partial w}{\partial x} + \dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial^5 w}{\partial x^5} + a\dfrac{\partial^7 w}{\partial x^7} = 0.$

Seventh-order KdV-type equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - 6C_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions with $a = \dfrac{769}{2500}$:

$$w = \frac{86625}{591361}\operatorname{sech}^6\left[\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right) + C\right],$$
$$w = -\frac{86625}{591361}\operatorname{csch}^6\left[\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right) + C\right],$$
$$w = -\frac{60000}{591361} + \frac{86625}{591361}\operatorname{sech}^6\left[\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right) + C\right],$$
$$w = -\frac{60000}{591361} - \frac{86625}{591361}\operatorname{csch}^6\left[\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right) + C\right],$$
$$w = -\frac{86625}{591361}\sec^6\left[\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right) + C\right],$$
$$w = -\frac{86625}{591361}\csc^6\left[\frac{5}{\sqrt{1538}}\left(x + \frac{180000}{591361}t\right) + C\right],$$
$$w = \frac{60000}{591361} - \frac{86625}{591361}\sec^6\left[\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right) + C\right],$$
$$w = \frac{60000}{591361} - \frac{86625}{591361}\csc^6\left[\frac{5}{\sqrt{1538}}\left(x - \frac{180000}{591361}t\right) + C\right],$$

where C is an arbitrary constant.

⊙ *References*: Y. Pomeau, A. Ramani, and B. Grammaticos (1988), W.-X. Ma (1993), P. G. Drazin and R. S. Johnson (1996), B. R. Duffy and E. J. Parkes (1996), A. M. Wazwaz (2008).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + bw^k\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.2 with $n = 7$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.3 with $n = 7$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.4 with $n = 7$.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + [b\,\operatorname{arcsinh}(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.5 with $n = 3$.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + [b\,\operatorname{arccosh}(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.6 with $n = 3$.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + [b\arcsin(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.7 with $n = 3$.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^7 w}{\partial x^7} + [b\arccos(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.8 with $n = 3$.

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + bw\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.1 with $n = 9$.

15. $\dfrac{\partial w}{\partial t} + 6w\dfrac{\partial w}{\partial x} + \dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial^5 w}{\partial x^5} + a\dfrac{\partial^7 w}{\partial x^7} + b\dfrac{\partial^9 w}{\partial x^9} = 0.$

Ninth-order KdV-type equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - 6C_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solutions with $a = \dfrac{212648}{506527}$ and $b = -\dfrac{11304792976}{180266374449}$:

$$w = \frac{3816888075}{22609585952}\operatorname{sech}^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = \frac{3816888075}{22609585952}\operatorname{csch}^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = -\frac{83040300}{706549561} + \frac{3816888075}{22609585952}\operatorname{sech}^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = -\frac{83040300}{706549561} + \frac{3816888075}{22609585952}\operatorname{csch}^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}\,t\right) + C\right],$$

where C is an arbitrary constant.

⊙ *Reference*: A. M. Wazwaz (2008).

3°. Traveling-wave solutions with $a = \dfrac{212648}{506527}$ and $b = \dfrac{11304792976}{180266374449}$:

$$w = -\frac{3816888075}{22609585952}\sec^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = -\frac{3816888075}{22609585952}\csc^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x + \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = \frac{83040300}{706549561} - \frac{3816888075}{22609585952}\sec^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}\,t\right) + C\right],$$

$$w = \frac{83040300}{706549561} - \frac{3816888075}{22609585952}\csc^8\left[\frac{1}{4}\sqrt{\frac{5649}{26581}}\left(x - \frac{249120900}{706549561}\,t\right) + C\right],$$

where C is an arbitrary constant.

⊙ *Reference*: A. M. Wazwaz (2008).

16. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + bw^k\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.2 with $n = 9$.

17. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.3 with $n = 9$.

18. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.4 with $n = 9$.

19. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + [b\,\mathrm{arcsinh}(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.5 with $n = 4$.

20. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + [b\,\mathrm{arccosh}(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.6 with $n = 4$.

21. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + [b\arcsin(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.7 with $n = 4$.

22. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^9 w}{\partial x^9} + [b\arccos(kw) + c]\dfrac{\partial w}{\partial x}.$

This is a special case of equation 15.1.4.8 with $n = 4$.

15.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x+bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$aw_\xi^{(n)} - bw'_\xi + f(\xi, w) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + f(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{aA^n}{b(n-1)}e^{nbt} + e^{bt}\int e^{-bt}f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt}f(t)\,dt\right]\varphi(z), \quad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$a\varphi_z^{(n)} - \lambda\varphi'_z + b\varphi\ln\varphi = 0,$$

whose order can be reduced by one.

3°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}f(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\ln u.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + \big[f(x) + g(t)\big]w.$

1°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}g(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + b\varphi\ln\varphi + f(x)\varphi = 0.$$

2°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}g(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\ln u + f(x)u.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)w\ln w + g(t)w.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\bigl[\varphi(t)x + \psi(t)\bigr].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F, \quad \psi(t) = Be^F + e^F\int e^{-F}(aA^n e^{nF} + g)\,dt, \qquad F = \int f\,dt,$$

where A and B are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)w\ln w + \bigl[g(t)x + h(t)\bigr]w.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\bigl[\varphi(t)x + \psi(t)\bigr].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F + e^F\int e^{-F}g\,dt, \quad F = \int f\,dt,$$
$$\psi(t) = Be^F + e^F\int e^{-F}(a\varphi^n + h)\,dt,$$

where A and B are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)w\ln w + \bigl[bf(x)t + g(x)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi\ln\varphi + \bigl[g(x) + b\bigr]\varphi = 0.$$

15.1.4. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(w)\dfrac{\partial w}{\partial x}$

Preliminary remarks. Equations of this form admit traveling-wave solutions:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation (C is an arbitrary constant)

$$aw_z^{(n-1)} + \int f(w)\,dw - \lambda w = C.$$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x}.$

Generalized Burgers–Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-1} w(C_1 x + bC_1C_2 t + C_3, C_1^n t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = -\frac{x + C_1}{b(t + C_2)},$$

$$w(x,t) = (-1)^n \frac{a(2n-2)!}{b(n-1)!} \frac{1}{(x + bC_1 t + C_2)^{n-1}} + C_1.$$

The first solution is degenerate and the second one is a traveling-wave solution (a special case of the solution of Item 3°).

3°. Traveling-wave solution:

$$w = w(\xi), \quad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(\xi)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation

$$a w_\xi^{(n-1)} + \tfrac{1}{2} b w^2 = \lambda w + C.$$

4°. Self-similar solution:

$$w(x,t) = t^{\frac{1-n}{n}} u(\eta), \quad \eta = x t^{-\frac{1}{n}},$$

where the function $u(\eta)$ is determined by the ordinary differential equation

$$a u_\eta^{(n)} + b u u'_\eta + \frac{1}{n}\eta u'_\eta + \frac{n-1}{n} u = 0.$$

5°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \quad \zeta = x + bC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the $(n-1)$st-order ordinary differential equation

$$a U_\zeta^{(n-1)} + \tfrac{1}{2} b U^2 - C_2 U = 2C_1 \zeta + C_3.$$

6°. Solution:

$$w = \varphi^{n-1} F(z) + \frac{1}{b\varphi}(\varphi'_t x + \psi'_t), \quad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are defined by

$$\varphi(t) = (Ant + C_1)^{-\frac{1}{n}},$$

$$\psi(t) = C_2(Ant + C_1)^{\frac{n-1}{n}} + C_3(Ant + C_1)^{-\frac{1}{n}} + \frac{B}{A^2(n-1)},$$

where A, B, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$a F_z^{(n)} + bFF'_z + A(n-2)F + \frac{A^2}{b}(1-n)z + \frac{B}{b} = 0.$$

⊙ *References for equation* 15.1.4.1: A. D. Polyanin, V. F. Zaitsev (2004, pp. 635–636).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw^k\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-1} w(C_1^k x + C_2, C_1^{nk} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x,t) = t^{\frac{1-n}{nk}} U(z), \qquad z = xt^{-\frac{1}{n}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$aU_z^{(n)} + bU^k U_z' + \frac{1}{n} zU_z' + \frac{n-1}{nk} U = 0.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^n t + C_3) + \frac{n-1}{\lambda} \ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = U(z) + \frac{1-n}{n\lambda} \ln t, \qquad z = xt^{-\frac{1}{n}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$aU_z^{(n)} + be^{\lambda U} U_z' + \frac{1}{n} zU_z' + \frac{n-1}{n\lambda} = 0.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + (b \ln w + c)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = e^{C_1} w(x + bC_1 t + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{x + C_2}{C_1 - bt} + \frac{a}{b(n-2)} \frac{1}{(C_1 - bt)^{n-1}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w(x,t) = e^{\lambda t} u(z), \qquad z = x + \tfrac{1}{2} b\lambda t^2 + kt,$$

where k and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$au_z^{(n)} + (b \ln u + c - k)u_z' - \lambda u = 0.$$

To $\lambda = 0$ there corresponds a traveling-wave solution.

⊙ *References for equation* 15.1.4.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 636).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\,\operatorname{arcsinh}(kw) + c]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sinh\left[\frac{x+C_2}{C_1-bt} + \frac{a}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\,\operatorname{arccosh}(kw) + c]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cosh\left[\frac{x+C_2}{C_1-bt} + \frac{a}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 637).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\arcsin(kw) + c]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\left[\frac{x+C_2}{C_1-bt} + \frac{a(-1)^n}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 637).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\arccos(kw) + c]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\left[\frac{x+C_2}{C_1-bt} + \frac{a(-1)^n}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

15.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + (bx+c)\dfrac{\partial w}{\partial x} + f(w).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$a w_z^{(n)} + (bz + c)w_z' + f(w) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\dfrac{\partial w}{\partial x} + g(w).$

The transformation $w = u(z,t),\ z = x + \int f(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial z^n} + g(u),$$

which has a traveling-wave solution $u = u(kz + \lambda t)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + \big[bx + f(t)\big]\dfrac{\partial w}{\partial x} + g(w).$

Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-bt} + e^{-bt}\int e^{bt} f(t)\,dt,$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$aw_z^{(n)} + bzw_z' + g(w) = 0.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}h(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi_x' + b\varphi\ln\varphi + g(x)\varphi = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \quad z = x + b\int_{t_0}^{t}(t-\tau)f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 15.1.4.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\frac{\partial u}{\partial x}.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 638).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x} + cw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + bC_1e^{ct} + C_2,\ t + C_3) + C_1ce^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = U(z) + C_1ce^{ct}, \qquad z = x + bC_1e^{ct} + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + bUU_z' - C_2U_z' + cU = 0.$$

For $C_1 = 0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + \big[bw + f(t)\big]\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + \int_{t_0}^{t} f(\tau)\,d\tau + b\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 15.1.4.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\frac{\partial u}{\partial x}.$$

8. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^n w}{\partial x^n} + f(t)w\dfrac{\partial w}{\partial x} + g(t)w = 0.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + C_1\psi(t) + C_2,\ t\big) - C_1\varphi(t),$$

where

$$\varphi(t) = \exp\left[-\int g(t)\,dt\right], \qquad \psi(t) = \int f(t)\varphi(t)\,dt,$$

is also a solution of the equation (C_1 and C_2 are arbitrary constants).

Remark. This also remains true if a in the equation is an arbitrary function of time, $a = a(t)$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + [f(t)\ln w + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{n-1}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.*

* In equations 15.1.5.9 to 15.1.5.13 and their solutions, a can be an arbitrary function of time, $a = a(t)$.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\operatorname{arcsinh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sinh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\operatorname{arccosh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cosh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\arcsin(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a(-1)^n\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\arccos(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a(-1)^n\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

15.1.6. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-2} w(C_1 x + 2bC_1C_2 t + C_3, C_1^n t + C_4) + C_2 x + bC_2^2 t + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t + C_2 + \int \theta(z)\, dz, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation

$$a\theta_z^{(n-1)} + b\theta^2 - \lambda\theta - C_1 = 0.$$

To $C_1 = 0$ there corresponds a traveling-wave solution.

3°. Self-similar solution:

$$w(x,t) = t^{\frac{2-n}{n}} u(\zeta), \qquad \zeta = xt^{-\frac{1}{n}},$$

where the function $u(\zeta)$ is determined by the ordinary differential equation

$$au_\zeta^{(n)} + b(u'_\zeta)^2 + \frac{1}{n}\zeta u'_\zeta + \frac{n-2}{n}u = 0.$$

4°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

5°. The Bäcklund transformation

$$\frac{\partial w}{\partial x} = \frac{u}{2}, \qquad \frac{\partial w}{\partial t} = \frac{a}{2}\frac{\partial^{n-1} u}{\partial x^{n-1}} + \frac{b}{4}u^2 \tag{1}$$

connects the original equation with the generalized Burgers–Korteweg–de Vries equation 15.1.4.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\frac{\partial u}{\partial x}. \tag{2}$$

If $u = u(x,t)$ is a solution of equation (2), then the corresponding solution $w = w(x,t)$ of the original equation can be found from the linear system of first-order equations (1).

⊙ *References for equation* 15.1.6.1: A. D. Polyanin and V. F. Zaitsev (2004, p. 640).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

1°. Solution:

$$w(x,t) = C_1 t + C_2 + \int f(t)\,dt + \Theta(z), \quad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$a\Theta_z^{(n)} + b\left(\Theta_z'\right)^2 - \lambda\Theta_z' - C_1 = 0.$$

2°. The substitution $w = U(x,t) + \int f(t)\,dt$ leads to a simpler equation of the form 15.1.6.1:

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + b\left(\frac{\partial U}{\partial x}\right)^2.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, t) + C_2 e^{ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct}\int e^{-ct} f(t)\,dt + \theta(z), \quad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta_z^{(n)} + b\left(\theta_z'\right)^2 - \lambda\theta_z' + c\theta = 0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

4°. The substitution $w = U(x,t) + e^{ct}\int e^{-ct} f(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

1°. Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = k\varphi^2 + f(t)\varphi + g(t), \tag{1}$$

$$\psi_t' = \left[(c\lambda + 2k)\varphi + f(t) + a\lambda^n\right]\psi. \tag{2}$$

The Riccati equation (1) is integrable by quadrature in some special cases, for example,

(a) $k = 0$, (b) $g(t) \equiv 0$, (c) $f(t) = \text{const}$, $g(t) = \text{const}$.

See also Kamke (1977) and Polyanin and Zaitsev (2003). Whenever a solution of equation (1) is found, one can obtain the corresponding solution of the linear equation (2).

2°. For $c = 0$, there are generalized separable solutions of the form

$$w = \varphi(t)\exp\left(x\sqrt{|k/b|}\,\right) + \psi(t)\exp\left(-x\sqrt{|k/b|}\,\right) + \chi(t) \qquad \text{if } bk < 0,$$

$$w = \varphi(t)\cos\left(x\sqrt{k/b}\,\right) + \psi(t)\sin\left(x\sqrt{k/b}\,\right) + \chi(t) \qquad \text{if } bk > 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int h(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\left(\varphi_x'\right)^2 + g(x) - A = 0.$$

2°. The substitution $w = U(x,t) + \int h(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + g(x).$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt}h(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)(\varphi_x')^2 + b\varphi + g(x) = 0.$$

2°. The substitution $w = U(x,t) + e^{bt}\int e^{-bt}h(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + bU + g(x).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions involving exponentials of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp\left(\pm x\sqrt{-b}\,\right), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = bf\varphi^2 + g\varphi + h, \tag{2}$$

$$\psi_t' = \left[2bf\varphi + g + a(\pm\sqrt{-b}\,)^n\right]\psi. \tag{3}$$

The arguments of the functions f, g, and h are not specified.

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$ and, hence, can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f, g, and h.

Whenever a solution of equation (2) is known, the corresponding solution of equation (3) is computed by the formula

$$\psi(t) = C\exp\left[a(\pm\sqrt{-b}\,)^n t + \int (2bf\varphi + g)\,dt\right], \tag{4}$$

where C is an arbitrary constant.

Note two special integrable cases of equation (2).

Solution of equation (2) for $h \equiv 0$:

$$\varphi(t) = e^G\left(C_1 - b\int fe^G\,dt\right)^{-1}, \quad G = \int g\,dt,$$

where C_1 is an arbitrary constant.

If the functions f, g, and h are proportional,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

the solution of equation (2) is expressed as

$$\int \frac{d\varphi}{b\varphi^2 + \alpha\varphi + \beta} = \int f\,dt + C_2, \tag{5}$$

where C_2 is an arbitrary constant. On integrating the left-hand side of (5), one may obtain $\varphi = \varphi(t)$ in explicit form.

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t) = \varphi(t) + \psi(t)\exp\left(x\sqrt{-b}\,\right) + \chi(t)\exp\left(-x\sqrt{-b}\,\right), \qquad b < 0, \tag{6}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = bf\varphi^2 + g\varphi + h + 4bf\psi\chi, \tag{7}$$

$$\psi_t' = \left[2bf\varphi + g + a(\sqrt{-b}\,)^n\right]\psi, \tag{8}$$

$$\chi_t' = \left[2bf\varphi + g + a(-\sqrt{-b}\,)^n\right]\chi. \tag{9}$$

For equations of even order, with $n = 2m$, $m = 1, 2, \ldots$, it follows from (8) and (9) that $\psi(t)$ and $\chi(t)$ are proportional. Then, by setting $\psi(t) = A\theta(t)$ and $\chi(t) = B\theta(t)$, we can rewrite solution (6) in the form

$$w(x,t) = \varphi(t) + \theta(t)\big[A\exp\big(x\sqrt{-b}\big) + B\exp\big(-x\sqrt{-b}\big)\big], \qquad b < 0, \tag{10}$$

where the functions $\varphi(t)$ and $\theta(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = bf\big(\varphi^2 + 4AB\theta^2\big) + g\varphi + h, \tag{11}$$
$$\theta_t' = \big[2bf\varphi + g + (-1)^m ab^m\big]\theta. \tag{12}$$

The function φ can be expressed from (12) via θ and then substituted into (11) to obtain a second-order nonlinear equation for θ. For f, g, h = const, this equation is autonomous and its order can be reduced.

Note two special cases where solution (10) is expressed in terms of hyperbolic functions:

$$w(x,t) = \varphi(t) + \theta(t)\cosh\big(x\sqrt{-b}\big) \qquad \text{if} \quad A = \tfrac{1}{2},\ B = \tfrac{1}{2};$$
$$w(x,t) = \varphi(t) + \theta(t)\sinh\big(x\sqrt{-b}\big) \qquad \text{if} \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution involving trigonometric functions of x:

$$w(x,t) = \varphi(t) + \psi(t)\cos\big(x\sqrt{b}\big) + \chi(t)\sin\big(x\sqrt{b}\big), \qquad b > 0, \tag{13}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of ordinary differential equations (which is not written out here).

For equations of even order, with $n = 2m$, $m = 1, 2, \ldots$, there are exact solutions of the form

$$w(x,t) = \varphi(t) + \theta(t)\cos\big(x\sqrt{b} + c\big), \qquad b > 0, \tag{14}$$

where c is an arbitrary constant and the functions $\varphi(t)$ and $\theta(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = bf\big(\varphi^2 + \theta^2\big) + g\varphi + h, \tag{15}$$
$$\theta_t' = \big[2bf\varphi + g + (-1)^m ab^m\big]\theta. \tag{16}$$

The function φ can be expressed from (16) via θ and then substituted into (15) to obtain a second-order nonlinear equation for θ. For f, g, h = const, this equation is autonomous and its order can be reduced.

⊙ *References for equation* 15.1.6.7: V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2002, pp. 353–354).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Traveling-wave solutions with $bc < 0$:

$$w = C\exp\Big\{|c/b|^{1/2}x + \big(s + a|c/b|^{n/2}\big)t\Big\},$$
$$w = C\exp\Big\{-|c/b|^{1/2}x + \big[s + (-1)^n a|c/b|^{n/2}\big]t\Big\},$$

where C is an arbitrary constant.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\left(x\sqrt{|c/b|}\,\right) + \psi(t)\exp\left(-x\sqrt{|c/b|}\,\right),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = \left(s + a|c/b|^{n/2}\right)\varphi + 4c\varphi^2\psi,$$
$$\psi'_t = \left(s + (-1)^n a|c/b|^{n/2}\right)\psi + 4c\varphi\psi^2.$$

3°. For $bc > 0$, there are generalized separable solutions of the form

$$w = \varphi(t)\cos\left(x\sqrt{c/b}\,\right) + \psi(t)\sin\left(x\sqrt{c/b}\,\right).$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^n + \left[xg(t) + h(t)\right]\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$\tau = \int \varphi^n(t)\,dt, \quad z = \varphi(t)x + \int h(t)\varphi(t)\,dt, \quad \varphi(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives to the simpler equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^n w}{\partial z^n} + f(w)\left(\frac{\partial w}{\partial z}\right)^n,$$

which has a traveling-wave solution $w = u(kz + \lambda\tau)$ and a self-similar solution $w = v(z\tau^{-1/n})$.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f\left(x, \varphi'_x\right) - A = 0.$$

2°. The substitution

$$w = U(x,t) + \int g(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f\left(x, \frac{\partial U}{\partial x}\right).$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt} g(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f\left(x, \varphi_x'\right) + b\varphi = 0.$$

2°. The substitution

$$w = U(x,t) + e^{bt}\int e^{-bt} g(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + wf\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^n t + \int f(t,\lambda)\,dt\right],$$

where A and λ are arbitrary constants.

15.1.7. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2 + bw + c.$

1°. Generalized separable solution:

$$w = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)\exp[\lambda(t)x],$$

where the functions $\varphi_n = \varphi_n(t)$ and $\lambda = \lambda(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_1' &= -\varphi_2^2 + b\varphi_1 + c,\\
\varphi_2' &= b\varphi_2,\\
\varphi_3' &= (\lambda^2\varphi_1 - 2\lambda\varphi_2 + a\lambda^n + b)\varphi_3,\\
\lambda' &= \varphi_2\lambda^2.
\end{aligned}$$

Integrating the second, fourth, and first equations successively, we get

$$\text{for } b \neq 0: \quad \varphi_2 = C_1 e^{bt}, \quad \lambda = \frac{b}{C_2 - C_1 e^{bt}}, \quad \varphi_1 = C_3 e^{bt} - \frac{C_1^2}{b}e^{2bt} - \frac{c}{b};$$

$$\text{for } b = 0: \quad \varphi_2 = C_1, \qquad \lambda = \frac{1}{C_2 - C_1 t}, \qquad \varphi_1 = C_3 + (c - C_1^2)t,$$

where C_1, C_2, and C_3 are arbitrary constants. Substituting these expressions into the third equation yields a separable equation, which is easy to integrate.

2°. Generalized separable solution:

$$w = \varphi(t) + \psi_1(t)e^{\lambda x} + \psi_2(t)e^{-\lambda x},$$

where λ is an arbitrary constant and the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi' = 4\lambda^2\psi_1\psi_2 + b\varphi + c,$$
$$\psi_1' = (\lambda^2\varphi + a\lambda^n + b)\psi_1,$$
$$\psi_2' = [\lambda^2\varphi + (-1)^n a\lambda^n + b]\psi_2.$$

It is apparent that for equations of even order $n = 2m$, we have $\psi_2 = C\psi_1$, where C is an arbitrary constant.

3°. There is a generalized separable solution of the form

$$w = \varphi(t) + \psi_1(t)\cos(\lambda x) + \psi_2(t)\sin(\lambda x).$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\sum\limits_{i,j=0}^{i,j<n} b_{ij}\dfrac{\partial^i w}{\partial x^i}\dfrac{\partial^j w}{\partial x^j} + \sum\limits_{k=0}^{n-1} g_k(t)\dfrac{\partial^k w}{\partial x^k} + h(t).$

Here, we adopt the notation: $\frac{\partial^0 w}{\partial x^0} \equiv w$.

1°. In the general case, the equation has generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the algebraic equation $\sum\limits_{i,j=0}^{i,j<n} b_{ij}\lambda^{i+j} = 0$.

2°. Let n be an even number and let all coefficients b_{ij} be zero for odd $i + j$. In this case, the original equation has also generalized separable solutions of the form

$$w(x,t) = \varphi_1(t) + \psi_1(t)\big[A\cosh(\lambda x) + B\sinh(\lambda x)\big],$$
$$w(x,t) = \varphi_2(t) + \psi_2(t)\big[A\cos(\lambda x) + B\sin(\lambda x)\big],$$

where A and B are arbitrary constants, the parameter λ is determined by solving algebraic equations, and the functions $\varphi_1(t)$, $\psi_1(t)$ and $\varphi_2(t)$, $\psi_2(t)$ are found from appropriate systems of first-order ordinary differential equations.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\big(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\big) - A = 0,$$

whose order can be reduced with the substitution $U(x) = \varphi_x'$.

2°. The substitution $w = u(x,t) + \int g(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right).$$

⊙ *Reference for equation* 15.1.7.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 646).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt} g(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\big(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\big) + b\varphi = 0.$$

2°. The substitution $w = u(x,t) + e^{bt}\int e^{-bt} g(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right) + bu.$$

⊙ *Reference for equation* 15.1.7.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 646).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^n t + \int F(t, \lambda, \ldots, \lambda^{n-1})\,dt\right],$$

where A and λ are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n-2} w}{\partial x^{2n-2}}\right).$

Multiplicative separable solutions:

$$w(x,t) = \big[A\cosh(\lambda x) + B\sinh(\lambda x)\big]\exp\left[a\lambda^{2n}t + \int F(t, \lambda^2, \ldots, \lambda^{2n-2})\,dt\right],$$

$$w(x,t) = \big[A\cos(\lambda x) + B\sin(\lambda x)\big]\exp\big[(-1)^n a\lambda^{2n}t + \Phi(t)\big],$$

$$\Phi(t) = \int F\big(t, -\lambda^2, \ldots, (-1)^{n-1}\lambda^{2n-2}\big)\,dt,$$

where A, B, and λ are arbitrary constants.

15.1.8. Equations of the Form $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(t)w + g(t).$

1°. Degenerate solution:

$$w(x,t) = F(t)\big(A_{n-1}x^{n-1} + \cdots + A_1 x + A_0\big) + F(t)\int \frac{g(t)}{F(t)}dt, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where $A_0, A_1, \ldots, A_{n-1}$ are arbitrary constants.

2°. Generalized separable solution:

$$w(x,t)=\varphi(t)\big(x^n+A_{n-1}x^{n-1}+\cdots+A_1x+A_0\big)+\varphi(t)\int\frac{g(t)}{\varphi(t)}dt,$$
$$\varphi(t)=F(t)\left[C-an!\int F(t)\,dt\right]^{-1},\quad F(t)=\exp\left[\int f(t)\,dt\right],$$

where A_0, A_1, …, A_{n-1}, and C are arbitrary constants.

2. $\dfrac{\partial w}{\partial t}=aw\dfrac{\partial^n w}{\partial x^n}+f(x)w+\sum\limits_{k=0}^{n-1}b_kx^k.$

Generalized separable solution:

$$w(x,t)=t\sum_{k=0}^{n-1}b_kx^k+\sum_{k=0}^{n-1}C_kx^k-\frac{1}{a(n-1)!}\int_{x_0}^{x}(x-\xi)^{n-1}f(\xi)\,d\xi,$$

where C_0, C_1, …, C_{n-1}, and x_0 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t}=aw\dfrac{\partial^n w}{\partial x^n}+bw^2+f(t)w+g(t).$

Generalized separable solution:

$$w(x,t)=\varphi(t)\Theta(x)+\psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t=C\varphi^2+b\varphi\psi+f(t)\varphi,$$
$$\psi'_t=C\varphi\psi+b\psi^2+f(t)\psi+g(t),$$

where C is an arbitrary constant and the function $\Theta(x)$ satisfies the nth-order linear ordinary differential equation

$$a\Theta_x^{(n)}+b\Theta=C.$$

4. $\dfrac{\partial w}{\partial t}=aw\dfrac{\partial^{2n} w}{\partial x^{2n}}-ak^{2n}w^2+f(x)w+b_1\sinh(kx)+b_2\cosh(kx).$

Generalized separable solution linear in t:

$$w(x,t)=t\big[b_1\sinh(kx)+b_2\cosh(kx)\big]+\varphi(x).$$

Here, the function $\varphi(x)$ is determined from the constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\varphi_x^{(2n)}-ak^{2n}\varphi+f(x)=0.$$

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = H(t)u(z,\tau), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^n(t)H(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^n u}{\partial z^n},$$

which admits, for example, a traveling-wave solution $u = u(kz + \lambda\tau)$ and a self-similar solution of the form $u = u(\xi)$, $\xi = z\tau^{-1/n}$.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
&\varphi'_t = C\varphi^2 + g(t)\varphi,\\
&\psi'_t = \big[C\varphi + g(t)\big]\psi + h(t),\\
&a\Theta_x^{(n)} + f(x)\Theta'_x = C,
\end{aligned}$$

where C is an arbitrary constant. On integrating the first two equations successively, one obtains

$$\begin{aligned}
&\varphi(t) = G(t)\left[A - C\int G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right],\\
&\psi(t) = B\varphi(t) + \varphi(t)\int \frac{h(t)}{\varphi(t)}\,dt,
\end{aligned}$$

where A and B are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)H(t)\left[A + B\int H(t)\,dt\right]^{-1}, \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the linear ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi'_x + g(x)\varphi + B = 0.$$

15.1.9. Other Equations Involving Arbitrary Parameters and/or Arbitrary Functions

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}(w^2) + bw.$

There are exact solutions of the following forms:

$$w = \varphi_0(t) + \varphi_2(t)x + \cdots + \varphi_n(t)x^n,$$
$$w = \psi_1(t)x^{k/2} + \psi_2(t)x^{m/2} + \psi_3(t)x^n,$$

where $k < n$, $m < n$, and k, m, and $\frac{1}{2}(k + m)$ are nonnegative integers.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(w^m\dfrac{\partial^k w}{\partial x^k}\right).$

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^m C_2^{n+k} t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation $a[w^m w_z^{(k)}]_z^{(n)} - \lambda w_z' = 0$.

3°. Self-similar solution:

$$w(x, t) = t^{-\frac{(n+k)\beta+1}{m}} u(\xi), \qquad \xi = xt^\beta,$$

where β is an arbitrary constant and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$-[(n + k)\beta + 1]u + m\beta\xi u_\xi' = am[u^m u_\xi^{(k)}]_\xi^{(n)}.$$

4°. Solution:

$$w(x, t) = (C_1 t + C_2)^{-1/m} V(\zeta), \qquad \zeta = x + C_3 \ln|C_1 t + C_2|,$$

where the function $V = V(\zeta)$ is determined by the autonomous ordinary differential equation

$$am[V^m V_\zeta^{(k)}]_\zeta^{(n)} - mC_1 C_3 V_\zeta' + C_1 V = 0.$$

Remark. For a special case $C_3 = 0$, we have a solution in multiplicative separable form.

5°. Generalized self-similar solution:

$$w(x, t) = e^{-(n+k)\beta t}\varphi(\eta), \qquad \eta = xe^{m\beta t},$$

where β is an arbitrary constant and the function $\varphi = \varphi(\eta)$ is determined by the ordinary differential equation

$$-(n + k)\beta\varphi + m\beta\eta\varphi_\eta' = a[\varphi^m \varphi_\eta^{(k)}]_\eta^{(n)}.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(w^m\dfrac{\partial^k w}{\partial x^k}\right) + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{n+k}(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 15.1.9.2:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^n}{\partial z^n}\left(u^m\frac{\partial^k u}{\partial z^k}\right).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002, pp. 357–358).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(e^{\lambda w}\dfrac{\partial^k w}{\partial x^k}\right) + f(t).$

The transformation

$$w(x,t) = u(x,\tau) + F(t), \quad \tau = \int \exp\big[\lambda F(t)\big]\,dt, \qquad F(t) = \int f(t)\,dt,$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^n}{\partial x^n}\left(e^{\lambda u}\frac{\partial^k u}{\partial x^k}\right).$$

It admits, for example, exact solutions of the forms

$$\begin{aligned} &u = U(kx + \lambda\tau) && \text{(traveling-wave solution)},\\ &u = V\left(x\tau^{-1/(n+k)}\right) && \text{(self-similar solution)},\\ &u = \varphi(x) + \psi(\tau) && \text{(additive separable solution)}. \end{aligned}$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(e^{\lambda w}\dfrac{\partial^k w}{\partial x^k}\right) + f(x)e^{\lambda w}.$

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\frac{d^n}{dx^n}\left(e^{\lambda\varphi}\frac{d^k\varphi}{dx^k}\right) + f(x)e^{\lambda\varphi} + 1 = 0.$$

For $k = 1$, it is reduced with the change of variable $\psi = e^{\lambda\varphi}$ to a linear equation.

6. $\dfrac{\partial w}{\partial t} = \sum\limits_{k=0}^{n} [f_k(t) \ln w + g_k(t)] \dfrac{\partial^k w}{\partial x^k}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \sum_{k=0}^{n} f_k(t)\varphi^{k+1},$$
$$\psi'_t = \sum_{k=0}^{n} \varphi^k [f_k(t)\psi + g_k(t)].$$

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^{n-1}}{\partial x^{n-1}} \left[f(w) \dfrac{\partial w}{\partial x} \right] + \dfrac{a}{f(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\, dw = at - \frac{b}{n!} x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 650).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^{n-1}}{\partial x^{n-1}} \left[f(w) \dfrac{\partial w}{\partial x} \right] + \dfrac{g(t)}{f(w)} + h(x).$

Functional separable solution in implicit form:

$$\int f(w)\, dw = \int g(t)\, dt - \frac{1}{(n-1)!} \int_{x_0}^{x} (x-\xi)^{n-1} h(\xi)\, d\xi + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants and x_0 is any number (for which the integral exists).

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n} \left[f(w) \dfrac{\partial^k w}{\partial x^k} \right].$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^{n+k} t + C_3),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the forms

$$w(x,t) = u(\xi), \quad \xi = kx + \lambda t \quad \text{(traveling-wave solution)},$$
$$w(x,t) = z(\zeta), \quad \zeta = x^{n+k}/t \quad \text{(self-similar solution)}.$$

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n}\left[f(w)\dfrac{\partial^k w}{\partial x^k}\right] + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation of the independent variables

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{n+k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 15.1.9.9:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^n}{\partial z^n}\left[f(w)\frac{\partial^k w}{\partial z^k}\right].$$

11. $\dfrac{\partial w}{\partial t} = f(w)\left(\dfrac{\partial^n w}{\partial x^n}\right)^k + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{nk}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial w}{\partial \tau} = f(w)\left(\frac{\partial^n w}{\partial z^n}\right)^k.$$

It admits a traveling-wave solution and a self-similar solution.

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n}\big[f(x,w)\big] + \dfrac{g(t)}{f_w(x,w)} + h(x).$

Solution in implicit form:

$$f(x,w) = \int g(t)\,dt - \frac{1}{(n-1)!}\int_{x_0}^{x}(x-\xi)^{n-1}h(\xi)\,d\xi + C_{n-1}x^{n-1} + \cdots + C_1x + C_0,$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants and x_0 is any number (for which the integral exists).

15.1.10. Nonlinear Equations Involving Arbitrary Linear Differential Operators

1. $\dfrac{\partial w}{\partial t} = L[f(w)] + \dfrac{af(w) + b}{f'(w)} + c[af(w) + b].$

Here, L is an nth-order linear differential operator in x with coefficients independent of t, satisfying the condition $L[1] = 0$.

1°. Functional separable solution in implicit form with $a \neq 0$:

$$f(w) = \frac{1}{a}[\rho(x)e^{at} - b],$$

where $\rho = \rho(x)$ is an arbitrary solution of the linear equation

$$L[\rho] + ac\rho = 0.$$

2°. Functional separable solution in implicit form with $a = 0$:

$$f(w) = \rho(x) + bt,$$

where $\rho = \rho(x)$ is an arbitrary solution of the linear equation

$$L[\rho] + bc = 0.$$

Remark. The differential operator L and function ρ can depend on several coordinates $x = (x_1, \ldots, x_m)$.

2. $\dfrac{\partial w}{\partial t} = L[\sqrt{w}\,] + f(x) + g(x)\sqrt{w}.$

Here, L is an nth-order linear differential operator in x with coefficients independent of t, and $w \geq 0$.

Generalized separable solution:

$$w = [\varphi(x) + \psi(x)t]^2,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$L[\varphi] - 2\varphi\psi + g\varphi + f = 0,$$
$$L[\psi] - 2\psi^2 + g\psi = 0.$$

Remark. The differential operator L and functions f and g can depend on several coordinates $x = (x_1, \ldots, x_m)$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, pp. 280–281).

3. $\dfrac{\partial w}{\partial t} = L[\sqrt{w}\,] + f(x) + g(x)\sqrt{w} + cw.$

Here, L is an nth-order linear differential operator in x with coefficients independent of t, and $w \geq 0$. For $c = 0$, see equation 15.1.10.2.

Generalized separable solution:

$$w = \left[\varphi(x) + \psi(x)\exp\left(\tfrac{1}{2}ct\right)\right]^2,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$L[\varphi] + c\varphi^2 + g\varphi + f = 0,$$
$$L[\psi] + c\varphi\psi + g\psi = 0.$$

Remark. The differential operator L and functions f and g can depend on several coordinates $x = (x_1, \ldots, x_m)$.

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 282).

15.1.11. Equations of the Burgers and the Korteweg–de Vries Hierarchies and Related Equations

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial}{\partial x}B_n[w] = 0, \qquad B_n[w] \equiv \left(\dfrac{\partial}{\partial x} + w\right)^n w, \quad n = 1, 2, \ldots.$

Equation of the Burgers hierarchy.

1°. Solutions of the equation are expressed in terms of solutions $u = u(x,t)$ of the linear equation

$$\frac{\partial u}{\partial t} + a\frac{\partial^{n+1} u}{\partial x^{n+1}} = 0 \tag{1}$$

by the formula (the *Hopf–Cole transformation*)

$$w(x,t) = \frac{1}{u}\frac{\partial u}{\partial x}. \tag{2}$$

2°. Let $w = w(x,t)$ be a solution of one of the equations of the Burgers hierarchy. Then the function

$$\widetilde{w} = w + \frac{w_x}{w}$$

is also a solution of the same equation of the Burgers hierarchy.

3°. Let $w_1 = \dfrac{\psi_x}{\psi}$ be a solution of one of the equations of the Burgers hierarchy. Then the function

$$w_{m+1} = \frac{\psi_x^{(m+1)}}{\psi_x^{(m)}}, \qquad \psi_x^{(m)} = \frac{\partial^m \psi}{\partial x^m}, \tag{3}$$

is also a solution of the same equation of the Burgers hierarchy.

4°. Equation (1) has a solution

$$u = \sum_{i=1}^{N} C_i \exp\bigl(k_i x - a k_i^{n+1} t\bigr), \tag{4}$$

where C_i and k_i are arbitrary constants (N is any positive integer). Substituting (4) into (2) yields a solution to the original equation in the form

$$w = \frac{\sum_{i=1}^{N} C_i k_i \exp\bigl(k_i x - a k_i^{n+1} t\bigr)}{\sum_{i=1}^{N} C_i \exp\bigl(k_i x - a k_i^{n+1} t\bigr)}.$$

Using formula (3), we get the solutions

$$w = \frac{\sum_{i=1}^{N} C_i k_i^{m+1} \exp\bigl(k_i x - a k_i^{n+1} t\bigr)}{\sum_{i=1}^{N} C_i k_i^{m} \exp\bigl(k_i x - a k_i^{n+1} t\bigr)}, \quad m = 0, 1, 2, \ldots.$$

5°. Below are two special cases of an equation of the Burgers hierarchy with $n = 1$ and $n = 2$.

Special case 1. With $n = 1$, we have a unnormalized Burgers equation of the form 5.1.5.3:

$$\frac{\partial w}{\partial t} + a\frac{\partial^2 w}{\partial x^2} + 2aw\frac{\partial w}{\partial x}.$$

Special case 2. With $n = 2$ and $a = 1$, we have the *Sharma–Tasso–Olver equation* 13.1.7.5:

$$\frac{\partial w}{\partial t} + 3w^2\frac{\partial w}{\partial x} + 3\left(\frac{\partial w}{\partial x}\right)^2 + 3w\frac{\partial^2 w}{\partial x^2} + \frac{\partial^3 w}{\partial x^3} = 0.$$

⊙ *Reference for equation* 15.1.11.1: N. A. Kudryashov (2010 *d*).

2. $\dfrac{\partial w}{\partial t}+\dfrac{\partial}{\partial x}K_{n+1}[w]=0, \quad n=0,\ 1,\ \ldots.$

Here, $K_{n+1}[w]$ is the Lenard operator, defined by the recurrence relation

$$\frac{\partial}{\partial x}K_{n+1}[w]=\left(\frac{\partial^3}{\partial x^3}+4w\frac{\partial}{\partial x}+2\frac{\partial w}{\partial x}\right)K_n[w], \qquad K_1[w]=w.$$

It is the *equation of the Korteweg–de Vries hierarchy*.

1°. The one-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_1(x,t), \qquad F_1(x,t)=1+C\exp(kx-k^{2n+1}t),$$

where C and k are arbitrary constants.

2°. The two-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_2(x,t), \qquad F_2(x,t)=1+\exp(\theta_1)+\exp(\theta_2)+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\theta_1+\theta_2),$$
$$\theta_1=k_1x-k_1^{2n+1}t+C_1, \qquad \theta_2=k_2x-k_2^{2n+1}t+C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

3°. The three-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_3(x,t), \qquad F_3(x,t)=1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)$$
$$+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\theta_1+\theta_2)+\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\exp(\theta_1+\theta_3)+\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\exp(\theta_2+\theta_3)$$
$$+\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_3+k_3)^2}\exp(\theta_1+\theta_2+\theta_3),$$
$$\theta_1=k_1x-k_1^{2n+1}t+C_1, \quad \theta_2=k_2x-k_2^{2n+1}t+C_2, \quad \theta_3=k_3x-k_3^{2n+1}t+C_3,$$

where C_1, C_2, C_3, k_1, k_2, and k_3 are arbitrary constants.

4°. Particular cases of an equation of the Korteweg–de Vries hierarchy with $n=1$ and $n=2$.

Special case 1. With $n=1$, we have the Korteweg–de Vries equation of the form 13.1.1.1 (after substituting w by $-w$):

$$\frac{\partial w}{\partial t}+\frac{\partial^3 w}{\partial x^3}+6w\frac{\partial w}{\partial x}=0.$$

Special case 2. With $n=2$, we have the *Lax equation* (see Item 2° in equation 15.1.1.12):

$$\frac{\partial w}{\partial t}+30w^2\frac{\partial w}{\partial x}+20\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+10w\frac{\partial^3 w}{\partial x^3}+\frac{\partial^5 w}{\partial x^5}=0.$$

⊙ *References for equation* 15.1.11.2: P. D. Lax (1968), T. Chou (1987), M. J. Ablowitz and P. A. Clarkson (1991), N. A. Kudryashov (2010 *d*).

3. $$\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\sum_{n=1}^{N}a_nK_{n+1}[w]=0,\qquad n=0,\ 1,\ \ldots.$$

Here, $K_{n+1}[w]$ is the Lenard operator, defined by the recurrence relation

$$\frac{\partial}{\partial x}K_{n+1}[w]=\left(\frac{\partial^3}{\partial x^3}+4w\frac{\partial}{\partial x}+2\frac{\partial w}{\partial x}\right)K_n[w],\qquad K_1[w]=w.$$

It is the *equation of the complete Korteweg–de Vries hierarchy*.

1°. The one-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_1(x,t),\qquad F_1(x,t)=1+C\exp(\theta_1),\qquad \theta_1=kx-\sum_{n=1}^{N}a_nk^{n+1}t,$$

where C and k are arbitrary constants.

2°. The two-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_2(x,t),\qquad F_2(x,t)=1+\exp(\theta_1)+\exp(\theta_2)+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\theta_1+\theta_2),$$
$$\theta_1=k_1x-\Big(\sum_{n=1}^{N}a_nk_1^{n+1}\Big)t+C_1,\qquad \theta_2=k_2x-\Big(\sum_{n=1}^{N}a_nk_2^{n+1}\Big)t+C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

3°. The three-soliton solution:

$$w=2\frac{\partial^2}{\partial x^2}F_3(x,t),\qquad F_3(x,t)=1+\exp(\theta_1)+\exp(\theta_2)+\exp(\theta_3)$$
$$+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\theta_1+\theta_2)+\frac{(k_1-k_3)^2}{(k_1+k_3)^2}\exp(\theta_1+\theta_3)+\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\exp(\theta_2+\theta_3)$$
$$+\frac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_3+k_3)^2}\exp(\theta_1+\theta_2+\theta_3),$$
$$\theta_m=k_mx-\Big(\sum_{n=1}^{N}a_nk_m^{n+1}\Big)t+C_m,\qquad m=1,\ 2,\ 3,$$

where C_1, C_2, C_3, k_1, k_2, and k_3 are arbitrary constants.

⊙ *Reference for equation* 15.1.11.3: N. A. Kudryashov (2010 *d*).

4. $$\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+2w\right)K_n\left[\frac{\partial w}{\partial x}-w^2\right]=0.$$

It is the *equation of the modified Korteweg–de Vries hierarchy*, where $K_n[w]$ is the Lenard operator (see equation 15.1.10.3). For $n=2$, the equation is used for describing the motion of waves of water surface (see Olver, 1984).

All equations of the modified Korteweg–de Vries hierarchy can be integrated by the inverse problem method and have solutions in the form of solitons.

⊙ *Reference*: N. A. Kudryashov (2010 *d*).

15.2. General Form Equations Involving the First Derivative in t

15.2.1. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right). \tag{1}$$

1°. Suppose $w(x,t)$ is a solution of equation (1). Then the function $w(x + C_1, t + C_2)$, where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. In the general case, equation (1) admits a traveling-wave solution

$$w = w(\xi), \qquad \xi = kx + \lambda t, \tag{2}$$

where k and λ are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$F\big(w, kw'_\xi, \ldots, k^n w^{(n)}_\xi\big) - \lambda w'_\xi = 0.$$

Special cases of equation (1) that admit, apart from traveling-wave solutions (2), also other types of solutions are presented in this subsection.

1. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^n w}{\partial x^n}\right)$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-n} w(C_1 x + C_2, C_1^n t + C_3) + \sum_{k=0}^{n-1} A_k x^k,$$

where C_1, C_2, C_3, and the A_k are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = F(A)t + \frac{A}{n!}x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where A, C_0, C_1, …, C_{n-1} are arbitrary constants.

3°. Solution linear in t:

$$w(x,t) = t\sum_{k=0}^{n-1} A_k x^k + \sum_{k=0}^{n-1} B_k x^k + \int_0^x \frac{(x-\xi)^{n-1}}{(n-1)!}\Phi\left(\sum_{k=0}^{n-1} A_k \xi^k\right) d\xi,$$

where the A_k and B_k are arbitrary constants and $\Phi(u)$ is the inverse of the function $F(u)$.

4°. Solution:

$$w(x,t) = A_1 t + \frac{1}{n!}A_2 x^n + \sum_{m=0}^{n-1} B_m x^m + U(z), \qquad z = kx + \lambda t,$$

where A_1, A_2, the B_m, k, and λ are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$A_1 + \lambda U'_z = F\big(A_2 + k^n U_z^{(n)}\big).$$

5°. Self-similar solution:

$$w(x,t) = t\,\Theta(\zeta), \quad \zeta = xt^{-1/n},$$

where the function $\Theta(\zeta)$ is determined by the ordinary differential equation

$$nF\big(\Theta_\zeta^{(n)}\big) + \zeta\Theta'_\zeta - n\Theta = 0.$$

2. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(k\varphi'_\xi, k^2\varphi''_{\xi\xi}, \ldots, k^n\varphi_\xi^{(n)}\big) - \lambda\varphi'_\xi - A = 0.$$

3. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + aw.$

This is a special case of equation 15.2.2.1 with $g(t) = a$ and $F_t = 0$. There is a traveling-wave solution, $w = w(kx + \lambda t)$.

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + aC_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = -\frac{x + C_1}{a\tau} + \frac{1}{\tau}\int \tau F\Big(-\frac{1}{a\tau}, 0, \ldots, 0\Big)\,d\tau, \qquad \tau = t + C_2.$$

3°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$F\big(U'_\zeta, U''_{\zeta\zeta}, \ldots, U_\zeta^{(n)}\big) + aUU'_\zeta = C_2 U'_\zeta + 2C_1.$$

In the special case $C_1 = 0$, we have a traveling-wave solution.

⊙ *Reference for equation* 15.2.1.4: A. D. Polyanin and V. F. Zaitsev (2004, p. 653).

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + bw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + aC_1e^{bt} + C_2, t + C_3) + C_1be^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

3°. Traveling-wave solution:

$$w = w(\xi), \quad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(w'_\xi, w''_{\xi\xi}, \ldots, w^{(n)}_\xi\right) + aww'_\xi - \lambda w'_\xi + bw = 0.$$

6. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1}w(x + C_2, C_1t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = t\varphi(\xi), \quad \xi = kx + \lambda\ln|t|,$$

where k and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{k}{\varphi}\varphi'_\xi, \frac{k^2}{\varphi}\varphi''_{\xi\xi}, \ldots, \frac{k^n}{\varphi}\varphi^{(n)}_\xi\right) = \lambda\varphi'_\xi + \varphi.$$

7. $\dfrac{\partial w}{\partial t} = wF\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1w(x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = Ce^{\lambda t}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{\varphi'_x}{\varphi}, \frac{\varphi''_{xx}}{\varphi}, \ldots, \frac{\varphi^{(n)}_x}{\varphi}\right) = \lambda.$$

This equation has particular solutions of the form $\varphi(x) = e^{\alpha x}$, where α is a root of the algebraic (or transcendental) equation $F\left(\alpha, \alpha^2, \ldots, \alpha^n\right) - \lambda = 0$.

3°. Solution:

$$w(x,t) = Ce^{\lambda t}\psi(\xi), \quad \xi = kx + \beta t$$

where C, k, λ, and β are arbitrary constants, and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$\psi F\left(\frac{k}{\psi}\psi'_\xi,\ \frac{k^2}{\psi}\psi''_{\xi\xi},\ \ldots,\ \frac{k^n}{\psi}\psi^{(n)}_\xi\right) = \beta\psi'_\xi + \lambda\psi.$$

This equation has particular solutions of the form $\psi(\xi) = e^{\mu\xi}$.

8. $\dfrac{\partial w}{\partial t} = w^m F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

For $m = 0$, see equation 15.2.1.6, and for $m = 1$, see 15.2.1.7.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{m-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \big[(1-m)At + B\big]^{\frac{1}{1-m}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\varphi^{m-1}F\left(\frac{\varphi'_x}{\varphi},\ \frac{\varphi''_{xx}}{\varphi},\ \ldots,\ \frac{\varphi^{(n)}_x}{\varphi}\right) = A.$$

3°. Solution:

$$w(z,t) = (t+C)^{\frac{1}{1-m}}\Theta(z), \quad z = kx + \lambda\ln(t+C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$\Theta^m F\left(k\frac{\Theta'_z}{\Theta},\ k^2\frac{\Theta''_{zz}}{\Theta},\ \ldots,\ k^n\frac{\Theta^{(n)}_z}{\Theta}\right) = \lambda\Theta'_z + \frac{1}{1-m}\Theta.$$

9. $\dfrac{\partial w}{\partial t} = w^m F\left(w^k\dfrac{\partial w}{\partial x},\ w^{2k+1}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ w^{nk+n-1}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1^{-k-1}x + C_2, C_1^{m-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution with $m \neq 1$ and $k \neq -1$:

$$w(x,t) = t^{\frac{1}{1-m}} U(z), \quad z = xt^{\frac{k+1}{m-1}},$$

where the function $U(z)$ is determined by the ordinary differential equation

$$\frac{1}{1-m}U + \frac{k+1}{m-1}zU'_z = U^m F\big(U^k U'_z,\, U^{2k+1}U''_{zz},\, \ldots,\, U^{nk+n-1}U^{(n)}_z\big).$$

3°. Generalized self-similar solution with $m = 1$ and $k \neq -1$:

$$w(x,t) = \exp\left(-\frac{1}{k+1}t\right)u(\xi), \quad \xi = xe^t,$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$-\frac{1}{k+1}u + \xi u'_\xi = uF\big(u^k u'_\xi,\, u^{2k+1}u''_{\xi\xi}\ \ldots,\, u^{nk+n-1}u^{(n)}_\xi\big).$$

4°. For $k = -1$ see equation 11.2.1.8.

10. $\dfrac{\partial w}{\partial t} = e^{\beta w} F\left(\dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, C_2 t + C_3) + \frac{1}{\beta}\ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\varphi}F\big(\varphi'_x,\, \varphi''_{xx},\, \ldots,\, \varphi^{(n)}_x\big) + A = 0.$$

3°. Solution:

$$w(x,t) = -\frac{1}{\beta}\ln(t + C) + \Theta(\xi), \quad \xi = kx + \lambda\ln(t + C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\Theta}F\big(k\Theta'_\xi,\, k^2\Theta''_{\xi\xi},\, \ldots,\, k^n\Theta^{(n)}_\xi\big) = \lambda\Theta'_\xi - \frac{1}{\beta}.$$

11. $\dfrac{\partial w}{\partial t} = a\ln w\dfrac{\partial w}{\partial x} + wF\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = e^{C_1}w(x + aC_1t + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[-\frac{x}{at} + \frac{1}{t}\int tF\left(-\frac{1}{at}, \ldots, \frac{1}{(-at)^n}\right)dt\right].$$

3°. Solution:

$$w(x,t) = e^{\lambda t}u(z), \qquad z = x + \tfrac{1}{2}a\lambda t^2 + kt,$$

where k and λ are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$(a\ln u - k)u'_z - \lambda u + uF\left(\frac{u'_z}{u}, \frac{u''_{zz}}{u}, \ldots, \frac{u^{(n)}_z}{u}\right) = 0.$$

To $\lambda = 0$ there corresponds a traveling-wave solution.

12. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 15.2.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1}w(x + C_2, C_1t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k\varphi''_{\xi\xi}/\varphi'_\xi, \ldots, k^{n-1}\varphi^{(n)}_\xi/\varphi'_\xi\right) = \lambda\varphi'_\xi + A.$$

3°. Solution:

$$w(x,t) = (t + C_1)\Theta(z) + C_2, \quad z = kx + \lambda\ln|t + C_1|,$$

where C_1, C_2, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$F\left(k\Theta''_{zz}/\Theta'_z, \ldots, k^{n-1}\Theta^{(n)}_z/\Theta'_z\right) = \lambda\Theta'_z + \Theta.$$

13. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 15.2.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$k\varphi'_z F\big(k\varphi''_{zz}/\varphi'_z, \ldots, k^{n-1}\varphi^{(n)}_z/\varphi'_z\big) = \lambda\varphi'_z + A.$$

3°. Solution:

$$w(x,t) = Ae^{\beta t}\Theta(\xi) + B, \qquad \xi = kx + \lambda t,$$

where A, B, k, β, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$k\Theta'_\xi F\big(k\Theta''_{\xi\xi}/\Theta'_\xi, \ldots, k^{n-1}\Theta^{(n)}_\xi/\Theta'_\xi\big) = \lambda\Theta'_\xi + \beta\Theta.$$

14. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^\beta F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 15.2.1.2. For $\beta = 0$ and $\beta = 1$ see equations 15.2.1.12 and 15.2.1.13.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{\beta-1}t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,t) = \big[A(1-\beta)t + B\big]^{\frac{1}{1-\beta}}\varphi(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\big(\varphi'_x\big)^\beta F\big(\varphi''_{xx}/\varphi'_x, \ldots, \varphi^{(n)}_x/\varphi'_x\big) = A\varphi.$$

3°. Solution:

$$w(x,t) = (t + A)^{\frac{1}{1-\beta}}\Theta(z) + B, \qquad z = kx + \lambda\ln(t + A),$$

where A, B, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$k^\beta\big(\Theta'_z\big)^\beta F\big(k\Theta''_{zz}/\Theta'_z, \ldots, k^{n-1}\Theta^{(n)}_z/\Theta'_z\big) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

15.2.2. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right)$

1. $\dfrac{\partial w}{\partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x,t) + C\exp\left[\int g(t)\,dt\right],$$

where C is an arbitrary constant, is also a solution of the equation.

2. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + aw\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + a\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

where t_0 is any, leads to a simpler equation of the form 15.2.1.4:

$$\frac{\partial u}{\partial t} = au\frac{\partial u}{\partial x} + F\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 657).

3. $\dfrac{\partial w}{\partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + f(t)w\dfrac{\partial w}{\partial x} + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + C_1\psi(t) + C_2,\ t\big) + C_1\varphi(t),$$

where

$$\varphi(t) = \exp\left[\int g(t)\,dt\right], \quad \psi(t) = \int f(t)\varphi(t)\,dt, \quad C_1 \text{ and } C_2 \text{ are arbitrary constants,}$$

is also a solution of the equation.

4. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + \int F(t, \lambda, \ldots, \lambda^n)\,dt\right],$$

where A and λ are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solutions:

$$w(x,t) = A\exp\left[\lambda x + \int F\left(t, \lambda^2, \ldots, \lambda^{2n}\right) dt\right],$$

$$w(x,t) = \left[A\cosh(\lambda x) + B\sinh(\lambda x)\right]\exp\left[\int F\left(t, \lambda^2, \ldots, \lambda^{2n}\right) dt\right],$$

$$w(x,t) = \left[A\cos(\lambda x) + B\sin(\lambda x)\right]\exp\left[\int F\left(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\right) dt\right],$$

where A, B, and λ are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = f(t)w^\beta \Phi\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 15.2.1.8:

$$\frac{\partial u}{\partial \tau} = u^\beta \Phi\left(\frac{1}{u}\frac{\partial u}{\partial x}, \frac{1}{u}\frac{\partial^2 u}{\partial x^2}, \ldots, \frac{1}{u}\frac{\partial^n u}{\partial x^n}\right),$$

which has, for instance, a traveling-wave solution $u = u(ax + b\tau)$ and a multiplicative solution of the form $u = \varphi(x)\psi(\tau)$.

7. $\dfrac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\left[\beta G(t)\right] dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 15.2.1.10:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right),$$

which has, for instance, a traveling-wave solution $u = u(ax + b\tau)$ and an additive separable solution of the form $u = \varphi(x) + \psi(\tau)$.

8. $\dfrac{\partial w}{\partial t} = f(t)\Phi\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t)\dfrac{\partial w}{\partial x}.$

The transformation

$$w = u(z,\tau), \qquad z = x + \int g(t)\,dt, \qquad \tau = \int f(t)\,dt,$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = \Phi\left(u, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}, \ldots, \frac{\partial^n u}{\partial z^n}\right),$$

which has a traveling-wave solution $u = u(kz + \lambda\tau)$.

9. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + \sum\limits_{k=1}^{m} \dfrac{\partial^k w}{\partial x^k}\Phi_k\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

The equation has a multiplicative solution of the form

$$w(x,t) = Ae^{\lambda x}\Theta(t),$$

where A and λ are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + \sum\limits_{k=1}^{m} \dfrac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right).$

The equation has multiplicative solutions of the following forms:

$$w(x,t) = Ae^{\lambda x}\Theta_1(t),$$
$$w(x,t) = \big[A\cosh(\lambda x) + B\sinh(\lambda x)\big]\Theta_1(t),$$
$$w(x,t) = \big[A\cos(\lambda x) + B\sin(\lambda x)\big]\Theta_2(t),$$

where A, B, and λ are arbitrary constants.

15.2.3. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right)$

1. $\dfrac{\partial w}{\partial t} = F\left(x, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

Generalized separable solution linear in t:

$$w(x,t) = Axt + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x, \varphi''_{xx}, \ldots, \varphi_x^{(n)}\big) = Ax + B.$$

2. $\dfrac{\partial w}{\partial t} = F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x, \varphi'_x, \varphi''_{xx}, \ldots, \varphi_x^{(n)}\big) = A.$$

3. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, x\dfrac{\partial^2 w}{\partial x^2}, \ldots, x^{n-1}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x, C_1 t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(\varphi'_x, x\varphi''_{xx}, \ldots, x^{n-1}\varphi_x^{(n)}\big) = A.$$

3°. Solution:

$$w(x,t) = tU(z) + C, \quad z = x/t,$$

where C is an arbitrary constant and the function $U(z)$ is determined by the ordinary differential equation

$$F\big(U'_z, zU''_{zz}, \ldots, z^{n-1}U_z^{(n)}\big) + zU'_z - U = 0.$$

4. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial w}{\partial x} + F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$F\big(w, w'_z, w''_{zz}, \ldots, w_z^{(n)}\big) + azw'_z = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 659).

5. $\dfrac{\partial w}{\partial t} = F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right).$

The substitution $x = \pm e^z$ leads to an equation of the form 15.2.1.2.

6. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x, C_1^{-k} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x,t) = w(z), \quad z = xt^{1/k},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$kz^{k-1}F\big(w,\ zw'_z,\ \ldots,\ z^n w_z^{(n)}\big) - w'_z = 0.$$

7. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right) + ax\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xe^{at}, \quad \tau = \frac{1}{ak}\big(1 - e^{-akt}\big),$$

we obtain an equation of the form 15.2.3.6:

$$\frac{\partial w}{\partial \tau} = z^k F\left(w,\ z\frac{\partial w}{\partial z},\ \ldots,\ z^n\frac{\partial^n w}{\partial z^n}\right).$$

8. $\dfrac{\partial w}{\partial t} = e^{\lambda x} F\left(w,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, e^{-\lambda C_1} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = w(z), \quad z = \lambda x + \ln t,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$e^z F\big(w,\ \lambda w'_z,\ \ldots,\ \lambda^n w_z^{(n)}\big) - w'_z = 0.$$

9. $\dfrac{\partial w}{\partial t} = wF\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = Ae^{\mu t}\varphi(x),$$

where A and μ are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi,\ \ldots,\ \varphi_x^{(n)}/\varphi\big) = \mu.$$

10. $\dfrac{\partial w}{\partial t} = w^\beta F\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

For $\beta = 1$, see equation 15.2.3.9.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, C_1^{\beta-1}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \big[(1-\beta)At + B\big]^{\frac{1}{1-\beta}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi^{\beta-1}F\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi,\ \ldots,\ \varphi_x^{(n)}/\varphi\big) = A.$$

11. $\dfrac{\partial w}{\partial t} = e^{\beta w}F\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$e^{\beta\varphi}F\big(x,\ \varphi'_x,\ \varphi''_{xx},\ \ldots,\ \varphi_x^{(n)}\big) + A = 0.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

1°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x F\big(x,\ \varphi''_{xx}/\varphi'_x,\ \ldots,\ \varphi^{(n)}_x/\varphi'_x\big) = A.$$

2°. Generalized separable solution:

$$w(x,t) = Ae^{\mu t}\Theta(x) + B,$$

where A, B, and μ are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\Theta'_x F\big(x,\ \Theta''_{xx}/\Theta'_x\ \ldots,\ \Theta^{(n)}_x/\Theta'_x\big) = \mu\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

For $\beta = 1$, see equation 15.2.3.12.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, C_1^{\beta-1}t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\big(\varphi'_x\big)^{\beta} F\big(x,\ \varphi''_{xx}/\varphi'_x,\ \ldots,\ \varphi^{(n)}_x/\varphi'_x\big) = A.$$

3°. Generalized separable solution:

$$w(x,t) = \big[A(1-\beta)t + C_1\big]^{\frac{1}{1-\beta}}\big[\Theta(x) + B\big] + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\big(\Theta'_x\big)^{\beta} F\big(x,\ \Theta''_{xx}/\Theta'_x\ \ldots,\ \Theta^{(n)}_x/\Theta'_x\big) = A\Theta + AB.$$

15.2.4. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right)$

1. $\dfrac{\partial w}{\partial t} = F\left(x, t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x,t) + C\exp\left[\int g(t)\,dt\right],$$

where C are arbitrary constants, is also a solution of the equation.

2. $\dfrac{\partial w}{\partial t} = F\left(ax + bt,\ w,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w = w(\xi), \quad \xi = ax + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi,\ \ldots,\ a^n w^{(n)}_\xi\right) - bw'_\xi = 0.$$

3. $\dfrac{\partial w}{\partial t} = F\left(ax + bt,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w = \varphi(\xi) + Ct, \quad \xi = ax + bt,$$

where C is an arbitrary constant and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ a\varphi'_\xi,\ \ldots,\ a^n\varphi^{(n)}_\xi\right) - b\varphi'_\xi - C = 0,$$

whose order can be reduced with the substitution $U(\xi) = \varphi'_\xi$.

4. $\dfrac{\partial w}{\partial t} = f(t)x^k\Phi\left(w,\ x\dfrac{\partial w}{\partial x},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right) + xg(t)\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xG(t), \quad \tau = \int f(t)G^{-k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives at a simpler equation of the form 15.2.3.6:

$$\frac{\partial w}{\partial \tau} = z^k\Phi\left(w,\ z\frac{\partial w}{\partial z},\ \ldots,\ z^n\frac{\partial^n w}{\partial z^n}\right).$$

5. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + Be^{-\lambda x}E(t),$$

$$E(t) = \exp\left[\int \Phi\left(t, \lambda^2, \ldots, \lambda^{2n}\right)dt\right],$$

where A and B are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = w\Phi\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + e^{-\lambda x}E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\left(t, \lambda^2, \ldots, \lambda^{2n}\right)dt\right],$$

where A and B are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = w\Phi\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cosh(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sinh(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\left(t, \lambda^2, \ldots, \lambda^{2n}\right)dt\right],$$

where A and B are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = w\Phi\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + B\sin(\lambda x)E(t),$$
$$E(t) = \exp\left[\int \Phi\left(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\right)dt\right],$$

where A and B are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = w\Phi\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sin(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\left(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\right)dt\right],$$

where A and B are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = f(t)w^{\beta}\Phi\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 15.2.3.10:

$$\frac{\partial u}{\partial \tau} = u^{\beta}\Phi\left(x,\ \frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2},\ \frac{1}{u}\frac{\partial^n u}{\partial x^n}\right),$$

which has a multiplicative separable solution $u = \varphi(x)\psi(\tau)$.

11. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^{k}\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant)

$$\varphi'_t = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi'_t = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

and the function $\Theta(x)$ satisfies the nth-order ordinary differential equation

$$\left(\Theta'_x\right)^k\Phi\left(x,\ \Theta''_{xx}/\Theta'_x,\ \ldots,\ \Theta^{(n)}_x/\Theta'_x\right) = A\Theta + B.$$

The general solution of system (1), (2) is given by

$$\varphi(t) = G(t)\left[C + A(1-k)\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int\left[Bf(t)\varphi^k(t) + h(t)\right]\frac{dt}{G(t)},$$

where A, B, C, and D are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = \left[f_1(t)w + f_0(t)\right]\left(\dfrac{\partial w}{\partial x}\right)^{k}\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right) + g_1(t)w + g_0(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant):

$$\varphi'_t = Cf_1(t)\varphi^{k+1} + g_1(t)\varphi, \tag{1}$$
$$\psi'_t = \big[Cf_1(t)\varphi^k + g_1(t)\big]\psi + Cf_0(t)\varphi^k + g_0(t), \tag{2}$$

and the function $\Theta(x)$ satisfies the nth-order ordinary differential equation

$$\big(\Theta'_x\big)^k \Phi\big(x,\ \Theta''_{xx}/\Theta'_x,\ \ldots,\ \Theta^{(n)}_x/\Theta'_x\big) = C.$$

The general solution of system (1), (2) is given by

$$\varphi(t) = G(t)\left[A - kC\int f_1(t)G^k(t)\,dt\right]^{-1/k}, \quad G(t) = \exp\left[\int g_1(t)\,dt\right],$$
$$\psi(t) = B\varphi(t) + \varphi(t)\int \big[Cf_0(t)\varphi^k(t) + g_0(t)\big]\frac{dt}{\varphi(t)},$$

where A, B, and C are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\big[\beta G(t)\big]\,dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 15.2.3.11:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(x,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2},\ \ldots,\ \frac{\partial^n u}{\partial x^n}\right),$$

which has a solution in the additive separable form $u = \varphi(x) + \psi(\tau)$.

14. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right)$
$+ \displaystyle\sum_{k=1}^{m} \frac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

15. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right)$
$+ \displaystyle\sum_{k=1}^{m} \frac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

There is a generalized separable solution of the form

$$w(x,t) = \cosh(\lambda x)\varphi(t) + \sinh(\lambda x)\psi(t).$$

16. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right)$
$+\sum\limits_{k=1}^{m}\dfrac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

17. $\dfrac{\partial w}{\partial t} = wF(t,\zeta_0,\zeta_1,\ldots,\zeta_n), \quad \zeta_k = \sum\limits_{i=k}^{n}\dfrac{(-1)^{i+k}}{k!\,(i-k)!}x^{i-k}\dfrac{\partial^i w}{\partial x^i}, \quad k = 0, 1, \ldots, n.$

Multiplicative separable solution:

$$w(x,t) = (C_0 + C_1 x + \cdots + C_n x^n)\varphi(t),$$

where C_0, C_1, …, C_n are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t = \varphi F(t, C_0\varphi, C_1\varphi, \ldots, C_n\varphi).$$

⊙ *Reference*: Ph. W. Doyle (1996), the case $\partial_t F \equiv 0$ was treated.

15.3. Equations Involving the Second Derivative in t

15.3.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x+bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$aw_\xi^{(n)} - b^2 w''_{\xi\xi} + f(\xi, w) = 0.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + f(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b\ln\varphi + f(t) + C\big]\varphi = 0,$$
$$a\psi_x^{(n)} + \big(b\ln\psi - C\big)\psi = 0,$$

where C is an arbitrary constant.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + \big[f(x) + g(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b\ln\varphi + g(t) + C\big]\varphi = 0,$$
$$a\psi_x^{(n)} + \big[b\ln\psi + f(x) - C\big]\psi = 0,$$

where C is an arbitrary constant.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi\ln\varphi + \big[g(x) - b^2\big]\varphi = 0.$$

15.3.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b\ln\varphi + h(t) + C\big]\varphi = 0,$$
$$a\psi_x^{(n)} + f(x)\psi'_x + \big[b\ln\psi + g(x) - C\big]\psi = 0,$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, t) + C_2 t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t} (t-\tau) f(\tau)\, d\tau + \theta(z), \quad z = x + \lambda t,$$

where C_1, C_2, t_0, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta_z^{(n)} - \lambda^2 \theta''_{zz} + b\big(\theta'_z\big)^2 - 2C_1 = 0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

4°. The substitution $w = U(x,t) + \int_0^t (t-\tau) f(\tau)\, d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + b\left(\frac{\partial U}{\partial x}\right)^2,$$

which admits a self-similar solution of the form $U = t^{\frac{2(2-n)}{n}} u(\zeta)$, where $\zeta = xt^{-\frac{2}{n}}$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x + C_1, t) + C_2 \cosh(kt) + C_3 \sinh(kt) \quad \text{if } c = k^2 > 0,$$
$$w_2 = w(x + C_1, t) + C_2 \cos(kt) + C_3 \sin(kt) \quad \text{if } c = -k^2 < 0,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - c\varphi - f(t) = 0,$$
$$a\psi_z^{(n)} - \lambda^2 \psi''_{zz} + b\big(\psi'_z\big)^2 + c\psi = 0.$$

The general solution of the first equation is expressed as

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t f(\tau) \sinh\big[k(t-\tau)\big]\, d\tau \quad \text{if} \quad c = k^2 > 0,$$

$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t f(\tau) \sin\big[k(t-\tau)\big]\, d\tau \quad \text{if} \quad c = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

4°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 2°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

1°. Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ are roots of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = k\varphi^2 + f(t)\varphi + g(t), \tag{1}$$
$$\psi''_{tt} = \big[(c\lambda + 2k)\varphi + f(t) + a\lambda^n\big]\psi. \tag{2}$$

In the special case $f(t) = \text{const}$ and $g(t) = \text{const}$, equation (1) has particular solutions of the form $\varphi = \text{const}$ and, due to autonomy, can be integrated by quadrature. Equation (2) is linear in ψ, and, hence, for $\varphi = \text{const}$, its general solution is expressed in terms of exponentials or sine and cosine.

2°. For $c = 0$, there are generalized separable solutions of the form

$$w = \varphi(t)\exp\big(x\sqrt{|k/b|}\,\big) + \psi(t)\exp\big(-x\sqrt{|k/b|}\,\big) + \chi(t) \qquad \text{if } bk < 0,$$
$$w = \varphi(t)\cos\big(x\sqrt{k/b}\,\big) + \psi(t)\sin\big(x\sqrt{k/b}\,\big) + \chi(t) \qquad \text{if } bk > 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^3 + sw.$

1°. Multiplicative separable solutions with $bc < 0$:

$$w = \begin{cases} \exp\big(|c/b|^{1/2}x\big)\big[C_1\exp(kt) + C_2\exp(-kt)\big] & \text{if } s + a|c/b|^{n/2} = k^2 > 0, \\ \exp\big(|c/b|^{1/2}x\big)\big[C_1\cos(kt) + C_2\sin(kt)\big] & \text{if } s + a|c/b|^{n/2} = -k^2 < 0, \end{cases}$$
$$w = \begin{cases} \exp\big(-|c/b|^{1/2}x\big)\big[C_1\exp(kt) + C_2\exp(-kt)\big] & \text{if } s + (-1)^n a|c/b|^{n/2} = k^2 > 0, \\ \exp\big(-|c/b|^{1/2}x\big)\big[C_1\cos(kt) + C_2\sin(kt)\big] & \text{if } s + (-1)^n a|c/b|^{n/2} = -k^2 < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

Remark. The more general equation

$$w_{tt} = aw_x^{(n)} + f(w)\big(bw_x^2 + cw^2\big) + sw,$$

where $f(w)$ is an arbitrary function, has the same solutions.

2°. Generalized separable solution with $bc < 0$ (generalizes the solutions of Item 1°):

$$w = \varphi(t)\exp\big(x\sqrt{|c/b|}\,\big) + \psi(t)\exp\big(-x\sqrt{|c/b|}\,\big),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = \big(s + a|c/b|^{n/2}\big)\varphi + 4c\varphi^2\psi,$$
$$\psi''_{tt} = \big(s + (-1)^n a|c/b|^{n/2}\big)\psi + 4c\varphi\psi^2.$$

3°. For $bc > 0$, there are generalized separable solutions of the form

$$w = \varphi(t)\cos\big(x\sqrt{c/b}\,\big) + \psi(t)\sin\big(x\sqrt{c/b}\,\big).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t} (t-\tau)h(\tau)\,d\tau + \varphi(x).$$

Here, C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\big(\varphi_x'\big)^2 + g(x) - 2C_1 = 0.$$

2°. The substitution $w = U(x,t) + \displaystyle\int_0^t (t-\tau)h(\tau)\,d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + g(x).$$

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi_{tt}'' - b\varphi - h(t) = 0,$$
$$a\psi_x^{(n)} + f(x)(\psi_x')^2 + b\psi + g(x) = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t h(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$

$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t h(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions involving exponentials of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp\big(\pm x\sqrt{-b}\,\big), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi_{tt}'' = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi_{tt}'' = \big[2bf\varphi + g + (-1)^n ab^n\big]\psi; \tag{3}$$

the arguments of the functions f, g, and h are not specified.

In the special case of constant f, g, and h, equation (2) has particular solutions of the form $\varphi = \text{const}$. In this case, the general solution of equation (3) is expressed in terms of exponentials or sine and cosine.

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t)=\varphi(t)+\psi(t)\big[A\exp\big(x\sqrt{-b}\,\big)+B\exp\big(-x\sqrt{-b}\,\big)\big],\qquad b<0, \tag{4}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi''_{tt}=bf\big(\varphi^2+4AB\psi^2\big)+g\varphi+h, \tag{5}$$
$$\psi''_{tt}=\big[2bf\varphi+g+(-1)^n ab^n\big]\psi. \tag{6}$$

One can express φ via ψ in (6) and substitute the resulting expression into (5) to obtain a fourth-order nonlinear equation for ψ. For f, g, h = const, this equation is autonomous and its order can be reduced.

Note two special cases where solution (4) is expressed in terms of hyperbolic functions:

$$w(x,t)=\varphi(t)+\psi(t)\cosh\big(x\sqrt{-b}\,\big)\quad\text{if } A=\tfrac{1}{2},\ B=\tfrac{1}{2};$$
$$w(x,t)=\varphi(t)+\psi(t)\sinh\big(x\sqrt{-b}\,\big)\quad\text{if } A=\tfrac{1}{2},\ B=-\tfrac{1}{2}.$$

3°. Generalized separable solution involving trigonometric functions of x:

$$w(x,t)=\varphi(t)+\psi(t)\cos\big(x\sqrt{b}+c\big),\qquad b>0,$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi''_{tt}=bf\big(\varphi^2+\psi^2\big)+g\varphi+h,$$
$$\psi''_{tt}=\big[2bf\varphi+g+(-1)^n ab^n\big]\psi.$$

⊙ *References for equation* 15.3.2.8: V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2002, p. 371).

9. $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial^n w}{\partial x^n}+f\left(x,\dfrac{\partial w}{\partial x}\right)+g(t)$.

1°. Additive separable solution:

$$w(x,t)=C_1t^2+C_2t+\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau+\varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)}+f\big(x,\varphi'_x\big)-2C_1=0,$$

whose order can be reduced with the substitution $u(x)=\varphi'_x$.

2°. The substitution $w=U(x,t)+\displaystyle\int_0^t(t-\tau)g(\tau)\,d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2}=a\frac{\partial^n U}{\partial x^n}+f\left(x,\frac{\partial U}{\partial x}\right).$$

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi_x^{(n)} + f\left(x, \psi'_x\right) + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\left[k(t-\tau)\right]d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is specified Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + f\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + wf\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \left[a\lambda^n + f(t,\lambda)\right]\varphi.$$

15.3.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial^2 w}{\partial x^2} + c.$

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^n u_\xi^{(n)} + (bk^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + bUU''_{zz} - b^2C_2^2U''_{zz} - 2bC_1U'_z = 8bC_1^2 - c.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

⊙ *Reference for equation* 15.3.3.1: A. D. Polyanin and V. F. Zaitsev (2004, p. 671).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + c.$

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^n u_\xi^{(n)} + bk^2(uu'_\xi)'_\xi - \lambda^2 u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + b(UU'_z)'_z - b^2C_2^2U''_{zz} - 2bC_1U'_z = 8bC_1^2 - c.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

⊙ *Reference for equation* 15.3.3.2: A. D. Polyanin and V. F. Zaitsev (2004, p. 671).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\sum\limits_{i,j=0}^{i,j<n} b_{ij}\dfrac{\partial^i w}{\partial x^i}\dfrac{\partial^j w}{\partial x^j} + \sum\limits_{k=0}^{n-1} g_k(t)\dfrac{\partial^k w}{\partial x^k} + h(t).$

Here, we adopt the notation $\dfrac{\partial^0 w}{\partial x^0} \equiv w$.

1°. In the general case, the equation has generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the algebraic equation $\sum\limits_{i,j=0}^{i,j<n} b_{ij}\lambda^{i+j} = 0$.

2°. Let n be an even number and let all coefficients b_{ij} be zero for odd $i + j$. In this case, the original equation has also generalized separable solutions of the form

$$w(x,t) = \varphi_1(t) + \psi_1(t)\big[A\cosh(\lambda x) + B\sinh(\lambda x)\big],$$
$$w(x,t) = \varphi_2(t) + \psi_2(t)\big[A\cos(\lambda x) + B\sin(\lambda x)\big],$$

where A and B are arbitrary constants, the parameter λ is determined by solving algebraic equations, and the functions $\varphi_1(t)$, $\psi_1(t)$ and $\varphi_2(t)$, $\psi_2(t)$ are found from appropriate systems of second-order ordinary differential equations.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)g(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\big(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\big) - A = 0,$$

whose order can be reduced with the substitution $U(x) = \varphi_x'$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)g(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi_x^{(n)} + F\big(x, \psi'_x, \ldots, \psi_x^{(n-1)}\big) + b\psi = 0.$$

The general solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is specified in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right) + bu.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \big[a\lambda^n + F(t, \lambda, \ldots, \lambda^{n-1})\big]\varphi.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n-2} w}{\partial x^{2n-2}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \big[A\cosh(\lambda x) + B\sinh(\lambda x)\big]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \Phi(t)\varphi, \qquad \Phi(t) = a\lambda^{2n} + F\big(t, \lambda^2, \ldots, \lambda^{2n-2}\big).$$

3°. Multiplicative separable solution:

$$w(x,t) = \big[A\cos(\lambda x) + B\sin(\lambda x)\big]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \Phi(t)\varphi, \qquad \Phi(t) = (-1)^n a\lambda^{2n} + F\big(t, -\lambda^2, \ldots, (-1)^{n-1}\lambda^{2n-2}\big).$$

15.3.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w + \sum\limits_{k=0}^{n-1} b_k x^k.$

Generalized separable solution:

$$w(x,t) = \tfrac{1}{2}t^2\sum_{k=0}^{n-1} b_k x^k + t\sum_{k=0}^{n-1} A_k x^k + \sum_{k=0}^{n-1} B_k x^k - \frac{1}{a(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)\,d\xi,$$

where $A_0, A_1, \ldots, A_{n-1}$ and $B_0, B_1, \ldots, B_{n-1}$ are arbitrary constants.

2. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\big(A_n x^n + \cdots + A_1 x\big) + \psi(t),$$

where $A_1, \ldots, A_n$ are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = A_n an!\,\varphi^2 + f(t)\varphi,$$
$$\psi''_{tt} = A_n an!\,\varphi\psi + f(t)\psi + g(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations (C is an arbitrary constant):

$$\varphi''_{tt} = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi''_{tt} = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

and the function $\Theta(x)$ satisfies the nth-order constant-coefficient linear ordinary differential equation

$$a\Theta^{(n)}_x + b\Theta = C.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^{2n} w}{\partial x^{2n}} - ak^{2n}w^2 + f(x)w + b_1\sinh(kx) + b_2\cosh(kx).$

Generalized separable solution quadratic in t:

$$w(x,t) = \tfrac{1}{2}(t+C)^2\big[b_1\sinh(kx) + b_2\cosh(kx)\big] + \varphi(x).$$

Here, C is an arbitrary constant and the function $\varphi(x)$ is found from the constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\varphi_x^{(2n)} - ak^{2n}\varphi + f(x) = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
\varphi''_{tt} &= C\varphi^2 + g(t)\varphi,\\
\psi''_{tt} &= \big[C\varphi + g(t)\big]\psi + h(t),\\
a\Theta_x^{(n)} + f(x)\Theta'_x &= C,
\end{aligned}$$

and C is an arbitrary constant.

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the ordinary differential equations

$$\begin{aligned}
a\varphi_x^{(n)} + f(x)\varphi'_x + g(x)\varphi - C &= 0,\\
\psi''_{tt} - C\psi^2 - h(t)\psi &= 0,
\end{aligned}$$

and C is an arbitrary constant.

15.3.5. Other Equations Involving Arbitrary Functions

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^{n-1}}{\partial x^{n-1}}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{b}{n!}x^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0,$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2004, p. 675).

2. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-2n} w(C_1^2 x + C_2, C_1^n t + C_3) + \sum_{k=0}^{n-1} (A_k t + B_k) x^k,$$

where C_1, C_2, C_3, the A_k, and the B_k are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution in the form of an nth-degree polynomial in x:

$$w(x,t) = \frac{1}{n!}(C_1 t + C_2)x^n + \sum_{k=0}^{n-1} (A_k t + B_k)x^k + \int_0^t (t-\xi) F(C_1\xi + C_2)\, d\xi,$$

where C_1, C_2, the A_k, and the B_k are arbitrary constants.

3°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}t^2 \sum_{k=0}^{n-1} A_k x^k + t \sum_{k=0}^{n-1} B_k x^k + \sum_{k=0}^{n-1} C_k x^k + \int_0^x \frac{(x-\xi)^{n-1}}{(n-1)!} \Phi\left(\sum_{k=0}^{n-1} A_k \xi^k\right) d\xi,$$

where the A_k, B_k, and C_k are arbitrary constants, and $\Phi(u)$ is the inverse of the function $F(u)$.

4°. Solution:

$$w(x,t) = \frac{1}{2}A_1 t^2 + \frac{1}{n!}A_2 x^n + \sum_{m=0}^{n-1} (B_m t + C_m)x^m + \varphi(\zeta), \qquad \zeta = kx + \lambda t,$$

where A_1, A_2, the B_m, the C_m, k, and λ are arbitrary constants, and the function $\varphi = \varphi(\zeta)$ is determined by the autonomous ordinary differential equation

$$A_1 + \lambda^2 \varphi''_{\zeta\zeta} = F\left(A_2 + k^n \varphi^{(n)}_\zeta\right).$$

5°. Self-similar solution:

$$w = t^2 U(z), \qquad z = x t^{-2/n},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$2U + \frac{2(2-3n)}{n^2} z U'_z + \frac{4}{n^2} z^2 U''_{zz} = F(U^{(n)}_z).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Degenerate solution linear in x:

$$w = (C_1 t + C_2)x + C_3 t + C_4 + \int_0^t (t-\tau) F(C_1\tau + C_2, 0, \ldots, 0)\, d\tau.$$

2°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$(ak^2u - \lambda^2)u''_{\xi\xi} + F(ku'_\xi, k^2u''_{\xi\xi}, \ldots, k^nu^{(n)}_\xi) = 0.$$

3°. Solution:

$$w = U(z) + 4aC_1^2t^2 + 4aC_1C_2t, \qquad z = x + aC_1t^2 + aC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U''_{zz} - 2aC_1U'_z + F(U'_z, U''_{zz}, \ldots, U^{(n)}_z) = 8aC_1^2.$$

⊙ *Reference for equation* 15.3.5.3: A. D. Polyanin and V. F. Zaitsev (2004, p. 676).

4. $\dfrac{\partial^2 w}{\partial t^2} = (aw + bx)\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

The substitution $w = u - (b/a)x$ leads to an equation of the form 15.3.5.3:

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + F\left(\frac{\partial u}{\partial x} - \frac{b}{a}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + aw.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x,t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } a = k^2 > 0,$$
$$w_2 = w(x,t) + C_1\cos(kt) + C_2\sin(kt) \quad \text{if } a = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

15.3.6. Equations Involving Arbitrary Differential Operators

1. $\dfrac{\partial^2 w}{\partial t^2} = L[\sqrt{w}] + a(x) + b(x)\sqrt{w}.$

Here, L is an nth-order linear differential operator in x with coefficients independent of t, and $w \geq 0$.

Generalized separable solution:

$$w = [f(x)t^2 + g(x)t + h(x)]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$L[f] + bf - 12f^2 = 0,$$
$$L[g] + bg - 12fg = 0,$$
$$L[h] + bh + a - 4fh - 2g^2 = 0.$$

Remark. The differential operator L and functions a and b can depend on several coordinates $x = (x_1, \ldots, x_m)$.

2. $\dfrac{\partial^2 w}{\partial t^2} = L[\sqrt{w}] + a(x) + b(x)\sqrt{w} + cw.$

For $c = 0$, see equation 15.3.6.1.

1°. Generalized separable solution for $c > 0$:

$$w = \left[f(x)\exp\left(\tfrac{1}{2}\sqrt{c}\,t\right) + g(x)\exp\left(-\tfrac{1}{2}\sqrt{c}\,t\right) + h(x)\right]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &L[f] + \tfrac{3}{2}cfh + bf = 0,\\ &L[g] + \tfrac{3}{2}cgh + bg = 0,\\ &L[h] + ch^2 + bh + a + 2cfg = 0. \end{aligned}$$

⊙ *Reference*: V. A. Galaktionov and S. R. Svirshchevskii (2007, p. 282).

2°. Generalized separable solution for $c < 0$:

$$w = \left[f(x)\cos\left(\tfrac{1}{2}\sqrt{|c|}\,t\right) + g(x)\sin\left(\tfrac{1}{2}\sqrt{|c|}\,t\right) + h(x)\right]^2,$$

where the functions $f = f(x)$, $g = g(x)$, and $h = h(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &L[f] + \tfrac{3}{2}cfh + bf = 0,\\ &L[g] + \tfrac{3}{2}cgh + bg = 0,\\ &L[h] + ch^2 + bh + a + \tfrac{1}{2}c(\varphi^2 + \psi^2) = 0. \end{aligned}$$

Remark. The differential operator L and functions a and b can depend on several coordinates $x = (x_1, \ldots, x_m)$.

3. $\dfrac{\partial^2 w}{\partial t^2} = L[w^{1/3}] + a(x) + b(x)w^{1/3}.$

Here, L is an nth-order linear differential operator in x with coefficients independent of t.

Generalized separable solution:

$$w = [f(x)t + g(x)]^3,$$

where the functions $f = f(x)$ and $g = g(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &L[f] + bf - 6f^3 = 0,\\ &L[g] + bg + a - 6f^2 g = 0. \end{aligned}$$

Remark. The differential operator L and functions a and b can depend on several coordinates $x = (x_1, \ldots, x_m)$.

4. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial t^2} = L[w] + a(x).$

Here, L is an nth-order linear differential operator in x with coefficients independent of t.

1°. Generalized separable solution:

$$w = \varphi_0(x) + \varphi_1(x)t + \varphi_2(x)t^2 + \varphi_3(x)t^3,$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} L[\varphi_0] &= 2\varphi_1\varphi_2 - a,\\ L[\varphi_1] &= 4\varphi_2^2 + 6\varphi_1\varphi_3,\\ L[\varphi_2] &= 18\varphi_2\varphi_3,\\ L[\varphi_3] &= 18\varphi_3^2. \end{aligned}$$

2°. Generalized separable solution:

$$w = \psi_0(x) + \psi_1(x)t^{3/2} + \psi_2(x)t^3,$$

where the functions $\psi_k = \psi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} L[\varphi_0] &= \tfrac{9}{8}\psi_1^2 - a,\\ L[\varphi_1] &= \tfrac{45}{4}\psi_1\psi_2,\\ L[\varphi_2] &= 18\psi_2^2. \end{aligned}$$

3°. If $L[1] \equiv 0$ or $L[w]$ linear constant-coefficient differential operator, then there is an additive separable solution of the form $w = \varphi(t) + \psi(x)$.

Remark. The operator L and the function a can depend on several coordinates $x = (x_1, \ldots, x_m)$.

15.4. Other Equations

15.4.1. Equations Involving Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = f(t)\dfrac{\partial^n w}{\partial x^n}.$

This is a special case of equation 15.4.1.7.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1(x, t) = w\big(x + \varphi(t), t\big) + \varphi_t'(t),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized separable solution:

$$w = \varphi(t)e^{\lambda x} + \frac{1}{\lambda}\frac{\varphi_t'(t)}{\varphi(t)} - \lambda^{n-2}f(t),$$

where $\varphi(t)$ is an arbitrary function and λ is an arbitrary constant.

3°. For reducing the equation order by one, see equation 15.4.1.7.

Remark. This equation with $n = 3$ and $f(t) = \text{const}$ occurs in fluid dynamics; see 13.3.3.1, equation (2) and 14.3.3.1, equation (4) with $f_1(t) = 0$.

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^n w}{\partial y^n}.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1(x, y) = C_1^{n-2} w(x, C_1 y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution:

$$w(x, y) = \sum_{k=0}^{n-1} C_k \big[y + \varphi(x)\big]^k,$$

where $\varphi(x)$ is an arbitrary function and the C_k are arbitrary constants.

3°. Generalized separable solution:

$$w(x, y) = \varphi(x)e^{\lambda y} - \lambda^{n-2}\int f(x)\,dx + C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

4°. Generalized separable solution:

$$w(x, y) = \varphi(y)\int f(x)\,dx + \psi(y),$$

where the functions $\varphi = \varphi(y)$ and $\psi = \psi(y)$ are determined by the autonomous system of ordinary differential equations

$$(\varphi'_y)^2 - \varphi\varphi''_{yy} = \varphi_y^{(n)},$$
$$\varphi'_y\psi'_y - \varphi\psi''_{yy} = \psi_y^{(n)}.$$

5°. Generalized self-similar solution:

$$w(x, y) = \varphi(x)U(z), \qquad z = \psi(x)y$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $U = U(z)$ are determined by the system of ordinary differential equations

$$(\varphi\psi)'_x = C_1 f(x)\psi^{n-1},$$
$$\varphi'_x = C_2 f(x)\psi^{n-2},$$
$$C_1(U'_z)^2 - C_2 U U''_{zz} = U_z^{(n)}.$$

6°. See also equation 15.4.1.3 with $g(x) = 0$.

Remark. This equation with $n = 3$ and $f(x) = \text{const}$ occurs in fluid dynamics; see 13.3.1.1.

⊙ *Reference for equation* 15.4.1.2: A. D. Polyanin and V. F. Zaitsev (2002, p. 375).

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=f(x)\dfrac{\partial^{2n} w}{\partial y^{2n}}+g(x).$

This is a special case of equation 15.4.1.5.

Generalized separable solution:

$$w(x,y)=\varphi(x)e^{\lambda y}-\frac{1}{2\lambda^2\varphi(x)}\left[\int g(x)\,dx+C_1\right]e^{-\lambda y}-\lambda^{2n-2}\int f(x)\,dx+C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants.

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=f(x)\Big(\dfrac{\partial^2 w}{\partial y^2}\Big)^{k-1}\dfrac{\partial^n w}{\partial y^n}.$

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y)=C_1^{2k+n-4}w\big(x,C_1^{2-k}y+\varphi(x)\big)+C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w=U(z),\qquad z=y\left[\int f(x)\,dx+C\right]^{\frac{1}{4-2k-n}}+\varphi(x),$$

where $\varphi(x)$ is an arbitrary function and the function $U=U(z)$ is determined by the autonomous ordinary differential equation

$$(U_z')^2=(4-2k-n)(U_{zz}'')^{k-1}U_z^{(n)}.$$

3°. Multiplicative separable solution:

$$w(x,y)=\left[(2-k)\int f(x)\,dx+C\right]^{\frac{1}{2-k}}\theta(y),$$

where the function $\theta(y)$ is determined by the autonomous ordinary differential equation

$$(\theta_y')^2-\theta\theta_{yy}''=(\theta_{yy}'')^{k-1}\theta_y^{(n)}.$$

5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=F\left(x,w,\dfrac{\partial w}{\partial y},\ldots,\dfrac{\partial^n w}{\partial y^n}\right).$

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y)=w\big(x,y+\varphi(x)\big),$$

where $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Suppose the right-hand side of the equation is independent of x explicitly. Then there is a generalized traveling-wave solution of the form

$$w=w(z),\qquad z=y+\varphi(x),$$

where $\varphi(x)$ is an arbitrary function, and the function $w(z)$ is determined by the autonomous ordinary differential equation $F\big(w,w_z',\ldots,w_z^{(n)}\big)=0$.

3°. Suppose the right-hand side of the equation is independent of x and w explicitly. Then there is an exact solution of the form

$$w = Cx + g(z), \qquad z = y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function, C is an arbitrary constant, and the function $g(z)$ is determined by the autonomous ordinary differential equation $F\big(g'_z, \ldots, g_z^{(n)}\big) + Cg''_{zz} = 0$.

4°. In the original equation, let us change from the old variables to the von Mises variables:

$$x,\ y,\ w = w(x,y) \quad \Longrightarrow \quad x,\ w,\ \eta = \eta(x,w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial y}.$$

The derivatives are transformed as follows:

$$\frac{\partial}{\partial y} = \frac{\partial w}{\partial y}\frac{\partial}{\partial w} = \eta\frac{\partial}{\partial w}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial}{\partial w},$$
$$\frac{\partial^2 w}{\partial y^2} = \eta\frac{\partial \eta}{\partial w}, \quad \frac{\partial^3 w}{\partial y^3} = \eta\frac{\partial}{\partial w}\Big(\eta\frac{\partial \eta}{\partial w}\Big), \quad \ldots, \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \eta}{\partial w}.$$

As a result, the original nth-order equation is reduced to an $(n-1)$st-order equation

$$\eta\frac{\partial \eta}{\partial x} = F\left(x, w, \eta, \eta\frac{\partial \eta}{\partial w}, \ldots, \frac{\partial^{n-1}\eta}{\partial y^{n-1}}\right),$$

where the higher-order derivatives with respect to y are calculated as

$$\frac{\partial^k w}{\partial y^k} = \frac{\partial^{k-1}\eta}{\partial y^{k-1}} = \eta\frac{\partial}{\partial w}\frac{\partial^{k-2}\eta}{\partial y^{k-2}}, \qquad \frac{\partial}{\partial y} = \eta\frac{\partial}{\partial w}, \qquad k = 3, \ldots, n.$$

⊙ *Reference for equation* 15.4.1.5: A. D. Polyanin and V. F. Zaitsev (2002, p. 376).

6. $\dfrac{\partial w}{\partial y}\dfrac{\partial G}{\partial x} - \dfrac{\partial w}{\partial x}\dfrac{\partial G}{\partial y} = F\left(x, w, \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^n w}{\partial y^n}\right), \quad G = \left(x, w, \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^k w}{\partial y^k}\right).$

For the case $G = w_y$ we have equation 15.4.1.5.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y) = w\big(x, y + \varphi(x)\big),$$

where $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. The von Mises transformation

$$x,\ y,\ w = w(x,y) \quad \Longrightarrow \quad x,\ w,\ \eta = \eta(x,w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial y},$$

reduces the order of the equation by one

$$\eta\frac{\partial G}{\partial x} = F\left(x, w, \eta, \eta\frac{\partial \eta}{\partial w}, \ldots, \frac{\partial^{n-1}\eta}{\partial y^{n-1}}\right), \quad G = G\left(x, w, \eta, \eta\frac{\partial \eta}{\partial w}, \ldots, \frac{\partial^{k-1}\eta}{\partial y^{k-1}}\right).$$

Formulas for computing derivatives can be found in Item 4° of equation 15.4.1.5.

Remark. On the left-hand side of the equation in question, the term $w_y G_x - w_x G_y$ can be replaced by a more complex expression $H_y G_x - H_x G_y$ with $H = H(x, w, w_y, \ldots, w_y^{(m)})$. In this case also, the von Mises transformation reduces the order of the equation.

⊙ *Reference*: A. D. Polyanin (2010).

7. $\dfrac{\partial^2 w}{\partial x \partial t} + [a(t)w + b(t)x]\dfrac{\partial^2 w}{\partial x^2} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^3 w}{\partial x^3}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Let $a(t) \not\equiv 0$. Suppose $\widetilde{w}(t, x)$ is a solution of the equation in question. Then the function

$$w = \widetilde{w}(t, x + \varphi(t)) + \frac{1}{a(t)}\big[b(t)\varphi(t) - \varphi_t'(t)\big],$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation. For $a(t) \equiv 0$, the function $w = \widetilde{w}(t, x) + \varphi(t)$ solves the equation.

2°. In the original equation, denote

$$\eta = \frac{\partial w}{\partial x}, \qquad \Phi = \frac{\partial^2 w}{\partial x^2}, \tag{1}$$

divide the resulting equation by $w_{xx} = \Phi$, differentiate with respect to x, and take into account (1) to obtain

$$\frac{\Phi_t}{\Phi} - \frac{w_{xt}\Phi_x}{\Phi^2} + a(t)\eta + b(t) = \frac{\partial}{\partial x}\frac{1}{\Phi}F\left(t, \eta, \Phi, \frac{\partial \Phi}{\partial x}, \ldots, \frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right). \tag{2}$$

In (2), let us change from the old variables to the *Crocco variables*

$$t,\ x,\ u = u(t, x) \quad \Longrightarrow \quad t,\ \eta,\ \Phi = \Phi(t, \eta), \tag{3}$$

where η and Φ are given by (1). The derivatives are transformed as follows:

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = w_{xx}\frac{\partial}{\partial \eta} = \Phi\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + w_{xt}\frac{\partial}{\partial \eta}.$$

As a result, equation (2) is reduced to an $(n-1)$st-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi}F\Big(t, \eta, \Phi, \Phi\frac{\partial \Phi}{\partial \eta}, \ldots, \frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\Big)\right], \tag{4}$$

where the higher-order derivatives are calculated as

$$\frac{\partial^k w}{\partial x^k} = \frac{\partial^{k-2}\Phi}{\partial x^{k-2}} = \Phi\frac{\partial}{\partial \eta}\frac{\partial^{k-3}\Phi}{\partial x^{k-3}}, \qquad \frac{\partial}{\partial x} = \Phi\frac{\partial}{\partial \eta}, \qquad k = 3, \ldots, n.$$

Whenever a solution to the original equation is known, formulas (1) give a solution to equation (4) in parametric form.

Suppose $\Phi = \Phi(\eta, t)$ is a solution to equation (4). Then the corresponding solution of the original equation can also be represented in parametric form:

$$x = \int \frac{ds}{\Phi(s, t)} + \varphi(t), \quad w = \int \frac{s\,ds}{\Phi(s, t)} + \frac{1}{a(t)}\big[b(t)\varphi(t) - \varphi_t'(t)\big],$$

where $\varphi(t)$ is an arbitrary function (in the integration, t is treated as a parameter).

3°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function, and $\varphi(t)$ is determined by the first-order ordinary differential equation $\varphi'_t = F(t, \varphi, 0, \ldots, 0)$.

4°. For $a = \text{const}$, $b = 0$, and $F = F(w_x, w_{xx}, \ldots, w_x^{(n)})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$k\lambda U''_{zz} + ak^2 U U''_{zz} = F\big(kU'_z, k^2 U''_{zz}, \ldots, k^n U_z^{(n)}\big).$$

Remark. One could consider the more general nonlinear nth-order equation

$$c(t)u_{tx} + [a(t)u + b(t)x]u_{xx} + d(t)(u_x u_{tx} - u_t u_{xx}) = F(t, u_x, u_{xx}, \ldots, u_x^{(n)}), \tag{5}$$

which coincides with the original equation 5.4.1.7 for $c(t) = 1$ and $d(t) = 0$.

1. Equation (5) has the following general property: if $\widetilde{w}(t,x)$ is a solution, then the function

$$w = \widetilde{w}(t, x + \varphi(t)) + \psi(t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are linked by the relation $d(t)\psi'_t - a(t)\psi = c(t)\varphi'_t - b(t)\varphi$ (one of the functions can be taken arbitrary), is also a solution to equation (5).

2. Dividing equation (5) by w_{xx}, differentiating the resulting equation with respect to x, and changing to the Crocco variables (3) with (1), one arrives at an $(n-1)$st-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - [d(t)\eta + c(t)]\frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi}F\Big(t, \eta, \Phi, \Phi\frac{\partial \Phi}{\partial \eta}, \ldots, \frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\Big)\right].$$

⊙ *References for equation* 15.4.1.7: A. D. Polyanin and S. N. Aristov (2009), S. N. Aristov and A. D. Polyanin (2010).

8. $\dfrac{\partial^{n+1} w}{\partial x^n \partial y} = a e^{\lambda w}$.

Generalized Liouville equation.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(C_1 x + C_2,\ C_3 y + C_4\big) + \frac{1}{\lambda}\ln(C_1^n C_3),$$
$$w_2 = w\big(x, \varphi(y)\big) + \frac{1}{\lambda}\ln \varphi'_y(y),$$

where $C_1, \ldots, C_4$ are arbitrary constants, $\varphi(y)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized traveling-wave solution:

$$w(x,y) = -\frac{n+1}{\lambda}\ln z, \quad z = \varphi(y)x + \frac{a\lambda(-1)^{n+1}}{(n+1)!}\varphi(y)\int \frac{dy}{[\varphi(y)]^{n+1}},$$

where $\varphi(y)$ is an arbitrary function.

3°. Traveling-wave solution:

$$w = F(\xi), \qquad \xi = k_1 x + k_2 y,$$

where k_1 and k_2 are arbitrary constants, and the function $F(\xi)$ is determined by the ordinary differential equation $k_1^n k_2 F_\xi^{(n+1)} = ae^{\lambda F}$.

4°. Solution:

$$w = U(\eta) + \frac{n-m}{m\lambda}\ln y, \qquad \eta = xy^{1/m},$$

where m is an arbitrary constant and the function $U(\eta)$ is determined by the ordinary differential equation

$$\eta U_\eta^{(n+1)} + nU_\eta^{(n)} = ame^{\lambda U}.$$

To $m = n$ there corresponds a self-similar solution.

⊙ *Reference for equation* 15.4.1.8: A. D. Polyanin and V. F. Zaitsev (2004, p. 679).

9. $\dfrac{\partial^n w}{\partial x_1 \partial x_2 \ldots \partial x_n} = e^w.$

Liouville type equation of the n th-order.

Solution:

$$w = \ln \frac{(-1)^n n! \prod\limits_{k=1}^{n} \alpha_k'(x_k)}{\left[\sum\limits_{k=1}^{n} \alpha_k(x_k)\right]^n},$$

where $\alpha_k(x_k)$ $(k = 1, \ldots, n)$ are arbitrary functions; a prime denotes a derivative with respect to the argument of the function.

⊙ *Reference*: A. N. Mironov (2010).

10. $\dfrac{\partial^{n+1} w}{\partial x^n \partial y} = f\left(x, \dfrac{\partial w}{\partial x}\right) e^{\lambda w}.$

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x, \varphi(y)\big) + \frac{1}{\lambda}\ln \varphi_y'(y),$$

where $\varphi(y)$ is an arbitrary function, is also a solution of the equation.

2°. For $f(x, w_x) = f(w_x)$, there exists a traveling-wave solution

$$w = w(z), \qquad z = k_1 x + k_2 y,$$

where k_1 and k_2 are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation $k_1^n k_2 w_\xi^{(n+1)} = f(k_1 w_z')e^{\lambda w}$.

11. $\dfrac{\partial^{k+1} w}{\partial x^k \partial t} = a(t) w \dfrac{\partial^{k+1} w}{\partial x^{k+1}} + F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \varphi(t), t) + \frac{\varphi_t'(t)}{a(t)},$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation for $k = 1, 2, \ldots$

2°. For $a = \text{const}$ and $F = F(w_x, w_{xx}, \ldots, w_x^{(n)})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$\lambda \beta^k U_z^{(k+1)} = a\beta^{k+1} U U_z^{(k+1)} + F\big(\beta U_z', \beta^2 U_{zz}'', \ldots, \beta^n U_z^{(n)}\big).$$

12. $\dfrac{\partial^2 w}{\partial x \partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t) \dfrac{\partial^2 w}{\partial y^2}.$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t), \qquad z = x + C_1 y + C_1^2 \int g(t)\, dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial z \partial t} = F\left(t, u, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}, \ldots, \frac{\partial^n u}{\partial z^n}\right).$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi, t), \qquad \xi = x + \varphi(t)(y + C_1)^2, \qquad \varphi(t) = -\left[4 \int g(t)\, dt + C_2\right]^{-1},$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial \xi \partial t} = F\left(t, U, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2}, \ldots, \frac{\partial^n U}{\partial \xi^n}\right) + 2g(t)\varphi(t) \frac{\partial U}{\partial \xi}.$$

15.4.2. Equations Involving $\dfrac{\partial^n w}{\partial x^n}$ and $\dfrac{\partial^m w}{\partial y^m}$

1. $a \dfrac{\partial^n w}{\partial x^n} + b \dfrac{\partial^n w}{\partial y^n} = (ay^n + bx^n) f(w).$

Solution:

$$w = w(z), \qquad z = xy,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$w_z^{(n)} = f(w).$$

Remark. This remains true if the constants a and b in the equation are replaced by arbitrary functions $a = a(x, y, w, w_x, w_y, \ldots)$ and $b = b(x, y, w, w_x, w_y, \ldots)$.

2. $F\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \ldots, \frac{1}{w}\frac{\partial^n w}{\partial x^n}; \frac{1}{w}\frac{\partial w}{\partial y}, \ldots, \frac{1}{w}\frac{\partial^m w}{\partial y^m}\right) = 0.$

Multiplicative separable solution:

$$w(x, y) = Ae^{\lambda y}\varphi(x),$$

where A and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\left(x, \varphi'_x/\varphi, \ldots, \varphi_x^{(n)}/\varphi; \lambda, \ldots, \lambda^m\right) = 0.$$

3. $F\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \ldots, \frac{1}{w}\frac{\partial^n w}{\partial x^n}; \frac{1}{w}\frac{\partial^2 w}{\partial y^2}, \ldots, \frac{1}{w}\frac{\partial^{2m} w}{\partial y^{2m}}\right) = 0.$

1°. Multiplicative separable solution:

$$w(x, y) = \left[A\cosh(\lambda y) + B\sinh(\lambda y)\right]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\left(x, \varphi'_x/\varphi, \ldots, \varphi_x^{(n)}/\varphi; \lambda^2, \ldots, \lambda^{2m}\right) = 0.$$

2°. Multiplicative separable solution:

$$w(x, y) = \left[A\cos(\lambda y) + B\sin(\lambda y)\right]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\left(x, \varphi'_x/\varphi, \ldots, \varphi_x^{(n)}/\varphi; -\lambda^2, \ldots, (-1)^m\lambda^{2m}\right) = 0.$$

4. $F_1\left(x, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right) + F_2\left(y, \frac{\partial w}{\partial y}, \ldots, \frac{\partial^m w}{\partial y^m}\right) = kw.$

Additive separable solution:

$$w(x, y) = \varphi(x) + \psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\left(x, \varphi'_x, \ldots, \varphi_x^{(n)}\right) - k\varphi = C,$$
$$F_2\left(y, \psi'_y, \ldots, \psi_y^{(m)}\right) - k\psi = -C,$$

where C is an arbitrary constant.

5. $F_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \ldots, \frac{1}{w}\frac{\partial^n w}{\partial x^n}\right) + w^k F_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}, \ldots, \frac{1}{w}\frac{\partial^m w}{\partial y^m}\right) = 0.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi^{-k}F_1\left(x, \varphi'_x/\varphi, \ldots, \varphi_x^{(n)}/\varphi\right) = C,$$
$$\psi^k F_2\left(y, \psi'_y/\psi, \ldots, \psi_y^{(m)}/\psi\right) = -C,$$

where C is an arbitrary constant.

6. $F_1\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + e^{\lambda w} F_2\left(y, \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Additive separable solution:

$$w(x, y) = \varphi(x) + \psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$e^{-\lambda\varphi} F_1\left(x, \varphi'_x, \ldots, \varphi_x^{(n)}\right) = C,$$
$$e^{\lambda\psi} F_2\left(y, \psi'_y, \ldots, \psi_y^{(m)}\right) = -C,$$

where C is an arbitrary constant.

7. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \ldots, \dfrac{1}{w}\dfrac{\partial^m w}{\partial y^m}\right) = k \ln w.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\left(x, \varphi'_x/\varphi, \ldots, \varphi_x^{(n)}/\varphi\right) - k \ln \varphi = C,$$
$$F_2\left(y, \psi'_y/\psi, \ldots, \psi_y^{(m)}/\psi\right) - k \ln \psi = -C,$$

where C is an arbitrary constant.

8. $F\left(ax + by,\ w,\ \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n},\ \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Solution:

$$w = w(\xi), \quad \xi = ax + by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi, \ldots, a^n w_\xi^{(n)},\ bw'_\xi, \ldots, b^m w_\xi^{(m)}\right) = 0.$$

9. $F\left(ax + by,\ \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n},\ \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Solution:

$$w = \varphi(\xi) + Cx, \quad \xi = ax + by,$$

where C is an arbitrary constant and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ a\varphi'_\xi + C,\ a^2\varphi''_{\xi\xi}, \ldots, a^n\varphi_\xi^{(n)},\ b\varphi'_\xi, \ldots, b^m\varphi_\xi^{(m)}\right) = 0.$$

10. $\dfrac{\partial^n}{\partial x^n}\left\{\left[a_1x+b_1y+f(w)\right]\dfrac{\partial^m w}{\partial x^m}\right\}+\dfrac{\partial^n}{\partial y^n}\left\{\left[a_2x+b_2y+g(w)\right]\dfrac{\partial^m w}{\partial y^m}\right\}=0.$

Solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By,$$

where the constants A and B are evaluated from the algebraic system of equations

$$\begin{aligned} a_1A^{n+m}+a_2B^{n+m}&=A,\\ b_1A^{n+m}+b_2B^{n+m}&=B. \end{aligned}$$

The desired function $w(z)$ is determined by the mth-order ordinary differential equation

$$\left[z+A^{n+m}f(w)+B^{n+m}g(w)\right]w_z^{(m)}=C_0+C_1z+\cdots+C_{n-1}z^{n-1},$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants.

11. $(a_1x+b_1y)\dfrac{\partial^n w}{\partial x^n}+(a_2x+b_2y)\dfrac{\partial^n w}{\partial y^n}=F\left(w,\dfrac{\partial w}{\partial x},\ldots,\dfrac{\partial^m w}{\partial x^m},\dfrac{\partial w}{\partial y},\ldots,\dfrac{\partial^k w}{\partial y^k}\right).$

Solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By,$$

where the constants A and B are evaluated from the algebraic system of equations

$$\begin{aligned} a_1A^n+a_2B^n&=A,\\ b_1A^n+b_2B^n&=B, \end{aligned}$$

and the desired function $w(z)$ is determined by the ordinary differential equation

$$zw_z^{(n)}=F\big(w,\,Aw_z',\ldots,A^mw_z^{(m)},\,Bw_z',\ldots,B^kw_z^{(k)}\big).$$

Remark. If the right-hand side of the equation is also dependent on mixed derivatives, solutions are constructed likewise.

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Chapter 16
Systems of Two First-Order Partial Differential Equations

16.1. Systems of the Form $\dfrac{\partial u}{\partial x} = F(u, w)$, $\dfrac{\partial w}{\partial t} = G(u, w)$

Preliminary remarks. Such systems of equations arise in the theory of chemical reactors, in studying fluid flows through porous media, and in chromatography.

The more general system

$$\frac{\partial u}{\partial \tau} + a_1 \frac{\partial u}{\partial \xi} = F(u, w), \qquad \frac{\partial w}{\partial \tau} + a_2 \frac{\partial w}{\partial \xi} = G(u, w),$$

which describes convective mass transfer in a two-component medium with a volume chemical reaction where the diffusion of both components can be neglected, is reduced to the system in question by changing from ξ and τ to the characteristic variables x and t defined by

$$x = \frac{\xi - a_2 \tau}{a_1 - a_2}, \qquad t = \frac{\xi - a_1 \tau}{a_2 - a_1} \qquad (a_1 \neq a_2).$$

If the first (resp., second) component is quiescent, then $a_1 = 0$ (resp., $a_2 = 0$).

This kind of system is invariant under translations in the independent variables and, hence, admit traveling-wave solutions: $u = u(kx - \lambda t)$ and $w = w(kx - \lambda t)$. Neither these solutions nor degenerate solutions, where one of the desired functions is identically zero or constant, are considered in what follows.

Below $f(\varphi)$, $g(\varphi)$, $h(\varphi)$, and $r(\varphi)$ are arbitrary functions of their argument, $\varphi = \varphi(u, w)$. The equations are listed in order of complexity of the argument.

16.1.1. Systems Involving Arbitrary Parameters

1. $\dfrac{\partial u}{\partial x} = auw, \qquad \dfrac{\partial w}{\partial t} = buw.$

Simplest model of a chemical reactor with second-order kinetic functions.

General solution:

$$u = -\frac{\psi_t'(t)}{a\varphi(x) + b\psi(t)}, \qquad w = -\frac{\varphi_x'(x)}{a\varphi(x) + b\psi(t)},$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions.

⊙ *References*: V. I. Pen'kovskii (1969), P. Ya. Polubarinova–Kochina (1977), V. S. Berman, L. A. Galin, and O. M. Churmaev (1979).

Remark. The cited papers also considered some initial-boundary value problems arising in studying flows through porous media and processes in chemical reactors described by the system of equations in question.

2. $\dfrac{\partial u}{\partial x} = au\sqrt{w}, \quad \dfrac{\partial w}{\partial t} = bu\sqrt{w}.$

This is a special case of equation 16.1.1.5 for $k = 1$, $n = 1/2$.

⊙ *References*: V. I. Pen'kovskii (1969), V. S. Berman, L. A. Galin, and O. M. Churmaev (1979).

3. $\dfrac{\partial u}{\partial x} = a\sqrt{uw}, \quad \dfrac{\partial w}{\partial t} = b\sqrt{uw}.$

This is a special case of equation 16.1.1.6 for $k = n = 1/2$.

⊙ *References*: V. I. Pen'kovskii (1969), V. S. Berman, L. A. Galin, and O. M. Churmaev (1979).

4. $\dfrac{\partial u}{\partial x} = auw^n, \quad \dfrac{\partial w}{\partial t} = bu^k w.$

General solution:

$$u = \left(\frac{-\psi_t'(t)}{bn\psi(t) - ak\varphi(x)}\right)^{1/k}, \quad w = \left(\frac{\varphi_x'(x)}{bn\psi(t) - ak\varphi(x)}\right)^{1/n},$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions.

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

5. $\dfrac{\partial u}{\partial x} = auw^n, \quad \dfrac{\partial w}{\partial t} = bu^k w^{1-n}.$

General solution:

$$w = \left\{\varphi(x) + E(x)\left[\psi(t) - \frac{1}{2}ak\int E(x)\,dx\right]^{-1}\right\}^{1/n},$$

$$u = \left(\frac{1}{b}w^{n-1}\frac{\partial w}{\partial t}\right)^{1/k}, \quad E(x) = \exp\left[ak\int \varphi(x)\,dx\right],$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions.

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006); the case $n = 1$ was considered.

6. $\dfrac{\partial u}{\partial x} = au^{1-k}w^n, \quad \dfrac{\partial w}{\partial t} = bu^k w^{1-n}.$

The transformation $U = u^k$, $W = w^n$ leads to a linear system of equations with constant coefficients:

$$\frac{\partial U}{\partial x} = akW, \quad \frac{\partial W}{\partial t} = bnU.$$

7. $\dfrac{\partial u}{\partial x} = \dfrac{cw^n u}{a + bw^n}, \quad \dfrac{\partial w}{\partial t} = (aw + bw^{n+1})u^k.$

General solution with $b \neq 0$:

$$w = \left[\psi(t)e^{F(x)} - be^{F(x)}\int e^{-F(x)}\varphi(x)\,dx\right]^{-1/n},$$

$$u = \left(\frac{w_t}{aw + bw^{n+1}}\right)^{1/k}, \quad F(x) = \frac{ck}{b}x - a\int \varphi(x)\,dx,$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions.

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

8. $\dfrac{\partial u}{\partial x} = w, \quad \dfrac{\partial w}{\partial t} = ae^{\lambda u}.$

Eliminating w gives the Liouville equation

$$\frac{\partial^2 u}{\partial x \partial t} = ae^{\lambda u},$$

which can be completely integrated (see Eq. 7.5.1.2).

⊙ *Reference*: V. S. Berman (1981).

9. $\dfrac{\partial u}{\partial x} = w, \quad \dfrac{\partial w}{\partial t} = a \sin(\lambda u).$

Eliminating w and changing the variable, $v = \lambda u$, one arrives at the sine-Gordon equation

$$\frac{\partial^2 v}{\partial x \partial t} = a\lambda \sin v,$$

which can be completely integrated (see Eq. 7.5.1.5).

⊙ *Reference*: V. S. Berman (1981).

16.1.2. Systems Involving Arbitrary Functions

1. $\dfrac{\partial u}{\partial x} = w, \quad \dfrac{\partial w}{\partial t} = f(u).$

Eliminating w gives an equation of the form 7.5.3.1:

$$\frac{\partial^2 u}{\partial x \partial t} = f(w).$$

⊙ *Reference*: V. S. Berman (1981).

2. $\dfrac{\partial u}{\partial x} = uf(w), \quad \dfrac{\partial w}{\partial t} = u^k g(w).$

1°. The transformation of the dependent variables

$$U = u^k, \quad W = \int \frac{dw}{g(w)} \tag{1}$$

leads to the simple system

$$\frac{\partial U}{\partial x} = \Phi(W)U, \quad \frac{\partial W}{\partial t} = U, \tag{2}$$

where the function $\Phi(W)$ is defined parametrically:

$$\Phi = kf(w), \quad W = \int \frac{dw}{g(w)}, \tag{3}$$

with w playing the role of the parameter. By replacing U in the first equation of system (2) with the left-hand side of the second equation, one arrives at a second-order equation for W:

$$\frac{\partial^2 W}{\partial x \partial t} = \Phi(W)\frac{\partial W}{\partial t}.$$

Integrating with respect to t gives

$$\frac{\partial W}{\partial x} = \int \Phi(W)\, dW + \theta(x), \tag{4}$$

where $\theta(x)$ is an arbitrary function.

In (4), returning to the original variable w using formulas (1) and (3), one obtains

$$\frac{\partial w}{\partial x} = kg(w)\int \frac{f(w)}{g(w)}\, dw + \theta(x)g(w). \tag{5}$$

The first integral (5) may be treated as a first-order ordinary differential equation in x. On finding its general solution, one should replace the constant of integration C with an arbitrary function of time $\psi(t)$, since w is dependent on x and t.

2°. To the special case $\theta(x) = \text{const}$ in (5) there correspond special solutions of the form

$$w = w(z), \quad u = [\psi'(t)]^{1/k} v(z), \quad z = x + \psi(t)$$

involving one arbitrary function $\psi(t)$, with the prime denoting a derivative. The functions $w(z)$ and $v(z)$ are described by the autonomous system of ordinary differential equations

$$v'_z = f(w)v,$$
$$w'_z = g(w)v^k,$$

the general solution of which can be written in implicit form as

$$\int \frac{dw}{g(w)[kF(w) + C_1]} = z + C_2, \quad v = [kF(w) + C_1]^{1/k}, \quad F(w) = \int \frac{f(w)}{g(w)}\, dw.$$

⊙ *Reference for system* 16.1.2.2: A. D. Polyanin and E. A. Vyazmina (2006).

3. $\dfrac{\partial u}{\partial x} = f(au + bw), \quad \dfrac{\partial w}{\partial t} = g(au + bw).$

Solution:

$$u = b(k_1 x - \lambda_1 t) + y(\xi), \quad w = -a(k_1 x - \lambda_1 t) + z(\xi), \quad \xi = k_2 x - \lambda_2 t,$$

where k_1, k_2, λ_1, and λ_2 are arbitrary constants, and the functions $y(\xi)$ and $z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$k_2 y'_\xi + bk_1 = f(ay + bz),$$
$$-\lambda_2 z'_\xi + a\lambda_1 = g(ay + bz).$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

4. $\dfrac{\partial u}{\partial x} = f(a_1 u + b_1 w), \quad \dfrac{\partial w}{\partial t} = g(a_2 u + b_2 w).$

Let $\Delta = a_1 b_2 - a_2 b_1 \neq 0$.

Additive separable solution:

$$u = \frac{1}{\Delta}[b_2\varphi(x) - b_1\psi(t)], \quad w = \frac{1}{\Delta}[a_1\psi(t) - a_2\varphi(x)],$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the autonomous ordinary differential equations

$$\frac{b_2}{\Delta}\varphi'_x = f(\varphi), \quad \frac{a_1}{\Delta}\psi'_t = g(\psi).$$

Integrating yields

$$\frac{b_2}{\Delta}\int \frac{d\varphi}{f(\varphi)} = x + C_1, \quad \frac{a_1}{\Delta}\int \frac{d\psi}{g(\psi)} = t + C_2.$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

5. $\dfrac{\partial u}{\partial x} = f(au - bw), \quad \dfrac{\partial w}{\partial t} = ug(au - bw) + wh(au - bw) + r(au - bw).$

Here, $f(z)$, $g(z)$, $h(z)$, and $r(z)$ are arbitrary functions.

Generalized separable solution:

$$u = \varphi(t) + b\theta(t)x, \quad w = \psi(t) + a\theta(t)x.$$

Here, the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\theta = \theta(t)$ are determined by a system involving one algebraic (transcendental) and two ordinary differential equations:

$$\begin{aligned} b\theta &= f(a\varphi - b\psi),\\ a\theta'_t &= b\theta g(a\varphi - b\psi) + a\theta h(a\varphi - b\psi),\\ \psi'_t &= \varphi g(a\varphi - b\psi) + \psi h(a\varphi - b\psi) + r(a\varphi - b\psi). \end{aligned}$$

6. $\dfrac{\partial u}{\partial x} = f(au - bw) + cw, \quad \dfrac{\partial w}{\partial t} = ug(au - bw) + wh(au - bw) + r(au - bw).$

Here, $f(z)$, $g(z)$, $h(z)$, and $r(z)$ are arbitrary functions.

Generalized separable solution:

$$u = \varphi(t) + b\theta(t)e^{\lambda x}, \quad w = \psi(t) + a\theta(t)e^{\lambda x}, \quad \lambda = \frac{ac}{b}.$$

Here, the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\theta = \theta(t)$ are determined by a system involving one algebraic (transcendental) and two ordinary differential equations:

$$\begin{aligned} &f(a\varphi - b\psi) + c\psi = 0,\\ &\psi'_t = \varphi g(a\varphi - b\psi) + \psi h(a\varphi - b\psi) + r(a\varphi - b\psi),\\ &a\theta'_t = b\theta g(a\varphi - b\psi) + a\theta h(a\varphi - b\psi). \end{aligned}$$

7. $\dfrac{\partial u}{\partial x} = u^k f(u^n w^m), \quad \dfrac{\partial w}{\partial t} = w^s g(u^n w^m).$

Self-similar solution with $s \neq 1$ and $n \neq 0$:

$$u = t^{\frac{m}{n(s-1)}} y(\xi), \quad w = t^{-\frac{1}{s-1}} z(\xi), \quad \xi = x t^{\frac{m(k-1)}{n(s-1)}},$$

where the functions $y(\xi)$ and $z(\xi)$ are determined by the system of ordinary differential equations

$$y'_\xi = y^k f(y^n z^m),$$
$$m(k-1)\xi z'_\xi - nz = n(s-1) z^s g(y^n z^m).$$

8. $\dfrac{\partial u}{\partial x} = u^k f(u^n w^m), \quad \dfrac{\partial w}{\partial t} = w g(u^n w^m).$

1°. Solution:

$$u = e^{mt} y(\xi), \quad w = e^{-nt} z(\xi), \quad \xi = e^{m(k-1)t} x,$$

where the functions $y(\xi)$ and $z(\xi)$ are determined by the system of ordinary differential equations

$$y'_\xi = y^k f(y^n z^m),$$
$$m(k-1)\xi z'_\xi - nz = z g(y^n z^m).$$

2°. If $k \neq 1$, it is more beneficial to seek solutions in the form

$$u = x^{-\frac{1}{k-1}} \varphi(\zeta), \quad w = x^{\frac{n}{m(k-1)}} \psi(\zeta), \quad \zeta = t + a \ln|x|,$$

where a is an arbitrary constant, and the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are determined by the autonomous system of ordinary differential equations

$$a\varphi'_\zeta + \frac{1}{1-k}\varphi = \varphi^k f(\varphi^n \psi^m),$$
$$\psi'_\zeta = \psi g(\varphi^n \psi^m).$$

9. $\dfrac{\partial u}{\partial x} = u f(u^n w^m), \quad \dfrac{\partial w}{\partial t} = w g(u^n w^m).$

Solution:

$$u = e^{m(kx-\lambda t)} y(\xi), \quad w = e^{-n(kx-\lambda t)} z(\xi), \quad \xi = \alpha x - \beta t,$$

where k, α, β, and λ are arbitrary constants, and the functions $y(\xi)$ and $z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$\alpha y'_\xi + kmy = y f(y^n z^m),$$
$$-\beta z'_\xi + n\lambda z = z g(y^n z^m).$$

10. $\dfrac{\partial u}{\partial x} = uf(u^n w^m), \quad \dfrac{\partial w}{\partial t} = wg(u^k w^s).$

Let $\Delta = sn - km \neq 0$.

Multiplicative separable solutions:

$$u = \big[\varphi(x)\big]^{s/\Delta}\big[\psi(t)\big]^{-m/\Delta}, \quad w = \big[\varphi(x)\big]^{-k/\Delta}\big[\psi(t)\big]^{n/\Delta},$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the autonomous ordinary differential equations

$$\frac{s}{\Delta}\varphi'_x = \varphi f(\varphi), \quad \frac{n}{\Delta}\psi'_t = \psi g(\psi).$$

Integrating yields

$$\frac{s}{\Delta}\int \frac{d\varphi}{\varphi f(\varphi)} = x + C_1, \quad \frac{n}{\Delta}\int \frac{d\psi}{\psi g(\psi)} = t + C_2.$$

11. $\dfrac{\partial u}{\partial x} = uf(au^k + bw), \quad \dfrac{\partial w}{\partial t} = u^k.$

Solution:

$$w = \varphi(x) + C\exp\left[-\lambda t + k\int f(b\varphi(x))\,dx\right], \quad u = \left(\frac{\partial w}{\partial t}\right)^{1/k}, \quad \lambda = \frac{b}{a},$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

12. $\dfrac{\partial u}{\partial x} = uf(au^n + bw), \quad \dfrac{\partial w}{\partial t} = u^k g(au^n + bw).$

Solution:

$$u = (C_1 t + C_2)^{\frac{1}{n-k}}\theta(x), \quad w = \varphi(x) - \frac{a}{b}(C_1 t + C_2)^{\frac{n}{n-k}}[\theta(x)]^n,$$

where C_1 and C_2 are arbitrary constants, and the functions $\theta = \theta(x)$ and $\varphi = \varphi(x)$ are determined by the system of differential-algebraic equations

$$\theta'_x = \theta f(b\varphi),$$
$$\theta^{n-k} = \frac{b(k-n)}{aC_1 n}g(b\varphi).$$

13. $\dfrac{\partial u}{\partial x} = e^{\lambda u}f(\lambda u - \sigma w), \quad \dfrac{\partial w}{\partial t} = e^{\sigma w}g(\lambda u - \sigma w).$

Solutions:

$$u = y(\xi) - \frac{1}{\lambda}\ln(C_1 t + C_2), \quad w = z(\xi) - \frac{1}{\sigma}\ln(C_1 t + C_2), \quad \xi = \frac{x + C_3}{C_1 t + C_2},$$

where the functions $y(\xi)$ and $z(\xi)$ are determined by the system of ordinary differential equations

$$y'_\xi = e^{\lambda y}f(\lambda y - \sigma z),$$
$$-C_1\xi z'_\xi - \frac{C_1}{\sigma} = e^{\sigma z}g(\lambda y - \sigma z).$$

14. $\dfrac{\partial u}{\partial x} = au\ln u + uf(u^n w^m), \qquad \dfrac{\partial w}{\partial t} = wg(u^n w^m).$

Solution:

$$u = \exp\bigl(Cme^{ax}\bigr)y(\xi), \quad w = \exp\bigl(-Cne^{ax}\bigr)z(\xi), \quad \xi = kx - \lambda t,$$

where C, k, and λ are arbitrary constants, and the functions $y(\xi)$ and $z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$ky'_\xi = ay\ln y + yf(y^n z^m),$$
$$-\lambda z'_\xi = zg(y^n z^m).$$

16.2. Other Systems of Two Equations

16.2.1. Gas Dynamic Type Systems Linearizable with the Hodograph Transformation

1. $\dfrac{\partial u}{\partial y} = \dfrac{\partial w}{\partial x}, \qquad \dfrac{\partial w}{\partial y} = -u\dfrac{\partial u}{\partial x}.$

Equations of a stationary transonic plane-parallel gas flow. This is a special case of system 16.2.1.6, in which t must be substituted by y.

1°. Solutions in implicit form:

$$x = 3(w + A)[4u^3 + 9(w + A)^2]^{-5/6} + B, \quad y = -2u[4u^3 + 9(w + A)^2]^{-5/6} + C;$$
$$x = 2u^2[4u^3 + 9(w + A)^2]^{-7/6} + B, \qquad y = 3(w + A)[4u^3 + 9(w + A)^2]^{-7/6} + C,$$

where A, B, and C are arbitrary constants.

2°. Solutions:

$$u = -\frac{(x + A)^2}{(y + B)^2}, \qquad w = \frac{2}{3}\frac{(x + A)^3}{(y + B)^3} + C,$$
$$u = -\frac{1}{A^2}x^2\wp(Ay + B), \quad w = -\frac{1}{3A}x^3\wp'(Ay + B),$$

where A, B, and C are arbitrary constants, $\wp(z) = \wp(z, 0, 4)$ is the Weierstrass elliptic function; the prime designates the derivative with respect to the argument.

3°. Solution:

$$u = u(z), \quad w = -C\ln y - \int zu'_z(z)\,dz, \quad z = \frac{x}{y},$$

where C is an arbitrary constant and u is determined by the ordinary differential equation $(z^2 + u)u'_z = C$.

4°. The system is linearized using the hodograph transformation

$$\frac{\partial y}{\partial u} - \frac{\partial x}{\partial w} = 0, \qquad \frac{\partial x}{\partial u} + u\frac{\partial y}{\partial w} = 0, \tag{1}$$

where u and w are treated as the dependent variables and x and y as the independent variables.

Eliminating x from (1) gives

$$\frac{\partial^2 y}{\partial w^2} + \frac{1}{u}\frac{\partial^2 y}{\partial u^2} = 0. \tag{2}$$

This equation admits polynomial solutions in w: $y = \sum_{k=0}^{n} \varphi_k(u) w^k$ [for details, see Polyanin (2002, p. 337)]. Some polynomial solutions of equation (2) and the corresponding solutions of system (1) are listed below:

$$\begin{aligned}
&(a)\quad y = C_1 uw + C_2 u + C_3 w + C_4,\\
&\qquad x = C_1(\tfrac{1}{2}w^2 - \tfrac{1}{3}u^3) + C_2(w - \tfrac{1}{2}u^2) + C_5;\\
&(b)\quad y = C_1(w^2 - \tfrac{1}{3}u^3) + C_2 uw + C_3 u + C_4 w + C_5,\\
&\qquad x = -C_1 u^2 w + C_2(\tfrac{1}{2}w^2 - \tfrac{1}{3}u^3) + C_3 w - \tfrac{1}{2}C_4 u^2 + C_6;\\
&(c)\quad y = C_1(w^3 - u^3 w) + C_2 uw + C_3 u + C_4 w + C_5,\\
&\qquad x = C_1(\tfrac{1}{5}u^5 - \tfrac{3}{2}u^2 w^2) + C_2(\tfrac{1}{2}w^2 - \tfrac{1}{3}u^3) + C_3 w - \tfrac{1}{2}C_4 u^2 + C_6,
\end{aligned}$$

where $C_1, \ldots, C_6$ are arbitrary constants.

⊙ *Reference for system* 16.2.1.1, Items 1°–3°: L. V. Ovsiannikov (1962).

2. $\dfrac{\partial u}{\partial t} - \dfrac{\partial w}{\partial x} = 0, \quad \dfrac{\partial w}{\partial t} - \dfrac{\partial \sigma(u)}{\partial x} = 0.$

This is a special case of system 16.2.1.6. The system describes nonlinear one-dimensional longitudinal vibrations of an elastic rod, with u denoting the deformation gradient (strain), w the rate of deformation (strain rate), and $\sigma(u)$ the stress. The condition $\sigma'(u) > 0$ expresses the hyperbolicity of the system, with the prime denoting the derivative with respect to u.

1°. Let $u = u(x,t)$, $w = w(x,t)$ be a solution of the system in question. Then the pair of functions

$$u_1 = u(C_1 x + C_2,\, C_1 t + C_3), \qquad w_1 = w(C_1 x + C_2,\, C_1 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the system.

2°. Trivial solutions:

$$u = C_1, \qquad w = C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solutions dependent on the ratio x/t of the independent variables:

$$\begin{aligned}
&w - \int \sqrt{\sigma'(u)}\, du = C_1, \quad \sqrt{\sigma'(u)} = -\frac{x}{t};\\
&w + \int \sqrt{\sigma'(u)}\, du = C_2, \quad \sqrt{\sigma'(u)} = \frac{x}{t},
\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

4°. Solutions in implicit form:

$$w - \int \sqrt{\sigma'(u)}\,du = C_1, \quad x + t\sqrt{\sigma'(u)} = \Phi_1(u);$$
$$w + \int \sqrt{\sigma'(u)}\,du = C_2, \quad x - t\sqrt{\sigma'(u)} = \Phi_2(u),$$

where $\Phi_m(u)$ are arbitrary functions and C_m are arbitrary constants ($m = 1, 2$). These solutions describe Riemann simple waves and are characterized by a functional relationship between the unknowns, $u = u(w)$. In the special cases $\Phi_m(w) \equiv 0$, these formulas become self-similar solutions from Item 3°.

5°. The system can be linearized with the hodograph transformation (for details see Item 5° of system 16.2.1.6):

$$\frac{\partial t}{\partial u} - \frac{\partial x}{\partial w} = 0, \qquad \frac{\partial x}{\partial u} - \sigma'(u)\frac{\partial t}{\partial w} = 0, \tag{1}$$

where u and w are treated as the independent variables and x and t as the dependent variables.

Eliminating x from (1) gives

$$\frac{\partial^2 t}{\partial w^2} = \frac{1}{\sigma'(u)}\frac{\partial^2 t}{\partial u^2}. \tag{2}$$

This equation admits polynomial solutions in w: $y = \sum_{k=0}^{n} \varphi_k(u)w^k$ [for details, see Polyanin (2002, p. 337)]. Some polynomial solutions of equation (2) and the corresponding solutions of system (1) are listed below:

(a) $t = C_1uw + C_2u + C_3w + C_4,$
$x = \frac{1}{2}C_1w^2 + C_2w + (C_1u + C_3)\sigma(u) - C_1\int \sigma(u)\,du + C_5;$

(b) $t = C_1w^2 + C_2uw + C_3w + C_4u + 2C_1\int \sigma(u)\,du + C_5,$
$x = 2C_1\sigma(u)w + \frac{1}{2}C_2w^2 + C_4w + (C_2u + C_3)\sigma(u) - C_2\int \sigma(u)\,du + C_6;$

(c) $t = C_1w^3 + C_2uw + C_3u + C_4w + 6C_1w\int \sigma(u)\,du + C_5,$
$x = 3C_1w^2\sigma(u) + \frac{1}{2}C_2w^2 + C_3w + \int \Big[C_2u + C_4 + 6C_1\int \sigma(u)\,du\Big]\sigma'_u(u)\,du + C_6,$

where $C_1, \ldots, C_6$ are arbitrary constants.

6°. Eliminating w from the system leads to an equation of the form 7.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\sigma'(u)\frac{\partial u}{\partial x}\right].$$

3. $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + g\dfrac{\partial w}{\partial x} = 0, \qquad \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial u}{\partial x} = 0.$

Shallow water equations. This is a special case of equation 16.2.1.4 with $\rho = w$, $n = 2$, and $A = \frac{1}{2}g$, where u is the horizontal velocity averaged over the height w of the water level and g is the acceleration due to gravity.

4. $\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+An\rho^{n-2}\dfrac{\partial \rho}{\partial x}=0, \qquad \dfrac{\partial \rho}{\partial t}+\rho\dfrac{\partial u}{\partial x}+u\dfrac{\partial \rho}{\partial x}=0.$

This is a special case of system 16.2.1.5 with $p(\rho)=A\rho^n+B$. It describes *one-dimensional polytropic ideal gas flow*, where u is the gas velocity and ρ is the gas density.

For $n\neq 1$, the system is often written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2}{n-1}c\frac{\partial c}{\partial x}=0, \qquad \frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+\frac{n-1}{2}c\frac{\partial u}{\partial x}=0, \tag{1}$$

where $c=\sqrt{p'(\rho)}=\sqrt{An\rho^{n-1}}$ is the sound speed.

1°. Let $u=u(x,t)$, $\rho=\rho(x,t)$ be a solution of the system in question. Then the pair of functions

$$u_1=B_1^{n-1}u(B_1^{1-n}B_2x+B_1^{1-n}B_2B_3t+B_4,\ B_2t+B_5)-B_3,$$
$$\rho_1=B_1^2\rho(B_1^{1-n}B_2x+B_1^{1-n}B_2B_3t+B_4,\ B_2t+B_5),$$

where $B_1,\ldots,B_5$ are arbitrary constants, is also a solution of the system.

2°. Trivial solutions:

$$u=B_1, \qquad \rho=B_2,$$

where B_1, B_2 are arbitrary constants.

3°. Self-similar solutions dependent on the ratio of the dependent variables x/t:

$$u=\frac{2}{n+1}\frac{x}{t}+B_1, \quad c=\frac{n-1}{n+1}\frac{x}{t}-B_1, \quad c=\sqrt{An\rho^{n-1}};$$
$$u=\frac{2}{n+1}\frac{x}{t}+B_2, \quad c=B_2-\frac{n-1}{n+1}\frac{x}{t}, \quad c=\sqrt{An\rho^{n-1}},$$

where B_1, B_2 are arbitrary constants.

Remark. The solutions from Items 2° and 3°, which are appropriately "glued" together along the straight lines $x/t=$ const, allow one to construct solutions to many gas dynamics problems (see the references cited below).

4°. More general self-similar solutions:

$$u=t^{k(1-n)}U(z), \qquad \rho=t^{-2k}R(z), \qquad z=t^{nk-k-1}x,$$

where the functions $U(z)$ and $R(z)$ are determined by the system of ordinary differential equations

$$k(1-n)U+(nk-k-1)zU_z'+UU_z'+AnR^{n-2}R_z'=0,$$
$$-2kR+(nk-k-1)zR_z'+RU_z'+UR_z'=0.$$

5°. Solutions in implicit form:

$$u=\frac{2}{n-1}c+B_1, \qquad x-t\left(\frac{n+1}{n-1}c+B_1\right)=\Phi_1(\rho), \quad c=\sqrt{An\rho^{n-1}};$$
$$u=-\frac{2}{n-1}c-B_2, \quad x+t\left(\frac{n+1}{n-1}c+B_2\right)=\Phi_2(\rho), \quad c=\sqrt{An\rho^{n-1}},$$

where $\Phi_m(\rho)$ are arbitrary functions, B_m are arbitrary constants ($m=1,\ 2$). These solutions describe *Riemann simple waves* and are characterized by a functional relationship between the unknowns, $u=u(\rho)$. In the special cases $\Phi_m(\rho)\equiv 0$, these formulas become self-similar solutions from Item 3°.

Special case. For $n = 3$, the general solution of system (1) is expressed implicitly by the formulas

$$x = (u + c)t + F_1(u + c),$$
$$x = (u - c)t + F_2(u - c),$$

where $F_1(z_1)$ and $F_2(z_2)$ are arbitrary functions.

6°. The system can be linearized with the hodograph transformation (for details, see Item 5° of system 16.2.1.6 with $\rho \equiv w$):

$$u\frac{\partial t}{\partial \rho} - \frac{\partial x}{\partial \rho} - An\rho^{n-2}\frac{\partial t}{\partial u} = 0, \qquad \rho\frac{\partial t}{\partial \rho} - u\frac{\partial t}{\partial u} + \frac{\partial x}{\partial u} = 0. \tag{2}$$

Here u and ρ are taken to be the independent variables and x and t to be the dependent variables. For arbitrary n, the general solution of the system obtained can be expressed in terms of Gauss's hypergeometric function. There exist finitely many values of the exponent $n = \dfrac{2m+3}{2m+1}$, where $m = -1, 0, 1, 2, \ldots$, for which the general solution of system (2) can be represented in closed form.

7°. The system can be reduced to the canonical (characteristic) form

$$\frac{\partial \mathcal{R}_1}{\partial t} + (u + c)\frac{\partial \mathcal{R}_1}{\partial x} = 0, \qquad \frac{\partial \mathcal{R}_2}{\partial t} + (u - c)\frac{\partial \mathcal{R}_2}{\partial x} = 0,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are Riemann invariants,

$$\mathcal{R}_1 = u + \frac{2}{n-1}c, \qquad \mathcal{R}_2 = u - \frac{2}{n-1}c, \qquad c = \sqrt{p'(\rho)}.$$

8°. See also solutions of system 16.2.1.5 with $p(\rho) = A\rho^n$ (Items 3° and 5°).

9°. The system can be rewritten in the form of conservation laws (divergence form):

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2 + A\rho^n)}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

These equations are used for deriving the *Rankine–Hugoniot conditions*, which form the basis for constructing discontinuous solutions that describe *shock waves*. Let $x_f(t)$ denote the position of the front of a shock wave depending on time t. Then the shock wave speed is determined by the time derivative, $D = x'_f$, and the Rankine–Hugoniot conditions are expressed as

$$[\rho u]D = [\rho u^2 + A\rho^n],$$
$$[\rho]D = [\rho u].$$

Here $[A] = A^+ - A^-$ stands for the jump of the quantity A at the shock wave; the plus and minus superscripts indicate quantities ahead of (upstream) and behind (downstream) the shock. Solutions of various gas dynamics problems with shock waves can be found in the references cited below.

⊙ *References for system* 16.2.1.4: R. Courant (1964), K. P. Stanyukovich (1971), L. V. Ovsiannikov (1981), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. Courant and R. Friedrichs (1985), G. G. Chernyi (1988), I. Sh. Akhatov, R. K. Gazizov, N. Kh. Ibragimov (1989, pages 35–78).

5. $\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=0, \quad \dfrac{\partial \rho}{\partial t}+\rho\dfrac{\partial u}{\partial x}+u\dfrac{\partial \rho}{\partial x}=0, \quad p=p(\rho).$

This is a special case of system 16.2.1.6 with $\rho \equiv w$. The system describes *one-dimensional barotropic flows of an ideal compressible gas*, with u denoting the gas velocity, ρ the gas density, and $p(\rho)$ the pressure. The speed of sound is given by $c=\sqrt{p'(\rho)}$, where the prime stands for the derivative; the $c>0$ expresses the hyperbolicity of the system.

1°. Let $u=u(x,t)$, $\rho=\rho(x,t)$ be a solution of the system in question. Then the pair of functions

$$u_1=u(C_1x+C_1C_2t+C_3,\ C_1t+C_4)-C_2, \quad \rho_1=\rho(C_1x+C_1C_2t+C_3,\ C_1t+C_4),$$

where $C_1,\ldots,C_4$ are arbitrary constants, is also a solution of the system.

2°. Trivial solutions:

$$u=C_1, \quad \rho=C_2,$$

where C_1, C_2 are arbitrary constants.

3°. Solution:

$$u=\frac{x+C_1}{t+C_2}, \quad \rho=\frac{C_3}{t+C_2},$$

where C_1, C_2, and C_3 are arbitrary constants.

4°. Self-similar solutions, dependent on the ratio of the dependent variables x/t, in implicit form:

$$u=\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+A_1, \quad \int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+\sqrt{p'(\rho)}+A_1=\frac{x}{t};$$
$$u=-\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}-A_2, \quad \int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+\sqrt{p'(\rho)}+A_2=-\frac{x}{t},$$

where A_1 and A_2 are arbitrary constants.

Remark. The solutions from Items 2° and 4°, which are appropriately "glued" together along the straight lines $x/t=$ const, allow one to construct solutions to many gas dynamics problems and, in particular, make it possible to obtain solutions to the discontinuity decay problem (see the references cited below).

5°. Solution:

$$u=C_1t+C_2+\theta(z), \quad \rho=\frac{C_3}{\theta(z)}, \quad z=x-\frac{1}{2}C_1t^2-C_2t,$$

where the function $\theta=\theta(z)$ is defined implicitly by

$$C_1z+\frac{1}{2}\theta^2-\int f\left(\frac{C_3}{\theta}\right)\frac{d\theta}{\theta}=C_4, \quad f(\rho)=p'(\rho),$$

and C_1, C_2, C_3, and C_4 are arbitrary constants.

6°. Solutions in implicit form:

$$u=\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+A_1, \quad x-t\left[\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+\sqrt{p'(\rho)}+A_1\right]=\Phi_1(\rho);$$
$$u=-\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}-A_2, \quad x+t\left[\int\sqrt{p'(\rho)}\,\frac{d\rho}{\rho}+\sqrt{p'(\rho)}+A_2\right]=\Phi_2(\rho),$$

where $\Phi_m(\rho)$ are arbitrary functions and A_m are arbitrary constants ($m=1,2$). These solutions describe *Riemann simple waves* and are characterized by a functional relationship between the unknowns, $u=u(\rho)$. In the special cases $\Phi_m(\rho)\equiv 0$, these formulas determine the self-similar solutions from Item 4°.

7°. The system can be linearized with the hodograph transformation (for details, see Item 5° of system 16.2.1.6 with $\rho \equiv w$):

$$u\frac{\partial t}{\partial \rho} - \frac{\partial x}{\partial \rho} - \frac{p'(\rho)}{\rho}\frac{\partial t}{\partial u} = 0, \qquad \rho\frac{\partial t}{\partial \rho} - u\frac{\partial t}{\partial u} + \frac{\partial x}{\partial u} = 0.$$

Here u and ρ are taken to be the independent variables and x and t to be the dependent ones.

8°. The system can be reduced to the canonical (characteristic) form

$$\frac{\partial \mathcal{R}_1}{\partial t} + (u+c)\frac{\partial \mathcal{R}_1}{\partial x} = 0, \qquad \frac{\partial \mathcal{R}_2}{\partial t} + (u-c)\frac{\partial \mathcal{R}_2}{\partial x} = 0,$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are Riemann invariants

$$\mathcal{R}_1 = u + \int \frac{c}{\rho}\,d\rho, \qquad \mathcal{R}_2 = u - \int \frac{c}{\rho}\,d\rho, \qquad c = \sqrt{p'(\rho)}.$$

9°. The system can be represented as conservation laws (in divergence form):

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial[\rho u^2 + p(\rho)]}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

These equations are used for deriving the *Rankine–Hugoniot conditions*, which form the basis for constructing discontinuous solutions that describe *shock waves*. Let $x_f(t)$ denote the position of the front of a shock wave depending on time t. Then the shock wave speed is determined by the time derivative, $D = x'_f$, and the Rankine–Hugoniot conditions are expressed as

$$[\rho u]D = [\rho u^2 + A\rho^n],$$
$$[\rho]D = [\rho u].$$

Here $[A] = A^+ - A^-$ stands for the jump of the quantity A at the shock wave; the plus and minus superscripts indicate quantities ahead of (upstream) and behind (downstream) the shock. Solutions of various gas dynamics problems with shock waves can be found in the references cited below.

⊙ *References for system* 16.2.1.5: R. Courant (1964), K. P. Stanyukovich (1971), L. V. Ovsiannikov (1981), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. Courant and R. Friedrichs (1985), G. G. Chernyi (1988), I. Sh. Akhatov, R. K. Gazizov, N. Kh. Ibragimov (1989, pages 35–78).

6. $$\boldsymbol{f_1(u,w)\frac{\partial u}{\partial t} + g_1(u,w)\frac{\partial w}{\partial t} + h_1(u,w)\frac{\partial u}{\partial x} + k_1(u,w)\frac{\partial w}{\partial x} = 0,}$$
$$\boldsymbol{f_2(u,w)\frac{\partial u}{\partial t} + g_2(u,w)\frac{\partial w}{\partial t} + h_2(u,w)\frac{\partial u}{\partial x} + k_2(u,w)\frac{\partial w}{\partial x} = 0.}$$

1°. Let $u = u(x,t)$, $w = w(x,t)$ be a solution of the system. Then the pair of functions

$$u_1 = u(C_1x + C_2,\ C_1t + C_3), \qquad w_1 = w(C_1x + C_2,\ C_1t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the system.

2°. For any two functions $f_k(u,w)$ and $g_k(u,w)$, the system in question admits the trivial solutions

$$u = C_1, \qquad w = C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. The system admits self-similar solutions of the form

$$u = u(\xi), \qquad w = w(\xi), \qquad \xi = \frac{x}{t}.$$

4°. We look for exact solutions that would generalize the solutions from Item 3° and are characterized by a functional relationship between the unknowns:

$$w = w(u). \tag{1}$$

Substituting (1) into the system, one obtains two equations for the function $u = u(x, t)$:

$$\begin{aligned}(f_1 + g_1 w'_u)\frac{\partial u}{\partial t} + (h_1 + k_1 w'_u)\frac{\partial u}{\partial x} &= 0,\\ (f_2 + g_2 w'_u)\frac{\partial u}{\partial t} + (h_2 + k_2 w'_u)\frac{\partial u}{\partial x} &= 0.\end{aligned} \tag{2}$$

The relationship (1) must be chosen so that the equations of (2) are consistent. This condition results in the following nonlinear first-order ordinary differential equation for $w(u)$:

$$(g_1 k_2 - g_2 k_1)(w'_u)^2 + (f_1 k_2 + g_1 h_2 - f_2 k_1 - g_2 h_1)w'_u + f_1 h_2 - f_2 h_1 = 0. \tag{3}$$

Treating (3) as a quadratic equation for the derivative w'_u, let us require that its discriminant is positive (which corresponds to the hyperbolicity condition of the system):

$$(f_1 k_2 + g_1 h_2 - f_2 k_1 - g_2 h_1)^2 - 4(f_1 h_2 - f_2 h_1)(g_1 k_2 - g_2 k_1) > 0. \tag{4}$$

In this case, equation (3) has two real distinct roots and is equivalent to two different first-order ordinary differential equations solved for the derivative:

$$w'_u = \Lambda_m(u, w), \qquad m = 1, 2. \tag{5}$$

Having determined a solution of this equation (for each $m = 1, 2$, we have a different solution), $w = w(u)$, we substitute it into either equation of (2) to obtain a quasilinear first-order partial differential equation for $u = u(x, t)$:

$$(f_1 + g_1\Lambda_m)\frac{\partial u}{\partial t} + (h_1 + k_1\Lambda_m)\frac{\partial u}{\partial x} = 0, \qquad w = w(u). \tag{6}$$

The general solution of this equation can be obtained by the method of characteristics (see Subsection 24.3.5).

Solutions of equations (5) and (6), with $m = 1, 2$, that depend on an arbitrary function and an arbitrary constant are called *Riemann simple waves*.

5°. Let us perform the *hodograph transformation*

$$x = x(u, w), \qquad t = t(u, w), \tag{7}$$

so that u and w are treated as the independent variables and x and t as the dependent ones. Differentiating the relations of (7) with respect to x and t (as composite functions) and eliminating the partial derivatives u_t, w_t, u_x, and w_x from the resulting equations, we obtain

$$\frac{\partial u}{\partial t} = -J\frac{\partial x}{\partial w}, \quad \frac{\partial w}{\partial t} = J\frac{\partial x}{\partial u}, \quad \frac{\partial u}{\partial x} = J\frac{\partial t}{\partial w}, \quad \frac{\partial w}{\partial x} = -J\frac{\partial t}{\partial u}, \tag{8}$$

where $J = \frac{\partial u}{\partial x}\frac{\partial w}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial w}{\partial x}$ is the Jacobian of the functions $u = u(x,t)$ and $w = w(x,t)$. Replacing in the original system the derivatives by using (8) and dividing by J, we arrive at the linear system of equations

$$g_1(u,w)\frac{\partial x}{\partial u} - k_1(u,w)\frac{\partial t}{\partial u} - f_1(u,w)\frac{\partial x}{\partial w} + h_1(u,w)\frac{\partial t}{\partial w} = 0,$$
$$g_2(u,w)\frac{\partial x}{\partial u} - k_2(u,w)\frac{\partial t}{\partial u} - f_2(u,w)\frac{\partial x}{\partial w} + h_2(u,w)\frac{\partial t}{\partial w} = 0.$$

Remark. The hodograph transformation (7) is inapplicable if $J \equiv 0$. In this degenerate case, u and w will be functionally dependent and, hence, can be used as independent variables. In this case, relation (1), which determines Riemann simple waves, holds. Consequently, the use of the hodograph transformation (7) results in the loss of solutions (1), describing Riemann simple waves.

⊙ *References for system* 16.2.1.6: R. Courant (1964), K. P. Stanyukovich (1971), L. V. Ovsiannikov (1981), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. Courant and R. Friedrichs (1985), G. G. Chernyi (1988), A. D. Polyanin, A. V. Manzhirov (2007).

7. $$f_1(t,w)\frac{\partial w}{\partial t} + [f_2(t,w) + f_3(t,w)x + f_4(t,w)v]\frac{\partial w}{\partial x} + f_5(t,w)\frac{\partial v}{\partial x} = f_6(t,w),$$
$$g_1(t,w)\frac{\partial w}{\partial t} + [g_2(t,w) + g_3(t,w)x + g_4(t,w)v]\frac{\partial w}{\partial x} + g_5(t,w)\frac{\partial v}{\partial x} = g_6(t,w).$$

Consider the nonclassical hodograph transformation

$$x = x(w,t), \qquad v = v(w,t), \tag{1}$$

where w, t are treated as the independent variables and x, v as the dependent variables (only one of the independent variables is taken to be the dependent one). The "old" derivatives are expressed through the "new" ones as follows:

$$w_x = \frac{1}{x_w}, \quad w_t = -\frac{x_t}{x_w}, \quad v_x = \frac{v_w}{x_w}, \quad v_t = v_t - \frac{x_t}{x_w}v_w. \tag{2}$$

The nonclassical hodograph transformation (1)–(2) reduced the original nonlinear system to the linear system

$$f_1(t,w)\frac{\partial x}{\partial t} + f_6(t,w)\frac{\partial x}{\partial w} - f_5(t,w)\frac{\partial v}{\partial w} = f_2(t,w) + f_3(t,w)x + f_4(t,w)v,$$
$$g_1(t,w)\frac{\partial x}{\partial t} + g_6(t,w)\frac{\partial x}{\partial w} - g_5(t,w)\frac{\partial v}{\partial w} = g_2(t,w) + g_3(t,w)x + g_4(t,w)v.$$

⊙ *Reference*: H. L. Lei (2002).

16.2.2. Other Systems of Equations

1. $$\frac{\partial (u+w)}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad \frac{\partial w}{\partial t} = f(w)u.$$

This system describes deep filtration of a one-component suspension of particles in a porous medium taking into account the permeability variation (due to particle capture by the porous medium). The first equation of the system is the mass balance for the accumulated particles and the suspension. The second equation describes the accumulation kinetics; u is the suspension concentration, w is the concentration of the accumulated substance (sediment), and $f(w)$ is the filtration coefficient.

1°. Replacing $\frac{\partial w}{\partial t}$ in the first equation of the system with the right-hand side of the second equation and changing from x and t to the new, characteristic variable $z = -x$ and $\eta = x - t$, one arrives at the system

$$\frac{\partial u}{\partial z} = f(w)u, \qquad \frac{\partial w}{\partial \eta} = -f(w)u,$$

which is a special case of system 16.1.2.2 with $k = 1$ and $g(w) = -f(w)$.

2°. The problem of pumping a suspension into a particle-free reservoir is described by the system in question with initial and boundary conditions

$$u = w = 0 \quad \text{at} \quad t = 0, \qquad u = 1 \quad \text{at} \quad x = 0.$$

The solutions of the problem can be represented as

$$u = \frac{w}{\Phi^{-1}(t-x)}, \qquad \int_w^{\Phi^{-1}(t-x)} \frac{dz}{zf(z)} = x. \tag{1}$$

The second equation in (1) defines $w = w(x, t)$ in implicit form, and $\Phi^{-1}(w)$ is the inverse of the function

$$\Phi(w) = \int_0^w \frac{z}{f(z)}. \tag{2}$$

The above formulas describe an exact solution to the problem in question for $x < t$. For $x > t$, the functions u and w are identically zero.

Special case. For $f(w) = 1 - w$, it follows from (2) that $\Phi(w) = -\ln(1 - w)$ and $\Phi^{-1}(w) = 1 - e^{-w}$. Substituting the last expression into formulas (1), we find the solution

$$u = \frac{e^{t-x}}{e^x + e^{t-x} - 1}, \qquad w = \frac{e^{t-x} - 1}{e^x + e^{t-x} - 1}.$$

⊙ *References for system* 16.2.2.1: J. P. Herzig, D. M. Leclerc, P. Le Goff (1970), D. J. Logan (2001), A. C. Alvarez, P. Bedrikovetsky, G. Hime, D. Marchesin, J. R. Rodriguez (2006).

2. $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -w, \qquad \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} = w.$

Model *equations of nonlinear vibrations of a stratified medium*.

General solution:

$$u = a\cos t + b\sin t, \qquad w = a\sin t - b\cos t.$$

Here the amplitudes $a = a(x, t)$ and $b = b(x, t)$ are determined by solving the transcendental system of equations

$$x - a\sin t + b\cos t = f(a), \qquad b = g(a),$$

where $f(a)$ and $g(a)$ are arbitrary functions.

Special case. For $f(a) = c_1 a + c_2$ and $g(a) = c_3 a + c_4$, we have

$$a = \frac{x + c_4\cos t - c_2}{c_1 + \sin t - c_3\cos t}, \qquad b = c_3 a + c_4.$$

⊙ *Reference*: V. G. Baidulov (2010).

3. $\dfrac{\partial}{\partial t}g(u,w)+a\dfrac{\partial u}{\partial x}=f_1(w)u, \qquad \dfrac{\partial w}{\partial t}=f_2(w)u.$

Eliminating u from the first equation with the aid of the second equation and then integrating with respect to t, one arrives at a nonlinear first-order partial differential equation for $w(x,t)$:

$$g\left(\frac{1}{f_2(w)}\frac{\partial w}{\partial t},w\right)+\frac{a}{f_2(w)}\frac{\partial w}{\partial x}=\int\frac{f_1(w)}{f_2(w)}\,dw+\theta(x),$$

where $\theta(x)$ is an arbitrary function. For θ = const, a complete integral of the equation obtained is sought in the form $w=w(C_1x+C_2t+C_3)$ and allows one to find the corresponding general integral, which contains an arbitrary function in parametric form (see Subsection 24.1.1).

4. $\dfrac{\partial u}{\partial x}=f_1(x,t)u+g_1(x,t)u^{1-n}+h_1(x,t)u^{1-n}w^m,$

$\dfrac{\partial w}{\partial t}=f_2(x,t)w+g_2(x,t)w^{1-m}+h_2(x,t)u^nw^{1-m}.$

The transformation $U=u^n$, $W=w^m$ leads to the linear system

$$\frac{\partial U}{\partial x}=nf_1(x,t)U+nh_1(x,t)W+ng_1(x,t),$$
$$\frac{\partial W}{\partial t}=mf_2(x,t)W+mh_2(x,t)U+mg_2(x,t).$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

Chapter 17
Systems of Two Parabolic Equations

17.1. Systems of the Form

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(u, w), \quad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + G(u, w)$$

Preliminary remarks. Systems of this form often arise in the theory of heat and mass transfer in chemically reactive media, theory of chemical reactors, combustion theory, mathematical biology, and biophysics.

Such systems are invariant under translations in the independent variables (and under the change of x to $-x$) and admit traveling-wave solutions $u = u(kx - \lambda t)$, $w = w(kx - \lambda t)$. These solutions as well as those with one of the unknown functions being identically zero are not considered further in this section.

The functions $f(\varphi)$, $g(\varphi)$, $h(\varphi)$ appearing below are arbitrary functions of their argument, $\varphi = \varphi(u, w)$; the equations are arranged in order of complexity of this argument.

17.1.1. Arbitrary Functions Depend on a Linear Combination of the Unknowns

1. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u\exp\left(k\dfrac{w}{u}\right)f(u), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \exp\left(k\dfrac{w}{u}\right)[wf(u) + g(u)].$

Solution:

$$u = y(\xi), \quad w = -\frac{2}{k}\ln|bx|\, y(\xi) + z(\xi), \quad \xi = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, C_3, and b are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{1}{b^2\xi^2}y\exp\left(k\frac{z}{y}\right)f(y) = 0,$$

$$az''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi - \frac{4a}{k\xi}y'_\xi + \frac{2a}{k\xi^2}y + \frac{1}{b^2\xi^2}\exp\left(k\frac{z}{y}\right)[zf(y) + g(y)] = 0.$$

⊙ *Reference*: T. Barannyk (2002).

2. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + f(bu + cw), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + g(bu + cw).$

Solution:

$$u = c\theta(x, t) + y(\xi), \quad w = -b\theta(x, t) + z(\xi), \quad \xi = kx - \lambda t,$$

where the functions $y(\xi)$ and $z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$ak^2y''_{\xi\xi} + \lambda y'_\xi + f(by + cz) = 0,$$
$$ak^2z''_{\xi\xi} + \lambda z'_\xi + g(by + cz) = 0,$$

and the function $\theta = \theta(x, t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}.$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

3. $\dfrac{\partial u}{\partial t} = a_1\dfrac{\partial^2 u}{\partial x^2} + f(bu + cw), \quad \dfrac{\partial w}{\partial t} = a_2\dfrac{\partial^2 w}{\partial x^2} + g(bu + cw).$

Solution:

$$u = c(\alpha x^2 + \beta x + \gamma t) + y(\xi), \quad w = -b(\alpha x^2 + \beta x + \gamma t) + z(\xi), \quad \xi = kx - \lambda t,$$

where k, α, β, γ, and λ are arbitrary constants, and the functions $y(\xi)$ and $z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$a_1k^2y''_{\xi\xi} + \lambda y'_\xi + 2a_1c\alpha - c\gamma + f(by + cz) = 0,$$
$$a_2k^2z''_{\xi\xi} + \lambda z'_\xi - 2a_2b\alpha + b\gamma + g(by + cz) = 0.$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

4. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + c_2f(b_1u + c_1w) + c_1g(b_2u + c_2w),$

$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - b_2f(b_1u + c_1w) - b_1g(b_2u + c_2w).$

It is assumed that $b_1c_2 - b_2c_1 \neq 0$.

Multiplying the equations by suitable constants and adding together, one arrives at two independent equations of the form 5.6.1.1:

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + (b_1c_2 - b_2c_1)f(U), \qquad U = b_1u + c_1w;$$
$$\frac{\partial W}{\partial t} = a\frac{\partial^2 W}{\partial x^2} - (b_1c_2 - b_2c_1)g(W), \qquad W = b_2u + c_2w.$$

In the general case, these equations admit traveling-wave solutions

$$U = U(k_1x - \lambda_1t), \quad W = W(k_2x - \lambda_2t),$$

where k_m and λ_m are arbitrary constants. The corresponding solution of the original system is the superposition (linear combination) of two nonlinear traveling waves.

5. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(bu-cw) + g(bu-cw),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(bu-cw) + h(bu-cw).$$

To the special case of $f(z) = 0$, $g(z) = z$, and $h(z) = -z$, there corresponds a reversible first-order chemical reaction (see Danckwerts, 1970). With $f(z) = z + k$ and $g(z) = h(z) = 0$, the system is a special case of the *Lotka–Volterra system*, which describes the dynamics of biological systems in which two species interact by fighting for the same food; see Romanovskii, Stepanova, and Chernavskii (1984, pp. 35, 57).

1°. Solution:

$$u = \varphi(t) + c\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t), \quad w = \psi(t) + b\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi'_t = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

2°. Solution:

$$u = \widetilde{\varphi}(x) + c\widetilde{\theta}(x,t), \quad w = \widetilde{\psi}(x) + b\widetilde{\theta}(x,t),$$

where the functions $\widetilde{\varphi} = \widetilde{\varphi}(x)$ and $\widetilde{\psi} = \widetilde{\psi}(x)$ are determined by the system of ordinary differential equations

$$a\varphi''_{xx} + \widetilde{\varphi} f(b\widetilde{\varphi} - c\widetilde{\psi}) + g(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$
$$a\psi''_{xx} + \widetilde{\psi} f(b\widetilde{\varphi} - c\widetilde{\psi}) + h(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$

and the function $\widetilde{\theta} = \widetilde{\theta}(x,t)$ solves the linear Schrödinger equation

$$\frac{\partial \widetilde{\theta}}{\partial t} = a\frac{\partial^2 \widetilde{\theta}}{\partial x^2} + F(x)\widetilde{\theta}, \quad F(x) = f(b\widetilde{\varphi} - c\widetilde{\psi}).$$

About exact solutions to equations of this form see Polyanin (2002, Subsections 1.8.1 and 1.8.9).

3°. Let us multiply the first equation by b and the second one by $-c$ and add the results together to obtain

$$\frac{\partial \zeta}{\partial t} = a\frac{\partial^2 \zeta}{\partial x^2} + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(\zeta) + g(\zeta). \tag{2}$$

Equation (1) can be treated separately. About exact solutions to equations of this form for various kinetic functions $F(\zeta) = \zeta f(\zeta) + bg(\zeta) - ch(\zeta)$ see 5.6.1.1. Given a solution $\zeta = \zeta(x,t)$ to equation (1), the function $u = u(x,t)$ can be determined by solving the linear equation (2) and the function $w = w(x,t)$ is found as $w = (bu - \zeta)/c$.

Note two important solutions to equation (1):

(i) In the general case, equation (1) admits traveling-wave solutions $\zeta = \zeta(z)$, where $z = kx - \lambda t$. Then the corresponding exact solutions to equation (2) are expressed as $u = u_0(z) + \sum e^{\beta_n t} u_n(z)$.

(ii) If the condition $\zeta f(\zeta) + bg(\zeta) - ch(\zeta) = k_1\zeta + k_0$ holds, equation (1) is linear,

$$\frac{\partial \zeta}{\partial t} = a\frac{\partial^2 \zeta}{\partial x^2} + k_1\zeta + k_0,$$

and, hence, can be reduced to the linear heat equation.

⊙ *References for system* 17.1.1.5: A. D. Polyanin (2004 *b*, 2005).

6. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + e^{\lambda u} f(\lambda u - \sigma w), \qquad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + e^{\sigma w} g(\lambda u - \sigma w).$

1°. Solution:

$$u = y(\xi) - \frac{1}{\lambda}\ln(C_1 t + C_2), \qquad w = z(\xi) - \frac{1}{\sigma}\ln(C_1 t + C_2), \qquad \xi = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{\lambda} + e^{\lambda y} f(\lambda y - \sigma z) = 0,$$
$$bz''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi + \frac{C_1}{\sigma} + e^{\sigma z} g(\lambda y - \sigma z) = 0.$$

2°. Solution with $b = a$:

$$u = \theta(x,t), \qquad w = \frac{\lambda}{\sigma}\theta(x,t) - \frac{k}{\sigma},$$

where k is a root of the algebraic (transcendental) equation

$$\lambda f(k) = \sigma e^{-k} g(k),$$

and the function $\theta = \theta(x,t)$ is determined by the differential equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + f(k)e^{\lambda\theta}.$$

For exact solutions to this equation, see Polyanin and Zaitsev (2004).

⊙ *Reference for system* 17.1.1.6: A. D. Polyanin (2004 *b*).

7. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u-w) + e^{u+w}g(u-w) - 2e^{2(u+w)},$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} - f(u-w) + e^{u+w}g(u-w) - 2ae^{2(u+w)}.$$

Solution:

$$u = \tfrac{1}{2}[\varphi(t) - \ln(2x + \psi(t)],$$
$$w = \tfrac{1}{2}[-\varphi(t) - \ln(2x + \psi(t)],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = 2f(\varphi), \qquad \psi'_t = -2g(\varphi).$$

⊙ *Reference*: O. V. Kaptsov (1995 *b*).

17.1.2. Arbitrary Functions Depend on the Ratio of the Unknowns

1. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right), \qquad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right).$$

To the special case of $f(z) = k_1 - k_2 z^{-1}$ and $g(z) = k_2 - k_1 z$ there corresponds a reversible first-order chemical reaction (see Danckwerts, 1973). The *Eigen–Schuster model*, which describes the dynamics of biological systems in which two species interact by fighting for food at constant multiplication factors, leads to the system in question for $f(z) = \dfrac{k}{z+1}$ and $g(z) = -\dfrac{kz}{z+1}$, where k is the difference in the multiplication factors [see Romanovskii, Stepanova, and Chernavskii (1984, pp. 31, 32)].

1°. Multiplicative separable solution:

$$u = [C_1 \sin(kx) + C_2 \cos(kx)]\varphi(t),$$
$$w = [C_1 \sin(kx) + C_2 \cos(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = -ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = -bk^2\psi + \psi g(\varphi/\psi).$$

2°. Multiplicative separable solution:

$$u = [C_1 \exp(kx) + C_2 \exp(-kx)]U(t),$$
$$w = [C_1 \exp(kx) + C_2 \exp(-kx)]W(t),$$

where C_1, C_2, and k are arbitrary constants, and the functions $U = U(t)$ and $W = W(t)$ are determined by the autonomous system of ordinary differential equations

$$U'_t = ak^2U + Uf(U/W),$$
$$W'_t = bk^2W + Wg(U/W).$$

3°. Degenerate solution:

$$u = (C_1x + C_2)U(t),$$
$$w = (C_1x + C_2)W(t),$$

where C_1 and C_2 are arbitrary constants, and the functions $U = U(t)$ and $W = W(t)$ are determined by the autonomous system of ordinary differential equations

$$U'_t = Uf(U/W),$$
$$W'_t = Wg(U/W).$$

This autonomous system can be integrated since it is reduced, after eliminating t, to a homogeneous first-order equation. The systems presented in Items 1° and 2° can be integrated likewise.

4°. Multiplicative separable solution:

$$u = e^{-\lambda t}y(x), \qquad w = e^{-\lambda t}z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the autonomous system of ordinary differential equations

$$ay''_{xx} + \lambda y + yf(y/z) = 0,$$
$$bz''_{xx} + \lambda z + zg(y/z) = 0.$$

5°. Solution (generalizes the solution of Item 4°):

$$u = e^{kx-\lambda t}y(\xi), \qquad w = e^{kx-\lambda t}z(\xi), \qquad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$a\beta^2 y''_{\xi\xi} + (2ak\beta + \gamma)y'_\xi + (ak^2 + \lambda)y + yf(y/z) = 0,$$
$$b\beta^2 z''_{\xi\xi} + (2bk\beta + \gamma)z'_\xi + (bk^2 + \lambda)z + zg(y/z) = 0.$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution. If $k = \gamma = 0$ and $\beta = 1$, we have the solution of Item 4°.

⊙ *References for system* 17.1.2.1: A. D. Polyanin (2004 *b*, 2005).

2. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\left(\dfrac{u}{w}\right), \qquad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wg\left(\dfrac{u}{w}\right).$

This system is a special case of system 17.1.2.1 with $b = a$ and hence it admits the above solutions given in Items 1°–5°. In addition, it has some interesting properties and other solutions, which are given below.

Suppose $u = u(x, t)$, $w = w(x, t)$ is a solution of the system. Then the functions

$$u_1 = Au(\pm x + C_1,\ t + C_2), \qquad w_1 = Aw(\pm x + C_1,\ t + C_2);$$
$$u_2 = \exp(\lambda x + a\lambda^2 t)u(x + 2a\lambda t,\ t), \qquad w_2 = \exp(\lambda x + a\lambda^2 t)w(x + 2a\lambda t,\ t),$$

where A, C_1, C_2, and λ are arbitrary constants, are also solutions of these equations.

1°. Point-source solution:

$$u = \exp\left(-\frac{x^2}{4at}\right)\varphi(t), \qquad w = \exp\left(-\frac{x^2}{4at}\right)\psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the nonautonomous system of ordinary differential equations

$$\varphi'_t = -\frac{1}{2t}\varphi + \varphi f\left(\frac{\varphi}{\psi}\right),$$
$$\psi'_t = -\frac{1}{2t}\psi + \psi g\left(\frac{\varphi}{\psi}\right).$$

2°. Functional separable solution:

$$u = \exp\left(kxt + \tfrac{2}{3}ak^2t^3 - \lambda t\right)y(\xi),$$
$$w = \exp\left(kxt + \tfrac{2}{3}ak^2t^3 - \lambda t\right)z(\xi), \qquad \xi = x + akt^2,$$

where k and λ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the nonautonomous system of ordinary differential equations

$$ay''_{\xi\xi} + (\lambda - k\xi)y + yf(y/z) = 0,$$
$$az''_{\xi\xi} + (\lambda - k\xi)z + zg(y/z) = 0.$$

3°. Let k be a root of the algebraic (transcendental) equation

$$f(k) = g(k). \tag{1}$$

Solution:

$$u = ke^{\lambda t}\theta, \quad w = e^{\lambda t}\theta, \quad \lambda = f(k),$$

where the function $\theta = \theta(x, t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

4°. Periodic solution:

$$u = Ak\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B),$$
$$w = A\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B), \qquad \beta = \sqrt{\mu^2 + \frac{1}{a}f(k)},$$

where A, B, and μ are arbitrary constants, and k is a root of the algebraic (transcendental) equation (1).

5°. Solution:

$$u = \varphi(t)\exp\left[\int g(\varphi(t))\,dt\right]\theta(x, t), \quad w = \exp\left[\int g(\varphi(t))\,dt\right]\theta(x, t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \tag{2}$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}.$$

To the particular solution $\varphi = k = \text{const}$ of equation (2) there corresponds the solution given in Item 3°. The general solution of equation (2) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

6°. Solution for the special case $g(z) = -z^2 f(z)$:

$$u = r(x,t)\sin\varphi(t), \quad w = r(x,t)\cos\varphi(t),$$

where the function $\varphi = \varphi(t)$ satisfies the separable first-order ordinary differential equation

$$\varphi_t' = f(\tan\varphi)\tan\varphi$$

and the function $r = r(x,t)$ satisfies the linear heat equation

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2}. \tag{3}$$

7°. Solution for the special case $g(z) = z^2 f(z)$:

$$u = r(x,t)\sinh\varphi(t), \quad w = r(x,t)\cosh\varphi(t),$$

where the function $\varphi = \varphi(t)$ satisfies the separable first-order ordinary differential equation

$$\varphi_t' = f(\tanh\varphi)\tanh\varphi,$$

and the function $r = r(x,t)$ satisfies the linear heat equation (3).

Remark. In this case, also there is a solution of the form

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t).$$

8°. The transformation

$$u = a_1 U + b_1 W, \quad w = a_2 U + b_2 W,$$

where a_n and b_n are arbitrary constants ($n = 1, 2$), leads to an equation of similar form for U and W.

⊙ *References for system* 17.1.2.2: A. D. Polyanin (2004 *b*, 2005).

3. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\left(\dfrac{u}{w}\right) + g\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Let k be a root of the algebraic (transcendental) equation

$$g(k) = kh(k).$$

1°. Solution with $f(k) \neq 0$:

$$u(x,t) = k\left(\exp[f(k)t]\theta(x,t) - \frac{h(k)}{f(k)}\right), \quad w(x,t) = \exp[f(k)t]\theta(x,t) - \frac{h(k)}{f(k)},$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}. \tag{1}$$

2°. Solution with $f(k) = 0$:

$$u(x,t) = k[\theta(x,t) + h(k)t], \quad w(x,t) = \theta(x,t) + h(k)t,$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation (1).

⊙ *Reference for system* 17.1.2.3: A. D. Polyanin (2004 *b*).

4. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf\left(\dfrac{u}{w}\right) + \dfrac{u}{w}h\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + wg\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Solution:

$$u = \varphi(t)G(t)\left[\theta(x,t) + \int \frac{h(\varphi)}{G(t)}\,dt\right], \quad w = G(t)\left[\theta(x,t) + \int \frac{h(\varphi)}{G(t)}\,dt\right], \quad G(t) = \exp\left[\int g(\varphi)\,dt\right],$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi_t' = [f(\varphi) - g(\varphi)]\varphi, \tag{1}$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

The general solution of equation (1) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

5. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf_1\left(\dfrac{w}{u}\right) + wg_1\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + uf_2\left(\dfrac{w}{u}\right) + wg_2\left(\dfrac{w}{u}\right).$

Solution:

$$u = \exp\left\{\int [f_1(\varphi) + \varphi g_1(\varphi)]\,dt\right\}\theta(x,t), \quad w(x,t) = \varphi(t)\exp\left\{\int [f_1(\varphi) + \varphi g_1(\varphi)]\,dt\right\}\theta(x,t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi_t' = f_2(\varphi) + \varphi g_2(\varphi) - \varphi[f_1(\varphi) + \varphi g_1(\varphi)],$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

6. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^3 f\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + u^3 g\left(\dfrac{u}{w}\right).$

Solution:

$$u = (x + A)\varphi(z), \quad w = (x + A)\psi(z), \quad z = t + \frac{1}{6a}(x + A)^2 + B,$$

where A and B are arbitrary constants, and the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_{zz}'' + 9a\varphi^3 f(\varphi/\psi) = 0,$$
$$\psi_{zz}'' + 9a\varphi^3 g(\varphi/\psi) = 0.$$

⊙ *References*: T. A. Barannyk (2002), T. A. Barannyk, A. G. Nikitin (2004).

7. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + au - u^3 f\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw - u^3 g\left(\dfrac{u}{w}\right).$

1°. Solution with $a > 0$:

$$u = \left[C_1 \exp\left(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) - C_2 \exp\left(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right)\right]\varphi(z),$$
$$w = \left[C_1 \exp\left(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) - C_2 \exp\left(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right)\right]\psi(z),$$
$$z = C_1 \exp\left(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) + C_2 \exp\left(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are determined by the autonomous system of ordinary differential equations

$$a\varphi''_{zz} = 2\varphi^3 f(\varphi/\psi),$$
$$a\psi''_{zz} = 2\varphi^3 g(\varphi/\psi).$$

2°. Solution with $a < 0$:

$$u = \exp\left(\tfrac{3}{2}at\right)\sin\left(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\right)U(\xi),$$
$$w = \exp\left(\tfrac{3}{2}at\right)\sin\left(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\right)W(\xi),$$
$$\xi = \exp\left(\tfrac{3}{2}at\right)\cos\left(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\right) + C_2,$$

where C_1 and C_2 are arbitrary constants, and the functions $U = U(\xi)$ and $W = W(\xi)$ are determined by the autonomous system of ordinary differential equations

$$aU''_{\xi\xi} = -2U^3 f(U/W),$$
$$aW''_{\xi\xi} = -2U^3 g(U/W).$$

⊙ *References for system* 17.1.2.7: T. A. Barannyk (2002), A. D. Polyanin (2004 *b*).

8. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^n f\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + w^n g\left(\dfrac{u}{w}\right).$

If $f(z) = kz^{-m}$ and $g(z) = -kz^{n-m}$, the system describes an nth-order chemical reaction (of order $n - m$ in the component u and of order m in the component w); to $n = 2$ and $m = 1$ there corresponds a fairly common second-order reaction (see Danckwerts, 1973).

1°. Self-similar solution with $n \neq 1$:

$$u = (C_1 t + C_2)^{\frac{1}{1-n}} y(\xi), \quad w = (C_1 t + C_2)^{\frac{1}{1-n}} z(\xi), \quad \xi = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{n-1}y + y^n f\left(\frac{y}{z}\right) = 0,$$
$$bz''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi + \frac{C_1}{n-1}z + z^n g\left(\frac{y}{z}\right) = 0.$$

2°. Solution with $b = a$:

$$u(x,t) = k\theta(x,t), \quad w(x,t) = \theta(x,t),$$

where k is a root of the algebraic (transcendental) equation

$$k^{n-1} f(k) = g(k),$$

and the function $\theta = \theta(x,t)$ satisfies the heat equation with a power-law nonlinearity of the form 5.1.3.1:

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + g(k)\theta^n.$$

⊙ *Reference for system* 17.1.2.8: A. D. Polyanin (2004 *b*).

9. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right)\ln u + ug\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{u}{w}\right)\ln w + wh\left(\frac{u}{w}\right).$$

Solution:

$$u(x,t) = \varphi(t)\psi(t)\theta(x,t), \quad w(x,t) = \psi(t)\theta(x,t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by solving the first-order autonomous ordinary differential equations

$$\varphi'_t = \varphi[g(\varphi) - h(\varphi) + f(\varphi)\ln\varphi], \tag{1}$$
$$\psi'_t = \psi[h(\varphi) + f(\varphi)\ln\psi], \tag{2}$$

and the function $\theta = \theta(x,t)$ is determined by the differential equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + f(\varphi)\theta\ln\theta. \tag{3}$$

The separable equation (1) can be solved to obtain a solution in implicit form. Equation (2) is easy to integrate—with the change of variable $\psi = e^\zeta$, it is reduced to a linear equation. Equation (3) admits exact solutions of the form

$$\theta = \exp\left[\sigma_2(t)x^2 + \sigma_1(t)x + \sigma_0(t)\right],$$

where the functions $\sigma_n(t)$ are described by the equations

$$\sigma'_2 = f(\varphi)\sigma_2 + 4a\sigma_2^2,$$
$$\sigma'_1 = f(\varphi)\sigma_1 + 4a\sigma_1\sigma_2,$$
$$\sigma'_0 = f(\varphi)\sigma_0 + a\sigma_1^2 + 2a\sigma_2.$$

This system can be integrated directly, since the first equation is a Bernoulli equation and the second and third ones are linear in the unknown. Note that the first equation has a particular solution $\sigma_2 = 0$.

Remark. Equation (1) has a special solution $\varphi = k = \text{const}$, where k is a root of the algebraic (transcendental) equation $g(k) - h(k) + f(k)\ln k = 0$.

⊙ *Reference*: A. D. Polyanin (2004 *b*).

10. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\Big(\frac{w}{u}\Big) - wg\Big(\frac{w}{u}\Big) + \frac{u}{\sqrt{u^2+w^2}}h\Big(\frac{w}{u}\Big),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\Big(\frac{w}{u}\Big) + ug\Big(\frac{w}{u}\Big) + \frac{w}{\sqrt{u^2+w^2}}h\Big(\frac{w}{u}\Big).$$

Solution:

$$u = r(x,t)\cos\varphi(t), \qquad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined from the separable first-order ordinary differential equation

$$\varphi'_t = g(\tan\varphi),$$

and the function $r = r(x,t)$ satisfies the linear equation

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + rf(\tan\varphi) + h(\tan\varphi). \tag{1}$$

The change of variable

$$r = F(t)\Big[Z(x,t) + \int \frac{h(\tan\varphi)\,dt}{F(t)}\Big], \qquad F(t) = \exp\Big[\int f(\tan\varphi)\,dt\Big]$$

brings (1) to the linear heat equation

$$\frac{\partial Z}{\partial t} = a\frac{\partial^2 Z}{\partial x^2}.$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

11. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\Big(\frac{w}{u}\Big) + wg\Big(\frac{w}{u}\Big) + \frac{u}{\sqrt{u^2-w^2}}h\Big(\frac{w}{u}\Big),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\Big(\frac{w}{u}\Big) + ug\Big(\frac{w}{u}\Big) + \frac{w}{\sqrt{u^2-w^2}}h\Big(\frac{w}{u}\Big).$$

Solution:

$$u = r(x,t)\cosh\varphi(t), \qquad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined from the separable first-order ordinary differential equation

$$\varphi'_t = g(\tanh\varphi),$$

and the function $r = r(x,t)$ satisfies the linear equation

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + rf(\tanh\varphi) + h(\tanh\varphi). \tag{1}$$

The change of variable

$$r = F(t)\Big[Z(x,t) + \int \frac{h(\tanh\varphi)\,dt}{F(t)}\Big], \qquad F(t) = \exp\Big[\int f(\tanh\varphi)\,dt\Big]$$

brings (1) to the linear heat equation

$$\frac{\partial Z}{\partial t} = a\frac{\partial^2 Z}{\partial x^2}.$$

17.1.3. Arbitrary Functions Depend on the Product of Powers of the Unknowns

1. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^n w^m), \qquad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + wg(u^n w^m).$

Solution:

$$u = e^{m(kx-\lambda t)}y(\xi), \qquad w = e^{-n(kx-\lambda t)}z(\xi), \qquad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$a\beta^2 y''_{\xi\xi} + (2akm\beta + \gamma)y'_\xi + m(ak^2 m + \lambda)y + yf(y^n z^m) = 0,$$
$$b\beta^2 z''_{\xi\xi} + (-2bkn\beta + \gamma)z'_\xi + n(bk^2 n - \lambda)z + zg(y^n z^m) = 0.$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution.

⊙ *References*: T. A. Barannyk (2002), A. D. Polyanin (2004 *b*).

2. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^{1+kn}f\big(u^n w^m\big), \qquad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + w^{1-km}g\big(u^n w^m\big).$

Self-similar solution:

$$u = (C_1 t + C_2)^{-\frac{1}{kn}}y(\xi), \qquad w = (C_1 t + C_2)^{\frac{1}{km}}z(\xi), \qquad \xi = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{kn}y + y^{1+kn}f\big(y^n z^m\big) = 0,$$
$$bz''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi - \frac{C_1}{km}z + z^{1-km}g\big(y^n z^m\big) = 0.$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

3. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + cu\ln u + uf(u^n w^m), \qquad \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2} + cw\ln w + wg(u^n w^m).$

Solution:

$$u = \exp(Ame^{ct})y(\xi), \qquad w = \exp(-Ane^{ct})z(\xi), \qquad \xi = kx - \lambda t,$$

where A, k, and λ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$ak^2 y''_{\xi\xi} + \lambda y'_\xi + cy\ln y + yf(y^n z^m) = 0,$$
$$bk^2 z''_{\xi\xi} + \lambda z'_\xi + cz\ln z + zg(y^n z^m) = 0.$$

To the special case $A = 0$ there corresponds a traveling-wave solution. For $\lambda = 0$, we have a solution in the form of the product of two functions dependent on time t and the coordinate x.

⊙ *Reference*: A. D. Polyanin (2004 *b*).

17.1.4. Arbitrary Functions Depend on Sum or Difference of Squares of the Unknowns

1. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg(u^2+w^2),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + ug(u^2+w^2) + wf(u^2+w^2).$$

1°. A periodic solution in the spatial coordinate:

$$u = \psi(t)\cos\varphi(x,t), \qquad w = \psi(t)\sin\varphi(x,t), \qquad \varphi(x,t) = C_1 x + \int g(\psi^2)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the separable first-order ordinary differential equation

$$\psi_t' = \psi f(\psi^2) - aC_1^2\psi,$$

whose general solution can be represented in implicit form as

$$\int \frac{d\psi}{\psi f(\psi^2) - aC_1^2\psi} = t + C_3.$$

2°. A periodic solution in time:

$$u = r(x)\cos\big[\theta(x) + C_1 t + C_2\big], \qquad w = r(x)\sin\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta = \theta(x)$ are determined by the autonomous system of ordinary differential equations

$$ar_{xx}'' - ar(\theta_x')^2 + rf(r^2) = 0,$$
$$ar\theta_{xx}'' + 2ar_x'\theta_x' - C_1 r + rg(r^2) = 0.$$

3°. Solution (generalizes the solution of Item 2°):

$$u = r(z)\cos\big[\theta(z) + C_1 t + C_2\big], \qquad w = r(z)\sin\big[\theta(z) + C_1 t + C_2\big], \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the functions $r = r(z)$ and $\theta = \theta(z)$ are determined by the system of ordinary differential equations

$$ar_{zz}'' - ar(\theta_z')^2 - \lambda r_z' + rf(r^2) = 0,$$
$$ar\theta_{zz}'' + 2ar_z'\theta_z' - \lambda r\theta_z' - C_1 r + rg(r^2) = 0.$$

⊙ *References for system* 17.1.4.1: T. A. Barannyk (2002), A. D. Polyanin (2004 *b*).

2. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\big(u^2+w^2\big) - wg\big(u^2+w^2\big) - w\arctan\Big(\frac{w}{u}\Big)h\big(u^2+w^2\big),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\big(u^2+w^2\big) + ug\big(u^2+w^2\big) + u\arctan\Big(\frac{w}{u}\Big)h\big(u^2+w^2\big).$$

Functional separable solution (for fixed t, it defines a structure periodic in x):

$$u = r(t)\cos\big[\varphi(t)x + \psi(t)\big], \qquad w = r(t)\sin\big[\varphi(t)x + \psi(t)\big],$$

where the functions $r = r(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$r_t' = -ar\varphi^2 + rf(r^2),$$
$$\varphi_t' = h(r^2)\varphi,$$
$$\psi_t' = h(r^2)\psi + g(r^2).$$

⊙ *References*: A. D. Polyanin (2004 *b*, 2005).

3. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg(u^2 - w^2),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + ug(u^2 - w^2) + wf(u^2 - w^2).$$

1°. Solution:

$$u = \psi(t)\cosh\varphi(x,t), \qquad w = \psi(t)\sinh\varphi(x,t), \qquad \varphi(x,t) = C_1 x + \int g(\psi^2)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $\psi = \psi(t)$ is described by the separable first-order ordinary differential equation

$$\psi_t' = \psi f(\psi^2) + aC_1^2\psi,$$

whose general solution can be represented in implicit form as

$$\int \frac{d\psi}{\psi f(\psi^2) + aC_1^2\psi} = t + C_3.$$

2°. Solution:

$$u = r(x)\cosh\big[\theta(x) + C_1 t + C_2\big], \qquad w = r(x)\sinh\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta = \theta(x)$ are determined by the autonomous system of ordinary differential equations

$$ar_{xx}'' + ar(\theta_x')^2 + rf(r^2) = 0,$$
$$ar\theta_{xx}'' + 2ar_x'\theta_x' + rg(r^2) - C_1 r = 0.$$

3°. Solution (generalizes the solution of Item 2°):

$$u = r(z)\cosh\big[\theta(z) + C_1 t + C_2\big], \qquad w = r(z)\sinh\big[\theta(z) + C_1 t + C_2\big], \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the functions $r = r(z)$ and $\theta = \theta(z)$ are determined by the autonomous system of ordinary differential equations

$$ar_{zz}'' + ar(\theta_z')^2 - \lambda r_z' + rf(r^2) = 0,$$
$$ar\theta_{zz}'' + 2ar_z'\theta_z' - \lambda r\theta_z' - C_1 r + rg(r^2) = 0.$$

⊙ *Reference for system* 17.1.4.3: A. D. Polyanin (2004 *b*).

4. $$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\big(u^2 - w^2\big) + wg\big(u^2 - w^2\big) + w\operatorname{arctanh}\Big(\frac{w}{u}\Big)h\big(u^2 - w^2\big),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\big(u^2 - w^2\big) + ug\big(u^2 - w^2\big) + u\operatorname{arctanh}\Big(\frac{w}{u}\Big)h\big(u^2 - w^2\big).$$

Functional separable solution:

$$u = r(t)\cosh\big[\varphi(t)x + \psi(t)\big], \qquad w = r(t)\sinh\big[\varphi(t)x + \psi(t)\big],$$

where the functions $r = r(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$r_t' = ar\varphi^2 + rf(r^2),$$
$$\varphi_t' = h(r^2)\varphi,$$
$$\psi_t' = h(r^2)\psi + g(r^2).$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

17.1.5. Arbitrary Functions Depend on the Unknowns in a Complex Way

1. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u^{k+1} f(\varphi), \quad \varphi = u\exp\left(-\dfrac{w}{u}\right),$

$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + u^{k+1}[f(\varphi)\ln u + g(\varphi)].$

Solution:

$$u = (C_1 t + C_2)^{-\frac{1}{k}} y(\xi), \quad w = (C_1 t + C_2)^{-\frac{1}{k}}\left[z(\xi) - \frac{1}{k}\ln(C_1 t + C_2) y(\xi)\right], \quad \xi = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$a y''_{\xi\xi} + \frac{1}{2} C_1 \xi y'_\xi + \frac{C_1}{k} y + y^{k+1} f(\varphi) = 0, \qquad \varphi = y\exp\left(-\frac{z}{y}\right),$$
$$a z''_{\xi\xi} + \frac{1}{2} C_1 \xi z'_\xi + \frac{C_1}{k} z + \frac{C_1}{k} y + y^{k+1}[f(\varphi)\ln y + g(\varphi)] = 0.$$

⊙ *Reference*: T. A. Barannyk (2002).

2. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + u f(bu + cw) - cw g\left(\dfrac{w}{u}\right),$

$\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + w f(bu + cw) + bw g\left(\dfrac{w}{u}\right).$

Solution:

$$u = cr(x,t)\cos^2\varphi(t), \quad w = br(x,t)\sin^2\varphi(t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order ordinary differential equation

$$\varphi'_t = \frac{1}{2} b\tan\varphi\, g\left(\frac{b}{c}\tan^2\varphi\right) \tag{1}$$

and the function $r = r(x,t)$ satisfies an equation of the form 5.6.1.1:

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + r f(bcr). \tag{2}$$

The general solution to equation (1) can be written in implicit form as

$$\int \frac{d\varphi}{\tan\varphi\, g(bc^{-1}\tan^2\varphi)} = \frac{1}{2} bt + C.$$

Note that equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

3. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2+w^2) - wg\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + ug\left(\dfrac{w}{u}\right) + wf(u^2+w^2).$

Solution:

$$u = r(x,t)\cos\varphi(t), \quad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi'_t = g(\tan\varphi), \tag{1}$$

and the function $r = r(x,t)$ is determined by a differential equation of the form 5.6.1.1:

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + rf(r^2). \tag{2}$$

The general solution of equation (1) is expressed in implicit form as

$$\int \frac{d\varphi}{g(\tan\varphi)} = t + C.$$

Equation (2) admits an exact, traveling-wave solution $r = r(z)$, where $z = kx - \lambda t$ with arbitrary constants k and λ, and the function $r(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 r''_{zz} + \lambda r'_z + rf(r^2) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *b*, 2005).

4. $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2-w^2) + wg\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + ug\left(\dfrac{w}{u}\right) + wf(u^2-w^2).$

Solution:

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi'_t = g(\tanh\varphi), \tag{1}$$

and the function $r = r(x,t)$ is determined by a differential equation of the form 5.6.1.1:

$$\frac{\partial r}{\partial t} = a\frac{\partial^2 r}{\partial x^2} + rf(r^2). \tag{2}$$

The general solution of equation (1) is expressed in implicit form as

$$\int \frac{d\varphi}{g(\tanh\varphi)} = t + C.$$

Equation (2) admits an exact, traveling-wave solution $r = r(z)$, where $z = kx - \lambda t$ with arbitrary constants k and λ, and the function $r(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 r''_{zz} + \lambda r'_z + rf(r^2) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *b*, 2005).

17.1.6. Some Systems Depending on Arbitrary Parameters

1. $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+\beta_1 u\big(u^{-\lambda_2}w^{\lambda_1}\big)^{\frac{4}{\lambda_1-\lambda_2}},\qquad \dfrac{\partial w}{\partial t}=\dfrac{\partial^2 w}{\partial x^2}+\beta_2 w\big(u^{-\lambda_2}w^{\lambda_1}\big)^{\frac{4}{\lambda_1-\lambda_2}}.$

This is a special case of equation 17.1.3.1 with $n=-\lambda_1$, $m=\lambda_1$, $f(z)=\beta_1 z^{\frac{4}{\lambda_1-\lambda_2}}$, and $g(z)=\beta_2 z^{\frac{4}{\lambda_1-\lambda_2}}$.

1°. Suppose $u_0(t,x)$, $w_0(t,x)$ is an arbitrary solution of the system. Then the six-parameter family of functions

$$\begin{aligned} u&=f(t)\exp\left[\lambda_1\frac{pm^2x^2+2mqx+m^2\varepsilon^2t+b_0}{4(d_0-pm^2t)}\right]u_0\left(\frac{m^2t+s}{d_0-pm^2t},\frac{mx+m^2\varepsilon t+d}{d_0-pm^2t}\right),\\ w&=f(t)\exp\left[\lambda_2\frac{pm^2x^2+2mqx+m^2\varepsilon^2t+b_0}{4(d_0-pm^2t)}\right]w_0\left(\frac{m^2t+s}{d_0-pm^2t},\frac{mx+m^2\varepsilon t+d}{d_0-pm^2t}\right)\end{aligned}\tag{1}$$

is also a solution of the system, with

$$f(t)=\frac{m_0^{\lambda_1}\sqrt{m}}{\sqrt{d_0-pm^2t}},\quad d_0=1-pr,\quad d=s+\varepsilon r,\quad q=\varepsilon+ps,\quad b_0=ps^2+2\varepsilon s+\varepsilon^2 r,$$

where ε, p, $m_0>0$, $m>0$, r, and s are arbitrary constants.

Interesting particular cases of this formulas are

$$\begin{aligned} u&=u_0(t,x+\varepsilon t)\exp\left[\frac{\lambda_1}{2}\left(\varepsilon x+\frac{\varepsilon^2 t}{2}\right)\right],\\ w&=w_0(t,x+\varepsilon t)\exp\left[\frac{\lambda_2}{2}\left(\varepsilon x+\frac{\varepsilon^2 t}{2}\right)\right],\end{aligned}\tag{2}$$

and

$$\begin{aligned} u&=t^{-1/2}\exp\left(-\frac{\lambda_1 x^2}{4t}\right)u_0\left(-\frac{1}{t},\frac{x}{t}\right),\\ w&=t^{-1/2}\exp\left(-\frac{\lambda_2 x^2}{4t}\right)w_0\left(-\frac{1}{t},\frac{x}{t}\right).\end{aligned}\tag{3}$$

The formula (2) is generated by the Galilei transformation and the ε parameter can be thought of as representing a velocity. Formula (3) can be obtained by passing to the limits $p\to\infty$, $m\to 0$, and $pm\to -1$ (the other parameters being zero). Both formulas can be used to convert time-independent (stationary) solutions of the original system into time-dependent (nonstationary) ones.

2°. Let us illustrate the usage of formulas (1) by a specific example. Consider a time-dependent solution only, with $u=\varphi_1(t)$ and $w=\varphi_2(t)$, where the functions $\varphi_1(t)$ and $\varphi_2(t)$ satisfy the system of equations

$$\begin{aligned}\frac{d\varphi_1}{dt}&=\beta_1\varphi_1^{1-\gamma\lambda_2}\varphi_2^{\gamma\lambda_1},\\ \frac{d\varphi_2}{dt}&=\beta_2\varphi_2^{1+\gamma\lambda_1}\varphi_1^{-\gamma\lambda_2},\end{aligned}$$

with $\gamma=4/(\lambda_1-\lambda_2)$.

The general solution of this system is

$$u = A^{\lambda_1}(\pm\gamma k)^{-1/4}(t_0 \pm t)^{\frac{\beta_1}{\gamma k}}, \qquad w = A^{\lambda_2}(\pm\gamma k)^{-1/4}(t_0 \pm t)^{\frac{\beta_2}{\gamma k}} \tag{4}$$

if $k = \beta_1\lambda_2 - \beta_2\lambda_1 \neq 0$, and

$$u = A^{\lambda_1}\left(\frac{\lambda_1\alpha}{\beta_1}\right)^{1/4}\exp(\lambda_1\alpha t), \qquad w = A^{\lambda_2}\left(\frac{\lambda_2\alpha}{\beta_2}\right)^{1/4}\exp(\lambda_2\alpha t) \tag{5}$$

if $k = 0$; the constants t_0, A, and α are arbitrary. Using formula (1), one can convert solutions (4) and (5) into solutions of the form:

$$u = A^{\lambda_1}(\pm\gamma k)^{-1/4}t^{-1/2}\left(t_0 \mp \frac{1}{t}\right)^{\frac{\beta_1}{\gamma k}}\exp\left(-\frac{\lambda_1 x^2}{4t}\right),$$

$$w = A^{\lambda_2}(\pm\gamma k)^{-1/4}t^{-1/2}\left(t_0 \mp \frac{1}{t}\right)^{\frac{\beta_2}{\gamma k}}\exp\left(-\frac{\lambda_2 x^2}{4t}\right),$$

and

$$u = A^{\lambda_1}\left(\frac{\lambda_1\alpha}{\beta_1}\right)^{1/4}t^{-1/2}\exp\left[-\frac{\lambda_1}{t}\left(\alpha + \frac{x^2}{4}\right)\right],$$

$$w = A^{\lambda_2}\left(\frac{\lambda_2\alpha}{\beta_2}\right)^{1/4}t^{-1/2}\exp\left[-\frac{\lambda_2}{t}\left(\alpha + \frac{x^2}{4}\right)\right].$$

3°. Solution:

$$u = \varphi_1(x)\exp(-\alpha\lambda_1 t), \quad w = \varphi_2(x)\exp(-\alpha\lambda_2 t),$$

where the functions $\varphi_1 = \varphi_1(x)$ and $\varphi_2 = \varphi_2(x)$ are determined by the system of ordinary differential equations

$$\frac{d^2\varphi_1}{dx^2} + \alpha\lambda_1^2\varphi_1 + \beta_1\varphi_1^{1-\gamma\lambda_2}\varphi_2^{\gamma\lambda_1} = 0,$$

$$\frac{d^2\varphi_2}{dx^2} + \alpha\lambda_2^2\varphi_2 + \beta_2\varphi_2^{1+\gamma\lambda_1}\varphi_1^{-\gamma\lambda_2} = 0,$$

and $\gamma = 4/(\lambda_1 - \lambda_2)$.

4°. Solution:

$$u = \exp\left[\frac{\lambda_1 t}{2}\left(\frac{t^2}{3} \mp x\right)\right]\varphi_1(\xi), \quad w = \exp\left[\frac{\lambda_2 t}{2}\left(\frac{t^2}{3} \mp x\right)\right]\varphi_2(\xi), \quad \xi = 2x \mp t^2,$$

where the functions $\varphi_1 = \varphi_1(\xi)$ and $\varphi_2 = \varphi_2(\xi)$ are determined by the system of ordinary differential equations

$$4\frac{d^2\varphi_1}{d\xi^2} \pm \frac{\lambda_1^2}{4}\xi\varphi_1 + \beta_1\varphi_1^{1-\gamma\lambda_2}\varphi_2^{\gamma\lambda_1} = 0,$$

$$4\frac{d^2\varphi_2}{d\xi^2} \pm \frac{\lambda_2^2}{4}\xi\varphi_2 + \beta_2\varphi_2^{1+\gamma\lambda_1}\varphi_1^{-\gamma\lambda_2} = 0,$$

with $\gamma = 4/(\lambda_1 - \lambda_2)$.

5°. Self-similar solution:

$$u = t^{-(2\alpha\lambda_1)/4}\varphi_1(\eta), \quad w = t^{-(2\alpha\lambda_2)/4}\varphi_2(\eta), \quad \eta = \frac{x}{\sqrt{t}},$$

where the functions $\varphi_1 = \varphi_1(\eta)$ and $\varphi_2 = \varphi_2(\eta)$ are determined by the system of ordinary differential equations

$$\frac{d^2\varphi_1}{d\eta^2} + \frac{\lambda_1}{2}\eta\frac{d\varphi_1}{d\eta} + \frac{\lambda_1}{4}(2\alpha\lambda_1 + 1)\varphi_1 + \beta_1\varphi_1^{1-\gamma\lambda_2}\varphi_2^{\gamma\lambda_1} = 0,$$
$$\frac{d^2\varphi_2}{d\eta^2} + \frac{\lambda_2}{2}\eta\frac{d\varphi_2}{d\eta} + \frac{\lambda_2}{4}(2\alpha\lambda_2 + 1)\varphi_2 + \beta_2\varphi_2^{1+\gamma\lambda_1}\varphi_1^{-\gamma\lambda_2} = 0,$$

and $\gamma = 4/(\lambda_1 - \lambda_2)$.

6°. Solution:

$$u = (t^2+1)^{-1/4}\exp\left[-\frac{\lambda_1}{4}\left(\frac{tx^2}{1+t^2} + 4\alpha\arctan t\right)\right]\varphi_1(\zeta), \quad \zeta = \frac{x}{\sqrt{1+t^2}},$$
$$w = (t^2+1)^{-1/4}\exp\left[-\frac{\lambda_2}{4}\left(\frac{tx^2}{1+t^2} + 4\alpha\arctan t\right)\right]\varphi_2(\zeta),$$

where the functions $\varphi_1 = \varphi_1(\zeta)$ and $\varphi_2 = \varphi_2(\zeta)$ are determined by the system of ordinary differential equations

$$\frac{d^2\varphi_1}{d\zeta^2} + \frac{\lambda_1^2}{4}(4\alpha + \zeta^2)\varphi_1 + \beta_1\varphi_1^{1-\gamma\lambda_2}\varphi_2^{\gamma\lambda_1} = 0,$$
$$\frac{d^2\varphi_2}{d\zeta^2} + \frac{\lambda_2^2}{4}(4\alpha + \zeta^2)\varphi_2 + \beta_2\varphi_2^{1+\gamma\lambda_1}\varphi_1^{-\gamma\lambda_2} = 0,$$

and $\gamma = 4/(\lambda_1 - \lambda_2)$.

⊙ *Reference for system* 17.1.6.1: R. Cherniha and J. R. King (2000).

2. $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (u^2+w^2)^{k/2}(a_1u - a_2w) + b_1u - b_2w,$$
$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + (u^2+w^2)^{k/2}(a_2u + a_1w) + b_2u + b_1w.$$

1°. Solution with $a_1 \neq 0$:

$$u = \left[\frac{\gamma^2 - b_1}{a_1 + c_0\exp[(\gamma^2 - b_1)kt]}\right]^{1/k}\cos\left(\gamma x + \beta t - \frac{a_2}{a_1 k}\ln|a_1 + c_0\exp[(\gamma^2 - b_1)kt]|\right),$$
$$w = \left[\frac{\gamma^2 - b_1}{a_1 + c_0\exp[(\gamma^2 - b_1)kt]}\right]^{1/k}\sin\left(\gamma x + \beta t - \frac{a_2}{a_1 k}\ln|a_1 + c_0\exp[(\gamma^2 - b_1)kt]|\right),$$
$$\beta = b_2 + \frac{a_2}{a_1}(\gamma^2 - b_1),$$

where c_0 and γ are arbitrary constants, and $\gamma^2 - b_1 \neq 0$.

Remark. In the case $\gamma^2 - b_1 = 0$, we have the solution

$$u = [ka_1(t_0 - t)]^{-1/k} \cos\Big(\sqrt{b_1} + b_2 t - \frac{a_2}{a_1 k} \ln|t_0 - t|\Big),$$
$$w = [ka_1(t_0 - t)]^{-1/k} \sin\Big(\sqrt{b_1} + b_2 t - \frac{a_2}{a_1 k} \ln|t_0 - t|\Big),$$

where t_0 is an arbitrary constant.

2°. Solution with $a_1 = 0$:

$$u = c_0 \exp[(b_1 - \gamma^2)kt] \cos\left(\gamma x + b_2 t + \frac{a_2 c_0^k}{k(b_1 - \gamma^2)} \exp[(b_1 - \gamma^2)kt]\right),$$
$$w = c_0 \exp[(b_1 - \gamma^2)kt] \sin\left(\gamma x + b_2 t + \frac{a_2 c_0^k}{k(b_1 - \gamma^2)} \exp[(b_1 - \gamma^2)kt]\right),$$

where c_0 and γ are arbitrary constants.

3°. Solution with $a_2 = 0$:

$$u = I_k^{-1}(x_0 \pm x) \cos(\Psi(x) + b_2 t),$$
$$w = I_k^{-1}(x_0 \pm x) \sin(\Psi(x) + b_2 t),$$

where

$$\Psi(x) = c_0 \int \frac{dx}{[I_k^{-1}(x_0 \pm x)]^2};$$

x_0 and c_0 are arbitrary constants and the function I_k^{-1} is the inverse of the integral

$$x_0 \pm x = \int \frac{\tau\, d\tau}{\sqrt{\frac{2a_1}{k+2}\tau^{k+4} - b_1\tau^4 + c_1\tau^2 - c_0^2}} \equiv I_k(\tau) \qquad \text{if} \quad k \neq -2,$$
$$x_0 \pm x = \int \frac{\tau\, d\tau}{\sqrt{2a_1\tau^2 \ln\tau - b_1\tau^4 + c_1\tau^2 - c_0^2}} \equiv I_k(\tau)\big|_{k=-2} \quad \text{if} \quad k = -2.$$

⊙ *Reference for system* 17.1.6.2: R. Cherniha and J. R. King (2003).

3. $\boldsymbol{\lambda_1 \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u(a_1 + u + w),}$
$\boldsymbol{\lambda_2 \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + w(a_2 + u + w), \quad \lambda_1 \neq \lambda_2, \quad a_1 \neq a_2.}$

1°. Solution:

$$u = \varphi_1(x) - \varphi_2(x) \exp\left(\frac{a_1 - a_2}{\lambda_1 - \lambda_2} t\right),$$
$$w = \varphi_2(x) \exp\left(\frac{a_1 - a_2}{\lambda_1 - \lambda_2} t\right),$$

where the functions $\varphi_1(x)$ and $\varphi_2(x)$ are determined by the ordinary differential equations

$$\varphi_1'' + \varphi_1^2 + a_1\varphi_1 = 0,$$
$$\varphi_2'' + \frac{a_2\lambda_1 - a_1\lambda_2}{\lambda_1 - \lambda_2}\varphi_2 + \varphi_1\varphi_2 = 0.$$

Special case 1. If $\varphi_1 = -a_1$ and $\beta = \dfrac{a_1 - a_2}{\lambda_1 - \lambda_2} > 0$ we have

$$u = -a_1 + \frac{1}{a_2 - a_1}\Big[C_1 \exp\big(\sqrt{\beta\lambda_1}\,x\big) + C_2 \exp\big(-\sqrt{\beta\lambda_1}\,x\big)\Big]e^{\beta t},$$
$$w = \frac{1}{a_1 - a_2}\Big[C_1 \exp\big(\sqrt{\beta\lambda_1}\,x\big) + C_2 \exp\big(-\sqrt{\beta\lambda_1}\,x\big)\Big]e^{\beta t},$$

where C_1 and C_2 are arbitrary constants.

Special case 2. If $\varphi_1 = -a_1$ and $\beta = \dfrac{a_1 - a_2}{\lambda_1 - \lambda_2} < 0$ we have

$$u = -a_1 + \frac{1}{a_2 - a_1}\Big[C_1 \cos\big(\sqrt{-\beta\lambda_1}\,x\big) + C_2 \sin\big(\sqrt{-\beta\lambda_1}\,x\big)\Big]e^{\beta t},$$
$$w = \frac{1}{a_1 - a_2}\Big[C_1 \cos\big(\sqrt{-\beta\lambda_1}\,x\big) + C_2 \sin\big(\sqrt{-\beta\lambda_1}\,x\big)\Big]e^{\beta t}.$$

2°. Solution with $\lambda_1 = \frac{9}{5}\lambda_2$ and $b = a_1 - a_2 > 0$:

$$u = \frac{1}{2}a_1 - \frac{3}{2}a_1 \tanh^2\left(\frac{1}{2}\sqrt{b}\,x\right) - \frac{1}{b}\cosh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{5b}{4\lambda_2}t\right),$$
$$w = -\frac{3}{2}a_2\left[1 - \tanh^2\left(\frac{1}{2}\sqrt{b}\,x\right)\right] + \frac{1}{b}\cosh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{5b}{4\lambda_2}t\right).$$

3°. Solution with $\lambda_1 = \frac{9}{5}\lambda_2$ and $b = a_1 - a_2 > 0$:

$$u = \frac{1}{2}a_1 - \frac{3}{2}a_1 \coth^2\left(\frac{1}{2}\sqrt{b}\,x\right) - \frac{1}{b}\sinh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{5b}{4\lambda_2}t\right),$$
$$w = -\frac{3}{2}a_2\left[1 - \coth^2\left(\frac{1}{2}\sqrt{b}\,x\right)\right] + \frac{1}{b}\sinh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{5b}{4\lambda_2}t\right).$$

4°. Solution with $\lambda_1 = \frac{9}{5}\lambda_2$ and $c = a_2 - a_1 > 0$:

$$u = \frac{1}{2}a_1 + \frac{3}{2}a_1 \tan^2\left(\frac{1}{2}\sqrt{c}\,x\right) - \frac{1}{a_1 - a_2}\cos^3\left(\frac{1}{2}\sqrt{c}\,x\right)\exp\left(-\frac{5c}{4\lambda_2}t\right),$$
$$w = -\frac{3}{2}a_2\left[1 + \tan^2\left(\frac{1}{2}\sqrt{c}\,x\right)\right] + \frac{1}{a_1 - a_2}\cos^3\left(\frac{1}{2}\sqrt{c}\,x\right)\exp\left(-\frac{5c}{4\lambda_2}t\right).$$

5°. Solution with $\lambda_1 = \frac{4}{3}\lambda_2$ and $b = a_1 - a_2 > 0$:

$$u = \frac{1}{2}a_1 - \frac{3}{2}a_1 \tanh^2\left(\frac{1}{2}\sqrt{b}\,x\right) - \frac{1}{b}\sinh\left(\frac{1}{2}\sqrt{b}\,x\right)\cosh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{3b}{\lambda_2}t\right),$$
$$w = -\frac{3}{2}a_2\left[1 - \tanh^2\left(\frac{1}{2}\sqrt{b}\,x\right)\right] + \frac{1}{b}\sinh\left(\frac{1}{2}\sqrt{b}\,x\right)\cosh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{3b}{\lambda_2}t\right).$$

6°. Solution with $\lambda_1 = \frac{4}{3}\lambda_2$ and $b = a_1 - a_2 > 0$:

$$u = \frac{1}{2}a_1 - \frac{3}{2}a_1 \coth^2\left(\frac{1}{2}\sqrt{b}\,x\right) - \frac{1}{b}\cosh\left(\frac{1}{2}\sqrt{b}\,x\right)\sinh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{3b}{\lambda_2}t\right),$$
$$w = -\frac{3}{2}a_2\left[1 - \coth^2\left(\frac{1}{2}\sqrt{b}\,x\right)\right] + \frac{1}{b}\cosh\left(\frac{1}{2}\sqrt{b}\,x\right)\sinh^3\left(\frac{1}{2}\sqrt{b}\,x\right)\exp\left(\frac{3b}{\lambda_2}t\right),$$

⊙ *References for system* 17.1.6.3: R. Cherniha (2010), R. Cherniha and V. Davydovych (2010).

4. $\lambda_1 \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u(a\lambda_1 + u + w), \qquad \lambda_2 \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + w(a\lambda_2 + u + w), \qquad \lambda_1 \neq \lambda_2.$

1°. Solution with $a\lambda_2 < 0$:

$$u = C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right]e^{at},$$
$$w = -\tfrac{3}{2}a\lambda_2\left[1 - \tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right] - C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

2°. Solution with $a\lambda_2 < 0$:

$$u = C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right]e^{at},$$
$$w = -\tfrac{3}{2}a\lambda_2\left[1 - \coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right] - C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_2}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

3°. Solution with $a\lambda_2 > 0$:

$$u = -C_1\left[\tfrac{1}{2} + \tfrac{3}{2}\tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_2}\,x + C_2\right)\right]e^{at},$$
$$w = -\tfrac{3}{2}a\lambda_2\left[1 + \tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_2}\,x + C_2\right)\right] + C_1\left[\tfrac{1}{2} + \tfrac{3}{2}\tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_2}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

4°. Solution with $a\lambda_1 < 0$:

$$u = -\tfrac{3}{2}a\lambda_1\left[1 - \tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right] - C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right]e^{at},$$
$$w = C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\tanh^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

5°. Solution with $a\lambda_1 < 0$:

$$u = -\tfrac{3}{2}a\lambda_1\left[1 - \coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right] - C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right]e^{at},$$
$$w = C_1\left[-\tfrac{1}{2} + \tfrac{3}{2}\coth^2\left(\tfrac{1}{2}\sqrt{-a\lambda_1}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

6°. Solution with $a\lambda_1 > 0$:

$$u = -\tfrac{3}{2}a\lambda_1\left[1 + \tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_1}\,x + C_2\right)\right] + C_1\left[\tfrac{1}{2} + \tfrac{3}{2}\tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_1}\,x + C_2\right)\right]e^{at},$$
$$w = -C_1\left[\tfrac{1}{2} + \tfrac{3}{2}\tan^2\left(\tfrac{1}{2}\sqrt{a\lambda_1}\,x + C_2\right)\right]e^{at},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference for system* 17.1.6.4: R. Cherniha and V. Davydovych (2010).

5. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u(a_1 - b_1 u - c_1 w), \qquad \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + w(a_2 - b_2 u - c_2 w).$

Solutions:

$$u = \frac{a}{4b}\left[1 - \tanh\left(\sqrt{\frac{a}{24}}\,x - \frac{5a}{12}t + C\right)\right]^2, \quad w = \beta_0 + \beta_1 u;$$
$$u = \frac{a}{4b}\left[1 - \coth\left(\sqrt{\frac{a}{24}}\,x - \frac{5a}{12}t + C\right)\right]^2, \quad w = \beta_0 + \beta_1 u,$$

where C is an arbitrary constant and the coefficients a, b, β_0, and β_1 are defined by the formulas

$$a = a_1 = a_2, \quad b = \frac{c_1 b_2 - b_1 c_2}{c_1 - c_2}, \quad \beta_0 = 0, \qquad \beta_1 = \frac{b_1 - b_2}{c_2 - c_1} \quad \text{if} \quad b_1 \neq b_2,\ c_1 \neq c_2;$$
$$a = a_1 - a_2, \quad b = b_1 + c_1\beta_1, \qquad \beta_0 = \frac{a_2}{c_1}, \quad \beta_1 = -\frac{a_2 b_1}{a_1 c_1} \quad \text{if} \quad b_1 = b_2,\ c_1 = c_2.$$

⊙ *Reference for system* 17.1.6.5: R. M. Cherniha and V. A. Dutka (2004).

6. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u(1 - u^2 - w^2), \qquad \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} - 2u^2 w.$

Solutions:

$$u = \frac{a + \exp(x + t)}{a + ax - 2at + \exp(x + t)}, \qquad w = \pm\frac{ax - 2at}{a + ax - 2at + \exp(x + t)};$$
$$u = \frac{a + \exp(-x + t)}{a - ax - 2at + \exp(-x + t)}, \qquad w = \pm\frac{-ax - 2at}{a - ax - 2at + \exp(-x + t)},$$

where a is an arbitrary constant.

⊙ *Reference*: K. A. Volosov (2007).

17.2. Systems of the Form

$$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + F(u, w),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + G(u, w)$$

Preliminary remarks. Such systems of equations often arise in the theory of mass transfer in reacting media, the theory of chemical reactors, combustion theory, mathematical biology, and biophysics. For $n = 0$, see Section 17.1; the values $n = 1$ and $n = 2$ correspond to plane and spatial problems in radially symmetric cases (x plays the role of the radial coordinate).

The functions $f(\varphi)$, $g(\varphi)$, and $h(\varphi)$ used throughout this section are arbitrary functions of an argument $\varphi = \varphi(u, w)$; the equations are arranged in order of complexity of this argument.

17.2.1. Arbitrary Functions Depend on a Linear Combination of the Unknowns

1. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + c_2 f(b_1u + c_1w) + c_1 g(b_2u + c_2w),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) - b_2 f(b_1u + c_1w) - b_1 g(b_2u + c_2w).$$

It is assumed that $b_1c_2 - b_2c_1 \neq 0$.

By multiplying the equations by suitable constants and adding together, one obtains two independent equations

$$\frac{\partial U}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial U}{\partial x}\right) + (b_1c_2 - b_2c_1)f(U), \qquad U = b_1u + c_1w;$$
$$\frac{\partial W}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial W}{\partial x}\right) - (b_1c_2 - b_2c_1)g(W), \qquad W = b_2u + c_2w.$$

In general, both equations admit simple solutions of the forms

$$U = U(t), \quad W = W(x); \qquad U = U(x), \quad W = W(t).$$

2. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u f(bu - cw) + g(bu - cw),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w f(bu - cw) + h(bu - cw).$$

1°. Solution:

$$u = \varphi(t) + c\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t),$$
$$w = \psi(t) + b\exp\left[\int f(b\varphi - c\psi)\,dt\right]\theta(x,t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi'_t = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x,t)$ satisfies linear heat equation

$$\frac{\partial \theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \theta}{\partial x}\right). \tag{1}$$

For solutions to this equation, see Polyanin (2002, Section 1.2).

2°. Solution:

$$u = \widetilde{\varphi}(x) + c\widetilde{\theta}(x,t), \qquad w = \widetilde{\psi}(x) + b\widetilde{\theta}(x,t),$$

where the functions$\widetilde{\varphi} = \widetilde{\varphi}(x)$ and $\widetilde{\psi} = \widetilde{\psi}(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n\widetilde{\varphi}'_x)'_x + \widetilde{\varphi} f(b\widetilde{\varphi} - c\widetilde{\psi}) + g(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$
$$ax^{-n}(x^n\widetilde{\psi}'_x)'_x + \widetilde{\psi} f(b\widetilde{\varphi} - c\widetilde{\psi}) + h(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$

and the function $\widetilde{\theta} = \widetilde{\theta}(x,t)$ satisfies the linear equation

$$\frac{\partial\widetilde{\theta}}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\widetilde{\theta}}{\partial x}\right) + f(b\widetilde{\varphi} - c\widetilde{\psi})\widetilde{\theta}.$$

3°. Let us multiply the first equation by b and the second one by $-c$ and add the results together to obtain

$$\frac{\partial\zeta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\zeta}{\partial x}\right) + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{2}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(\zeta) + g(\zeta). \tag{3}$$

Equation (2) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ to equation (2), the function $u = u(x,t)$ can be determined by solving the linear equation (3) and the function $w = w(x,t)$ is found as $w = (bu - \zeta)/c$.

Note two important solutions to equation (2):

(i) In the general case, equation (2) admits steady-state solutions $\zeta = \zeta(x)$. The corresponding exact solutions to equation (3) are expressed as $u = u_0(x) + \sum_n e^{\beta_n t}u_n(x)$.

(ii) If the condition $\zeta f(\zeta) + bg(\zeta) - ch(\zeta) = k_1\zeta + k_0$ holds, equation (2) is linear,

$$\frac{\partial\zeta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\zeta}{\partial x}\right) + k_1\zeta + k_0,$$

and hence can be reduced to the linear heat equation (1) with the substitution $\zeta = e^{k_1 t}\bar{\zeta} - k_0 k_1^{-1}$.

⊙ *References for system* 17.2.1.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2201.pdf).

3. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + e^{\lambda u}f(\lambda u - \sigma w),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + e^{\sigma w}g(\lambda u - \sigma w).$$

Solution:

$$u = y(\xi) - \frac{1}{\lambda}\ln(C_1 t + C_2), \qquad w = z(\xi) - \frac{1}{\sigma}\ln(C_1 t + C_2), \qquad \xi = \frac{x}{\sqrt{C_1 t + C_2}},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$a\xi^{-n}(\xi^n y'_\xi)'_\xi + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{\lambda} + e^{\lambda y}f(\lambda y - \sigma z) = 0,$$
$$b\xi^{-n}(\xi^n z'_\xi)'_\xi + \frac{1}{2}C_1\xi z'_\xi + \frac{C_1}{\sigma} + e^{\sigma z}g(\lambda y - \sigma z) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2202.pdf).

17.2.2. Arbitrary Functions Depend on the Ratio of the Unknowns

1. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial u}{\partial x}\right) + uf\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + wg\left(\dfrac{u}{w}\right).$

This system is a special case of system 17.2.2.2 with $b = a$ and, hence, admits the solutions given there in Items 1°–3°. In addition, the system has other solutions, which are listed below.

1°. Let k be a root of the algebraic (transcendental) equation

$$f(k) = g(k).$$

Solution:

$$u = ke^{\lambda t}\theta, \quad w = e^{\lambda t}\theta, \quad \lambda = f(k),$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial \theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial \theta}{\partial x}\right). \tag{1}$$

For solutions to this equation, see Polyanin (2002, Section 1.2).

2°. Solution:

$$u = \varphi(t)\exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t), \quad w = \exp\left[\int g(\varphi(t))\,dt\right]\theta(x,t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \tag{2}$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation (1).

To the particular solution $\varphi = k = \text{const}$ of equation (2), there corresponds the solution presented in Item 1°. The general solution of equation (2) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

3°. Solution for the special case $g(z) = -z^2 f(z)$:

$$u = r(x,t)\sin\varphi(t), \quad w = r(x,t)\cos\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = f(\tan\varphi)\tan\varphi,$$

and the function $r = r(x,t)$ satisfies the linear heat equation

$$\frac{\partial r}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial r}{\partial x}\right). \tag{3}$$

4°. Solution for the special case $g(z) = z^2 f(z)$:

$$u = r(x,t)\sinh\varphi(t), \quad w = r(x,t)\cosh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = f(\tanh\varphi)\tanh\varphi,$$

and the function $r = r(x,t)$ satisfies the linear heat equation (3).

Remark. In this case, there are also solutions of the form

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t).$$

⊙ *References for system* 17.2.2.1: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2203.pdf), A. D. Polyanin and E. A. Vyazmina (2006).

2. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial u}{\partial x}\right) + uf\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + wg\left(\dfrac{u}{w}\right).$

1°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = -ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = -bk^2\psi + \psi g(\varphi/\psi).$$

2°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi'_t = ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi'_t = bk^2\psi + \psi g(\varphi/\psi).$$

3°. Multiplicative separable solution:

$$u = e^{-\lambda t}y(x), \quad w = e^{-\lambda t}z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x + \lambda y + yf(y/z) = 0,$$
$$bx^{-n}(x^n z'_x)'_x + \lambda z + zg(y/z) = 0.$$

⊙ *References for system* 17.2.2.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2204.pdf).

3. $\dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial u}{\partial x}\right) + uf\left(\dfrac{u}{w}\right) + \dfrac{u}{w}h\left(\dfrac{u}{w}\right),$
$\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + wg\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Solution:

$$u = \varphi(t)G(t)\left[\theta(x,t) + \int \frac{h(\varphi)}{G(t)}\,dt\right], \quad w = G(t)\left[\theta(x,t) + \int \frac{h(\varphi)}{G(t)}\,dt\right], \quad G(t) = \exp\left[\int g(\varphi)\,dt\right],$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \tag{1}$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right).$$

The general solution of equation (1) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

⊙ *Reference*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2205.pdf).

4. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf_1\left(\frac{w}{u}\right) + wg_1\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + uf_2\left(\frac{w}{u}\right) + wg_2\left(\frac{w}{u}\right).$$

Solution:

$$u = \exp\left\{\int [f_1(\varphi) + \varphi g_1(\varphi)]\,dt\right\}\theta(x,t), \quad w(x,t) = \varphi(t)\exp\left\{\int [f_1(\varphi) + \varphi g_1(\varphi)]\,dt\right\}\theta(x,t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi'_t = f_2(\varphi) + \varphi g_2(\varphi) - \varphi[f_1(\varphi) + \varphi g_1(\varphi)],$$

and the function $\theta = \theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right).$$

5. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u^k f\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w^k g\left(\frac{u}{w}\right).$$

Self-similar solution:

$$u = (C_1 t + C_2)^{\frac{1}{1-k}} y(\xi), \quad w = (C_1 t + C_2)^{\frac{1}{1-k}} z(\xi), \quad \xi = \frac{x}{\sqrt{C_1 t + C_2}},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$a\xi^{-n}(\xi^n y'_\xi)'_\xi + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{k-1}y + y^k f(y/z) = 0,$$
$$b\xi^{-n}(\xi^n z'_\xi)'_\xi + \frac{1}{2}C_1\xi z'_\xi + \frac{C_1}{k-1}z + z^k g(y/z) = 0.$$

⊙ *Reference*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2206.pdf).

6. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right)\ln u + ug\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf\left(\frac{u}{w}\right)\ln w + wh\left(\frac{u}{w}\right).$$

Solution:

$$u = \varphi(t)\psi(t)\theta(x,t), \quad w = \psi(t)\theta(x,t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by solving the autonomous ordinary differential equations

$$\begin{aligned}\varphi'_t &= \varphi[g(\varphi) - h(\varphi) + f(\varphi)\ln\varphi],\\ \psi'_t &= \psi[h(\varphi) + f(\varphi)\ln\psi],\end{aligned} \tag{1}$$

and the function $\theta = \theta(x,t)$ is determined by the differential equation

$$\frac{\partial\theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right) + f(\varphi)\theta\ln\theta. \tag{2}$$

The first equation in (1) is a separable equation; its solution can be written out in implicit form. The second equation in (1) can be solved using the change of variable $\psi = e^\zeta$ (it is reduced to a linear equation for ζ).

Equation (2) admits exact solutions of the form

$$\theta = \exp\bigl[\sigma_2(t)x^2 + \sigma_0(t)\bigr],$$

where the functions $\sigma_n(t)$ are described by the equations

$$\begin{aligned}\sigma'_2 &= f(\varphi)\sigma_2 + 4a\sigma_2^2,\\ \sigma'_0 &= f(\varphi)\sigma_0 + 2a(n+1)\sigma_2.\end{aligned}$$

This system can be successively integrated, since the first equation is a Bernoulli equation and the second one is linear in the unknown.

If $f = \text{const}$, equation (2) also has a traveling-wave solution $\theta = \theta(kx - \lambda t)$.

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2207.pdf).

17.2.3. Arbitrary Functions Depend on the Product of Powers of the Unknowns

1. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(x, u^k w^m),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wg(x, u^k w^m).$$

Multiplicative separable solution:

$$u = e^{-m\lambda t}y(x), \quad w = e^{k\lambda t}z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}ax^{-n}(x^n y'_x)'_x + m\lambda y + yf(x, y^k z^m) &= 0,\\ bx^{-n}(x^n z'_x)'_x - k\lambda z + zg(x, y^k z^m) &= 0.\end{aligned}$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2208.pdf).

2. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u^{1+kn} f\left(u^n w^m\right),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w^{1-km} g\left(u^n w^m\right).$$

Self-similar solution:

$$u = (C_1 t + C_2)^{-\frac{1}{kn}} y(\xi), \qquad w = (C_1 t + C_2)^{\frac{1}{km}} z(\xi), \qquad \xi = \frac{x}{\sqrt{C_1 t + C_2}},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$a\xi^{-n}(\xi^n y'_\xi)'_\xi + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{kn}y + y^{1+kn} f\left(y^n z^m\right) = 0,$$
$$b\xi^{-n}(\xi^n z'_\xi)'_\xi + \frac{1}{2}C_1\xi z'_\xi - \frac{C_1}{km}z + z^{1-km} g\left(y^n z^m\right) = 0.$$

3. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + cu\ln u + uf(x, u^k w^m),$$
$$\frac{\partial w}{\partial t} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + cw\ln w + wg(x, u^k w^m).$$

Multiplicative separable solution:

$$u = \exp(Ame^{ct})y(x), \qquad w = \exp(-Ake^{ct})z(x),$$

where A is an arbitrary constant, and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x + cy\ln y + yf(x, y^k z^m) = 0,$$
$$bx^{-n}(x^n z'_x)'_x + cz\ln z + zg(x, y^k z^m) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2209.pdf).

17.2.4. Arbitrary Functions Depend on Sum or Difference of Squares of the Unknowns

1. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2+w^2) - wg(u^2+w^2),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2+w^2) + ug(u^2+w^2).$$

Time-periodic solution:

$$u = r(x)\cos\left[\theta(x) + C_1 t + C_2\right], \qquad w = r(x)\sin\left[\theta(x) + C_1 t + C_2\right],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$ar''_{xx} - ar(\theta'_x)^2 + \frac{an}{x}r'_x + rf(r^2) = 0,$$
$$ar\theta''_{xx} + 2ar'_x\theta'_x + \frac{an}{x}r\theta'_x + rg(r^2) - C_1 r = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2211.pdf).

2. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2-w^2) + wg(u^2-w^2),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2-w^2) + ug(u^2-w^2).$$

Solution:

$$u = r(x)\cosh\big[\theta(x) + C_1 t + C_2\big], \qquad w = r(x)\sinh\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$ar''_{xx} + ar(\theta'_x)^2 + \frac{an}{x}r'_x + rf(r^2) = 0,$$
$$ar\theta''_{xx} + 2ar'_x\theta'_x + \frac{an}{x}r\theta'_x + rg(r^2) - C_1 r = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2212.pdf).

17.2.5. Arbitrary Functions Have Different Arguments

1. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(bu+cw) - cwg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(bu+cw) + bwg\left(\frac{w}{u}\right).$$

Solution:

$$u = cr(x,t)\cos^2\varphi(t), \qquad w = br(x,t)\sin^2\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = \frac{1}{2}b\tan\varphi\, g\left(\frac{b}{c}\tan^2\varphi\right), \tag{1}$$

and the function $r = r(x,t)$ satisfies the equation

$$\frac{\partial r}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial r}{\partial x}\right) + rf(bcr). \tag{2}$$

The general solution to equation (1) can be written in implicit form as

$$\int \frac{d\varphi}{\tan\varphi\, g(bc^{-1}\tan^2\varphi)} = \frac{1}{2}bt + C.$$

Note that equation (2) admits a stationary solution $r = r(x)$.

2. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2+w^2) - wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2+w^2) + ug\left(\frac{w}{u}\right).$$

Solution:

$$u = r(x,t)\cos\varphi(t), \qquad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi'_t = g(\tan\varphi), \tag{1}$$

and the function $r = r(x,t)$ is determined by the differential equation

$$\frac{\partial r}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial r}{\partial x}\right) + rf(r^2). \tag{2}$$

The general solution of equation (1) is expressed in implicit form as

$$\int \frac{d\varphi}{g(\tan\varphi)} = t + C.$$

Equation (2) admits a time-independent exact solution $r = r(x)$.

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2213.pdf).

3. $$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(u^2-w^2) + wg\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(u^2-w^2) + ug\left(\frac{w}{u}\right).$$

Solution:

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi'_t = g(\tanh\varphi), \tag{1}$$

and the function $r = r(x,t)$ is determined by the differential equation

$$\frac{\partial r}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial r}{\partial x}\right) + rf(r^2). \tag{2}$$

The general solution of equation (1) is expressed in implicit form as

$$\int \frac{d\varphi}{g(\tanh\varphi)} = t + C.$$

Equation (2) admits a time-independent exact solution $r = r(x)$.

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde2214.pdf).

17.3. Other Systems of Two Parabolic Equations

17.3.1. Second-Order Equations Involving Real Functions of Real Variables

1. $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(au+bw)\frac{\partial u}{\partial x}\right] + g(au+bw),$$
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(au+bw)\frac{\partial w}{\partial x}\right] + h(au+bw).$$

Solution:

$$u = y(\xi) + b\theta(\xi,t), \quad w = z(\xi) - a\theta(\xi,t), \quad \xi = kx - \lambda t,$$

where the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$k^2[f(ay+bz)y'_\xi]'_\xi + \lambda y'_\xi + g(ay+bz) = 0,$$
$$k^2[f(ay+bz)z'_\xi]'_\xi + \lambda z'_\xi + h(ay+bz) = 0,$$

and the function $\theta = \theta(\xi, t)$ satisfies the heat equation with variable coefficients

$$\frac{\partial\theta}{\partial t} = k^2\frac{\partial}{\partial\xi}\Big[f(ay+bz)\frac{\partial\theta}{\partial\xi}\Big] + \lambda\frac{\partial\theta}{\partial\xi}.$$

Once the functions $y = y(\xi)$ and $z = z(\xi)$ have been determined, the equation becomes linear in θ; its exact solutions can be obtained using the procedure outlined in Polyanin (2002).

2. $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(t, au+bw)\dfrac{\partial u}{\partial x}\Big] + g(t, au+bw),$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(t, au+bw)\dfrac{\partial w}{\partial x}\Big] + h(t, au+bw).$

Solution:

$$u = \varphi(t) - b\theta(x,\tau), \qquad w = \psi(t) + a\theta(x,\tau), \qquad \tau = \int f(t, a\varphi+b\psi)\,dt,$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = g(t, a\varphi+b\psi),$$
$$\psi'_t = h(t, a\varphi+b\psi),$$

and the function $\theta = \theta(x, \tau)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial x^2}.$$

3. $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big[f\Big(t, \dfrac{u}{w}\Big)\dfrac{\partial u}{\partial x}\Big] + ug\Big(t, \dfrac{u}{w}\Big),$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big[f\Big(t, \dfrac{u}{w}\Big)\dfrac{\partial w}{\partial x}\Big] + wh\Big(t, \dfrac{u}{w}\Big).$

Solution:

$$u = \varphi(t)\exp\Big[\int h(t,\varphi(t))\,dt\Big]\theta(x,\tau), \quad w = \exp\Big[\int h(t,\varphi(t))\,dt\Big]\theta(x,\tau), \quad \tau = \int f(t,\varphi(t))\,dt,$$

where the function $\varphi = \varphi(t)$ is described by the ordinary differential equation

$$\varphi'_t = [g(t,\varphi) - h(t,\varphi)]\varphi,$$

and the function $\theta = \theta(x, \tau)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial x^2}.$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

4. $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(au+bw)\dfrac{\partial u}{\partial x}\Big] + ug(au+bw) - bwh\Big(\dfrac{w}{u}\Big),$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(au+bw)\dfrac{\partial w}{\partial x}\Big] + wg(au+bw) + awh\Big(\dfrac{w}{u}\Big).$

1°. Solution:

$$u = br(x,t)\cos^2\varphi(t), \quad w = ar(x,t)\sin^2\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = \frac{1}{2}a\tan\varphi\, h\left(\frac{a}{b}\tan^2\varphi\right), \tag{1}$$

and $r = r(x,t)$ satisfies an equation of the form 5.6.15.2:

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\Big[f(abr)\frac{\partial r}{\partial x}\Big] + rg(abr). \tag{2}$$

The general solution to equation (1) can be represented in implicit form. Equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

2°. Solution:

$$u = br(x,t)\cosh^2\psi(t), \quad w = -ar(x,t)\sinh^2\psi(t),$$

where the function $\psi = \psi(t)$ is determined by the separable first-order ordinary differential equation

$$\psi'_t = \frac{1}{2}a\tanh\psi\, h\left(-\frac{a}{b}\tanh^2\psi\right),$$

and the function $r = r(x,t)$ satisfies equation (2).

3°. Solution:

$$u = -br(x,t)\sinh^2\theta(t), \quad w = ar(x,t)\cosh^2\theta(t),$$

where the function $\theta = \theta(t)$ is determined by the separable first-order ordinary differential equation

$$\theta'_t = \frac{1}{2}a\coth\theta\, h\left(-\frac{a}{b}\coth^2\theta\right),$$

and the function $r = r(x,t)$ satisfies equation (2).

5. $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(u^2+w^2)\dfrac{\partial u}{\partial x}\Big] + ug(u^2+w^2) - wh\Big(\dfrac{w}{u}\Big),$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big[f(u^2+w^2)\dfrac{\partial w}{\partial x}\Big] + wg(u^2+w^2) + uh\Big(\dfrac{w}{u}\Big).$

Solution:

$$u = r(x,t)\cos\varphi(t), \quad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = h(\tan\varphi), \tag{1}$$

and $r = r(x,t)$ satisfies an equation of the form 5.6.15.2:

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\Big[f(r^2)\frac{\partial r}{\partial x}\Big] + rg(r^2). \tag{2}$$

The general solution to equation (1) can be written in implicit form as

$$\int \frac{d\varphi}{h(\tan\varphi)} = t + C.$$

Equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

6. $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big[f(u^2 - w^2)\frac{\partial u}{\partial x}\Big] + ug(u^2 - w^2) + wh\Big(\frac{w}{u}\Big),$$
$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\Big[f(u^2 - w^2)\frac{\partial w}{\partial x}\Big] + wg(u^2 - w^2) + uh\Big(\frac{w}{u}\Big).$$

Solution:

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi'_t = h(\tanh\varphi), \tag{1}$$

and $r = r(x,t)$ satisfies an equation of the form 5.6.15.2:

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\Big[f(r^2)\frac{\partial r}{\partial x}\Big] + rg(r^2). \tag{2}$$

The general solution to equation (1) can be written in implicit form as

$$\int \frac{d\varphi}{h(\tanh\varphi)} = t + C.$$

Equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

7. $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(w\frac{\partial u}{\partial x}\right) + u - 2w + 1, \qquad \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) - 2u + w.$$

Generalized separable solution quadratic in x:

$$u = \tfrac{1}{2}x^2 + ax + ce^{-t} + de^{2t} - \tfrac{1}{18}b^2e^{-4t} + \tfrac{1}{2}a^2,$$
$$w = -\tfrac{1}{2}x^2 + (be^{-2t} - a)x + 2ce^{-t} - de^{2t} - \tfrac{5}{18}b^2e^{-4t} + abe^{-2t} - \tfrac{1}{2}a^2 + 1,$$

where a, b, c, and d are arbitrary constants.

⊙ *Reference*: O. V. Kaptsov (2009, p. 155).

8. $$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\big[(d_1 + d_{11}u + d_{12}w)u\big] + u(a_1 - b_1u - c_1w),$$
$$\frac{\partial w}{\partial t} = \frac{\partial^2}{\partial x^2}\big[(d_2 + d_{21}u + d_{22}w)w\big] + w(a_2 - b_2u - c_2w).$$

1°. Let

$$b_1 = b_2 = b, \quad c_1 = c_2 = c, \quad a_2 = -\frac{a_1c}{b}, \quad d_{12} = d_{22} = \frac{cd_{11}}{b}, \quad d_{21} = d_{11}, \quad d_2 = d_1 + 4\frac{a_1d_{11}}{b}\left(1 + \frac{c}{b}\right),$$

then the solution is

$$u = \frac{a_1}{b}\frac{e^{\lambda t}}{e^{\lambda t}+C_1} + e^{At}\big(e^{\lambda t}+C_1\big)^{-3/4}\big(C_2e^{\gamma x}+C_3e^{-\gamma x}\big),$$
$$w = -\frac{a_1}{b}\frac{C_1}{e^{\lambda t}+C_1} - \frac{b}{c}e^{At}\big(e^{\lambda t}+C_1\big)^{-3/4}\big(C_2e^{\gamma x}+C_3e^{-\gamma x}\big),$$

where

$$\lambda = \frac{a_1}{b}(b+c), \quad A = a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c, \quad \gamma^2 = \frac{b_1}{4d_{11}} = \frac{c_1}{4d_{12}} = \frac{b_2}{4d_{21}} = \frac{c_2}{4d_{22}} > 0,$$

and C_1, C_2, and C_3 are arbitrary constants.

2°. Let

$$b_1 = b_2 = b, \quad c_1 = c_2 = c, \quad d_{12} = d_{22} = \frac{cd_{11}}{b}, \quad d_{21} = d_{11},$$

then the solution is

$$u = \frac{a_1}{b}\frac{C_1}{e^{(a_2-a_1)t}+C_1} + e^{At}\big[e^{(a_2-a_1)t}+C_1\big]^k\big(C_2e^{\gamma x}+C_3e^{-\gamma x}\big),$$
$$w = \frac{a_2}{c}\frac{e^{(a_2-a_1)t}}{e^{(a_2-a_1)t}+C_1} - \frac{b}{c}e^{At}\big[e^{(a_2-a_1)t}+C_1\big]^k\big(C_2e^{\gamma x}+C_3e^{-\gamma x}\big),$$

where

$$A = \Big(d_1 + \frac{a_1}{b}d_{11}\Big)\gamma^2, \quad k = \frac{1}{b}d_{11}\gamma^2 - 1, \quad \gamma^2 = \frac{a_2-a_1}{d_1-d_2} > 0,$$

and C_1, C_2, and C_3 are arbitrary constants.

3°. Let

$$b_1{=}b_2{=}b, \quad c_1{=}c_2{=}c, \quad a_2{=}{-}\frac{a_1c}{b}, \quad d_{12}{=}d_{22}{=}\frac{cd_{11}}{b}, \quad d_{21}{=}d_{11}, \quad d_2{=}d_1{+}4\frac{a_1d_{11}}{b}\Big(1{+}\frac{c}{b}\Big),$$

then the solution is

$$u = \frac{a_1}{b}\frac{e^{\lambda t}}{e^{\lambda t}+C_1} + e^{At}\big(e^{\lambda t}+C_1\big)^{-3/4}\big[C_2\cos(\gamma x)+C_3\sin(\gamma x)\big],$$
$$w = -\frac{a_1}{b}\frac{C_1}{e^{\lambda t}+C_1} - \frac{b}{c}e^{At}\big(e^{\lambda t}+C_1\big)^{-3/4}\big[C_2\cos(\gamma x)+C_3\sin(\gamma x)\big],$$

where

$$\lambda = \frac{a_1}{b}(b+c), \quad A = a_1 + \frac{bd_1}{4d_{11}} + \frac{3}{4}\frac{a_1}{b}c, \quad \frac{b}{4d_{11}} = -\gamma^2 < 0,$$

and C_1, C_2, and C_3 are arbitrary constants.

4°. Let

$$b_1 = b_2 = b, \quad c_1 = c_2 = c, \quad d_{12} = d_{22} = \frac{cd_{11}}{b}, \quad d_{21} = d_{11},$$

then the solution is

$$u = \frac{a_1}{b}\frac{C_1}{e^{(a_2-a_1)t}+C_1} + e^{-At}\big[e^{(a_2-a_1)t}+C_1\big]^{-k}\big[C_2\cos(\gamma x)+C_3\sin(\gamma x)\big],$$
$$w = \frac{a_2}{c}\frac{e^{(a_2-a_1)t}}{e^{(a_2-a_1)t}+C_1} - \frac{b}{c}e^{-At}\big[e^{(a_2-a_1)t}+C_1\big]^{-k}\big[C_2\cos(\gamma x)+C_3\sin(\gamma x)\big],$$

where

$$A = \left(d_1 + \frac{a_1}{b}d_{11}\right)\gamma^2, \quad k = \frac{1}{b}d_{11}\gamma^2 + 1, \quad \gamma^2 = \frac{a_1 - a_2}{d_1 - d_2} > 0,$$

and C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference for system* 17.3.1.8: R. M. Cherniha and L. Myroniuk (2008).

9. $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(w\dfrac{\partial u}{\partial x} - u\dfrac{\partial w}{\partial x}\right), \qquad \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}.$

This system is a simple model for the solid-state diffusion of a substitutional impurity by a vacancy mechanism.

1°. Solution:

$$u = A_1(t) + A_2(t)x + A_3(t)x^2 + A_4(t)x^3,$$
$$w = B_1(t) + B_2(t)x + B_3x^2 + B_4x^3,$$

where B_3 and B_4 are arbitrary constants, and the functions $A_n = A_n(t)$, $B_1 = B_1(t)$, and $B_2 = B_2(t)$ are determined by the system of ordinary differential equations

$$A_1' = 2(B_1A_3 - A_1B_3),$$
$$A_2' = 2(B_2A_3 - A_2B_3) + 6(B_1A_4 - A_1B_4),$$
$$A_3' = 6(B_2A_4 - A_2B_4),$$
$$A_4' = 4(B_3A_4 - A_3B_4),$$
$$B_1' = 2A_3, \quad B_2' = 6A_4.$$

2°. Solution:

$$u = A_1(t) + A_2(t)x + A_3(t)x^2 + A_4(t)x^3 + A_5(t)x^4,$$
$$w = B_1(t) + B_2(t)x + B_3(t)x^2,$$

where the functions $A_n = A_n(t)$ and $B_m = B_m(t)$ are determined by the system of ordinary differential equations

$$A_1' = 2(B_1A_3 - A_1B_3),$$
$$A_2' = 2(B_2A_3 - A_2B_3) + 6B_1A_4,$$
$$A_3' = 6B_2A_4 + 12B_1A_5,$$
$$A_4' = 4B_3A_4 + 12B_2A_5,$$
$$A_5' = 10B_3A_5,$$
$$B_1' = 2A_3, \quad B_2' = 6A_4, \quad B_3' = 12A_5.$$

⊙ *Reference for system* 17.3.1.9: J. R. King (1990 *a*).

17.3.2. Second-Order Nonlinear Equations of Laser Systems

Preliminary remarks. The phenomena of the wave front correction for optical radiations in laser systems are simulated by nonlinear equations known as the system of phase conjugate reflection (wave front reversal) equations (Zel'dovitch, Pilipetskii, and Shkurov (1985) and Bespalov and Pasmannik (1986)). Here, we shall consider a simplified (by choosing the medium parameters) variant of this system and restrict the discussion of the stationary case. In what follows, E_1 and E_2 denote the complex amplitudes of the incident and phase conjugate (amplified) light waves, respectively, and $i^2 = -1$.

1. $\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial^2}{\partial y^2}\right)\boldsymbol{E_1} = |\boldsymbol{E_2}|^2\boldsymbol{E_1}, \qquad \left(\dfrac{\partial}{\partial x} + i\dfrac{\partial^2}{\partial y^2}\right)\boldsymbol{E_2} = |\boldsymbol{E_1}|^2\boldsymbol{E_2}.$

1°. Solution:

$$u = u(x)\exp[i(-C_1^2x + C_1y + C_2)], \qquad w = w(x)\exp[i(C_3^2x + C_3y + C_4)],$$

where C_1, C_2, C_3, and C_4 are arbitrary real constants, and the real functions $u = u(x)$ and $w = w(x)$ are determined by the system of ordinary differential equations

$$u'_x = w^2u, \qquad w'_x = u^2w.$$

The general solution of this system is

$$u = \pm\sqrt{\frac{B}{1 - Ae^{2Bx}}}, \qquad w = \pm\sqrt{\frac{ABe^{2Bx}}{1 - Ae^{2Bx}}},$$

where A and B are arbitrary constants; the plus or minus signs are selected arbitrarily.

2°. Solution:

$$E_1 = u(x)\exp\left(\frac{iy^2}{4x}\right), \qquad E_2 = w(x)\exp\left(-\frac{iy^2}{4x}\right),$$

where the real functions $u = u(x)$ and $w = w(x)$ are determined by the system of ordinary differential equations

$$\frac{du}{dx} = w^2u - \frac{u}{2x}, \qquad \frac{dw}{dx} = u^2w - \frac{w}{2x}.$$

The general solutions of this system is given by

$$u = \pm\sqrt{\frac{k}{x(1-\zeta^2)}}, \qquad w = \pm\zeta\sqrt{\frac{k}{x(1-\zeta^2)}}, \qquad \zeta = Cx^k,$$

where C and k are arbitrary constants; the plus or minus signs are selected arbitrarily.

3°. About other exact solutions see equation 17.3.2.2 with $a = -1$, $b = 1$, $f(\xi,\eta) = |\eta|^2$, and $g(\xi,\eta) = |\xi|^2$.

⊙ *Reference for system* 17.3.2.1: N. H. Ibragimov (1995).

2. $\left(\dfrac{\partial}{\partial x} + ia\dfrac{\partial^2}{\partial y^2}\right)\boldsymbol{E_1} = \boldsymbol{f}(|\boldsymbol{E_1}|, |\boldsymbol{E_2}|)\boldsymbol{E_1},$

$\left(\dfrac{\partial}{\partial x} + ib\dfrac{\partial^2}{\partial y^2}\right)\boldsymbol{E_2} = \boldsymbol{g}(|\boldsymbol{E_1}|, |\boldsymbol{E_2}|)\boldsymbol{E_2}.$

This system is a considerable generalization of the preceding system of equations; a and b are arbitrary real constants, while $f(\xi,\eta)$ and $g(\xi,\eta)$ are real functions of real variables.

It is convenient to seek solutions to the original system in the form

$$E_1 = u(x,y)\exp[i\varphi(x,y)], \qquad E_2 = w(x,y)\exp[i\psi(x,y)], \tag{1}$$

where $u = u(x,y)$, $w = w(x,y)$, $\varphi = \varphi(x,y)$, and $\psi = \psi(x,y)$ are real functions of real variables that satisfy the system of equations

$$\begin{aligned} u_x - au\varphi_{yy} - 2au_y\varphi_y &= f(|u|,|w|)u,\\ u\varphi_x + au_{yy} - au\varphi_y^2 &= 0,\\ w_x - bw\psi_{yy} - 2bw_y\psi_y &= g(|u|,|w|)w,\\ w\psi_x + bw_{yy} - bw\psi_y^2 &= 0. \end{aligned} \tag{2}$$

Listed below are some exact solutions to system (2) that allow one to obtain exact solutions to the original system by using formulas (1).

1°. Solution:

$$u = u(x), \quad w = w(x), \quad \varphi = aC_1^2 x + C_1 y + C_2, \quad \psi = bC_3^2 x + C_3 y + C_4,$$

where C_1, C_2, C_3, and C_4 are arbitrary real constants, and the functions $u = u(x)$ and $w = w(x)$ are determined by the system of ordinary differential equations

$$u'_x = f(|u|, |w|)u, \quad w'_x = g(|u|, |w|)w.$$

2°. Solution:

$$u = u(x), \quad w = w(x), \quad \varphi = -\frac{(y + A_1)^2}{4a(x + B_1)} + C_1, \quad \psi = -\frac{(y + A_2)^2}{4b(x + B_2)} + C_2,$$

where A_n, B_n, and C_n are arbitrary constants, and the functions $u = u(x)$ and $w = w(x)$ are determined by the system of ordinary differential equations

$$u'_x + \frac{1}{2(x + B_1)}u = f(|u|, |w|)u, \quad w'_x + \frac{1}{2(x + B_2)}w = g(|u|, |w|)w.$$

3°. Solution:

$$u = u(y), \quad w = w(y), \quad \varphi = C_1 x + \xi(y), \quad \psi = C_2 x + \eta(y),$$

where C_1 and C_2 are arbitrary constants, and the functions $u = u(y)$, $w = w(y)$, $\xi = \xi(y)$, and $\eta = \eta(y)$ are determined by the system of ordinary differential equations

$$\begin{aligned} -au\xi''_{yy} - 2au'_y\xi'_y &= f(|u|, |w|)u, \\ C_1 u + au''_{yy} - au(\xi'_y)^2 &= 0, \\ -bw\eta''_{yy} - 2bw'_y\eta'_y &= g(|u|, |w|)w, \\ C_2 w + bw''_{yy} - bw(\eta'_y)^2 &= 0. \end{aligned}$$

4°. There is a traveling-wave solution of the form

$$u = u(z), \quad w = w(z), \quad \varphi = \varphi(z), \quad \psi = \psi(z), \quad z = k_1 x + k_2 y.$$

Remark. For the solutions given in Items 1° and 2°, the functions f and g in the equation concerned can depend on an additional, third argument x. For the solution from Item 3°, the functions f and g can depend on an additional, third argument y.

17.3.3. Systems Involving Third-Order Evolution Equations

1. $\dfrac{\partial u}{\partial t} + a\dfrac{\partial w}{\partial x} + a\dfrac{\partial}{\partial x}(uw) + \dfrac{\partial^3 w}{\partial x^3} = 0, \qquad \dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial x} + aw\dfrac{\partial w}{\partial x} = 0.$

Traveling-wave solution:

$$u = \frac{2\lambda^2}{a}\Big[1 - \tanh^2\big(\lambda x + 2\lambda^2 t + C\big)\Big] - 1, \quad w = -\frac{2\lambda}{a}\Big[1 - \tanh\big(\lambda x + 2\lambda^2 t + C\big)\Big],$$

where C and λ are arbitrary constants.

⊙ *Reference*: L. Zhang (2011).

2. $\dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} + 2au\dfrac{\partial u}{\partial x} - bw\dfrac{\partial w}{\partial x} = 0, \qquad \dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + au\dfrac{\partial w}{\partial x} = 0.$

A modification of the coupled KdV equation.

Traveling-wave solution with $ab > 0$:

$$u = \frac{3\lambda^2}{a\cosh^2\big(\frac{1}{2}\lambda x - \frac{1}{2}\lambda^3 t + C\big)}, \quad w = \frac{3\lambda^2}{\sqrt{ab}\,\cosh^2\big(\frac{1}{2}\lambda x - \frac{1}{2}\lambda^3 t + C\big)}.$$

⊙ *Reference*: L. Zhang (2011).

Chapter 18
Systems of Two Second-Order Klein–Gordon Type Hyperbolic Equations

18.1. Systems of the Form $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + F(u, w),\ \dfrac{\partial^2 w}{\partial t^2} = b\dfrac{\partial^2 w}{\partial x^2} + G(u, w)$

18.1.1. Arbitrary Functions Depend on a Linear Combination of the Unknowns

1. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + c_2 f(b_1 u + c_1 w) + c_1 g(b_2 u + c_2 w),$

$\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} - b_2 f(b_1 u + c_1 w) - b_1 g(b_2 u + c_2 w).$

The condition $b_1 c_2 - b_2 c_1 \neq 0$ is assumed to hold.

Multiplying the equations by appropriate constants and then adding together, one arrives at two independent equations:

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^2 U}{\partial x^2} + (b_1 c_2 - b_2 c_1) f(U), \qquad U = b_1 u + c_1 w;$$
$$\frac{\partial^2 W}{\partial t^2} = a\frac{\partial^2 W}{\partial x^2} - (b_1 c_2 - b_2 c_1) g(W), \qquad W = b_2 u + c_2 w.$$

For solutions of these equations, see 7.4.1.1.

2. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + u f(bu - cw) + g(bu - cw),$

$\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + w f(bu - cw) + h(bu - cw).$

1°. Solution:

$$u = \varphi(t) + c\theta(x, t), \qquad w = \psi(t) + b\theta(x, t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi''_{tt} = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x,t)$ satisfies linear equation

$$\frac{\partial^2\theta}{\partial t^2} = a\frac{\partial^2\theta}{\partial x^2} + f(b\varphi - c\psi)\theta.$$

For f = const, this equation can be solved by separation of variables.

2°. Solution:

$$u = \widetilde{\varphi}(x) + c\widetilde{\theta}(x,t), \qquad w = \widetilde{\psi}(x) + b\widetilde{\theta}(x,t),$$

where the functions $\widetilde{\varphi} = \widetilde{\varphi}(x)$ and $\widetilde{\psi} = \widetilde{\psi}(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n\widetilde{\varphi}'_x)'_x + \widetilde{\varphi}f(b\widetilde{\varphi} - c\widetilde{\psi}) + g(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$
$$ax^{-n}(x^n\widetilde{\psi}'_x)'_x + \widetilde{\psi}f(b\widetilde{\varphi} - c\widetilde{\psi}) + h(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$

and the function $\widetilde{\theta} = \widetilde{\theta}(x,t)$ satisfies the linear equation

$$\frac{\partial^2\widetilde{\theta}}{\partial t^2} = a\frac{\partial^2\widetilde{\theta}}{\partial x^2} + f(b\widetilde{\varphi} - c\widetilde{\psi})\widetilde{\theta}.$$

For f = const, this equation can be solved by separation of variables.

3°. Let us multiply the first equation by b and the second one by $-c$ and add the results together to obtain

$$\frac{\partial^2\zeta}{\partial t^2} = a\frac{\partial^2\zeta}{\partial x^2} + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + uf(\zeta) + g(\zeta). \tag{2}$$

Equation (1) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ to equation (1), the function $u = u(x,t)$ can be determined by solving equation (2) and the function $w = w(x,t)$ is found as $w = (bu - \zeta)/c$.

Note three important solutions to equation (1):

(i) In the general case, equation (1) admits a spatially homogeneous solution $\zeta = \zeta(t)$. The corresponding solution to the original system is given in Item 1° in another form.

(ii) In the general case, equation (1) admits a steady-state solution $\zeta = \zeta(x)$. The corresponding exact solutions to equation (2) are expressed as $u = u_0(x) + \sum e^{-\beta_n t}u_n(x)$ and $u = u_0(x) + \sum \cos(\beta_n t)u_n^{(1)}(x) + \sum \sin(\beta_n t)u_n^{(2)}(x)$.

(iii) If the condition $\zeta f(\zeta) + bg(\zeta) - ch(\zeta) = k_1\zeta + k_0$ holds, equation (1) is linear,

$$\frac{\partial^2\zeta}{\partial t^2} = a\frac{\partial^2\zeta}{\partial x^2} + k_1\zeta + k_0,$$

and, hence, can be solved by separation of variables.

For other solutions to equation (1), see equation 7.4.1.1.

⊙ *References for system* 18.1.1.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4101.pdf).

3. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + e^{\lambda u} f(\lambda u - \sigma w), \qquad \dfrac{\partial^2 w}{\partial t^2} = b\dfrac{\partial^2 w}{\partial x^2} + e^{\sigma w} g(\lambda u - \sigma w).$

1°. Solution:

$$u = y(\xi) - \frac{2}{\lambda}\ln(C_1 t + C_2), \qquad w = z(\xi) - \frac{2}{\sigma}\ln(C_1 t + C_2), \qquad \xi = \frac{x}{C_1 t + C_2},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$C_1^2(\xi^2 y'_\xi)'_\xi + 2C_1^2\lambda^{-1} = a\xi^{-n}(\xi^n y'_\xi)'_\xi + e^{\lambda y} f(\lambda y - \sigma z),$$
$$C_1^2(\xi^2 z'_\xi)'_\xi + 2C_1^2\sigma^{-1} = b\xi^{-n}(\xi^n z'_\xi)'_\xi + e^{\sigma z} g(\lambda y - \sigma z).$$

2°. Solution with $b = a$:

$$u = \theta(x,t), \qquad w = \frac{\lambda}{\sigma}\theta(x,t) - \frac{k}{\sigma},$$

where k is a root of the algebraic (transcendental) equation

$$\lambda f(k) = \sigma e^{-k} g(k),$$

and the function $\theta = \theta(x,t)$ is described by the equation

$$\frac{\partial^2 \theta}{\partial t^2} = a\frac{\partial^2 \theta}{\partial x^2} + f(k)e^{\lambda\theta}.$$

It is a solvable equation; see 7.2.1.1.

⊙ *References for system* 18.1.1.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4102.pdf).

18.1.2. Arbitrary Functions Depend on the Ratio of the Unknowns

1. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + u f\left(\dfrac{u}{w}\right), \qquad \dfrac{\partial^2 w}{\partial t^2} = b\dfrac{\partial^2 w}{\partial x^2} + w g\left(\dfrac{u}{w}\right).$

1°. Periodic multiplicative separable solution:

$$u = [C_1\cos(kt) + C_2\sin(kt)]y(x), \qquad w = [C_1\cos(kt) + C_2\sin(kt)]z(x), \tag{1}$$

where C_1, C_2, and k are arbitrary constants and the functions $y = y(x)$ and $z = z(x)$ are determined by the autonomous system of ordinary differential equations

$$ay''_{xx} + k^2 y + y f(y/z) = 0,$$
$$bz''_{xx} + k^2 z + z g(y/z) = 0.$$

2°. Multiplicative separable solution:

$$u = [C_1\exp(kt) + C_2\exp(-kt)]y(x), \qquad w = [C_1\exp(kt) + C_2\exp(-kt)]z(x), \tag{2}$$

where C_1, C_2, and k are arbitrary constants and the functions $y = y(x)$ and $z = z(x)$ are determined by the autonomous system of ordinary differential equations

$$ay''_{xx} - k^2 y + yf(y/z) = 0,$$
$$bz''_{xx} - k^2 z + zg(y/z) = 0.$$

3°. Degenerate multiplicative separable solution:

$$u = (C_1 t + C_2)y(x), \qquad w = (C_1 t + C_2)z(x), \tag{3}$$

where the functions $y = y(x)$ and $z = z(x)$ are determined by the autonomous system of ordinary differential equations

$$ay''_{xx} + yf(y/z) = 0,$$
$$bz''_{xx} + zg(y/z) = 0.$$

Remark. In solutions (1)–(3), the variables t and x can be swapped (this will result in slight changes in the determining systems of ordinary differential equations for y and z).

4°. Solution:

$$u = e^{kx-\lambda t}y(\xi), \qquad w = e^{kx-\lambda t}z(\xi), \qquad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$(a\beta^2 - \gamma^2)y''_{\xi\xi} + 2(ak\beta - \gamma\lambda)y'_\xi + (ak^2 - \lambda^2)y + yf(y/z) = 0,$$
$$(b\beta^2 - \gamma^2)z''_{\xi\xi} + 2(bk\beta - \gamma\lambda)z'_\xi + (bk^2 - \lambda^2)z + zg(y/z) = 0.$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution.

5°. Solution with $b = a$:

$$u = k\theta(x,t), \qquad w = \theta(x,t),$$

where k is a root of the algebraic (transcendental) equation $f(k) = g(k)$, and the function $\theta = \theta(x,t)$ is described by the linear Klein–Gordon equation

$$\frac{\partial^2\theta}{\partial t^2} = a\frac{\partial^2\theta}{\partial x^2} + f(k)\theta.$$

For details about this equation, see Polyanin (2002, Subsection 4.1.3).

⊙ *References for system* 18.1.2.1: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4103.pdf).

2. $$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right) + \frac{u}{w}h\left(\frac{u}{w}\right),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial w}\right) + wg\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right).$$

Solution:

$$u = k\theta(x,t), \qquad w = \theta(x,t),$$

where k is a root of the algebraic (transcendental) equation $f(k) = g(k)$, and the function $\theta = \theta(x,t)$ is described by the linear equation

$$\frac{\partial^2\theta}{\partial t^2} = a\frac{\partial^2\theta}{\partial x^2} + f(k)\theta + h(k).$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4104.pdf).

3. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + u^k f\left(\dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial t^2} = b\dfrac{\partial^2 w}{\partial x^2} + w^k g\left(\dfrac{u}{w}\right).$

Self-similar solution:

$$u = (C_1 t + C_2)^{\frac{2}{1-k}} y(\xi), \qquad w = (C_1 t + C_2)^{\frac{2}{1-k}} z(\xi), \qquad \xi = \frac{x}{C_1 t + C_2},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$C_1^2\xi^2 y''_{\xi\xi} + \frac{2C_1^2(k+1)}{k-1}\xi y'_\xi + \frac{C_1^2(k+1)}{(k-1)^2}y = \frac{a}{\xi^n}(\xi^n y'_\xi)'_\xi + y^k f\Big(\frac{y}{z}\Big),$$
$$C_1^2\xi^2 z''_{\xi\xi} + \frac{2C_1^2(k+1)}{k-1}\xi z'_\xi + \frac{C_1^2(k+1)}{(k-1)^2}z = \frac{b}{\xi^n}(\xi^n z'_\xi)'_\xi + z^k g\Big(\frac{y}{z}\Big).$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4105.pdf).

18.1.3. Other Systems

1. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + u f(x, u^k w^m), \qquad \dfrac{\partial^2 w}{\partial t^2} = b\dfrac{\partial^2 w}{\partial x^2} + w g(x, u^k w^m).$

Multiplicative separable solution:

$$u = e^{-m\lambda t} y(x), \quad w = e^{k\lambda t} z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$a y''_{xx} - m^2\lambda^2 y + y f(x, y^k z^m) = 0,$$
$$b z''_{xx} - k^2\lambda^2 z + z g(x, y^k z^m) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4106.pdf).

2. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + u f(u^2 + w^2) - w g(u^2 + w^2),$
$\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + w f(u^2 + w^2) + u g(u^2 + w^2).$

1°. Periodic solution in t:

$$u = r(x)\cos\big[\theta(x) + C_1 t + C_2\big], \qquad w = r(x)\sin\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta(x)$ are determined by the system of ordinary differential equations

$$a r''_{xx} - a r(\theta'_x)^2 + C_1^2 r + r f(r^2) = 0,$$
$$a r\theta''_{xx} + 2a r'_x\theta'_x + r g(r^2) = 0.$$

2°. There is an exact solution of the form (generalizes the solution of Item 1°):

$$u = r(z)\cos\big[\theta(z) + C_1 t + C_2\big], \qquad w = r(z)\sin\big[\theta(z) + C_1 t + C_2\big], \qquad z = kx - \lambda t.$$

⊙ *References for system* 18.1.3.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4107.pdf).

3. $\dfrac{\partial^2 u}{\partial t^2} = a\dfrac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg(u^2 - w^2),$

$\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + wf(u^2 - w^2) + ug(u^2 - w^2).$

1°. Solution:

$$u = r(x)\cosh\big[\theta(x) + C_1 t + C_2\big], \qquad w = r(x)\sinh\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta(x)$ are determined by the system of ordinary differential equations

$$ar''_{xx} + ar(\theta'_x)^2 - C_1^2 r + rf(r^2) = 0,$$
$$ar\theta''_{xx} + 2ar'_x\theta'_x + rg(r^2) = 0.$$

2°. There is an exact solution of the form (generalizes the solution of Item 1°):

$$u = r(z)\cosh\big[\theta(z) + C_1 t + C_2\big], \qquad w = r(z)\sinh\big[\theta(z) + C_1 t + C_2\big], \qquad z = kx - \lambda t.$$

⊙ *References for system* 18.1.3.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4108.pdf).

18.2. Systems of the Form

$$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + F(u, w),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + G(u, w)$$

18.2.1. Arbitrary Functions Depend on a Linear Combination of the Unknowns

1. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial u}{\partial x}\right) + c_2 f(b_1 u + c_1 w) + c_1 g(b_2 u + c_2 w),$

$\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) - b_2 f(b_1 u + c_1 w) - b_1 g(b_2 u + c_2 w).$

The condition $b_1 c_2 - b_2 c_1 \neq 0$ is assumed to hold.

Multiplying these equations by appropriate constants and then adding together, one arrives at two independent equations

$$\frac{\partial^2 U}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial U}{\partial x}\right) + (b_1 c_2 - b_2 c_1)f(U), \qquad U = b_1 u + c_1 w;$$
$$\frac{\partial^2 W}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial W}{\partial x}\right) - (b_1 c_2 - b_2 c_1)g(W), \qquad W = b_2 u + c_2 w.$$

In the general case, these equations admit simple particular solutions of the form

$$U = U(t), \quad W = W(x); \qquad U = U(x), \quad W = W(t).$$

2. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(bu-cw) + g(bu-cw),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + wf(bu-cw) + h(bu-cw).$$

1°. Solution:

$$u = \varphi(t) + c\theta(x,t), \qquad w = \psi(t) + b\theta(x,t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$
$$\psi''_{tt} = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x,t)$ satisfies linear equation

$$\frac{\partial^2\theta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right) + f(b\varphi - c\psi)\theta.$$

For f = const, this equation can be solved by separation of variables.

2°. Solution:

$$u = \widetilde{\varphi}(x) + c\widetilde{\theta}(x,t), \qquad w = \widetilde{\psi}(x) + b\widetilde{\theta}(x,t),$$

where the functions $\widetilde{\varphi} = \widetilde{\varphi}(x)$ and $\widetilde{\psi} = \widetilde{\psi}(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n\widetilde{\varphi}'_x)'_x + \widetilde{\varphi}f(b\widetilde{\varphi} - c\widetilde{\psi}) + g(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$
$$ax^{-n}(x^n\widetilde{\psi}'_x)'_x + \widetilde{\psi}f(b\widetilde{\varphi} - c\widetilde{\psi}) + h(b\widetilde{\varphi} - c\widetilde{\psi}) = 0,$$

and the function $\widetilde{\theta} = \widetilde{\theta}(x,t)$ satisfies the linear equation

$$\frac{\partial^2\widetilde{\theta}}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\widetilde{\theta}}{\partial x}\right) + f(b\widetilde{\varphi} - c\widetilde{\psi})\widetilde{\theta}.$$

For solutions of this equation with f = const, see Polyanin (2002).

3°. Let us multiply the first equation by b and the second one by $-c$ and add the results together to obtain

$$\frac{\partial^2\zeta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\zeta}{\partial x}\right) + \zeta f(\zeta) + bg(\zeta) - ch(\zeta), \qquad \zeta = bu - cw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf(\zeta) + g(\zeta). \tag{2}$$

Equation (1) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ to equation (1), the function $u = u(x,t)$ can be determined by solving equation (2) and the function $w = w(x,t)$ is found as $w = (bu - \zeta)/c$.

Note three important solutions to equation (1):

(i) In the general case, equation (1) admits a spatially homogeneous solution $\zeta = \zeta(t)$. The corresponding solution to the original system is given in Item 1° in another form.

(ii) In the general case, equation (1) admits a steady-state solution $\zeta = \zeta(x)$. The corresponding exact solutions to equation (2) are expressed as $u = u_0(x) + \sum_n e^{-\beta_n t} u_n(x)$ and $u = u_0(x) + \sum_n \cos(\beta_n t)\, u_n^{(1)}(x) + \sum_n \sin(\beta_n t)\, u_n^{(2)}(x)$.

(iii) If the condition $\zeta f(\zeta) + bg(\zeta) - ch(\zeta) = k_1\zeta + k_0$ holds, equation (1) is linear,

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial \zeta}{\partial x}\right) + k_1\zeta + k_0,$$

and, hence, can be solved by separation of variables.

⊙ *References for system* 18.2.1.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4101.pdf).

3. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + e^{\lambda u} f(\lambda u - \sigma w),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + e^{\sigma w} g(\lambda u - \sigma w).$$

1°. Solution:

$$u = y(\xi) - \frac{2}{\lambda}\ln(C_1 t + C_2), \qquad w = z(\xi) - \frac{2}{\sigma}\ln(C_1 t + C_2), \qquad \xi = \frac{x}{C_1 t + C_2},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$C_1^2(\xi^2 y'_\xi)'_\xi + 2C_1^2\lambda^{-1} = a\xi^{-n}(\xi^n y'_\xi)'_\xi + e^{\lambda y} f(\lambda y - \sigma z),$$
$$C_1^2(\xi^2 z'_\xi)'_\xi + 2C_1^2\sigma^{-1} = b\xi^{-n}(\xi^n z'_\xi)'_\xi + e^{\sigma z} g(\lambda y - \sigma z).$$

2°. Solution with $b = a$:

$$u = \theta(x,t), \qquad w = \frac{\lambda}{\sigma}\theta(x,t) - \frac{k}{\sigma},$$

where k is a root of the algebraic (transcendental) equation

$$\lambda f(k) = \sigma e^{-k} g(k),$$

and the function $\theta = \theta(x,t)$ is described by the equation

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial \theta}{\partial x}\right) + f(k)e^{\lambda\theta}.$$

This equation is solvable for $n = 0$; for its exact solutions, see equation 7.4.1.1.

⊙ *References for system* 18.2.1.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4102.pdf).

18.2.2. Arbitrary Functions Depend on the Ratio of the Unknowns

1. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial u}{\partial x}\right) + uf\left(\dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial t^2} = \dfrac{b}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + wg\left(\dfrac{u}{w}\right).$

1°. Periodic multiplicative separable solution:

$$u = [C_1 \cos(kt) + C_2 \sin(kt)]y(x), \qquad w = [C_1 \cos(kt) + C_2 \sin(kt)]z(x),$$

where C_1, C_2, and k are arbitrary constants and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x + k^2 y + yf(y/z) = 0,$$
$$bx^{-n}(x^n z'_x)'_x + k^2 z + zg(y/z) = 0.$$

2°. Multiplicative separable solution:

$$u = [C_1 \exp(kt) + C_2 \exp(-kt)]y(x), \qquad w = [C_1 \exp(kt) + C_2 \exp(-kt)]z(x),$$

where C_1, C_2, and k are arbitrary constants and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x - k^2 y + yf(y/z) = 0,$$
$$bx^{-n}(x^n z'_x)'_x - k^2 z + zg(y/z) = 0.$$

3°. Degenerate multiplicative separable solution:

$$u = (C_1 t + C_2)y(x), \qquad w = (C_1 t + C_2)z(x),$$

where the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x + yf(y/z) = 0,$$
$$bx^{-n}(x^n z'_x)'_x + zg(y/z) = 0.$$

4°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 J_\nu(kx) + C_2 Y_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = -ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi''_{tt} = -bk^2\psi + \psi g(\varphi/\psi).$$

5°. Multiplicative separable solution:

$$u = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\varphi(t), \quad \nu = \tfrac{1}{2}|n-1|,$$
$$w = x^{\frac{1-n}{2}}[C_1 I_\nu(kx) + C_2 K_\nu(kx)]\psi(t),$$

where C_1, C_2, and k are arbitrary constants, $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions, and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = ak^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi''_{tt} = bk^2\psi + \psi g(\varphi/\psi).$$

6°. Solution with $b = a$:

$$u = k\theta(x,t), \qquad w = \theta(x,t),$$

where k is a root of the algebraic (transcendental) equation $f(k) = g(k)$, and the function $\theta = \theta(x,t)$ is described by the linear Klein–Gordon equation

$$\frac{\partial^2\theta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right) + f(k)\theta.$$

For solutions of this equation, see Polyanin (2002).

⊙ *References for system* 18.2.2.1: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4103.pdf).

2. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + uf\left(\frac{u}{w}\right) + \frac{u}{w}h\left(\frac{u}{w}\right),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial w}\right) + wg\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right).$$

Solution:

$$u = k\theta(x,t), \qquad w = \theta(x,t),$$

where k is a root of the algebraic (transcendental) equation $f(k) = g(k)$, and the function $\theta = \theta(x,t)$ is described by the linear equation

$$\frac{\partial^2\theta}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial\theta}{\partial x}\right) + f(k)\theta + h(k).$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4104.pdf).

3. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial u}{\partial x}\right) + u^k f\left(\frac{u}{w}\right), \quad \frac{\partial^2 w}{\partial t^2} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + w^k g\left(\frac{u}{w}\right).$$

Self-similar solution:

$$u = (C_1t + C_2)^{\frac{2}{1-k}}y(\xi), \qquad w = (C_1t + C_2)^{\frac{2}{1-k}}z(\xi), \qquad \xi = \frac{x}{C_1t + C_2},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$C_1^2\xi^2 y''_{\xi\xi} + \frac{2C_1^2(k+1)}{k-1}\xi y'_\xi + \frac{C_1^2(k+1)}{(k-1)^2}y = \frac{a}{\xi^n}(\xi^n y'_\xi)'_\xi + y^k f\left(\frac{y}{z}\right),$$
$$C_1^2\xi^2 z''_{\xi\xi} + \frac{2C_1^2(k+1)}{k-1}\xi z'_\xi + \frac{C_1^2(k+1)}{(k-1)^2}z = \frac{b}{\xi^n}(\xi^n z'_\xi)'_\xi + z^k g\left(\frac{y}{z}\right).$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4105.pdf).

18.2.3. Other Systems

1. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + uf(x, u^k w^m),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{b}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + wg(x, u^k w^m).$$

Multiplicative separable solution:

$$u = e^{-m\lambda t} y(x), \quad w = e^{k\lambda t} z(x),$$

where λ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are determined by the system of ordinary differential equations

$$ax^{-n}(x^n y'_x)'_x - m^2\lambda^2 y + yf(x, y^k z^m) = 0,$$
$$bx^{-n}(x^n z'_x)'_x - k^2\lambda^2 z + zg(x, y^k z^m) = 0.$$

⊙ *References*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4106.pdf).

2. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + uf(u^2 + w^2) - wg(u^2 + w^2),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + wf(u^2 + w^2) + ug(u^2 + w^2).$$

1°. Periodic solution in t:

$$u = r(x)\cos\big[\theta(x) + C_1 t + C_2\big], \quad w = r(x)\sin\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta(x)$ are determined by the system of ordinary differential equations

$$ar''_{xx} - ar(\theta'_x)^2 + \frac{an}{x}r'_x + C_1^2 r + rf(r^2) = 0,$$
$$ar\theta''_{xx} + 2ar'_x\theta'_x + \frac{an}{x}r\theta'_x + rg(r^2) = 0.$$

2°. For $n = 0$, there is an exact solution of the form

$$u = r(z)\cos\big[\theta(z) + C_1 t + C_2\big], \quad w = r(z)\sin\big[\theta(z) + C_1 t + C_2\big], \quad z = kx - \lambda t.$$

⊙ *References for system* 18.2.3.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4107.pdf).

3. $$\frac{\partial^2 u}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + uf(u^2 - w^2) + wg(u^2 - w^2),$$
$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + wf(u^2 - w^2) + ug(u^2 - w^2).$$

1°. Solution:

$$u = r(x)\cosh\big[\theta(x) + C_1 t + C_2\big], \quad w = r(x)\sinh\big[\theta(x) + C_1 t + C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r = r(x)$ and $\theta(x)$ are determined by the system of ordinary differential equations

$$ar''_{xx} + ar(\theta'_x)^2 + \frac{an}{x}r'_x - C_1^2 r + rf(r^2) = 0,$$
$$ar\theta''_{xx} + 2ar'_x\theta'_x + \frac{an}{x}r\theta'_x + rg(r^2) = 0.$$

2°. For $n = 0$, there is an exact solution of the form

$$u = r(z)\cosh\big[\theta(z) + C_1 t + C_2\big], \qquad w = r(z)\sinh\big[\theta(z) + C_1 t + C_2\big], \qquad z = kx - \lambda t.$$

⊙ *References for system* 18.2.3.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde4108.pdf).

Chapter 19
Systems of Two Elliptic Equations

19.1. Systems of the Form $\Delta u = F(u, w)$, $\Delta w = G(u, w)$

Preliminary remarks. Such systems of equations frequently arise in the theory of mass/heat transfer of reacting media, the theory of chemical reactors, and combustion theory; these describe steady-state processes.

The systems in question are invariant under translations in the independent variables and rotations in the xy-plane. These admit traveling-wave solutions $u = u(k_1x + k_2y)$, $w = w(k_1x + k_2y)$. Neither these solutions nor degenerate solutions with one of the unknowns being constant are considered in this section.

The functions $f(\varphi)$, $g(\varphi)$, and $h(\varphi)$ that are used throughout this section are arbitrary functions of an argument $\varphi = \varphi(u, w)$; the equations considered below are arranged in order of complexity of this argument.

19.1.1. Arbitrary Functions Depend on a Linear Combination of the Unknowns

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = b_2 f(a_1u + b_1w) + b_1 g(a_2u + b_2w),$$
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -a_2 f(a_1u + b_1w) - a_1 g(a_2u + b_2w).$$

It is assumed that $a_1b_2 - a_2b_1 \neq 0$.

Multiplying the equations by suitable constants and then adding together, one arrives at two independent equations of the form 9.4.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = (a_1b_2 - a_2b_1) f(U), \qquad U = a_1u + b_1w;$$
$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = -(a_1b_2 - a_2b_1) g(W), \qquad W = a_2u + b_2w.$$

In general, these equations admit traveling-wave solutions

$$U = U(k_1x + k_2y), \quad W = W(k_3x + k_4y),$$

where $k_1, \ldots, k_4$ are arbitrary constants. The corresponding solution of the original system will be a superposition (a linear combination) of two nonlinear traveling waves.

2. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(au-bw)+g(au-bw),$

$\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(au-bw)+h(au-bw).$

1°. Solution:

$$u=\varphi(x)+b\theta(x,y),\quad w=\psi(x)+a\theta(x,y),$$

where $\varphi=\varphi(x)$ and $\psi=\psi(x)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{xx}=\varphi f(a\varphi-b\psi)+g(a\varphi-b\psi),$$
$$\psi''_{xx}=\psi f(a\varphi-b\psi)+h(a\varphi-b\psi),$$

and the function $\theta=\theta(x,y)$ satisfies the *linear Schrödinger equation* of the special form

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=F(x)\theta,\qquad F(x)=f(au-bw).$$

For solutions of this equation with some specific $F(x)$, see Polyanin (2002).

2°. Let us multiply the first equation by a and the second one by $-b$ and add the results together to obtain

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}=\zeta f(\zeta)+ag(\zeta)-bh(\zeta),\qquad \zeta=au-bw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=uf(\zeta)+g(\zeta). \tag{2}$$

Equation (1) can be treated separately. Note two important solutions to equation (1):

(i) In the general case, equation (1) admits an exact, traveling-wave solution $\zeta=\zeta(z)$, where $z=k_1x+k_2y$ with arbitrary constants k_1 and k_2.

(ii) If the condition $\zeta f(\zeta)+ag(\zeta)-bh(\zeta)=c_1\zeta+c_0$ holds, equation (1) is a linear Helmholtz equation.

For other exact solutions of (1), see 9.4.1.1.

Given a solution $\zeta=\zeta(x,y)$ to equation (1), the function $u=u(x,y)$ can be determined by solving the linear equation (2) and the function $w=w(x,y)$ is found as $w=(bu-\zeta)/c$.

⊙ *Reference for system* 19.1.1.2: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde3101.pdf).

3. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^{\lambda u}f(\lambda u-\sigma w),\quad \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=e^{\sigma w}g(\lambda u-\sigma w).$

1°. Solution:

$$u=U(\xi)-\frac{2}{\lambda}\ln|x+C_1|,\quad w=W(\xi)-\frac{2}{\sigma}\ln|x+C_1|,\quad \xi=\frac{y+C_2}{x+C_1},$$

where C_1 and C_2 are arbitrary constants, and the functions $U = U(\xi)$ and $W = W(\xi)$ are determined by the system of ordinary differential equations

$$(1+\xi^2)U''_{\xi\xi} + 2\xi U'_\xi + \frac{2}{\lambda} = e^{\lambda U} f(\lambda U - \sigma W),$$
$$(1+\xi^2)W''_{\xi\xi} + 2\xi W'_\xi + \frac{2}{\sigma} = e^{\sigma W} g(\lambda U - \sigma W).$$

2°. Solution:

$$u = \theta(x, y), \qquad w = \frac{\lambda}{\sigma}\theta(x, y) - \frac{k}{\sigma},$$

where k is a root of the algebraic (transcendental) equation

$$\lambda f(k) = \sigma e^{-k} g(k),$$

and the function $\theta = \theta(x, y)$ is described by the solvable equation of the form 9.2.1.1:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = f(k)e^{\lambda\theta}.$$

⊙ *Reference for system* 19.1.1.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde3102.pdf).

19.1.2. Arbitrary Functions Depend on the Ratio of the Unknowns

1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u f\left(\dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = w g\left(\dfrac{u}{w}\right).$

1°. A space-periodic solution in multiplicative form (another solution is obtained by interchanging x and y):

$$u = [C_1 \sin(kx) + C_2 \cos(kx)]\varphi(y),$$
$$w = [C_1 \sin(kx) + C_2 \cos(kx)]\psi(y),$$

where C_1, C_2, and k are arbitrary constants and the functions $\varphi = \varphi(y)$ and $\psi = \psi(y)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{yy} = k^2\varphi + \varphi f(\varphi/\psi),$$
$$\psi''_{yy} = k^2\psi + \psi g(\varphi/\psi).$$

2°. Solution in multiplicative form:

$$u = [C_1 \exp(kx) + C_2 \exp(-kx)]U(y),$$
$$w = [C_1 \exp(kx) + C_2 \exp(-kx)]W(y),$$

where C_1, C_2, and k are arbitrary constants and the functions $U = U(y)$ and $W = W(y)$ are determined by the autonomous system of ordinary differential equations

$$U''_{yy} = -k^2 U + U f(U/W),$$
$$W''_{yy} = -k^2 W + W g(U/W).$$

3°. Degenerate solution in multiplicative form:

$$u = (C_1x + C_2)U(y),$$
$$w = (C_1x + C_2)W(y),$$

where C_1 and C_2 are arbitrary constants and the functions $U = U(y)$ and $W = W(y)$ are determined by the autonomous system of ordinary differential equations

$$U''_{yy} = Uf(U/W),$$
$$W''_{yy} = Wg(U/W).$$

Remark. The functions f and g in Items 1°–3° can be dependent on y.

4°. Solution in multiplicative form:

$$u = e^{a_1x+b_1y}\xi(z), \qquad w = e^{a_1x+b_1y}\eta(z), \qquad z = a_2x + b_2y,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants, and the functions $\xi = \xi(z)$ and $\eta = \eta(z)$ are determined by the autonomous system of ordinary differential equations

$$(a_2^2 + b_2^2)\xi''_{zz} + 2(a_1a_2 + b_1b_2)\xi'_z + (a_1^2 + b_1^2)\xi = \xi f(\xi/\eta),$$
$$(a_2^2 + b_2^2)\eta''_{zz} + 2(a_1a_2 + b_1b_2)\eta'_z + (a_1^2 + b_1^2)\eta = \eta g(\xi/\eta).$$

5°. Solution:

$$u = k\theta(x, y), \qquad w = \theta(x, y),$$

where k is a root of the algebraic (transcendental) equation $f(k) = g(k)$, and the function $\theta = \theta(x, y)$ is described by the linear Helmholtz equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = f(k)\theta.$$

For its exact solutions, see books A. N. Tikhonov and A. A. Samarskii (1990) and A. D. Polyanin (2002).

⊙ *Reference for system* 19.1.2.1: A. D. Polyanin (2004 *b*).

2. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = uf\left(\dfrac{u}{w}\right) + \dfrac{u}{w}h\left(\dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = wg\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Solution:

$$u = kw, \qquad w = \theta(x, y) - \frac{h(k)}{f(k)},$$

where k is a root of the algebraic (transcendental) equation

$$f(k) = g(k),$$

and the function $\theta = \theta(x, y)$ satisfies the linear Helmholtz equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = f(k)\theta.$$

For solutions of this equation, see Polyanin (2002).

⊙ *Reference*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde3104.pdf).

3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u^n f\Big(\dfrac{u}{w}\Big), \quad \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = w^n g\Big(\dfrac{u}{w}\Big).$

For $f(z) = kz^{-m}$ and $g(z) = -kz^{n-m}$, the system describes an nth-order chemical reaction (of order $n-m$ in the component u and of order m in the component w); to $n = 2$ and $m = 1$ there corresponds a second-order reaction, which often occurs in applications.

1°. Solution:

$$u = r^{\frac{2}{1-n}} U(\theta), \quad w = r^{\frac{2}{1-n}} W(\theta), \quad r = \sqrt{(x + C_1)^2 + (y + C_2)^2}, \quad \theta = \frac{y + C_2}{x + C_1},$$

where C_1 and C_2 are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the autonomous system of ordinary differential equations

$$U''_{\theta\theta} + \frac{4}{(1-n)^2} U = U^n f\left(\frac{U}{W}\right),$$
$$W''_{\theta\theta} + \frac{4}{(1-n)^2} W = W^n g\left(\frac{U}{W}\right).$$

2°. Solution:

$$u = k\zeta(x, y), \quad w = \zeta(x, y),$$

where k is a root of the algebraic (transcendental) equation

$$k^{n-1} f(k) = g(k),$$

and the function $\zeta = \zeta(x, y)$ satisfies the equation with a power-law nonlinearity

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = g(k)\zeta^n.$$

⊙ *Reference for system* 19.1.2.3: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde3105.pdf).

19.1.3. Other Systems

1. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u f(u^n w^m), \quad \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = w g(u^n w^m).$

Solution in multiplicative form:

$$u = e^{m(a_1 x + b_1 y)} \xi(z), \quad w = e^{-n(a_1 x + b_1 y)} \eta(z), \quad z = a_2 x + b_2 y,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants, and the functions $\xi = \xi(z)$ and $\eta = \eta(z)$ are determined by the autonomous system of ordinary differential equations

$$(a_2^2 + b_2^2)\xi''_{zz} + 2m(a_1 a_2 + b_1 b_2)\xi'_z + m^2(a_1^2 + b_1^2)\xi = \xi f(\xi^n \eta^m),$$
$$(a_2^2 + b_2^2)\eta''_{zz} - 2n(a_1 a_2 + b_1 b_2)\eta'_z + n^2(a_1^2 + b_1^2)\eta = \eta g(\xi^n \eta^m).$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

2. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(u^2+w^2)-wg(u^2+w^2),$

$\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(u^2+w^2)+ug(u^2+w^2).$

1°. A periodic solution in y:

$$u=r(x)\cos\big[\theta(x)+C_1y+C_2\big],\qquad w=r(x)\sin\big[\theta(x)+C_1y+C_2\big],$$

where C_1 and C_2 are arbitrary constants, and the functions $r=r(x)$ and $\theta=\theta(x)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}r''_{xx}&=r(\theta'_x)^2+C_1^2r+rf(r^2),\\ r\theta''_{xx}&=-2r'_x\theta'_x+rg(r^2).\end{aligned}$$

2°. Solution (generalizes the solution of Item 1°):

$$u=r(z)\cos\big[\theta(z)+C_1y+C_2\big],\qquad w=r(z)\sin\big[\theta(z)+C_1y+C_2\big],\qquad z=k_1x+k_2y,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants, and the functions $r=r(z)$ and $\theta=\theta(z)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}(k_1^2+k_2^2)r''_{zz}&=k_1^2r(\theta'_z)^2+r(k_2\theta'_z+C_1)^2+rf(r^2),\\ (k_1^2+k_2^2)r\theta''_{zz}&=-2\big[(k_1^2+k_2^2)\theta'_z+C_1k_2\big]r'_z+rg(r^2).\end{aligned}$$

⊙ *Reference for system* 19.1.3.2: A. D. Polyanin (2004 *b*).

3. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=uf(u^2-w^2)+wg(u^2-w^2),$

$\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=wf(u^2-w^2)+ug(u^2-w^2).$

Solution:

$$u=r(z)\cosh\big[\theta(z)+C_1y+C_2\big],\qquad w=r(z)\sinh\big[\theta(z)+C_1y+C_2\big],\qquad z=k_1x+k_2y,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants, and the functions $r=r(z)$ and $\theta=\theta(z)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}(k_1^2+k_2^2)r''_{zz}+k_1^2r(\theta'_z)^2+r(k_2\theta'_z+C_1)^2&=rf(r^2),\\ (k_1^2+k_2^2)r\theta''_{zz}+2\big[(k_1^2+k_2^2)\theta'_z+C_1k_2\big]r'_z&=rg(r^2).\end{aligned}$$

⊙ *Reference*: A. D. Polyanin (2004 *c*, http://eqworld.ipmnet.ru/en/solutions/syspde/spde3108.pdf).

19.2. Other Systems of Two Second-Order Elliptic Equations

1. $$a_1x\frac{\partial u}{\partial x}+a_1y\frac{\partial u}{\partial y}=b_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-f(u,w),$$
$$a_2x\frac{\partial w}{\partial x}+a_2y\frac{\partial w}{\partial y}=b_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-g(u,w).$$

Solution:

$$u(x,y)=U(z),\qquad w(x,y)=W(z),\qquad z=k_1x+k_2y,$$

where k_1 and k_2 are arbitrary constants, and the functions $U=U(z)$ and $W=W(z)$ are described by the system of ordinary differential equations

$$a_1zU'_z=b_1(k_1^2+k_2^2)U''-f(U,W),\qquad a_2zW'_z=b_2(k_1^2+k_2^2)W''-g(U,W).$$

2. $$\frac{\partial}{\partial x}\left(ax^m\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^n\frac{\partial u}{\partial y}\right)=F(u,w),$$
$$\frac{\partial}{\partial x}\left(ax^m\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^n\frac{\partial w}{\partial y}\right)=G(u,w).$$

Solution:

$$u=u(\zeta),\quad w=w(\zeta),\quad \zeta^2=C\left[\frac{x^{2-m}}{a(2-m)^2}+\frac{y^{2-n}}{b(2-n)^2}\right],$$

where the functions $u(\zeta)$ and $w(\zeta)$ are described by the system of ordinary differential equations

$$u''_{\zeta\zeta}+\frac{4-mn}{(2-m)(2-n)}\frac{1}{\zeta}u'_\zeta=\frac{4}{C}F(u,w),$$
$$w''_{\zeta\zeta}+\frac{4-mn}{(2-m)(2-n)}\frac{1}{\zeta}w'_\zeta=\frac{4}{C}G(u,w).$$

3. $$\frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial u}{\partial y}\right)=F(u,w),$$
$$\frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial w}{\partial y}\right)=G(u,w).$$

Solution:

$$u=u(\xi),\quad w=w(\xi),\quad \xi=\left(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\right)^{1/2},$$

where the functions $u(\xi)$ and $w(\xi)$ are described by the system of ordinary differential equations

$$u''_{\xi\xi}-\frac{1}{\xi}u'_\xi=\frac{4}{ab\beta^2\mu^2}F(u,w),$$
$$w''_{\xi\xi}-\frac{1}{\xi}w'_\xi=\frac{4}{ab\beta^2\mu^2}G(u,w).$$

4. $$\frac{\partial}{\partial x}\left(ax^n\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial u}{\partial y}\right)=F(u,w),$$
$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial w}{\partial y}\right)=G(u,w).$$

Solution:

$$u=u(\xi),\quad w=w(\xi),\quad \xi=\left[b\mu^2x^{2-n}+a(2-n)^2e^{-\mu y}\right]^{1/2},$$

where the functions $u(\xi)$ and $w(\xi)$ are described by the system of ordinary differential equations

$$u''_{\xi\xi}+\frac{n}{2-n}\frac{1}{\xi}u'_\xi=\frac{4}{ab\mu^2(2-n)^2}F(u,w),$$
$$w''_{\xi\xi}+\frac{n}{2-n}\frac{1}{\xi}w'_\xi=\frac{4}{ab\mu^2(2-n)^2}G(u,w).$$

19.3. Von Kármán Equations (Fourth-Order Elliptic Equations)

The *von Kármán equations* have the form

$$\Delta\Delta u=a\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right],\qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$
$$\Delta\Delta w=b\left(\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)+c.$$

These are used in nonlinear elasticity for describing large deformations of plates, with u denoting a stress function and w the displacement perpendicular to the midplane of the plate.

⊙ *References*: T. von Kármán (1910), E. H. Mansfield (1955), D. G. Ashwell (1957), S. Timoshenko and S. Woinowsky-Krieger (1959).

Exact solutions in the rectangular Cartesian coordinates.

1°. Suppose $u=u(x,y)$, $w=w(x,y)$ is a solution of the system. Then the functions

$$u_1=u(\pm x+A_1,\pm y+A_2)+B_1x+B_2y+B_3,$$
$$w_1=w(\pm x+A_3,\pm y+A_4)+C_1x+C_2y+C_3;$$
$$u_2=u(x\cos\beta-y\sin\beta,\ x\sin\beta+y\cos\beta),$$
$$w_2=w(x\cos\beta-y\sin\beta,\ x\sin\beta+y\cos\beta),$$

where A_n, B_m, C_m, and β are arbitrary constants, are also solutions of the system. The respective plus or minus signs in front of the independent variables in the expressions of u_1 and w_1 can be either the same or different.

If u, w is a solution of the system with $c=0$, then u, $-w$ also is a solution of the system.

2°. Solution:

$$u=\tfrac{1}{2}(A_3x^3+A_2x^2+A_1x+A_0)y^2+(B_3x^3+B_2x^2+B_1x+B_0)y$$
$$-\tfrac{1}{10}A_3x^5-\tfrac{1}{6}A_2x^4+C_3x^3+C_2x^2+C_1x+C_0,\qquad(1)$$
$$w=\int_0^x(x-t)f(t)\,dt+D_1x+D_2y+D_3,$$

where A_n, B_n, C_n, and D_m are arbitrary constants, and the function $f = f(x)$ is determined by the second-order linear ordinary differential equation

$$f''_{xx} = b(A_3x^3 + A_2x^2 + A_1x + A_0)f + c. \tag{2}$$

The general solution of equation (2) can be found if a nontrivial particular solution of the corresponding homogeneous equation (with $c = 0$) is known. For $A_1 = A_2 = A_3 = 0$, equation (2) becomes a constant-coefficient equation, which is easy to solve. For $A_2 = A_3 = 0$, the general solution of the homogeneous equation (2) is expressed in terms of the *Airy functions*; see Polyanin and Zaitsev (2003, pp. 213–214). For $A_3 = 0$, the homogeneous equation(2) is reduced to the *Weber equation*; see Polyanin and Zaitsev (2003, p. 214). For $A_0 = A_1 = A_2 = 0$, the general solution of the homogeneous equation (2) is expressed in terms of the Bessel functions or modified Bessel functions of order $1/5$ depending on the sign of A_3; see Polyanin and Zaitsev (2003, p. 215).

Remark. For the special case $A_1 = A_2 = A_3 = 0$, equation (2) with $f = w''_{xx}$ is considered in Timoshenko and Woinowsky-Krieger (1959).

3°. Generalized separable solution linear in y:

$$u = f(x)y + g(x), \qquad w = h(x)y + p(x),$$

where

$$\begin{aligned}
f(x) &= A_3x^3 + A_2x^2 + A_1x + A_0,\\
h(x) &= B_3x^3 + B_2x^2 + B_1x + B_0,\\
g(x) &= \frac{a}{6}\int_0^x (x-t)^3[h'_t(t)]^2\,dt + C_3x^3 + C_2x^2 + C_1x + C_0,\\
p(x) &= -\frac{b}{3}\int_0^x (x-t)^3 f'_t(t)h'_t(t)\,dt + D_3x^3 + D_2x^2 + D_1x + D_0,
\end{aligned}$$

where A_n, B_n, C_n, and D_n are arbitrary constants.

4°. Solution:

$$\begin{aligned}
u &= \frac{1}{2}A_1x^2 + \frac{1}{2}A_2y^2 + A_3xy + A_4x + A_5y + \int_0^z (z-t)U(t)\,dt, \qquad z = k_1x + k_2y,\\
w &= \frac{1}{2}B_1x^2 + \frac{1}{2}B_2y^2 + B_3xy + B_4x + B_5y + \int_0^z (z-t)W(t)\,dt,
\end{aligned}$$

where A_n, B_n, k_1, and k_2 are arbitrary constants, and the functions $U = U(z)$ and $W = W(z)$ are determined by the system of linear ordinary differential equations with constant coefficients

$$\begin{aligned}
(k_1^2 + k_2^2)^2U''_{zz} &= a(2B_3k_1k_2 - B_1k_2^2 - B_2k_1^2)W + a(B_3^2 - B_1B_2),\\
(k_1^2 + k_2^2)^2W''_{zz} &= b(B_1k_2^2 + B_2k_1^2 - 2B_3k_1k_2)U + b(A_1k_2^2 + A_2k_1^2 - 2A_3k_1k_2)W\\
&\quad + b(A_1B_2 + A_2B_1 - 2A_3B_3) + c.
\end{aligned}$$

5°. Generalized separable solution quadratic in y (generalizes the solution from Item 3°):

$$u = f_2(x)y^2 + f_1(x)y + f_0(x), \qquad w = g_2(x)y^2 + g_1(x)y + g_0(x),$$

where the functions $f_n = f_n(x)$ and $g_n = g_n(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
f_2'''' &= 2a[2(g_2')^2 - g_2 g_2''],\\
g_2'''' &= 2b(f_2 g_2'' + g_2 f_2'' - 4f_2' g_2'),\\
f_1'''' &= 2a[2g_1' g_2' - g_2 g_1''],\\
g_1'''' &= 2b(f_2 g_1'' + g_2 f_1'' - 2f_2' g_1' - 2f_1' g_2'),\\
f_0'''' &= a[(g_1')^2 - 2g_2 g_0''] - 4f_2'',\\
g_0'''' &= 2b(f_2 g_0'' + g_2 f_0'' - f_1' g_1') - 4g_2'' + c.
\end{aligned} \tag{3}$$

Special case 1. The first and second equations can be satisfied by setting $f_2 = \alpha$ and $g_2 = \beta$, where α and β are arbitrary constants. Then the third and fourth equations will form a closed system of homogeneous linear equations with constant coefficients and, hence, are easily integrable. Then the fifth and sixth equations will form a closed system of nonhomogeneous linear equations with constant coefficients and are also easily integrable.

In the degenerate case $f_2 = g_2 = 0$, we have the solution from Item 3°.

If $ab < 0$, there are solutions of the form $f_2 = A(x+C)^{-2}$, $g_2 = B(x+C)^{-2}$ and also more general solutions of the form $g_2(x) = \pm\sqrt{-b/(2a)}\, f_2(x)$.

6°. There is an exact solution of the form

$$u = f_2(z)y^2 + f_1(z)y + f_0(z), \qquad w = g_2(z)y^2 + g_1(z)y + g_0(z), \qquad z = k_1 x + k_2 y,$$

which generalizes the solution from Item 4°.

7°. The transformation

$$\begin{aligned}
u(x,y) &= \frac{1}{2}A_1 x^2 + \frac{1}{2}A_2 y^2 + A_3 xy + A_4 x + A_5 y + U(x,y),\\
w(x,y) &= \frac{1}{2}B_1 x^2 + \frac{1}{2}B_2 y^2 + B_3 xy + B_4 x + B_5 y + W(x,y),
\end{aligned}$$

where A_n and B_n are arbitrary constants, leads to the system of equations

$$\begin{aligned}
\Delta\Delta U &= a(W_{xy}^2 - W_{xx}W_{yy}) + a(B_3 W_{xy} - B_2 W_{xx} - B_1 W_{yy}) + a(B_3^2 - B_1 B_2),\\
\Delta\Delta W &= b(U_{yy}W_{xx} + U_{xx}W_{yy} - 2U_{xy}W_{xy})\\
&\quad + b(B_2 U_{xx} + B_1 U_{yy} + A_2 W_{xx} + A_1 W_{yy} - 2B_3 U_{xy} - 2A_3 W_{xy})\\
&\quad + b(A_1 B_2 + A_2 B_1 - 2A_3 B_3) + c,
\end{aligned}$$

which differs from the original system in the presence of additional linear terms.

The system obtained also admits exact solutions of the form

$$U = f_2(z)y^2 + f_1(z)y + f_0(z), \qquad W = g_2(z)y^2 + g_1(z)y + g_0(z), \qquad z = k_1 x + k_2 y.$$

Exact solutions in the cylindrical coordinates. In polar coordinates r, φ, the system in question becomes

$$\Delta\Delta u = -\tfrac{1}{2}aL(w,w), \qquad \Delta\Delta w = bL(u,w) + c, \tag{4}$$

where

$$L(u,w) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) + \frac{\partial^2 u}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) - 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \varphi}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right),$$

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

8°. System (4) has radially symmetric solutions of the form

$$u = u(r), \qquad w = w(r),$$

where the functions $u = u(r)$ and $w = w(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\Delta_r(\Delta_r u) &= -ar^{-1}w'_r w''_{rr},\\
\Delta_r(\Delta_r w) &= br^{-1}(u'_r w''_{rr} + w'_r u''_{rr}) + c,\\
\Delta_r F &\equiv F''_{rr} + r^{-1}F'_r = r^{-1}(rF'_r)'_r.
\end{aligned}$$

The order of the system obtained can be reduced by two by integrating the equations with respect to r and then by performing the change of variables $U = u'_r$, $W = w'_r$. As a result, one obtains

$$\begin{aligned}
r[r^{-1}(rU)'_r]'_r &= -\tfrac{1}{2}aW^2 + A,\\
r[r^{-1}(rW)'_r]'_r &= bUW + \tfrac{1}{2}cr^2 + B,
\end{aligned}$$

where A and B are arbitrary constants.

9°. Additive separable solution (generalizes the solution from Item 8°):

$$u = A\varphi + f(r), \qquad w = B\varphi + g(r),$$

where A and B are arbitrary constants, and the functions $f = f(r)$ and $g = g(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\Delta_r(\Delta_r f) &= -ar^{-1}g'_r g''_{rr} + aB^2 r^{-4},\\
\Delta_r(\Delta_r g) &= br^{-1}(f'_r g'_r)'_r - 2bABr^{-4} + c.
\end{aligned}$$

On integrating the equations with respect to r and the performing the change of variables $F = f'_r$, $G = g'_r$, one obtains

$$\begin{aligned}
r[r^{-1}(rF)'_r]'_r &= -\tfrac{1}{2}aG^2 - \tfrac{1}{2}aB^2 r^{-2} + C_1,\\
r[r^{-1}(rG)'_r]'_r &= bFG + bABr^{-2} + \tfrac{1}{2}cr^2 + C_2,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

10°. For $c = 0$, there is a solution of the form

$$u = u(\varphi), \qquad w = w(\varphi),$$

where the functions $u = u(\varphi)$ and $w = w(\varphi)$ are determined by the system of ordinary differential equations

$$u^{(4)}_\varphi + 4u''_{\varphi\varphi} = a(w'_\varphi)^2, \qquad w^{(4)}_\varphi + 4w''_{\varphi\varphi} = -2bu'_\varphi w'_\varphi. \tag{5}$$

The order of this system can be reduced with the change of variables $U = u'_\varphi$, $W = w'_\varphi$. This results in

$$U'''_{\varphi\varphi\varphi} + 4U'_\varphi = aW^2, \qquad W'''_{\varphi\varphi\varphi} + 4W'_\varphi = -2bUW.$$

11°. For $c = 0$, there is a generalized separable solution linear in r (generalization of the solution from Item 10°):

$$u = f_1(\varphi)r + g_1(\varphi), \qquad w = f_2(\varphi)r + g_2(\varphi),$$

where the functions $f_n = f_n(\varphi)$ and $g_n = g_n(\varphi)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f_1'''' + 2f_1'' + f_1 &= 0, \\ f_2'''' + 2f_2'' + f_2 &= 0, \\ g_1'''' + 4g_1'' &= a(g_2')^2, \\ g_2'''' + 4g_2'' &= -2bg_1'g_2'. \end{aligned} \tag{6}$$

The general solutions of the first two equations are

$$\begin{aligned} f_1 &= \varphi(A_1\cos\varphi + A_2\sin\varphi) + A_3\cos\varphi + A_4\sin\varphi, \\ f_2 &= \varphi(B_1\cos\varphi + B_2\sin\varphi) + B_3\cos\varphi + B_4\sin\varphi, \end{aligned}$$

where A_m and B_m are arbitrary constants. The last two equations in (6) form a system that coincides, up to renaming of variables, with system (5).

12°. For $c = 0$, there is a solution of the form

$$u = A\varphi + U(\xi), \quad w = B\varphi + W(\xi), \quad \xi = \varphi + k\ln r,$$

where A, B, and k are arbitrary constants, and the functions $U = U(\xi)$ and $W = W(\xi)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &(k^2+1)^2U'''' - 4k(k^2+1)U''' + 4(k^2+1)U'' \\ &\quad = -a[k(k^2+1)W'W'' - (k^2+1)(W')^2 + k(A+B)W'' - (A+B)W' - AB], \\ &(k^2+1)^2W'''' - 4k(k^2+1)W''' + 4(k^2+1)W'' \\ &\quad = b[k(k^2+1)(U'W')' - 2(k^2+1)U'W' + 2k(BU'' + AW'') - 2(BU' + AW') - 2AB]. \end{aligned}$$

The order of the system is reduced with the change of variables $F(\xi) = U'$, $G(\xi) = W'$.

Chapter 20
First-Order Hydrodynamic and Other Systems Involving Three or More Equations

20.1. Equations of Motion of Ideal Fluid (Euler Equations)

20.1.1. Euler Equations in Various Coordinate Systems

Vector representation of the Euler equations. The closed system of equations for an *ideal* (*inviscid*) *incompressible fluid* has the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure referred to the fluid density, and t is time.

The vector equation (1) is a short form of representation of three scalar *Euler equations* and Eq. (2) is the *continuity equation*. In equation (1), it is assumed that the mass forces (e.g., the gravitational force) are all potential and included into the pressure. For brevity, we will call the system of equations (1)–(2) the Euler equations (without distinguishing the continuity equation).

Euler equations in the orthogonal Cartesian coordinates. In the orthogonal Cartesian coordinates (x, y, z) the Euler equations (1)–(2) are written as:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x},\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y},\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z},\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\end{aligned} \tag{3}$$

where u, v, and w are the components of the fluid velocity.

Euler equations in the cylindrical coordinates. The cylindrical coordinates (r, φ, z) are related to the orthogonal Cartesian coordinates (x, y, z) by

$$r = \sqrt{x^2 + y^2}, \quad \tan\varphi = y/x, \quad z = z \quad (\sin\varphi = y/r);$$
$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z,$$

where $0 \le \varphi \le 2\pi$.

The fluid velocity components in the cylindrical coordinates are expressed in terms of those in the rectangular Cartesian coordinates as follows:

$$u_r = u\cos\varphi + v\sin\varphi, \quad u_\varphi = v\cos\varphi - u\sin\varphi, \quad u_z = w.$$

Remark. For $u_z = 0$, the cylindrical coordinates r and φ are also used as polar coordinates on the plane xy.

The Euler equations (1)–(2) in the cylindrical coordinates become

$$\begin{aligned}
&\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\varphi^2}{r} = -\frac{\partial p}{\partial r},\\
&\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_r u_\varphi}{r} = -\frac{1}{r}\frac{\partial p}{\partial \varphi},\\
&\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z\frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z},\\
&\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0,
\end{aligned} \tag{4}$$

where u_r, u_φ, and u_z are components of the fluid velocity.

Euler equations in the spherical coordinates. The spherical coordinates (R, θ, φ) are related to the orthogonal Cartesian coordinates (x, y, z) by

$$R = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos\frac{z}{R}, \quad \tan\varphi = \frac{y}{x} \quad \left(\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}\right);$$
$$x = R\sin\theta\cos\varphi, \quad y = R\sin\theta\sin\varphi, \quad z = R\cos\theta,$$

where $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

The Euler equations (1)–(2) in the spherical coordinates become

$$\begin{aligned}
&\frac{\partial u_R}{\partial t} + u_R\frac{\partial u_R}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_R}{\partial \theta} + \frac{u_\varphi}{R\sin\theta}\frac{\partial u_R}{\partial \varphi} - \frac{u_\theta^2 + u_\varphi^2}{R} = -\frac{\partial p}{\partial R},\\
&\frac{\partial u_\theta}{\partial t} + u_R\frac{\partial u_\theta}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\varphi}{R\sin\theta}\frac{\partial u_\theta}{\partial \varphi} + \frac{u_R u_\theta - u_\varphi^2\cot\theta}{R} = -R\frac{\partial p}{\partial \theta},\\
&\frac{\partial u_\varphi}{\partial t} + u_R\frac{\partial u_\varphi}{\partial R} + \frac{u_\theta}{R}\frac{\partial u_\varphi}{\partial \theta} + \frac{u_\varphi}{R\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_R u_\varphi + u_\theta u_\varphi\cot\theta}{R} = -\frac{1}{R\sin\theta}\frac{\partial p}{\partial \varphi},\\
&\frac{\partial}{\partial R}\big(R^2\sin\theta u_R\big) + \frac{\partial}{\partial \theta}\big(R\sin\theta u_\theta\big) + \frac{\partial}{\partial \varphi}\big(R u_\varphi\big) = 0,
\end{aligned} \tag{5}$$

where u_R, u_θ, and u_φ are components of the fluid velocity.

20.1.2. Two-Dimensional Euler Equations for Incompressible Ideal Fluid (Plane Flows)

For the two-dimensional unsteady Euler equations, describing plane flows, the fluid velocity components and pressure have the form

$$u = u(x, y, t), \quad v = v(x, y, t), \quad w = 0, \quad p = p(x, y, t).$$

1°. Solution in implicit form:

$$y - xf_u'(u) + t[uf_u'(u) - f(u)] = g(u), \quad v = f(u), \quad p = p(t),$$

where $f(u)$, $g(u)$, and $p(t)$ are arbitrary functions.

2°. Generalized separable solution:

$$u = a_1(t)x + b_1(t)y + c_1(t), \quad v = a_2(t)x + b_2(t)y + c_2(t),$$
$$p = \varphi_1(t)x^2 + \varphi_2(t)xy + \varphi_3(t)y^2 + \psi_1(t)x + \psi_2(t)y + \psi_3(t),$$

where the functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\varphi_n = \varphi_n(t)$, and $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
a_1' + a_1^2 + a_2b_1 + 2\varphi_1 &= 0,\\
b_1' + a_1b_1 + b_1b_2 + \varphi_2 &= 0,\\
c_1' + a_1c_1 + b_1c_2 + \psi_1 &= 0,\\
a_2' + a_1a_2 + a_2b_2 + \varphi_2 &= 0,\\
b_2' + a_2b_1 + b_2^2 + 2\varphi_3 &= 0,\\
c_2' + a_2c_1 + b_2c_2 + \psi_2 &= 0,\\
a_1 + b_2 &= 0.
\end{aligned}$$

The general solution of this system is expressed as

$$b_1 = a_2 + C, \quad b_2 = -a_1,$$
$$\varphi_1 = -\tfrac{1}{2}(a_1' + a_1^2 + a_2^2 + Ca_2), \quad \varphi_2 = -a_2', \quad \varphi_3 = -\tfrac{1}{2}(-a_1' + a_1^2 + a_2^2 + Ca_2),$$
$$\psi_1 = -c_1' - a_1c_1 - a_2c_2 - Cx_2, \quad \psi_2 = -c_2' + a_1c_2 - a_2c_1,$$

and $a_1 = a_1(t)$, $a_2 = a_2(t)$, $c_1 = c_1(t)$, $c_2 = c_2(t)$, and $\psi_3 = \psi_3(t)$ are arbitrary functions, and C is an arbitrary constant.

3°. "Two-dimensional" multiplicative separable solution:

$$u = \frac{\varphi(\xi,\eta)}{t + C_3}, \quad v = \frac{\psi(\xi,\eta)}{t + C_3}, \quad p = \frac{\theta(\xi,\eta)}{t + C_3},$$
$$\xi = x + C_1 \ln|t + C_3|, \quad \eta = y + C_2 \ln|t + C_3|,$$

where C_1, C_2, and C_3 are arbitrary constants and the functions $\varphi = \varphi(\xi,\eta)$, $\psi = \psi(\xi,\eta)$, $\theta = \theta(\xi,\eta)$ are determined by the system of differential equations

$$\begin{aligned}
\varphi\varphi_\xi + \psi\varphi_\eta + C_1\varphi_\xi + C_2\varphi_\eta + \theta_\xi - \varphi &= 0,\\
\varphi\psi_\xi + \psi\psi_\eta + C_1\psi_\xi + C_2\psi_\eta + \theta_\eta - \psi &= 0,\\
\varphi_\xi + \psi_\eta &= 0.
\end{aligned}$$

▷ *For other exact solutions of the Euler equations, see Section 13.4.*

20.1.3. Other Solutions with Two Nonzero Components of the Fluid Velocity

1°. Solution in the cylindrical coordinates:

$$u_r = Ar\sin[2(kz + \varphi + B)]\Psi, \quad u_\varphi = -2Ar\sin^2(kz + \varphi + B)\Psi, \quad u_z = 0,$$
$$p = p_0 - \frac{2A}{C}\Psi^2\ln\Psi - \frac{1}{2}(u_r^2 + u_\varphi^2), \quad \Psi = \exp\bigl[-ACr^2\sin^2(kz + \varphi + B)\bigr],$$

where A, B, C, k, and p_0 are arbitrary constants.

⊙ *Reference*: S. N. Aristov and D. V. Knyazev (2010).

20.1.4. Rotationally Symmetric Motions of Fluid

The rotationally symmetric motions of an incompressible ideal fluid in the cylindrical coordinate system $(r,\ \varphi,\ z)$ are independent of the angular coordinate φ and obey the equations

$$u_t + uu_r + wu_z - \frac{v^2}{r} + p_r = 0,$$
$$v_t + uv_r + wv_z + \frac{uv}{r} = 0,$$
$$w_t + uw_r + ww_z + p_z = 0,$$
$$u_r + \frac{1}{r}u + w_z = 0,$$

where u, v, and w are the projections of the fluid velocity $\mathbf{u}$ onto the axes of the cylindrical coordinate system r, φ, z.

1°. Solution:

$$u = -\frac{f_t'(t)}{2f(t)}r + \frac{a_t'(t)}{2rf(t)}, \quad v = \frac{1}{r}V(\xi), \quad w = -\frac{f_t'(t)}{f(t)}z + \frac{1}{f(t)}W(\xi), \quad \xi = r^2 f(t) - a(t),$$
$$p = -\frac{f_t'(t)}{2f(t)}z^2 - \frac{u^2}{2} - \frac{\partial}{\partial t}\int u\,dr + \int \frac{v^2}{r}\,dr + \varphi(t),$$

where $f = f(t)$, $a = a(t)$, $\varphi = \varphi(t)$, $V = V(\xi)$, and $W = W(\xi)$ are arbitrary functions.

2°. Solutions:

$$u = u(r), \quad w = -\left(u_r' + \frac{u}{r}\right)z + f(r), \quad v = \pm\sqrt{\frac{2}{ar^2}\big[ag(r) - t\big]}, \quad p = h(r,z) - \frac{ag(r) - t}{ar^2}.$$

Here, the functions $u = u(r)$ and $f = f(r)$ are determined by the ordinary differential equations

$$u\left(u_r' + \frac{u}{r}\right)_r' - \left(u_r' + \frac{u}{r}\right)^2 = C_1 = \text{const},$$
$$uf_r' - \left(u_r' + \frac{u}{r}\right)f = C_2 = \text{const}.$$

The form of the general solution to the equation for $u = u(r)$ depends on the value of the constant of integration C_1:

1. If $C_1 > 0$, there is one solution:

$$u = \frac{1}{ar}\cosh\left(\frac{a\sqrt{C_1}}{2}r^2 + b\right).$$

2. If $C_1 = 0$, there is also one solution:

$$u = \frac{b}{r}\exp(ar^2).$$

3. If $C_1 < 0$, there are four distinct solutions:

$$u = \frac{1}{ar}\sinh\left(\frac{a\sqrt{|C_1|}}{2}r^2 + b\right), \quad u = \frac{1}{ar}\sin\left(\frac{a\sqrt{|C_1|}}{2}r^2 + b\right), \quad u = \pm\frac{\sqrt{|C_1|}}{2}r + \frac{b}{r}.$$

The functions $f(r)$, $g(r)$, and $h(r,z)$ are found as

$$f(r) = ru\left(C_2\int\frac{dr}{ru^2}+C_3\right),$$
$$g(r) = \frac{1}{a}\int\frac{dr}{u}+C_5,$$
$$h(r,z) = -\frac{u^2}{2}+\frac{C_1}{2}z^2-C_2z+\frac{1}{a}\int\frac{dr}{r^2u}+C_4.$$

In the above formulas, a, b, $C_1,\ldots,C_5$ are arbitrary constants.

⊙ *Reference for Subsection* 20.1.4: V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998).

20.1.5. Euler Equations for Barotropic Gas Flow

Two forms of the Euler equations. A barotropic gas flow is described by the Euler and continuity equations

$$\mathbf{u}_t+(\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \tag{1}$$
$$\rho_t+(\mathbf{u}\nabla)\rho+\rho\,\mathrm{div}\,\mathbf{u} = 0, \tag{2}$$
$$p = p(\rho), \tag{3}$$

where $\mathbf{u}=(u,v,w)$ is the gas velocity, ρ is the gas density, and p is the pressure. It is assumed that the dependence $p=p(\rho)$ is given and there are no mass forces.

The Euler equations (1) *in the Gromeka–Lamb form* is

$$\mathbf{u}_t+\frac{1}{2}\nabla\mathbf{u}^2+\mathrm{curl}\,\mathbf{u}\times\mathbf{u}+\frac{1}{\rho}\nabla p = 0. \tag{4}$$

Potential gas flows. Cauchy–Lagrange integral. In what follows, we will be considering *irrotational* (*potential*) *gas flows*

$$\mathbf{u} = \nabla\varphi, \tag{5}$$

where $\varphi=\varphi(\mathbf{r},t)$ is the velocity potential. As a result, Eqs. (4) and (5) lead to the *Cauchy–Lagrange integral*

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}|\nabla\varphi|^2+P(\rho) = 0, \tag{6}$$

where

$$P(\rho) = \int\frac{dp}{\rho}. \tag{7}$$

From equations (2) and (6) for the velocity potential φ and density ρ, one can obtain, using relations (3), (5), and (7), a single second-order equation for the potential

$$\varphi_{tt}+2\varphi_x\varphi_{xt}+2\varphi_y\varphi_{yt}+2\varphi_z\varphi_{zt}+2\varphi_x\varphi_y\varphi_{xy}+2\varphi_x\varphi_z\varphi_{xz}+2\varphi_y\varphi_z\varphi_{yz}$$
$$+(\varphi_x^2-c^2)\varphi_{xx}+(\varphi_y^2-c^2)\varphi_{yy}+(\varphi_z^2-c^2)\varphi_{zz} = 0,$$

where $c = \sqrt{p'(\rho)}$ is the speed of sound; the prime denotes a derivative. The sound speed c is expressed through the potential φ using formulas (6) and (7).

Two-dimensional steady-state potential gas flow. The stream function. For two-dimensional steady-state gas flows, the continuity equation (2) is written as

$$\frac{\partial}{\partial x}(\rho u y^{n-1}) + \frac{\partial}{\partial y}(\rho v y^{n-1}) = 0, \tag{8}$$

where $n = 1$ corresponds to plane motions and $n = 2$, to axisymmetric motions; y is the radial coordinate here.

Equation (8) can be satisfied identically by introducing a stream function ψ such as

$$\rho u y^{n-1} = \rho_0 \frac{\partial \psi}{\partial y}, \qquad \rho v y^{n-1} = -\rho_0 \frac{\partial \psi}{\partial x}, \tag{9}$$

where $\rho_0 \neq 0$ is an arbitrary constant.

For formula (5) for potential gas motions it follows that

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}. \tag{10}$$

Eliminating the velocity components from (9) and (10), one arrives at the following system of equations for φ and ψ:

$$\frac{\rho}{\rho_0} y^{n-1} \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\rho}{\rho_0} y^{n-1} \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{11}$$

Eliminating the stream function ψ from (11), one obtains a single equation for the potential

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0} y^{n-1} \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0} y^{n-1} \frac{\partial \varphi}{\partial y}\right) = 0. \tag{12}$$

In a similar way, one can obtain the equation for the stream function

$$\frac{\partial}{\partial x}\left(\frac{\rho_0}{\rho} y^{1-n} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho_0}{\rho} y^{1-n} \frac{\partial \psi}{\partial y}\right) = 0. \tag{13}$$

The density ratio ρ/ρ_0 in (12) and (13) can be expressed, using relations (3) and (6) (with $\varphi_t \equiv 0$ in the steady-state case), in terms of the sum of squares $\varphi_x^2 + \varphi_y^2$ or $\psi_x^2 + \psi_y^2$. Therefore, system (11) and equations (12) and (13) are nonlinear.

Potential plane gas flow ($n = 1$). Hodograph transformation. In system (11) with $n = 1$, one can choose $V = |\mathbf{u}| = \sqrt{u^2 + v^2}$ and $\tan\theta = v/u$ to be the new independent variables instead of the coordinates x and y. By doing so, one obtains the linear system of equations

$$\frac{1}{V}\frac{\partial \varphi}{\partial V} = \frac{d}{dV}\left(\frac{1}{V}\frac{\rho_0}{\rho}\right)\frac{\partial \psi}{\partial \theta}, \qquad \frac{1}{V}\frac{\partial \varphi}{\partial \theta} = \frac{\rho_0}{\rho}\frac{\partial \psi}{\partial V}. \tag{14}$$

By virtue of (3)–(7) (with $\varphi_t \equiv 0$), the ratio ρ_0/ρ is the function of the variable V alone.

Eliminating the potential from (14), one arrives at the equation for the stream function

$$\frac{\partial}{\partial V}\left(\frac{\rho_0}{\rho} V \frac{\partial \psi}{\partial V}\right) - V \frac{d}{dV}\left(\frac{\rho_0}{\rho}\frac{1}{V}\right)\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{15}$$

In a similar way, one can obtain the second equation for the potential.

Equations (14) and (15) are known as *Chaplygin's equations*.

Simplest exact solutions of Chaplygin's equations (14).

1°. *The first solution*:

$$\varphi = -A\theta, \qquad \psi = -A\int_{V_{\max}}^{V} \frac{\rho\, dV}{\rho_0 V},$$

where A is an arbitrary constant. If the polar coordinates (r, φ) are introduced at the point (x_0, y_0) of the plane (x, y), the resulting velocity field is defined by

$$V = \frac{A}{r}, \qquad \theta = \varphi - \frac{\pi}{2}.$$

The streamlines of this motion are concentric circles with common center at (x_0, y_0). The gas particles move in these circular paths with constant speeds (counterclockwise if $A > 0$). The speeds at the streamlines are inversely proportional to the radius, with the constant A connected to the velocity circulation, Γ, around a closed streamline by the obvious relation $\Gamma = 2\pi A$. This kind of flow represents the motion of a concentrated vortex in a compressible gas.

2°. *The second solution*:

$$\psi = A\theta, \qquad \varphi = A\int_{V_0}^{V} V\frac{d}{dV}\left(\frac{\rho_0}{\rho V}\right) dV,$$

where A is an arbitrary constant. With the polar coordinates (r, φ) introduced at the point (x_0, y_0) of the plane (x, y), the resulting velocity field is defined by

$$r = A\frac{\rho_0}{\rho V}, \qquad \theta = \varphi. \tag{16}$$

The second relation in (16) shows that the streamlines go along rays issuing from the point (x_0, y_0). The speed on the streamlines is the same along each circle centered at this point and varies with the circle radius r according to the first relation in (15), which expresses the law of conservation of mass. The constant A is connected with the gas rate of flow Q by the relation $Q = 2\pi\rho_0 A$.

Mach angle and number. Prandtl–Meyer solutions (simple waves). The *Mach angle* μ is defined by

$$\sin\mu = \frac{c}{V} = \frac{1}{\mathsf{M}},$$

where $c = \sqrt{p'(\rho)}$ the sound of speed and M is the Mach number; $\mathsf{M} < 1$ for subsonic flows and $\mathsf{M} > 1$ for supersonic flows.

In the variables $V = \sqrt{u^2 + v^2}$ and $\tan\theta = v/u$, two families of *Prandtl–Meyer solutions* for supersonic flows are expressed as

$$\theta + \int_{V_0}^{V} \cot\mu\frac{dV}{V} = A_1, \qquad y - x\tan(\theta + \mu) = f_1(\theta);$$
$$\theta - \int_{V_0}^{V} \cot\mu\frac{dV}{V} = A_2, \qquad y - x\tan(\theta - \mu) = f_2(\theta),$$

where A_1 and A_2 are arbitrary constants and $f_1(\theta)$ and $f_2(\theta)$ are arbitrary functions. The characteristics of these families are straight lines and the gas parameters are constant along each characteristic.

Two-dimensional unsteady Euler equations. The two-dimensional unsteady Euler equations for the motion of a barotropic gas are

$$\begin{aligned}\rho(u_t + uu_x + vu_y) + p_x &= 0,\\ \rho(v_t + uv_x + vv_y) + p_y &= 0,\\ \rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) &= 0,\end{aligned}$$

where u and v are the fluid velocity components and $p = p(\rho)$.

1°. "Two-dimensional" generalized separable solution:

$$u = x\varphi(y,t) + \psi(y,t), \quad v = v(y,t), \quad \rho = \rho(y,t),$$

where the functions $\varphi = \varphi(y,t)$, $\psi = \psi(y,t)$, $v = v(y,t)$, and $\rho = \rho(y,t)$ are determined by the system of differential equations

$$\begin{aligned}\varphi_t + \varphi^2 + v\varphi_y &= 0,\\ \psi_t + \varphi\psi + v\psi_y &= 0,\\ \rho(v_t + vv_y) + p'_\rho\rho_y &= 0,\\ \rho_t + v\rho_y + \rho(\varphi + v_y) &= 0.\end{aligned}$$

2°. There is an exact solution of the form

$$u = U(\xi,\eta,\zeta), \quad v = V(\xi,\eta,\zeta), \quad w = w(\xi,\eta,\zeta), \quad \xi = \frac{x}{t}, \quad \eta = \frac{y}{t}, \quad \zeta = \frac{z}{t}.$$

⊙ *References for Subsection* 20.1.5: S. A. Chaplygin (1904), R. Courant (1962), L. V. Ovsiannikov (1981), G. G. Chernyi (1988), L. G. Loitsyanskiy (1996).

▷ *In equations 16.2.1.4 and 16.1.2.5 and Section 38.4, one can find one-dimensional exact solutions and some problems for the Euler equations for general barotropic and polytropic gas flows.*

20.2. Adiabatic Gas Flow

20.2.1. Preliminary Remarks

An adiabatic (nonisentropic) gas flow is governed by the equations

$$\rho[\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u}] + \nabla p = \mathbf{0}, \tag{1}$$

$$\rho_t + (\mathbf{u}\nabla)\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \tag{2}$$

$$S_t + (\mathbf{u}\nabla)S = 0, \qquad S = S(p,\rho), \tag{3}$$

where $\mathbf{u}$ is the gas velocity, ρ is the gas density, p is the pressure, and S is the entropy (assumed to be a given function of p and ρ).

Equation (3) can be rewritten as

$$p_t + (\mathbf{u}\nabla)p + F(p,\rho)\,\mathrm{div}\,\mathbf{u} = 0, \tag{3a}$$

where

$$F(p,\rho) = -\rho S_\rho / S_p. \tag{4}$$

⊙ *References*: L. V. Ovsiannikov (1981), N. H. Ibragimov (1994).

20.2.2. One-Dimensional Case

One-dimensional adiabatic gas flows are governed by the equations

$$\begin{aligned}\rho(u_t + uu_x) + p_x &= 0,\\ \rho_t + u\rho_x + \rho u_x &= 0,\\ p_t + up_x + F(p,\rho)u_x &= 0.\end{aligned}$$

1. Similarity formulas.

Suppose $u = u(x,t)$, $\rho = \rho(x,t)$, and $p = p(x,t)$ is a solution of the above system. Then the two sets of functions

$$\begin{aligned}&u_1 = u(Ax+B, At+C), \quad \rho_1 = \rho(Ax+B, At+C), \quad p_1 = p(Ax+B, At+C);\\ &u_2 = u(x-Dt,t) + D, \qquad \rho_2 = \rho(x-Dt,t), \qquad\quad p_2 = p(x-Dt,t),\end{aligned}$$

where A, B, C, and D are arbitrary constants, are also solutions of the system.

2. Let $F(p,\rho)$ be an arbitrary function.

2.1. Solution:

$$u = C_1, \quad p = C_2, \quad \rho = \varphi(x - C_1 t),$$

where C_1 and C_2 are arbitrary constants and $\varphi(z)$ is an arbitrary function.

2.2. Solution (generalizes the solution from Item 2.1):

$$u = C_1 t + C_2, \quad p = C_1\varphi(z) + C_3, \quad \rho = -\varphi'_z(z), \quad z = x - \tfrac{1}{2}C_1 t^2 - C_2 t,$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(z)$ is an arbitrary function.

2.3. Solution:

$$u = \frac{x + C_1}{t}, \quad \rho = \frac{C_2}{t}, \quad p = p(t),$$

where the function $p = p(t)$ is determined by the ordinary differential equation

$$p'_t + \frac{1}{t}F\Big(p, \frac{C_2}{t}\Big) = 0.$$

2.4. Self-similar solution:

$$u = u(z), \quad \rho = \rho(z), \quad p = p(z), \quad z = \frac{x}{t},$$

where the functions $u = u(z)$, $\rho = \rho(z)$, and $p = p(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\rho(u-z)u'_z + p'_z &= 0,\\ (u-z)\rho'_z + \rho u'_z &= 0,\\ (u-z)p'_z + F(p,\rho)u'_z &= 0.\end{aligned}$$

By analyzing this system, one can represent its solutions in the implicit form

$$\begin{aligned}&u = z + \sqrt{\frac{F}{\rho}}, \quad u + \int\sqrt{\frac{F}{\rho}}\,\frac{d\rho}{\rho} = C, \quad z = \frac{x}{t};\\ &u = z - \sqrt{\frac{F}{\rho}}, \quad u - \int\sqrt{\frac{F}{\rho}}\,\frac{d\rho}{\rho} = C, \quad z = \frac{x}{t},\end{aligned}$$

where C is an arbitrary constant, $F = F(p, \rho)$, and the function $p = p(\rho)$ is determined by solving the ordinary differential equations

$$\frac{dp}{d\rho} = \frac{F}{\rho}.$$

2.5. Solution:

$$u = C_1 t + C_2 + \theta(z), \quad \rho = \frac{C_3}{\theta(z)}, \quad p = p(z), \quad z = x - \frac{1}{2}C_1 t^2 - C_2 t,$$

where the functions $\theta = \theta(z)$ and $p = p(z)$ are determined by the system of ordinary differential equations

$$\theta p'_z + C_3 \theta \theta'_z + C_1 C_3 = 0,$$
$$\theta p'_z + F(p, C_3/\theta)\theta'_z = 0.$$

2.6. Let $F(p_0, \rho) = 0$, where $p_0 = \text{const}$.
Solution in implicit form:

$$x - ut = \varphi(u), \quad \rho = \frac{\psi(u)}{t + \varphi'_u(u)}, \quad p = p_0,$$

where $\varphi(u)$ and $\psi(u)$ are arbitrary functions. The second relation can be represented in the form $\rho = \psi(u)u_x$.

2.7. Solution:

$$u = u(x, t), \quad \rho = \rho(u), \quad p = p(u).$$

Here, the functions $\rho = \rho(u)$ and $p = p(u)$ are determined by the system of ordinary differential equations

$$\frac{\rho}{\rho'_u} = \frac{p'_u}{\rho} = \frac{F(p, \rho)}{p'_u} \tag{1}$$

and the function $u = u(x, t)$ is defined implicitly by

$$x - t\left(u + \frac{p'_u(u)}{\rho(u)}\right) = \varphi(u), \tag{2}$$

where $\varphi(u)$ is an arbitrary function.

First, the function $p = \bar{p}(\rho)$ can be found from system (1) by solving the equation

$$\frac{d\bar{p}}{d\rho} = \frac{1}{\rho}F(\bar{p}, \rho),$$

and then $\rho = \rho(u)$ can be determined from

$$\frac{d\rho}{du} = \pm\frac{\rho^2}{\sqrt{\rho F(\bar{p}(\rho), \rho)}}.$$

Remark. To the degenerate case $\varphi(u) \equiv 0$ in (2) there corresponds the self-similar solution from Item 2.4.

2.8. Solution:

$$u = u(x, t), \quad \rho = \rho(x, t), \quad p = p(\rho),$$

where the function $p = p(\rho)$ is determined by the ordinary differential equation

$$\frac{dp}{d\rho} = \frac{1}{\rho}F(p, \rho), \tag{3}$$

and the functions u and ρ are determined by the system of differential equations

$$\begin{aligned} u_t + uu_x + \frac{1}{\rho}p_x = 0, \quad p = p(\rho), \\ \rho_t + u\rho_x + \rho u_x = 0. \end{aligned} \tag{4}$$

This system appears to coincide with system 16.2.1.5, which describes barotropic gas flows. The only difference is that the pressure function $p = p(\rho)$ in system 16.2.1.4 is known in advance and that in this case has to be determined from equation (3).

System (4) can be linearized with the hodograph transformation.

3. Let $F(p, \rho) = pf(p/\rho)$, where $f(z)$ is an arbitrary function.

3.1. Multiplicative separable solution:

$$u = u(x), \quad \rho = e^{at}\varphi(x), \quad p = e^{at}\psi(x),$$

where a is an arbitrary constant and the functions $u = u(x)$, $\varphi = \varphi(x)$, and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \psi'_x + \varphi u u'_x = 0, \\ u\varphi'_x + (u'_x + a)\varphi = 0, \\ u\psi'_x + \psi f(\psi/\varphi)u'_x + a\psi = 0. \end{aligned}$$

3.2. Solution (generalizes the solution from Item 3.1):

$$u = u(z), \quad \rho = e^{at+bx}\varphi(z), \quad p = e^{at+bx}\psi(z), \quad z = kx - ct,$$

where a, b, c, and k are arbitrary constants and the functions $u = u(z)$, $\varphi = \varphi(z)$, and $\psi = \psi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \varphi(ku - c)u'_z + b\psi + k\psi'_z = 0, \\ a\varphi - c\varphi'_z + u(b\varphi + k\varphi'_z) + k\varphi u'_z = 0, \\ a\psi - c\psi'_z + u(b\psi + k\psi'_z) + k\psi f(\psi/\varphi)u'_z = 0. \end{aligned}$$

3.3. Solution (generalizes the solution from Item 3.1):

$$u = C_1 t + C_2 + \theta(z), \quad \rho = e^{at}\varphi(z), \quad p = e^{at}\psi(z), \quad z = x - \tfrac{1}{2}C_1 t^2 - C_2 t,$$

where C_1, C_2, and a are arbitrary constants and the functions $\theta = \theta(z)$, $\varphi = \varphi(z)$, and $\psi = \psi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \psi'_z + (\theta\theta'_z + C_1)\varphi = 0, \\ \theta\varphi'_z + (\theta'_z + a)\varphi = 0, \\ \theta\psi'_z + \psi f(\psi/\varphi)\theta'_z + a\psi = 0. \end{aligned}$$

3.4. Solution:

$$u = \frac{x}{t} + \theta(t), \quad \rho = \exp\Big(\frac{x}{t}\Big)\varphi(t), \quad p = \exp\Big(\frac{x}{t}\Big)\psi(t),$$

where the functions $\theta = \theta(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} t\varphi\theta'_t + \varphi\theta + \psi &= 0,\\ t\varphi'_t + (\theta+1)\varphi &= 0,\\ t\psi'_t + \psi f(\psi/\varphi) + \psi\theta &= 0. \end{aligned}$$

The substitution $t = \pm e^{\tau}$ reduces this system to the autonomous form.

3.5. Solution:

$$u = \frac{x}{t} + a, \quad \rho = Ct^m \exp\Big(-\frac{ax}{kt}\Big), \quad p = Ckt^m \exp\Big(-\frac{ax}{kt}\Big), \quad m = \frac{a^2}{k} - 1,$$

where a and C are arbitrary constants and k is a root of the algebraic (transcendental) equation

$$f(k) - 1 = 0.$$

4. Let $F(p, \rho) = pf(p\rho)$, where $f(z)$ is an arbitrary function.

4.1. Traveling-wave solution:

$$u = U(z) + C, \quad \rho = \frac{k}{U(z)}, \quad p = -kU(z), \quad z = x - Ct,$$

where C is an arbitrary constant, $U(z)$ is an arbitrary function, and k is a root of the algebraic (transcendental) equation

$$f(-k^2) + 1 = 0.$$

4.2. Solutions:

$$u = \lambda x, \quad \rho = \pm\frac{k}{\sqrt{Ce^{2\lambda t} - k\lambda^2 x^2}}, \quad p = \pm\sqrt{Ce^{2\lambda t} - k\lambda^2 x^2},$$

where C and λ are arbitrary constants and k is a root of the algebraic (transcendental) equation

$$f(k) + 1 = 0.$$

4.3. Solution (generalizes the solutions from Item 4.2):

$$u = e^{\lambda t}\theta(z), \quad \rho = e^{-\lambda t}\theta(z), \quad p = e^{\lambda t}\psi(z), \quad z = xe^{-\lambda t},$$

where λ is an arbitrary constant and the functions $\theta = \theta(z)$, $\varphi = \varphi(z)$, and $\psi = \psi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \psi'_z + \varphi(\theta - \lambda z)\theta'_z + \lambda\varphi\theta &= 0,\\ (\theta - \lambda z)\varphi'_z + \varphi\theta'_z - \lambda\varphi &= 0,\\ (\theta - \lambda z)\psi'_z + \psi f(\varphi\psi)\theta'_z + \lambda\psi &= 0. \end{aligned}$$

By setting $\theta(z) = \lambda z$, we obtain the solutions from Item 4.2.

5. Let $F(p,\rho)=pf(\rho)$, where $f(\rho)$ is an arbitrary function.

5.1. Generalized separable solution:

$$u=\frac{x}{t}+\theta(t),\quad \rho=\frac{C}{t},\quad p=\varphi(t)x+\psi(t),$$

where C is an arbitrary constant and the functions $\theta=\theta(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\varphi_t'+\frac{1}{t}\left[1+f\left(\frac{C}{t}\right)\right]\varphi&=0,\\ \psi_t'+\frac{1}{t}f\left(\frac{C}{t}\right)\psi+\varphi\theta&=0,\\ \theta_t'+\frac{1}{t}\theta+\frac{1}{C}t\varphi&=0.\end{aligned}$$

This system is easy to integrate: the first equation is linear in φ, the third equation is linear in θ, and the second one is linear in ψ.

5.2. Multiplicative separable solution:

$$u=x\theta(t),\quad \rho=\rho(t),\quad p=x^2\varphi(t),$$

where the functions $\theta=\theta(t)$, $\varphi=\varphi(t)$, and $\varphi=\varphi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\rho(\theta_t'+\theta^2)+2\varphi&=0,\\ \rho_t'+\rho\theta&=0,\\ \varphi_t'+[f(\rho)+2]\theta\varphi&=0.\end{aligned}\tag{5}$$

This system can be solved. Eliminating θ from the last two equations and integrating, one obtains

$$\varphi=C\rho^2\exp\left(\int\frac{f(\rho)}{\rho}\,d\rho\right)\equiv Cg(\rho).\tag{6}$$

Eliminating θ from the first and second equations and taking into account (6), one arrives at the autonomous equation

$$\rho\rho_{tt}''-2(\rho_t')^2-2C\rho g(\rho)=0,$$

which can be reduced to a linear first-order equation with the substitution $w(\rho)=(\rho_t')^2$.

5.3. Generalized separable solution (generalizes the solutions from Items 5.1 and 5.2):

$$u=x\theta(t)+\lambda(t),\quad \rho=\rho(t),\quad p=\varphi(t)x^2+\psi(t)x+\chi(t),$$

where the functions $\theta=\theta(t)$, $\lambda=\lambda(t)$, $\rho=\rho(t)$, $\varphi=\varphi(t)$, $\psi=\psi(t)$, and $\chi=\chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\rho(\theta_t'+\theta^2)+2\varphi&=0,\\ \rho(\lambda_t'+\theta\lambda)+\psi&=0,\\ \rho_t'+\theta\rho&=0,\\ \varphi_t'+[f(\rho)+2]\theta\varphi&=0,\\ \psi_t'+[f(\rho)+1]\theta\psi+2\lambda\varphi&=0,\\ \chi_t'+f(\rho)\theta\chi+\lambda\psi&=0.\end{aligned}$$

5.4. Multiplicative separable solution:

$$u = \frac{1}{t}\varphi(x), \quad \rho = \frac{C_1}{\varphi(x)}, \quad p = \frac{1}{t^2}[C_1 x - C_1\varphi(x) + C_2],$$

where C_1 and C_2 are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\left[(C_1 x - C_1\varphi + C_2) f\left(\frac{C_1}{\varphi}\right) - C_1\varphi\right]\varphi'_x + 3C_1\varphi - 2C_1 x - 2C_2 = 0.$$

5.5. Solution (generalizes the solution from Item 5.2):

$$u = x\theta(z), \quad \rho = \rho(z), \quad p = x^2\varphi(z), \quad z = t + C\ln x,$$

where C is an arbitrary constant and the functions $\theta = \theta(z)$, $\rho = \rho(z)$, and $\varphi = \varphi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\rho(C\theta\theta'_z + \theta'_z + \theta^2) + C\varphi'_z + 2\varphi &= 0,\\
C(\theta\rho'_z + \rho\theta'_z) + \rho'_z + \rho\theta &= 0,\\
(C\theta + 1)\varphi'_z + \varphi(C\theta'_z + \theta)f(\rho) + 2\varphi\theta &= 0.
\end{aligned}$$

5.6. Solution (generalizes the solution from Item 5.4):

$$u = \frac{1}{t}\theta(z), \quad \rho = \rho(z), \quad p = \frac{1}{t^2}\varphi(z), \quad z = x + C\ln t,$$

where C is an arbitrary constant and the functions $\theta = \theta(z)$, $\rho = \rho(z)$, and $\varphi = \varphi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\rho(\theta\theta'_z + C\theta'_z - \theta) + \varphi'_z &= 0,\\
\theta\rho'_z + \rho\theta'_z + C\rho'_z &= 0,\\
(\theta + C)\varphi'_z + \varphi f(\rho)\theta'_z - 2\varphi &= 0.
\end{aligned}$$

Integrating the second equation gives the relation

$$\theta = \frac{B}{\rho} - C,$$

where B is an arbitrary constant.

6. Let $F(p, \rho) = pf(\rho) + g(\rho)$, where $f(\rho)$ and $g(\rho)$ are arbitrary functions.

Generalized separable solution (generalizes the solution from Item 5.3):

$$u = x\theta(t) + \lambda(t), \quad \rho = \rho(t), \quad p = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\theta = \theta(t)$, $\lambda = \lambda(t)$, $\rho = \rho(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\rho(\theta'_t + \theta^2) + 2\varphi &= 0,\\
\rho(\lambda'_t + \theta\lambda) + \psi &= 0,\\
\rho'_t + \theta\rho &= 0,\\
\varphi'_t + [f(\rho) + 2]\theta\varphi &= 0,\\
\psi'_t + [f(\rho) + 1]\theta\psi + 2\lambda\varphi &= 0,\\
\chi'_t + [f(\rho)\chi + g(\rho)] + \lambda\psi &= 0.
\end{aligned}$$

7. Let $F(p, \rho) = f(p)/\rho$, where $f(p)$ is an arbitrary function.

Solution in implicit form:

$$x - ut = \varphi(u), \quad \rho = \frac{C_1}{t + \varphi'_u(u)}, \quad C_1\int\frac{dp}{f(p)} = -t + C_2,$$

where $\varphi(u)$ is an arbitrary function and C_1 and C_2 are arbitrary constants.

8. Let $F(p,\rho)=pf(p\rho^{\alpha})$, where $f(z)$ is an arbitrary function.

8.1. Multiplicative separable solution:

$$u=x\theta(t),\quad \rho=x^{-\frac{2}{\alpha+1}}\varphi(t),\quad p=x^{\frac{2\alpha}{\alpha+1}}\psi(t),$$

where the functions $\theta=\theta(t)$, $\varphi=\varphi(t)$, and $\psi=\psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\varphi\theta_t'+\varphi\theta^2+\frac{2\alpha}{\alpha+1}\psi&=0,\\ \varphi_t'+\frac{\alpha-1}{\alpha+1}\varphi\theta&=0,\\ \psi_t'+\psi\theta f(\psi\varphi^{\alpha})+\frac{2\alpha}{\alpha+1}\psi\theta&=0.\end{aligned}$$

8.2. Multiplicative separable solution:

$$u=\frac{1}{t}\theta(x),\quad \rho=t^{\frac{2}{\alpha+1}}\varphi(x),\quad p=t^{-\frac{2\alpha}{\alpha+1}}\psi(x),$$

where the functions $\theta=\theta(x)$, $\varphi=\varphi(x)$, and $\psi=\psi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\theta\varphi(\theta_x'-1)+\psi_x'&=0,\\ \theta\varphi_x'+\varphi\theta_x'+\frac{2}{\alpha+1}\varphi&=0,\\ \theta\psi_x'+\psi f(\psi\varphi^{\alpha})\theta_x'-\frac{2\alpha}{\alpha+1}\psi&=0.\end{aligned}$$

8.3. Solution (generalizes the solution from Item 8.1):

$$u=x\theta(z),\quad \rho=x^{-\frac{2}{\alpha+1}}\varphi(z),\quad p=x^{\frac{2\alpha}{\alpha+1}}\psi(z),\quad z=t+C\ln x,$$

where C is an arbitrary constant and the functions $\theta=\theta(z)$, $\varphi=\varphi(z)$, and $\psi=\psi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\varphi(C\theta\theta_z'+\theta_z'+\theta^2)+C\psi_z'+\frac{2\alpha}{\alpha+1}\psi&=0,\\ C(\theta\varphi_z'+\varphi\theta_z')+\varphi_z'+\frac{\alpha-1}{\alpha+1}\varphi\theta&=0,\\ (C\theta+1)\psi_z'+\psi(C\theta_z'+\theta)f(\psi\varphi^{\alpha})+\frac{2\alpha}{\alpha+1}\psi\theta&=0.\end{aligned}$$

8.4. Solution (generalizes the solution from Item 8.2):

$$u=\frac{1}{t}\theta(z),\quad \rho=t^{\frac{2}{\alpha+1}}\varphi(z),\quad p=t^{-\frac{2\alpha}{\alpha+1}}\psi(z),\quad z=x+C\ln t,$$

where C is an arbitrary constant and the functions $\theta=\theta(z)$, $\varphi=\varphi(z)$, and $\psi=\psi(z)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\varphi(\theta\theta_z'+C\theta_z'-\theta)+\psi_z'&=0,\\ \theta\varphi_z'+\varphi\theta_z'+C\varphi_z'+\frac{2}{\alpha+1}\varphi&=0,\\ (\theta+C)\psi_z'+\psi f(\psi\varphi^{\alpha})\theta_z'-\frac{2\alpha}{\alpha+1}\psi&=0.\end{aligned}$$

9. Let $F(p,\rho) = pf(ap+b\rho)$, where $f(z)$ is an arbitrary function that satisfies the condition $f(0) = 1$.

Solution:

$$u = \frac{x}{t} + C_1, \quad \rho = C_2 a t^k \exp\left(\lambda\frac{x}{t}\right), \quad p = -C_2 b t^k \exp\left(\lambda\frac{x}{t}\right),$$
$$k = -\frac{a}{b}C_1^2 - 1, \quad \lambda = \frac{a}{b}C_1,$$

where C_1 and C_2 are arbitrary constants.

Examples of admissible functions include $f(z) = \exp(\beta z)$, $f(z) = \cosh(\beta z)$, and $f(z) = \cos(\beta z)$. If $f(z) \equiv 1$, the parameters a and b in the solution can regarded as arbitrary.

10. Let $F(p,\rho) = \gamma p$, where γ is a constant. In this special case, many exact solutions can be obtained from the more general solution presented in Items 2–9. Listed below are some solutions having a fairly simple form.

10.1. Solution:

$$u = C_1 t + C_2, \quad \rho = -\varphi_z'(z), \quad p = C_1\varphi(z) + C_3, \quad z = x - \tfrac{1}{2}C_1 t^2 - C_2 t,$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(z)$ is an arbitrary function.

10.2. Solution in implicit form:

$$x - ut = \varphi(u), \quad \rho = \psi(u)u_x, \quad p = 0,$$

where $\varphi(u)$ and $\psi(u)$ are arbitrary functions.

10.3. Solution:

$$u = \frac{x + C_1}{t}, \quad \rho = \frac{C_2}{t}, \quad p = C_3 t^{-\gamma},$$

where C_1, C_2, and C_3 are arbitrary constants.

10.4. Solution:

$$u = \frac{x}{t} + \frac{1}{\gamma^2}t^{1-\gamma}, \quad \rho = \frac{C\gamma^2}{(\gamma-2)t}\exp\left(\frac{x}{t} + \frac{t^{1-\gamma}}{\gamma^2(\gamma-1)}\right), \quad p = Ct^{-\gamma}\exp\left(\frac{x}{t} + \frac{t^{1-\gamma}}{\gamma^2(\gamma-1)}\right),$$

where C is an arbitrary constant and $\gamma \neq 1, 2$.

10.5. Solution:

$$u = \frac{2}{\gamma+1}\frac{x}{t}, \quad \rho = \frac{1}{x}\varphi(z), \quad p = x^{-\gamma}\psi(z), \quad z = tx^{-(\gamma+1)/2},$$

where $\psi = \psi(z)$ is an arbitrary function and $\varphi = \varphi(z)$ is defined by formula

$$\varphi = -\frac{(\gamma+1)^2}{2(\gamma-1)}z^2\left(\frac{\gamma+1}{2}z\psi_z' + \gamma\psi\right).$$

10.6. Solution:

$$u = u(x), \quad \rho = u^{\gamma-1}\exp\left(t - \frac{u^{\gamma+1}}{\gamma+1}\right), \quad p = \exp\left(t - \frac{u^{\gamma+1}}{\gamma+1}\right), \quad \gamma \neq 1,$$

where the function $u = u(x)$ is defined implicitly by

$$\frac{1}{\gamma+2}u^{\gamma+2} - \gamma u = x \quad \text{if} \quad \gamma \neq -2;$$
$$\ln u + 2u = x \quad \text{if} \quad \gamma = -2.$$

10.7. Solution:

$$u = t - U(z), \quad \rho = \frac{1}{U(z)}, \quad p = U^{-\gamma}(z), \quad z = x - \frac{1}{2}t^2,$$

where the function $U = U(z)$ is defined implicitly by

$$2z + U^2 + \frac{2\gamma}{\gamma-1}U^{1-\gamma} = 0 \quad \text{if} \quad \gamma \neq 1,$$
$$2z + U^2 + 2\ln U = 0 \quad \text{if} \quad \gamma = 1.$$

10.8. Solution:

$$u = t + U(z), \quad \rho = U^{-1}\exp\left(at - a\int \frac{dz}{U}\right), \quad p = U^{-\gamma}\exp\left(at - a\int \frac{dz}{U}\right), \quad z = x - \frac{1}{2}t^2,$$

where the function $U = U(z)$ is defined implicitly by

$$z = \int \frac{U^{\gamma+1} - \gamma}{a - U^{\gamma}}\,dU + C,$$

and a and C are arbitrary constants.

10.9. Traveling-wave solution with $\gamma = -1$:

$$u = U(z) + C, \quad \rho = \frac{k}{U(z)}, \quad p = -kU(z), \quad z = x - Ct,$$

where C and k are arbitrary constants and $U(z)$ is an arbitrary function.

10.10. Solutions with $\gamma = -1$:

$$u = \lambda x, \quad \rho = \pm\frac{k}{\sqrt{Ce^{2\lambda t} - k\lambda^2 x^2}}, \quad p = \pm\sqrt{Ce^{2\lambda t} - k\lambda^2 x^2},$$

where C, k, and λ are arbitrary constants.

10.11. Solution with $\gamma = -1$:

$$u = \frac{x}{1-a}, \quad \rho = x^{a-2}e^t, \quad p = -\frac{x^a}{a(a-1)^2}e^t,$$

where a is an arbitrary constant.

10.12. Solutions with $\gamma = -1$:

$$u = t \pm \sqrt{a(t^2 - 2x)}, \quad \rho = \pm\frac{1}{\sqrt{a(t^2-2x)}}, \quad p = \pm\frac{1-a}{a}\sqrt{a(t^2-2x)},$$

where a is an arbitrary constant.

10.13. Solution with $\gamma = -1$:

$$u = x, \quad \rho = x^{-1}\varphi(z), \quad p = -e^t \int \varphi(z)\,dz, \quad z = xe^{-t},$$

where $\varphi(z)$ is an arbitrary function.

10.14. Solution with $\gamma = 1$:

$$u = \frac{x}{t} + a, \quad \rho = ct^m \exp\Big(-\frac{ax}{bt}\Big), \quad p = bct^m \exp\Big(-\frac{ax}{bt}\Big), \quad m = \frac{a^2}{b} - 1,$$

where a, b, and c are arbitrary constants.

10.15. Solution with $\gamma = 1$:

$$u = \frac{x}{t} + 1, \quad \rho = \exp\left(-\frac{x}{t}\right)\varphi'_z(z), \quad p = \frac{1}{t}\varphi(z), \quad z = t\exp\left(-\frac{x}{t}\right),$$

where $\varphi(z)$ is an arbitrary function.

10.16. Solution with $\gamma = 1$:

$$u = \frac{x}{t}, \quad \rho = \frac{1}{t}\varphi\left(\frac{x}{t}\right), \quad p = \frac{1}{t},$$

where $\varphi(z)$ is an arbitrary function.

10.17. Solution with $\gamma = 1$:

$$u = C_1 t + C_2 + \theta(z), \quad \rho = \frac{C_3}{\theta(z)}, \quad p = p(\rho), \quad z = x - \frac{1}{2}C_1 t^2 - C_2 t,$$

where $p = p(\rho)$ is an arbitrary function and the function $\theta = \theta(z)$ is defined implicitly by

$$C_1 z + \frac{1}{2}\theta^2 - \int f\left(\frac{C_3}{\theta}\right)\frac{d\theta}{\theta} = C_4, \qquad f(\rho) = p'(\rho),$$

and C_1, C_2, C_3, and C_4 are arbitrary constants.

10.18. Solutions in implicit form with $\gamma = 1$:

$$u = \int \sqrt{p'(\rho)}\,\frac{d\rho}{\rho} + A_1, \qquad x - t\left[\int \sqrt{p'(\rho)}\,\frac{d\rho}{\rho} + \sqrt{p'(\rho)} + A_1\right] = \Phi_1(\rho), \quad p = p(\rho);$$

$$u = -\int \sqrt{p'(\rho)}\,\frac{d\rho}{\rho} - A_2, \quad x + t\left[\int \sqrt{p'(\rho)}\,\frac{d\rho}{\rho} + \sqrt{p'(\rho)} + A_2\right] = \Phi_2(\rho), \quad p = p(\rho);$$

where $p = p(\rho)$, $\Phi_1(\rho)$, and $\Phi_2(\rho)$ are arbitrary functions, and A_m are arbitrary constants ($m = 1, 2$). In the special cases of $\Phi_m(\rho) \equiv 0$, these formulas determine self-similar solutions.

10.19. Solution with $\gamma = 1$:

$$u = \varphi(t)x, \quad \rho = \frac{C}{x}, \quad p = Ax + B,$$

where A, B, and C are arbitrary constants, and the function $\varphi(t)$ is

$$\varphi(x) = \begin{cases} -k\tan(kt + D) & \text{if } A/C = k^2 > 0, \\ k\tanh(kt + D) & \text{if } A/C = -k^2 < 0, \\ 1/(t + D) & \text{if } A = 0, \end{cases}$$

with D being an arbitrary constant.

10.20. Solution with $\gamma = 2$:

$$u = \frac{x}{t} - \frac{\ln t}{t}, \quad \rho = \frac{1}{t}\exp\left(\frac{x}{t} - \frac{\ln t + 1}{t}\right), \quad p = \frac{1}{t^2}\exp\left(\frac{x}{t} - \frac{\ln t + 1}{t}\right).$$

10.21. Solution with $\gamma = 3$:

$$u = \frac{1}{2}tx\omega, \quad \rho = x^{a-2}\omega^{(a-1)/2}, \quad p = x^a\omega^{(a+3)/2}, \quad \omega = \frac{2}{t^2 - 4a},$$

where a is an arbitrary constant.

⊙ *References for Subsection* 20.2.2, Item 10: L. V. Ovsiannikov (1962, 1981), N. H. Ibragimov (1994).

20.2.3. Two-Dimensional Case

Two-dimensional motions of an adiabatic gas are described by the equations

$$\begin{aligned}
\rho(u_t + uu_x + vu_y) + p_x &= 0,\\
\rho(v_t + uv_x + vv_y) + p_y &= 0,\\
\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) &= 0,\\
p_t + up_x + vp_y + F(p,\rho)(u_x + v_y) &= 0.
\end{aligned}$$

1. Similarity formulas.

Suppose $u = u(x, y, t)$, $v = v(x, y, t)$, $\rho = \rho(x, y, t)$, and $p = p(x, y, t)$ is a solution to the above system of equations. Then the two sets of functions

$$\begin{aligned}
u_1 &= u(Ax + B_1, Ay + B_2, At + B_3), & v_1 &= v(Ax + B_1, Ay + B_2, At + B_3),\\
\rho_1 &= \rho(Ax + B_1, Ay + B_2, At + B_3), & p_1 &= p(Ax + B_1, Ay + B_2, At + B_3);\\
u_2 &= u(x - C_1t, y - C_2t) + C_1, & \rho_2 &= \rho(x - C_1t, y - C_2t) + C_2,\\
\rho_2 &= \rho(x - C_1t, y - C_2t), & p_2 &= p(x - C_1t, y - C_2t),
\end{aligned}$$

where A, B_1, B_2, B_3, C_1, and C_2 are arbitrary constants, are also solutions of the system.

2. Let $F(p, \rho)$ be an arbitrary function.

2.1. Solution:

$$u = f(y - C_1t), \quad v = C_1, \quad p = C_2, \quad \rho = C_3,$$

where C_1, C_2, and C_3 are arbitrary constants and $f = f(z)$ is an arbitrary function.

⊙ *Reference*: N. H. Ibragimov (1967) (for a case $C_1 = 0$).

2.2. Solution:

$$p = C_1, \quad \rho = C_2,$$

with u and v defined from the equations

$$\begin{aligned}
xu + yv - C_3t &= F(v),\\
u^2 + v^2 &= C_3,
\end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants and $F(v)$ is an arbitrary function.

2.3. Solution:

$$u = \frac{tx - y - a}{t^2} + \frac{b}{t}, \quad v = \frac{y + a}{t}, \quad \rho = \frac{c}{t^2}, \quad p = p(t),$$

where a, b, and c are arbitrary constants and $p = p(t)$ satisfies the equation

$$\frac{dp}{dt} + \frac{2}{t} A\left(p, \frac{c}{t^2}\right) = 0.$$

⊙ *Reference*: V. M. Men'shikov (1969).

2.4. Solution:

$$u = \frac{x + C_1}{t}, \quad v = \frac{y + C_2}{t}, \quad \rho = \frac{C_3}{t^2}, \quad p = p(t),$$

where C_1, C_2, and C_3 are arbitrary constants and the function $p = p(t)$ is determined by the ordinary differential equation

$$p'_t + \frac{2}{t} F\left(p, \frac{C_3}{t^2}\right) = 0.$$

2.5. Generalized separable solution (generalizes the solutions from the two preceding items):

$$u = a_1(t)x + b_1(t)y + c_1(t), \quad v = a_2(t)x + b_2(t)y + c_2(t), \quad \rho = \rho(t), \quad p = p(t),$$

where the functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\rho = \rho(t)$, and $p = p(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
a'_1 + a_1^2 + a_2 b_1 &= 0,\\
b'_1 + a_1 b_1 + b_1 b_2 &= 0,\\
c'_1 + a_1 c_1 + b_1 c_2 &= 0,\\
a'_2 + a_1 a_2 + a_2 b_2 &= 0,\\
b'_2 + a_2 b_1 + b_2^2 &= 0,\\
c'_2 + a_2 c_1 + b_2 c_2 &= 0,\\
\rho' + (a_1 + b_2)\rho &= 0,\\
p' + (a_1 + b_2)F(p, \rho) &= 0.
\end{aligned}$$

It follows from the second and fourth equations that $b_1 = C_1 a_2$ (if $b_2 \neq -a_1$), where C_1 is an arbitrary constant; the last two equations reduce to a single ordinary differential equation for $p = p(\rho)$:

$$\rho p'_\rho = F(p, \rho).$$

2.6. Solution:

$$u = \frac{1}{t}[x + \varphi(z)], \quad v = \frac{C_1}{t}[x + \varphi(z)], \quad \rho = \frac{C_2}{t}, \quad p = p(t), \quad z = y - C_1 x,$$

where $\varphi(z)$ is an arbitrary function and the function $p = p(t)$ is determined by the ordinary differential equation

$$p'_t = -\frac{1}{t} F\left(p, \frac{C_2}{t}\right).$$

2.7. Solution:

$$u = A_1 t + A_2, \quad v = B_1 t + B_2, \quad \rho = -p'_z(z), \quad p = p(z),$$
$$z = A_1 x + B_1 y - \tfrac{1}{2}(A_1^2 + B_1^2)t^2 - (A_1 A_2 + B_1 B_2)t,$$

where $p = p(z)$ is an arbitrary function and A_1, A_2, B_1, and B_2 are arbitrary constants.

2.8. "Two-dimensional" solution (generalizes the solution from Item 2.7):

$$u = A_1 t + A_2 + \varphi(\xi,\eta), \quad v = B_1 t + B_2 + \psi(\xi,\eta), \quad \rho = \rho(\xi,\eta), \quad p = p(\xi,\eta),$$
$$\xi = x - \tfrac{1}{2}A_1 t^2 - A_2 t, \quad \eta = y - \tfrac{1}{2}B_1 t^2 - B_2 t,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants, and the functions $\varphi = \varphi(\xi,\eta)$, $\psi = \psi(\xi,\eta)$, $\rho = \rho(\xi,\eta)$, and $p = p(\xi,\eta)$ are determined by the system of differential equations

$$\begin{aligned}
\rho(A_1 + \varphi\varphi_\xi + \psi\varphi_\eta) + p_\xi &= 0,\\
\rho(B_1 + \varphi\psi_\xi + \psi\psi_\eta) + p_\eta &= 0,\\
\varphi\rho_\xi + \psi\rho_\eta + \rho(\varphi_\xi + \psi_\eta) &= 0,\\
\varphi p_\xi + \psi p_\eta + F(p,\rho)(\varphi_\xi + \psi_\eta) &= 0.
\end{aligned}$$

2.9+. Solution in implicit form:

$$y - x f'_u(u) + t[u f'_u(u) - f(u)] = g(u), \quad v = f(u), \quad p = C,$$

where $f(u)$ and $g(u)$ are arbitrary functions, C is an arbitrary constant, and the function $\rho = \rho(x,t)$ is determined by the linear first-order partial differential equation

$$\rho_t + u\rho_x + f(u)\rho_y = 0.$$

Special case. For $f(u) = au$, we have the solution

$$u = u(z), \quad v = au(z), \quad \rho = \Phi(z, x - u(z)t), \quad p = C, \quad z = y - ax,$$

where $u(z)$ and $\Phi(z,\xi)$ are arbitrary functions, and a and C are arbitrary constants.

2.10+. "Two-dimensional" generalized separable solution:

$$u = x\varphi(y,t) + \psi(y,t), \quad v = v(y,t), \quad \rho = \rho(y,t), \quad p = p(y,t),$$

where the functions $\varphi = \varphi(y,t)$, $\psi = \psi(y,t)$, $v = v(y,t)$, $\rho = \rho(y,t)$, and $p = p(y,t)$ are determined by the system of differential equations

$$\begin{aligned}
\varphi_t + \varphi^2 + v\varphi_y &= 0,\\
\psi_t + \varphi\psi + v\psi_y &= 0,\\
\rho(v_t + vv_y) + p_y &= 0,\\
\rho_t + v\rho_y + \rho(\varphi + v_y) &= 0,\\
p_t + vp_y + F(p,\rho)(\varphi + v_y) &= 0.
\end{aligned}$$

3. Let $F(p,\rho) = pf(\rho) + g(\rho)$, where $f(\rho)$ and $g(\rho)$ are arbitrary functions.

Generalized separable solution:

$$u = a_1(t)x + b_1(t)y + c_1(t), \quad v = a_2(t)x + b_2(t)y + c_2(t),$$
$$\rho = \rho(t), \quad p = \varphi_1(t)x^2 + \varphi_2(t)xy + \varphi_3(t)y^2 + \psi_1(t)x + \psi_2(t)y + \psi_3(t),$$

where the functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\rho = \rho(t)$, $\varphi_n = \varphi_n(t)$, and $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\rho(a_1' + a_1^2 + a_2 b_1) + 2\varphi_1 &= 0,\\
\rho(b_1' + a_1 b_1 + b_1 b_2) + \varphi_2 &= 0,\\
\rho(c_1' + a_1 c_1 + b_1 c_2) + \psi_1 &= 0,\\
\rho(a_2' + a_1 a_2 + a_2 b_2) + \varphi_2 &= 0,\\
\rho(b_2' + a_2 b_1 + b_2^2) + 2\varphi_3 &= 0,\\
\rho(c_2' + a_2 c_1 + b_2 c_2) + \psi_2 &= 0,\\
\rho' + (a_1 + b_2)\rho &= 0,\\
\varphi_1' + 2a_1\varphi_1 + a_2\varphi_2 + (a_1 + b_2)f(\rho)\varphi_1 &= 0,\\
\varphi_2' + (a_1 + b_2)\varphi_2 + 2b_1\varphi_1 + 2a_2\varphi_3 + (a_1 + b_2)f(\rho)\varphi_2 &= 0,\\
\varphi_3' + b_1\varphi_2 + 2b_2\varphi_3 + (a_1 + b_2)f(\rho)\varphi_3 &= 0,\\
\psi_1' + a_1\psi_1 + a_2\psi_2 + 2c_1\varphi_1 + c_2\varphi_2 + (a_1 + b_2)f(\rho)\psi_1 &= 0,\\
\psi_2' + b_1\psi_1 + b_2\psi_2 + c_1\varphi_2 + 2c_2\varphi_3 + (a_1 + b_2)f(\rho)\psi_2 &= 0,\\
\psi_3' + c_1\psi_1 + c_2\psi_2 + (a_1 + b_2)[f(\rho)\psi_3 + g(\rho)] &= 0.
\end{aligned}$$

4. Let $F(p, \rho) = f(p)/\rho$.
"Two-dimensional" generalized separable solution (special case of solution 2.10):

$$u = x\varphi(y, t) + \psi(y, t), \quad v = v(y, t), \quad \rho = \rho(y, t), \quad p = p(t).$$

Here the functions $\varphi = \varphi(y, t)$, $\psi = \psi(y, t)$, $v = v(y, t)$, $\rho = \rho(y, t)$, and $p = p(t)$ are defined implicitly by the formulas

$$y - vt = A(v), \quad \varphi = \frac{1}{t - A_v'(v) + C_1}, \quad \psi = \frac{B(v)}{t - A_v'(v) + C_1},$$
$$\rho = \frac{C_2}{[t + A_v'(v)][t - A_v'(v) + C_1]}, \quad \int \frac{dp}{f(p)} + \frac{1}{C_2}t^2 + \frac{C_1}{C_2}t + C_3 = 0,$$

where $A(v)$ and $B(v)$ is an arbitrary function and C_1, C_2, and C_3 are arbitrary constants.

5. Let $F(p, \rho) = 2p$. The two solutions given below are written in polar coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x); \quad u_r = v\sin\varphi + u\cos\varphi, \quad u_\varphi = v\cos\varphi - u\sin\varphi.$$

5.1. Solution:

$$u_r = \frac{rt}{1 + t^2}, \quad u_\varphi = 0, \quad \rho = \frac{1}{2(1 + t^2)}\left(C - \frac{r^2}{1 + t^2}\right), \quad p = \frac{1}{2}\rho^2,$$

where C is an arbitrary constant.

5.2. Solution:

$$u_r = \frac{rt}{1 + t^2}, \quad u_\varphi = \frac{a}{\sqrt{1 + t^2}}, \quad \rho = \frac{1}{1 + t^2}\left(C - \frac{r^2}{2(1 + t^2)} + a^2 \ln\frac{r}{\sqrt{1 + t^2}}\right), \quad p = \frac{1}{2}\rho^2,$$

where a and C are arbitrary constants.

⊙ *References for Subsection* 20.2.3, Item 5: N. H. Ibragimov (1966, 1967).

20.2.4. Three-Dimensional Case

Three-dimensional motion of an adiabatic gas is described by the equations

$$\begin{aligned}
\rho(u_t + uu_x + vu_y + wu_z) + p_x &= 0,\\
\rho(v_t + uv_x + vv_y + wv_z) + p_y &= 0,\\
\rho(w_t + uw_x + vw_y + ww_z) + p_z &= 0,\\
\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) &= 0,\\
p_t + up_x + vp_y + w\rho_z + F(p,\rho)(u_x + v_y + w_z) &= 0.
\end{aligned}$$

1. Let $F(p, \rho)$ be an arbitrary function.

1.1. Solution:

$$u = f(y - C_1 t, z - C_2 t), \quad v = C_1, \quad w = C_2, \quad p = C_3, \quad \rho = C_4,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants and $f = f(\xi, \eta)$ is an arbitrary function.

1.2. Solution:

$$\begin{gathered}
u = A_1 t + A_2, \quad v = B_1 t + B_2, \quad w = C_1 t + C_2, \quad \rho = -p_z'(z), \quad p = p(z),\\
z = A_1 x + B_1 y + C_1 z - \tfrac{1}{2}(A_1^2 + B_1^2 + C_1^2)t^2 - (A_1 A_2 + B_1 B_2 + C_1 C_2)t,
\end{gathered}$$

where $p = p(z)$ is an arbitrary function and A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants.

1.3. Solution:

$$u = \frac{x + C_1}{t}, \quad v = \frac{y + C_2}{t}, \quad w = \frac{z + C_3}{t}, \quad \rho = \frac{C_4}{t^3}, \quad p = p(t),$$

where C_1, C_2, C_3, and C_4 are arbitrary constants and the function $p = p(t)$ is determined by the ordinary differential equation

$$p_t' + \frac{3}{t} F\left(p, \frac{C_3}{t^3}\right) = 0.$$

1.4. Generalized separable solution (generalizes the solution from Item 1.3):

$$\begin{aligned}
u &= a_1(t)x + b_1(t)y + c_1(t)z + d_1(t),\\
v &= a_2(t)x + b_2(t)y + c_2(t)z + d_2(t),\\
w &= a_3(t)x + b_3(t)y + c_3(t)z + d_3(t),\\
\rho &= \rho(t), \quad p = p(t),
\end{aligned}$$

where the functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\rho = \rho(t)$, and $p = p(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
a_1' + a_1^2 + a_2 b_1 + a_3 c_1 &= 0,\\
b_1' + a_1 b_1 + b_1 b_2 + b_3 c_1 &= 0,\\
c_1' + a_1 c_1 + b_1 c_2 + c_1 c_3 &= 0,\\
d_1' + a_1 d_1 + b_1 d_2 + c_1 d_3 &= 0,\\
a_2' + a_1 a_2 + a_2 b_2 + a_3 c_2 &= 0,\\
b_2' + a_2 b_1 + b_2^2 + b_3 c_2 &= 0,\\
c_2' + a_2 c_1 + b_2 c_2 + c_2 c_3 &= 0,\\
d_2' + a_2 d_1 + b_2 d_2 + c_2 d_3 &= 0,
\end{aligned}
\qquad
\begin{aligned}
a_3' + a_1 a_3 + a_2 b_3 + a_3 c_3 &= 0,\\
b_3' + a_3 b_1 + b_2 b_3 + b_3 c_3 &= 0,\\
c_3' + a_3 c_1 + b_3 c_2 + c_3^2 &= 0,\\
d_3' + a_3 d_1 + b_3 d_2 + c_3 d_3 &= 0,\\
\rho' + (a_1 + b_2 + c_3)\rho &= 0,\\
p' + (a_1 + b_2 + c_3)F(p,\rho) &= 0.
\end{aligned}$$

1.5. There is a "three-dimensional" solution of the form (generalizes the solution from Item 1.2):

$$u = A_1 t + A_2 + \varphi(\xi,\eta,\zeta), \quad v = B_1 t + B_2 + \psi(\xi,\eta,\zeta), \quad w = C_1 t + C_2 + \chi(\xi,\eta,\zeta),$$
$$\rho = \rho(\xi,\eta,\zeta), \quad p = p(\xi,\eta,\zeta),$$
$$\xi = x - \tfrac{1}{2}A_1 t^2 - A_2 t, \quad \eta = y - \tfrac{1}{2}B_1 t^2 - B_2 t, \quad \zeta = z - \tfrac{1}{2}C_1 t^2 - C_2 t,$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants, and the functions $\varphi = \varphi(\xi,\eta,\zeta)$, $\psi = \psi(\xi,\eta,\zeta)$, $\chi = \chi(\xi,\eta,\zeta)$, $\rho = \rho(\xi,\eta,\zeta)$, and $p = p(\xi,\eta,\zeta)$ are determined by the system of differential equations

$$\begin{aligned}
\rho(A_1 + \varphi\varphi_\xi + \psi\varphi_\eta + \chi\varphi_\zeta) + p_\xi &= 0,\\
\rho(B_1 + \varphi\psi_\xi + \psi\psi_\eta + \chi\psi_\zeta) + p_\eta &= 0,\\
\rho(C_1 + \varphi\chi_\xi + \psi\chi_\eta + \chi\chi_\zeta) + p_\zeta &= 0,\\
\varphi\rho_\xi + \psi\rho_\eta + \chi\rho_\zeta + \rho(\varphi_\xi + \psi_\eta + \chi_\zeta) &= 0,\\
\varphi p_\xi + \psi p_\eta + \chi p_\zeta + F(p,\rho)(\varphi_\xi + \psi_\eta + \chi_\zeta) &= 0.
\end{aligned}$$

1.6. "Three-dimensional" generalized separable solution:

$$u = x\varphi(y,z,t) + \psi(y,z,t), \quad v = v(y,z,t), \quad w = w(y,z,t), \quad \rho = \rho(y,z,t), \quad p = p(y,z,t),$$

where the functions $\varphi = \varphi(y,z,t)$, $\psi = \psi(y,z,t)$, $v = v(y,z,t)$, $w = w(y,z,t)$, $\rho = \rho(y,z,t)$, and $p = p(y,z,t)$ are determined by the system of differential equations

$$\begin{aligned}
\varphi_t + \varphi^2 + v\varphi_y + w\varphi_z &= 0,\\
\psi_t + \varphi\psi + v\psi_y + w\psi_z &= 0,\\
\rho(v_t + vv_y + wv_z) + p_y &= 0,\\
\rho(w_t + vw_y + ww_z) + p_z &= 0,\\
\rho_t + v\rho_y + w\rho_z + \rho(\varphi + v_y + w_z) &= 0,\\
p_t + vp_y + wp_z + F(p,\rho)(\varphi + v_y + w_z) &= 0.
\end{aligned}$$

2. Let $F(p,\rho) = pf(\rho) + g(\rho)$, where $f(\rho)$ and $g(\rho)$ are arbitrary functions.

There is a generalized separable solution of the form

$$\begin{aligned}
&u = a_1(t)x + b_1(t)y + c_1(t)z + d_1(t), \quad v = a_2(t)x + b_2(t)y + c_2(t)z + d_2(t),\\
&w = a_3(t)x + b_3(t)y + c_3(t)z + d_3(t), \quad \rho = \rho(t),\\
&p = \varphi_{11}(t)x^2 + \varphi_{22}(t)y^2 + \varphi_{33}(t)z^2 + \varphi_{12}(t)xy + \varphi_{13}(t)xz + \varphi_{23}(t)yz\\
&\quad + \psi_1(t)x + \psi_2(t)y + \psi_3(t)z + \psi_4(t),
\end{aligned}$$

where the functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\rho = \rho(t)$, $\varphi_{nm} = \varphi_{nm}(t)$, and $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations (which is not written out here).

3. Let $F(p,\rho) = \gamma p$, where $\gamma = \text{const}$.

3.1. Solution:

$$\mathbf{u} = \frac{dM}{dt}M^{-1}\mathbf{x}, \quad \rho = \frac{1}{|M|}F(M^{-1}\mathbf{x}), \quad p = \frac{1}{|M|^\gamma}G(M^{-1}\mathbf{x}),$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u, v, w)$; the functions $F = F(\boldsymbol{\xi})$ and $G = G(\boldsymbol{\xi})$ and the fixed square third-order matrix L satisfy the equality

$$\nabla_{\boldsymbol{\xi}} G = FL\boldsymbol{\xi}, \qquad \boldsymbol{\xi} \in \mathbb{R}^3;$$

and the nonsingular square third-order matrix $M = M(t)$ is the solution of the equation

$$M^* \frac{d^2 M}{dt^2} + |M|^{1-\gamma} L = 0$$

where M^* is the transposed matrix M, and $|M|$ is the modulus of the determinant of M.

3.2. Solutions (a)–(c) are written in the cylindrical coordinate system (x, r, φ):

$$r = \sqrt{y^2 + z^2}, \quad \varphi = \arctan(y/z); \qquad v_r = v \sin\varphi + w \cos\varphi, \quad v_\varphi = v \cos\varphi - w \sin\varphi.$$

(a) Solution:

$$u = t, \quad v_r = 0, \quad v_\varphi = \sqrt{x - \frac{1}{2}t^2}, \quad \rho = \frac{2}{r} f'\left(\frac{t^2 - 2x}{r}\right), \quad p = f\left(\frac{t^2 - 2x}{r}\right),$$

where $f(z)$ is an arbitrary function.

(b) Solution:

$$u = \pm e^{\alpha\varphi \mp x}\left(\frac{\alpha^2}{r}\right)^{1/2}\left(\frac{P'}{R}\right)^{1/2}, \quad v_r = 0, \quad v_\varphi = \pm e^{\alpha\varphi \mp x}\left(\frac{rP'}{R}\right)^{1/2},$$
$$\rho = e^{-2(\alpha\varphi \mp x)} R(r), \quad p = P(r),$$

where $R(r)$ and $P(r)$ are arbitrary functions.

(c) Solution:

$$u = \frac{C_1 r}{t(t + C_2)} + \frac{x}{t}, \quad v_r = \frac{r}{t + C_2}, \quad v_\varphi = 0, \quad \rho = \frac{C_3}{tr^2}, \quad p = C_4 t^{-\gamma}(t + C_2)^{-2\gamma},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

(d) Solution:

$$u = \frac{1}{t}(x + C_1), \quad v = \frac{1}{t}(y + C_2), \quad w = \frac{1}{t}(z + C_3), \quad \rho = C_4 t^{-3}, \quad p = C_5 t^{-3\gamma},$$

where $C_1, \ldots, C_5$ are arbitrary constants.

(e) Solution:

$$u = \frac{1}{t}(x + C_1), \quad v = \frac{y}{t} + \frac{\ln z}{At} - \frac{1}{t}\ln t, \quad w = A\frac{z}{t}, \quad \rho = C_2 t^2 z^{\frac{\gamma+4}{\gamma-2}}, \quad p = B z^{\frac{3\gamma}{\gamma-2}},$$

where C_1 and C_2 are arbitrary constants and $A = \dfrac{2(2-\gamma)}{\gamma+1}$ and $B = \dfrac{2C_2(1-\gamma)(2-\gamma)^2}{\gamma(1+\gamma)^2}$.

3.3. For $\gamma = \frac{5}{3}$, the system has (in addition to those in Item 3.2) the following solutions [solutions (a)–(d) and (g) are written in cylindrical coordinates (x, r, φ)]:

(a) Solution:

$$u = C_1\frac{r}{t^2} + \frac{x}{t}, \quad v_r = \frac{r}{t}, \quad v_\varphi = 0, \quad \rho = \frac{C_2}{tr^2}, \quad p = C_3 t^{-5},$$

where C_1, C_2, and C_3 are arbitrary constants.

(b) Solution:

$$u = \frac{1}{t}U\left(\frac{r}{t}\right) + \frac{x}{t}, \quad v_r = \frac{r}{t}, \quad v_\varphi = \left(\frac{rP'}{tR}\right)^{1/2}\frac{r}{t^2}, \quad \rho = \frac{1}{tr^2}R\left(\frac{r}{t}\right), \quad p = t^{-5}P\left(\frac{r}{t}\right),$$

where $U(\xi)$, $R(\xi)$, and $P(\xi)$ are arbitrary functions.

(c) Solution:

$$u = \frac{1}{t}U\left(\frac{t}{r}\right) + \frac{x}{t} - \frac{\varphi}{t}, \quad v_r = \frac{r}{t}, \quad v_\varphi = 0, \quad \rho = t^{-3}R\left(\frac{t}{r}\right), \quad p = Ct^{-5},$$

where $U(\xi)$ and $R(\xi)$ are arbitrary functions and C is an arbitrary constant.

(d) Solution:

$$u = \frac{1}{t}\left[U\left(\frac{t}{r}\right) + x + \frac{1}{t} - \varphi\right], \quad v_r = \frac{r}{t}, \quad v_\varphi = -\frac{r}{t^2}, \quad \rho = -\frac{1}{r^3}P'\left(\frac{t}{r}\right), \quad p = t^{-5}P\left(\frac{t}{r}\right),$$

where $U(\xi)$ and $P(\xi)$ are arbitrary functions.

(e) Solution:

$$u = C_1\frac{z}{t^2} + \frac{x}{t}, \quad v = \frac{y}{t}, \quad w = \frac{z}{t}, \quad \rho = C_2 t^{-3}, \quad p = C_3 t^{-5},$$

where C_1, C_2, and C_3 are arbitrary constants.

(f) Solution:

$$u = \frac{U(\xi) + x}{t}, \quad v = \frac{V(\xi) + y}{t}, \quad w = \frac{z}{t} + \frac{1}{t^2}, \quad \rho = \frac{P'(\xi)}{t^3}, \quad p = \frac{P(\xi)}{t^5}, \quad \xi = \frac{1}{t}\left(z + \frac{1}{2t}\right),$$

where $U(\xi)$, $V(\xi)$, and $P(\xi)$ are arbitrary functions.

(g) Solution:

$$u = \frac{xt}{1+t^2}, \quad v_r = \frac{rt}{1+t^2}, \quad v_\varphi = \pm\frac{(r^2+x^2)^{1/2}}{1+t^2},$$
$$\rho = -\frac{1}{xr(1+t^2)^{1/2}}P'\left(\frac{x}{r}\right), \quad p = \frac{1}{(1+t^2)^{5/2}}P\left(\frac{x}{r}\right),$$

where $P = P(\xi)$ is an arbitrary function.

⊙ *References for Subsection* 20.2.4, Items 3.1–3.3: L. V. Ovsiannikov (1956), B. V. Lapko (1978), N. H. Ibragimov (1994).

20.3. Systems Describing Fluid Flows in the Atmosphere, Seas, and Oceans

20.3.1. Equations of Breezes and Monsoons

The following equations describe flows in breezes and monsoons [see Gutman (1947)]:

$$\begin{aligned} u_t &= u_{yy} - p_x, \\ T_t &= T_{yy} - uT_x, \\ p_y &= T, \end{aligned}$$

where t is the time, x and y are horizontal and vertical coordinates, u is the horizontal velocity, p is the pressure, and T is the temperature.

Remark. Eliminating the pressure yields the system of two equations

$$u_{yt} = u_{yyy} - T_x, \quad T_t = T_{yy} - uT_x.$$

1°. Multiplicative separable solution:

$$u = x\varphi(y), \qquad p = x^2\psi(y), \qquad T = x^2\theta(y),$$

where functions $\varphi = \varphi(y)$, $\psi = \psi(y)$, and $\theta = \theta(y)$ are defined from the system of ordinary differential equations

$$\varphi''_{yy} = 2\psi, \quad \theta''_{yy} = 2\varphi\theta, \quad \psi'_y = \theta.$$

From this system, we have the equation $\varphi^{(5)}_y = 2\varphi\varphi^{(3)}_y$.

⊙ *Reference*: Katkov (1964).

2°. Traveling-wave solution:

$$u = u(z), \quad p = p(z), \quad T = T(z), \quad z = x - \frac{y}{C},$$

where functions $u = u(z)$, $p = p(z)$, and $T = T(z)$ are defined from the system of ordinary differential equations

$$u''_{zz} = C^3T, \quad T''_{zz} = C^2uT'_z, \quad p'_z = -CT. \tag{1}$$

From this system, we have the equation $u^{(4)}_z = Cuu^{(3)}_z$.

⊙ *Reference*: Katkov (1964).

3°. Other simple solutions:

$$\begin{aligned} &u = f(t-x), \quad p = f(t-x) + C, \quad T = 0; \\ &u = f(x), \quad p = C, \quad T = 0; \\ &u = C, \quad p = f(t), \quad T = 0, \end{aligned}$$

where $f(z)$ is an arbitrary function and C is an arbitrary constant.

⊙ *Reference*: Katkov (1964).

4°. Solution (generalizes the solution from Item 1°):

$$u = x\varphi(z), \quad p = x^2\psi(z), \quad T = x^2\theta(z), \quad z = y + C\ln|x|,$$

where C is an arbitrary constant and the functions $\varphi = \varphi(z)$, $\psi = \psi(z)$, and $\theta = \theta(z)$ are defined from the system of ordinary differential equations

$$\varphi''_{zz} = C\psi'_z + 2\psi, \quad \theta''_{zz} = \varphi(C\theta'_z + 2\theta), \quad \psi'_z = \theta.$$

5°. Solution:

$$u = \frac{\partial f}{\partial x}, \quad p = -\frac{\partial f}{\partial t}, \quad T = 0,$$

where $f = f(x,t)$ is an arbitrary function.

6°. Solution:

$$u = w(y,t) - \int p(x,t)\,dt, \quad p = p(x,t), \quad T = 0,$$

where $p = p(x,t)$ is an arbitrary function and the function $w = w(y,t)$ is determined by the linear heat equation

$$w_t = w_{xx}.$$

7°. Solution:

$$u = u(x), \quad p = \int T(t,y)\,dy, \quad T = T(t,y),$$

where $u(x)$ is an arbitrary function and the function $T = T(t,y)$ is determined by the linear heat equation

$$T_t = T_{yy}.$$

8°. Solution:

$$u = u(t), \quad p = -u'_t(t)x + Cy + f(t), \quad T = C,$$

where $u(t)$ and $f(t)$ are arbitrary functions and C is an arbitrary constant.

9°. Solution:

$$u = u(t), \quad p = -u'_t(t)x + \int T(t,y)\,dy, \quad T = T(t,y),$$

where $u(t)$ is an arbitrary constant and the function $T = T(t,y)$ is determined by the linear heat equation

$$T_t = T_{yy}.$$

10°. Generalized separable solution:

$$\begin{aligned} u &= xa(y,t) + b(y,t),\\ p &= x^2c(y,t) + xd(y,t) + e(y,t),\\ T &= x^2f(y,t) + xg(y,t) + h(y,t), \end{aligned}$$

where the functions $a = a(y,t)$, $b = b(y,t)$, $c = c(y,t)$, $d = d(y,t)$, $e = e(y,t)$, $f = f(y,t)$, $g = g(y,t)$, and $h = h(y,t)$ are determined by the system of differential equations

$$\begin{aligned} a_t &= a_{yy} - 2c,\\ b_t &= b_{yy} - d,\\ f_t &= f_{yy} - 2af,\\ g_t &= g_{yy} - ag - 2bf,\\ h_t &= h_{yy} - bg,\\ c_y &= f,\\ d_y &= g,\\ e_y &= h. \end{aligned}$$

11°. Solution:

$$u = -\frac{\partial}{\partial x}\int \varphi(x,t)\,dt + w(x,y,t), \qquad p = \int T(y,t)\,dy + \varphi(x,t), \qquad T = T(y,t),$$

where $\varphi = \varphi(x,t)$ is an arbitrary function and the functions $w = w(x,y,t)$ and $\varphi = \varphi(x,t)$ are determined by the linear heat equations

$$w_t = w_{yy}, \qquad T_t = T_{yy}.$$

It is noteworthy that the first equation is independent of x, although $w = w(x,y,t)$. Hence, particular solutions of this equation will contain arbitrary functions of x.

12°. There is a "two-dimensional" solution of the form:

$$u = u(\xi,\eta), \quad p = p(\xi,\eta), \quad T = T(\xi,\eta), \quad \xi = a_1x + a_2y + a_3t, \quad \eta = b_1x + b_2y + b_3,$$

where a_n and b_n are arbitrary constants.

13°. There is a "two-dimensional" solution of the form:

$$u = t^{\frac{k}{2}}\varphi(\xi,\eta), \quad p = t^k\psi(\xi,\eta), \quad T = t^{\frac{2k-1}{2}}\theta(\xi,\eta), \quad \xi = yt^{-\frac{1}{2}}, \quad \eta = xt^{-\frac{k+2}{2}},$$

where k is an arbitrary constant.

14°. "Two-dimensional" solution (generalizes the solutions from Items 1° and 4°):

$$u = x\varphi(z,t), \qquad p = x^2\psi(z,t), \qquad T = x^2\theta(z,t), \qquad z = y + C\ln|x|,$$

where C is an arbitrary constant and the functions $\varphi = \varphi(z,t)$, $\psi = \psi(z,t)$, and $\theta = \theta(z,t)$ are defined from the system of partial differential equations

$$\varphi_t = \varphi_{zz} - C\psi_z - 2\psi, \qquad \theta_t = \theta_{zz} - \varphi(C\theta_z + 2\theta), \qquad \psi_z = \theta.$$

20.3.2. Equations of Atmospheric Circulation in the Equatorial Region

The following equations describe a nonstationary hydrodynamic model of atmospheric circulation in the equatorial region (Dobryshman (1964), see also Fal'kovich (1968)):

$$\begin{aligned} u_t + vu_y + wu_z &= yv - w,\\ v_t + vv_y + wv_z &= -yu - \Phi_y,\\ v_y + w_z &= 0, \end{aligned}$$

where u, v, and w are components of velocity in Cartesian coordinates (x,y,z), with the x-axis pointing west, y-axis pointing north, and z-axis pointing up; $\Phi = \Phi(t,y,z)$ is a geopotential.

1°. Let

$$\Phi_y = \left[y + a(y)e^{-kt}\right]z + b(y),$$

where

$$\begin{aligned} a(y) &= kB_1(1+y^2) - 2B_0B_1y\sqrt{1+y^2}\\ &\quad + 2kB_1y\sqrt{1+y^2}\ln\bigl(y+\sqrt{1+y^2}\,\bigr) - A_1y\sqrt{1+y^2},\\ b(y) &= kB_0\sqrt{1+y^2} - k^2\sqrt{1+y^2}\ln\bigl(y+\sqrt{1+y^2}\,\bigr) - B_0^2y\\ &\quad + 2kB_0y\ln\bigl(y+\sqrt{1+y^2}\,\bigr) - k^2y\ln^2\bigl(y+\sqrt{1+y^2}\,\bigr) - A_0y - \tfrac{1}{2}y^3, \end{aligned}$$

and A_0, A_1, B_0, B_1, and k are arbitrary constants.

Solution:

$$u = -z + \frac{y^2}{2}A_0 + A_1 z\sqrt{1+y^2}e^{-kt} - B_1^2 z^2(1+y^2)e^{-2kt},$$
$$v = \sqrt{1+y^2}\Big[B_0 - k\ln\big(y+\sqrt{1+y^2}\,\big)\Big] + B_1 z(1+y^2)e^{-kt},$$
$$w = z\left[k - \frac{y}{\sqrt{1+y^2}}\Big(B_0 - k\ln\big(y+\sqrt{1+y^2}\,\big)\Big)\right] - B_1 z^2 y e^{-kt}.$$

⊙ *References*: A. I. Fal'kovich (1968), N. H. Ibragimov (1995).

2°. Let

$$\Phi = -\tfrac{1}{2}u(t)y^2 - v_t'(t)y + \varphi(z,t),$$

where $u(t)$, $v(t)$, and $\varphi(z,t)$ are arbitrary functions.

Solution:

$$u = u(t), \quad v = v(t), \quad w = v(t)y - u_t'(t).$$

3°. Let

$$\Phi_y = -yu(y,t) - v_t'(t),$$

where $u(y,t)$ and $v(t)$ are arbitrary functions.

Solution (generalizes the solution from Item 2°):

$$u = u(y,t), \quad v = v(t), \quad w = v(t)y - \frac{\partial u}{\partial t} - v(t)\frac{\partial u}{\partial y}.$$

4°. Generalized separable solution:

$$u = a(y,t)z + b(y,t), \quad v = c(y,t)z + d(y,t), \quad w = f(y,t)z^2 + g(y,t)z + h(y,t),$$

where the functions $a = a(y,t)$, $b = b(y,t)$, $c = c(y,t)$, $d = d(y,t)$, $f = f(y,t)$, $g = g(y,t)$, and $h = h(y,t)$ are determined by the underdetermined system of differential equations (the system consists of five equations for seven unknowns)

$$\begin{aligned} ca_y + af &= -f, \\ a_t + cb_y + da_y + ag &= -cy - g, \\ b_t + db_y + ah &= -dy - h, \\ c_y + 2f &= 0, \\ d_y + g &= 0. \end{aligned}$$

The geopotential is determined by interacting

$$\Phi_y = -v_t - vv_y - wv_z - yu$$

with respect to y.

20.3.3. Equations of Dynamic Convection in the Sea

The equations

$$\begin{aligned} u_x + v_y + w_z &= 0,\\ \rho_t + u\rho_x + v\rho_y + w\rho_z &= 0,\\ \rho(u_t + uu_x + vu_y + wu_z + v) &= -p_x,\\ \rho(v_t + uv_x + vv_y + wv_z - u) &= -p_y,\\ \rho &= p_z \end{aligned}$$

describe dynamic convection in the sea; u, v, and w are components of velocity vector of relative motion of fluid in Cartesian coordinates (x, y, z), ρ is the density of fluid, and p is the pressure.

1°. Solution:

$$u = \frac{x-y}{2} - ny\sqrt{k - \frac{2z}{x^2+y^2}}, \quad v = \frac{x+y}{2} + nx\sqrt{k - \frac{2z}{x^2+y^2}}, \quad w = k(x^2+y^2) - z,$$
$$p = \varphi(t) - \frac{c}{n^2-1}\left\{2n^2[k(x^2+y^2)-2z] + 4z - x^2 - y^2\right\}(x^2+y^2)^{-n^2}e^{n^2t}, \quad \rho = c(x^2+y^2)^{-n^2}e^{n^2t}.$$

where $\varphi(t)$ is an arbitrary function, and k and n are arbitrary constants.

⊙ *References*: L. V. Ovsiannikov (1967) and N. H. Ibragimov (1995); these papers had errors in the formulas for p and ρ.

Remark. This solution is a special case of the more general class of solutions having the form

$$u = u(x, y, z), \quad v = v(x, y, z), \quad w = w(x, y, z), \quad \rho = e^{-\lambda t}R(x, y, z), \quad p = e^{-\lambda t}P(x, y, z);$$

note that here time-dependent density and pressure distributions correspond to a time-invariant velocity field.

2°. Solution:

$$u = a'(t), \quad v = 0, \quad w = a'''x - a''y + \tfrac{1}{2}(a'a'' - aa'''), \quad \rho = 1, \quad p = z - a''x + a'y + \tfrac{1}{2}aa'',$$

where $a = a(t)$ is an arbitrary function and the primes denote its derivatives.

⊙ *References*: L. V. Ovsiannikov (1967) and N. H. Ibragimov (1995).

3°. Solution:

$$\begin{aligned} u = u(t), \quad v = v(t), \quad w = w(x, y, t), \quad \rho = a,\\ p = az - a[u_t'(t) + v(t)]x - a[v_t'(t) - u(t)]y + \varphi(t), \end{aligned}$$

where $u(t)$, $v(t)$, $w(x, y, t)$, $\varphi(t)$ are arbitrary functions, and a is an arbitrary constant.

4°. Solution (generalizes the solution from Item 3°):

$$u = u(x, y, t), \quad v = v(x, y, t), \quad w = \xi(x, y, t)z + \eta(x, y, t), \quad \rho = a, \quad p = az + a\theta(x, y, t),$$

where $\eta(x, y, t)$ is an arbitrary function, a is an arbitrary constant, and the functions $u = u(x, y, t)$, $v = v(x, y, t)$, $\xi = \xi(x, y, t)$, and $\theta = \theta(x, y, t)$ are described by the underdetermined system of differential equations (the system consists of three equations for four unknowns):

$$\begin{aligned} u_x + v_y + \xi &= 0,\\ u_t + uu_x + vu_y + v &= -\theta_x,\\ v_t + uv_x + vv_y - u &= -\theta_y. \end{aligned} \tag{1}$$

4.1. System (1) has a generalized separable solution of the form

$$u = a_1(t)x + b_1(t)y + c_1(t), \quad v = a_2(t)x + b_2(t)y + c_2(t), \quad \xi = -a_1(t) - b_2(t),$$
$$\theta = \varphi_1(t)x^2 + \varphi_2(t)xy + \varphi_3(t)y^2 + \psi_1(t)x + \psi_2(t)y + \psi_3(t),$$

where $\psi_3(t)$ is an arbitrary function and other functions $a_k = a_k(t)$, $b_k = b_k(t)$, $c_k = c_k(t)$, $\varphi_n = \varphi_n(t)$, and $\psi_n = \psi_n(t)$ are described by the underdetermined system of ordinary differential equations:

$$\begin{aligned}
a_1' + a_1^2 + a_2b_1 + a_2 + 2\varphi_1 &= 0,\\
b_1' + a_1b_1 + b_1b_2 + b_2 + \varphi_2 &= 0,\\
c_1' + a_1c_1 + b_1c_2 + c_2 + \psi_1 &= 0,\\
a_2' + a_1a_2 + a_2b_2 - a_1 + \varphi_2 &= 0,\\
b_2' + a_2b_1 + b_2^2 - b_1 + 2\varphi_3 &= 0,\\
c_2' + a_2c_1 + b_2c_2 - c_1 + \psi_2 &= 0.
\end{aligned}$$

This is a system of six equations for eleven unknowns; five out of them, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$, $c_1 = c_1(t)$, and $c_2(t)$, can be set arbitrarily, provided that $a_2 \neq b_1 + 1$, and the other functions are found as

$$\begin{aligned}
b_2 &= \frac{b_1' - a_2' + a_1b_1 - a_1a_2 + a_1}{a_2 - b_1 - 1},\\
\varphi_1 &= -\tfrac{1}{2}(a_1' + a_1^2 + a_2b_1 + a_2),\\
\varphi_2 &= -(b_1' + a_1b_1 + b_1b_2 + b_2),\\
\varphi_3 &= -\tfrac{1}{2}(b_2' + a_2b_1 + b_2^2 - b_1),\\
\psi_1 &= -(c_1' + a_1c_1 + b_1c_2 + c_2),\\
\psi_2 &= -(c_2' + a_2c_1 + b_2c_2 - c_1).
\end{aligned}$$

4.2. System (1) has solutions of the form

$$v = \pm\sqrt{C^2 - u^2}, \quad \theta = \theta(t), \quad \xi = -u_x \pm \frac{u}{\sqrt{C^2 - u^2}}u_y, \tag{2}$$

where C is an arbitrary constant, $\theta(t)$ is an arbitrary function, and the function $u = u(x, y, t)$ is determined by the first-order partial differential equation

$$u_t + uu_x \pm \sqrt{C^2 - u^2}\,u_y \pm \sqrt{C^2 - u^2} = 0. \tag{3}$$

In formulas (2)–(3), one takes either the upper or the lower sign. The general solution of equation (3) is

$$\Phi\left(y + u,\ x \mp \sqrt{C^2 - u^2},\ t \pm \arcsin\frac{u}{C}\right) = 0,$$

where $\Phi(z_1, z_2, z_3)$ is an arbitrary function.

20.3.4. Equations of Flows in the Baroclinic Layer of the Ocean

Preliminary remarks. Steady flows in the baroclinic layer of the unlimited ocean by the wind activity and climatic factors are described by the system of equations (Lineikin, 1963):

$$\begin{aligned} u_x + v_y + w_z &= 0,\\ u\rho_x + v\rho_y + w\rho_z &= \nu_1(\rho_{xx} + \rho_{yy}) + \nu_2\rho_{zz},\\ p_x &= \rho_0(f_0 - \beta x)v,\\ p_y &= -\rho_0(f_0 - \beta x)u,\\ p_z &= g\rho, \end{aligned}$$

where $\nu_1, \nu_2, \rho_0, f_0, \beta,$ and g are constants. Here, the x-axis is pointed south, the y-axis is pointed east, and the z-axis is pointed vertically down, and u, v, and w are the respective components of the velocity.

After the changes of variables

$$\bar{x} = \frac{x\beta}{f_0}, \quad \bar{y} = \frac{y\beta}{f_0}, \quad \bar{z} = \frac{z\beta}{f_0}\sqrt{\frac{\nu_1}{\nu_2}}, \quad \bar{u} = \frac{uf_0}{\nu_1\beta}, \quad \bar{v} = \frac{vf_0}{\nu_1\beta}, \quad \bar{w} = \frac{wf_0}{\beta\sqrt{\nu_1\nu_2}},$$
$$\bar{p} = \frac{p}{\rho_0 f_0 \nu_1}, \quad \bar{g} = \frac{g}{\rho_0\beta\nu_1}\sqrt{\frac{\nu_1}{\nu_2}},$$

the equations in question are rewritten in the same form with $\nu_1 = \nu_2 = \rho_0 = f_0 = \beta = 1$.

Notation. In the above solutions, a, b, c, a_1, b_1, c_1, c_2, c_3, c_4, d_1, and d_2 are arbitrary constants, and

$$\varphi = \frac{\beta(ay + bz)}{f_0 - \beta x}, \qquad k = \sqrt{a^2 + b^2\frac{\nu_2}{\nu_1}}.$$

1°. Case $\nu_1 \neq 0$.

1.1. Solution:

$$\begin{aligned} u &= -\frac{ag}{b\rho_0(f_0 - \beta x)}\left[\frac{c}{k}\arctan\frac{\varphi}{k} + d_1\right],\\ v &= \frac{1}{f_0 - \beta x}\left[\frac{cg}{2b\rho_0}\ln\left(k^2 + \varphi^2\right) - d_2\nu_1\right],\\ w &= \frac{ag}{b^2\rho_0(f_0 - \beta x)}\left[\frac{c\varphi}{k}\arctan\frac{\varphi}{k} - \frac{c}{2}\ln\left(k^2 + \varphi^2\right) + d_1\varphi + \frac{d_2 b\rho_0\nu_1}{g}\right],\\ p &= \frac{g(f_0 - \beta x)}{b\beta}\left[\frac{c\varphi}{k}\arctan\frac{\varphi}{k} - \frac{c}{2}\ln\left(k^2 + \varphi^2\right) + d_1\varphi + \frac{d_2 b\rho_0\nu_1}{g}\right] + gc_1 z,\\ \rho &= \frac{c}{k}\arctan\frac{\varphi}{k} + d_1 + c_1. \end{aligned}$$

1.2. Solution:

$$\begin{aligned} u &= -\frac{ag}{b\rho_0(f_0 - \beta x)}\left[\frac{c}{k}\arctan\frac{\varphi}{k} + d_1\right] - \frac{\nu_1\beta}{f_0 - \beta x},\\ v &= \frac{\nu_1\beta}{f_0 - \beta x}\left[\frac{cg}{2b\rho_0\beta\nu_1}\ln\left(k^2 + \varphi^2\right) - d_2\right], \end{aligned}$$

$$w=\frac{ag}{b(f_0-\beta x)}\left[\frac{\varphi}{b\rho_0}\left(\frac{c}{k}\arctan\frac{\varphi}{k}+d_1\right)-\frac{c}{2b\rho_0}\ln\bigl(k^2+\varphi^2\bigr)\,\frac{\beta\nu_1}{ag}(\varphi+d_2a)\right],$$
$$p=\frac{g(f_0-\beta x)}{b\beta}\left[\frac{c\varphi}{k}\arctan\frac{\varphi}{k}-\frac{c}{2}\ln\bigl(k^2+\varphi^2\bigr)+d_1\varphi+\frac{d_2\beta b\rho_0\nu_1}{g}\right]+y\beta\rho_0\nu_1+gc_1z,$$
$$\rho=\frac{c}{k}\arctan\frac{\varphi}{k}+d_1+c_1.$$

1.3. Solution:

$$u=-\frac{ag}{b\rho_0(f_0-\beta x)}\left[\frac{c_2}{c_1}\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)+c_3\right],$$
$$v=\frac{gc_2}{b\rho_0c_1(f_0-\beta x)}\left[\varphi\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)-\int\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)d\varphi-\frac{b\rho_0c_1c_4\beta\nu_1}{gc_2}\right],$$
$$w=\frac{ag}{f_0-\beta x}\left[\frac{c_2}{\rho_0b^2c_1}\int\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)d\varphi+\frac{c_3\varphi}{\rho_0b^2}+\frac{(ac_4+c_1)\beta\nu_1}{abg}\right],$$
$$p=\frac{g(f_0-\beta x)}{b\beta}\left[\frac{c_2}{c_1}\int\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)d\varphi+c_3\varphi+\frac{c_4b\beta\rho_0\nu_1}{g}\right]+a_1gz,$$
$$\rho=\frac{c_2}{c_1}\exp\left(\frac{c_1}{k}\arctan\frac{\varphi}{k}\right)+c_3+a_1.$$

1.4. Solution:

$$u=-\frac{ag}{b\rho_0(f_0-\beta x)}\left(\frac{c_2}{c_1}F(\varphi)+\frac{\rho_0\beta\nu_1b+agc_3}{ag}\right),$$
$$v=\frac{gc_2}{b\rho_0(f_0-\beta x)}\left(\frac{\varphi}{c_1}F(\varphi)-\frac{G(\varphi)}{c_1}\right),$$
$$w=\frac{1}{b(f_0-\beta x)}\left(\frac{gc_2a}{\rho_0bc_1}G(\varphi)+\frac{b\rho_0\beta\nu_1+agc_3}{b\rho_0}\varphi+c_1\beta\nu_1\right),$$
$$p=\frac{g(f_0-\beta x)}{\beta b}\left(\frac{c_2}{c_1}G(\varphi)+c_3\varphi\right)+\rho_0\beta\nu_1y+a_1gz,$$
$$\rho=\frac{c_2}{c_1}F(\varphi)+c_3+a_1,$$

where

$$F(\varphi)=\exp\left[\frac{c_1}{k}\arctan\left(\frac{\varphi}{k}\right)\right],\quad G(\varphi)=\int F(\varphi)d\varphi,\quad \varphi=\frac{\beta(ay+bz)}{f_0-\beta x}.$$

2°. Case $\nu_1=0$, i.e., the terms defining the horizontal advection of density are absent.

2.1. Solution:

$$u=-\frac{a\beta}{f_0-\beta x}[2a_1\varphi+b_1],$$
$$v=\frac{\beta}{f_0-\beta x}[a_1\varphi^2-c_1],$$
$$w=\frac{a\beta}{b(f_0-\beta x)}[a_1\varphi^2+b_1\varphi+c_1],$$
$$p=\rho(f_0-\beta x)[a_1\varphi^2+b_1\varphi+c_1]+c_2gz,$$
$$\rho=\frac{\rho_0\beta b}{g}[2a_1\varphi+b_1]+c_2.$$

2.2. Solution:

$$u = -\frac{a\beta}{f_0 - \beta x}\left[\frac{c_2 b\sqrt{\nu_2}}{c_1}\exp\left(\frac{c_1\varphi}{b\sqrt{\nu_2}}\right) + c_3\right],$$

$$v = \frac{\beta}{f_0 - \beta x}\left[\frac{c_2 b\sqrt{\nu_2}}{c_1}\left(\varphi - \frac{b\sqrt{\nu_2}}{c_1}\right)\exp\left(\frac{c_1\varphi}{b\sqrt{\nu_2}}\right) - c_4\right],$$

$$w = \frac{\beta}{f_0 - \beta x}\left[\frac{ac_2 b\nu_2}{c_1^2}\exp\left(\frac{c_1\varphi}{b\sqrt{\nu_2}}\right) + c_1\sqrt{\nu_2} + c_4\frac{a}{b} + c_3\frac{a\varphi}{b}\right],$$

$$p = \rho_0(f_0 - \beta x)\left[\frac{c_2 b^2\nu_2}{c_1^2}\exp\left(\frac{c_1\varphi}{b\sqrt{\nu_2}}\right) + c_3\varphi + c_4\right] + a_1 gz,$$

$$\rho = \frac{c_2 b^2\rho_0\beta\sqrt{\nu_2}}{c_1 g}\exp\left(\frac{c_1\varphi}{b\sqrt{\nu_2}}\right) + \frac{c_3 b\rho_0\beta}{g} + a_1.$$

Remark. Other solutions of the system under consideration are discussed in Lineikin (1963), Robinson and Welander (1963), and Kozlov (1966).

⊙ *Reference for Subsection* 20.3.4: Filippov (1968) (except for solution 1.4).

20.4. Chromatography Equations

Preliminary remarks. The chromatography equations have the form

$$c\frac{\partial u_i}{\partial x} + \frac{\partial}{\partial t}[u_i + a_i(\mathbf{u})] = 0, \qquad i = 1, \ldots, n,$$

where $u_i = u_i(x, t)$ is the concentration of the ith component of the compound in the moving phase, $a_i(\mathbf{u}) = a_i(u_1, \ldots, u_n)$ is its concentration in the sorbent, and $c = \text{const}$ is the velocity of the moving phase. After introducing $\tau = ct - x$, the chromatography equations can be rewritten in the form

$$\frac{\partial u_i}{\partial x} + \frac{\partial a_i(\mathbf{u})}{\partial \tau} = 0.$$

Different models are obtained after different choices of $a_i(\mathbf{u}) = a_i(u_1, \ldots, u_n)$, called *isotherms of adsorption*; for example, see Kuznetsov (1967) and Rozhdestvenskii and Yanenko (1983).

Considered below are isotherms of adsorption, for which the chromatography equations admit *Riemann invariants*—new variables, $R_i = R_i(u_1, \ldots, u_n)$, in which the system matrix becomes diagonal:

$$\frac{\partial R_i}{\partial x} + \lambda_i(\mathbf{R})\frac{\partial R_i}{\partial \tau} = 0, \qquad i = 1, 2, \ldots, n,$$

where $\lambda_i(\mathbf{R}) = \lambda_i(R_1, \ldots, R_n)$. For conditions of integrability of such diagonal systems, see Subsection 20.5.1.

20.4.1. Langmuir Isotherm

1°. The chromatography equations for the Langmuir isotherm are

$$\frac{\partial u_i}{\partial x} + \frac{\partial a_i(\mathbf{u})}{\partial \tau} = 0, \qquad a_i(\mathbf{u}) = \frac{\Gamma_i u_i}{1 + \sum_{s=1}^{n}\Gamma_s u_s},$$

where the $\Gamma_i = \text{const}$ are the *Henry* (or *adsorption*) *coefficients*.

Remark. Similar equations arise also in the theory of electrophoresis (see Zhukov and Yudovich, 1982).

The characteristic values λ_i are the roots of the equation

$$\sum_{s=1}^{n} \frac{\Gamma_s^2 u_s}{\Gamma_s - \lambda V} = V, \quad \text{where} \quad V = 1 + \sum_{s=1}^{n} \Gamma_s u_s.$$

The Riemann invariants R_i are given by

$$R_i = \lambda_i V.$$

There exist simple formulas that connect Riemann invariants R_i with "physical" variables u_i:

$$u_i = \frac{1}{\Gamma_i} \prod_{s=1}^{n} \left(\frac{\Gamma_i}{R_s} - 1 \right) \Big/ \prod_{s \neq i} \left(\frac{\Gamma_i}{\Gamma_s} - 1 \right).$$

The diagonal form of the chromatography equations for the Langmuir isotherm is

$$\frac{\partial R_i}{\partial x} + R_i \left(\prod_{s=1}^{n} R_s \Big/ \prod_{s=1}^{n} \Gamma_s \right) \frac{\partial R_i}{\partial \tau} = 0. \tag{1}$$

2°. The general solution of the chromatography equations (1) can be represented in the implicit form

$$\frac{\prod_{s=1}^{n} \Gamma_s}{R_i \prod_{s=1}^{n} R_s} \tau - x = w_i(\mathbf{R}). \tag{2}$$

Here,

$$w_i(\mathbf{R}) = R_i \prod_{s=1}^{n} R_s \frac{\partial}{\partial R_i} \left[\sum_{k=1}^{n} \frac{f_k(R_k)}{\prod_{m \neq k} (R_m - R_k)} \right], \tag{3}$$

where $f_k(R_k)$ are arbitrary functions ($k = 1, \ldots, n$).

3°. Formulas (2)–(3) are too general to be effective. There exist some particular solutions of equations (1) given in a much more convenient and explicit form (Ferapontov (1991)):

$$R_i(x, \tau) = \frac{\varphi(\tau) - A_i}{f(x) - B_i},$$

where A_i and B_i are arbitrary constants and the functions $f(x)$ and $\varphi(\tau)$ are solutions of the ordinary differential equations

$$\frac{df}{dx} = \frac{C}{\prod_{s=1}^{n} \Gamma_s \prod_{s=1}^{n} (f - B_s)},$$
$$\frac{d\varphi}{d\tau} = \frac{C}{\prod_{s=1}^{n} (\varphi - A_s)}, \quad C = \text{const}.$$

It is possible to write down some explicit solutions even in the "physical" variables u_i. In the n-component case for arbitrary $n \geq 2$, there are exact solutions of the form:

$$u_i = \frac{\varphi'_\tau(\tau)}{\Gamma_i} P_i^{(n)}(z_i), \quad z_i = \Gamma_i f(x) - \varphi(\tau),$$

where $P_i^{(n)}(z_i)$ are some polynomials of nth order in z_i with certain restrictions on their coefficients, and the functions $f(x)$ and $\varphi(\tau)$ satisfy some ordinary differential equations.

Special case. Consider the two-component case ($n = 2$). We have

$$\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial \tau}\left(\frac{\Gamma_1 u_1}{1 + \Gamma_1 u_1 + \Gamma_2 u_2}\right) = 0,$$
$$\frac{\partial u_2}{\partial x} + \frac{\partial}{\partial \tau}\left(\frac{\Gamma_2 u_2}{1 + \Gamma_1 u_1 + \Gamma_2 u_2}\right) = 0.$$

The desired solutions are given by

$$u_1 = \frac{\varphi'_\tau(\tau)}{\Gamma_1}\Big[A_0 + A_1\big(\Gamma_1 f(x) - \varphi(\tau)\big) + A_2\big(\Gamma_1 f(x) - \varphi(\tau)\big)^2\Big],$$
$$u_2 = \frac{\varphi'_\tau(\tau)}{\Gamma_2}\Big[B_0 + B_1\big(\Gamma_2 f(x) - \varphi(\tau)\big) + B_2\big(\Gamma_2 f(x) - \varphi(\tau)\big)^2\Big],$$

where A_0, A_1, A_2, B_0, B_1, and B_2 are arbitrary constants subject to one constraint

$$A_2\Gamma_1 + B_2\Gamma_2 = 0,$$

whereas the functions $f = f(x)$ and $\varphi = \varphi(\tau)$ are determined by the separable first-order ordinary differential equations

$$\frac{df}{dx} = \frac{1}{C + A_0 + B_0 + (A_1\Gamma_1 + B_1\Gamma_2)f + (A_2\Gamma_1^2 + B_2\Gamma_2^2)f^2},$$
$$\frac{d\varphi}{d\tau} = \frac{1}{C + (A_1 + B_1)\varphi - (A_2 + B_2)\varphi^2},$$

with C being an arbitrary constant.

⊙ *References for Subsection* 20.4.1: N. N. Kuznetsov (1967), M. Yu. Zhukov and V. I. Yudovich (1982), B. L. Rozhdestvenskii and N. N. Yanenko (1983), S. P. Tsarev (1989), E. V. Ferapontov and S. P. Tsarev (1991), N. H. Ibragimov (1995).

20.4.2. Generalized Langmuir Isotherm

1°. The chromatography equations for the generalized Langmuir isotherm are

$$\frac{\partial u_i}{\partial x} + \frac{\partial a_i(\mathbf{u})}{\partial \tau} = 0, \qquad a_i(\mathbf{u}) = \frac{\Gamma_i u_i}{\big[1 + \sum(\Gamma_s u_s)^{1/r}\big]^r},$$

where $r = \text{const}$. For $r = 1$ we recover the classical Langmuir case; see Subsection 20.4.1.

The characteristic values λ_i are the roots of the equation

$$\sum_{s=1}^{n} \frac{\Gamma_s(\Gamma_s u_s)^{1/r}}{\Gamma_s - \lambda V} = V^{1/r}, \qquad \text{where} \qquad V = \left[1 + \sum_{s=1}^{n}(\Gamma_s u_s)^{1/r}\right]^r.$$

The Riemann invariants R_i are given by $R_i = \lambda_i V$. There exist simple formulas that connect Riemann invariants R_i with the "physical" variables u_i:

$$(u_i)^{1/r} = \frac{1}{\Gamma_i}\prod_{s=1}^{n}\left(\frac{\Gamma_i}{R_s} - 1\right)\Big/\prod_{s\neq i}\left(\frac{\Gamma_i}{\Gamma_s} - 1\right).$$

The diagonal form of the chromatography equations for the generalized Langmuir isotherm is

$$\frac{\partial R_i}{\partial x} + R_i\left(\prod_{s=1}^{n} R_s\Big/\prod_{s=1}^{n}\Gamma_s\right)^r\frac{\partial R_i}{\partial \tau} = 0.$$

2°. Solution:

$$R_i(x,\tau) = \frac{\varphi(\tau) - A_i}{f(x) - B_i},$$

where A_1 and B_i are arbitrary constants and the functions $f(x)$ and $\varphi(\tau)$ are solutions of the ordinary differential equations

$$\frac{df}{dx} = \frac{C}{\left[\prod_{s=1}^{n} \Gamma_s \prod_{s=1}^{n} (f - B_i)\right]^r},$$
$$\frac{d\varphi}{d\tau} = \frac{C}{\left[\prod_{s=1}^{n} (\varphi - A_s)\right]^r}.$$

For $r = -\frac{1}{2}$, both $f(x)$ and $\varphi(\tau)$ become hyperelliptic integrals.

⊙ *References for Subsection* 20.4.2: E. V. Ferapontov and S. P. Tsarev (1991), N. H. Ibragimov (1995).

20.4.3. Power Isotherm

1°. The chromatography equations for the power isotherm are

$$\frac{\partial u_i}{\partial x} + \frac{\partial a_i(\mathbf{u})}{\partial \tau} = 0, \qquad a_i(\mathbf{u}) = \frac{\Gamma_i u_i}{\prod_{s=1}^{n} (\Gamma_s u_s)^{\beta_s}},$$

where $\beta_s = \text{const}$.

The characteristic values λ_i are the roots of the equation

$$\sum_{s=1}^{n} \frac{\Gamma_s \beta_s}{\Gamma_s - \lambda V} = 1, \qquad \text{where} \qquad V = \prod_{s=1}^{n} (\Gamma_s u_s)^{\beta_s}.$$

Hence λ_i can be represented in the form $\lambda_i = \mu_i / V$, where the constants μ_i are the solutions of the algebraic equation

$$\sum_{s=1}^{n} \frac{\Gamma_s \beta_s}{\Gamma_s - \mu} = 1.$$

The Riemann invariants R_i are given by

$$R_i = \sum_{s=1}^{n} \frac{\beta_s}{\Gamma_s - \mu_i} \ln(\Gamma_s u_s).$$

The diagonal form of the chromatography equations for the power isotherm is

$$\frac{\partial R_i}{\partial x} + \mu_i \exp\left(-\sum_{s=1}^{n} p_s R_s\right) \frac{\partial R_i}{\partial \tau} = 0,$$

where the constants p_s are solutions of the system of linear equations

$$\sum_{s=1}^{n} \frac{p_s}{\Gamma_j - \mu_s} - 1 = 0, \qquad j = 1, \ldots, n.$$

2°. The general solution can be represented in the implicit form

$$\tau - \mu_i \exp\left(-\sum_{s=1}^{n} p_s R_s\right) x = w_i(\mathbf{R}),$$

with $w_i = w_i(\mathbf{R})$ being an arbitrary solution of the linear system of partial differential equations

$$\frac{\partial w_i}{\partial R_j} = C_{ij}(w_j - w_i), \qquad C_{ij} = \frac{p_j \mu_j}{\mu_i - \mu_j},$$

where C_{ij} are constants.

3°. There are exact solutions of the form

$$R_i(x,\tau) = P_i^{(n)}(z_i), \qquad z_i = \varphi(\tau) - \mu_i f(x),$$

where $P_i^{(n)}(z_i)$ are certain polynomials of nth order and the functions $\varphi(\tau)$ and $f(x)$ satisfy ordinary differential equations.

Special case. Consider the two-component case ($n = 2$). We have

$$\frac{\partial R_1}{\partial x} + \mu_1 \exp\bigl(-p_1 R_1 - p_2 R_2\bigr)\frac{\partial R_1}{\partial \tau} = 0,$$
$$\frac{\partial R_2}{\partial x} + \mu_2 \exp\bigl(-p_1 R_1 - p_2 R_2\bigr)\frac{\partial R_2}{\partial \tau} = 0.$$

The desired solutions are given in the form

$$R_1 = A_0 + A_1\bigl[\varphi(\tau) - \mu_1 f(x)\bigr] + A_2\bigl[\varphi(\tau) - \mu_1 f(x)\bigr]^2,$$
$$R_2 = B_0 + B_1\bigl[\varphi(\tau) - \mu_2 f(x)\bigr] + B_2\bigl[\varphi(\tau) - \mu_2 f(x)\bigr]^2,$$

where the constants A_0, A_1, A_2, B_0, B_1, and B_2 are subject to one constraint

$$A_2 p_1 \mu_1 + B_2 p_2 \mu_2 = 0.$$

Here, the functions $\varphi = \varphi(\tau)$ and $f = f(x)$ satisfy the ordinary differential equations

$$\varphi'_\tau = C \exp\Bigl[\bigl(A_2 p_1 \mu_1^2 + B_2 p_2 \mu_2^2\bigr)\varphi^2 - \bigl(A_1 p_1 \mu_1 + B_1 p_2 \mu_2\bigr)\varphi + A_0 p_1 + B_0 p_2\Bigr],$$
$$f'_x = C \exp\Bigl[-\bigl(A_2 p_1 + B_2 p_2\bigr) f^2 - \bigl(A_1 p_1 + B_1 p_2\bigr) f\Bigr],$$

where C is an arbitrary constant. Both functions $\varphi(\tau)$ and $f(x)$ can be expressed in terms of the integral $\int \exp(-\xi^2)\,d\xi$.

⊙ *References for Subsection* 20.4.3: E. V. Ferapontov and S. P. Tsarev (1991), N. H. Ibragimov (1995).

20.4.4. Exponential Isotherm

The chromatography equations for the power isotherm are

$$\frac{\partial u_i}{\partial x} + \frac{\partial a_i(\mathbf{u})}{\partial \tau} = 0, \qquad a_i(\mathbf{u}) = \frac{k_i \exp(\Gamma_i u_i)}{1 + \sum_{s=1}^{n} k_s \exp(\Gamma_s u_s)},$$

where $k_s = \text{const}$. By changing to the new variables $v_i = k_i \exp(\Gamma_i u_i)$, one obtains

$$\frac{\partial v_i}{\partial x} + v_i \frac{\partial}{\partial \tau}\left(\frac{\Gamma_i v_i}{1 + \sum_{s=1}^{n} v_s}\right) = 0.$$

The eigenvalues λ_i are roots of the equation

$$\sum_{s=1}^{n} \frac{\Gamma_s v_s^2}{\Gamma_s v_s - \lambda V} = V, \quad \text{where} \quad V = 1 + \sum_{s=1}^{n} v_s.$$

The Riemann invariant R_i corresponding to an eigenvalue λ_i is expressed as

$$R_i = \sum_{s=1}^{n} \frac{1}{\Gamma_s} \left(\ln \frac{\lambda_i V}{\Gamma_s v_s - \lambda_i V} + \frac{\lambda_i V}{\Gamma_s v_s - \lambda_i V} \right).$$

⊙ *References for Subsection* 20.4.4: E. V. Ferapontov and S. P. Tsarev (1991), N. H. Ibragimov (1995).

20.5. Other Hydrodynamic-Type Systems

20.5.1. Hydrodynamic-Type Systems of Diagonal Form

Consider a system of diagonal form

$$\frac{\partial w_i}{\partial t} + f_i(\mathbf{w}) \frac{\partial w_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, n,$$

where the $f_i(\mathbf{w}) = f_i(w_1, w_2, \ldots, w_n)$ are given functions.

The system is said to be *semi-Hamiltonian*, if for any three different indices, $i \neq j \neq k$, the relations

$$\frac{\partial}{\partial w_k} \left(\frac{1}{f_j - f_i} \frac{\partial f_i}{\partial w_j} \right) = \frac{\partial}{\partial w_j} \left(\frac{1}{f_k - f_i} \frac{\partial f_i}{\partial w_k} \right) \tag{1}$$

hold.

Suppose the functions $\varphi_i = \varphi_i(w_1, w_2, \ldots, w_n)$ are determined by solving the linear system of equations

$$\frac{\partial \varphi_i}{\partial w_j} = g_{ij}(\mathbf{w})(\varphi_j - \varphi_i), \qquad g_{ij}(\mathbf{w}) = \frac{1}{f_j - f_i} \frac{\partial f_i}{\partial w_j}, \qquad i \neq j. \tag{2}$$

By virtue of (1), system (2) is consistent and determines a solution φ_i ($i = 1, 2, \ldots, n$), which depends on n arbitrary functions of a single argument.

The general solution of the system, with conditions (1) satisfied, can be represented in the implicit form

$$\varphi_i(\mathbf{w}) = x - f_i(\mathbf{w})t, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where $\varphi_i(\mathbf{w}) = \varphi_i(w_1, w_2, \ldots, w_n)$ is the general solution of the linear system (2).

⊙ *References for Subsection* 20.5.1: B. A. Dubrovin and S. P. Novikov (1983, 1989), S. P. Tsarev (1985, 1991, 2000), E. V. Ferapontov and S. P. Tsarev (1991), M. V. Pavlov (2007).

20.5.2. Hydrodynamic-Type Systems of Nondiagonal Form

Consider a system of nondiagonal form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} v_{ij}(\mathbf{u}) \frac{\partial u_j}{\partial x} = 0, \qquad i = 1, 2, \ldots, n,$$

where the $v_{ij}(\mathbf{u}) = v_{ij}(u_1, u_2, \ldots, u_n)$ are given functions.

Suppose that the roots λ_p of the characteristic equation

$$\det[v_{ij}(\mathbf{u}) - \lambda\delta_{ij}] = 0,$$

where δ_{ij} is the Kronecker delta, are all real and distinct, and $\mathbf{a}_p = (a_{p1}, \ldots, a_{pn})$ are the corresponding left eigenvectors of the matrix v_{ij}:

$$a_{pi}v_{ij} = \lambda_p a_{pj}, \qquad p = 1, 2, \ldots, n.$$

Here and henceforth, summation is assumed over repeated indices (in this case, over the subscript i). Suppose that for each eigenvector $\mathbf{a}_p$ there exists an integrating factor c_p such that the vector $c_p\mathbf{a}_p$ is a gradient:

$$c_p a_{pj} = \frac{\partial R_p}{\partial u_j}.$$

The functions $R_p = R_p(\mathbf{u})$ ($p = 1, 2, \ldots, n$) are called *Riemann invariants*.

Remark. For $n = 2$, there always exists an integrating factor; however, this is not the case for $n > 2$.

In the variables R_p, the system becomes diagonal

$$\frac{\partial R_i}{\partial t} + \lambda_i(\mathbf{R})\frac{\partial R_i}{\partial x} = 0 \tag{1}$$

and its condition of integrability (the property of being semi-Hamiltonian) has the form

$$\frac{\partial}{\partial R_k}\left(\frac{1}{\lambda_j - \lambda_i}\frac{\partial \lambda_i}{\partial R_j}\right) = \frac{\partial}{\partial R_j}\left(\frac{1}{\lambda_k - \lambda_i}\frac{\partial \lambda_i}{\partial R_k}\right), \qquad i \neq j \neq k. \tag{2}$$

For the method for constructing the general solution of system (1) provided that conditions (2) are satisfied, see the system from Subsection 20.5.1. The corresponding general solution of the original system is given in implicit form by the algebraic system

$$x\delta_{ik} - tv_{ik}(\mathbf{u}) = w_{ik}(\mathbf{u}), \qquad i, k = 1, 2, \ldots, n,$$

where the functions $w_{ik}(\mathbf{u})$ are described by a suitable linear system of first-order partial differential equations.

Below is a simple criterion for the existence of Riemann invariants. First, using the matrix v_{ij}, one constructs the *Nijenhuis tensor* (see Nijenhuis, 1951)

$$N^i_{jk} = v_{pj}\frac{\partial v_{ik}}{\partial u_p} - v_{pk}\frac{\partial v_{ij}}{\partial u_p} - v_{ip}\left(\frac{\partial v_{pk}}{\partial u_j} - \frac{\partial v_{pj}}{\partial u_k}\right),$$

and then the tensor

$$T^i_{jk} = N^i_{pr}v_{pj}v_{rk} - N^p_{jr}v_{ip}v_{rk} - N^p_{rk}v_{ip}v_{rj} + N^p_{jk}v_{ir}v_{rp}.$$

THEOREM. *The matrix v_{ij} is (locally) diagonalizable in a suitable coordinate system R_i if and only if $T^j_{jk} \equiv 0$.*

⊙ *References for Subsection* 20.5.2: A. Haantjes (1955), B. A. Dubrovin and S. P. Novikov (1983, 1989), S. P. Tsarev (1985, 1991, 2000), E. V. Ferapontov and S. P. Tsarev (1991), M. V. Pavlov (2007).

20.6. Ideal Plasticity with the von Mises Yield Criterion

20.6.1. Two-Dimensional Equations. Plane Case

Equations of ideal plasticity. The steady-state two-dimensional *equations of ideal plasticity* with the *von Mises yield criterion* have the form

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau}{\partial y}=0, \quad \frac{\partial \tau}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{1}$$

$$(\sigma_x-\sigma_y)^2+4\tau^2=4K_s^2, \tag{2}$$

$$\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\tau=\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(\sigma_x-\sigma_y), \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3}$$

where x and y are the rectangular Cartesian coordinates, (u, v) are the velocity components, $(\sigma_x, \sigma_y, \tau)$ are the stress tensor components, and K_s is the yield strength in simple shear.

1°. *Prandtl solution*:

$$\sigma_x=-P-K_s\left(\frac{x}{h}-2\sqrt{1-\frac{y^2}{h^2}}\right), \quad \sigma_y=-P-K_s\frac{x}{h}, \quad \tau=\frac{K_s}{h}y, \tag{4}$$

where γ, h, and P are arbitrary constants. With this solution, equations (3) become a first-order linear system of equations for u and v:

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\frac{2\sqrt{h^2-y^2}}{y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{5}$$

Below are two exact solutions of equations (5).

1.1. *Nadai multiplicative separable solution*:

$$u=f(y)e^{ax}, \quad v=g(y)e^{ax},$$

where a is an arbitrary constant and the functions $f(y)$ and $g(y)$ are determined by the linear system of ordinary differential equations

$$g''_{yy}-\frac{ay}{\sqrt{h^2-y^2}}g'_y-a^2g=0, \quad af+g'_y=0.$$

⊙ *Reference*: V. V. Sokolovskii (1969).

1.2. Generalized separable solution:

$$u=-\alpha xy+\beta x-\frac{\alpha h^2}{2}\arcsin\frac{y}{h}+\frac{\alpha y}{2}\sqrt{h^2-y^2}-\beta\sqrt{h^2-y^2}+C_1,$$
$$v=\frac{\alpha}{2}(x^2+y^2)-\beta y+C_2,$$

where C_1, C_2, α, and β are arbitrary constants.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

20.6.2. Axisymmetric Case

In the axially symmetric case, the steady-state equations of ideal plasticity with the *von Mises yield criterion* have the following form in the cylindrical coordinates r, θ, z:

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0, \quad \frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\sigma_{rz}}{r}=0,$$
$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\sigma_{rz}^2=6K_s^2,$$
$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0,$$
$$\frac{\partial u}{\partial r}=\lambda(2\sigma_r-\sigma_\theta-\sigma_z), \quad u=\lambda r(2\sigma_\theta-\sigma_r-\sigma_z),$$
$$\frac{\partial w}{\partial z}=\lambda(2\sigma_z-\sigma_r-\sigma_\theta), \quad \frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}=2\lambda\sigma_{rz},$$
$$-3p=\sigma_r+\sigma_\theta+\sigma_z.$$

It has been assumed that

$$u_r=u(r,z), \quad u_z=w(r,z), \quad u_\theta=0.$$

1°. Solution:

$$u=AJ_0'(r)\exp z, \quad w=-AJ_0(r)\exp z,$$
$$\sigma_r=-K_s\int_r^R[f(r)+2]r^{-1}F(r)\,dr, \quad f(r)=-r\frac{J_0(r)}{J_0'(r)}, \quad F(r)=(1+f+f^2)^{1/2},$$
$$\sigma_\theta=\sigma_r+K_s[f(r)+2]F(r), \quad \sigma_z=\sigma_r+K_s[2f+1]F(r), \quad \sigma_{rz}=0,$$

where A and R are arbitrary constants and $J_0(r)$ is the Bessel function.

⊙ *References*: B. D. Annin (1973), B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

2°. Solution:

$$u=n_1r+\frac{n_2}{r}, \quad w=-2n_1z+6\int\frac{\sigma_{rz}u\,dr}{r(2\sigma_\theta-\sigma_z-\sigma_r)},$$
$$\sigma_z=\frac{3n_1r^2}{2n_2}(\sigma_r-\sigma_\theta)+\frac{1}{2}(\sigma_r-\sigma_\theta), \quad \sigma_\theta=\sigma_r-\frac{2\mu n_2}{r}\sqrt{\frac{r-(m_1r^2+m_2)^2}{n_2^2+3n_1^2r^4}},$$
$$\sigma_r=-2\mu n_1\int\sqrt{\frac{r^2-(m_1r^2+m_2)^2}{n_2^2+3n_1^2r^4}}\,\frac{dr}{r^2}-2m_1z+C_1, \quad \sigma_{rz}=m_1r+\frac{m_2}{r},$$

where n_1, n_2, m_1, m_2, and C_1 are arbitrary constants and $\mu=\operatorname{sign}(\sigma_r-\sigma_\theta)$. With $n_2=m_2=0$, one obtains the solution discussed in Hill (1954).

⊙ *Reference*: D. D. Ivlev (1966).

3°. Solution:

$$u=A\frac{(1+\xi^2)^{3/2}}{\xi^2}, \quad w=3A\left[\frac{(1+\xi^2)^{3/2}}{\xi}-\xi(1+\xi^2)^{1/2}-\ln\left|\xi+\sqrt{1+\xi^2}\,\right|\right]+B,$$
$$\sigma_r=K_s\frac{\xi(2-\xi^2)}{(1+\xi^2)^{3/2}}, \quad \sigma_\theta=K_s\frac{\xi}{(1+\xi^2)^{3/2}}, \quad \sigma_z=-3K_s\frac{\xi}{(1+\xi^2)^{3/2}}, \quad \sigma_{rz}=K_s\frac{2\xi^2-1}{(1+\xi^2)^{3/2}},$$

where A and B are arbitrary constants and $\xi=z/r$.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

20.6.3. Three-Dimensional Equations. Steady-State Case

Three-dimensional equations of ideal plasticity. The steady-state three-dimensional equations of ideal plasticity with the *von Mises yield criterion* have the form

$$\begin{gathered}
\frac{\partial s_{11}}{\partial x_1}+\frac{\partial s_{12}}{\partial x_2}+\frac{\partial s_{13}}{\partial x_3}=\frac{\partial p}{\partial x_1},\\
\frac{\partial s_{12}}{\partial x_1}+\frac{\partial s_{22}}{\partial x_2}+\frac{\partial s_{23}}{\partial x_3}=\frac{\partial p}{\partial x_2},\\
\frac{\partial s_{13}}{\partial x_1}+\frac{\partial s_{23}}{\partial x_2}+\frac{\partial s_{33}}{\partial x_3}=\frac{\partial p}{\partial x_3},\\
s_{11}+s_{22}+s_{33}=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0,\\
s_{11}^2+s_{22}^2+s_{33}^2+2(s_{12}^2+s_{13}^2+s_{23}^2)=2K_s^2,\\
s_{11}=\lambda\frac{\partial u_1}{\partial x_1},\quad s_{22}=\lambda\frac{\partial u_2}{\partial x_2},\quad s_{33}=\lambda\frac{\partial u_3}{\partial x_3},\\
s_{12}=\frac{\lambda}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right),\quad s_{13}=\frac{\lambda}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right),\quad s_{23}=\frac{\lambda}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right).
\end{gathered}\tag{1}$$

Here, (x_1, x_2, x_3) are (dimensionless) Cartesian coordinates, (u_1, u_2, u_3) are the components of the velocity vector, the s_{ij} are the components of the stress deviator tensor, with $s_{ij} = \sigma_{ij} - p\delta_{ij}$, where the σ_{ij} are the stress tensor components and δ_{ij} is the Kronecker delta, $p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is the pressure, K_s is the yield strength in simple shear, and λ is a positive constant (or function).

Solutions in the Cartesian coordinates.

1°. Three-parameter family of solutions:

$$\begin{gathered}
u_1=ax_1,\quad u_2=bx_2,\quad u_3=-(a+b)x_3\mp 2\sqrt{(a^2+b^2+ab)(1-x_1^2)},\\
\sigma_{11}=K_sx_3-C_1,\quad \sigma_{13}=-K_sx_1,\quad \sigma_{12}=\sigma_{23}=0,\\
\sigma_{22}=K_sx_3-C_1\pm\frac{K_s(a-b)}{\sqrt{a^2+b^2+ab}}\sqrt{1-x_1^2},\\
\sigma_{33}=K_sx_3-C_1\pm\frac{K_s(2a+b)}{\sqrt{a^2+b^2+ab}}\sqrt{1-x_1^2},\quad \lambda=\frac{K_s}{\sqrt{a^2+b^2+ab}}\sqrt{1-x_1^2},
\end{gathered}$$

where a, b, and C_1 are arbitrary constants.

⊙ *Reference*: D. D. Ivlev (1958).

2°. Solution family with two arbitrary functions:

$$\begin{gathered}
u_1=ax_1,\quad u_2=ax_2,\quad u_3=-2ax_3+\psi(x_1,x_2),\\
\sigma_{11}=2x_3C_0+C_1,\quad \sigma_{22}=2x_3C_0+C_2,\quad \sigma_{12}=0,\\
\sigma_{33}=-3a\lambda+2x_3C_0+C_1,\quad 2\sigma_{13}=\lambda\frac{\partial\psi}{\partial x_1},\quad 2\sigma_{23}=\lambda\frac{\partial\psi}{\partial x_2},\\
\lambda=\sqrt{2}\,K_s\left[6a^2+\frac{1}{2}\left(\frac{\partial\psi}{\partial x_1}\right)^2+\frac{1}{2}\left(\frac{\partial\psi}{\partial x_2}\right)^2\right]^{-1/2},
\end{gathered}\tag{2}$$

where a, C_0, C_1, and C_2 are arbitrary constants, and $\psi(x_1, x_2)$ and $P(x_1, x_2)$ are arbitrary functions.

We will use the second-order equation for the function ψ obtained from system (1). By setting $\psi = \sqrt{12}\,a\varphi$, we can write it as the following equation for φ:

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{H}}\frac{\partial\varphi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{H}}\frac{\partial\varphi}{\partial x_2}\right)=2\frac{C_0}{K_s}=2C,\quad H=1+\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\left(\frac{\partial\varphi}{\partial x_2}\right)^2. \quad (3)$$

2.1. For equation (3) with $C = 0$, an exact solution is given by

$$\varphi=\operatorname{arccosh}\frac{r}{k},\quad r^2=(x_1^2+x_2^2),\quad k=\text{const}.$$

In this case, solution (2) can be rewritten in the cylindrical coordinates as

$$u=ar,\quad v=0,\quad w=-2az+\sqrt{12}\,a\operatorname{arccosh}\frac{r}{k},$$
$$\sigma_r=C_1,\quad \sigma_\theta=C_2,\quad \sigma_z=-3a\lambda+C_1,\quad \sigma_{r\theta}=\sigma_{\theta z}=0,$$
$$2\sigma_{rz}=\lambda\frac{\sqrt{12}\,ak}{\sqrt{r^2-k^2}},\quad \lambda=\frac{k}{\sqrt{3}\,a}\sqrt{\frac{r^2-k^2}{2r^2-k^2}},$$

where C_1 and C_2 are arbitrary constants and $r > k > 0$.

2.2. For equation (3) with $C = 0$, another exact solution is

$$\varphi=\beta\theta+k\ln\bigl(\sqrt{r^2+\beta^2}+\sqrt{r^2-k^2}\bigr)+\frac{1}{2\beta}\arcsin\frac{(\beta^2-k^2)r^2-2\beta^2k^2}{r^2(\beta^2+k^2)},$$

where $r^2 = (x_1^2 + x_2^2)$ and $\theta = \arctan(x_1/x_2)$. In this case, solution (2) can be rewritten in the form

$$u=ar,\quad v=0,\quad w=-2az+\sqrt{12}\,a\varphi,$$
$$\sigma_r=C_1,\quad \sigma_\theta=C_1,\quad \sigma_z=-3a\lambda+C_1,$$
$$\sigma_{r\theta}=0,\quad \sigma_{\theta z}=\sqrt{12}\,\frac{\beta a}{r},\quad 2\sigma_{rz}=\lambda\frac{\sqrt{12}\,ak}{r}\sqrt{\frac{r^2+a^2}{r^2-k^2}},$$
$$\lambda=\frac{kr\sqrt{r^2-k^2}}{\sqrt{3}\,a\sqrt{\bigl(2r^2+\beta^2\bigr)\,(r^2-k^2)+k^2(r^2-\beta^2)}},$$

where a, C_1, k, and β are arbitrary constants and $r > k$.

2.3. One more exact solution of equation (3) with $C = 0$ is given by

$$\varphi=\ln\left(\frac{\cos x_1}{\cos x_2}\right).$$

In this case, solution (2) becomes

$$u_1=-\frac{A}{h}x_1,\quad u_2=-\frac{A}{h}x_2,\quad u_3=B+\frac{2A}{h}x_3+\frac{\sqrt{12}\,A}{\pi}\left(\ln\cos\frac{\pi x_1}{2h}-\ln\cos\frac{\pi x_2}{2h}\right),$$
$$\sigma_{11}=\sigma_{22}=C_1,\quad \sigma_{33}=K_s\frac{3A}{h}+C_1,\quad \sigma_{12}=0,$$
$$\sigma_{13}=K_s\tan\frac{\pi x_1}{2h}\omega(x_1,x_2),\quad \sigma_{23}=K_s\tan\frac{\pi x_2}{h}\omega(x_1,x_2),$$
$$\omega(x_1,x_2)=\left(1+\tan^2\frac{\pi x_1}{2h}+\tan^2\frac{\pi x_2}{2h}\right)^{-1/2},$$

where A, B, C_1, and h are arbitrary constants.

2.4. Exact solution of equation (3) with $C \neq 0$:

$$\varphi = \alpha\theta + \int \frac{(Cr^2+k)\sqrt{r^2+\alpha^2}}{r\sqrt{r^2-(Cr^2+k)^2}}\,dr,$$

where $r^2 = (x_1^2 + x_2^2)$ and $\theta = \arctan(x_1/x_2)$. In this case, solution (2) can be represented in the form

$$u = ar, \quad v = 0, \quad w = -2az + \sqrt{12}\,a\left[2a + \int \frac{(Cr^2+k)\sqrt{r^2+\alpha^2}}{r\sqrt{r^2-(Cr^2+k)^2}}\,dr\right],$$
$$\sigma_r = 2C_0 z + C_1, \quad \sigma_\theta = 2C_0 z + C_1, \quad \sigma_z = \sigma_r - 3\lambda,$$
$$\sigma_{r\theta} = 0, \quad 2\sigma_{\theta z} = \sqrt{12}\,\frac{\alpha a\lambda}{r}, \quad 2\sigma_{rz} = \sqrt{12}\,a\lambda\frac{(Cr^2+k)\sqrt{r^2+\alpha^2}}{r\sqrt{r^2-(Cr^2+k)^2}},$$
$$\lambda = \frac{kr}{\sqrt{3}\,a}\left[\frac{r^2-(Cr^2+k)^2}{(2r^2+\alpha^2)\big(r^2-(Cr^2+k)^2\big)+(Cr^2+k^2)(r^2+\alpha^2)}\right]^{1/2},$$
$$\frac{\sqrt{1-2Ck-\sqrt{1-4Ck}}}{C} < r < \frac{\sqrt{1-2Ck+\sqrt{1-4Ck}}}{C},$$

where a, C_0, C_1, and k are arbitrary constants and $C = C_0/K_s$.

⊙ *Reference for Item* 2°: R. Hill (1948).

3°. Six-parameter solution family of system (1):

$$u_1 = C(kx_1x_3 + px_2x_3 + rx_3^2), \quad u_2 = C(px_1x_3 + lx_2x_3 + qx_3^2),$$
$$u_3 = -\frac{1}{2}C(kx_1^2 + lx_2^2 + 2px_1x_2 - mx_3^2),$$
$$s_{11} = k, \quad s_{22} = l, \quad s_{33} = m, \quad s_{12} = p, \quad s_{13} = r, \quad s_{23} = q,$$

where C, k, l, p, r, and q are arbitrary constants and $m = -k - l$.

⊙ *Reference*: V. Prager (1958).

4°. Six-parameter solution family of system (1):

$$u_1 = 2\alpha x_1x_3 + 2\int \frac{\sigma_{13}}{\omega}\,dx_3, \quad u_2 = -\beta x_2x_3 \mp 2x_1x_3 + \int \frac{\sigma_{23}}{\omega}\,dx_3,$$
$$u_3 = \left(\frac{1}{2} - \frac{\alpha}{2}\right)x_1^2 \pm 2x_1x_3 + \frac{1}{2}\beta x_2^2 + \frac{1}{2}\beta x_3^2,$$
$$\sigma_{13} = \alpha x_3 + C_1, \quad \sigma_{23} = \gamma x_3 + C_2, \quad \sigma_{33} = \alpha x_1 + \gamma x_2 + c_3,$$
$$\sigma_{11} = \sigma_{33} + (2e_{11} + 2e_{22})\omega, \quad \sigma_{12} = e_{12}\omega, \quad \sigma_{22} = \sigma_{33} + (e_{11} + 2e_{22})\omega,$$
$$\omega^2 = \frac{K_s^2 - \sigma_{13}^2 - \sigma_{23}^2}{e_{11}^2 + e_{11}e_{33} + e_{22}^2 + e_{12}^2},$$

where C_1, C_2, C_3, α, β, and γ are arbitrary constants and e_{ij} are the strain rate tensor components, $e_{ij} = \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)$.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

Solutions in the spherical coordinates r, θ, φ:

$$u_r = r^\alpha u(\theta,\varphi), \quad u_\theta = r^\alpha v(\theta,\varphi), \quad u_\varphi = r^\alpha w(\theta,\varphi), \quad p = p(\theta,\varphi). \tag{4}$$

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

1°. Solution (4) with $\alpha = 1$:

$$u = 0, \quad \beta v = w\cos\theta - \frac{\partial w}{\partial\theta}\sin^2\theta, \quad w = r\left(\beta_1\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right|^{a/2} + \beta_2\left|\frac{\cos\theta + 1}{\cos\theta - 1}\right|^{a/2}\right)e^{\beta\varphi},$$
$$s_r = s_{r\theta} = s_{r\varphi} = s_{\theta\varphi} = 0, \quad s_\varphi = K_s, \quad 0 \le \theta \le \pi,$$

where β, β_1, and β_2 are arbitrary constants, $a = \sqrt{1+\beta^2}$, and the component σ_θ of the stress tensor is

$$\sigma_\theta = K_s \ln|2K_s\sin^2\theta| + C,$$

with C being an arbitrary constant.

⊙ *Reference*: A. A. Il'ushin (1978).

2°. Solution (4) with $\alpha = 1$:

$$u = 0, \quad v = \frac{rC_3}{\sin\theta} + rB, \quad w = -rB\varphi\sin\theta + r\sin\theta\left[\int\frac{M(\theta)}{\sin\theta}\,d\theta + C_4\right],$$
$$\sigma_\theta = -\int F(\theta)\cot\theta\,d\theta - A\varphi + C_2, \quad \sigma_\varphi = -\int F(\theta)\cot\theta\,d\theta - F(\theta) - A\varphi + C_2,$$
$$s_{\theta\varphi} = (C_1 - A\cos\theta)/\sin^2\theta, \quad s_{r\theta} = s_{r\varphi} = 0,$$

where

$$F(\theta) = 2\kappa\sqrt{k^2 - s_{\theta\varphi}} \quad M(\theta) = \frac{(B\cos\theta - 2C_3\cos\theta)s_{\theta\varphi}}{F(\theta)\sin\theta}, \quad \kappa = \operatorname{sign}(\sigma_\theta - \sigma_\varphi),$$

and A, B, C_1, C_2, C_3, C_4, and k are arbitrary constants.

⊙ *Reference*: L. V. Ershov, D. D. Ivlev, and A. V. Romanov (1982).

3°. Solution (4) with $\alpha = 0$:

$$u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\exp\left(-\frac{\sin\theta}{\beta^2+1} + \beta\varphi\right), \quad v = 0,$$
$$w = \frac{2\sin\theta}{\beta}\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\exp\left(-\frac{\sin\theta}{\beta^2+1} + \beta\varphi\right),$$
$$s_r = 0, \quad s_\theta = \frac{\beta\sin\theta}{\beta^2 + \sin^2\theta}s_{r\varphi}, \quad s_\varphi = -s_\theta,$$
$$s_{r\theta} = \frac{C}{\sin\theta}, \quad s_{\theta\varphi} = -\frac{2C}{\beta}, \quad s_{r\varphi} = \frac{4C}{3\beta}\cot\theta,$$

where C is an arbitrary constant and $\beta^2 + 1 = 4C/3$.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

Solutions in the cylindrical coordinates r, θ, z:

$$u_r = u(r,\xi), \quad u_\theta = v(r,\xi), \quad u_z = w(r,\xi), \quad p = p(r,\xi), \quad \xi = z + k\theta,$$

where k is an arbitrary constant.

1°. Two-parameter exact solution:

$$u = AJ_\nu(r)e^\xi, \quad w = -A\left(\int J_\nu(r)\,dr\right)e^\xi, \quad v = -\frac{1}{k}\left[rw + \frac{\partial(ru)}{\partial r}\right],$$
$$s_r = -(f+1)s_\theta, \quad s_\theta = \frac{K_s}{\sqrt{1+f+f^2+\varphi^2}}, \quad s_z = fs_\theta,$$
$$s_{r\theta} = 0, \quad s_{rz} = 0, \quad s_{\theta z} = \varphi s_\theta, \quad p = s_r - \int \frac{f+2}{r}s_\theta\,dr,$$

where A is an arbitrary constant, $J_\nu(r)$ is the Bessel function, $\nu = \sqrt{1+k^2}$, and $f = f(r)$, and $\varphi = \varphi(r)$.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

2°. Four-parameter exact solution:

$$u = C_1 r + \frac{C_2}{r}, \quad w = \alpha\xi, \quad v = \gamma\xi r,$$
$$s_r = \lambda\left(C_1 - \frac{C_2}{r^2}\right), \quad s_\theta = \lambda\left(k\gamma + C_1 + \frac{C_2}{r^2}\right), \quad s_z = \alpha\lambda,$$
$$s_{r\theta} = 0, \quad s_{rz} = 0, \quad s_{\theta z} = \frac{\lambda}{2}\left(\frac{\alpha k}{r} + \gamma r\right), \quad p = s_r + \int_{r_1}^{r}\lambda\left(-k\gamma - \frac{2C_2}{r^2}\right)dr,$$
$$2K_s^2 r^4 = \lambda^2\left[(C_1 r^2 - C_2)^2 + \left((k\gamma + C_1)r^2 + C_2\right)^2 + \frac{1}{2}(\alpha k r + \gamma r^3)^2 + \alpha^2 r^4\right],$$

where α, γ, and C_2 are arbitrary constants and $C_1 = -\frac{1}{2}(\alpha + \gamma k)$.

These solutions can be used for describing plastic flows possessing a spiral-helical symmetry; with $k = 0$, they become axisymmetric.

⊙ *Reference*: B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

20.6.4. Dynamic Case. Two-Dimensional Equations

The unsteady two-dimensional equations of ideal plasticity with the von Mises yield criterion have the form

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + \frac{\partial p}{\partial x_1} = \frac{\partial s_{11}}{\partial x_1} + \frac{\partial s_{12}}{\partial x_2},$$
$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + \frac{\partial p}{\partial x_2} = \frac{\partial s_{12}}{\partial x_1} + \frac{\partial s_{22}}{\partial x_2},$$
$$s_{11} = \lambda\frac{\partial u_1}{\partial x_1}, \quad s_{22} = \lambda\frac{\partial u_2}{\partial x_2}, \quad 2s_{12} = \lambda\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),$$
$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \quad s_{11}^2 + s_{22}^2 + s_{12}^2 = 2K_s^2.$$

For the notation, see Subsection 20.6.1.

1°. Two four-parameter solutions:

$$u_1 = \frac{ax_1}{H-ta} + 2f(\eta), \quad u_2 = \eta,$$
$$s_{11} = k[mf^2(\eta) + C], \quad s_{22} = -s_{11}, \quad s_{12} = k\sqrt{1 - [mf^2(\eta) + C]},$$

where a, C, k, and H are arbitrary constants, $\eta = ax_2/(H - ta)$, and $f(\eta)$ is given by the equation

$$\frac{df(\eta)}{d\eta} = \frac{\pm\sqrt{K_s^2 - (f^2(\eta) + C)^2}}{f^2(\eta) + C}.$$

2°. Solution with four arbitrary functions:

$$u_1 = a(t)x_1 + G(x_2, t), \quad u_2 = -a(t)x_2 + b(t),$$
$$p = -\frac{1}{2}[a^2(t) + a'(t)]x_1^2 - \frac{1}{2}[a^2(t) - a'(t)]x_2^2 - [b'(t) - a(t)b(t)]x_2 + m(t) - c(t)x_1 - a(t)F(x_2, t),$$
$$\sigma_{11} = -p + a(t)F(x_2, t), \quad \sigma_{22} = -p - a(t)F(x_2, t), \quad \sigma_{12} = \frac{1}{2}F(x_2, t)G(x_2, t),$$
$$F(x_2, t) = 2K_s\left[4a^2 + \left(\frac{\partial G}{\partial x_2}\right)^2\right]^{-1/2},$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, and $m = m(t)$ are arbitrary functions, and the function $G = G(x, t)$ is determined by the equation

$$\frac{\partial G}{\partial t} + aG + (-ax_2 + b)\frac{\partial G}{\partial x_2} + c - \frac{1}{2}\frac{\partial}{\partial x_2}\left(F(x_2, t)\frac{\partial G}{\partial x_2}\right) = 0.$$

⊙ *References*: B. D. Annin (1975), B. D. Annin, V. O. Bytev, and S. I. Senashov (1985).

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Chapter 21
Navier–Stokes and Related Equations

21.1. Navier–Stokes Equations

21.1.1. Navier–Stokes Equations in Various Coordinate Systems

Vector form of the Navier–Stokes equations. The closed system of equations for a *viscous incompressible Newtonian fluid* has the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, ν is the kinematic viscosity of the fluid, p is the ratio of the pressure to the fluid density, t is time, Δ is the Laplace operator, and ∇ is the Hamilton operator.

The vector equation (1) is a short representation of the *Navier–Stokes equations* and (2) is the *continuity equation*. In equation (1), it is assumed that the mass forces (e.g., the gravitational force) are potential and included into the pressure. In what follows, the system of equations (1)–(2) will be called, for brevity, the Navier–Stokes equations (without distinguishing the continuity equation) and *the ratio of the pressure to the fluid density, p, will be called the pressure*.

Equations (1) can be rewritten in the equivalent form

$$\mathbf{u}_t + (\operatorname{curl}\mathbf{u})\times\mathbf{u} = -\nabla\left(p + \tfrac{1}{2}|\mathbf{u}|^2\right) - \nu\operatorname{curl}\operatorname{curl}\mathbf{u}, \qquad \operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u},$$

where the symbol $\times$ stands for the vector product.

Remark. System (1)–(2) only applies to *laminar fluid flows*, although some authors try to use it also for numerical simulation of turbulent flows.

Navier–Stokes equations in the rectangular Cartesian coordinates. In the rectangular Cartesian coordinates (x, y, z) the Navier–Stokes equations (1)–(2) are written as

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right),\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right),\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\end{aligned} \tag{3}$$

where u, v, and w are the components of the fluid velocity.

Remark 1. The scaling transformation

$$t = \frac{\nu}{a^2}\bar{t}, \quad x = \frac{\nu}{a}\bar{x}, \quad y = \frac{\nu}{a}\bar{y}, \quad z = \frac{\nu}{a}\bar{z}, \quad u = a\bar{u}, \quad v = a\bar{v}, \quad w = a\bar{w}, \quad p = a^2\bar{p},$$

where a is an arbitrary (nonzero) constant, reduces system (3) to the same form, with $\nu = 1$.

Remark 2. The system of four equations (3) can be reduced to a system of two equations by introducing two *stream functions*, φ and ψ, in terms of which the fluid velocity components are expressed as

$$\begin{aligned} u &= a_1\varphi_y - a_3\varphi_z + b_1\psi_y - b_3\psi_z, \\ v &= a_2\varphi_z - a_1\varphi_x + b_2\psi_z - b_1\psi_x, \\ w &= a_3\varphi_x - a_2\varphi_y + b_3\psi_x - b_2\psi_y, \end{aligned} \tag{4}$$

where a_i and b_i are free parameters. Substituting (4) into the continuity equation (the last equation in system (3)) turns it into an identity. Eliminating the pressure p from the first three equations (by cross differentiation), we get two equations

$$\begin{aligned} [u_t + (\mathbf{u}\nabla)u]_y - [v_t + (\mathbf{u}\nabla)v]_x &= \nu(\Delta u)_y - \nu(\Delta v)_x, \\ [u_t + (\mathbf{u}\nabla)u]_z - [w_t + (\mathbf{u}\nabla)w]_x &= \nu(\Delta u)_z - \nu(\Delta w)_x. \end{aligned} \tag{5}$$

Substituting (4) into (5) yields two equations, for φ and ψ.

To the special case of $a_1 = a_2 = a_3 = b_2 = b_3 = 0$ and $b_1 = 1$ in (4) there corresponds the well-known representation of the fluid velocity components in terms of a single stream function for two-dimensional plane flows:

$$u = \psi_y, \quad v = -\psi_x \qquad (w = 0, \quad \psi = \psi(x, y, t)).$$

By setting $a_1 = a_2 = b_1 = b_3 = 0$, $a_3 = -1$, and $b_2 = 1$ in (4), we arrive at the following simple representation of the fluid velocity components for three-dimensional flows in terms of two stream functions:

$$u = \varphi_z, \quad v = \psi_z, \quad w = -\varphi_x - \psi_y.$$

Remark 3. Either equation in (5) can be replaced by the equation

$$[v_t + (\mathbf{u}\nabla)v]_z - [w_t + (\mathbf{u}\nabla)w]_y = \nu(\Delta v)_z - \nu(\Delta w)_y, \tag{6}$$

which is obtained by a different way of eliminating the pressure.

Equation (6) should be used for the verification of exact solutions obtained with the aid of equation (5) (since, in general, exact solutions to equation (5) can contain extra arbitrary constants and functions).

Navier–Stokes equations in the cylindrical coordinates. The cylindrical coordinates (r, φ, z) are related to the rectangular Cartesian coordinates (x, y, z) by

$$\begin{gathered} r = \sqrt{x^2 + y^2}, \quad \tan\varphi = y/x, \quad z = z \quad (\sin\varphi = y/r); \\ x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \end{gathered}$$

where $0 \le \varphi \le 2\pi$.

The fluid velocity components in the cylindrical coordinates are expressed in terms of those in the rectangular Cartesian coordinates as follows:

$$u_r = u\cos\varphi + v\sin\varphi, \quad u_\varphi = v\cos\varphi - u\sin\varphi, \quad u_z = w.$$

Remark. For $u_z = 0$, the cylindrical coordinates r and φ also used as the polar coordinates in the xy-plane.

The Navier–Stokes equations (1)–(2) in the cylindrical coordinates become:

$$\begin{aligned} &\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\varphi^2}{r} = -\frac{\partial p}{\partial r} + \nu\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right), \\ &\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_r u_\varphi}{r} = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \nu\left(\Delta u_\varphi - \frac{u_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right), \\ &\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z\frac{\partial u_z}{\partial z} = -\frac{\partial p}{\partial z} + \nu\Delta u_z, \\ &\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} = 0, \end{aligned} \tag{7}$$

where u_r, u_φ, and u_z are components of the fluid velocity and the Laplace operator Δ is given by the formula

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

Navier–Stokes equations in the spherical coordinates. The spherical coordinates (R, θ, φ) are related to the rectangular Cartesian coordinates (x, y, z) by

$$R = \sqrt{x^2+y^2+z^2}, \quad \theta = \arccos\frac{z}{R}, \quad \tan\varphi = \frac{y}{x} \quad \left(\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}\right);$$
$$x = R\sin\theta\cos\varphi, \quad y = R\sin\theta\sin\varphi, \quad z = R\cos\theta,$$

where $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

The Navier–Stokes equations (1)–(2) in the spherical coordinates become

$$\begin{aligned}
&\mathbb{M}\,u_R - \frac{u_\theta^2 + u_\varphi^2}{R} = -\frac{\partial p}{\partial R} + \nu\Delta u_R - \frac{2\nu}{R^2}\left(\frac{\partial u_\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + u_R + \cot\theta\, u_\theta\right),\\
&\mathbb{M}\,u_\theta + \frac{u_R u_\theta - u_\varphi^2\cot\theta}{R} = -\frac{1}{R}\frac{\partial p}{\partial\theta} + \nu\Delta u_\theta\\
&\qquad\qquad + \frac{\nu}{R^2\sin^2\theta}\left(2\sin^2\theta\,\frac{\partial u_R}{\partial\theta} - 2\cos\theta\,\frac{\partial u_\varphi}{\partial\varphi} - u_\theta\right),\\
&\mathbb{M}\,u_\varphi + \frac{u_R u_\varphi + u_\theta u_\varphi\,\cot\theta}{R} = -\frac{1}{R\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\Delta u_\varphi\\
&\qquad\qquad + \frac{\nu}{R^2\sin^2\theta}\left(2\sin\theta\,\frac{\partial u_R}{\partial\varphi} + 2\cos\theta\,\frac{\partial u_\theta}{\partial\varphi} - u_\varphi\right),\\
&\frac{\partial}{\partial R}\big(R^2\sin\theta u_R\big) + \frac{\partial}{\partial\theta}\big(R\sin\theta u_\theta\big) + \frac{\partial}{\partial\varphi}\big(R u_\varphi\big) = 0,
\end{aligned} \tag{8}$$

where u_R, u_θ, and u_φ are components of the fluid velocity and the differential operators $\mathbb{M}$ and Δ are given by the formulas

$$\mathbb{M} \equiv \frac{\partial}{\partial t} + u_R\frac{\partial}{\partial R} + \frac{u_\theta}{R}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{R\sin\theta}\frac{\partial}{\partial\varphi},$$
$$\Delta \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

⊙ *References for Subsection* 21.1.1: C.-L.-M.-H. Navier (1827), G. G. Stokes (1845, 1846), G. K. Batchelor (1970), R. Temam (1979), H. Schlichting (1981), L. D. Landau and E. M. Lifshitz (1987), Z. U. A. Warsi (1993), T. E. Faber (1995), L. G. Loitsyanskiy (1996), D. Paneli and C. Gutfinger (1997), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002), P. G. Drazin and N. Riley (2006).

21.1.2. General Properties of the Navier–Stokes Equations

Outlined below are the general properties of the Navier–Stokes equations that allow one to multiply exact solutions.

Suppose the set of functions $\bar{u}(x, y, y, z)$, $\bar{v}(x, y, z, t)$, $\bar{v}(x, y, z, t)$, and $\bar{p}(x, y, z, t)$ is some solution to the Navier–Stokes equations (3). Then the following propositions hold.

1°. The set of functions

$$u = A\bar{u}(Ax + B_1, Ay + B_2, Az + B_3, A^2t + B_4),$$
$$v = A\bar{v}(Ax + B_1, Ay + B_2, Az + B_3, A^2t + B_4),$$
$$w = A\bar{w}(Ax + B_1, Ay + B_2, Az + B_3, A^2t + B_4),$$
$$p = A^2\bar{p}(Ax + B_1, Ay + B_2, Az + B_3, A^2t + B_4) + B_5,$$

where A, B_1, B_2, B_3, B_4, and B_5 are arbitrary constants, also solves equations (3); this property is a consequence of the invariance of the equations under translations by arbitrary constants in all independent variables and the pressure as well as under coordinated scaling in the independent and dependent variables.

2°. The set of functions

$$u = \bar{u}(X, Y, z, t)\cos\alpha - \bar{v}(X, Y, z, t)\sin\alpha,$$
$$v = \bar{u}(X, Y, z, t)\sin\alpha + \bar{v}(X, Y, z, t)\cos\alpha,$$
$$w = \bar{w}(X, Y, z, t), \qquad p = \bar{p}(X, Y, z, t),$$
$$X = x\cos\alpha + y\sin\alpha, \quad Y = y\cos\alpha - x\sin\alpha,$$

where α is an arbitrary constant, also solves equations (3); this property is a consequence of the invariance of the equations under the rotation of the coordinate system about the z-axis through an arbitrary angle α in the xy-plane with coordinated rotation of the fluid velocity components; similar functions are obtained using rotations in the yz- and xz-planes.

3°. The set of functions

$$u = \bar{u}(x - x_0, y - y_0, z - z_0, t) + x_0',$$
$$v = \bar{v}(x - x_0, y - y_0, z - z_0, t) + y_0',$$
$$w = \bar{w}(x - x_0, y - y_0, z - z_0, t) + z_0',$$
$$p = \bar{p}(x - x_0, y - y_0, z - z_0, t) - x_0''x - y_0''y - z_0''z + p_0,$$

where $x_0 = x_0(t)$, $y_0 = y_0(t)$, $z_0 = z_0(t)$, and $p_0 = p_0(t)$ are arbitrary functions of time (a prime denotes differentiation with respect to t), also solves equations (3); this property is a consequence of the invariance of the equations to the transition to another coordinate system whose origin moves according to the law $x_0 = x_0(t)$, $y_0 = y_0(t)$, and $z_0 = z_0(t)$ and the axes remain parallel to the original axes, with the pressure varying accordingly.

4°. For two-dimensional plane motions (with u, v, and p independent of z, and $w = 0$), the velocity components can be expressed in terms of the stream function ψ as

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}.$$

In this case, the set of functions

$$u = \bar{u}(X, Y, t)\cos(\omega t) - \bar{v}(X, Y, t)\sin(\omega t) - \omega Y,$$
$$v = \bar{u}(X, Y, t)\sin(\omega t) + \bar{v}(X, Y, t)\cos(\omega t) - \omega X,$$
$$p = \bar{p}(X, Y, t) - 2\omega\bar{\psi}(X, Y, t) + \tfrac{1}{2}\omega^2(x^2 + y^2),$$
$$X = x\cos(\omega t) + y\sin(\omega t), \quad Y = y\cos(\omega t) - x\sin(\omega t),$$

where ω is an arbitrary constant, will also solve equations (3); this property is a consequence of the invariance of the equations under the rotation of the coordinate system about the z-axis with a constant angular velocity ω with coordinated rotation of the velocity components and an appropriate variation of the pressure.

Remark. Properties 3° and 4° allow one to obtain some nonstationary solutions from stationary ones.

⊙ *References for Subsection* 21.1.2: V. V. Pukhnachov (1960, 2006), G. K. Batchelor (1970), V. O. Bytev (1972), N. H. Ibragimov (1995), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

21.2. Solutions with One Nonzero Component of the Fluid Velocity

21.2.1. Unidirectional Plane Flows

Form of exact solutions and the determining equation. The Navier–Stokes equations admit solutions with one nonzero velocity component:

$$u = u(y,t), \quad v = w = 0, \quad p = p_0(t) - f(t)x, \tag{1}$$

where $f(t)$ and $p_0(t)$ are arbitrary functions and the function $u = u(y,t)$ satisfies the linear nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + f(t). \tag{2}$$

⊙ *References*: G. G. Stokes (1851), Lord Rayleigh (1911).

Formulas (1) together with equation (2) contain exact solutions that describe a number of classical pressure and shear plane flows.

Steady-state flows. For the *Couette–Poiseuille plane flow* we have

$$u = \frac{Ah^2}{2\nu}\left[1 - \left(\frac{y}{h}\right)^2\right] + \frac{U_1}{2}\left(1 + \frac{y}{h}\right) + \frac{U_2}{2}\left(1 - \frac{y}{h}\right), \qquad f(t) = A. \tag{3}$$

Formulas (1) and (3) describe a steady-state flow of a viscous fluid between two planes, $y = \pm h$, along the x-axis under a constant pressure gradient, equal to $-A$, and a shear caused by the motion of the upper plane, $y = h$, with a constant speed U_1 and the motion of the lower plane, $y = -h$, with a constant speed U_2. Solution (1), (3) satisfies the no-slip conditions at both planes, which corresponds to the boundary conditions $u|_{y=h} = U_1$ and $u|_{y=-h} = U_2$.

⊙ *References*: J. Poiseuille (1840), M. Couette (1890).

Unsteady flows.

1°. *The transient motion of a flat plate* (the *first Stokes problem*). At the initial moment, the plane $y = 0$, above which there is a quiescent viscous fluid, starts moving suddenly with a constant speed U_0 along the x-axis. In this case, one should set $f(t) \equiv 0$ in equation (2) and add the initial and boundary conditions $u|_{t=0} = 0$ and $u|_{y=0} = U_0$. The evolution of the perturbation brought into the medium occurs according to the self-similar law

$$u = U_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\xi} \exp(-\eta^2)\,d\eta\right), \qquad \xi = \frac{y}{2\sqrt{\nu t}}.$$

The first Stokes problem admits a generalization to the case in which the plate velocity is a given function of time, which corresponds to the following conditions: $u|_{t=0} = 0$ and $u|_{y=0} = U(t)$. The solution of the problem is given by

$$u = \frac{2}{\sqrt{\pi}}\int_{\xi}^{\infty} U\left(t - \frac{y^2}{4\nu\eta^2}\right)\exp(-\eta^2)\,d\eta, \qquad \xi = \frac{y}{2\sqrt{\nu t}}.$$

Special case. For $U(t) = Ae^{\omega t}$, the solution is

$$u = A\exp(\omega t - \sqrt{\omega/\nu}\,y).$$

⊙ *References*: G. G. Stokes (1851), Lord Rayleigh (1911), H. Görtler (1944), E. Watson (1955).

2°. *Applied shear stress at the boundary*. Consider the problem in which a stagnant viscous incompressible fluid occupying the half-space $y > 0$ is set in motion for $t > 0$ by a constant tangential stress τ_0 acting on the fluid surface (the simplest model of flow in the near-surface layer of water under the action of wind). In this case, the initial and boundary conditions for Eq. (2) with $f(t) \equiv 0$ are written as follows:

$$u = 0 \quad \text{at} \quad t = 0, \qquad \mu \frac{\partial u}{\partial y} = \tau_0 \quad \text{at} \quad y = 0, \qquad u \to 0 \quad \text{as} \quad y \to \infty. \tag{4}$$

The solution of problem (2), (4) is given by the formula

$$u = \frac{2\tau_0}{\mu}\sqrt{\frac{\nu t}{\pi}}\,\big[\sqrt{\pi}\,\xi \operatorname{erfc}\xi + \exp(-\xi^2)\big], \qquad \xi = \frac{y}{2\sqrt{\nu t}}. \tag{5}$$

Note that on the free boundary ($y = 0$) the expression in the square brackets in (5) is equal to unity.

⊙ *Reference*: R. Berker (1963).

3°. *A plane boundary suddenly set in motion relative to another, fixed boundary.* Suppose there is a quiescent fluid confined between two parallel plane solid boundaries at $y = 0$ and $y = h$. At an initial instant, the lower boundary is suddenly set in motion with a constant velocity U directed along the plane, while the upper boundary remains fixed. The resulting fluid motion is described by equation (2) with $f(t) \equiv 0$ and the initial and boundary conditions

$$u = 0 \quad \text{at} \quad t = 0, \qquad u = U \quad \text{at} \quad y = 0, \qquad u = 0 \quad \text{at} \quad y = h.$$

The solution of this problem is

$$u = U\left[1 - \frac{y}{h} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\pi^2 n^2 \frac{\nu t}{h^2}\right)\sin\left(\pi n \frac{y}{h}\right)\right].$$

⊙ *Reference*: G. K. Batchelor (1970).

4°. The *oscillatory motion of a flat plate* (the *second Stokes problem*). The plane $y = 0$ oscillates harmonically along the x-axis, which causes motion of the fluid above it. This corresponds to $f(t) \equiv 0$ in equation (2) and the boundary conditions $u|_{y=0} = U_0 \cos(\omega t)$ and $u|_{y\to\infty} \to 0$ (this is a *problem without initial data*). The solution of the problem has the form

$$u = U_0 \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right).$$

If the fluid is confined by an upper, stationary boundary at $y = h$, then the boundary conditions are $u|_{y=0} = U_0 \cos(\omega t)$ and $u|_{y=h} = 0$ and the corresponding solution of equation (2) is

$$u = \frac{U_0}{2[\cosh(2\lambda h) - \cos(2\lambda h)]}\big[e^{-\lambda(y-2h)}\cos(\omega t - \lambda y) + e^{\lambda(y-2h)}\cos(\omega t + \lambda y)$$
$$+ e^{-\lambda y}\cos(\omega t - \lambda y + 2\lambda h) - e^{\lambda y}\cos(\omega t + \lambda y - 2\lambda h)\big], \qquad \lambda = \sqrt{\frac{\omega}{2\nu}}.$$

⊙ *References*: G. G. Stokes (1851), P. G. Drazin and N. Riley (2006, p. 90).

5°. Equation (2) often arises in heat and mass transfer theory and is called the *linear heat equation* with a source. Various solutions of this equation can be found, for example, in the books by Carslaw and Jaeger (1984) and Polyanin (2002).

21.2.2. Unidirectional Flows in Tubes of Various Cross-Sections. External Flow Around a Cylinder

Form of exact solutions and the determining equations. The Navier–Stokes equations admit solutions with one nonzero velocity component in the form (generalization of solution (1) from Subsection 21.2.1):

$$u = u(y, z, t), \quad v = w = 0, \quad p = p_0(t) - f(t)x, \tag{1}$$

where the function $u = u(y, z, t)$ satisfies the linear two-dimensional nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f(t). \tag{2}$$

Formulas (1) together with equation (2) describe laminar fluid flows in rectilinear tubes of constant cross-section. The *fluid streamlines* in such systems are strictly parallel (we neglect the influence of the tube ends on the flow). Here, we shall use the Cartesian coordinates x, y, z with x-axis directed along the flow.

For fluid flows in rectilinear tubes, equation (2) must be supplemented by the *no-slip condition*

$$u = 0 \qquad \text{on the tube surface.} \tag{3}$$

Steady-state flows. For steady-state flows, the *pressure gradient* p_x is constant along the tube length and we have

$$f(t) = \frac{\Delta p}{L} = \text{const}, \tag{4}$$

where $\Delta p > 0$ is the total pressure drop along a tube part of length L referred to the fluid density.

Considered in Items 1°–5° below is equation (2) with the function (4).

1°. *A circular tube.* For a circular tube of radius a, the solution of equation (2) under the no-slip condition (3) on the surface of the tube describes an axisymmetric *Hagen–Poiseuille flow* with parabolic velocity profile:

$$u = \frac{\Delta p}{4\nu L}(a^2 - r^2), \quad r = \sqrt{y^2 + z^2}.$$

Remark. The total rate of flow of the fluid through the tube cross-section is given by

$$Q = 2\pi \int_0^a ru\, dr = \frac{\pi a^4 \Delta p}{8\nu L}.$$

This result was first found empirically by Hagen (1839) and Poiseuille (1840), who independently measured the flow of water along capillary tubes. Stokes (1845) first solved the equations of fluid motion for pipe flow and confirmed theoretically the results of Hagen and Poiseuille.

2°. *A channel between two coaxial circular cylinders.* The flow in an annular channel between two coaxial circular cylinders of radii a and b ($a < b$) is described by

$$u = \frac{\Delta p}{4\nu L}\left[b^2 - r^2 + \frac{a^2 - b^2}{\ln(b/a)} \ln\frac{b}{r}\right], \quad r = \sqrt{y^2 + z^2}, \tag{5}$$

and satisfies the no-slip conditions on the cylinder surfaces, $u|_{r=a} = u|_{r=b} = 0$.

If the internal cylinder moves along its axis with a constant speed U and the external surface remains stationary, which corresponds to the boundary conditions $u|_{r=a} = U$ and $u|_{r=b} = 0$, then the fluid flow is described by

$$u = \frac{\Delta p}{4\nu L}(b^2 - r^2) + \left[U + \frac{\Delta p}{4\nu L}(a^2 - b^2)\right]\frac{1}{\ln(b/a)} \ln\frac{b}{r}, \quad r = \sqrt{y^2 + z^2}.$$

⊙ *References*: J. Boussinesq (1868), P. G. Drazin and N. Riley (2006).

3°. *A tube of elliptic cross-section.* Consider a tube whose cross-section is an ellipse with semiaxes a and b. Let the surface of this tube be given by the equation

$$\left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1. \tag{6}$$

The solution of equation (2) under the no-slip condition (3) on the surface (6) has the form

$$u = \frac{a^2 b^2 \Delta p}{2\nu L(a^2+b^2)}\left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right).$$

⊙ *References*: J. Boussinesq (1868), J. Happel and H. Brenner (1965).

4°. *A tube of rectangular cross-section.* Consider a tube of rectangular cross-section with sides a and b. We assume that the flow region is described by the inequalities $0 \le y \le a$ and $0 \le z \le b$. The solution of equation (2) under the no-slip conditions on the tube surface has the form

$$u = \frac{\Delta p}{2\nu L} y(a-y) + \sum_{m=1}^{\infty} \sin\left(\frac{\pi m y}{a}\right)\left(A_m \cosh\frac{\pi m z}{a} + B_m \sinh\frac{\pi m z}{a}\right),$$

$$A_m = \frac{a^2 \Delta p}{\pi^3 m^3 \nu L}[\cos(\pi m) - 1], \quad B_m = -A_m \frac{\cosh(\pi m k) - 1}{\sinh(\pi m k)}, \quad k = \frac{b}{a}.$$

⊙ *References*: J. Boussinesq (1868), J. Happel and H. Brenner (1965).

5°. *A tube of triangular cross-section.* Now we suppose that the cross-section of the tube is an equilateral triangle with side b. We place the origin at the center of the cross-section and measure the coordinate y along one of the sides of the triangle. In this case, the solution of equation (2) under the boundary condition (3) has the form

$$u = \frac{\sqrt{3}\,\Delta p}{6\nu b L}\left(z - \frac{b}{2\sqrt{3}}\right)\left(z + \sqrt{3}\,y - \frac{b}{\sqrt{3}}\right)\left(z - \sqrt{3}\,y - \frac{b}{\sqrt{3}}\right).$$

⊙ *References*: J. Boussinesq (1868), J. Happel and H. Brenner (1965).

The fluid flow in the tube whose cross-section is the right-angled isosceles triangle with the sides determined by the formulas $y = \pi$ and $y \pm z = 0$ is described by

$$u = \frac{\Delta p}{2L\nu}(y+z)(\pi - y) - \frac{\Delta p}{\pi L \nu}\sum_{n=0}^{\infty} \frac{F_n(y,z)}{m_n^3 \sinh(2m_n\pi)}, \quad m_n = n + \frac{1}{2},$$

$$F_n(y,z) = \sinh[m_n(2\pi - y + z)]\sin[m_n(y+z)] - \sinh[m_n(y+z)]\sin[m_n(y-z)].$$

⊙ *References*: J. Proudman (1914), P. G. Drazin and N. Riley (2006).

Unsteady flows.

6°. *Transient flow in a tube under instantaneously applied pressure gradient.* Consider the problem on a transient laminar viscous flow in an infinite circular tube of radius a under a constant pressure gradient instantaneously applied at $t = 0$. The equation of motion has the form (2) with

$$f(t) = \begin{cases} 0 & \text{for } t \le 0, \\ \Delta p/L & \text{for } t > 0, \end{cases} \tag{7}$$

where Δp is the pressure drop on a part of length L of the tube. In this case the initial and boundary conditions can be written as follows:

$$u = 0 \quad \text{at} \quad t = 0, \qquad u = 0 \quad \text{at} \quad r = a,$$

where $r = \sqrt{y^2 + z^2}$.

The solution of this problem has the form

$$u = \frac{a^2}{4\nu}\frac{\Delta p}{L}\left[1 - \frac{r^2}{a^2} - 8\sum_{k=1}^{\infty}\frac{1}{{\lambda_k}^3 J_1(\lambda_k)}\exp\left(-\lambda_k^2\frac{\nu t}{a^2}\right)J_0\left(\lambda_k\frac{r}{a}\right)\right], \tag{8}$$

where $J_0(\xi)$ and $J_1(\xi)$ are the Bessel functions of the first kind and the λ_k are roots of the transcendental equation $J_0(\lambda) = 0$.

⊙ *Reference*: P. Szymański (1932).

7°. *Transient flow in an annular channel with a homogeneous initial velocity field.* For $t > 0$ and $a \le r \le b$, we consider the equation of motion (2) with function (7) and the initial and boundary conditions

$$u = U = \text{const} \quad \text{at} \quad t = 0, \qquad u = 0 \quad \text{at} \quad r = a, \qquad u = 0 \quad \text{at} \quad r = b.$$

The solution of this problem is

$$u = \frac{\Delta p}{4\nu L}\left[b^2 - r^2 - \frac{(b^2 - a^2)}{\ln(b/a)}\ln\frac{b}{r}\right] + 2\sum_{k=1}^{\infty}\left(U - \frac{b^2\Delta p}{\nu L\lambda_k^2}\right)\frac{\exp\left(-\lambda_k^2\nu t/b^2\right)Z_0\left(\lambda_k r/b\right)}{\lambda_k Z_1(\lambda_k) + q\lambda_k Z_1(q\lambda_k)}, \tag{9}$$

where

$$q = \frac{a}{b}, \quad Z_n\left(\lambda\frac{r}{b}\right) = Y_0(q\lambda)J_n\left(\lambda\frac{r}{b}\right) - J_0(q\lambda)Y_n\left(\lambda\frac{r}{b}\right), \quad n = 0,\ 1.$$

Here $J_n(\xi)$ and $Y_n(\xi)$ are the Bessel functions of the first and second kind, respectively, and the λ_k are roots of the transcendental equation

$$J_0(\lambda)Y_0(q\lambda) - J_0(q\lambda)Y_0(\lambda) = 0.$$

The solution (9) tends to the steady-state solution (5) for annular tubes as $t \to \infty$.

⊙ *Reference*: W. Müller (1936).

8°. *Pulsating laminar flow in a circular tube.* Consider the problem about a laminar viscous flow in an infinite circular tube of radius a with the axial pressure gradient varying according to a harmonic law. The equation of motion has the form (2) with

$$f(t) = A\cos\omega t + B\sin(\omega t) + C \tag{10}$$

and the boundary condition

$$u = 0 \quad \text{at} \quad r = \sqrt{y^2 + z^2} = a. \tag{11}$$

The exact solution of problem (2), (10), (11) (this is a *problem without initial data*) has the form

$$\begin{aligned} u &= \frac{C}{4\nu}(a^2 - r^2) + \frac{1}{\omega}\{A\Phi_2(kr) + B[\Phi_1(kr) - 1]\}\cos(\omega t) \\ &\quad + \frac{1}{\omega}\{B\Phi_2(kr) + A[\Phi_1(kr) - 1]\}\sin(\omega t), \quad k = \sqrt{\frac{\omega}{\nu}}, \\ \Phi_1(kr) &= \frac{\mathrm{bei}(ka)\mathrm{bei}(kr) + \mathrm{ber}(ka)\mathrm{ber}(kr)}{\mathrm{bei}^2(ka) + \mathrm{ber}^2(ka)}, \\ \Phi_2(kr) &= \frac{\mathrm{bei}(ka)\mathrm{ber}(kr) - \mathrm{ber}(ka)\mathrm{bei}(kr)}{\mathrm{bei}^2(ka) + \mathrm{ber}^2(ka)}. \end{aligned} \tag{12}$$

The *Kelvin functions* $\mathrm{ber}\,(x) = \mathrm{Re}\, J_0(x\sqrt{i}\,)$ and $\mathrm{bei}\,(x) = -\mathrm{Im}\, J_0(x\sqrt{i}\,)$, where $J_0(x)$ is the Bessel function of the first kind and order zero, are tabulated, for example, in the book by Janke, Emde, and Lösch (1960).

For $A = B = 0$, we have the steady-state axisymmetric Poiseuille flow.

With $B = C = 0$ for small frequencies $a\sqrt{\omega/\nu} \ll 1$, we have the asymptotic expression

$$u = \frac{A}{4\nu}(a^2 - r^2)\cos\omega t.$$

With $B = C = 0$ for large frequencies $a\sqrt{\omega/\nu} \gg 1$, we have the asymptotic expression

$$u = \frac{A}{\omega}\left[\sin\omega t - \sqrt{\frac{a}{r}}\,\exp(-\zeta)\sin(\omega t - \zeta)\right], \quad \zeta = \sqrt{\frac{\omega}{2\nu}}\,(a - r).$$

It follows from this expression that near the tube axis the fluid and pressure oscillations are opposite in phase, and the viscosity effects are noticeable only in the near-wall boundary layer with characteristic size $\sqrt{\nu/\omega}$.

⊙ *References*: T. Sexl (1930), P. Lambossy (1952), J. R. Womersley (1955 *a, b*), S. Uchida (1956).

9°. *External periodic fluid flow around a cylinder.* Consider the solution

$$u = U\frac{K_0(r\sqrt{i\omega/\nu}\,)}{K_0(a\sqrt{i\omega/\nu}\,)}\,e^{i\omega t}, \quad v = w = 0, \quad r = \sqrt{y^2 + x^2},$$

where $K_0(\zeta)$ is the modified Bessel function and $i^2 = -1$.

The solution describes a fluid flow around an impermeable circular cylinder of radius a that performs an oscillatory motion parallel to its generators with $u|_{r=a} = Ue^{i\omega t}$.

Far from the cylinder, the following asymptotic formula holds:

$$u \sim Cr^{-1/2}\exp\Big\{-\sqrt{\omega/(2\nu)}\,r + i\big[\omega t - \sqrt{\omega/(2\nu)}\,r\big]\Big\} \quad \text{as} \quad r \to \infty.$$

⊙ *References*: H. Hasimoto (1956), P. G. Drazin and N. Riley (2006, pp. 141–142).

10°. *Other solutions.* Equation (2) often arises in heat and mass transfer theory and is called the *two-dimensional linear heat equation* with a source. Various solutions of this equation can be found, for example, in the books by Carslaw and Jaeger (1984) and Polyanin (2002).

21.2.3. One-Dimensional Rotation Fluid Flows

Form of exact solutions and the determining equation. The Navier–Stokes equations in the cylindrical coordinates (r, φ, z) admit solutions with one nonzero angular velocity component in the form

$$u_r = u_z = 0, \quad u_\varphi = u_\varphi(r,t), \quad p = p_1(r,t) - f(t)\varphi + p_0(t), \quad r = \sqrt{x^2+y^2}, \tag{1}$$

where $f(t)$ and $p_0(t)$ are arbitrary functions, the function $u_\varphi = u_\varphi(r,t)$ is determined by the equation

$$\frac{\partial u_\varphi}{\partial t} = \frac{f(t)}{r} + \nu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2}\right), \tag{2}$$

and the function $p_1(r,t)$, appearing in the pressure, is found by quadrature

$$p_1(r,t) = \int u_\varphi^2(r,t)\frac{dr}{r}. \tag{3}$$

For $f(t) \equiv 0$, formulas (1) in conjunction with equation (2) describe, for example, laminar circular motion of a fluid inside a rotating hollow cylinder (or between two coaxial rotating cylinders). The *fluid streamlines* in such systems are circles.

Steady-state flows. Further, we assume $u_\varphi = u_\varphi(r)$, $p_1 = p_1(r)$, and $f(t) \equiv 0$ in (1)–(3).

1°. *Fluid flow around a rotating circular cylinder.* Consider a fluid flow around a circular cylinder that rotates about its axis with an angular velocity ω_a. The fluid is assumed to be unbounded and quiescent far away from the cylinder. Let a denote the radius of the cylinder. The solution of equation (2) that satisfies the no-slip condition on the cylinder surface, $u_\varphi|_{r=a} = \omega_a$, and no-motion condition at infinity, $u_\varphi|_{r=\infty} = 0$, is expressed as

$$u_\varphi = \frac{a^2\omega_a}{r}, \quad p = p_\infty - \frac{a^4\omega_a^2}{2r^2},$$

where p_∞ is the unperturbed pressure at infinity.

⊙ *Reference*: M. Couette (1890).

2°. *Fluid flow between rotating coaxial circular cylinder.* Let a and b denote the radii of the inner and outer cylinders, respectively ($a < b$), and let ω_a and ω_b denote their angular velocities. The solution of equation (2) that satisfies the no-slip conditions on the cylinder surfaces, $u_\varphi|_{r=a} = \omega_a$ and $u_\varphi|_{r=b} = \omega_b$, is expressed as

$$u_\varphi = Ar + \frac{B}{r}, \quad A = \frac{b^2\omega_b - a^2\omega_a}{b^2 - a^2}, \quad B = \frac{a^2b^2(\omega_a - \omega_b)}{b^2 - a^2}, \tag{4}$$

and the pressure distribution is given by

$$p = p_a + \frac{1}{(b^2-a^2)^2}\left[\frac{1}{2}(b^2\omega_b - a^2\omega_a)^2(r^2-a^2) - 2a^2b^2(\omega_b-\omega_a)(b^2\omega_b - a^2\omega_a)\ln\frac{r}{a} + \frac{1}{2}a^4b^4(\omega_b-\omega_a)^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\right]. \tag{5}$$

The constant of integration has been chosen such that the pressure p_a on the inner cylinder wall is constant.

Special case. If both cylinders rotate in the same direction with the same angular velocity, $\omega = \omega_a = \omega_b$, then formulas (4) and (5) become

$$u_\varphi = \omega r, \quad p = p_a + \tfrac{1}{2}\omega^2(r^2 - a^2),$$

which corresponds to a rigid rotation of the fluid.

⊙ *Reference*: M. Couette (1890).

Unsteady flows.

3°. *Transient fluid motion caused by angular momentum.* One of the simplest solutions of equation (2) with $f(t) \equiv 0$ has the form

$$u_\varphi = \frac{Cr}{4\pi\nu t^2}\exp\left(-\frac{r^2}{4\nu t}\right).$$

The solution describes a fluid flow caused by bringing a point angular momentum C into the fluid at the initial time $t = 0$.

⊙ *Reference*: G. I. Taylor (1908).

4°. *Transient fluid motion for a whirlwind.* The classical solution of equation (2) with $f(t) \equiv 0$ that describes diffusion of a vortex line has the form

$$u_\varphi = \frac{C}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right],$$

where C is the intensity of the vortex line.

⊙ *Reference*: C. W. Oseen (1911).

5°. *Diffusion of a vortex with arbitrary initial velocity profile.* The solution of equation (2) with $f(t) \equiv 0$ and the general initial condition $u_\varphi(r, 0) = V_0(r)$ is given by

$$u_\varphi(r,t) = \frac{1}{r}\int_0^r \Omega(\xi,t)\xi\,d\xi, \quad \Omega(r,t) = \frac{1}{2\nu t}\int_0^\infty \Omega_0(\eta)\exp\left(-\frac{r^2+\eta^2}{4\nu t}\right) I_0\left(\frac{r\eta}{2\nu t}\right)\eta\,d\eta,$$
$$\Omega_0(r) = \Omega(r,0) = \frac{1}{r}\frac{d}{dr}[rV_0(r)],$$

where $I_0(z)$ is the modified Bessel function and $\Omega(r,t)$ is the vorticity.

⊙ *Reference*: N. E. Kochin, I. A. Kibel', and N. V. Roze (1963).

6°. *Transient fluid motion inside a hollow rotating cylinder.* The transient motion of a fluid inside a rotating cylinder (in the region $r \le a$) is described by equation (2) with $f(t) \equiv 0$ and the initial and boundary conditions

$$u_\varphi = 0 \quad \text{at} \quad t = 0, \qquad u_\varphi = \omega a \quad \text{at} \quad r = a. \tag{6}$$

The solution of this problem has the form

$$u_\varphi = \omega a\left[\frac{r}{a} + 2\sum_{k=1}^{\infty}\frac{1}{\lambda_k J_0(\lambda_k)}\exp\left(-\lambda_k^2\frac{\nu t}{a^2}\right)J_1\left(\lambda_k\frac{r}{a}\right)\right],$$

where $J_0(\lambda)$ and $J_1(\lambda)$ are the Bessel functions of the first kind and the λ_k are positive roots of the transcendental equation $J_1(\lambda) = 0$.

⊙ *Reference*: N. A. Slezkin (1955).

7°. *Transient fluid motion outside a rotating cylinder.* The transient motion of a fluid outside a rotating cylinder (in the region $r \ge a$) is described by equation (2) with $f(t) \equiv 0$, the initial and boundary conditions (6), and the additional condition at infinity $u_\varphi|_{r\to\infty} \to 0$.

The solution of the problem has the form

$$u_\varphi = \frac{\omega a^2}{r} + \frac{2\omega a}{\pi}\int_0^\infty \exp(-\nu\beta^2 t)F(r,\beta)\,d\beta,$$
$$F(r,\beta) = \frac{J_1(r\beta)Y_1(a\beta) - J_1(a\beta)Y_1(r\beta)}{\beta\left[J_1^2(a\beta) + Y_1^2(a\beta)\right]}.$$

It should be noted that the spectrum of eigenvalues of this problem is continuous, $0 \le \beta < \infty$.

⊙ *Reference*: P. A. Lagerstrom (1996).

8°. *Transient fluid motion between rotating cylinders.* The transient motion of an initially stagnant fluid in the gap between two coaxial cylinders of radii a and b ($a < b$) that suddenly begin to rotate at angular velocities ω_a and ω_b is described by equation (2) with $f(t) \equiv 0$ and the initial and boundary conditions

$$u_\varphi = 0 \quad \text{at} \quad t = 0, \qquad u_\varphi = a\omega_a \quad \text{at} \quad r = a, \qquad u_\varphi = b\omega_b \quad \text{at} \quad r = b.$$

The solution of the problem has the form

$$\frac{u_\varphi}{b(\omega_b - \omega_a)} = \frac{1-\omega(1-q^2)}{1-q^2}\frac{r}{b} - \frac{q^2}{1-q^2}\frac{b}{r}$$
$$- \pi\sum_{k=1}^{\infty} A_k \exp\left(-\frac{\nu\lambda_k^2 t}{b^2}\right)\left[Y_1(q\lambda_k)J_1\left(\lambda_k\frac{r}{b}\right) - J_1(q\lambda_k)Y_1\left(\lambda_k\frac{r}{b}\right)\right], \qquad (7)$$
$$q = \frac{a}{b}, \quad \omega = \frac{\omega_a}{\omega_a - \omega_b}, \quad A_k = J_1(\lambda_k)\frac{(1-\omega)J_1(q\lambda_k) + q\omega J_1(\lambda_k)}{J_1^2(q\lambda_k) - J_1^2(\lambda_k)},$$

where $J_1(\lambda)$ and $Y_1(\lambda)$ are respectively the Bessel functions of the first and second kind and the λ_k are real roots of the transcendental equation

$$J_1(\lambda)Y_1(q\lambda) - J_1(q\lambda)Y_1(\lambda) = 0;$$

these roots are tabulated in Bird and Curtiss (1959).

For small values of t, we have the asymptotic expression

$$u_\varphi = \sqrt{\frac{b}{r}}\left\{a\omega_a\sqrt{q}\left[\operatorname{erfc}\left(\frac{r-bq}{2\sqrt{\nu t}}\right) - \operatorname{erfc}\left(\frac{2b-bq-r}{2\sqrt{\nu t}}\right)\right]\right.$$
$$\left. + b\omega_b\left[\operatorname{erfc}\left(\frac{b-r}{2\sqrt{\nu t}}\right) - \operatorname{erfc}\left(\frac{b-bq+r}{2\sqrt{\nu t}}\right)\right]\right\}.$$

The first two terms on the right-hand side in (7) determine the steady-state solution (4) obtained by passing to the limit as $t \to \infty$.

⊙ *Reference*: R. B. Bird and C. F. Curtiss (1959).

9°. *Torsional oscillations of a cylinder.* A solution that describes the fluid flow around an impermeable circular cylinder of radius a that performs torsional oscillations with the circumferential velocity $u_\varphi|_{r=a} = Ve^{i\omega t}$ is given by

$$u_\varphi = V\frac{K_1\big(r\sqrt{i\omega/\nu}\,\big)}{K_1\big(a\sqrt{i\omega/\nu}\,\big)}\,e^{i\omega t}, \quad u_r = u_z = 0,$$

where $K_1(\zeta)$ is the modified Bessel function and $i^2 = -1$.

The following asymptotic formula holds at large radial distances from the cylinder:

$$u \sim Cr^{-1/2}\exp\Big\{-\sqrt{\omega/(2\nu)}\,r + i\big[\omega t - \sqrt{\omega/(2\nu)}\,r\big]\Big\} \quad \text{as} \quad r \to \infty.$$

⊙ *Reference*: P. G. Drazin and N. Riley (2006, p. 142).

21.2.4. Purely Radial Fluid Motions

Steady-state flows.

1°. *The simplest plane purely radial fluid motion* is described by the formulas

$$u_r = \frac{C}{r}, \quad u_\varphi = u_z = 0, \quad p = p_0 - \frac{C^2}{2r^2}, \quad r = \sqrt{x^2+y^2},$$

where C and p_0 are arbitrary constants.

2°. *The Jeffery–Hamel flow between nonparallel walls.* The Navier–Stokes equations admit solutions of the form

$$u_r = \frac{\nu}{r}f(\varphi), \quad u_\varphi = u_z = 0, \quad p = \frac{\nu^2}{r^2}[2f(\varphi) - C_1],$$

where C_1 is an arbitrary constant and the function $f = f(\varphi)$ is determined by the ordinary differential equation

$$f''_{\varphi\varphi} + f^2 + 4f - 2C_1 = 0. \tag{1}$$

The equation has a simple solution of the form $f = \text{const}$. The general solution of equation (1) can be represented in the implicit form

$$\varphi = \pm\int \frac{df}{\sqrt{-\frac{2}{3}f^3 - 4f^2 + 4C_1 f + C_2}} + C_3,$$

where C_2 and C_3 are arbitrary constants.

Special case 1. Consider the flow between nonparallel plane walls of included angle 2α. The walls intersect at the origin where there is a source or sink leading to either a diverging or converging flow. In problems with stationary impermeable walls, the function f must satisfy the no-slip conditions

$$f(\alpha) = f(-\alpha) = 0, \tag{2}$$

which are supplemented by prescribing the flow rate of the fluid:

$$\int_{-\alpha}^{\alpha} f(\varphi)\,d\varphi = \mathrm{Re}, \tag{3}$$

where $\mathrm{Re} = Q/\nu$ is the Reynolds number and Q is the integral flow rate of the fluid.

The *Jeffery–Hamel problem* consists in solving problem (1)–(3), in which the constant C is to be determined. The parameter Re can assume positive or negative values. The former case corresponds to a fluid *flow in a diffuser*, and the latter case corresponds to a *flow in a confuser*.

It has been found that, for $\mathrm{Re} < 0$, there can exist both symmetric and nonsymmetric multi-modal solutions of the Jeffery–Hamel problem (also including small angles α for sufficiently large values of the product $\alpha\,\mathrm{Re}$). It is noteworthy that, for $\mathrm{Re} > 0$, the solution of problem (1)–(3) can be nonunique.

A detailed analysis of the solution to problem (1)–(3) as well as various generalizations of the problem can be found in the literature cited below.

⊙ *References*: G. B. Jeffery (1915), G. Hamel (1916), L. Rosenhead (1940), K. Millsaps and K. Pohlhausen (1953), R. Berker (1963), G. B. Whitham (1963), M. A. Goldshtik and V. N. Shtern (1989), V. Putkaradze and P. Dimon (2000), L. D. Akulenko, S. A. Kumakshev, and S. V. Nesterov (2002), L. D. Akulenko and S. A. Kumakshev (2004), L. D. Akulenko, S. A. Kumakshev, and D. V. Georgievskii (2004), V. V. Pukhnachov (2006), P. G. Drazin and N. Riley (2006).

Special case 2. Consider equation (1) with $C_1 = 0$ in the domain $0 < \varphi < \beta$. Supplement the equation with the boundary conditions

$$f(0) = f'_\varphi(\beta) = 0 \tag{4}$$

and the condition for the flow rate

$$\int_0^\beta f(\varphi)\,d\varphi = \mathrm{Re}. \tag{5}$$

Problem (1), (4), (5) with $C_1 = 0$ describes a steady-state flow in the sector $0 \le \varphi \le \beta$, where $\varphi = 0$ corresponds to a solid wall and $\varphi = \beta$ corresponds to a free boundary with no surface tension. In the problem, the quantity β is unknown.

For $\mathrm{Re} > \mathrm{Re}_1 \approx -22.65$, the problem can have one to four solutions, depending on the value of Re. For $\mathrm{Re} > \mathrm{Re}_2 \approx 1.66$, there is a unique solution, with $f > 0$ for $0 < \varphi \le \beta$.

⊙ *Reference*: V. V. Pukhnachov (2006).

Special case 3. Equation (1) with $C_1 = 0$ and the boundary conditions

$$f'_\varphi(0) = f'_\varphi(\beta) = 0$$

as well as the flow rate condition (5) describes a steady-state flow in the sector $0 \le \varphi \le \beta$ with two free boundaries. In this problem, β is also unknown.

It turns out that the problem has at least one solution within the range $-8\pi < \mathrm{Re} \le 0$ and does not have solutions for $\mathrm{Re} \le -8\pi$ and $\mathrm{Re} > 0$. For low negative Re, the problem has seven solutions; as $|\mathrm{Re}|$ increases, solutions disappear in pairs until there is only one solution, $f \approx -4$, for Re close to -8π (this is a unique solution, in which $f(\varphi)$ does not change sign).

⊙ *References*: V. V. Pukhnachov (1971, 1972, 2006), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998).

3°. *Spherical radially symmetric fluid motion.* There are solutions that describe spherically symmetric fluid motions, which depend on the radial coordinate alone:

$$u_R = \frac{C}{R^2}, \quad u_\theta = u_\varphi = 0, \quad p = p_0 - \frac{C^2}{2R^4}, \quad R = \sqrt{x^2 + y^2 + z^2},$$

where C and p_0 are arbitrary constants.

Unsteady flows.

4°. *Plane radially symmetric unsteady fluid motion.* There are solutions that describe plane fluid motions dependent on the radial coordinate alone:

$$u_r = \frac{a(t)}{r}, \quad u_\varphi = u_z = 0, \quad p = p_0(t) - a'_t(t)\ln r - \frac{a^2(t)}{2r^2}, \quad r = \sqrt{x^2 + y^2},$$

where $a(t)$ and $p_0(t)$ are arbitrary functions.

5°. *Spherical radially symmetric unsteady fluid motion.* There are solutions that describe spherically symmetric fluid motions dependent on the radial coordinate alone:

$$u_R = \frac{\varphi(t)}{R^2}, \quad u_\theta = u_\varphi = 0, \quad p = \frac{\varphi_t'(t)}{R} - \frac{\varphi^2(t)}{2R^4} + p_\infty(t), \quad R = \sqrt{x^2 + y^2 + z^2}, \tag{6}$$

where $\varphi(t)$ and $p_\infty(t)$ are arbitrary functions.

Special case. *Rayleigh–Zababakhin–Plesset equation*. The above formulas can be used for describing the dynamics of the boundary of a spherical bubble in a viscous fluid. Suppose that the bubble boundary varies according to a law $R = a(t)$, so that the bubble pulsates in the surrounding fluid as the pressure at infinity $p_\infty(t)$ changes.

If there is no mass transfer across the bubble boundary, the fluid velocity on the boundary is equal to the velocity of the boundary,

$$u_R = a_t' \quad \text{at} \quad R = a. \tag{7}$$

Substituting the expression of the radial velocity component (6) into (7), we get $\varphi(t) = a^2 a_t'$ and then

$$u_R = \frac{a^2 a_t'}{R^2}, \quad p = \frac{a^2 a_{tt}'' + 2a(a_t')^2}{R} - \frac{a^4 (a_t')^2}{2R^4} + p_\infty(t). \tag{8}$$

Substituting $R = a$ into (8), we have

$$p|_{R=a} = a a_{tt}'' + \tfrac{3}{2}(a_t')^2 + p_\infty(t). \tag{9}$$

The fluid pressure $p|_{R=a}$ at the bubble boundary can be determined from the condition on the jump of normal stresses on the discontinuity surface (see Barenblatt and Chernyi (1963) and Sedov (1973)), that is, the bubble boundary. Under the homobaric assumptions, the gas in the bubble does not move, which implies that

$$p = p_{\mathrm{b}} - \frac{2\sigma}{a} + 2\nu \frac{\partial u_R}{\partial R} \quad \text{at} \quad R = a, \tag{10}$$

where σ is the ratio of the surface tension coefficient at the bubble boundary to the fluid density and p_{b} is the pressure inside the bubble.

With (8)–(10), we obtain the *Rayleigh–Zababakhin–Plesset equation*

$$a a_{tt}'' + \frac{3}{2}(a_t')^2 + 4\nu \frac{a_t'}{a} + \frac{2\sigma}{a} = p_{\mathrm{b}} - p_\infty(t), \tag{11}$$

which describes the dynamics of the bubble radius variation under the action of pressure that varies at infinity. The initial conditions for this equation are usually posed as

$$a = a_0, \quad a_t' = 0 \quad \text{at} \quad t = 0. \tag{12}$$

If the gas in the bubble expands and contracts adiabatically, then the gas pressure in the bubble is related to the initial pressure p_0 by the adiabatic equation

$$p_{\mathrm{b}} = p_0 \Big(\frac{a_0}{a} \Big)^{3\gamma}, \tag{13}$$

where γ is the adiabatic exponent.

In Zaitsev and Polyanin (1992, 1994), it is shown that problem (11)–(13) with $p_\infty(t) = \text{const}$ can be solved in terms of integrals for $\gamma = 1,\ \frac{2}{3},\ \frac{5}{6}$ and can be reduced to the Bessel equation for $\gamma = \frac{11}{12},\ \frac{7}{6}$.

Remark. The shorter name Rayleigh–Plesset equation for equation (11), which is prevalent in the literature, poorly reflects the history of the issue.

⊙ *References*: E. I. Zababakhin (1960), A. D. Pernik (1966), M. S. Plesset and A. Prosperetti (1977), A. Prosperetti (1982), L. I. Sedov (1984), R. I. Nigmatulin (1987), C. E. Brennen (2005).

21.3. Solutions with Two Nonzero Components of the Fluid Velocity

21.3.1. Two-Dimensional Solutions in the Rectangular Cartesian Coordinates (Plane Flows)

Steady-state solutions.

1°. Solution:

$$u = \frac{A}{v_0}y + B\exp\left(\frac{v_0}{\nu}y\right) + C, \quad v = v_0, \quad p = p_0 - Ax, \tag{1}$$

where A, B, C, p_0, and v_0 are arbitrary constants ($v_0 \neq 0$).

Special case 1. For the boundary conditions

$$u = 0 \quad \text{at} \quad y = 0, \qquad u = u_0 \quad \text{at} \quad y = h,$$

one should set

$$B = -C = \frac{u_0 - (Ah/v_0)}{\exp(v_0 h/\nu) - 1} \tag{2}$$

in solution (1). Solution (2) describes the flow in a plane tube with parallel and porous walls $y = 0$ and $y = h$ with a constant pressure gradient along the tube and with a constant transverse velocity v_0, with the upper wall $y = h$ moving with a constant velocity u_0 and the lower wall being stationary.

In the limit as $v_0 \to 0$, when the boundaries are impermeable, from (1)–(2) we recover the *classical Couette–Poiseuille solution*

$$u = u_0\frac{y}{h} + \frac{Ah^2}{2\nu}\frac{y}{h}\left(1 - \frac{y}{h}\right), \quad v = 0, \quad p = p_0 - Ax.$$

Special case 2. For $A = 0$, $B = -U$, and $C = U$, solution (1) corresponds to a uniform flow over an infinite flat porous plate through the surface of which fluid is injected (for $v_0 > 0$) or sucked away (for $v_0 < 0$).

⊙ *References*: J. L. M. Poiseuille (1840), M. Couette (1890).

2°. Solution:

$$u = k(Ax + B)e^{-ky}, \quad v = Ae^{-ky} - k\nu, \quad p = \tfrac{1}{2}A^2e^{-2ky} + p_0,$$

where A, B, k, and p_0 are arbitrary constants.

Special case 3. Setting $A = U$, $B = 0$, and $k = U/\nu$, we obtain the solution

$$u = (U^2/\nu)x\exp(-Uy/\nu), \quad v = U[\exp(-Uy/\nu) - 1],$$

that describes the steady-state motion of a fluid due to the motion of the surface particles at $y = 0$ with a velocity $u|_{y=0} = (U^2/\nu)x$. A possible interpretation of the solution includes: the fluid flow is due to stretching of the flat surface.

⊙ *References*: D. Riabouchinsky (1924), J. T. Stuart (1966 *a*), L. J. Crane (1970), J. E. Danberg and K. S. Fansler (1976), C. Y. Wang (1984).

3°. Solution:

$$u = Ae^{-\lambda y} - \nu\lambda, \quad v = Be^{-\lambda x} - \nu\lambda, \quad p = -ABe^{-\lambda(x+y)} + p_0,$$

where A, B, λ, and p_0 are arbitrary constants.

This solution describes a steady-state fluid motion in the quadrant $x \geq x_0$, $y \geq y_0$ with

$$x_0 = -\frac{1}{\lambda}\ln\frac{\nu\lambda}{A}, \quad y_0 = -\frac{1}{\lambda}\ln\frac{\nu\lambda}{B} \qquad (A\lambda > 0, \ \ B\lambda > 0).$$

At the porous boundaries of the quadrant, fluid is injected (or sucked away) in the normal direction to the surface.

⊙ *References*: R. Berker (1963, Section 15), A. D. Polyanin (2001 *c*).

4°. Solution:

$$u = Ae^{-\lambda y} + \nu\lambda, \quad v = Be^{\lambda x} - \nu\lambda, \quad p = -ABe^{\lambda(x-y)} + p_0,$$

where A, B, λ, and p_0 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

5°. Solution:

$$u = \lambda(Ae^{\beta x} + Be^{-\beta x})e^{-\lambda y}, \quad v = \beta(Ae^{\beta x} - Be^{-\beta x})e^{-\lambda y} - \frac{\nu}{\lambda}(\beta^2 + \lambda^2), \quad p = 2AB\beta^2 e^{-2\lambda y} + p_0,$$

where A, B, β, λ, and p_0 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

6°. Solutions:

$$u = A\lambda e^{\lambda y + \beta x} + \nu\gamma, \quad v = -A\beta e^{\lambda y + \beta x} - B\gamma e^{\gamma x} + \frac{\nu}{\lambda}\gamma(\gamma - \beta),$$
$$p = AB\gamma(\beta - \gamma)e^{\lambda y + (\beta+\gamma)x} + p_0, \quad \gamma = \pm\sqrt{\beta^2 + \lambda^2},$$

where A, B, β, λ, and p_0 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

7°. Periodic solution in x:

$$u = A\lambda \sin(\beta x + B)e^{-\lambda y}, \quad v = A\beta \cos(\beta x + B)e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2 - \lambda^2), \quad p = -\frac{1}{2}A^2\beta^2 e^{-2\lambda y} + p_0,$$

where A, B, β, λ, and p_0 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

8°. Solution:

$$u = u(x), \quad v = -yu'_x, \quad p = p_0 - \tfrac{1}{2}k^2y^2 - \tfrac{1}{2}u^2 + \nu u'_x, \tag{3}$$

where k and p_0 are arbitrary constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$\nu u'''_{xxx} - uu''_{xx} + (u'_x)^2 - k^2 = 0. \tag{4}$$

Special case 1. The *Hiemenz problem.* If u satisfies the boundary conditions

$$u(0) = u'_x(0) = 0, \quad u'_x(\infty) = -k \qquad (k > 0), \tag{5}$$

then solution (3)–(5) describes a steady-state flow of a viscous fluid in the half-plane $x > 0$ with a critical point $x = y = 0$. If there is no viscosity ($\nu = 0$), the velocity field is potential: $u = -kx$ and $v = kx$. The effect of viscosity results in the appearance of a boundary layer near the wall; the effective thickness of the boundary layer (the value $x = \delta$ at which $v = 0.99\,ky$) is independent of y and equal to $\delta = 2.4\sqrt{\nu/k}$.

Special case 2. If one sets $k = 0$ in (3) and (4) and imposes the boundary conditions

$$u(0) = u''_{xx}(0) = 0, \quad u'_x(\infty) = -m \qquad (m > 0), \tag{6}$$

then solution (3), (4), (6) will describe a steady-sate flow of a viscous fluid near the critical point at the free boundary.

Special case 3. If one sets $k = 0$ in equation (4) and imposes the boundary conditions

$$u(0) = 0, \quad u'_x(0) = -m, \quad u'_x(\infty) = 0 \qquad (m > 0), \tag{7}$$

then the solution of problem is given by

$$u = A[\exp(-mx/A) - 1], \quad A = \sqrt{-\nu m}. \tag{8}$$

Solution (3), (8) with $k = 0$ models the extrusion process in manufacturing sheet products.

⊙ *References*: K. Hiemenz (1911), L. J. Crane (1970), C. Y. Wang (1991), V. V. Pukhnachov (2006).

Unsteady solutions.

9°. Solution:

$$u = U_0 e^{-ay}\cos(\omega t + by), \qquad v = -V_0, \qquad p = p_0,$$

where U_0, V_0, p_0, and ω are arbitrary constants, and

$$a = \left(\frac{\omega}{2\nu}\right)^{1/2}\left\{\left[1 + \left(\frac{V_0^2}{4\omega\nu}\right)^2\right]^{1/2} - \frac{V_0^2}{4\omega\nu}\right\}^{-1/2} + \frac{V_0}{2\nu},$$

$$b = -\left(\frac{\omega}{2\nu}\right)^{1/2}\left\{\left[1 + \left(\frac{V_0^2}{4\omega\nu}\right)^2\right]^{1/2} - \frac{V_0^2}{4\omega\nu}\right\}^{1/2}.$$

The solution describes *oscillatory motion of a flat porous plate*, where V_0 is the suction velocity. The longitudinal fluid velocity component satisfies the boundary conditions $u|_{y=0} = U_0\cos(\omega t)$ and $u|_{y\to\infty} \to 0$.

⊙ *References*: J. T. Stuart (1955), W. R. Debler and R. D. Montgomery (1971).

10°. Solution:

$$u = U_1(1 - e^{-V_0 y/\nu}) + U_0[\cos(\omega t) - e^{-ay}\cos(\omega t + by)], \quad v = -V_0, \quad p = p_0 + U_0\omega\sin(\omega t)x,$$

where U_1, U_0, V_0, p_0, and ω are arbitrary constants, and the constants a and b are the same as in the previous solution from Item 9°.

⊙ *Reference*: J. T. Stuart (1955).

11°. Solution (generalizes the previous solutions from Items 9° and 10°):

$$u = u(y, t), \quad v = v(t), \quad p = -f(t)x - v'_t(t)y + g(t),$$

where $f(t)$, $g(t)$, and $v(t)$ are arbitrary functions, and the function $u = u(y, t)$ is a solution of the linear partial differential equation

$$\frac{\partial u}{\partial t} + v(t)\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + f(t).$$

The transformation

$$u = U(\xi, t) + \int f(t)\,dt, \qquad \xi = y - \int v(t)\,dt$$

leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial \xi^2}.$$

Special case 1. Solution:

$$u = \frac{ah}{V}\left(\frac{e^{\mathrm{Re}\,\eta} - 1}{e^{\mathrm{Re}} - 1} - \eta\right) + \frac{ibh^2}{\nu M^2}\left[1 + \frac{(1 - e^{\lambda_2})e^{\lambda_1\eta} - (1 - e^{\lambda_1})e^{\lambda_2\eta}}{e^{\lambda_2} - e^{\lambda_1}}\right]e^{i\omega t}, \quad v = V, \quad p = p_0 + (a + be^{i\omega t})x,$$

$$\eta = \frac{y}{h}, \quad \mathrm{Re} = \frac{Vh}{\nu}, \quad M = \frac{\omega h^2}{\nu}, \quad \lambda_{1,2} = \frac{1}{2}\left(\mathrm{Re} \pm \sqrt{\mathrm{Re}^2 + 4iM^2}\right), \quad i^2 = -1.$$

This solution describes a flow in a plane channel with fixed, but porous walls at a distance h apart, across which fluid is drawn with a uniform speed V and between which there is a fluctuating pressure gradient $p_x = a + be^{i\omega t}$. In order to obtain the solution corresponding to the pressure gradient $p_x = a + b\cos\omega t$, one should take the real part of the velocity component u.

⊙ *Reference*: C. Y. Wang (1971).

Special case 2. Solution:

$$u = U_0\left[1 + g(t) - e^{-\eta} - \frac{1}{4\sqrt{\pi}}\eta e^{-\eta/2}\int_0^\infty g(\tau - s)\exp\left(-s - \frac{\eta^2}{16s}\right)\frac{ds}{s^{3/2}}\right], \quad v = -V_0, \quad \eta = \frac{V_0 y}{\nu}, \quad \tau = \frac{V_0^2 t}{\nu},$$

with uniform suction, $v = -V_0$, and a free stream, $U(t) = U_0[1 + g(t)]$ at $y \to \infty$.

⊙ *References*: H. Hasimoto (1957), J. Watson (1958), R. E. Kelly (1965).

Special case 3. Solution:

$$u = \sqrt{\frac{\nu}{t}}\,\exp\left(-\tfrac{1}{4}Y^2\right)\left[C_2\int \exp\left(\tfrac{1}{4}Y^2\right)dY + C_3\right],$$
$$v = C_1\sqrt{\frac{\nu}{t}}, \quad p = \frac{C_1\sqrt{\nu}\,y}{t\sqrt{t}} + \frac{C_4\nu}{t}, \quad Y = \frac{y}{\sqrt{\nu t}} - 2C_1,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

⊙ *Reference*: V. V. Pukhnachov (1960).

12°. Solution:

$$u = \frac{x}{t} + C_2 f_1(\xi) + C_3 f_2(\xi), \quad v = -\frac{y}{t} + \frac{C_1}{\sqrt{t}},$$
$$p = \frac{1}{t}\left(-\frac{y^2}{2t} + \frac{3C_1 y}{2\sqrt{t}} + C_4\right), \quad \xi = \frac{y}{\sqrt{\nu t}}.$$

Here $f_1(\xi)$ and $f_2(\xi)$ are any two linearly independent solutions of the ordinary differential equation

$$f''_{\xi\xi} + \left(\tfrac{3}{2}\xi - C_1\right)f'_\xi - f = 0,$$

and C_1, C_2, C_3, and C_4 are arbitrary constants.

⊙ *Reference*: V. V. Pukhnachov (1960).

13°. Solution:

$$u = \frac{x}{2t} + \sqrt{\frac{\nu}{t}}\,f(\eta), \quad v = -\frac{y}{2t} + A\sqrt{\frac{\nu}{t}},$$
$$p = \frac{1}{t}\left[\frac{x^2 - 3y^2}{8t} + \sqrt{\frac{\nu}{t}}\,(Bx + Ay) + C_1\right],$$

where

$$\eta = \frac{y}{\sqrt{\nu t}} - A, \quad f(\eta) = C_2\int_0^\eta \exp\left(-\tfrac{1}{2}z^2\right)dz + B\int_0^\eta \exp\left(-\tfrac{1}{2}z^2\right)\left[\int_z^\eta \exp\left(-\tfrac{1}{2}\zeta^2\right)d\zeta\right]dz + C_3,$$

and A, B, C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: V. V. Pukhnachov (1960).

14°. Solution (generalizes the previous solutions from Items 12° and 13°):

$$\begin{aligned} &u = a(t)x + f(y,t), \quad v = -a(t)y + b(t),\\ &p = -\tfrac{1}{2}[a'_t(t) + a^2(t)]x^2 + \tfrac{1}{2}[a'_t(t) - a^2(t)]y^2 + c(t)x + [a(t)b(t) - b'_t(t)]y + p_0(t), \end{aligned} \tag{9}$$

where $a(t)$, $b(t)$, and $p_0(t)$ are arbitrary functions, and the function $f = f(y,t)$ is determined by the linear partial differential equation

$$f_t + [b(t) - a(t)y]f_y + a(t)f + c(t) = \nu f_{yy}. \tag{10}$$

The transformation

$$f = \frac{1}{A(t)}\left(C_1 F(z,\tau) - \int A(t)c(t)\,dt + C_2\right), \quad A(t) = C_3 \exp\left(\int a(t)\,dt\right),$$
$$\tau = \int A^2(t)\,dt + C_4, \quad z = A(t)y - \int A(t)b(t)\,dt + C_5,$$

where C_1, C_2, C_3, C_4, and C_5 are arbitrary constants ($C_1 \neq 0$ and $C_3 \neq 0$), leads (10) to the classical linear heat equation

$$F_\tau = \nu F_{zz}.$$

Special case. With $b(t) = c(t) \equiv 0$, $C_1 = C$, $C_2 = C_4 = C_5 = 0$, and $C_3 = 1$, we have, for example, the following particular solutions of equation (2):

$$f = \frac{C}{A(t)} \exp(-\nu\mu^2\tau)\cos[\mu A(t)y],$$
$$f = \frac{C}{A(t)} \exp(-\nu\mu^2\tau)\sin[\mu A(t)y],$$
$$f = \frac{C}{A(t)\sqrt{\tau}} \exp\left(\frac{A^2(t)y^2}{4\nu\tau}\right),$$
$$f = \frac{C}{A(t)} \operatorname{erf}\left(\frac{A(t)y}{2\sqrt{\nu\tau}}\right),$$

where $A(t) = \exp\left(\int_0^t a(t)\,dt\right)$ and $\tau = \int_0^t A^2(t)\,dt$.

The case $b(t) = c(t) \equiv 0$ may be interpreted as superposed shear flows; see Kambe (1983).

15°. Solution:

$$u = \frac{ax}{2(1-at)} f'_\eta(\eta), \quad v = -\frac{ah}{2\sqrt{1-at}} f(\eta), \quad \eta = \frac{y}{h\sqrt{1-at}},$$
$$p = p_0 + \frac{ba^2x^2}{8(1-at)^2} - \frac{a^2h^2}{8(1-at)}\left[f^2(\eta) - 2\eta f(\eta) + \frac{4\operatorname{sign} a}{M^2} f'_\eta(\eta)\right], \quad M^2 = \frac{|a|h^2}{\nu}, \tag{11}$$

where a, b, h, and p_0 are arbitrary constants, and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$2M^{-2} \operatorname{sign} a\, f'''_{\eta\eta\eta} + (f-\eta) f''_{\eta\eta} - (f'_\eta)^2 - 2f'_\eta = b. \tag{12}$$

If one supplements equation (12) with the boundary conditions

$$f(0) = f''_{\eta\eta}(0) = 0, \quad f(1) = 1, \quad f'_\eta(1) = 0, \tag{13}$$

the formulas (11) together with (12) and (13) will describe a special mode of squeezing of a fluid between two plates (it is a model for unsteady loading of mechanical parts as, for example, in thrust bearings). In this case, the constant b is unknown and has to be determined in the course of solving problem (12)–(13).

⊙ *References*: C. Y. Wang (1976), E. C. Dauenhauer and J. Majdalani (2003).

16°. Solution:

$$u = \sqrt{\frac{\nu}{t}}\, f(\xi,\eta) + \frac{x}{2t}, \quad v = \sqrt{\frac{\nu}{t}}\, g(\xi,\eta) + \frac{y}{2t},$$
$$p = \frac{\nu}{t} h(\xi,\eta) + \frac{x^2+y^2}{8t^2}, \quad \xi = \frac{x}{\sqrt{\nu t}}, \quad \eta = \frac{y}{\sqrt{\nu t}},$$

where the functions $f = f(\xi,\eta)$, $g = g(\xi,\eta)$, and $h = h(\xi,\eta)$ are determined by the system of differential equations

$$f f_\xi + g h_\eta = -h_\xi + f_{\xi\xi} + f_{\eta\eta},$$
$$f g_\xi + g g_\eta = -h_\eta + g_{\xi\xi} + g_{\eta\eta},$$
$$f_\xi + g_\eta = -1.$$

⊙ *Reference*: V. V. Pukhnachov (1960), N. N. Ibragimov (1995).

17°. Solution:

$$\begin{aligned} &u = u(\xi,t), \quad v = a_t'(t)x + a(t)u(\xi,t), \quad \xi = y - a(t)x, \\ &p = \tfrac{1}{2}a(t)a_{tt}''(t)x^2 - a_{tt}''(t)xy + \psi(\xi,t), \end{aligned} \tag{14}$$

where $a = a(t)$ is an arbitrary function and the function $u = u(\xi,t)$ is determined by the linear partial differential equation

$$u_t + \frac{2aa_t'}{a^2+1}u - \frac{a_{tt}''}{a^2+1}\xi = \nu(a^2+1)u_{\xi\xi}. \tag{15}$$

The function $\psi = \psi(\xi,t)$, appearing in the formula (14) for the pressure, is expressed in terms of u as

$$\psi(\xi,t) = -\frac{aa_{tt}''}{2(a^2+1)}\xi^2 - \frac{2a_t'}{a^2+1}\int u(\xi,t)\,d\xi.$$

The transformation

$$U(\xi,\tau) = [1+a^2(t)]u(\xi,t) - a_t'(t)\xi, \qquad \tau = \nu\int [1+a^2(t)]\,dt,$$

leads (15) to the classical linear heat equation

$$U_\tau = U_{\xi\xi}.$$

⊙ *References*: V. O. Bytev (1972).

18°. Solution:

$$\begin{aligned} &u = t + f(\xi,t), \quad v = af(\xi,t) + g(t), \quad \xi = a\big(x - \tfrac{1}{2}t^2\big) - y, \\ &p = h(t)x + [ah(t) + a - g_t'(t)]y + p_0(t), \end{aligned} \tag{16}$$

where $g(t)$, $h(t)$, and $p_0(t)$ are arbitrary functions, a is an arbitrary constant, and the function $f = f(\xi,t)$ is a solution of the linear partial differential equation

$$\frac{\partial f}{\partial t} - g(t)\frac{\partial f}{\partial \xi} + h(t) + 1 = \nu(a^2+1)\frac{\partial^2 f}{\partial \xi^2}. \tag{17}$$

The transformation

$$f = U(\eta,\tau) - t - \int h(t)\,dt, \quad \eta = \xi + \int g(t)\,dt, \quad \tau = \nu(a^2+1)t,$$

brings (17) to the classical linear heat equation

$$U_\tau = U_{\eta\eta}.$$

Remark. With $f(\xi,t) = f(\xi)$, $g(t) = b = \text{const}$, and $h(t) = c = \text{const}$, solution (16) becomes the solution obtained by Pukhnachov (1960). In this case, the general solution of equation (17) has the form

$$f = \begin{cases} \dfrac{c+1}{2\nu(a^2+1)}\xi^2 + C_1\xi + C_2 & \text{if } b = 0, \\ \dfrac{c+1}{b}\xi + C_1\exp\left(-\dfrac{b\xi}{\nu(a^2+1)}\right) + C_2 & \text{if } b \neq 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants and $\xi = a\big(x - \frac{1}{2}t^2\big) - y$.

19°. Solution:

$$u = u(x,t), \qquad v = -y\frac{\partial u}{\partial x} + f(x,t),$$
$$p = \frac{1}{2}a(t)y^2 + b(t)y + \nu\frac{\partial u}{\partial x} - \frac{1}{2}u^2 - \frac{\partial}{\partial t}\int_{x_0}^{x} u(\xi,t)\,d\xi + p_0(t), \tag{18}$$

where $a(t)$, $b(t)$, and $p_0(t)$ are arbitrary functions, and the functions $u = u(x,t)$ and $f = f(x,t)$ are determined by the system of differential equations

$$u_{xt} + uu_{xx} - u_x^2 = \nu u_{xxx} + a(t), \tag{19}$$
$$f_t + uf_x - u_x f = \nu f_{xx} - b(t). \tag{20}$$

Equation (18) is solved independently. Equation (19) depends on u an is linear in f.

Exact solutions of equation (19). For $a(t) = a_0 = \text{const}$, equation (19) admits a traveling-wave solution of the form

$$u = u(z), \quad z = x - \lambda t; \qquad -(u_z')^2 + (u-\lambda)u_{zz}'' = \nu u_{zzz}''' + a_0,$$

where λ is an arbitrary constant.

For $a(t) = a_0(t+c)^{-2}$, equation (19) admits a self-similar solution of the form

$$u = (t+c)^{-1/2}U(\zeta), \quad \zeta = x(t+c)^{-1/2}; \qquad -U_\zeta' - (U_\zeta')^2 + (U - \tfrac{1}{2}\zeta)U_{\zeta\zeta}'' = \nu U_{\zeta\zeta\zeta}''' + a_0.$$

The third-order equation (19) admits many other exact solutions and can be reduced to a second-order equation; for details, see 13.5.1.2 (where one should set $w = -u$) and 13.5.1.3 with $q(t) \equiv 0$.

Exact solutions of equation (20). Given an exact solution $u = u(x,t)$ of equation (18), the formula

$$f = A(t)u_x + A_t'(t),$$

where the function $A = A(t)$ satisfies the second-order linear ordinary differential equation

$$A_{tt}'' + a(t)A + b(t) = 0,$$

produces an exact solution equation to (20).

Two problems with a free boundary. From now on, we set $a(t) = b(t) = f(t) = 0$ in equations (19)–(20) and introduce the new variable

$$u = -\int_0^x U(\xi,t)\,d\xi \qquad [v = yU(x,t)].$$

As a result, we arrive at the integro-differential equation

$$U_t + U^2 - U_x\int_0^x U(\xi,t)\,d\xi = \nu U_{xx}. \tag{21}$$

Consider equation (21) in the range $0 < x < s(t)$, where $s(t)$ is an unknown function. We supplement equation (21) with the initial condition

$$U(x,0) = U_0(x), \qquad s(0) = s_0 > 0, \tag{22}$$

the condition at the free boundary

$$U_x(s(t),t)=0, \qquad \frac{ds}{dt}=-\int_0^{s(t)} U(\xi,t)\,d\xi, \tag{23}$$

and one of the following two conditions on the straight line $x=0$:

$$U_x(0,t)=0 \qquad \text{(problem A)}, \tag{24}$$

$$U(0,t)=0 \qquad \text{(problem B)}. \tag{25}$$

Problem A describes a symmetric deformation of a viscous fluid band with two free boundaries, $x=\pm s(t)$; problem B corresponds to the case where the line $x=0$ is a rigid wall. The condition $U_x=0$ at $x=s(t)$ ensures that there are no tangential stresses on the free boundary. The condition of vanishing of the normal stress can be ensured by a suitable choice of the function $p_0(t)$ in (18).

For some sufficiently general conditions on the function $U_0(x)$, there exists a unique solution to both problems A and B for $0<t<T$ and sufficiently small T.

Special case. Let us illustrate different possibilities for the evolution of the solution to problem (21)–(24) (problem A). To this end, let us consider the exact solution of the form

$$U=k(t)+m(t)\cos[\pi n x/s(t)], \tag{26}$$

where n is a positive integer and the functions $k=k(t)$, $m=m(t)$, and $s=s(t)$ solve the Cauchy problem for the system of ordinary differential equations

$$k_t'=-k^2-m^2, \quad m_t'=-[2k+\nu(\pi n/s)^2]m, \quad s_t'=-ks \tag{27}$$

with the initial conditions

$$k(0)=k_0, \quad m(0)=m_0, \quad s(0)=s_0. \tag{28}$$

Here k_0 and m_0 are arbitrary constants.

If $k_0>0$, there exists a solution of problem (27)–(28) for any $t>0$. If $k_0<0$ (or $k_0=0$ and $m_0\neq 0$), there exists a finite positive number t_* such that $s\to\infty$ and $k\to-\infty$ as $t\to t_*$.

⊙ *References*: V. V. Pukhnachov (1998, 1999, 2006), V. A. Galaktionov and J. L. Vazques (1999), A. D. Polyanin (2001 *c*), S. N. Aristov and A. D. Polyanin (2009, 2010), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

▷ *Some other two-dimensional steady-state and unsteady solutions of the Navier–Stokes equations in the rectangular Cartesian coordinates (plane flows) can be found in Subsections 14.3.2 (see Eq. 1) and 14.3.3.*

21.3.2. Two-Dimensional Solutions in the Cylindrical Coordinates (Plane Flows)

The fluid velocity components in the cylindrical (polar) coordinates are expressed in terms of those in the rectangular Cartesian coordinates as follows (the two-dimensional case with $u_z=0$ is considered):

$$u_r=u\cos\varphi+v\sin\varphi, \quad u_\varphi=v\cos\varphi-u\sin\varphi;$$

$$r=\sqrt{x^2+y^2}, \quad \tan\varphi=y/x \quad (\sin\varphi=y/r).$$

Steady-state solutions.

1°. Solution:

$$u_r = \frac{\nu k}{r}, \quad u_\varphi = \frac{A}{r} + Br^{k+1}, \quad p = -\frac{\nu^2 k^2 + A^2}{2r^2} + \frac{2AB}{k} r^k + \frac{B^2}{2(k+1)} r^{2(k+1)} + C, \quad (1)$$

where A, B, C, and k are arbitrary constants ($k \neq -2, -1, 0$).

In the exceptional case $k = -2$, the solution is

$$u_r = \frac{\nu k}{r}, \quad u_\varphi = \frac{A + B\ln r}{r}, \quad p = -\frac{\nu^2 k^2}{2r^2} + \int \frac{(A + B\ln r)^2}{r^3}\, dr + C.$$

Special case 1. For

$$A = \frac{a^2 b^2 (b^k \omega_a - a^k \omega_b)}{b^{k+2} - a^{k+2}}, \quad B = \frac{b^2 \omega_b - a^2 \omega_a}{b^{k+2} - a^{k+2}}, \quad k \neq -2,$$

solution (1) satisfies the boundary conditions

$$u_\varphi = a\omega_a \quad \text{at} \quad r = a, \qquad u_\varphi = b\omega_b \quad \text{at} \quad r = b,$$

which corresponds to a fluid flow in the gap between two coaxial cylinders rotating about their common axis with angular velocities ω_a and ω_b.

Special case 2. For

$$A = a^2 \omega_a - B a^{k+2}, \quad k < -1, \quad (2)$$

where B is an arbitrary constant, solution (1) satisfies the boundary conditions

$$u_\varphi = a\omega_a \quad \text{at} \quad r = a, \qquad u_\varphi \to 0 \quad \text{as} \quad r \to \infty, \quad (3)$$

which corresponds to a fluid flow around a porous circular cylinder rotating with an angular velocity ω_a. This case is an example of nonuniqueness of the steady-state solution satisfying the boundary conditions (3) provided that the fluid is sucked away ($k < -1$) through the porous surface sufficiently intensively.

⊙ *References*: D. E. Dolidze (1937), L. M. Hocking (1963).

2°. Solution:

$$u_r = \frac{C_1 \nu}{r}, \quad u_\varphi = \nu f(r), \quad p = A\nu^2 \varphi - \frac{C_1^2 \nu^2}{2r^2} + \nu^2 \int \frac{f^2(r)}{r}\, dr + C_2,$$

where A, C_1, and C_2 are arbitrary constants and

$$f(r) = \begin{cases} -\dfrac{A}{2C_1} r + \dfrac{C_3}{r} + C_4 r^{C_1+1} & \text{if } C_1 \neq 0 \text{ and } C_1 \neq -2, \\ \dfrac{A}{2} r \ln r + \dfrac{C_3}{r} + C_4 r & \text{if } C_1 = 0, \\ \dfrac{A}{4} r + \dfrac{C_3}{r} + C_4 \dfrac{\ln r}{r} & \text{if } C_1 = -2. \end{cases}$$

The constants C_3 and C_4 can be chosen so that, for given a and b, one has

$$f(a) = f(b) = 0.$$

This solution describes the flow in a channel formed by two arcs of concentric circles with radii a and b if the pressure is a linear function of the polar angle.

3°. Solution:

$$u_r = \frac{\nu}{r} f(\varphi), \qquad u_\varphi = \frac{\nu C_1}{r}, \qquad p = \frac{\nu^2}{r^2}[2f(\varphi) + C_2],$$

where C_1 and C_2 are arbitrary constants the function $f = f(\varphi)$ is determined by the ordinary differential equation

$$f''_{\varphi\varphi} - C_1 f'_\varphi + f^2 + 4f + C_1^2 + 2C_2 = 0.$$

This solution describes a fluid flow in a diffusor with two porous walls from one of which the fluid is winding while from the other one the fluid is drawing off with the normal velocity $C_1 r^{-1}$.

If $C_1 = 0$, we have the *Jeffery–Hamel solution* (see Subsection 21.2.4, Item 2°).

⊙ *Reference*: V. V. Pukhnachov (1960).

4°. The transformation

$$u_r = \frac{1}{r} f(\xi, \varphi), \quad u_\varphi = \frac{1}{r} g(\xi, \varphi), \quad p = \frac{1}{r^2} h(\xi, \varphi), \quad \xi = \ln r,$$

leads to the equations

$$\begin{aligned} f f_\xi + g f_\varphi - f^2 - g^2 &= 2h - h_\xi + \nu(f_{\xi\xi} + f_{\varphi\varphi}), \\ f g_\xi + g g_\varphi &= -h_\varphi + \nu(g_{\xi\xi} + g_{\varphi\varphi} - 2g_\xi + 2f_\varphi), \\ f_\xi + g_\varphi &= 0. \end{aligned} \tag{4}$$

Here the third equation has been used to simplify the right-hand side of the first one. Equations (4) do not contain the dependent variables explicitly and so admit a traveling-wave solution of the form

$$f = C_1 - a\Theta(\eta), \quad g = b\Theta(\eta), \quad h = h(\eta), \quad \eta = a\varphi + b\xi,$$

where a, b, and C_1 are arbitrary constants, and the functions $\Theta = \Theta(\eta)$ and $h(\eta)$ are determined by the system of ordinary differential equations

$$\begin{aligned} C_1 ab\Theta' + (C_1 - a\Theta)^2 + b^2\Theta^2 &= bh' - 2h + \nu a(a^2 + b^2)\Theta'', \\ C_1 b^2\Theta' &= -ah' + \nu(a^2 + b^2)(b\Theta'' - 2\Theta'). \end{aligned} \tag{5}$$

Integrating the second equation in (5), eliminating h from the first equation, and rearranging, we obtain

$$\begin{aligned} &\nu(a^2 + b^2)^2\Theta'' - b(a^2 + b^2)(C_1 + 4\nu)\Theta' \\ &\qquad - a(a^2 + b^2)\Theta^2 + 2(a^2 + b^2)(2\nu + C_1)\Theta - a(C_1^2 + 2C_2) = 0, \\ &h = \nu\frac{b}{a}(a^2 + b^2)\Theta' - \frac{1}{a}[2\nu(a^2 + b^2) + C_1 b^2]\Theta + C_2. \end{aligned}$$

The first equation serves to determine Θ and then h can be determined from the second equation. The first equation is integrable by quadrature in some cases, for example, if $C_1 = -4\nu$.

Unsteady solutions.

5°. Solution:

$$u_r = \frac{a\nu}{r}, \quad u_\varphi = \frac{b\nu}{r} f(\xi) + c, \quad p = \nu^2 \int \frac{f^2(\xi)}{r^3}\, dr - \frac{a^2\nu^2}{2r^2} + p_0(t), \quad \xi = \frac{r}{\sqrt{\nu t}},$$

where a, b, and c are arbitrary constants, $p_0(t)$ is an arbitrary function, and

$$f(\xi) = \int \xi^{a+1} \exp\left(-\tfrac{1}{4}\xi^2\right) d\xi,$$

When $p_0(t) = c = 0$, this solution describes the flow where a source of intensity $2\pi a\nu$ is located at the origin of the coordinate system and, at the moment $t = 0$, there exists a whirlwind with circulation $2\pi b\nu$ at the point $r = 0$.

⊙ *Reference*: V. V. Pukhnachov (1960).

6°. Solution:

$$u_r = \frac{a\nu}{r}, \quad u_\varphi = \frac{\nu}{r} f(\xi), \quad p = \frac{b\nu\varphi}{t} - \frac{a^2\nu^2}{2r^2} + \frac{\nu}{t} \int_b^\xi \frac{f^2(\theta)}{\theta}\, d\theta + p_0(t), \quad \xi = \frac{r}{\sqrt{\nu t}},$$

where

$$f(\xi) = C_1 \int_b^\xi z^{a+1} \exp\left(-\tfrac{1}{4}z^2\right) dz + C_2 + b \int_b^\xi z^{a+1} \exp\left(-\tfrac{1}{4}z^2\right) \left(\int_z^\xi \zeta^{-a-1} \exp\left(\tfrac{1}{4}\zeta^2\right) d\zeta \right) dz,$$

and a, b, C_1, and C_2 are arbitrary constants and $p_0(t)$ is an arbitrary function.

⊙ *Reference*: V. V. Pukhnachov (1960).

7°. Solution (generalizes the solutions from Items 5° and 6°):

$$u_r = \frac{a(t)}{r}, \quad u_\varphi = f(r,t), \quad p = p_1(r,t) - b(t)\varphi + p_0(t), \quad r = \sqrt{x^2 + y^2}, \tag{6}$$

where $a(t)$, $b(t)$ and $p_0(t)$ are arbitrary functions, the function $u_\varphi = f(r,t)$ is described by the linear partial differential equation

$$\frac{\partial f}{\partial t} + \frac{a(t)}{r}\left(\frac{\partial f}{\partial r} + \frac{f}{r}\right) = \frac{b(t)}{r} + \nu\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2}\right), \tag{7}$$

and the function $p_1(r,t)$, entering the pressure, is calculated as

$$p_1(r,t) = -a_t'(t) \ln r - \frac{a^2(t)}{2r^2} + \int f^2(r,t) \frac{dr}{r}.$$

For $b(t) \equiv 0$, formulas (6) together with equation (7) describes, for example, a motion of a rotating viscous hoop with a free boundary (Bytev, 1970).

▷ *Some other two-dimensional steady-state and unsteady solutions of the Navier–Stokes equations in the cylindrical coordinates (plane flows) can be found in Subsections 14.3.2 (see Eq. 4) and 14.3.3 (see Eq. 3).*

21.3.3. Axisymmetric Fluid Flows

Stream function in the cylindrical and spherical coordinates.

1°. For axisymmetric fluid flows, it is often reasonable to use the cylindrical coordinates (r, φ, z); in this case, $u_\varphi = 0$ and the other velocity components and the pressure are independent of φ. The velocity components can be expressed in terms of the stream function ψ as follows:

$$u_r = \frac{\alpha}{r}\frac{\partial \psi}{\partial z}, \qquad u_z = -\frac{\alpha}{r}\frac{\partial \psi}{\partial r}, \qquad r = \sqrt{x^2 + y^2},$$

where $\alpha \neq 0$ is an arbitrary constant. Then the continuity equation is satisfied identically. The constant α is usually taken to be equal to 1 or –1.

For axisymmetric fluid flows, the Navier–Stokes equations, and some exact solutions written in terms of the stream function in the cylindrical coordinates, see also equations 14.3.2.5 and 14.3.3.5.

2°. For axisymmetric fluid flows, the spherical coordinates (R, θ, φ) can also be used; in this case, $u_\varphi = 0$ and the other velocity components and the pressure are independent of φ. The velocity components can be expressed in terms of the stream function ψ as follows:

$$u_R = \frac{\alpha}{R^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\alpha}{R \sin\theta}\frac{\partial \psi}{\partial R}, \qquad R = \sqrt{x^2 + y^2 + z^2},$$

where $\alpha \neq 0$ is an arbitrary constant. The continuity equation is satisfied identically. The constant α is usually taken to be equal to 1 or –1.

For axisymmetric fluid flows, the Navier–Stokes equations, and some exact solutions written in terms of the stream function in the spherical coordinates, see also equation 14.3.2.6.

Steady-state solutions.

1°. Solution in the cylindrical coordinates:

$$u_r = -\frac{k\nu}{r}, \qquad u_\varphi = 0, \qquad u_z = A + Br^{-k}, \qquad p = p_0 - \frac{k^2\nu^2}{2r^2},$$

where A, B, k, and p_0 are arbitrary constants.

For $A = U$ and $B = -Ua^k$, the solution describes a uniform axial flow $(0, 0, U)$ external to a porous cylinder of radius a.

⊙ *Reference*: J. T. Stuart (1966 *b*).

2°. Solution in the cylindrical coordinates:

$$\begin{gathered} u_r = -\frac{U}{2h} r f'_\eta(\eta), \qquad u_\varphi = 0, \qquad u_z = U f(\eta), \qquad \eta = \frac{z}{h}, \\ p = p_0 - \frac{U^2\beta}{2h^2} r^2 + \frac{U\nu}{h}\left[f'_\eta(\eta) - \frac{1}{2}\mathrm{Re}\, f^2(\eta)\right], \qquad \mathrm{Re} = \frac{Uh}{\nu}, \end{gathered} \tag{1}$$

where the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\frac{1}{\mathrm{Re}} f'''_{\eta\eta\eta} - f f''_{\eta\eta} + \frac{1}{2}(f'_\eta)^2 - 2\beta = 0 \tag{2}$$

with boundary conditions

$$f(0) = 1, \qquad f'_\eta(0) = 0, \qquad f(1) = f'_\eta(1) = 0. \tag{3}$$

Formulas (1) together with (2) and (3) describes the solution to the problem of two thin infinite disks floating above an air table through which air is blown. It is assumed that the two planes are at a distance h apart, the upper is impermeable and lower is porous, through which fluid is injected normally with a uniform speed U.

Remark. Four boundary conditions (3) for the third-order equation (2) reflect the fact that β is unknown *a priori*.

⊙ *References*: E. J. Hinch and J. Lemaître (1994), S. M. Cox (2002).

3°. Solution in the cylindrical coordinates:

$$\begin{aligned} u_r &= -ka\eta^{-1/2} f(\eta), \quad u_\varphi = 0, \quad u_z = 2kz f'_\eta(\eta), \quad \eta = (r/a)^2, \\ p &= p_0 - \tfrac{1}{2}k^2a^2\eta^{-1} f^2(\eta) - 2\nu k f'_\eta(\eta) - 2k^2z^2, \end{aligned} \tag{4}$$

where a, k, and p_0 are arbitrary constants and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \mathrm{Re}[f f''_{\eta\eta} - (f'_\eta)^2 + c] = 0, \qquad \mathrm{Re} = \frac{ka^2}{2\nu}. \tag{5}$$

Equation (5) can be supplemented with the boundary conditions

$$f(1) = f'_\eta(1) = 0, \qquad f'_\eta(\infty) = c. \tag{6}$$

For $c = 1$, formulas (4) together with (5) and (6) describe a stagnation flow on a circular cylinder, namely, that on a circle of radius a, circumscribing a circular cylinder of the same radius whose generators lie in the z-direction.

The exact solution of problem (5)–(6) with $c = 1$ and $\mathrm{Re} = 2$ is

$$f(\eta) = \eta + \tfrac{1}{2}e^{2(1-\eta)} - \tfrac{3}{2}.$$

Remark. There are boundary conditions other than (6) that also have a physical meaning (see the literature cited below).

⊙ *References*: C. Y. Wang (1974), G. I. Burde (1989, 1994), P. G. Drazin and N. Riley (2006, pp. 56–59).

4°. Solution in the cylindrical coordinates:

$$\begin{aligned} u_r &= -U\eta^{-1/2} f(\eta), \quad u_\varphi = 0, \quad u_z = 2U\xi f'_\eta(\eta), \quad \xi = \frac{z}{a}, \quad \eta = \left(\frac{r}{a}\right)^2, \\ p &= \frac{\nu U}{a}\left[4\lambda\xi^2 - 2f'_\eta(\eta) - \frac{\mathrm{Re}}{2\eta} f^2(\eta)\right], \quad \mathrm{Re} = \frac{aU}{\nu}, \end{aligned} \tag{7}$$

where a, U, and λ are arbitrary constants, Re is the Reynolds number, and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \tfrac{1}{2}\mathrm{Re}\,[f f''_{\eta\eta} - (f'_\eta)^2] - \lambda = 0. \tag{8}$$

If equation (8) is supplemented with the boundary conditions

$$f(0) = 0, \quad \lim_{\eta\to 0} \eta^{1/2} f''_{\eta\eta} = 0, \quad f(1) = 0, \quad f'_\eta(1) = 1, \tag{9}$$

then formulas (7) together with (8) and (9) describe a stretching tube flow with a moving boundary. The four boundary conditions (9) are required, since λ in equation (8) has to be determined.

⊙ *References*: J. F. Brady and A. Acrivos (1981), P. G. Drazin and N. Riley (2006, pp. 67–68).

5°. Solution in the cylindrical coordinates:

$$u_r = V\eta^{-1/2} f(\eta), \quad u_\varphi = 0, \quad u_z = (U - 2V\xi) f'_\eta(\eta), \quad \xi = \frac{z}{a}, \quad \eta = \left(\frac{r}{a}\right)^2,$$
$$p = \frac{\nu V}{a}\left[2f'_\eta(\eta) - \frac{\mathrm{Re}}{2\eta} f^2(\eta) - 4\lambda\xi^2 + 4\lambda\frac{U}{V}\xi\right], \quad \mathrm{Re} = \frac{aV}{\nu}, \tag{10}$$

where a, U, V, and λ are arbitrary constants, Re is the Reynolds number, and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \tfrac{1}{2}\mathrm{Re}\,[(f'_\eta)^2 - f f''_{\eta\eta}] - \lambda = 0. \tag{11}$$

Equation (11) can be supplemented with the boundary conditions

$$f(0) = 0, \quad \lim_{\eta\to 0} \eta^{1/2} f''_{\eta\eta} = 0, \quad f(1) = 1, \quad f'_\eta(1) = 0. \tag{12}$$

Formulas (10) together with (11) and (12) describe a flow in a porous circular tube of radius a through the lateral surface of which fluid is sucked away or injected. The four boundary conditions (12) are required, since λ in equation (11) has to be determined.

Remark. Terrill and Thomas (1969) and Skalak and Wang (1977) obtained the following principal numerical results:

(i) for $2.3 < \mathrm{Re} < 9.1$, there cannot be a fully developed flow in the tube (in this range of the Reynolds numbers, problem (11)–(12) does not have any solutions),

(ii) for $\mathrm{Re} < 2.3$ (Re can be positive or negative) and $9.1 < \mathrm{Re} < 20.6$, there are two solutions, and

(iii) for $\mathrm{Re} > 20.6$, there are four solutions.

⊙ *References*: A. S. Berman (1958), F. M. White (1962), R. M. Terril and P. W. Thomas (1969), F. M. Skalak and C. Y. Wang (1977), P. G. Drazin and N. Riley (2006, pp. 62–66).

6°. Solution in the cylindrical coordinates:

$$u_r = -ka\eta^{-1/2} f(\eta), \quad u_\varphi = 0, \quad u_z = 2kz f'_\eta(\eta) + bg(\eta), \quad \eta = (r/a)^2,$$
$$p = p_0 - \tfrac{1}{2}k^2a^2\eta^{-1} f^2(\eta) - 2\nu k f'_\eta(\eta) - 2k^2z^2 + \frac{4b\nu}{a^2}z,$$

where a, b, k, and p_0 are arbitrary constants and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \mathrm{Re}\,[f f''_{\eta\eta} - (f'_\eta)^2 + c] = 0, \qquad \mathrm{Re} = \frac{ka^2}{2\nu},$$
$$\eta g''_{\eta\eta} + g'_\eta + \mathrm{Re}\,(f'_\eta g'_\eta - f'_\eta g) = 1.$$

⊙ *References*: P. D. Weidman and V. Putkaradze (2003, 2005).

7°. The *Slezkin solution* in the spherical coordinates:

$$u_R = -\frac{\nu}{R} f'_\eta(\eta), \quad u_\theta = -\frac{\nu}{R}\frac{f(\eta)}{\sqrt{1-\eta^2}}, \quad \eta = \cos\theta,$$
$$p = p_0 - \frac{\nu^2}{R^2}\left(\frac{1}{2}\frac{f^2(\eta)}{1-\eta^2} + f'_\eta(\eta) - C_1\right) \qquad [\psi = \nu R f(\theta), \quad \alpha = 1],$$

where C_1 and p_0 are arbitrary constants, and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$(1-\eta^2)f'_\eta - \tfrac{1}{2}f^2 + 2\eta f = C_1\eta^2 + 2C_2\eta + 2C_3, \tag{13}$$

and C_2 and C_3 are arbitrary constants.

The solution is used for describing a *round jet*. For the radial velocity on the axis to be finite, it is necessary to set

$$C_1 = -C_2 = 2C_3 = k, \tag{14}$$

so that the three terms on the right-hand side of equation (13) become $k(1-\eta)^2$. Consider a few special cases.

(i) *Case* $k = 0$. The *Landau solution* of equation (13)–(14) is

$$f(\eta) = \frac{2(1-\eta^2)}{a+1-\eta}, \tag{15}$$

where a is an arbitrary constant. Solution (13) with (15) is appropriate to a round jet in an unbounded fluid due to a point force at the origin.

(ii) *Case* $k = \frac{1}{2}(1+4b^2)$. A solution of equation (13)–(14) is

$$f(\eta) = \frac{(1+4b^2)(1-\eta)}{2b\cot[b\ln(1+\eta)] - 1}.$$

The solution satisfies the boundary conditions $f|_{\eta=0} = f|_{\eta=1} = 0$; therefore, the plane $\varphi = \pi/2$ may be considered as a solid boundary.

(iii) *Case of* $C_1 = C_2 = 0$ *and* $C_3 = 1 - b^2$. The solution of equation (13) is

$$f = 2\left[\frac{A(1-\eta)^b(b+\eta) - (1+\eta)^b(b-\eta)}{A(1-\eta)^b + (1+\eta)^b}\right],$$

where A is an arbitrary constant. In this case, unlike the two previous examples, it does not suppress the singularity of the solution.

Remark. Equation (13) is a special case of the general *Riccati equation*, which is reducible to a linear second-order equation.* For this reason, the general solution of equation (13) can be expressed in terms hypergeometric functions; see Yatseev (1950).

⊙ *References*: N. A. Slezkin (1934), L. D. Landau (1944), V. I. Vatseyev (1950), Yu. B. Rumer (1952), H. B. Squire (1951, 1952, 1955), P. G. Drazin and N. Riley (2006, pp. 78–81), G. I. Broman and O. V. Rudenko (2010).

8°. By introducing a stream function $\psi = \psi(r, z)$ such that

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (\alpha = -1),$$

one can reduce, after eliminating the pressure, the equations of steady-state axisymmetric fluid flows to a single equation:

$$r\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{1}{r^2}E\psi\right) - r\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r^2}E\psi\right) = \nu E^2\psi, \tag{16}$$

* For details on reducing the general Riccati equation to a linear second-order equation, see, for example, Polyanin and Zaitsev (2003, pp. 6–8).

where E is the Stokes operator,

$$E = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Equation (16) can be satisfied if one sets $E\psi = Ar^2$, so that

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = Ar^2, \tag{17}$$

where A is an arbitrary constant. Under condition (17), both sides of equation (16) vanish independently; therefore, the class of solutions in question satisfies simultaneously both the Navier–Stokes equations and the Euler equations for inviscid fluids.

1. Polynomial exact solution of equation (17):

$$\psi = C_1 r^4 + (\tfrac{1}{2}A - 4C_1)r^2z^2 + C_2r^2 + C_3r^2z + C_4(r^6 - 12r^4z^2 + 8r^2z^4) + C_5z, \tag{18}$$

where $C_1, \ldots, C_5$ are arbitrary constants.

Consider some other special cases of solution (18) that admit simple physical interpretation.

(i) *Case of* $C_1 = \cdots = C_5 = 0$ and $A = 2a \neq 0$. The solution $\psi = ar^2z^2$ may be interpreted as a rotational flow against the plane boundary $z = 0$ at which the no-slip condition is satisfied (see Agrawal, 1957).

(ii) *Case of* $C_2 = C_4 = C_5 = 0$, $C_1 = ab$, $C_3 = -a$, and $A = 8C_1$. The solution $\psi = ar^2(br^2 - z)$ corresponds to two opposing rotational streams divided by the paraboloidal stream surface $z = br^2$ (see Berker, 1963).

(iii) *Case of* $C_3 = C_4 = C_5 = 0$. Without loss of generality, the solution can be rewritten as

$$\psi = ar^2\left(\frac{r^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

With $b \neq c$, it represents the *ellipsoidal vortex* of O'Brien (1961). With $b = c$, we have Hill's (1894) *spherical vortex* with radius b. The spherical vortex may be immersed in a stream, uniform at infinity, with the velocities continuous across the spherical interface (in applications, the solution is used for modeling a translational fluid flow around a spherical bubble or drop).

(iv) *Case of* $A = 8C_1$ *and* $C_3 = C_5 = 0$. The solution can be written as

$$\psi = r^2[ar^2 + b + c(r^4 - 12r^2z^2 + 8z^4)]$$

and represents the *toroidal vortex* of Wang (1990).

Remark. The term containing the logarithm $C_6(z^2 - r^2\ln r)$ can also be added to the right-hand side of formula (18).

2. *Shcheprov's solution*:

$$\psi = Ar^2\left[\frac{r^2 + z^2}{10} - \frac{b^2}{6} + \frac{b^5}{15(r^2 + z^2)^{3/2}}\right], \tag{19}$$

where b is an arbitrary constant. This solution satisfies the no-slip conditions on the surface of a spherical solid particle, defined by the equation $r^2 + z^2 = b^2$, and holds for any Reynolds numbers (independent of the viscosity ν).

⊙ *Reference*: A. V. Shcheprov (2004).

3. Equation (15) has an exact solution of the form (generalization of the solution from Item 1):

$$\psi = \frac{1}{8}Ar^4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2(n+1)} r^{2n+2} \frac{d^{2n}}{dz^{2n}} f(z). \tag{20}$$

If $f(z)$ is a polynomial, then the series is finite and formula (20) gives an exact solution to equations (16) and (17) that satisfies the condition $\psi|_{r=0} = f(z)$ at the symmetry axis. If $f(z)$ is not a polynomial, then the series in (20) is absolutely convergent for finite r and z if the function $f(z)$ is infinitely differentiable; in this case, the series in (20) can be differentiated with respect to r and z as many times as required.

4. Solution to equation (17) obtained by separation of variables:

$$\psi = \frac{1}{8}Ar^4 + Br^2 + r\sum_{n=1}^{\infty}[\alpha_n \cos(\lambda_n z) + \beta_n \sin(\lambda_n z)]Z_1(\lambda_n r),$$
$$Z_1(\xi) = \gamma_n I_1(\xi) + \delta_n K_1(\xi),$$

where B, α_n, β_n, γ_n, and δ_n are arbitrary constants, and $I_1(\xi)$ and $K_1(\xi)$ are modified Bessel functions.

5. Solution to equation (17) obtained by separation of variables:

$$\psi = \frac{1}{8}Ar^4 + Br^2 + r\sum_{n=1}^{\infty}[\alpha_n \exp(\lambda_n z) + \beta_n \exp(-\lambda_n z)]Z_2(\lambda_n r),$$
$$Z_2(\xi) = \gamma_n J_1(\xi) + \delta_n Y_1(\xi),$$

where B, α_n, β_n, γ_n, and δ_n are arbitrary constants, and $J_1(\xi)$ and $Y_1(\xi)$ are Bessel functions.

Special case. Consider a special case of the last solution

$$\psi = \frac{\Delta p}{16\nu}\left[2\Big(\frac{r}{a}\Big)^2 - \Big(\frac{r}{a}\Big)^4\right] + \sum_{k=1}^{\infty} c_k \Big(\frac{r}{a}\Big) J_1\Big(\lambda_k \frac{r}{a}\Big) \exp\Big(\lambda_k \frac{z}{a}\Big). \tag{21}$$

The first term of formula (21) represents a Poiseuille pipe flow. The second term, determined by an infinite series, will satisfy the no-slip condition at $r = a$ provided that the constants λ_k are the zeros of the Bessel function, $J_0(\lambda) = 0$. Solution (21) describes various situations with the transpiration velocity across the porous pipe.

⊙ *References*: U. Crudeli (1927 *a,b,c*), R. Berker (1963), S. Weinbaum and V. O'Brien (1967), R. M. Terrill (1982), Yu. D. Shmyglevskii (1999), A. V. Shcheprov (2004), V. N. Koterov, Yu. D. Shmyglevskii, and A. V. Shcheprov (2005), P. G. Drazin and N. Riley (2006, pp. 50–52).

Unsteady solutions.

9°. Solution in the cylindrical coordinates:

$$u_r = \frac{A\nu}{r}, \quad u_\varphi = 0, \quad u_z = Bf(\eta) + C, \quad p = p_0(t) - \frac{A^2}{2r^2},$$
$$\eta = \frac{r}{2\sqrt{\nu(t+D)}}, \quad f(\eta) = \int \exp(-\eta^2)\eta^{A-1}\,d\eta,$$

where A, B, C, and D are arbitrary constants and $p_0(t)$ is an arbitrary function.

With a suitable choice of the free parameters, this solution describes a fluid flow in a porous cylinder whose radius increases according to the law $a(t) = \sqrt{\nu t + a_0^2}$.

In the special case $A = 2$, we have $f(\eta) = -\frac{1}{2}\exp(-\eta^2)$.

⊙ *References*: P. A. Lagerstrom and J. D. Cole (1955), P. G. Drazin and N. Riley (2006, pp. 140–141).

10°. Solution in the cylindrical coordinates:

$$u_r = -\frac{aV}{r}, \quad u_z = W\Big(\frac{a}{r}\Big)^k\bigg[\Big(\frac{a}{r}\Big)^k + \frac{2}{\pi}\int_0^\infty e^{-\nu\zeta^2 t}\,\frac{J_k(r\zeta)Y_k(a\zeta) - Y_k(r\zeta)J_k(a\zeta)}{\zeta[J_k^2(a\zeta) + Y_k^2(a\zeta)]}\,d\zeta\bigg],$$

where a, V, and W are arbitrary constants, and $k = aV/(2\nu)$ is the modified Reynolds number.

The solution describes an impulsive flow around a porous cylinder of radius a with fluid seeping through the surface of the cylinder with a radial velocity $u_r|_{r=a} = -V$ and the cylinder moving along its axis with a constant velocity W.

Note that the longitudinal velocity component u_z satisfies the following initial and boundary conditions:

$$u_z = 0 \ \text{ at } \ t = 0 \ \ (r > a), \qquad u_z = W \ \text{ at } \ r = a \ \ (t > 0), \qquad u_z \to 0 \ \text{ as } \ r \to \infty.$$

For the special values $k = n - \frac{1}{2}$ with integer n, the formula expressing the longitudinal velocity component u_z can be considerably simplified. Below are two examples.

Special case 1. For the suction case with $k = 1/2$, we have

$$u_z = \frac{aW}{r}\operatorname{erfc}\Big(\frac{r-a}{2\sqrt{\nu t}}\Big),$$

so that $u_z \to aW/r$ as $t \to \infty$ for any fixed r.

Special case 2. For the transpiration case with $k = -1/2$, we have

$$u_z = W\operatorname{erfc}\Big(\frac{r-a}{2\sqrt{\nu t}}\Big),$$

so that $u_z \to W$ as $t \to \infty$ for any fixed r.

⊙ *References*: H. Hasimoto (1956), P. G. Drazin and N. Riley (2006, pp. 139–140).

11°. Solution (generalizes the solutions from Items 9° and 10°):

$$u_r = \frac{a(t)}{r}, \quad u_\varphi = 0, \quad u_z = f(r,t), \quad p = b(t)z - a_t'(t)\ln r - \frac{a^2(t)}{2r^2} + p_0(t),$$

where $a(t)$, $b(t)$, and $p_0(t)$ are arbitrary functions, and the function $f = f(r,t)$ is described by the linear partial differential equation

$$\frac{\partial f}{\partial t} + \frac{a(t)}{r}\frac{\partial f}{\partial r} = -b(t) + \nu\Big(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\Big).$$

Here are two exact solutions of this equation:

$$f = \frac{A}{t}\exp\bigg[-\frac{r^2}{4\nu t} + \frac{1}{2\nu}\int\frac{a(t)}{t}\,dt\bigg] - \int b(t)\,dt + B,$$

$$f = r^2 + 4\nu t - \int[2a(t) + b(t)]\,dt + A,$$

where A and B are arbitrary constants. The second solution is a special case of solutions having the form

$$f = r^{2n} + A_{2n-2}(t)r^{n-2} + \cdots + A_2(t)r^2 + A_0(t),$$

with n arbitrary constants.

12°. Solution:

$$u_r = \frac{A}{r}\exp\left(-\frac{r^2}{4\nu t}\right), \quad u_\varphi = 0, \quad u_z = \frac{Az + B}{2\nu t}\exp\left(-\frac{r^2}{4\nu t}\right),$$

where A and B are arbitrary constants.

One can see that at $t = 0$ the fluid is everywhere at rest. For $t > 0$, a radial flow develops from the axis $r = 0$, where $u_r|_{r=0} = \infty$ and $u_z = (Az + B)/(2\nu t)$. As $t \to \infty$ $(r > 0)$, we have $u_z \to 0$ and the flow becomes irrotational due to a steady-state line source, $u_r \to A/r$.

⊙ *Reference*: S. Irmay and M. Zuzovsky (1970).

13°. Solution in the cylindrical coordinates:

$$\begin{aligned} &u_r = -\frac{a_0 k f(\eta)}{2\sqrt{(1-kt)\eta}}, \quad u_\varphi = 0, \quad u_z = \frac{kz f'_\eta(\eta)}{1-kt}, \quad \eta = \frac{r^2}{a_0^2(1-kt)}, \\ &p = p_0(t) + \frac{2Ak\nu z^2}{a_0^2(1-kt)^2} - \frac{a_0^2 k^2}{4(1-kt)}\left[\frac{\operatorname{sign} k}{M} f'_\eta(\eta) + \frac{1}{2\eta} f^2(\eta) - f(\eta)\right], \quad M = \frac{a_0^2|k|}{4|\nu|}, \end{aligned} \tag{22}$$

where a_0, k, and A are arbitrary constants, $p_0(t)$ is an arbitrary function, and the function $f = f(\eta)$ is determined by the ordinary differential equation

$$\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} - \operatorname{sign} k\, M\left[(\eta - f) f''_{\eta\eta} + f'_\eta + (f'_\eta)^2\right] = A. \tag{23}$$

If equation (23) is supplemented with the boundary conditions

$$\lim_{\eta\to 0}(\eta^{-1/2} f) = 0, \quad \lim_{\eta\to 0}(\eta^{-1/2} f''_{\eta\eta}) = 0, \quad f'_\eta(1) = 0, \quad f(1) = 1, \tag{24}$$

then formulas (22) together with (23)–(24) describe a fluid flow in a pipe with the radius varying according to the law $a(t) = a_0\sqrt{1-kt}$. In this case, the first two conditions in (24) express the symmetry of the solution about the pipe centerline $r = 0$, while the other two conditions represent no slip at the pipe wall and no flow through it, and the constant A in (22) and (23) is determined in the course of solving the problem as $A = -\operatorname{sign} k\, M\{f'_\eta(0) + [f'_\eta(0)]^2\}$.

⊙ *References*: S. Uchida and H. Aoki (1977), F. M. Skalak and C. Y. Wang (1979).

Remark. To solution (22) there corresponds the stream function $\psi = \frac{1}{2}a_0^2 kz f(\eta)$. The papers cited below treated a more general solution with the stream function

$$\psi = \nu z f(\xi, t), \quad \xi = r/a(t),$$

where $a(t)$ is an arbitrary function. The case of a periodically pulsating radius, $a(t) = a_0[1 + b\cos(\omega t)]$ with $0 < b < 1$, was considered in detail.

⊙ *References*: S. L. Waters (2001), M. G. Bluth, P. Hall, and D. T. Parageorgiu (2003).

14°. Solution:

$$\begin{aligned} &u_r = \frac{Ur}{h_0}\frac{\partial F}{\partial \eta}, \quad u_\varphi = 0, \quad u_z = -2UF, \quad F = F(\eta, \tau), \quad \eta = \frac{z}{h_0}, \quad \tau = \frac{Ut}{h_0}, \\ &p = p_0(\tau) + \frac{U^2}{2h_0^2} r^2 A(\tau) - 2U^2\left(k\frac{\partial F}{\partial \eta} + F^2 + \frac{\partial}{\partial \tau}\int_\eta^1 F\, d\eta\right), \quad k = \frac{\nu}{Uh_0} = \frac{1}{\mathrm{Re}}, \end{aligned} \tag{25}$$

where U and h_0 are arbitrary constants, $A(t)$ and $p_0(t)$ are arbitrary functions, and the function $F = F(\eta, \tau)$ is determined by the differential equation

$$F_{\eta\tau} = kF_{\eta\eta\eta} + 2FF_{\eta\eta} - F_\eta^2 - A(\tau). \tag{26}$$

Formulas (25) and equation (26) are used for describing unsteady squeezing of a viscous fluid between two parallel disks (planes), a distance $h = h(t)$ apart with $h_0 = h(0)$. In this case, equation (26) is supplemented with the initial and boundary conditions

$$F = 0 \quad \text{at} \quad \tau = 0; \qquad F = F_\eta = 0 \quad \text{at} \quad \eta = 0; \qquad F = -\tfrac{1}{2}\bar{h}'_\tau, \quad F_\eta = 0 \quad \text{at} \quad \eta = \bar{h}(\tau) = h/h_0,$$

and the functions $A(\tau)$ and $p_0(\tau)$ are determined in the course of solving the problem.

In the papers cited below, the following two special cases were considered:

(i) the fluid motion is induced by a constant downward force squeezing the fluid out radially;

(ii) a time-dependent gap is prescribed.

⊙ *References*: C. Y. Wang (1976), C. Y. Wang and L. T. Watson (1979), S. Weinbaum, C. J. Lawrence, and Y. Kuang (1985), C. J. Lawrence, Y. Kuang, and S. Weinbaum (1985), P. G. Drazin and N. Riley (2006, pp. 151–156).

▷ *For other axisymmetric steady-state and unsteady solutions of the Navier–Stokes equations, see also in Subsections 14.3.2 (Eq. 5) and 14.3.3 (Eq. 5).*

21.3.4. Other Fluid Flows with Two-Nonzero Velocity Components

Steady-state solutions.

1°. Solution:

$$\begin{aligned} &u = U(z) - \Omega y, \quad v = V(z) + \Omega x, \quad w = 0, \\ &p = -ax - by + \tfrac{1}{2}\Omega^2(x^2 + y^2) + p_0, \end{aligned} \tag{1}$$

where a, b, p_0, and Ω are arbitrary constants, and the functions $U = U(z)$ and $V = V(z)$ described by the formulas

$$\begin{aligned} U &= \cos(\omega z)[C_1 \cosh(\omega z) + C_2 \sinh(\omega z)] \\ &\quad + \sin(\omega z)[C_3 \cosh(\omega z) + C_4 \sinh(\omega z)] + \frac{b}{\Omega}, \qquad \omega = \sqrt{\frac{\Omega}{2\nu}}, \\ V &= -\cos(\omega z)[C_4 \cosh(\omega z) + C_3 \sinh(\omega z)] \\ &\quad + \sin(\omega z)[C_2 \cosh(\omega z) + C_1 \sinh(\omega z)] - \frac{a}{\Omega}, \end{aligned} \tag{2}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Consider two special cases in more detail.

1. *The Ekman flow* is described by formulas (1) where

$$\begin{aligned} U &= U_0 - \exp(-\omega z)[U_0 \cos(\omega z) + V_0 \sin(\omega z)], \\ V &= V_0 - \exp(-\omega z)[V_0 \cos(\omega z) - U_0 \sin(\omega z)], \\ a &= -V_0\Omega, \quad b = U_0\Omega. \end{aligned}$$

This solution describes a layered fluid flow arising as the plane $z = 0$ that rotates with a constant angular velocity Ω about the axis through the origin. It is assumed that there is constant wind blowing far away from the rotating plane with the velocity components $(U_0, V_0, 0)$.

2. *Pressure flow in a rotating layer.* In a fluid layer, whose solid boundaries $z = 0$ and $z = h$ rotate with a constant angular velocity Ω about the z-axis, there is a uniform longitudinal pressure gradient $\nabla p = (-a, -b, 0)$. This results in a shear flow with velocity $(U, V, 0)$ expressed by formulas (1)–(2), in which the constants of integration, determined by the boundary conditions

$$U(0) = V(0) = U(h) = V(h) = 0,$$

are

$$C_1 = -\frac{b}{\Omega}, \quad C_2 = \frac{a\sin(\omega h) + b\sinh(\omega h)}{\Omega[\cosh(\omega h) + \cos(\omega h)]}, \quad C_3 = -\frac{b\sin(\omega h) - a\sinh(\omega h)}{\Omega[\cosh(\omega h) + \cos(\omega h)]}, \quad C_4 = -\frac{a}{\Omega}.$$

3. Formulas (1)–(2) allow one to describe also the flow between two planes rotating about different axes; see Berker (1981) and the solution presented in Item 2° below.

⊙ *References*: V. W. Ekman (1905), R. A. Berker (1979, 1981), C.-Y. Lai, K. R. Rajagopal, and A. Z. Szeri (1984).

2°. Solution in the cylindrical coordinates:

$$u_r = \omega l[f(\eta)\cos\varphi + g(\eta)\sin\theta], \quad u_\varphi = \omega r + \omega l[g(\eta)\cos\varphi - f(\eta)\sin\theta], \quad u_z = 0,$$
$$p = p_0 + \tfrac{1}{2}\omega^2 r^2, \quad \eta = z/h,$$

where h, l, p_0, and ω are arbitrary constants, and the functions $f(\eta)$ and $g(\eta)$ have the form

$$f(\eta) = \frac{\sinh\alpha\cos\alpha}{\sinh^2\alpha\cos^2\alpha + \cosh^2\alpha\sin^2\alpha}\{\sinh(\alpha\eta)\cos(\alpha\eta) + \sinh[\alpha(\eta-1)]\cos[\alpha(\eta-1)]\}$$
$$+ \frac{\cosh\alpha\sin\alpha}{\sinh^2\alpha\cos^2\alpha + \cosh^2\alpha\sin^2\alpha}\{\cosh(\alpha\eta)\sin(\alpha\eta) + \cosh[\alpha(\eta-1)]\sin[\alpha(\eta-1)]\},$$
$$g(\eta) = \frac{\sinh\alpha\cos\alpha}{\sinh^2\alpha\cos^2\alpha + \cosh^2\alpha\sin^2\alpha}\{\cosh(\alpha\eta)\sin(\alpha\eta) + \cosh[\alpha(\eta-1)]\sin[\alpha(\eta-1)]\}$$
$$- \frac{\cosh\alpha\sin\alpha}{\sinh^2\alpha\cos^2\alpha + \cosh^2\alpha\sin^2\alpha}\{\sinh(\alpha\eta)\cos(\alpha\eta) + \sinh[\alpha(\eta-1)]\cos[\alpha(\eta-1)]\},$$
$$\alpha = \left(\frac{1}{2}\mathrm{Re}\right)^{1/2}, \quad \mathrm{Re} = \frac{\omega h^2}{\nu}.$$

The solution describes the fluid flow between two infinite parallel planes that each rotate with an angular velocity ω about different axes. It is assumed that the planes are separated by a distance h, with the lower plane, at $z = 0$, rotating about the axis through the point $x = 0$, $y = -l$ and the upper plane, at $z = h$, rotating about the axis through the point $x = 0$, $y = l$.

⊙ *References*: R. Berker (1963, 1982), T. N. J. Abbot and K. Walters (1970).

3°. The *Couette–Poiseuille type solution* in the cylindrical coordinates:

$$u_r = 0, \quad u_\varphi = \frac{a^2\omega}{b^2 - a^2}\frac{b^2 - r^2}{r}, \quad u_z = \frac{\Delta p}{4\nu L}\left[b^2 - r^2 + \frac{a^2 - b^2}{\ln(b/a)}\ln\frac{b}{r}\right],$$
$$p = p_0 - \frac{\Delta p}{L}z + \frac{a^4\omega^2}{(b^2 - a^2)^2}\left(\frac{1}{2}r^2 - 2b^2\ln\frac{r}{b} - \frac{b^4}{2r^2}\right).$$

These formulas describe a fluid flow in an annular channel between two coaxial circular cylinders, with the inner cylinder of radius a rotating at a constant angular velocity ω and the outer cylinder of radius b being at rest. In addition, there is a constant pressure gradient along the common axis with $\Delta p > 0$ being the total pressure drop along a tube part of length L.

Unsteady solutions.

4°. Solution:

$$\begin{aligned} &u = U(z,t) - \omega y, \quad v = V(z,t) + \omega x, \quad w = 0, \\ &p = -a(t)x - b(t)y + \tfrac{1}{2}\omega^2(x^2+y^2) + p_0(t), \end{aligned} \tag{3}$$

where $a(t)$, $b(t)$, and $p_0(t)$ are arbitrary functions, and the functions $U = U(z,t)$ and $V = V(z,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned} U_t - \omega V &= \nu U_{zz} + a(t), \\ V_t + \omega U &= \nu V_{zz} + b(t). \end{aligned} \tag{4}$$

Formulas (3) and equations (4) describe unsteady layered flows in a rotating fluid.

In the general case, the transformation of the desired functions

$$\begin{aligned} U &= \widetilde{U}\cos(\omega t) + \widetilde{V}\sin(\omega t) + A(t), \\ V &= -\widetilde{U}\sin(\omega t) + \widetilde{V}\cos(\omega t) + B(t), \end{aligned}$$

where

$$A(t) = \cos(\omega t)\int [a(t)\cos(\omega t) - b(t)\sin(\omega t)]\,dt + \sin(\omega t)\int [a(t)\sin(\omega t) + b(t)\cos(\omega t)]\,dt,$$

$$B(t) = -\sin(\omega t)\int [a(t)\cos(\omega t) - b(t)\sin(\omega t)]\,dt + \cos(\omega t)\int [a(t)\sin(\omega t) + b(t)\cos(\omega t)]\,dt,$$

brings system (4) to two independent linear heat equations

$$\widetilde{U}_t = \nu\widetilde{U}_{zz}, \quad \widetilde{V}_t = \nu\widetilde{V}_{zz}.$$

Remark. In the special case $a(t) = a = \text{const}$ and $b(t) = b = \text{const}$ we have $A(t) = b/\omega = \text{const}$ and $B(t) = -a/\omega = \text{const}$.

⊙ *Reference*: S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

5°. Solution in the cylindrical coordinates:

$$\begin{aligned} &u_r = \omega l[f(\eta,\tau)\cos\varphi + g(\eta,\tau)\sin\varphi], \quad u_\varphi = \omega r + \omega l[g(\eta,\tau)\cos\varphi - f(\eta,\tau)\sin\varphi], \\ &u_z = 0, \quad p = p_0(\tau) + \tfrac{1}{2}\omega^2 r^2 + \omega^2 lr[A(\tau)\cos\varphi + B(\tau)\sin\varphi], \quad \eta = z/h, \quad \tau = \omega t, \end{aligned} \tag{5}$$

where $A(\tau)$, $B(\tau)$, and $p_0(\tau)$ are arbitrary functions, h, l, and ω are arbitrary constants, and the functions $f = f(\eta,\tau)$ and $g = g(\eta,\tau)$ are determined by the system of linear partial differential equations

$$\begin{aligned} f_\tau - g - kf_{\eta\eta} &= -A(\tau), \\ g_\tau + f - kg_{\eta\eta} &= -B(\tau), \end{aligned} \tag{6}$$

where $k = \nu/(\omega h^2)$.

Formulas (5) and equations (6) describe the unsteady nonaxisymmetric or eccentric flows between two parallel disks, separated by a distance h, that rotate with the same angular velocity ω about different axes.

Special case. Consider the case where the upper disk is absent and the fluid in the half-space $z \ge 0$. Let the disk and fluid be in rigid-body rotation about the common axis $r = 0$ for $\tau < 0$ and let the axis of rotation of the disk be moved instantly from $r = 0$ to $r = l$, $\varphi = \frac{1}{2}\pi$ at time $\tau = 0$. In this problem, one should set $A(\tau) = B(\tau) = 0$ in (5) and (6); the initial and boundary conditions for equations (6) are

$$f = g = 0 \quad \text{at} \quad \tau = 0; \qquad f = 1, \quad g = 0 \quad \text{at} \quad \eta = 0; \qquad f \to 0, \quad g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{7}$$

The solution of the problem is

$$f = \tfrac{1}{2}\operatorname{Re}\Phi, \quad g = \tfrac{1}{2}\operatorname{Im}\Phi,$$

$$\Phi = \exp\left(-\frac{1+i}{\sqrt{2}}\zeta\right)\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} - \frac{1+i}{2\sqrt{2}}\sqrt{\tau}\right) + \exp\left(\frac{1+i}{\sqrt{2}}\zeta\right)\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} + \frac{1+i}{2\sqrt{2}}\sqrt{\tau}\right), \quad \zeta = \frac{\eta}{\sqrt{k}},$$

where $i^2 = -1$ and $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ are the real and imaginary part of the complex quantity Φ, respectively.

⊙ *Reference*: S. H. Smith (1987), A. R. Rao and S. R. Kasiviswanathan (1987).

21.4. Solutions with Three Nonzero Fluid Velocity Components Dependent on Two Space Variables

21.4.1. Quasi-plane Flows (with the Fluid Velocity Components Independent of z)

Structure of exact solutions. The determining system of equations. There are exact solutions of the Navier–Stokes equations of the form

$$u = u(x, y, t), \quad v = v(x, y, t), \quad w = w(x, y, t), \quad p = \widetilde{p}(x, y, t) + q(t)z,$$

where the fluid velocity components are all independent of z and described by the system of equations

$$u_t + uu_x + vu_y = -\widetilde{p}_x + \nu(u_{xx} + u_{yy}), \tag{1}$$
$$v_t + uv_x + vv_y = -\widetilde{p}_y + \nu(v_{xx} + v_{yy}), \tag{2}$$
$$u_x + v_y = 0, \tag{3}$$
$$w_t + uw_x + vw_y = -q(t) + \nu(w_{xx} + w_{yy}). \tag{4}$$

Equations (1)–(3) describe plane fluid flows. Equation (4), for the third fluid velocity component, does not affect the other two components but depends on them and the arbitrary function $q(t)$. This equation determines the z-axis velocity component, which is the same for all cross-sections $z = \text{const}$, as a function of the coordinates x and y. We will call such flows *quasi-plane*.

⊙ *Reference*: V. V. Pukhnachov (2006).

Note that equation (4) always admits a simple exact solution uniform across the space:

$$w = -\int q(t)\,dt + C.$$

In what follows, we will be looking for exact solutions to equation (4) while assuming that solutions to equations (1)–(3) are known. The function $q(t)$ can be eliminated from equation (4) with the change of variable $w = W - \int q(t)\,dt$. The equation for W will be linear and homogeneous and admit the principle of superposition: its solutions multiplied by arbitrary constants can be added together.

Considered below are some special cases where exact solutions to equation (4) can be expressed in terms solutions to equations (1)–(3) explicitly.

General formulas for constructing exact solutions and examples of their application. We look for solutions to equation (4) in the form

$$w = A(t)u + B(t)v + C(t) + a(t)u_y + b(t)v_x. \tag{5}$$

In what follows, we will need two corollaries of equations (1) and (2) obtained by differentiating with respect to y and x, respectively, and taking into account the continuity equation (3):

$$\begin{aligned} u_{yt} + uu_{xy} + vu_{yy} &= \nu(\Delta u)_y - \widetilde{p}_{xy}, \\ v_{xt} + uv_{xx} + vv_{xy} &= \nu(\Delta v)_x - \widetilde{p}_{xy}, \end{aligned} \tag{6}$$

Substituting (5) into (4) and using (1), (2), and (6), we obtain

$$A'_t u + B'_t v + a'_t u_y + b'_t v_x - A\widetilde{p}_x - B\widetilde{p}_y - (a + b)\widetilde{p}_{xy} + C'_t + q = 0. \tag{7}$$

Equation (7) can be satisfied in the following three cases.

(i) For an arbitrary pressure function $\widetilde{p} = \widetilde{p}(x, y, t)$ and with $A = B = 0$ and $b = -a = \text{const}$, we have a two-parameter exact solution of equation (4):

$$w = au_y - av_x - \int q(t)\,dt + C_1, \tag{8}$$

where a and C_1 are arbitrary constants. The condition $q = 0$ must hold in the steady-state case.

In the cylindrical coordinate system, solution (8) becomes

$$w = a\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} - \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) - \int q(t)\,dt + C_1. \tag{9}$$

(ii) Suppose the pressure function has the form

$$\widetilde{p} = \gamma(t)xy + f(x, t) + g(y, t), \tag{10}$$

where $f(x, t)$, $g(y, t)$, and $\gamma(t)$ are arbitrary functions. Then, by setting $A = B = 0$, $a = \text{const}$, and $b = \text{const}$ in (7), we obtain a three-parameter family of solutions to equation (4):

$$w = au_y + bv_x + \int [(a + b)\gamma(t) - q(t)]\,dt + C_1, \tag{11}$$

where a, b, and C_1 are arbitrary constants. The condition $(a + b)\gamma - q = 0$ must hold in the steady-state case.

⊙ *Reference*: A. D. Polyanin (2009 *b*).

Special case. For the pressure function (10) of the special form

$$\widetilde{p} = \gamma(t)xy - \frac{B}{2A}\gamma(t)x^2 + \alpha(t)x - \frac{A}{2B}\gamma(t)y^2 + \beta(t)y + p_0(t), \tag{12}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $p_0(t)$ are arbitrary functions, we get a five-parameter family of solutions to equation (4):

$$w = Au + Bv + au_y + bv_x + \int [A\alpha(t) + B\beta(t) + (a + b)\gamma(t) - q(t)]\,dt + C_1, \tag{13}$$

where A, B, C_1, a, and b are arbitrary constants (if $\gamma(t) \not\equiv 0$ then $A \neq 0$ and $B \neq 0$). The condition $A\alpha + B\beta + (a + b)\gamma - q = 0$ must hold in the steady-state problems.

(iii) Suppose that the pressure gradient in the plane problem (1)–(3) is constant or dependent on time alone, so that

$$\widetilde{p} = \alpha(t)x + \beta(t)y + p_0(t), \tag{14}$$

where $\alpha(t)$, $\beta(t)$, and $p_0(t)$ are arbitrary functions. Such expressions of the pressure arise sometimes in applications (pipe flows, jets, flows past flat plates, etc.). In this case, we get a five-parameter family of solutions to equation (4):

$$w = Au + Bv + au_y + bv_x + \int [A\alpha(t) + B\beta(t) - q(t)]\,dt + C_1, \tag{15}$$

where A, B, C_1, a, and b are arbitrary constants. Formulas (14) and (15) result from (12) and (13) if $\gamma(t) \equiv 0$; in this case, the constraints imposed on the constants A and B are removed. For steady-state problems, one should set $q = A\alpha + B\beta$ in equation (4) and formula (15).

⊙ *Reference*: A. D. Polyanin (2009 *b*).

The general formulas (8), (9), and (11) allow one to obtain three-dimensional quasi-plane solutions automatically by using the two-dimensional plane solutions presented in Subsections 21.3.1 and 21.3.2.

Remark. For some solutions of a plane system (1)–(3) of special form, relation (7) allows one to obtain more exact solutions of equation (4).

Example 1. As the original solution with two fluid velocity components, let us take the steady-state solution from Subsection 21.3.2, Item 1°. In order to construct a more complicated solution with three nonzero velocity components, we use formula (9) with $q(t) \equiv 0$. Taking into account that $w = u_z$, we finally get

$$\begin{aligned} u_r &= \frac{\nu k}{r}, \quad u_\varphi = \frac{A}{r} + Br^{k+1}, \quad u_z = C_1 + C_2 r^k, \\ p &= -\frac{\nu^2 k^2 + A^2}{2r^2} + \frac{2AB}{k} r^k + \frac{B^2}{2(k+1)} r^{2(k+1)} + C_3, \end{aligned} \tag{16}$$

where A, B, C_1, C_2, C_3, and k are arbitrary constants ($k \neq -1, -2$). For convenience, C_2 has been renamed $-aB(k+2)$ in p. Let us consider a few special cases in more detail.

(i) For

$$A = a^2\omega_a - Ba^{k+2}, \quad C_2 = -C_1 a^{-k}, \quad k < -1,$$

where B is an arbitrary constant, solution (16) satisfies the boundary conditions

$$u_\varphi = a\omega_a, \quad u_z = 0 \quad \text{at} \quad r = a; \qquad u_\varphi \to 0, \quad u_z \to C_1 \quad \text{as} \quad r \to \infty, \tag{17}$$

which corresponds to a fluid flow around a porous circular cylinder rotating at an angular velocity ω_a about its axis; the flow is directed along the axis and has the velocity C_1 far from the cylinder. This case gives an example of nonuniqueness of a steady-state solution satisfying the boundary conditions (17), provided that the fluid is sucked away ($k < -1$) through the porous surface at a sufficiently large intensity.

(ii) For

$$A = a^2\omega_a - Ba^{k+2}, \quad C_1 = 0, \quad k < -1,$$

where B is an arbitrary constant, solution (16) satisfies the boundary conditions

$$u_\varphi = a\omega_a, \quad u_z = U_a \quad \text{at} \quad r = a; \qquad u_\varphi \to 0, \quad u_z \to 0 \quad \text{as} \quad r \to \infty, \tag{18}$$

which corresponds to a fluid flow around a porous circular cylinder rotating at an angular velocity ω_a about its axis and moving at a constant velocity $U_a = C_2 a^k$ along its axis. This case gives an example of nonuniqueness of a steady-state solution satisfying the boundary conditions (18).

(iii) For

$$A = \frac{a^2 b^2 (b^k \omega_a - a^k \omega_b)}{b^{k+2} - a^{k+2}}, \qquad B = \frac{b^2\omega_b - a^2\omega_a}{b^{k+2} - a^{k+2}}, \qquad k \neq -2,$$

solution (16) satisfies the boundary conditions

$$u_\varphi = a\omega_a, \quad u_z = U_a \quad \text{at} \quad r = a; \qquad u_\varphi = b\omega_b, \quad u_z = U_b \quad \text{at} \quad r = b,$$

where $U_a = C_1 + C_2 a^k$ and $U_b = C_1 + C_2 b^k$, which corresponds to a fluid flow in the gap between two coaxial porous cylinders rotating at angular velocities ω_a and ω_b about the common axis and moving at constant velocities along the common axis.

Example 2. As the original solution with two fluid velocity components, let us take the steady-state solution 7° from Subsection 21.3.1, where the pressure depends on y alone and, hence, is a special case of (10). Substituting this solution in formula (11) with $\gamma(t) = q(t) = 0$, we find the third velocity component

$$w = D\sin(\beta x + B)e^{-\lambda y},$$

where $D = -A\lambda(a\lambda + b\beta)$ is an arbitrary constant. As a result, we obtain a solution with three velocity components

$$\begin{aligned} u &= A\lambda\sin(\beta x + B)e^{-\lambda y}, \quad v = A\beta\cos(\beta x + B)e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2 - \lambda^2), \\ w &= D\sin(\beta x + B)e^{-\lambda y}, \qquad p = -\frac{1}{2}A^2\beta^2 e^{-2\lambda y} + p_0, \end{aligned}$$

where A, B, C, D, β, λ, and p_0 are arbitrary constants.

Case of $u = u(x, t)$ and $v = v(x, y, t)$. In this case, exact solutions of equation (4) can be sought in the form $w = w(x, t)$ with

$$w_t + u(x,t)w_x = -q(t) + \nu w_{xx}.$$

Given a steady-state dependence $u = u(x)$, the corresponding steady-state solution $w = w(x)$ of the last equation ($q(t) = q = \text{const}$) is written as

$$w = C_1 \int \varphi(x)\,dx + C_2 + \frac{q}{\nu}\int \varphi(x)\left(\int \frac{dx}{\varphi(x)}\right)dx, \qquad \varphi(x) = \exp\left(\frac{1}{\nu}\int u(x)\,dx\right),$$

where C_1 and C_2 are arbitrary constants.

In the case of $u = u(x, y, t)$ and $v = v(y, t)$, exact solutions to equation (4) can be sought likewise, in the form $w = w(y, t)$.

Example 3. As the original solution with two fluid velocity components, let us take the steady-state solution 2 from Section 21.3.1, in which $v = Ae^{-ky} - k\nu$ (depends on y alone). The exact solution to equation (4) of the form $w = w(y)$ is expressed as

$$w = C_1 \int \varphi(y)\,dx + C_2 + \frac{q}{\nu}\int \varphi(y)\left(\int \frac{dy}{\varphi(y)}\right)dy, \qquad \varphi(y) = \exp\left(-\frac{A}{k\nu}e^{-ky} - ky\right),$$

where C_1 and C_2 are arbitrary constants. In the simplest case of $A = 0$, this results in the following solution with three velocity components:

$$u = B_1 e^{-ky}, \quad v = -k\nu, \quad w = C_3 e^{-ky} + C_2 + \frac{q}{k\nu}y, \quad p = p_0 + qz,$$

where $B_1 = Bk$ and $C_3 = -C_1/k$ are arbitrary constants.

Note that in solution 2 from Section 21.3.1, the pressure depends on y alone. Therefore, another solution for w can be obtained by using formula (11) with $\gamma(t) = q(t) = 0$. As a result, this yields the following three-component solution:

$$u = k(Ax + B)e^{-ky}, \quad v = Ae^{-ky} - k\nu, \quad w = -ak^2(Ax + B)e^{-ky} + C, \quad p = \tfrac{1}{2}A^2e^{-2ky} + p_0,$$

where A, B, C, a, k, and p_0 are arbitrary constants.

Some other examples of quasi-plane flows. Let us restrict ourselves to a few interesting examples of exact solutions to equation (4); the more detailed analysis (e.g., with the aid of formulas (8), (9), and (11)) is left to the interested reader.

1°. Solution:

$$u = -ky, \quad v = -kx, \quad w = A + Be^{-kxy/\nu}, \quad p = \tfrac{1}{2}k^2(x^2 + y^2) + p_0,$$

where A, B, k, and p_0 are arbitrary constants. For $B = -A$, the solution corresponds to a uniform flow (in z) along the corner ($x = 0$, $y \ge 0$; $y = 0$, $x \ge 0$) with porous walls.

⊙ *Reference*: J. T. Stuart (1966 *b*).

2°. Consider the following solution of problem (1)–(3):

$$u = ax + by, \quad v = cx - ay, \quad \widetilde{p} = \tfrac{1}{2}(a^2 + bc)(x^2 + y^2) + p_0,$$

which describes an arbitrary steady-state plane linear flow of an inviscid fluid (with a, b, c, and p_0 being arbitrary constants).

The steady-state equation (4) admits two families of exact solutions:

$$w = w(z), \quad z = (a + k)x + by, \quad k = \pm\sqrt{a^2 + bc},$$

where the function $w = w(z)$ is determined by the ordinary differential equation

$$kzw'_z = -q + \nu[(a + k)^2 + b^2]w''_{zz},$$

which is easy to integrate. In particular, if $q = 0$, the general solution of this equation is expressed as

$$w = C_1 \int \exp(\sigma z^2)\,dz + C_2, \qquad \sigma = \frac{k}{2\nu[(a + k)^2 + b^2]},$$

where C_1 and C_2 are arbitrary constants.

3°. Consider the solution of equations (1)–(3) that is independent of ν and describes an arbitrary steady-state potential plane flow of an inviscid fluid. In this case, the fluid velocity components u and v can be expressed in terms of the potential $\varphi = \varphi(x, y)$ and stream function $\psi = \psi(x, y)$ as follows:

$$u = \varphi_x = \psi_y, \qquad v = \varphi_y = -\psi_x. \tag{19}$$

The function φ is determined by solving the Laplace equation $\Delta\varphi = 0$. In specific problems, the potential φ and stream function ψ may be identified by invoking the complex variable theory [e.g., see Lavrent'ev and Shabat (1973) and Sedov (1980)].

By changing, in the steady-state equation (4) with $q(t) = 0$, from x and y to the new variables φ and ψ (it is the *Boussinesq transformation*) and taking into account (19), we arrive at a simpler equation with constant coefficients

$$w_\varphi = \nu(w_{\varphi\varphi} + w_{\psi\psi}). \tag{20}$$

The substitution $w = \exp\left(\frac{1}{2}\varphi/\nu\right)U$ brings (2) to the *Helmholtz equation*

$$U_{\varphi\varphi} + U_{\psi\psi} = \tfrac{1}{4}\nu^{-2}U. \tag{21}$$

Exact solutions of equations (20)–(21) can be found, for example, in the handbook by Polyanin (2002). Below are a few simple solutions of equation (20):

$$w = A\exp(\varphi/\nu) + B,$$
$$w = 2A\nu\varphi + A\psi^2 + B,$$
$$w = \exp(\tfrac{1}{2}\varphi/\nu)(A\cos\alpha\varphi + B\sin\alpha\varphi)(C\cosh\beta\psi + D\sinh\beta\psi), \quad \beta = \sqrt{\alpha^2 + \tfrac{1}{4}\nu^{-2}},$$

where A, B, C, D, and α are arbitrary constants.

Remark. For an ideal fluid ($\nu = 0$), the general solution to equation (20) has the form $w = F(\psi)$, where $F(\psi)$ is an arbitrary function.

4°. Solution:

$$u = U_0 e^{-ay}\cos(\omega t + by), \qquad v = -V_0, \qquad w = W_0(1 - e^{-V_0 y/\nu}), \qquad p = p_0,$$

where U_0, V_0, p_0, and ω are arbitrary constants, and

$$a = \left(\frac{\omega}{2\nu}\right)^{1/2}\left\{\left[1 + \left(\frac{V_0^2}{4\omega\nu}\right)^2\right]^{1/2} - \frac{V_0^2}{4\omega\nu}\right\}^{-1/2} + \frac{V_0}{2\nu},$$
$$b = -\left(\frac{\omega}{2\nu}\right)^{1/2}\left\{\left[1 + \left(\frac{V_0^2}{4\omega\nu}\right)^2\right]^{1/2} - \frac{V_0^2}{4\omega\nu}\right\}^{1/2}.$$

⊙ *Reference*: W. R. Debler and R. D. Montgomery (1971).

5°. Solution:

$$u = -a\exp(-2b^2\nu t)\cos(bx + c_1)\sin(by + c_2),$$
$$v = a\exp(-2b^2\nu t)\sin(bx + c_1)\cos(by + c_2),$$
$$w = \sqrt{2}\,a\exp(-2b^2\nu t)\cos(bx + c_1)\cos(by + c_2),$$
$$p = p_0(t) - \tfrac{1}{4}a\exp(-4b^2\nu t)[a\cos(2bx + 2c_1) + b\cos(2by + 2c_2)],$$

where a, b, c_1, and c_2 are arbitrary constants and $p_0(t)$ is an arbitrary function.

⊙ *Reference*: R. Berker (1963).

21.4.2. Cylindrical and Conical Vortex Flows. Von Kármán-Type Rotationally Symmetric Motions

Steady-state solutions.

1°. Solution in the cylindrical coordinates:

$$u_r = -kr, \quad u_\varphi = \frac{1}{r}\left[A + B\exp\left(-\frac{kr^2}{2\nu}\right)\right], \quad u_z = 2kz,$$
$$p = p_0 - 2k^2z^2 - \frac{1}{2}k^2r^2 + \int \frac{u_\varphi^2(r)}{r}\,dr,$$

where k, A, B, and p_0 are arbitrary constants.

For $B = -A$, the solution describes a cylindrical *Burgers vortex* with circulation $\Gamma = 2\pi A$ at large distances r.

⊙ *Reference*: J. M. Burgers (1948).

2°. Solution in the cylindrical coordinates:

$$u_r = -kr + \frac{1}{r}f(\eta), \quad u_\varphi = \frac{\Gamma}{2\pi r}\frac{g(\eta)}{g(\infty)}, \quad u_z = 2kz\left[1 - \frac{f'_\eta(\eta)}{2\nu}\right], \quad \eta = \frac{kr^2}{2\nu},$$

where k and Γ are arbitrary constants, and

$$f(\eta) = 6\nu(1 - e^{-\eta}), \quad g(\eta) = \frac{\nu}{k}\int_0^\eta \exp\left(-\xi + 3\int_0^\xi \frac{1 - e^{-s}}{s}\,ds\right)d\xi.$$

The solution describes a two-cell vortex.

⊙ *Reference*: R. D. Sullivan (1959).

3°. Solution in the cylindrical coordinates:

$$\begin{aligned} &u_r = \omega r f'_\eta(\eta), \quad u_\varphi = \omega r g(\eta), \quad u_z = -2\sqrt{\nu\omega}\, f(\eta), \\ &p = p_0 + \tfrac{1}{2}\lambda\omega^2 r^2 - 2\nu\omega[f'_\eta(\eta) + f^2(\eta)], \quad \eta = \sqrt{\omega/\nu}\, z, \end{aligned} \tag{1}$$

where p_0, ω, and λ are arbitrary constants, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f'''_{\eta\eta\eta} + 2f f''_{\eta\eta} - (f'_\eta)^2 + g^2 &= \lambda, \\ g''_{\eta\eta} + 2(f g'_\eta - f'_\eta g) &= 0. \end{aligned} \tag{2}$$

The one-disk problem. Let us supplement equations (2) with the boundary conditions

$$f(0) = a_1, \quad f'_\eta(0) = 0, \quad g(0) = 1, \quad f'_\eta(\infty) = a_2, \quad g(\infty) = a_3. \tag{3}$$

For $\lambda = a_1 = a_2 = a_3 = 0$, solution (1)–(3) describes the classical *von Kármán flow* in the half-space $z > 0$, caused by uniform rotation of an infinite disk in a fluid. In this problem, the fluid motion far from the disk tends a uniform flow with nonzero z-axis velocity component; specifically, $u_r \to 0$, $u_\varphi \to 0$, and $u_z \to -0.883\sqrt{\nu\omega}$ as $z \to \infty$. A numerical analysis

shows that $g'_\eta(0) = -0.616$. The conditional thickness of the boundary layer, $z = \delta$, at which $g = 0.01$ is equal to $\delta = 5.4\sqrt{\nu/\omega}$.

To $\lambda = a_2 = a_3 = 0$ and $a_1 \neq 0$ there corresponds the effect of suction in the case of a rotating porous disk.

For $\lambda = -(k/\omega)^2$, $a_1 = a_3 = 0$, and $a_2 = k/\omega$, the solution describes a flow towards a rotating disk such that $u_r = kr$, $u_\varphi = 0$, and $u_z = -2kz$ at large distances from the disk, just as in the classical stagnation-point flow.

The existence, uniqueness, and nonuniqueness issues for solutions of problem (2)–(3) depending on the values of the parameters λ and a_m ($m = 1, 2, 3$) as well as numerical solutions and various applications are discussed in the literature cited below.

⊙ *References*: T. von Kármán (1921), W. G. Cochran (1934), D. M. Hannah (1947), G. K. Batchelor (1951), J. T. Stuart (1954), M. H. Rogers and G. N. Lance (1960), J. B. McLeod (1971), H. Ockendon (1972), P. J. Zandbergen and D. Dijkstra (1977, 1987), P. J. Zandbergen (1980), P. G. Drazin and N. Riley (2006, pp. 68–72).

The two-disk problem. For the problem of two disks rotated coaxially about the common axis $r = 0$ and separated by a distance h, formulas (1) and equations (2) remain valid. But two boundary conditions are now applied at a finite value of the variable $\eta = \sqrt{\omega/\nu}\, h = \sqrt{\mathrm{Re}}$, where Re is the Reynolds number. With the new variables

$$\zeta = \mathrm{Re}^{-1/2}\eta, \quad F = \mathrm{Re}^{-1/2} f, \quad G = g,$$

one can rewrite equations (2) in the form

$$\begin{aligned} \mathrm{Re}^{-1} F'''_{\zeta\zeta\zeta} + 2FF''_{\zeta\zeta} - (F'_\zeta)^2 + G^2 = \lambda, \\ \mathrm{Re}^{-1} G''_{\zeta\zeta} + 2(FG'_\zeta - F'_\zeta G) = 0. \end{aligned} \tag{4}$$

For impermeable disks, the equations must be supplemented with the boundary conditions

$$F(0) = F'_\zeta(0) = 0, \quad G(0) = 1, \quad F(1) = F'_\zeta(1) = 0, \quad G(1) = s. \tag{5}$$

There are six boundary conditions (5) for the fifth-order system (4); therefore, the constant λ is not known in advance and has to be determined from the solution procedure.

For $s \geq 0$ and $\mathrm{Re} \gg 1$, Batchelor (1951) argued that the fluid in the core would rotate with constant angular velocity, with boundary layers at each disk (as in the appropriate one-disk family solutions). For counter-rotating disks with $s = -1$ and $\mathrm{Re} \gg 1$, Batchelor argued that in the core would be two regions of uniform, counter-rotating flow, separated by a shear layer at $\zeta = 1/2$, with boundary layers at each disk. Subsequent studies revealed the existence of other fluid flow modes as well (see Stewartson (1953) and the literature cited below). Kreiss and Parter (1983), by rigorous analytical methods, established existence and nonuniqueness of solutions for all s when Re is sufficiently large.

⊙ *References*: G. K. Batchelor (1951), K. Stewartson (1953), G. N. Lance and M. H. Rogers (1962), C. E. Pearson (1965), G. L. Melor, P. J. Chapple, and V. K. Stokes (1968), K. K. Tam (1969), J. B. McLeod and S. V. Parter (1974), H. B. Keller and R. K. H. Szeto (1980), M. Holodniok, M. Kubicek, and V. Hlavacek (1981), H. O. Kreiss and S. V. Parter (1983), P. J. Zandbergen and D. Dijkstra (1987), P. G. Drazin and N. Riley (2006, pp. 73–77).

4°. Solution in the cylindrical coordinates:

$$\begin{aligned} u_r = -ka\eta^{-1/2} f(\eta), \quad u_\varphi = a\omega\eta^{-1/2} g(\eta), \quad u_z = 2kz f'_\eta(\eta), \quad \eta = (r/a)^2, \\ p = p_0 - \frac{1}{2}k^2a^2\eta^{-1} f^2(\eta) - 2\nu k f'_\eta(\eta) - 2k^2z^2 + \frac{1}{2}a^2\omega^2 \int_1^\eta \frac{g^2(s)}{s^2}\, ds, \end{aligned} \tag{6}$$

where a, k, p_0, and ω are arbitrary constants and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$\begin{gathered}\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \mathrm{Re}[f f''_{\eta\eta} - (f'_\eta)^2 + c] = 0,\\ \eta g''_{\eta\eta} + \mathrm{Re}\, f g'_\eta = 0, \qquad \mathrm{Re} = \frac{ka^2}{2\nu}.\end{gathered} \tag{7}$$

The general solution of the second equation (7) is expressed in term of $f(\eta)$ as

$$g(\eta) = C_1 \int e^{-\mathrm{Re}\,\Phi(\eta)} d\eta + C_2, \qquad \Phi(\eta) = \int \frac{f(\eta)}{\eta}\, d\eta,$$

where C_1 and C_2 are arbitrary constants.

Equations (7) can be supplemented with the boundary conditions

$$f(1) = -\lambda, \quad f'_\eta(1) = 0, \quad f'_\eta(\infty) = c, \quad g(1) = 1, \quad g(\infty) = 0. \tag{8}$$

For $c = 1$ and $\lambda = V_0/(ak)$, formulas (6) together with (7) and (8) describe a fluid flow generated by a porous circular cylinder of radius a rotating about its axis with an angular velocity ω, with uniform transpiration with a velocity V_0 through the surface of the cylinder.

For the case in which there is no rotation, $\omega = 0$, the exact solution of the problem (7)–(8) with $c = 1$ and $\mathrm{Re} = 2/(1 + \lambda)$ for the function f is given by (see Burde, 1994)

$$f(\eta) = \eta + \tfrac{1}{2}(1 + \lambda)[e^{2(1-\eta)/(1+\lambda)} - 3].$$

⊙ *References*: G. M. Cunning, A. M. J. Davis, and P. D. Weidman (1998), P. G. Drazin and N. Riley (2006, p. 61).

5°. Solution in the cylindrical coordinates:

$$\begin{gathered}u_r = V\eta^{-1/2} f(\eta), \quad u_\varphi = (U - 2V\xi)\eta^{-1/2} g(\eta), \quad u_z = (U - 2V\xi) f'_\eta(\eta),\\ p = \frac{\nu V}{a}\left[2f'_\eta(\eta) - \frac{\mathrm{Re}}{2\eta} f^2(\eta) - 4\lambda\xi^2 + 4\lambda\frac{U}{V}\xi\right], \quad \xi = \frac{z}{a}, \quad \eta = \left(\frac{r}{a}\right)^2, \quad \mathrm{Re} = \frac{aV}{\nu},\end{gathered} \tag{9}$$

where a, U, V, and λ are arbitrary constants, Re is the Reynolds number, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$\begin{gathered}\eta f'''_{\eta\eta\eta} + f''_{\eta\eta} + \tfrac{1}{2}\mathrm{Re}\,[(f'_\eta)^2 - f f''_{\eta\eta}] - \lambda = 0,\\ \eta g''_{\eta\eta} + \tfrac{1}{2}\mathrm{Re}\,(f'_\eta g - f g'_\eta) = 0.\end{gathered} \tag{10}$$

Equations (10) can be supplemented with the boundary conditions

$$f(0) = 0, \quad \lim_{\eta\to 0} \eta^{1/2} f''_{\eta\eta} = 0, \quad f(1) = 1, \quad f'_\eta(1) = 0, \quad g(0) = 0, \quad g(1) = 1. \tag{11}$$

Formulas (9) together with (10) and (11) describe a flow in a porous circular tube of radius a with fluid sucked away or injected through the lateral surface. The four boundary conditions (11) are required, since λ in equation (10) has to be determined.

⊙ *References*: S. Prager (1964), R. M. Terril and P. W. Thomas (1973).

6°. Solution in the spherical coordinates:

$$u_R = \frac{\nu}{R} f'_\eta(\eta), \quad u_\theta = \frac{\nu}{R}\frac{f(\eta)}{\sqrt{1-\eta^2}}, \quad u_\varphi = \frac{k\nu}{R}\frac{g(\eta)}{\sqrt{1-\eta^2}},$$
$$p = p_0 + \frac{h(\eta)}{R^2(1-\eta^2)}, \quad \eta = \cos\theta, \tag{12}$$

where the constant k is related to the circulation, the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$(1-\eta^2)f''''_{\eta\eta\eta\eta} + (f-4\eta)f'''_{\eta\eta\eta} + 3f'_\eta f''_{\eta\eta} = -2k^2 g g'_\eta/(1-\eta^2),$$
$$(1-\eta^2)g''_{\eta\eta} + f g'_\eta = 0, \tag{13}$$

and

$$h(\eta) = -\tfrac{1}{2}(f^2 + k^2 g^2) - \tfrac{1}{2}(1-\eta^2)[f f''_{\eta\eta} + (f'_\eta)^2 + (1-\eta^2)f'''_{\eta\eta\eta} - 2\eta f''_{\eta\eta}].$$

Formulas (12) and equations (13) are used for describing conical vortex flows taking into account the influence of boundaries.

If the no-slip condition is to be satisfied at the bounding plane $z = 0$, that is $\eta = 0$, then

$$f(0) = f'_\eta(0) = g(0) = 0;$$

whilst if the vortex, aligned with the axis, is to be neither a line source nor a sink, then

$$f \to 0, \quad g \to 0 \quad \text{as} \quad \eta \to 1,$$

corresponding to a line vortex of circulation $\Gamma = 2\pi k\nu$.

For numerical solutions, the uniqueness and nonuniqueness issues for solutions, various applications, and possible modifications and generalizations of the conical vortex flows, see the literature below.

⊙ *References*: N. A. Slezkin (1934), R. R. Long (1958, 1961), M. A. Goldshtik (1960, 1990), J. Serrin (1972), C.-S. Yih, F. Wu, A. K. Garg, and S. Leibovich (1982), A. F. Pillow and R. Paul (1985), R. Paul and A. F. Pillow (1985), M. A. Goldshtik and V. N. Shtern (1990), C. F. Stein (2001), V. V. Pukhnachov (2006).

Unsteady solutions.

7°. Solution in the cylindrical coordinates:

$$u_r = -\gamma r, \quad u_\varphi = \frac{C}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2\nu\zeta}\right)\right], \quad u_z = 2\gamma z, \quad \gamma = \gamma(t),$$
$$p = p_0(t) + \frac{1}{2}(\gamma'_t - \gamma^2)r^2 - (\gamma'_t + 2\gamma^2)z^2 - \frac{aC^2}{16\pi\nu\varphi}\int_{ar^2/(2\nu\zeta)}^{\infty}\left(\frac{1-e^{-s}}{s}\right)^2 ds,$$
$$\zeta = 2a\exp\left(-2\int \gamma\,dt\right)\left[\int \exp\left(2\int \gamma\,dt\right)dt + B\right],$$

where $\gamma(t)$ and $p_0(t)$ are arbitrary functions, and B, C, and a are arbitrary constants.

Special case. For $\gamma = \text{const}$, we have

$$\zeta = \frac{a}{\gamma}(1 + De^{-2\gamma t}),$$

where $D = 2B\gamma$ is an arbitrary constant.

⊙ *References*: N. Rott (1958), H. K. Moffat (2000).

8°. Solution in the cylindrical coordinates (generalizes the previous solution from Item 7°):

$$u_r = -\gamma r, \quad u_\varphi = \frac{C}{2\pi r}\left[1 - \exp\left(-\frac{ar^2}{2\nu\zeta}\right)\right], \quad u_z = 2\gamma z + W(r,t), \quad \gamma = \gamma(t),$$

$$p = p_0(t) + \frac{1}{2}(\gamma_t' - \gamma^2)r^2 - (\gamma_t' + 2\gamma^2)z^2 - \frac{aC^2}{16\pi\nu\varphi}\int_{ar^2/(2\nu\zeta)}^{\infty}\left(\frac{1-e^{-s}}{s}\right)^2 ds,$$

$$\zeta = 2a\exp\left(-2\int \gamma\,dt\right)\left[\int \exp\left(2\int \gamma\,dt\right)dt + B\right],$$

where $\gamma(t)$ and $p_0(t)$ are arbitrary functions, B, C, and a are arbitrary constants, and the function $W = W(r,t)$ is described by the linear partial differential equation

$$W_t - \gamma r W_r + 2\gamma W = \nu(W_{rr} + r^{-1}W_r).$$

An exact solution of the last equation is

$$W = A\exp\left[-\frac{ar^2}{2\nu\zeta(t)} - 2\int_0^t\left(\gamma(\tau) + \frac{a}{\zeta(\tau)}\right)d\tau\right],$$

where A is an arbitrary constant.

Special case. For $\gamma = \text{const}$, we have

$$W = \frac{A}{e^{4\gamma t} + De^{2\gamma t}}\exp\left[-\frac{\gamma r^2}{2\nu(1 + De^{-2\gamma t})}\right],$$

where $D = 2B\gamma$ is an arbitrary constant.

⊙ *References*: J. D. Gibbon, A. S. Fokas, and C. R. Doering (1999), P. G. Drazin and N. Riley (2006, pp. 171–172).

9°. Solution in the cylindrical coordinates:

$$u_r = -\frac{2\nu}{r}f(\eta), \quad u_\varphi = \frac{C}{2\pi r}\frac{g(\eta)}{g(\infty)}, \quad u_z = \frac{z}{t}f_\eta'(\eta), \quad \eta = \frac{r^2}{4\nu t},$$

$$p = p_0(t) - \frac{1}{2}a(a-1)\left(\frac{z}{t}\right)^2 - \frac{\nu}{t}\left[f(\eta) + f_\eta'(\eta) + \frac{1}{2\eta}f^2(\eta)\right] + \frac{1}{2}\int_0^\eta \frac{u_\varphi^2}{\eta}\,d\eta,$$

where C is an arbitrary constant, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$(\eta f_{\eta\eta}'' + \eta f_\eta')_\eta' + f f_{\eta\eta}'' - (f_\eta')^2 + a(a-1) = 0, \tag{14}$$

$$\eta g_{\eta\eta}'' + (\eta + f)g_\eta' = 0. \tag{15}$$

The general solution of equation (15) is

$$g = A_1\int \exp\left[-\eta - \int \frac{f(\eta)}{\eta}\,d\eta\right]d\eta + A_2, \tag{16}$$

where A_1 and A_2 are arbitrary constants.

Equations (14)–(15) are supplemented with the boundary conditions

$$f(0) = 0, \quad f_{\eta\eta}''(0) = (\beta - a)(\beta + a - 1), \quad f_\eta'(\infty) = a, \quad g(0) = 0, \quad g(\infty) = 1, \tag{17}$$

where the constant $\beta = f'_\eta(0)$ is unknown *a priori*. The second boundary condition (16) implies that $f'''_{\eta\eta\eta}(0)$ is finite at $\eta = 0$.

There are two exact solutions of problem (14), (15), (17).

(i) In the case $\beta = a$, we have

$$f = a\eta, \quad g = 1 - e^{-(1+a)\eta}.$$

This is an example of one-cell vortices with $u_r \sim r/t$.

(ii) In the case $\beta = -2a$, we have

$$f = a\eta - \frac{3a}{1+a}[1 - e^{-(1+a)\eta}] \tag{18}$$

and the function g is found from formula (16) taking into account (18) and the last two boundary conditions in (17). This solution corresponds to two-cell vortices with $a = 2k - 1$.

⊙ *References*: P. G. Bellamy-Knights (1971), P. G. Drazin and N. Riley (2006, pp. 174–177).

10°. Solution in the cylindrical coordinates:

$$\begin{gathered} u_r = \frac{r}{2t} f'_\eta(\eta), \quad u_\varphi = \frac{\omega r}{4t} g(\eta), \quad u_z = -2\sqrt{\frac{\nu}{t}}\, f(\eta), \quad \eta = \frac{z}{2\sqrt{\nu t}}, \\ p = p_0(t) + \frac{1}{32}(4+\omega^2)\left(\frac{r}{t}\right)^2 - \frac{2\nu}{t}\left[f^2(\eta) + \eta f(\eta) + \frac{1}{2} f'_\eta(\eta)\right], \end{gathered} \tag{19}$$

where the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f'''_{\eta\eta\eta} + 2(2f+\eta) f''_{\eta\eta} + 4f'_\eta - 2(f'_\eta)^2 + \tfrac{1}{2}\omega^2 g^2 - \tfrac{1}{2}(4+\omega^2) = 0, \\ g''_{\eta\eta} + 4(f g'_\eta - g f'_\eta) + 2\eta g'_\eta + 4g = 0. \end{aligned} \tag{20}$$

If equations (20) are supplemented with the boundary conditions

$$f(0) = f'_\eta(0) = g(0) = 0, \quad f'_\eta(\infty) = g(\infty) = 1, \tag{21}$$

then formulas (19) together with (20) and (21) describe a flow established over the boundary which is in rigid-body rotation with instantaneous angular velocity $\frac{1}{4}\omega/t$ together with a radial and axial flow.

⊙ *References*: L. Hatton (1975), A. Öztekin, B. B. Seymour, and E. Varley (2001), P. G. Drazin and N. Riley (2006, pp. 177–180).

11°. Solution in the cylindrical coordinates:

$$u_r = \frac{\omega r}{1-kt} f'_\eta(\eta), \quad u_\varphi = \frac{\omega r}{1-kt} g(\eta), \quad u_z = -\frac{2\sqrt{\nu\omega}}{\sqrt{1-kt}} f(\eta), \quad \eta = \frac{\sqrt{\omega/\nu}\, z}{\sqrt{1-kt}}, \tag{22}$$

$$p = p_0(t) + \frac{\nu\omega}{1-kt}\left[A\eta f(\eta) - 2f^2(\eta) - 2f'_\eta(\eta)\right], \quad A = \frac{k}{\omega}, \tag{23}$$

where k and ω are arbitrary constants, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f'''_{\eta\eta\eta} + 2f f''_{\eta\eta} - (f'_\eta)^2 + g^2 - A\left(f'_\eta + \tfrac{1}{2}\eta f''_{\eta\eta}\right) = 0, \\ g''_{\eta\eta} + 2f g'_\eta - 2f'_\eta g - A\left(g + \tfrac{1}{2}\eta g'_\eta\right) = 0. \end{aligned} \tag{24}$$

(i) If equations (24) are supplemented with the boundary conditions

$$f(0) = f'_\eta(0) = 0, \quad g(0) = 1, \tag{25}$$

$$f'_\eta(\infty) = g(\infty) = 0, \tag{26}$$

then formulas (22)–(23) with (24)–(26) describe a fluid flow around an infinite disk rotating with angular velocity $\Omega(t) = \omega/(1 - kt)$.

(ii) If equations (24) are supplemented with the boundary conditions

$$f(\eta_0) = \tfrac{1}{4}A\eta_0, \quad f''_{\eta\eta}(\eta_0) = g'_\eta(\eta_0) = 0, \tag{27}$$

then formulas (22) for the velocity components and the modified formula for the pressure

$$p = p_0 + \frac{\nu\omega}{1 - kt}\left[A\eta f(\eta) - 2f^2(\eta) - 2f'_\eta(\eta) + 2f'_\eta(\eta_0) - \frac{1}{8}A^2\eta_0^2\right], \quad A = \frac{k}{\omega}, \tag{28}$$

with (24), (25), and (27) describe a liquid film flow with a free surface over an infinite disk rotating with an angular velocity $\Omega(t) = \omega/(1 - kt)$. In (28), p_0 is the pressure on the free surface.

(iii) Slightly modified formulas and equations together with suitable boundary conditions describe a fluid flow between two rotating disks.

⊙ *References*: L. T. Watson and C. Y. Wang (1979), E. A. Hamza and D. A. MacDonald (1984), C. Y. Wang, L. T. Watson and K. A. Alexander (1991), P. G. Drazin and N. Riley (2006, pp. 156–161).

12°. Solution in the cylindrical coordinates:

$$\begin{gathered} u_r = \omega r f_\eta(\eta,\tau), \quad u_\varphi = \omega r g(\eta,\tau), \quad u_z = -2\sqrt{\nu\omega}\, f(\eta,\tau), \quad \eta = \sqrt{\omega/\nu}\, z, \quad \tau = \omega t, \\ p = p_0(\tau) + \frac{1}{2}a(\tau)\omega^2 r^2 + 2\nu\omega\left[\frac{\partial}{\partial\tau}\int_0^\eta f(\xi,\tau)\,d\xi - f^2(\eta,\tau) - \frac{\partial}{\partial\eta}f(\eta,\tau)\right], \end{gathered} \tag{29}$$

where the functions $f = f(\eta,\tau)$ and $g = g(\eta,\tau)$ are determined by the system of differential equations

$$\begin{aligned} f_{\eta\tau} - f_{\eta\eta\eta} - 2ff_{\eta\eta} - (f_\eta)^2 - g^2 + a(\tau) = 0, \\ g_\tau - g_{\eta\eta} - 2fg_\eta + 2f_\eta g = 0. \end{aligned} \tag{30}$$

For $a(\tau) = \text{const}$, equations (30) admit traveling-wave solutions of the form

$$f = f(\xi), \quad g = g(\xi), \quad \xi = \eta - \lambda\tau,$$

where λ is an arbitrary constant, and the functions $f = f(\xi)$ and $g = g(\xi)$ are determined by an appropriate system of ordinary differential equations. To $\lambda = 0$ there corresponds a steady-state solution.

For $a(\tau) = a_0(\tau + b)^{-2}$, equations (30) admit self-similar solutions of the form

$$f = (\tau + b)^{-1/2}F(\zeta), \quad g = (\tau + b)^{-1}G(\zeta), \quad \zeta = (\tau + b)^{-1/2}(\eta + c);$$

the functions $F = F(\zeta)$ and $G = G(\zeta)$ are determined by an appropriate system of ordinary differential equations.

For $a(\tau) \equiv 1$, formulas (29) and equations (30) with suitable initial and boundary conditions were used for describing an unsteady flow caused by a rotating disk in a counter-rotating fluid.

For $a(\tau) \equiv 0$, formulas (29) and equations (30) with suitable initial and boundary conditions allow one to describe the process of spreading a liquid layer with a free surface on a rotating plane. A characteristic feature of this problem is that the free surface remains flat for all $\tau \geq 0$.

⊙ *References*: R. J. Bodonyi and K. Stewartson (1977), L. T. Watson and C. Y. Wang (1979), C. Y. Wang (1989), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1998), P. G. Drazin and N. Riley (2006, pp. 161–162), V. V. Pukhnachov (2006).

▷ *For von Kármán-type rotationally symmetric flows and related flows, see also Subsection 21.5.2, formulas (62)–(68).*

21.4.3. Rotationally Symmetric Motions of General Form

Equations describing rotationally symmetric fluid motions. In the Navier–Stokes equations written in the cylindrical coordinates (see equations (7) in Subsection 21.1.1)), we set

$$u_r = u_r(r,z,t), \quad u_\varphi = u_\varphi(r,z,t), \quad u_z = u_z(r,z,t), \quad p = p(r,z,t).$$

So we assume that the unknowns are all independent of the angle φ. As a result, we arrive at the following equations for describing rotationally symmetric fluid motions:

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\varphi^2}{r} &= -\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right),\\
\frac{\partial u_\varphi}{\partial t} + u_r\frac{\partial u_\varphi}{\partial r} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_r u_\varphi}{r} &= \nu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} + \frac{\partial^2 u_\varphi}{\partial z^2}\right),\\
\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} &= -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right),\\
\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} &= 0.
\end{aligned}$$

To the special case $u_\varphi \equiv 0$ there correspond axisymmetric motions.

Remark. A wider class of rotationally symmetric fluid motions can be described using

$$u_r = u_r(r,z,t), \quad u_\varphi = u_\varphi(r,z,t), \quad u_z = u_z(r,z,t), \quad p = \widetilde{p}(r,z,t) - b(t)\varphi,$$

where a term linear in φ has been added to the pressure. In this case, p in the first and third equations must be replaced by $\widetilde{p}$ and the term $b(t)/r$ must be added to the right-hand side of the second equation. An example of such a solution is given in Item 3°.

Steady-state solutions.

1°. Solution in the cylindrical coordinates:

$$\begin{aligned}
u_r &= \frac{\nu}{r}f(\zeta), \quad u_\varphi = \sqrt{2}\,\frac{\nu z}{ar}g(\zeta), \quad u_z = -\frac{2\nu z}{a^2}f'_\zeta,\\
p &= p_0 + \frac{2\nu^2}{a^2}\left[f'_\zeta - \frac{1}{4\zeta}f^2 - 2\frac{z^2}{a^2}h(\zeta)\right], \quad \zeta = \frac{r^2}{a^2},
\end{aligned} \tag{1}$$

where the functions $f = f(\zeta)$, $g = g(\zeta)$, and $h = h(\zeta)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
2(\zeta f''_{\zeta\zeta})'_\zeta &= f f''_{\zeta\zeta} - (f'_\zeta)^2 + 2h,\\
2\zeta g''_{\zeta\zeta} &= f g'_\zeta - g f'_\zeta,\\
4\zeta^2 h'_\zeta &= -g^2.
\end{aligned} \tag{2}$$

Let us supplement system (2) with the boundary conditions

$$f=g=0 \quad \text{at} \quad \zeta=0, \qquad f''=-\tfrac{1}{2}(f'_\zeta)^2+h \quad \text{at} \quad \zeta=0, \qquad f=f'_\zeta=g=0 \quad \text{at} \quad \zeta=1, \tag{3}$$

which means that the no-slip condition holds at the lateral surface of a hollow cylinder of radius a (at $\zeta = 1$), the solution is bounded at the symmetry axis (at $\zeta = 0$), and partial no-slip conditions ($u_r = u_\varphi = 0$ but $u_z \neq 0$) hold at the base of the cylinder (at $z = 0$).

Formulas (1) together with (2) and (3) describe quite well the vortex flow arising in a teapot when tea is stirred. The rotation of the liquid near the free surface causes an intensive vertical flow, due to which tea leaves move upward. Increasing the rotation speed leads to the formation of a downward flow near the axis of rotation. A similar phenomenon is characteristic of powerful atmospheric vortices.

⊙ *Reference*: S. N. Aristov (2001).

2°. Solution in the cylindrical coordinates:

$$\begin{aligned}
&u_r=\frac{\nu}{b}\frac{f(\xi)}{\sqrt{\xi}}, \quad u_\varphi=\frac{\nu}{b}\sqrt{\frac{2}{\xi}}\,[V(\xi)+\eta g(\xi)], \quad u_z=-2\frac{\nu}{b}[W(\xi)+\eta f'_\xi(\xi)],\\
&p=p_0+\frac{\nu^2}{b^2}\Big[u'_\xi(\xi)-\frac{f^2(\xi)}{4\xi}+\frac{1}{2}\int V^2(\xi)\frac{d\xi}{\xi^2}-4\eta F'_\xi(\xi)-2\eta^2 G(\xi)\Big],\\
&\xi=(r/b)^2, \quad \eta=z/b,
\end{aligned} \tag{4}$$

where a, b, and p_0 are arbitrary constants, and the functions $f = f(\xi)$, $g = g(\xi)$, $V = V(\xi)$, $W = W(\xi)$, $F = F(\xi)$, and $G = G(\xi)$ are determined by the system of ordinary differential equations

$$2\xi f'''_{\xi\xi\xi}=(f-2)f''_{\xi\xi}-(f'_\xi)^2+2G, \quad 2\xi g''_{\xi\xi}=fg'_\xi-gf'_\xi, \quad 4\xi^2 G'_\xi=-g^2; \tag{5}$$

$$2\xi W''_{\xi\xi}=(f-2)W'_\xi-f'_\xi W+2F'_\xi, \quad 2\xi V''_{\xi\xi}=fV'_\xi-gW, \quad 4\xi^2 F''_{\xi\xi}=-gV. \tag{6}$$

The first three equations in (5) represent a closed system for the unknowns f, g, and G. The last three equations in (6) are linear in the unknowns V, W, and F.

Formulas (4) and equations (5)–(6) enable one to solve the problem of a rotationally symmetric fluid flow in the gap between two coaxial cylindrical surfaces. Let the inner cylinder of radius a rotate at a constant angular velocity ω and the outer cylinder of radius b be at rest. We assume that the difference between the average pressures in two cross-sections of the gap, S_1 and S_2, separated by a distance h, is known, so that the quantity

$$\Delta p=\frac{1}{\pi(b^2-a^2)}\int_a^b\int_0^{2\pi}\Big(p|_{z=0}-p|_{z=h}\Big)r\,d\varphi\,dr \tag{7}$$

is given.

In this problem, equations (5)–(6) must be supplemented with the boundary conditions

$$\begin{aligned}
&f=f'_\xi=g=W=0, \quad V=\Omega, \quad F=F_a \quad \text{at} \quad \xi=\xi_a \quad \text{(on the inner surface)},\\
&f=f'_\xi=g=V=W=F=0 \qquad\qquad\quad \text{at} \quad \xi=1 \quad \text{(on the outer surface)},
\end{aligned} \tag{8}$$

where $\xi_a = (a/b)^2$ and $\Omega = a^2\omega/(\sqrt{2}\,\nu)$. Furthermore, the integral condition

$$\Delta p=\frac{4\nu^2 H}{b^2-a^2}\left(H\int_{\xi_a}^1 G(\xi)\,d\xi-2F_0\right), \quad H=\frac{h}{b},$$

which follows from (7), must be added.

Numerical analysis of problem (5)–(6), (8) shows that, in particular, the trivial solution of the nonlinear system (5) with the homogeneous conditions $f = f'_\xi = g = 0$ at $\xi = \xi_a$ and $\xi = 1$ is not unique.

⊙ *Reference*: S. N. Aristov and D. V. Knyazev (2007).

Unsteady solutions.

3°. There is a solution where the velocity components only depend on the single spatial coordinate r:

$$u_r = \frac{a(t)}{r}, \qquad u_\varphi = u_\varphi(r,t), \qquad u_z = u_z(r,t),$$
$$p = p_0(t) - b(t)\varphi - c(t)z - a'_t(t)\ln r - \frac{a^2(t)}{2r^2} + \int u_\varphi^2(r,t)\frac{dr}{r},$$

where $a(t)$, $b(t)$, $c(t)$, and $p_0(t)$ are arbitrary functions, and the functions $u_\varphi = u_\varphi(r,t)$ and $u_z = u_z(r,t)$ satisfy two independent linear partial differential equations

$$\frac{\partial u_\varphi}{\partial t} + \frac{a(t)-\nu}{r}\frac{\partial u_\varphi}{\partial r} + \frac{a(t)+\nu}{r^2}u_\varphi - \frac{b(t)}{r} = \nu\frac{\partial^2 u_\varphi}{\partial r^2},$$
$$\frac{\partial u_z}{\partial t} + \frac{a(t)-\nu}{r}\frac{\partial u_z}{\partial r} - c(t) = \nu\frac{\partial^2 u_z}{\partial r^2}.$$

4°. By using a stream function $\psi = \psi(r,z)$ such that

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

and eliminating the pressure, one can reduce the equations of rotationally symmetric motions to the following two equations:

$$E\psi_t + r\frac{\partial(\psi, r^{-2}E\psi)}{\partial(r,z)} + 2r^{-2}JJ_z = \nu E^2\psi, \tag{9}$$
$$J_t - \frac{1}{r}\psi_z J_r + \frac{1}{r}\psi_r J_z = \nu E J, \qquad J = ru_\varphi, \tag{10}$$

where E is the Stokes operator

$$E = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

System (9)–(10) admits solutions of the form

$$\psi = [a(t)r^2 + b(t)]z + F(r,t), \qquad J = J(r,t),$$

where $a(t)$ and $b(t)$ are arbitrary functions, and the functions $F = F(r,t)$ and $J = J(r,t)$ are described by the independent linear equations

$$\widetilde{E}F_t - r[a(t)r^2 + b(t)](r^{-2}\widetilde{E}F)_r = \nu\widetilde{E}^2F, \qquad \widetilde{E}F = F_{rr} - r^{-1}F_r,$$
$$J_t - r^{-1}[a(t)r^2 + b(t) - \nu]J_r = \nu J_{rr}.$$

System (9)–(10) also admits more complicated, generalized separable solutions of the form

$$\psi = A(r,t)z + B(r,t), \qquad J = C(r,t)z + D(r,t)$$

(generalize the previous solutions).

5°. Rotationally symmetric fluid motions can also be described by introducing, apart from the stream function, a new additional (nonlocal) dependent variable Φ:

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad u_\varphi = \frac{1}{r}J, \quad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad p = -\frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial\Phi}{\partial r},$$

where the functions $\psi = \psi(r, z)$, $J = J(r, z)$, and $\Phi = \Phi(r, z)$ are determined by the system of differential equations

$$J_t - \frac{1}{r}\psi_z J_r + \frac{1}{r}\psi_r J_z = \nu E J, \tag{11}$$

$$\psi_t - \frac{1}{r}\psi_r\psi_z + \Phi_z = \nu E\psi, \tag{12}$$

$$E\Phi = \frac{1}{r^2}(J^2 + \psi_z^2) + \frac{2}{r}\psi_r E\psi. \tag{13}$$

Remark. System (9)–(10) admits the infinitesimal operators

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = 2t\partial_t + r\partial_r + z\partial_z + \psi\partial_\psi, \quad \mathrm{X}_3 = 2\alpha\partial_z + \alpha_t' r^2\partial_\psi, \quad \mathrm{X}_4 = \beta\partial_\psi,$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are arbitrary constants.

System (11)–(13) admits more infinitesimal operators (the result obtained by S. V. Golovin and included, with his consent, into the paper by Aristov and Pukhnachov, 2004):

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = 2t\partial_t + r\partial_r + z\partial_z + \psi\partial_\psi, \quad \mathrm{X}_3 = 2\alpha\partial_t + \alpha_t' r^2\partial_\psi + (4\alpha_t' - \alpha_{tt}'' r^2 z)\partial_\Phi,$$
$$\mathrm{X}_4 = \beta\partial_\psi - \beta_t' z\partial_\Phi, \quad \mathrm{X}_5 = \gamma\partial_\Phi, \quad \mathrm{X}_6 = \delta r^2\partial_\Phi,$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, and $\delta = \delta(t)$ are arbitrary constants. One can see that changing to the new unknowns has resulted in a system that possesses richer group properties.

⊙ *Reference*: S. N. Aristov and V. V. Pukhnachov (2004).

6°. System (11)–(13) has exact solutions of the form

$$\begin{aligned}
&\psi = \frac{z}{2\alpha}(\alpha_t' r^2 + 2\beta) + f(r, t), \quad J = g(r, t),\\
&\Phi = \frac{z^2}{2\alpha^2}\left\{\left[(\alpha_t')^2 - \frac{1}{2}\alpha\alpha_{tt}''\right]r^2 + 2\alpha_t'\beta - \alpha\beta_t'\right\} + \frac{2\alpha_t'}{\alpha}z f(r, t) + h(r, t),
\end{aligned} \tag{14}$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are arbitrary functions, and the functions $f = f(r, t)$, $g = g(r, t)$, and $h = h(r, t)$ are determined by the linear differential equations

$$f_t - \frac{\alpha_t' r^2 + 2\beta}{2\alpha r}f_r + 2\frac{\alpha_t'}{\alpha}f = \nu\left(f_{rr} - \frac{1}{r}f_r\right), \tag{15}$$

$$g_t - \frac{\alpha_t' r^2 + 2\beta}{2\alpha r}g_r = \nu\left(g_{rr} - \frac{1}{r}g_r\right), \tag{16}$$

$$\begin{aligned}
h_{rr} - \frac{1}{r}h_r = &-\frac{1}{\alpha^2}\left\{\left[(\alpha_t')^2 - \frac{1}{2}\alpha\alpha_{tt}''\right]r^2 + 2\alpha_t'\beta - \alpha\beta_t'\right\}\\
&+ \frac{1}{r^2}g^2 + \frac{(\alpha_t' r^2 + 2\beta)^2}{4\alpha^2 r^2} + \frac{2}{r}f_r\left(f_{rr} - \frac{1}{r}f_r\right).
\end{aligned} \tag{17}$$

Equations (15) and (16) are independent. In addition, equation (15) is reduced, with the change of variable $f = \alpha^{-2}\widetilde{f}$, to the equation

$$\widetilde{f}_t - \frac{\alpha_t' r^2 + 2\beta}{2\alpha r}\widetilde{f}_r = \nu\left(\widetilde{f}_{rr} - \frac{1}{r}\widetilde{f}_r\right),$$

which coincides, up to obvious renaming, with equation (16). Equations of this form were considered in the handbook by Polyanin (2002, p. 133) and a number of their exact solutions were obtained. It is noteworthy that the transformation of the independent variables $\tau = \int |\alpha(t)|\,dt$, $\xi = \sqrt{|\alpha(t)|}\,r$ reduces the last equation to the simpler form

$$\widetilde{f}_\tau + \big[\nu - k(\tau)\big]\frac{1}{\xi}\widetilde{f}_\xi = \nu f_{\xi\xi},$$

where $k(\tau) = \alpha(t)/\beta(t)$.

The last equation in (17) represents a linear ordinary differential equation for h, where time t appears as a parameter; the equation admits order reduction with the change of variable $Z = h_r$ and is easy to integrate.

⊙ *Reference*: S. N. Aristov and V. V. Pukhnachov (2004).

7°. System (11)–(12) admits generalized separable solutions in z (generalization of the solution from Item 6°) such that

$$\begin{aligned}&\psi = A(r,t)z + B(r,t), \quad J = C(r,t)z + D(r,t),\\ &\Phi = F(r,t)z^2 + G(r,t)z + H(r,t),\end{aligned} \tag{18}$$

where the functions A, B, C, D, F, G, and H are determined by the system of differential equations

$$\begin{aligned}
A_t - \frac{1}{r}AA_r + 2F &= \nu\Big(A_{rr} - \frac{1}{r}A_r\Big),\\
C_t + \frac{1}{r}(CA_r - AC_r) &= \nu\Big(C_{rr} - \frac{1}{r}C_r\Big),\\
F_{rr} - \frac{1}{r}F_r - \frac{1}{r^2}C^2 &= \frac{2}{r}A_r\Big(A_{rr} - \frac{1}{r}A_r\Big),\\
B_t - \frac{1}{r}AB_r + G &= \nu\Big(B_{rr} - \frac{1}{r}B_r\Big),\\
D_t + \frac{1}{r}(CB_r - AD_r) &= \nu\Big(D_{rr} - \frac{1}{r}D_r\Big),\\
G_{rr} - \frac{1}{r}G_r - \frac{2}{r^2}CD &= \frac{2}{r}A_r\Big(B_{rr} - \frac{1}{r}B_r\Big) + \frac{2}{r}B_r\Big(A_{rr} - \frac{1}{r}A_r\Big),\\
H_{rr} - \frac{1}{r}H_r + 2F - \frac{1}{r^2}(A^2 + D^2) &= \frac{2}{r}B_r\Big(B_{rr} - \frac{1}{r}B_r\Big).
\end{aligned}$$

The first three equations form a closed quasi-linear system for A, C, and F. The next three equations are linear in B, D, and G and do not involve H. The last equation is a linear ordinary differential equation for H, where time t appears as a parameter; the equation admits order reduction with the change of variable $Z = H_r$ and is easy to integrate. Note that the system admits solutions with $B = D = G \equiv 0$.

Special case 1. The solutions of the form

$$\psi = zU(r), \quad J = zV(r), \quad \Phi = z^2W(r) \tag{19}$$

are a special case of solution (18) and describe steady-state cylindrical vortices; for details about these solutions, see Item 1° in this subsection.

Special case 2. The self-similar solutions

$$\psi = zU(\xi), \quad J = \frac{z}{\sqrt{t}}V(\xi), \quad \Phi = \frac{z^2}{t}W(\xi), \quad \xi = \frac{r}{\sqrt{\nu t}},$$

also belong to the class of solutions (18) and are somewhat analogous to the steady-state solutions (19).

⊙ *References*: S. N. Aristov and V. V. Pukhnachov (2004), V. V. Pukhnachov (2006).

21.5. Solutions with Three Nonzero Fluid Velocity Components Dependent on Three Space Variables

21.5.1. Three-Dimensional Stagnation-Point Type Flows

1°. Solution:

$$u = kxf'(\eta), \quad v = kyg'(\eta), \quad w = -\sqrt{k\nu}\,[f(\eta) + g(\eta)], \quad \eta = \sqrt{k/\nu}\,z,$$
$$p = p_0 - \tfrac{1}{2}(kx^2 + ly^2) - k\nu[f'(\eta) + g'(\eta)] - \tfrac{1}{2}k\nu[f(\eta) + g(\eta)]^2, \tag{1}$$

where the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$f'''_{\eta\eta\eta} + (f + g)f''_{\eta\eta} - (f'_\eta)^2 + 1 = 0,$$
$$g'''_{\eta\eta\eta} + (f + g)g''_{\eta\eta} - (g'_\eta)^2 + a^2 = 0, \quad a = l/k. \tag{2}$$

If equations (2) are supplemented with the boundary conditions

$$f(0) = g(0) = f'_\eta(0) = g'_\eta(0) = 0, \quad f'_\eta(\infty) = 1, \quad g'_\eta(\infty) = a, \tag{3}$$

then formulas (1) together with (2) and (3) describe a three-dimensional stagnation-point flow on a plane boundary.

Analysis has shown that problem (2)–(3) has two solutions for $-1 \le a \le 1$.

⊙ *References*: L. Howarth (1951), A. Davey (1961), P. L. Libby (1967), A. Davey and D. Schofield (1967), D. Schofield and A. Davey (1967), R. E. Hewitt, P. W. Duck, and S. R. Stow (2002).

2°. Solution in the cylindrical coordinates:

$$u_r = krf'_\eta(\eta) + ag(\eta)\cos\varphi, \quad u_\varphi = -ag(\eta)\sin\varphi, \quad u_z = -2\sqrt{k\nu}\,f(\eta),$$
$$p = p_0 - \tfrac{1}{2}k^2r^2 - 2k\nu[f^2(\eta) + f'_\eta(\eta)], \quad \eta = \sqrt{k/\nu}\,z, \tag{4}$$

where a, k, and p_0 are arbitrary constants and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$f'''_{\eta\eta\eta} + 2ff''_{\eta\eta} - (f'_\eta)^2 + 1 = 0,$$
$$g''_{\eta\eta} + 2fg'_\eta - f'_\eta g = 0. \tag{5}$$

Equations (5) can be supplemented with the boundary conditions

$$f(0) = b, \quad f'_\eta(0) = 0, \quad f'_\eta(\infty) = 1, \quad g(0) = 1, \quad g(\infty) = 0. \tag{6}$$

For $a = b = 0$, formulas (4) together with (5) and (6) describe a classical stagnation-point flow (the *Homann flow*) on a plane boundary.

⊙ *References*: F. Homann (1936), C. Y. Wang (1973), P. A. Libby (1974, 1976).

3°. Solution:

$$u = \frac{kx}{1+akt} f'_\eta(\eta), \qquad v = \frac{ky}{1+akt} g'_\eta(\eta), \qquad w = -\frac{\sqrt{k\nu}}{\sqrt{1+akt}}[f(\eta)+g(\eta)],$$
$$p = p_0 - \frac{k^2(1-a)}{2(1+akt)^2}(x^2+b^2y^2) - \frac{k\nu}{2(1+akt)}\big\{2[f'_\eta(\eta)+g'_\eta(\eta)] \tag{7}$$
$$+ [f(\eta)+g(\eta)]^2 + a\eta[f(\eta)+g(\eta)]\big\}, \qquad \eta = \frac{\sqrt{k/\nu}\, z}{\sqrt{1+akt}},$$

where a, b, and k are arbitrary constants, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$f'''_{\eta\eta\eta} + (f+g)f''_{\eta\eta} - (f'_\eta)^2 + a\big(f'_\eta + \tfrac{1}{2}\eta f''_{\eta\eta}\big) = a-1,$$
$$g'''_{\eta\eta\eta} + (f+g)g''_{\eta\eta} - (g'_\eta)^2 + a\big(g'_\eta + \tfrac{1}{2}\eta g''_{\eta\eta}\big) = ab-b^2, \tag{8}$$

with boundary conditions

$$f(0) = f'_\eta(0) = g(0) = g'_\eta(0) = 0, \quad f'_\eta(\infty) = 1, \quad g'_\eta(\infty) = b. \tag{9}$$

The solution describes an unsteady three-dimensional stagnation-point flow. The case $a = 0$ corresponds to a steady-state flow and the case $b = 1$ ($a \neq 0$) corresponds to an axisymmetric unsteady flow.

Remark. The same formulas for the velocity components (7), however, with a different pressure, describe an unsteady plate stretching for a particular time dependence. In this case, the constant terms on the right-hand sides of equations (8) must be omitted and the boundary conditions

$$f(0) = g(0) = 0, \quad f'_\eta(0) = 1, \quad g'_\eta(0) = b, \quad f'_\eta(\infty) = g'_\eta(\infty) = 0$$

must be used instead of conditions (9).

⊙ *References*: K. T. Yang (1958), J. C. Williams III (1968), E. H. W. Cheng, M. N. Özisik, and J. C. Williams III (1971), N. R. Rajappa (1979), C. D. S. Devi, H. S. Takhar, and G. Nath (1986).

21.5.2. Solutions with Linear Dependence of the Velocity Components on Two Space Variables. Axial Flows

Structure of exact solutions. The determining system of equations. The Navier–Stokes equations admit a wide class of exact solutions with linear dependence of the fluid velocity components on two space variables:

$$u = x\Big(f - \frac{1}{2}\frac{\partial w}{\partial z}\Big) + yg, \quad v = xh - y\Big(f + \frac{1}{2}\frac{\partial w}{\partial z}\Big), \quad w = w(z,t),$$
$$p = p_0 - \frac{1}{2}\alpha x^2 - \frac{1}{2}\beta y^2 - \gamma xy - \frac{1}{2}w^2 + \nu\frac{\partial w}{\partial z} - \frac{\partial}{\partial t}\int w\,dz, \tag{1}$$

where $p_0 = p_0(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are arbitrary functions determining the pressure distribution across the domain of flow, $f = f(z,t)$, $g = g(z,t)$, $h = h(z,t)$, and $w = w(z,t)$ are unknown functions, which are described by the system of equations

$$\frac{\partial^2 w}{\partial t\partial z} + w\frac{\partial^2 w}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} + 2(f^2+gh) - \alpha - \beta, \tag{2}$$
$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial z} - f\frac{\partial w}{\partial z} = \nu\frac{\partial^2 f}{\partial z^2} + \frac{\alpha-\beta}{2}, \tag{3}$$
$$\frac{\partial g}{\partial t} + w\frac{\partial g}{\partial z} - g\frac{\partial w}{\partial z} = \nu\frac{\partial^2 g}{\partial z^2} + \gamma, \tag{4}$$
$$\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial z} - h\frac{\partial w}{\partial z} = \nu\frac{\partial^2 h}{\partial z^2} + \gamma. \tag{5}$$

Remark 1. The original idea to look for exact solutions of the Navier–Stokes equations in a form where the velocity components are linearly dependent on two space variables,

$$u = f_1(z,t)x + g_1(z,t)y, \quad v = f_2(z,t)x + g_2(z,t)y, \qquad w = w(z,t), \tag{6}$$

should, apparently, be credited to Lin (1958).

Remark 2. With $\gamma = 0$, the solution structure (1) and system (2)–(5) were obtained by Meleshko (2004) when studying a class of partially invariant solutions; the case $\alpha = \beta = \gamma = 0$ was considered by Meleshko and Pukhnachov (1999). In Meleshko (2004), a group classification of system (2)–(5) with $\gamma = 0$ was carried out, which resulted in singling out two sorts of time dependence of the determining functions: (i) α and β are constant and (ii) α and β are both proportional to t^{-2}. To these cases there correspond traveling-wave and self-similar solutions, respectively.

Presented below are wide classes of exact solutions to system (2)–(5) where the determining functions α, β, and γ contain functional arbitrariness. The main idea of the subsequent analysis is to isolate the equation for the z-axis velocity component, w, from system (2)–(5).

General property of system (2)–(5). Suppose a set of functions $w_0(z,t)$, $f_0(z,t)$, $g_0(z,t)$, and $h_0(z,t)$ solve system (2)–(5). Then the set of functions

$$\begin{gathered} w = w_0(z+\psi(t),t) - \psi_t'(t), \\ f = f_0(z+\psi(t),t), \quad g = g_0(z+\psi(t),t), \quad h = h_0(z+\psi(t),t), \end{gathered} \tag{7}$$

where $\psi(t)$ is an arbitrary function, will also solve system (2)–(5).

Reduction of system (2)–(5) to a single equation. Consider a special class of exact solutions described by a single equation. In (2)–(5), we set

$$f = k\frac{\partial w}{\partial z} + C, \quad g = n\frac{\partial w}{\partial z} + B, \quad h = m\frac{\partial w}{\partial z} + A, \tag{8}$$

where m, n, k, A, B, and C are unknown functions of time t. Let us demand that the four equations (2)–(5) all coincide after substituting expressions (8) into them. As a result, we obtain a nonlinear system for the unknown functions, consisting of one algebraic equation and six ordinary differential equations:

$$mn + k^2 = \frac{1}{4}, \tag{9}$$

$$\frac{A - m_t'}{m} = \frac{B - n_t'}{n} = \frac{C - k_t'}{k} = 2(An + Bm + 2Ck), \tag{10}$$

$$\frac{\gamma - A_t'}{m} = \frac{\gamma - B_t'}{n} = \frac{\alpha - \beta - 2C_t'}{2k} = -\alpha - \beta + 2AB + 2C^2. \tag{11}$$

This system contains seven equations for nine unknowns—six functions m, n, k, A, B, and C from (8) and three functions α, β, and γ from (2)–(5) (which are also considered unknown here). It can be shown that the last equation in (10) follows from the other three equations (9)–(10). Therefore, three unknown functions in system (9)–(11) can, in general, be regarded as arbitrary.

In view of (8)–(11), system (2)–(5) is reduced to a single equation:

$$\frac{\partial^2 w}{\partial t \partial z} + w\frac{\partial^2 w}{\partial z^2} - \left(\frac{\partial w}{\partial z}\right)^2 = \nu\frac{\partial^3 w}{\partial z^3} + q\frac{\partial w}{\partial z} + p, \tag{12}$$

where the functions $p = p(t)$ and $q = q(t)$ are given by

$$p = \frac{\gamma - A_t'}{m}, \quad q = \frac{A - m_t'}{m}. \tag{13}$$

Equation (12) has the following general property: if $w_0(z,t)$ is a solution, then the function

$$w = w_0(z + \psi(t), t) - \psi_t'(t), \tag{14}$$

where $\psi(t)$ is an arbitrary function, will also be a solution of equation (12).

In constructing solutions to system (9)–(11), one should distinguish the following two cases.

(i) *Solution of system* (9)–(11) *with* $m = n$. In this case, the general solution of system (9)–(11) can be represented in the form

$$\begin{aligned}
&m = n = \tfrac{1}{2}\sin\varphi, \quad k = \tfrac{1}{2}\cos\varphi,\\
&A = B = \tfrac{1}{2}(q\sin\varphi + \varphi_t'\cos\varphi), \quad C = \tfrac{1}{2}(q\cos\varphi - \varphi_t'\sin\varphi),\\
&\alpha = \tfrac{1}{4}q^2 + \frac{1}{4}(\varphi_t')^2 - \tfrac{1}{2}p(1-\cos\varphi) + C_t',\\
&\beta = \tfrac{1}{4}q^2 + \frac{1}{4}(\varphi_t')^2 - \tfrac{1}{2}p(1+\cos\varphi) - C_t',\\
&\gamma = \tfrac{1}{2}p\sin\varphi + A_t',
\end{aligned} \tag{15}$$

where $p = p(t)$, $q = q(t)$, and $\varphi = \varphi(t)$ are arbitrary functions. For convenience, the free functions p and q in (15) have been chosen so as to reduce system (2)–(5), with the transformation (8), (15), to a single equation (12) with the same $p = p(t)$ and $q = q(t)$.

Hereby, we have proved the important proposition that any solution of equation (12) for any functions $p = p(t)$ and $q = q(t)$ generates an exact solution of the Navier–Stokes equations. This solution is described by the function $w = w(z,t)$ and formulas (1), (8), and (15).

(ii) *Solution of system* (9)–(11) *with* $m \neq n$. In this case, the general solution of system (9)–(11) can be represented as follows. The functions $m = m(t)$, $k = k(t)$, and $q = q(t)$ are taken arbitrarily with the only restriction that $m^2 + k^2 \neq \frac{1}{4}$. The other unknowns appearing in system (9)–(11) and equation (12) are calculated successively by the formulas

$$\begin{aligned}
&n = \frac{1-4k^2}{4m},\\
&A = mq + m_t', \quad B = nq + n_t', \quad C = kq + k_t',\\
&p = \frac{A_t' - B_t'}{n-m}, \quad \alpha = AB + C^2 + C_t' - \frac{1}{2}p(1-2k),\\
&\beta = AB + C^2 - C_t' - \frac{1}{2}p(1+2k), \quad \gamma = pm + A_t'.
\end{aligned} \tag{16}$$

In this case, the coefficient $p = p(t)$ in equation (12) is determined by the functions $m = m(t)$, $k = k(t)$, and $q = q(t)$ and their derivatives (unlike the case $m = n$, where it was arbitrary). The attempt to specify $p = p(t)$ instead of m (or k) leads to a nonlinear second-order ordinary differential equation for m (or k) with arbitrary $q = q(t)$.

Let us figure out how one should choose the function $q = q(t)$ in order that $p \equiv 0$. It follows from the expression of p in (16) that $A = B + s_0$, where s_0 is an arbitrary constant. Then, in view of formulas (16) for A, B, and n, we find q:

$$q = \frac{4s_0 m - 8(mm_t' + kk_t')}{4(m^2+k^2) - 1} + \frac{m_t'}{m} \qquad (\text{at } p = 0). \tag{17}$$

Exact solutions of the nonlinear third-order equation (12).

1°. Periodic solutions in the form of a product of functions with different arguments:

$$w = a(t)\sin(\sigma z + B), \quad a(t) = C\exp\Big[-\nu\sigma^2 t + \int q(t)\,dt\Big],$$
$$p = -\sigma^2 a^2(t), \quad q = q(t) \text{ is an arbitrary function,} \tag{18}$$

where B, C, and σ are arbitrary constants. In (18), setting $q(t) = \nu\sigma^2 + \varphi_t'(t)$, where $\varphi(t)$ is a periodic function, one obtains a periodic solution in both arguments z and t.

Special case. Consider the steady-state case. In formulas (15) and (18), we set

$$\varphi = 0, \quad a = -\frac{a_1 + a_2}{\sigma}, \quad q = \nu\sigma^2 = 2a_1, \quad p = -a^2\sigma^2, \quad \sigma = (2a_1/\nu)^{1/2}.$$

As a result, using the expressions in (1) and (8), we obtain the solution

$$u = a_1 x, \quad v = [(a_1 + a_2)\cos(\sigma z) - a_1]y, \quad w = -\frac{a_1 + a_2}{\sigma}\sin(\sigma z),$$

which describes a three-dimensional flow of a fluid layer between two plane elastic films (the locations of the films are determined by the values $z = 0$ and $z = 2\pi/\sigma$) whose surfaces are extended according to the law $u = a_1 x$, $v = a_2 y$.

2°. Generalized separable solutions exponential in z:

$$w = a(t)e^{-\sigma z} + b(t), \quad p = 0, \quad q = \frac{a_t'}{a} - \sigma b - \sigma^2\nu, \tag{19}$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions. By taking $a(t)$ and $b(t)$ to be periodic, we obtain a periodic solution in time.

Formulas (18) and (19) together with relations (1), (8), and (15) determine two classes of solutions of the Navier–Stokes equations dependent on several arbitrary functions.

3°. Solution (19) can be represented in the form

$$w = a_0\exp\Big[-\sigma z + \sigma^2\nu t + \int (q + \sigma b)\,dt\Big] + b(t), \quad p = 0, \tag{20}$$

where $b = b(t)$ and $q = q(t)$ are arbitrary functions, and a_0 is an arbitrary constant. Formula (20) together with relations (1) and (16) with $p = 0$ and (17) define a new class of exact solutions to the Navier–Stokes equations.

4°. Solution in the form of a product of functions with different arguments:

$$w = a(t)(C_1 e^{\sigma z} + C_2 e^{-\sigma z}), \quad p = 4C_1 C_2\sigma^2 a^2(t), \quad q = \frac{a_t'}{a} - \sigma^2\nu,$$

where $a = a(t)$ is an arbitrary function, and C_1, C_2, and σ are arbitrary constants.

5°. Monotonic solution and unbounded periodic traveling-wave solution:

$$w = -6\nu\sigma\tanh[\sigma(z - \lambda t) + B] + \lambda, \quad p = 0, \quad q = 8\nu\sigma^2;$$
$$w = 6\nu\sigma\tan[\sigma(z - \lambda t) + B] + \lambda, \quad p = 0, \quad q = -8\nu\sigma^2.$$

6°. Functional separable solution:

$$w = \frac{a(t)}{\lambda(t)}\exp[-\lambda(t)z] + b(t) + c(t)z,$$

where the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, and $\lambda = \lambda(t)$ satisfy the system of ordinary differential equations

$$\lambda_t' = -c\lambda, \quad a_t' = (\nu\lambda^2 + q + 2c + b\lambda)a, \quad c_t' = c^2 + qc + p.$$

Here, three out of the six functions $a(t)$, $b(t)$, $c(t)$, $\lambda(t)$, $p(t)$, and $q(t)$ can be taken arbitrarily. If, for example, $a(t)$, $b(t)$, and $\lambda(t)$ are specified (in an arbitrary way), then $c(t)$, $p(t)$, and $q(t)$ are determined elementarily (without integration).

7°. Functional separable solution:

$$w = \omega(t)z + \frac{\xi(t)}{\theta(t)}\sin[\theta(t)z + a], \tag{21}$$

where a is an arbitrary constant, and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, $\theta = \theta(t)$ are described by the system of ordinary differential equations

$$\theta_t' = -\omega\theta, \quad \omega_t' = \omega^2 + q(t)\omega + p(t) + \xi^2, \quad \xi_t' = [2\omega - \nu\theta^2 + q(t)]\xi. \tag{22}$$

In this system, the functions $\theta(t)$ and $\xi(t)$ can be treated as given (in an arbitrary way). Then the functions $\omega(t)$, $p(t)$, and $q(t)$ are determined elementarily (without integration). To periodic $\theta(t)$ and $\xi(t)$ there corresponds a periodic solution (21).

8°. Functional separable solution:

$$w = \omega(t)z + \frac{\xi(t)}{\theta(t)}\Big[C_1 e^{\theta(t)z} + C_2 e^{-\theta(t)z}\Big], \tag{23}$$

where C_1 and C_2 are arbitrary constants, and the functions $\omega = \omega(t)$, $\xi = \xi(t)$, and $\theta = \theta(t)$ are described by the system of ordinary differential equations

$$\theta_t' = -\omega\theta, \quad \omega_t' = \omega^2 + q(t)\omega + p(t) - 4C_1C_2\xi^2, \quad \xi_t' = [2\omega + \nu\theta^2 + q(t)]\xi.$$

9°. Generalized separable solution:

$$w = -a_t'(t) + b(t)[z + a(t)] - \frac{6\nu}{z + a(t)}, \quad q = -4b, \quad p = b_t' + 3b^2, \tag{24}$$

where $a = a(t)$ and $b = b(t)$ are arbitrary functions.

Remark 1. The obvious degenerate solution $w = a(t)z + b(t)$ has been omitted here. Traveling-wave and self-similar solutions have also been omitted (which correspond to special forms of p and q).

Remark 2. Property (14) of equation (12) allows us to generalize the solutions from Items 1°, 5°, 7°, and 8°.

Reduction of system (2)–(5) to two equations. Presented below are two cases in which system (2)–(5) is reducible to two equations: an isolated nonlinear equation for the velocity component w and an equation for a new auxiliary function.

(i) ***First case.*** By setting

$$f = abG, \quad g = -b^2G, \quad h = a^2G, \quad \alpha = \beta, \quad \gamma = 0, \tag{25}$$

where a and b are arbitrary constants, we reduce system (2)–(5) to an isolated equation for the velocity component w with an additional equation for $G = G(z, t)$:

$$\frac{\partial^2 w}{\partial t\partial z} + w\frac{\partial^2 w}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} - 2\alpha, \tag{26}$$

$$\frac{\partial G}{\partial t} + w\frac{\partial G}{\partial z} - G\frac{\partial w}{\partial z} = \nu\frac{\partial^2 G}{\partial z^2}. \tag{27}$$

1°. For arbitrary $\alpha = \alpha(t)$, system (26)–(27) has the solution

$$w = kz^2 + m(t)z + n(t), \quad n(t) = \frac{1}{4k}[m^2(t) - 2m'_t(t) - 4\alpha(t)],$$
$$G = A(t)z^2 + B(t)z + C(t),$$

where k is an arbitrary constant ($k \neq 0$), $m(t)$ is an arbitrary function, and $A = A(t)$, $B = B(t)$, and $C = C(t)$ are arbitrary functions described by the system of ordinary differential equations

$$A'_t + m(t)A - kB = 0,$$
$$B'_t + 2n(t)A - 2kC = 0,$$
$$C'_t + n(t)B - m(t)C = 2\nu A.$$

2°. If $\alpha(t) \equiv 0$, system (26)–(27) admits the solution

$$w = -\frac{4\nu}{z+\psi} - \psi'_t,$$
$$G = |z+\psi|^{-3/2}[C_1 \sin(\mu \ln|z+\psi|) + C_2 \cos(\mu \ln|z+\psi|)], \quad \mu = \frac{\sqrt{7}}{2},$$

where $\psi = \psi(t)$ is an arbitrary function, and C_1 and C_2 are arbitrary constants.

3°. Let $w = w(z)$ be a steady-state solution of equation (26). Then equation (27) admits an unsteady exponential solution

$$G(t,z) = e^{-\sigma t} r(z), \quad wr'_z - rw'_z = \nu r''_{zz} + \sigma r, \tag{28}$$

where σ is an arbitrary constant, and a periodic solution

$$G(t,z) = \sin(\sigma t)\Phi(z) + \cos(\sigma t)\Psi(z), \tag{29}$$

where the functions $\Phi = \Phi(z)$ and $\Psi = \Psi(z)$ are described by the system of ordinary differential equations

$$\begin{aligned} w\Phi'_z - \Phi w'_z &= \nu\Phi''_{zz} + \sigma\Psi, \\ w\Psi'_z - \Psi w'_z &= \nu\Psi''_{zz} - \sigma\Phi. \end{aligned} \tag{30}$$

Solutions (29) obtained for different values $\sigma = \sigma_1, \ldots, \sigma = \sigma_n$ (with one and the same $w = w(z)$) can be added together to give more solutions due to the linearity of equation (27).

Equation (27) for the steady-state velocity component $w(z)$ also has more complex solutions of the form $G(t,z) = e^{\mu t}[\sin(\sigma t)\Phi(z) + \cos(\sigma t)\Psi(z)]$.

With the argument z in G, r, Φ, and Ψ transformed in the same way as in h, property (7) enables us to generalize the above formulas (28)–(30).

(ii) ***Second case.*** In equations (2)–(5), we set

$$\begin{aligned} f &= \frac{1}{2}\cos\varphi\Big(\frac{\partial w}{\partial z} + q\Big) + ab\Theta, \\ g &= \frac{1}{2}\sin\varphi\Big(\frac{\partial w}{\partial z} + q\Big) - b^2\Theta, \\ h &= \frac{1}{2}\sin\varphi\Big(\frac{\partial w}{\partial z} + q\Big) + a^2\Theta, \\ \alpha &= \frac{1}{4}q^2 - \frac{1}{2}p(1-\cos\varphi) + \frac{1}{2}q'_t\cos\varphi, \\ \beta &= \frac{1}{4}q^2 - \frac{1}{2}p(1+\cos\varphi) - \frac{1}{2}q'_t\cos\varphi, \\ \gamma &= \frac{1}{2}p\sin\varphi + \frac{1}{2}q'_t\sin\varphi, \end{aligned} \tag{31}$$

where $p = p(t)$ and $q = q(t)$ are arbitrary functions, a and b are arbitrary constants, $\Theta = \Theta(z, t)$ is an unknown function, and φ is a constant determined from the transcendental equation

$$(a^2 - b^2)\sin\varphi + 2ab\cos\varphi = 0. \tag{32}$$

As a result, system (2)–(5) can be reduced to two equations

$$\frac{\partial^2 w}{\partial t\partial z} + w\frac{\partial^2 w}{\partial z^2} - \left(\frac{\partial w}{\partial z}\right)^2 = \nu\frac{\partial^3 w}{\partial z^3} + q\frac{\partial w}{\partial z} + p, \tag{33}$$

$$\frac{\partial \Theta}{\partial t} + w\frac{\partial \Theta}{\partial z} - \Theta\frac{\partial w}{\partial z} = \nu\frac{\partial^2 \Theta}{\partial z^2}. \tag{34}$$

The nonlinear equation (33) for w coincides with (12) and can be treated independently (some of its solutions have been described previously). Equation (34) is linear in the unknown Θ and admits the trivial solution $\Theta = 0$.

For the case of steady-state functional parameters α, β, γ, p, and q, it follows from the last three relations in (31) that

$$p = \pm\sqrt{(\alpha - \beta)^2 + 4\gamma^2}, \quad q = \pm\sqrt{2(\alpha + \beta + p)}, \quad \sin\varphi = 2\gamma/p.$$

It is apparent that each set of the pressure parameters α, β, and γ in (13) generates two sets of p and four sets of q in (33).

For the steady-state equations (33) and (26) (with constant p, q, and α), the unsteady equations (34) and (27) are linear separable equations, whose solutions can be sought using the Laplace transform with respect to time.

Equation (33), and also equation (26), admits the obvious degenerate solution $w = a(t)z + b(t)$; in this case, the corresponding equation (34), and also equation (27), can be reduced to the linear heat equation.

Equation (33) has solution (24). With $b(t) = 0$, the appropriate equation (34) is reduced, with the transformation $\Theta = \zeta^{-3}F(\zeta, t)$, $\zeta = z - a(t)$, to the classical heat equation $F_t = \nu F_{\zeta\zeta}$.

Representation of solutions to equation (34) in terms of solutions to equation (33). Suppose $w = w(z, t)$ is a solution to equation (33). Then equation (34) is solved by

$$\Theta = A'_t + Aq + A\frac{\partial w}{\partial z} + B\frac{\partial^2 w}{\partial z^2}, \tag{35}$$

where the functions $A = A(t)$ and $B = B(t)$ satisfy the ordinary differential equations

$$A''_{tt} + qA'_t + (p + q'_t)A = 0 \tag{36}$$

$$B'_t + qB = 0. \tag{37}$$

This fact is proved by eliminating Θ from (34) and (35) followed by comparing the resulting relation with both equation (33) and the equation resulting from the differentiation of (33) with respect to z.

The general solution of equation (37) is given by $B = C\exp\left(-\int q\,dt\right)$, where C is an arbitrary constant.

Stability/instability analysis of solutions, based on equation (34). For the stability/instability analysis of solutions, we use formula (35) and equations (36)–(37), which relate solutions of equations (33)–(34). It is significant that there is no need to know the explicit form of the function w in many cases.

First, let us consider problems with a steady-state longitudinal velocity component, which corresponds to $w = w(z)$ with $p = \text{const}$ and $q = \text{const}$. In this case, the solution of equation (36) depends on the sign of the discriminant $\Delta = q^2 - 4p$:

$$A(t) = \begin{cases} \exp\left(-\dfrac{qt}{2}\right)\left[C_1 \exp\left(\dfrac{t\sqrt{\Delta}}{2}\right) + C_2 \exp\left(-\dfrac{t\sqrt{\Delta}}{2}\right)\right] & \text{if } \Delta > 0, \\ \exp\left(-\dfrac{qt}{2}\right)\left[C_1 \sin\left(\dfrac{t\sqrt{|\Delta|}}{2}\right) + C_2 \cos\left(\dfrac{t\sqrt{|\Delta|}}{2}\right)\right] & \text{if } \Delta < 0, \\ \exp\left(-\dfrac{qt}{2}\right)(C_1 t + C_2) & \text{if } \Delta = 0, \end{cases} \tag{38}$$

where C_1 and C_2 are arbitrary constants. For simplicity, in what follows, we set $B = 0$ in (35) and (37). Two cases will be distinguished in the analysis.

(i) ***Nondegenerate case*** $\boldsymbol{w_z \not\equiv 0}$**.** For $q < 0$ (with any p) or $p < 0$ (with any q), solutions (35) and (38) increase exponentially as $t \to \infty$, provided that $C_1 \neq 0$. Therefore, the above values of p and q determine the domain of nonlinear instability of system (33)–(34) for any bounded steady-state profile of the longitudinal velocity component $w(z)$ (other than constant). The point $p = q = 0$ also belongs to the domain of instability of system (33)–(34).

Indeed, by virtue of (35) and (38), one can choose the constants C_1 and C_2 so as to make the initial value $|\Theta|_{t=0}$, which is treated as the initial perturbation relative to the trivial solution $\Theta = 0$ of equation (34), less than any given positive ε. However, for $q < 0$ (with any p) or $p < 0$ (with any q), we have $|\Theta| \to \infty$ as $t \to \infty$. This means that arbitrarily small initial perturbations of solutions to system (33)–(34) increase without bound with time.

Remark. If $w \to w_1$ as $z \to -\infty$ and $w \to w_2$ as $z \to +\infty$ ($w_1, w_2 = \text{const}$), then solution (35) with $A = 0$ and $B \neq 0$ vanishes as $z \to \pm\infty$.

If $q = 0$ with $p > 0$, solution (38) and, hence, solution (35) are periodic. The inequalities $q \geq 0$ and $p \geq 0$ (with $|p| + |q| \neq 0$) define the domain of conditional stability of the solutions under consideration.

It should be emphasized that *nonlinear instability* is discussed here and all results and solutions obtained above are exact rather than linearized, as this is the case in linear stability theory; also we have not used the various assumptions, expansions, and approximations characteristic of many nonlinear theories such as those presented by, for example, Goldshtik and Shtern (1977), Landau and Lifshitz (1987), and Godréche and Manneville (1998).

Example 1. The steady-state spatially periodic solution

$$w = a\sin(\sigma z + b), \quad \Theta = 0 \quad (p = -a^2\sigma^2, \quad q = \nu\sigma^2)$$

of system (33)–(34) is unstable for any values of a, b, and σ (with $a \neq 0$ and $\sigma \neq 0$).

Example 2. The steady-state monotonic bounded solution

$$w = -6\nu\sigma \tanh(\sigma z + b), \quad \Theta = 0 \quad (p = 0, \quad q = 8\nu\sigma^2)$$

of system (33)–(34) is unstable.

All above conclusions on stability and instability as well as formulas (35)–(38) remain valid for any unsteady solutions $w = w(z, t)$, $\Theta = \Theta(z, t)$ (provided that the derivative $w_z \not\equiv 0$ is bounded) of system (33)–(34) with $p = \text{const}$ and $q = \text{const}$.

In view of the aforesaid, three quarters of the plane of the parameters p and q correspond to unstable solutions. It is significant that the instability of solutions mentioned above is not connected with any specific velocity profile and is realized through equation (34), responsible for the transverse fluid velocity components. Since the fluid viscosity ν is not involved in equation (36) and formulas (38), the above results are independent of the Reynolds number, which means that the property of instability of these solutions occurs not only at sufficiently large but also small Reynolds numbers ($0 < \text{Re} < \infty$).

Remark. In a similar way, using equations (35)–(36), one can perform the stability analysis of unsteady solutions to equations (33)–(34) with variable $p = p(t)$ and $q = q(t)$.

(ii) ***Degenerate case*** $w_z \equiv 0$**.** Suppose

$$w = a = \text{const} \qquad (p = 0). \tag{39}$$

Then any solution of equation (34) reducible, with the change of variables from z and t to t and $\xi = z - at$, to the classical heat equation, is stable for any values of the parameters a and q.

Stability analysis of solutions $w = \text{const}$ *to equation (33) with* $p = 0$*.*

1°. First, let us perform the stability analysis of the trivial solution $w = 0$ of equation (33) with $p = 0$ for various values of $q = \text{const}$. Equation (33) admits the solution

$$w = \varepsilon e^{ikz+\lambda t}, \qquad \lambda = q - \nu k^2, \tag{40}$$

where ε, k, and λ are real numbers. This function also solves the linearized equation (33) in which the quadratic terms are dropped (extremely rare case). The absolute value of the difference between solution (40) and the trivial solution at the initial time is equal to $|\varepsilon|$ (through the choice of ε, this difference can be made arbitrarily small).

The trivial solution is unstable for $q - \nu k^2 > 0$ and stable for $q - \nu k^2 < 0$. The stability boundary is a parabola, $q = \nu k^2$, in the kq-plane. As the fluid viscosity decreases, $\nu \to 0$, which corresponds to increasing Reynolds numbers, the branches of the parabola tend to the straight line $q = 0$ and the domain of instability gets wider and, in the limit, covers the entire upper half-plane $q > 0$. Increasing ν or k results in widening the domain of stability.

Since the parameter k can be handled freely, one can achieve instability of the trivial solution through the choice of k for any $q > 0$.

2°. Now consider an arbitrary steady-state solution (39). Instead of (40), we now take the function

$$w = \varepsilon e^{ik(z-at)+\lambda t} + a, \qquad \lambda = q - \nu k^2, \tag{41}$$

which, by virtue of property (10) with $\psi(t) = -at$, is also a solution of equation (33). The absolute value of the difference between solutions (39) and (41) at the initial time can be made arbitrarily small through the choice of ε. All stability and instability criteria for solution (39) with respect to the parameters k and q remain the same as for the trivial solution.

Table 21.1 summarizes the results of the stability/instability analysis of bounded solutions to system (33)–(34) for various values of p and q. It is apparent that the domain of instability is significantly larger (it covers three out of four quadrants in the pq-plane) than the domain of stability.

Discussion of results. The above instability of solutions to the Navier–Stokes equations in wide ranges of variation of the parameters p and q ($q < 0$ with any $p \neq 0$ or $p < 0$ with any q) occurs for any shape of the profiles of the transverse velocity components and is independent of the Reynolds number; this means that the property of instability of the class of solutions in question manifests itself not only at sufficiently large but also arbitrarily small Reynolds numbers, $\mathrm{Re} \neq 0$. This effect is due to instability of the transverse velocity components, described by equation (34).

Solid walls, where no-slip conditions are imposed, usually serve as a stabilizing factor. As the Reynolds number increases, the fluid flow in the core gradually ceases to feel the influence of the boundary conditions. When a critical Reynolds number is attained, the mechanism of instability of solutions to the Navier–Stokes equations begins to dominate

TABLE 21.1
Domains of instability and stability of solutions to system (33)–(34)

Range of variation of parameters	Longitudinal velocity component w	Stability/instability
$p < 0$ and q is any; or $q < 0$ and p is any; or $p = q = 0$	w is any solution, except $w \equiv \text{const}$	Solutions of equation (34) are unstable
$p \geq 0$ and $q \geq 0$ (both inequalities hold together, with $\lvert p\rvert + \lvert q\rvert \neq 0$)	w is any solution, except $w \equiv \text{const}$	Solutions of equation (34) are conditionally stable
$p = 0$ and $q > 0$	$w \equiv \text{const}$	Solutions of equation (33) are unstable (although solutions of equation (34) are stable)
$p = 0$ and $q \leq 0$	$w \equiv \text{const}$	Solutions of both equations (33) and (34) are stable

and the flow becomes turbulent (for fluid flows in smooth pipes, critical Reynolds numbers can reach several thousands; see, for example, Loitsyanskiy (1996)).

If there are no solid walls, the instability of solutions to the Navier–Stokes equations arises significantly earlier. For example, laminar jets lose stability already at fairly low Reynolds numbers, $\mathrm{Re} \approx 5$; see, for example, Polyanin, Kutepov, et al. (2002).

Order reduction of the equation for the longitudinal velocity component using a Crocco-type transformation. Equations (12), (26), and (33) for the longitudinal velocity component w can be written as

$$\frac{\partial^2 w}{\partial t \partial z} + w\frac{\partial^2 w}{\partial z^2} - m\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} + q(t)\frac{\partial w}{\partial z} + p(t), \tag{42}$$

where the coefficient m can assume two values, $m = 1$ and $m = 1/2$.

Denote

$$\eta = \frac{\partial w}{\partial z}, \qquad \Phi = \frac{\partial^2 w}{\partial z^2}. \tag{43}$$

Transferring the term mw_z^2 (here and henceforth, we use a short notation for derivatives) to the right-hand side of equation (42), dividing the resulting equation by $w_{zz} = \Phi$, differentiating with respect to z, and taking into account (43), we obtain

$$\frac{\Phi_t}{\Phi} - \frac{w_{zt}\Phi_z}{\Phi^2} + \eta = \frac{\partial}{\partial z}\frac{\nu\Phi_z + m\eta^2 + q(t)\eta + p(t)}{\Phi}. \tag{44}$$

In (44), let us change from the old variables t, x, and $w = w(x, t)$ to the new variables t, η, and $\Phi = \Phi(t, \eta)$, with η and Φ defined by formulas (43). The derivatives are transformed as follows:

$$\frac{\partial}{\partial z} = \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} = w_{zz}\frac{\partial}{\partial \eta} = \Phi\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial t} + w_{zt}\frac{\partial}{\partial \eta}.$$

As a result, equation (44) is reduced to the second-order equation

$$\frac{\eta}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left(\frac{m\eta^2 + q\eta + p}{\Phi}\right) + \nu\frac{\partial^2 \Phi}{\partial \eta^2}$$

or

$$\frac{\partial \Phi}{\partial t} + (m\eta^2 + q\eta + p)\frac{\partial \Phi}{\partial \eta} = [(2m-1)\eta + q]\Phi + \nu\Phi^2\frac{\partial^2 \Phi}{\partial \eta^2}. \tag{45}$$

Note that in the degenerate case (inviscid fluid, $\nu = 0$), the original nonlinear second-order equation (42) reduces to the linear first-order equation (45), which can be integrated by the method of characteristics (see Subsection 24.3.5).

Given a solution of the original equation (42), formulas (43) define a solution of equation (45) in parametric form.

Suppose $\Phi = \Phi(\eta, t)$ is a solution of equation (45). Then the corresponding solution of the original equation (42) can also be represented in parametric form:

$$z = \int \frac{ds}{\Phi(s,t)} + \psi(t), \quad w = \int \frac{s\,ds}{\Phi(s,t)} - \psi_t'(t),$$

where $\psi(t)$ is an arbitrary function (in the integration, t is treated as a parameter).

Linear transformations of the equation for the velocity component w. Consider equation (42) again. Recall that if $m = 1$, both functions $p(t)$ and $q(t)$ in (42) can be chosen arbitrarily.

The linear transformation with respect to the desired function

$$w = a(t)F(\tau, \xi) + b(t)z + c(t), \quad \xi = \lambda(t)z + \sigma(t), \quad \tau = \int \lambda^2(t)\,dt + C_0, \tag{46}$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, $\lambda = \lambda(t)$, and $\sigma = \sigma(t)$ are arbitrary functions, reduces equation (42) to the form

$$\frac{\partial^2 F}{\partial \tau \partial \xi} + [\widetilde{a}(\tau)F + \widetilde{b}(\tau)\xi + \widetilde{c}(\tau)]\frac{\partial^2 F}{\partial \xi^2} - m\widetilde{a}(\tau)\Big(\frac{\partial F}{\partial \xi}\Big)^2 = \nu\frac{\partial^3 F}{\partial \xi^3} + \widetilde{q}(\tau)\frac{\partial F}{\partial \xi} + \widetilde{p}(\tau), \tag{47}$$

where

$$\begin{aligned} &\widetilde{a} = \frac{a}{\lambda}, \quad \widetilde{b} = \frac{1}{\lambda^3}(b\lambda + \lambda_t'), \quad \widetilde{c} = \frac{1}{\lambda^3}(c\lambda^2 - b\lambda\sigma + \lambda\sigma_t' - \sigma\lambda_t'), \\ &\widetilde{q} = \frac{1}{a\lambda^3}[aq\lambda + 2mab\lambda - (a\lambda)_t'], \quad \widetilde{p} = \frac{1}{a\lambda^3}(p + bq + mb^2 - b_t'). \end{aligned} \tag{48}$$

The argument of the functions on the left-hand sides of the equations is τ, while that on the right-hand sides is t; the variables τ and t are connected by the last relation in (46).

The presence of a large number of arbitrary functions (five to seven) in (42) and (46) enables one to construct various exact solutions of equation (42).

Example 1. By setting

$$\widetilde{a} = C_1, \quad \widetilde{b} = C_2, \quad \widetilde{c} = C_3, \quad \widetilde{q} = C_4, \quad \widetilde{p} = C_5, \tag{49}$$

where C_n are arbitrary constants, we obtain, from (47), an ordinary differential equation for $F = F(\xi)$. In this case, relations (48) under condition (49) represent a system of ordinary differential equations for the functional coefficients of the transformation (46). If $m = 1$, in equation (42) and transformation (46) subject to conditions (48)–(49), two functions can be set arbitrarily. The steady-state solution $F = F(\xi)$ of equation (47) generates an unsteady traveling-wave solution (46) of the original equation (42).

Example 2. Equation (47) admits the steady-state solutions

$$F(\xi) = \frac{1}{1 \pm e^{\xi}}, \tag{50}$$

which, in view of (46), lead to solutions of the original equation (42) of the form

$$w(z,t) = \frac{a(t)}{1 \pm \exp[\lambda(t)z + \sigma(t)]} + b(t)z + c(t). \tag{51}$$

To solution (50) there correspond the following coefficients in equation (47):

$$\widetilde{a} = \frac{6\nu}{2-m}, \quad \widetilde{b} = 0, \quad \widetilde{c} = -\frac{3\nu}{2-m}, \quad \widetilde{p} = 0, \quad \widetilde{q} = \frac{(1+m)}{2-m}\nu.$$

Substituting these expressions into (48), one obtains a system of ordinary differential equations for the functional coefficients of solution (51) (in this system, two functions can be set arbitrarily).

Example 3. Let us set

$$\widetilde{a} = C_1\tau^{-\frac{k+1}{2}}, \quad \widetilde{b} = C_2\tau^{-1}, \quad \widetilde{c} = C_3\tau^{-\frac{1}{2}}, \quad \widetilde{q} = C_4\tau^{-1}, \quad \widetilde{p} = C_5\tau^{\frac{k-3}{2}}, \quad \tau = \int \lambda^2(t)\,dt + C_0, \tag{52}$$

where k and C_n are arbitrary constants. In this case, equation (47) admits a self-similar solution of the form

$$F = \tau^{k/2}H(\zeta), \quad \zeta = \xi\tau^{-1/2},$$

where the function $H = H(\zeta)$ satisfies the ordinary differential equation

$$\tfrac{1}{2}\big(k - 1 - 2C_4\big)H'_\zeta + \big[C_1H + \big(C_2 - \tfrac{1}{2}\big)\zeta + C_3\big]H''_{\zeta\zeta} - mC_1(H'_\zeta)^2 = \nu H'''_{\zeta\zeta\zeta} + C_5. \tag{53}$$

Substituting (52) into (48) yields a system of integro-differential equations for determining the functional parameters of the original equation (42) and transformation (46). Note that the change of variable $\lambda = \sqrt{\varphi'_t}$ allows one to obtain, taking into account that $\tau = \varphi(t) + C_0$, a system of ordinary differential equations.

Example 4. With

$$\widetilde{a} = 1, \quad \widetilde{b} = \widetilde{c} = 0, \tag{54}$$

from the first three relations in (48) we obtain

$$\lambda = a, \quad b = -a'_t/a, \quad c = -\sigma'_t/a. \tag{55}$$

Then the last two relations in (48) become

$$\widetilde{q}(\tau) = \frac{1}{a^2}\left[q(t) - 2(m+1)\frac{a'_t}{a}\right], \quad \widetilde{p}(\tau) = \frac{1}{a^4}\left[p(t) - \frac{a'_t}{a}q(t) + (m-1)\Big(\frac{a'_t}{a}\Big)^2 + \frac{a'_{tt}}{a}\right], \quad \tau = \int a^2(t)\,dt + C_0. \tag{56}$$

Thus, transformation (46), into which expressions (55) must be substituted, reduces equation (42) to a similar equation (47) with coefficients defined by (54) and (56). Formulas (55)–(56) contain two arbitrary functions $a = a(t)$ and $\sigma = \sigma(t)$; in addition, if $m = 1$ in equation (42), then the functions $p(t)$ and $q(t)$ can also be arbitrary. By choosing appropriate $a(t)$, $\sigma(t)$, $p(t)$, and $q(t)$, one can construct and multiply exact solutions of the original equation.

In particular, with

$$q(t) = 2(m+1)\frac{a'_t}{a}, \quad p(t) = (m+3)\Big(\frac{a'_t}{a}\Big)^2 - \frac{a'_{tt}}{a}, \tag{57}$$

where $a = a(t)$ is an arbitrary function, formulas (56) give $\widetilde{q} = \widetilde{p} = 0$ and equation (47) reduces to the self-similar form

$$\frac{\partial^2 F}{\partial\tau\partial\xi} + F\frac{\partial^2 F}{\partial\xi^2} - m\Big(\frac{\partial F}{\partial\xi}\Big)^2 = \nu\frac{\partial^3 F}{\partial\xi^3}.$$

Any steady-state solution $F = F(\xi)$ of this equation generates an unsteady solution (46), (55) of equation (42) with coefficients (57).

Interpretation of solutions as leading terms of a Taylor series expansion. Let us consider "axial" flows of a viscous incompressible fluid where the fluid velocity vector at the z-axis is directed along the axis. The transverse velocity components are small near the z-axis and can be expanded as a Taylor series in the transverse coordinates x and y. If, in the velocity component expansions, we restrict ourselves to the leading terms in x and y, we obtain relations (6), which, after appropriate analysis, will lead to formulas (1).

The above property of solutions (1) to the Navier–Stokes equations enables one to use these solutions for approximate description of axial flows and, thus, expand the scope of their applicability.

Note that any fluid flows that have two planes of symmetry admit a representation in the form (6) near the line of intersection of these planes (in the above notation, the line of intersection defines the z-axis). Examples of such flows include axisymmetric flows, combined axisymmetric flows with rotation about the z-axis (in particular, von Kármán flows), plane flows symmetric about a straight line, flows in rectilinear impermeable and porous tubes with elliptic or rectangular cross-section, fluid jet emanating from elliptic or rectangular holes, and more.

Classification of axial flows. Axial flows described by formulas (1) can be conveniently treated as a nonlinear superposition of a translational (nonuniform) flow along the z-axis and a linear shear flow of a special form. In the vicinity of the point $z = z_0$, lying on the axis, the velocity components can be represented, by virtue of (1), in the form

$$\begin{gathered} V_k = w\delta_{k3} + G_{km}X_m \quad \text{(summation over } m\text{)}; \\ G_{11} = -\tfrac{1}{2}w_z + f, \quad G_{12} = g, \quad G_{21} = h, \quad G_{22} = -\tfrac{1}{2}w_z - f, \\ G_{13} = G_{23} = G_{31} = G_{32} = 0, \quad G_{33} = w_z; \quad X_1 = x, \quad X_2 = y, \quad X_3 = z - z_0. \end{gathered} \tag{58}$$

Here k, m = 1, 2, 3, G_{km} are the components of the shear matrix, δ_{km} is the Kronecker delta, and w_z is the partial derivative of w with respect to z. All quantities in (58) are taken at $z = z_0$. The fact that the sum of the diagonal entries $G_{11} + G_{22} + G_{33}$ is zero results from the incompressibility of the fluid.

Any matrix $[G_{km}]$ can be represented as the sum of its symmetric and antisymmetric parts:

$$\begin{gathered} [G_{km}] = [E_{km}] + [\Omega_{km}], \\ E_{km} = E_{mk} = \tfrac{1}{2}(G_{km} + G_{mk}), \quad \Omega_{km} = -\Omega_{mk} = \tfrac{1}{2}(G_{km} - G_{mk}). \end{gathered} \tag{59}$$

In turn, the symmetric matrix $[E_{km}]$ (in this case, it can be identified with the rate-of-deformation tensor) can be reduced, by rotations of the coordinate system, to a diagonal form with entries E_1, E_2, and E_3, which are roots of the cubic equation $\det[E_{km} - \lambda\delta_{km}] = 0$ for λ.

For flows defined by (58), the diagonal entries, which determine the intensity of tensile (compressive) motions along the respective axes, are given by the formulas

$$E_{1,2} = -\tfrac{1}{2}w_z \pm \tfrac{1}{2}\sqrt{4f^2 + (h+g)^2}, \quad E_3 = w_z. \tag{60}$$

The decomposition (59) of the shear matrix $[G_{km}]$ into the symmetric and antisymmetric parts is equivalent to the representation of the velocity field of a linear shear fluid flow as a superposition of a linear straining flow, with the principal strain rates E_1, E_2, and E_3 along the principal axes, and a rotational motion of the fluid as a rigid body, with the angular velocity $\boldsymbol{\omega} = (\Omega_{32}, \Omega_{13}, \Omega_{21})$.

For the flow (58), we have $\Omega_{32} = \Omega_{13} = 0$, and the fluid rotates about the z-axis with the angular velocity

$$\Omega_{21} = \tfrac{1}{2}(h - g). \tag{61}$$

It is not difficult to show that formulas (60) and (61) remain valid for any point (x_0, y_0, z_0) of the flow (1).

The analysis of formulas (60)–(61) allows us to single out several characteristic types of flow listed in Table 21.2.

TABLE 21.2
Classification of axial flows described by formulas (1)

No.	Type of flow	Unknown functions	Functions involved in pressure
1	Axisymmetric flow	$f = g = h = 0$	$\alpha = \beta,\ \gamma = 0$
2	Rotationally symmetric flow	$f = 0,\ g = -h$	$\alpha = \beta,\ \gamma = 0$
3	Purely straining flow (no rotation)	$g = h$	α, β, and γ are any
4	General axial flow	$g \neq h$	α, β, and γ are any

Von Kármán-type rotationally symmetric flows. Let us consider the nonlinear superposition of an axisymmetric flow and a rotational flow about the z-axis (second row in Table 21.2) in more detail. In this case, formulas (1) become

$$\begin{aligned} u &= -\frac{1}{2}x\frac{\partial w}{\partial z} + yg, \quad v = -xg - \frac{1}{2}y\frac{\partial w}{\partial z}, \quad w = w(z,t), \\ p &= p_0 - \frac{1}{2}\alpha(x^2+y^2) - \frac{1}{2}w^2 + \nu\frac{\partial w}{\partial z} - \frac{\partial}{\partial t}\int w\,dz, \end{aligned} \tag{62}$$

and system (2)–(5) is reduced to two equations

$$\frac{\partial^2 w}{\partial t\partial z} + w\frac{\partial^2 w}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} - 2g^2 - 2\alpha, \tag{63}$$

$$\frac{\partial g}{\partial t} + w\frac{\partial g}{\partial z} - g\frac{\partial w}{\partial z} = \nu\frac{\partial^2 g}{\partial z^2}. \tag{64}$$

Below are some exact solutions of system (63)–(64).

1°. Degenerate solution:

$$w = kz^2 + b(t)z + \frac{1}{4k}[b^2(t) - 2b_t'(t) - 4\alpha(t)], \quad g = 0, \tag{65}$$

where $b(t)$ and $\alpha(t)$ are arbitrary functions and k is an arbitrary constant.

2°. Solution with $\alpha \equiv 0$:

$$w = b_t'(t) + \frac{4\nu}{b(t) - z}, \quad g = 0, \tag{66}$$

where $b(t)$ is an arbitrary function.

3°. Solutions with arbitrary $\alpha = \alpha(t)$:

$$\begin{aligned} w &= a(t) + b(t)z + c(t)\sin[\omega(t)z + \theta(t)], \\ g &= A(t) + B(t)\sin[\omega(t)z + \theta(t)], \end{aligned} \tag{67}$$

where the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, $\omega = \omega(t)$, $\theta = \theta(t)$, $A = A(t)$, and $B = B(t)$ are determined by the system of ordinary differential equations (one of the functions can be set arbitrarily):

$$\begin{aligned} &b_t' - \tfrac{1}{2}b^2 + 2(A^2 - B^2 + \alpha) = 0, \quad c_t' + c(\nu\omega^2 - 2b) = 0, \\ &\omega_t' + b\omega = 0, \quad \theta_t' + a\omega = \pm 2A, \quad A_t' - bA = 0, \quad B = \pm\tfrac{1}{2}c\omega. \end{aligned} \tag{68}$$

By setting $A = a = \theta = 0$ in (67) and (68), we can satisfy equations (68) so that the no-slip conditions, $u = v = w = 0$, hold on the surface $z = 0$.

Remark. Some more information about von Kármán-type rotationally symmetric flows and related flows can be found in Subsections 21.4.2 and 21.4.3.

Fluid velocity components in cylindrical coordinates. The fluid velocity components (1) can be equivalently represented in the cylindrical coordinates as

$$\begin{aligned} u_r &= -\tfrac{1}{2}r\big[w_z - 2f\cos(2\varphi) - (g+h)\sin(2\varphi)\big], \\ u_\varphi &= \tfrac{1}{2}r\big[h - g + (g+h)\cos(2\varphi) - 2f\sin(2\varphi)\big], \\ u_z &= w. \end{aligned}$$

Special case 1. Solution in the cylindrical coordinates:

$$u_r = \Omega r[W'_\zeta(\zeta) + F(\zeta)\cos(2\varphi)], \quad u_\varphi = \Omega r[G(\zeta) - F(\zeta)\sin(2\varphi)], \quad \zeta = \sqrt{\Omega/\nu}\, z,$$
$$u_z = -2\sqrt{\nu\Omega}\, W(\zeta), \quad p = p_0 + \tfrac{1}{2}\lambda\Omega^2 r^2 - 2\nu\Omega[W'_\zeta(\zeta) + W^2(\zeta)],$$

where p_0, Ω, and λ are arbitrary constants, and the functions $W = W(\zeta)$, $F = F(\zeta)$, and $G = G(\zeta)$ are determined by the system of ordinary differential equations

$$W'''_{\zeta\zeta\zeta} + 2WW''_{\zeta\zeta} - (W'_\zeta)^2 + G^2 - F^2 = \lambda,$$
$$F''_{\zeta\zeta} + 2(WF'_\zeta - W'_\zeta F) = 0,$$
$$G''_{\zeta\zeta} + 2(WG'_\zeta - W'_\zeta G) = 0.$$

The solution describes a steady-state nonaxisymmetric flow over a rotating disk.

⊙ *Reference*: R. E. Hewitt, P. W. Duck, and M. R. Foster (1999), P. G. Drazin and N. Riley (2006, pp. 72–73).

Special case 2. Solution in the cylindrical coordinates:

$$u_r = \Omega r\Big[\frac{\partial}{\partial\zeta}W(\zeta,\tau) + F(\zeta,\tau)\cos(2\varphi)\Big], \quad \zeta = \sqrt{\Omega/\nu}\, z, \quad \tau = \Omega t,$$
$$u_\varphi = \Omega r[G(\zeta,\tau) - F(\zeta,\tau)\sin(2\varphi)], \quad u_z = -2\sqrt{\nu\Omega}\, W(\zeta,\tau),$$
$$p = p_0(t) + \Omega^2 r^2[A(\tau) + B(\tau)\cos(2\varphi)] - 2\nu\Omega\Big(\frac{\partial W}{\partial\zeta} + W^2 - \frac{\partial}{\partial\tau}\int W\,d\zeta\Big),$$

where $A(t)$ and $p_0(t)$ are arbitrary functions, Ω and λ are arbitrary constants, and the functions $W = W(\zeta,\tau)$, $F = F(\zeta,\tau)$, and $G = G(\zeta,\tau)$ are determined by the system of differential equations

$$W_{\zeta\tau} - W_{\zeta\zeta\zeta} - 2WW_{\zeta\zeta} + (W_\zeta)^2 - G^2 + F^2 + 2A(\tau) = 0,$$
$$F_\tau - F_{\zeta\zeta} - 2WF_\zeta + 2FW_\zeta + 2B(\tau) = 0,$$
$$G_\tau - G_{\zeta\zeta} - 2WG_\zeta + 2GW_\zeta = 0.$$

The solution describes an unsteady three-dimensional flow around a rotating disk.

⊙ *References*: P. Hall, P. Balakurmar, and D. Papageorgiu (1992).

Some generalizations. A combination of formulas (1) and the formulas from Item 3° from Subsection 21.1.2 (at $z_0 = 0$) determine an exact solution of the Navier–Stokes equations that can be treated as a generalized axial flow with the z-axis moving in the xy-plane according to the law $x = x_0(t)$, $y = y_0(t)$. This solution can be used for mathematical modeling of destructive atmospheric phenomena such as whirlwinds and tornadoes.

⊙ *References for Subsection* 21.5.2: S. N. Aristov and A. D. Polyanin (2009, 2010), A. D. Polyanin and S. N. Aristov (2009), A. D. Polyanin (2009 *a*), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

21.5.3. Solutions with Linear Dependence of the Velocity Components on Two Space Variables. General Analysis

Structure of exact solutions. The determining system of equations. The Navier–Stokes equations admit a wide class of exact solutions with linear dependence of the velocity components on two space variables:

$$u = U + x\Big(f - \frac{1}{2}\frac{\partial w}{\partial z}\Big) + yg, \quad v = V + xh - y\Big(f + \frac{1}{2}\frac{\partial w}{\partial z}\Big), \quad w = w(z,t),$$
$$p = p_0 - \frac{1}{2}\alpha x^2 - \frac{1}{2}\beta y^2 - \gamma xy - \varepsilon x - \delta y - \frac{1}{2}w^2 + \nu\frac{\partial w}{\partial z} - \frac{\partial}{\partial t}\int w\,dz, \tag{1}$$

where $p_0 = p_0(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\varepsilon = \varepsilon(t)$, and $\delta = \delta(t)$ are arbitrary functions determining the pressure and $U = V(z,t)$, $V = V(z,t)$, $f = f(z,t)$, $g = g(z,t)$, $h = h(z,t)$, and $w = w(z,t)$ are unknown functions, which are described by the system of equations

$$\frac{\partial^2 w}{\partial t \partial z} + w\frac{\partial^2 w}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} + 2(f^2 + gh) - \alpha - \beta, \tag{2}$$

$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial z} - f\frac{\partial w}{\partial z} = \nu\frac{\partial^2 f}{\partial z^2} + \frac{\alpha - \beta}{2}, \tag{3}$$

$$\frac{\partial g}{\partial t} + w\frac{\partial g}{\partial z} - g\frac{\partial w}{\partial z} = \nu\frac{\partial^2 g}{\partial z^2} + \gamma, \tag{4}$$

$$\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial z} - h\frac{\partial w}{\partial z} = \nu\frac{\partial^2 h}{\partial z^2} + \gamma, \tag{5}$$

$$\frac{\partial U}{\partial t} + w\frac{\partial U}{\partial z} - U\Big(\frac{1}{2}\frac{\partial w}{\partial z} - f\Big) + gV = \nu\frac{\partial^2 U}{\partial z^2} + \varepsilon, \tag{6}$$

$$\frac{\partial V}{\partial t} + w\frac{\partial V}{\partial z} - V\Big(\frac{1}{2}\frac{\partial w}{\partial z} + f\Big) + hU = \nu\frac{\partial^2 V}{\partial z^2} + \delta. \tag{7}$$

The first four equations (2)–(5) form a closed system, which was considered in detail in Subsection 21.5.2.

Remark. The fluid velocity components (1) can be equivalently represented in the cylindrical coordinates as

$$\begin{aligned} u_r &= U\cos\varphi + V\sin\varphi - \tfrac{1}{2}r\big[w_z - 2f\cos(2\varphi) - (g+h)\sin(2\varphi)\big],\\ u_\varphi &= V\cos\varphi - U\sin\varphi + \tfrac{1}{2}r\big[h - g + (g+h)\cos(2\varphi) - 2f\sin(2\varphi)\big],\\ u_z &= w. \end{aligned}$$

General property of system (2)–(5). Suppose a set of functions $w_0(z,t)$, $f_0(z,t)$, $g_0(z,t)$, $h_0(z,t)$, $U_0(z,t)$, and $V_0(z,t)$ is a solution of system (2)–(7). Then the set of functions

$$\begin{aligned} &w = w_0(z+\psi(t),t) - \psi_t'(t), \quad f = f_0(z+\psi(t),t), \quad g = g_0(z+\psi(t),t),\\ &h = h_0(z+\psi(t),t), \quad U = U_0(z+\psi(t),t), \quad V = V_0(z+\psi(t),t), \end{aligned} \tag{8}$$

where $\psi(t)$ is an arbitrary function, will also give a solution of system (2)–(7).

Traveling-wave solutions. Suppose the functions $w_0(z)$, $f_0(z)$, $g_0(z)$, $h_0(z)$, $U_0(z)$, and $V_0(z)$ determine the general solution of the steady-state system (2)–(7) with

$$\alpha = \text{const}, \quad \beta = \text{const}, \quad \gamma = \text{const}, \quad \varepsilon = \text{const}, \quad \delta = \text{const}. \tag{9}$$

Then all traveling-wave solutions of the unsteady system (2)–(7) under condition (9) are described by the formulas

$$\begin{aligned} &w = w_0(z-\lambda t) + \lambda, \quad f = f_0(z-\lambda t), \quad g = g_0(z-\lambda t),\\ &h = h_0(z-\lambda t), \quad U = U_0(z-\lambda t), \quad V = V_0(z-\lambda t), \end{aligned} \tag{10}$$

where λ is an arbitrary constant.

It follows from the first formula in (10) that if the function $w_0(z)$, determining a steady-state solution, vanishes as $z \to \infty$, then the corresponding wave traveling to the right at a speed $\lambda > 0$ is described by a function w that tends to λ as $z \to \infty$.

Self-similar solutions. For the determining functions

$$\alpha = \alpha_0 t^{-2}, \quad \beta = \beta_0 t^{-2}, \quad \gamma = \gamma_0 t^{-2}, \quad \varepsilon = \varepsilon_0 t^{-2}, \quad \delta = \delta_0 t^{-2}, \tag{11}$$

where α_0, β_0, γ_0, ε_0, and δ_0 are arbitrary constants, system (2)–(7) admits a self-similar solution of the form

$$w = \frac{\theta(\zeta)}{\sqrt{t}}, \quad f = \frac{\varphi_1(\zeta)}{t}, \quad g = \frac{\varphi_2(\zeta)}{t}, \quad h = \frac{\varphi_3(\zeta)}{t}, \quad U = \frac{\psi_1(\zeta)}{t}, \quad V = \frac{\psi_2(\zeta)}{t}, \quad \zeta = \frac{z}{\sqrt{t}},$$

where the functions $\theta = \theta(\zeta)$, $\varphi_n = \varphi_n(\zeta)$, and $\psi_m = \psi_m(\zeta)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
-\theta' + \tfrac{1}{2}(2\theta - \zeta)\theta'' - \tfrac{1}{2}(\theta')^2 &= \nu\theta''' + 2(\varphi_1^2 + \varphi_2\varphi_3) - \alpha_0 - \beta_0,\\
\tfrac{1}{2}(2\theta - \zeta)\varphi_1' - (\theta' + 1)\varphi_1 &= \nu\varphi_1'' + \tfrac{1}{2}(\alpha_0 - \beta_0),\\
\tfrac{1}{2}(2\theta - \zeta)\varphi_2' - (\theta' + 1)\varphi_2 &= \nu\varphi_2'' + \gamma_0,\\
\tfrac{1}{2}(2\theta - \zeta)\varphi_3' - (\theta' + 1)\varphi_3 &= \nu\varphi_3'' + \gamma_0,\\
\tfrac{1}{2}(2\theta - \zeta)\psi_1' - \tfrac{1}{2}(\theta' - 2\varphi_1 + 2)\psi_1 + \varphi_2\psi_2 &= \nu\psi_1'' + \varepsilon_0,\\
\tfrac{1}{2}(2\theta - \zeta)\psi_2' - \tfrac{1}{2}(\theta' + 2\varphi_1 + 2)\psi_2 + \varphi_3\psi_1 &= \nu\psi_2'' + \delta_0.
\end{aligned}$$

This solution with $\gamma_0 = \varepsilon_0 = \delta_0 = 0$ was obtained by Meleshko (2004). Note that with a translation in time $t \Longrightarrow t + C$, we can avoid the singularity in the solution at $t = 0$.

Two solutions in which the function w depends on time t alone.

1°. Consider solution (1) with $f = a(t)$, $g = h = b(t)$, and $w = w(t)$:

$$\begin{aligned}
&u = U(z,t) + a(t)x + b(t)y, \quad v = V(z,t) + b(t)x - a(t)y, \quad w = w(t),\\
&p = -b_t'(t)xy - \tfrac{1}{2}[a_t'(t) + a^2(t) + b^2(t)]x^2 + \tfrac{1}{2}[a_t'(t) - a^2(t) - b^2(t)]y^2\\
&\quad - A(t)x - B(t)y - w_t'(t)z + p_0(t),
\end{aligned}$$

where $a(t)$, $b(t)$, $A(t)$, $B(t)$, $w(t)$, and $p_0(t)$ are arbitrary functions, and the functions $U = U(z,t)$ and $V = V(z,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned}
U_t + a(t)U + b(t)V + w(t)U_z &= \nu U_{zz} + A(t),\\
V_t - a(t)V + b(t)U + w(t)V_z &= \nu V_{zz} + B(t).
\end{aligned} \tag{12}$$

The transformation

$$U = F(\xi,t) + \alpha(t), \quad V = G(\xi,t) + \beta(t), \quad \xi = x - \int w(t)\,dt,$$

where the functions $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are determined by the system of linear ordinary differential equations

$$\begin{aligned}
\alpha_t' + a(t)\alpha + b(t)\beta &= A(t),\\
\beta_t' - a(t)\beta + b(t)\alpha &= B(t),
\end{aligned} \tag{13}$$

leads (12) to the simpler system

$$\begin{aligned}
F_t + a(t)F + b(t)G &= \nu F_{zz},\\
G_t - a(t)G + b(t)F &= \nu G_{zz}.
\end{aligned} \tag{14}$$

The solution of system (13) can be represented in the following form (Polyanin and Manzhirov (2007, p. 1342)):

$$F = \varphi_1(t)R(z,t) + \varphi_2(t)S(z,t),$$
$$G = \psi_1(t)R(z,t) + \psi_2(t)S(z,t),$$

where the functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions to the system of linear ordinary differential equations

$$\begin{aligned} \varphi_t' + a(t)\varphi + b(t)\psi &= 0, \\ \psi_t' - a(t)\psi + b(t)\varphi &= 0, \end{aligned} \tag{15}$$

and the functions $R = R(z,t)$ and $S = S(z,t)$ satisfy the independent linear heat equations

$$R_t = \nu R_{zz}, \quad S_t = \nu S_{zz}.$$

Remark. Solutions of the nonhomogeneous system of linear ordinary differential equations (14) can be represented in terms of fundamental solutions of the homogeneous system of linear equations (15).

2°. Consider solution (1) with $f = a(t)$, $g = b$, $h = c$, and $w = w(t)$:

$$u = U(z,t) + a(t)x + by, \quad v = V(z,t) + cx - a(t)y, \quad w = w(t),$$
$$p = -\tfrac{1}{2}[a_t'(t) + a^2(t) + bc]x^2 + \tfrac{1}{2}[a_t'(t) - a^2(t) - bc]y^2 - A(t)x - B(t)y - w_t'(t)z + p_0(t),$$

where $a(t)$, $A(t)$, $B(t)$, $w(t)$, and $p_0(t)$ are arbitrary functions, b and c are arbitrary constants, and the functions $U = U(z,t)$ and $V = V(z,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned} U_t + a(t)U + bV + w(t)U_z &= \nu U_{zz} + A(t), \\ V_t - a(t)V + cU + w(t)V_z &= \nu V_{zz} + B(t). \end{aligned} \tag{16}$$

The transformation

$$U = F(\xi,t) + \alpha(t), \quad V = G(\xi,t) + \beta(t), \quad \xi = x - \int w(t)\,dt,$$

where the functions $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are determined by the system of linear ordinary differential equations

$$\alpha_t' + a(t)\alpha + b\beta = A(t), \quad \beta_t' - a(t)\beta + c\alpha = B(t),$$

brings (16) to the simpler system

$$\begin{aligned} F_t + a(t)F + bG &= \nu F_{zz}, \\ G_t - a(t)G + cF &= \nu G_{zz}. \end{aligned} \tag{17}$$

The solution of system (17) can be expressed in terms of solutions to two independent linear heat equations (see Polyanin and Manzhirov (2007, p. 1342)).

Solution with ten arbitrary functions. The system of the first four equations (2)–(5) admits the degenerate solution

$$w = a(t)z + b(t), \quad f = f(t), \quad g = g(t), \quad h = g(t) + C\exp\Big(\int a(t)\,dt\Big), \tag{18}$$

where $a = a(t)$, $b = b(t)$, $f = f(t)$, and $h = h(t)$ are arbitrary functions, in terms of which the free functions appearing in formula (4) for the pressure are expressed:

$$\begin{aligned}\alpha &= \tfrac{1}{4}a^2 - \tfrac{1}{2}a'_t + f^2 + gh - af + f'_t,\\ \beta &= \tfrac{1}{4}a^2 - \tfrac{1}{2}a'_t + f^2 + gh + af - f'_t,\\ \gamma &= g'_t - ag.\end{aligned} \tag{19}$$

Let us substitute expressions (18) into equations (6) and (7) and change the dependent variables with the formulas

$$\begin{aligned}U &= \varphi_1(t)R(z,t) + \varphi_2(t)S(z,t) + \varphi_0(t),\\ V &= \psi_1(t)R(z,t) + \psi_2(t)S(z,t) + \psi_0(t),\end{aligned} \tag{20}$$

where the two pairs of functions $\varphi_1 = \varphi_1(t)$, $\psi_1 = \psi_1(t)$ and $\varphi_2 = \varphi_2(t)$, $\psi_2 = \psi_2(t)$ are linearly independent (fundamental) solutions of the homogeneous system of linear differential equations

$$\varphi' = \left(\tfrac{1}{2}a - f\right)\varphi - g\psi, \quad \psi' = \left(\tfrac{1}{2}a + f\right)\psi - h\varphi, \tag{21}$$

and the functions $\varphi_0 = \varphi_0(t)$ and $\psi_0 = \psi_0(t)$ are particular solutions of the nonhomogeneous system of linear ordinary differential equations

$$\varphi'_0 = \left(\tfrac{1}{2}a - f\right)\varphi_0 - g\psi_0 + \varepsilon, \quad \psi'_0 = \left(\tfrac{1}{2}a + f\right)\psi_0 - h\varphi_0 + \delta. \tag{22}$$

With the transformations (20)–(22), equations (6) and (7) are reduced to two independent linear equations of the same form

$$\begin{aligned}\frac{\partial R}{\partial t} + (az+b)\frac{\partial R}{\partial z} &= \nu\frac{\partial^2 R}{\partial z^2},\\ \frac{\partial S}{\partial t} + (az+b)\frac{\partial S}{\partial z} &= \nu\frac{\partial^2 S}{\partial z^2}.\end{aligned} \tag{23}$$

The transformation of the independent variables (Polyanin, 2002)

$$\tau = \int A^2(t)\,dt + C_1, \quad \xi = A(t)z + \int A(t)b(t)\,dt + C_2, \quad A(t) = C_3\exp\left(\int a(t)\,dt\right), \tag{24}$$

where C_1, C_2, and C_3 are arbitrary constants, reduces the linear equations with variable coefficients (22) to the linear heat equations

$$\frac{\partial R}{\partial \tau} = \nu\frac{\partial^2 R}{\partial \xi^2}, \qquad \frac{\partial S}{\partial \tau} = \nu\frac{\partial^2 S}{\partial \xi^2}. \tag{25}$$

Remark 1. Formulas and equations (1), (18)–(25) describe a family of exact solutions to the Navier–Stokes equations; this family involves ten arbitrary functions: six functions of time a, b, f, g, ε, and δ and two pairs of arbitrary functions coming from the solutions of two heat equations (25), provided that these solutions are considered on a semi-infinite interval of ξ.

Remark 2. In order to construct a fundamental system of solutions to the linear homogeneous system (21), it suffices to know only one nontrivial particular solution of the system. The solution of the linear nonhomogeneous system (22) can be expressed in terms of the fundamental solutions of the homogeneous system (21).

Two special classes of exact solutions to the Navier–Stokes equations.

1°. Solution:

$$u = k\frac{xf'_\eta(\eta)}{1-kt} + a\frac{g(\eta)}{\sqrt{1-kt}}, \quad u = k\frac{yf'_\eta(\eta)}{1-kt}, \quad w = -2\sqrt{k\nu}\frac{f(\eta)}{\sqrt{1-kt}}, \quad \eta = \frac{\sqrt{k/\nu}\,z}{\sqrt{1-kt}},$$
$$p = p_0(t) - \frac{k^2(x^2+y^2)}{(1-kt)^2} + \frac{k\nu}{1-kt}\big[\eta f(\eta) - 2f^2(\eta) - 2f'_\eta(\eta)\big],$$

where k and a are arbitrary constants, $p_0(t)$ is an arbitrary function, and the functions $f = f(\eta)$ and $g = g(\eta)$ are determined by the system of ordinary differential equations

$$f'''_{\eta\eta\eta} + (2f - \tfrac{1}{2}\eta)f''_{\eta\eta} - (f'_\eta)^2 - f'_\eta + 2 = 0,$$
$$g''_{\eta\eta} + (2f - \tfrac{1}{2}\eta)g'_\eta - f'_\eta g - \tfrac{1}{2}g = 0.$$

⊙ *References*: C. Y. Wang (1985), P. G. Drazin and N. Riley (2006, pp. 146–148).

2°. In (1), we set

$$f = g = h = 0, \quad \alpha = \beta, \quad \gamma = 0.$$

As a result, system (2)–(7) is reduced to three equations

$$\frac{\partial^2 w}{\partial t\partial z} + w\frac{\partial^2 w}{\partial z^2} - \frac{1}{2}\Big(\frac{\partial w}{\partial z}\Big)^2 = \nu\frac{\partial^3 w}{\partial z^3} - 2\alpha, \tag{26}$$
$$\frac{\partial U}{\partial t} + w\frac{\partial U}{\partial z} - \frac{1}{2}U\frac{\partial w}{\partial z} = \nu\frac{\partial^2 U}{\partial z^2} + \varepsilon, \tag{27}$$
$$\frac{\partial V}{\partial t} + w\frac{\partial V}{\partial z} - \frac{1}{2}V\frac{\partial w}{\partial z} = \nu\frac{\partial^2 V}{\partial z^2} + \delta. \tag{28}$$

Here the nonlinear equation (26) for w is independent of the other two equations (27) and (28), which are linear in the unknowns U and V. Note that even with $V = \delta = 0$, the flow in question will be three-dimensional.

Equations (27) and (28) have a similar structure and are independent from each other. Therefore, we can restrict ourselves to the analysis of equations (26)–(27).

Suppose $w = w(z,t)$ solves equation (26). Then equation (27) is solved by the function

$$U = s(t)\frac{\partial w}{\partial z} + 2s'_t(t), \tag{29}$$

where the function $s = s(t)$ is described by the ordinary differential equation

$$s''_{tt} - \alpha(t)s - \tfrac{1}{2}\varepsilon(t) = 0. \tag{30}$$

Equation (26) coincides with equation (26) from Subsection 21.5.2; therefore, the solutions presented there can be used here. Formulas (29)–(30) enable one to construct solutions of equation (27).

⊙ *References for Subsection* 21.5.3: S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009), A. D. Polyanin (2009 *b*).

21.5.4. Solutions with the Linear Dependence of the Velocity Components on One Space Variable

1°. Generalized separable solution in z:

$$u = u(x,y,t), \quad v = v(x,y,t), \quad w = f(x,y,t)z + g(x,y,t), \quad p = \widetilde{p}(x,y,t) - \tfrac{1}{2}a(t)z^2 - b(t)z,$$

where $a(t)$ and $b(t)$ are arbitrary functions, and other functions are determined by the system of differential equations

$$u_t + uu_x + vu_y = -\widetilde{p}_x + \nu(u_{xx} + u_{yy}), \tag{1}$$
$$v_t + uv_x + vv_y = -\widetilde{p}_y + \nu(v_{xx} + v_{yy}), \tag{2}$$
$$u_x + v_y + f = 0, \tag{3}$$
$$f_t + uf_x + vf_y + f^2 = a(t) + \nu(f_{xx} + f_{yy}), \tag{4}$$
$$g_t + ug_x + vg_y + fg = b(t) + \nu(g_{xx} + g_{yy}). \tag{5}$$

The first four equations form a closed system for determining u, v, $\widetilde{p}_x$, and f.

Remark. For $g(x,y,t) \equiv 0$ and $a(t) = b(t) \equiv 0$, this solution can be used for describing unsteady fluid flows with a cylindrical (not necessarily circular) free boundary; see Pukhnachov (1998, 2006).

1.1. A solution of equation (5), which is expressed in terms of the function f, is given by

$$g = A(t)f + B(t), \quad A(t) = -\int B(t)\,dt + C_1, \quad B(t) = \int [b(t) - a(t)]\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

1.2. Equations (4) and (5) can be satisfied by assuming, for example, that $f = f(t)$ and $g = g(t)$, which are convenient to be treated as arbitrary and used to express the functions $a(t)$ and $b(t)$, involved in the pressure, by the formulas

$$a(t) = f_t'(t) + f^2(t), \quad b(t) = g_t'(t) + f(t)g(t).$$

Equation (3) can be satisfied by introducing a modified stream function $\psi = \psi(x,y,t)$ such that

$$u = \psi_y + h(t)x, \quad v = -\psi_x - [f(t) + h(t)]y,$$

where $h = h(t)$ is an arbitrary function (by choosing this function appropriately, one can simplify the subsequent equations). After the elimination of the pressure function $\widetilde{p}$, equations (1) and (2) reduce to a single fourth-order equation:

$$(\Delta\psi)_t + (\psi_y + hx)(\Delta\psi)_x - [\psi_x + (f+h)y](\Delta\psi)_y - f\Delta\psi = \nu\Delta\Delta\psi, \tag{6}$$

where $\Delta\psi = \psi_{xx} + \psi_{yy}$.

Equation (6) admits, for example, exact solutions of the form $\psi = \psi(x,t)$ and $\psi = \psi(y,t)$. In the former case, we have

$$\Psi_t + h(t)x\Psi_x - f(t)\Psi = \nu\Psi_{xx}, \qquad \Psi = \psi_{xx}. \tag{7}$$

This equation can be reduced to the classical linear heat equation; see Polyanin (2000, p. 134). Since the function $h(t)$ is arbitrary, we can simplify equation (7) by setting $h(t) \equiv 0$.

In this case, the substitution $\Psi = \exp\bigl(\int f\,dt\bigr)\zeta(x,t)$ leads to the linear heat equation, $\zeta_t = \nu\zeta_{xx}$.

Equation (6) admits generalized separable solutions of the form

$$\psi = F_1(x,t)y + G_1(x,t),$$
$$\psi = F_2(y,t)x + G_2(y,t).$$

Also there is a class of degenerate solutions, described by the equation

$$\Delta\psi = C_1\exp\Bigl(\int f\,dt\Bigr) + C_2,$$

where C_1 and C_2 are arbitrary constants, and the time t appears as a parameter. This solution satisfies the Euler equations for an inviscid fluid (with $\nu = 0$).

1.3. In what follows, we will need two corollaries of equations (1) and (2) obtained by differentiating them with respect to y and x, respectively:

$$\begin{aligned} u_{yt} + uu_{xy} + vu_{yy} + u_y(u_x + v_y) &= -p_{xy} + \nu(\Delta u)_y,\\ v_{xt} + uv_{xx} + vv_{xy} + v_x(u_x + v_y) &= -p_{xy} + \nu(\Delta v)_x, \end{aligned} \tag{8}$$

where Δ is the Laplace operator. Eliminating f from equations (3) and (4) and taking into account (8), we obtain

$$u_x^2 + v_y^2 + u_xv_y + u_yv_x = \tfrac{1}{2}a(t) - \tfrac{1}{2}\Delta\widetilde{p}. \tag{9}$$

Equations (1), (2), and (9) form a closed system for determining u, v, and $\widetilde{p}$. The function f is found from equation (3).

Special case 1. The system of equations (1), (2), and (9) admits solutions of the form

$$u = u(x,t), \quad v = \alpha y + \beta, \quad \widetilde{p} = -\tfrac{1}{2}(\alpha_t' + \alpha^2)y^2 - (\beta_t' + \alpha\beta)y + \theta(x,t),$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are arbitrary functions, and the functions $u = u(x,t)$ and $\theta = \theta(x,t)$ are determined by the system of differential equations

$$u_t + uu_x = -\theta_x + \nu u_{xx},$$
$$\theta_{xx} + 2u_x^2 + 2\alpha u_x = a(t) + \alpha_t' - \alpha^2, \quad \alpha = \alpha(t).$$

Eliminating θ yields a third-order equation for u:

$$u_{xt} + uu_{xx} - u_x^2 - 2\alpha u_x = \alpha^2 - a(t) - \alpha_t' + \nu u_{xxx}.$$

This equation coincides, up to the obvious renaming, with equation (12) from Subsection 21.5.2 (a considerable number of its exact solutions can also be found there).

Special case 2. The system of equations (1), (2), and (9) admits solutions of the form

$$u = \varphi(y,t)x + \psi(y,t), \quad v = v(y,t), \quad \widetilde{p} = A(t)x^2 + B(t)x + \zeta(y,t),$$

where $A(t)$ and $B(t)$ are arbitrary functions, and the functions φ, ψ, v, and ξ are determined by the system of differential equations

$$\begin{aligned} \varphi_t + \varphi^2 + v\varphi_y &= -2A(t) + \nu\varphi_{yy},\\ \psi_t + \varphi\psi + v\psi_y &= -B(t) + \nu\psi_{yy},\\ v_t + vv_y &= -\zeta_y + \nu v_{yy},\\ \varphi^2 + v_y^2 + \varphi v_y &= \tfrac{1}{2}a(t) - A(t) - \tfrac{1}{2}\zeta_{yy}. \end{aligned} \tag{10}$$

This system admits, for example, the traveling-wave solution

$$\varphi = \varphi(Y), \quad \psi = \psi(Y), \quad v = v(Y), \quad \zeta = \zeta(Y), \quad Y = x - \lambda t.$$

In this case, the third equation in (10) gives

$$\zeta = \nu v_Y' - \tfrac{1}{2}v^2 + \lambda v + C.$$

Special case 3. The system of equations (1), (2), and (9) with $a(t) = a_0 t^{-2}$ admits self-similar solutions of the form

$$u = \sqrt{\frac{\nu}{t}}\,U(\xi,\eta), \quad v = \sqrt{\frac{\nu}{t}}\,V(\xi,\eta), \quad \widetilde{p} = \frac{\nu}{t}\,P(\xi,\eta), \quad \xi = \frac{x}{\sqrt{\nu t}}, \quad \eta = \frac{y}{\sqrt{\nu t}}.$$

1.4. By setting $f = \varphi_x$, introducing a modified stream function such that

$$u = \psi_y - \varphi, \qquad v = -\psi_x,$$

and eliminating the pressure $\widetilde{p}$, one can reduce system (1)–(4) to two equations for φ and ψ.

2°. Solution:

$$u = x\left[f(z,t) - \frac{1}{2}\frac{\partial w}{\partial z}\right] + \varphi(y,z,t), \quad v = -y\left[f(z,t) + \frac{1}{2}\frac{\partial w}{\partial z}\right] - g(z,t), \quad w = w(z,t),$$

$$p = \frac{1}{2}a(t)x^2 + b(t)x + \frac{1}{2}c(t)y^2 + k(t)y + \nu\frac{\partial w}{\partial z} - \frac{1}{2}w^2 - \frac{\partial}{\partial t}\int w(z,t)\,dz + p_0(t),$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, $k = k(t)$, and $p_0(t)$ are arbitrary functions, and the functions $f = f(z,t)$, $g = g(z,t)$, $w = w(z,t)$, and $\varphi = \varphi(y,z,t)$ are determined by the system of differential equations

$$w_{zt} + ww_{zz} - \tfrac{1}{2}w_z^2 - 2f^2 = \nu w_{zzz} + a + c, \tag{11}$$

$$f_t + wf_z - fw_z = \nu f_{zz} + \tfrac{1}{2}(c-a), \tag{12}$$

$$g_t + wg_z - (f + \tfrac{1}{2}w_z)g = \nu g_{zz} - k, \tag{13}$$

$$\varphi_t - [y(f + \tfrac{1}{2}w_z) + g]\varphi_y + w\varphi_z + (f - \tfrac{1}{2}w_z)\varphi = \nu(\varphi_{yy} + \varphi_{zz}) - b. \tag{14}$$

Here the first two equations (11)–(12) form a closed subsystem for w and f. The last two equations (13)–(14) are linear in the unknowns g and φ.

⊙ *Reference*: S. V. Meleshko (2004).

2.1. Subsystem (11)–(13) admits the solution

$$f = \tfrac{1}{2}w_z + \beta(t), \quad g = \lambda(t)w_z + \sigma(t), \tag{15}$$

where the functions $\beta = \beta(t)$ and $w = w(z,t)$ are described by the equations

$$\beta_t' + \beta^2 = -a, \tag{16}$$

$$w_{zt} + ww_{zz} - w_z^2 - 2\beta w_z = \nu w_{zzz} + a + c + 2\beta^2, \tag{17}$$

and the functions $\lambda = \lambda(t)$ and $\sigma = \sigma(t)$ are given by the formulas

$$\lambda = C_1 \exp\Bigl(-\int \beta\,dt\Bigr), \quad \sigma = -\int\left(a + c + 2\beta^2 + \frac{k}{\lambda}\right)dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

2.2. Subsystem (11)–(13) admits another solution

$$f = -\tfrac{1}{2}w_z + \gamma(t), \quad g = g(t), \tag{18}$$

where the functions $\gamma = \gamma(t)$ and $w = w(z,t)$ are described by the equations

$$\gamma_t' - \gamma^2 = c, \tag{19}$$

$$w_{zt} + ww_{zz} - w_z^2 + 2\gamma w_z = \nu w_{zzz} + a + c + 2\gamma^2, \tag{20}$$

and the function $g = g(t)$ is given by

$$g = \Gamma(t)\left[C_1 - \int \frac{k(t)\,dt}{\Gamma(t)}\right], \quad \Gamma(t) = \exp\left(\int \gamma\,dt\right),$$

where C_1 is an arbitrary constant.

Equations (16) and (91) are Riccati equations, which are reduced to linear second-order equations. Instead of solving these equations, one can assume that $\beta = \beta(t)$ or $\gamma = \gamma(t)$ is an arbitrary function and find the function $a = a(t)$ (using (16)) or the function $c = c(t)$ (using (19)).

Equations (17) and (20) coincide, up to the obvious renaming, with equation (12) from Subsection 21.5.2, where a considerable number of its exact solutions can also be found.

Note that equation (14) admits a generalized separable solution of the form

$$\varphi = \varphi_2(z,t)y^2 + \varphi_1(z,t)y + \varphi_0(z,t). \tag{21}$$

Remark. One can also use the representation of the solution in the more symmetric form

$$u = xf_z(z,t) + \varphi(y,z,t), \quad v = yg_z(z,t) + h(z,t), \quad w = -f(z,t) - g(z,t),$$
$$p = \frac{1}{2}a(t)x^2 + b(t)x + \frac{1}{2}c(t)y^2 + k(t)y + \nu\frac{\partial w}{\partial z} - \frac{1}{2}w^2 - \frac{\partial}{\partial t}\int w(z,t)\,dz + p_0(t),$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, $k = k(t)$, and $p_0(t)$ are arbitrary functions, and the functions $f = f(z,t)$, $g = g(z,t)$, $w = w(z,t)$, and $\varphi = \varphi(y,z,t)$ are determined by the system of differential equations

$$\begin{gathered} f_{zt} - (f+g)f_{zz} + f_z^2 = \nu f_{zzz} - a, \\ g_{zt} - (f+g)g_{zz} + g_z^2 = \nu g_{zzz} - c, \\ h_t - (f+g)h_z + g_z h = \nu h_{zz} - k, \\ \varphi_t + (yg_z + h)\varphi_y - (f+g)\varphi_z + f_z\varphi = \nu(\varphi_{yy} + \varphi_{zz}) - b. \end{gathered} \tag{22}$$

The first two equations in (22) form a closed subsystem for f and g, which admits, for example, a generalized separable solution of the form

$$f = A_2(t)z^2 + A_1(t)z + A_0(t),$$
$$g = B_2(t)z^2 + B_1(t)z + B_0(t).$$

With $a = \text{const}$, $c = \text{const}$, and $\lambda = \text{const}$, the first two equations in (22) admit a traveling-wave solution.

By adding together and subtracting, one can reduce the first two equations in (22) to the form

$$\begin{gathered} F_{zt} - FF_{zz} + \tfrac{1}{2}F_z^2 + \tfrac{1}{2}G_z^2 = \nu F_{zzz} - a - c, \\ G_{zt} - FG_{zz} + F_z G_z = \nu G_{zzz} - a + c, \\ F = f + g, \quad G = f - g. \end{gathered}$$

With $a = c$, the second equation admits solutions $G = G(t)$, where $G(t)$ is an arbitrary function.

The last equation in (22) admits a generalized separable solution of the form (21).

21.5.5. Other Three-Dimensional Solutions

Steady-state solutions.

1°. Solutions in the spherical coordinates:

$$u_R = \frac{\nu}{R}\Delta_s\Phi, \quad u_\varphi = \frac{\nu}{R}\left(\frac{k}{\sin\varphi}\frac{\partial\Phi}{\partial\theta} - \frac{\partial\Phi}{\partial\varphi}\right), \quad u_\theta = -\frac{\nu}{R}\left(\frac{1}{\sin\varphi}\frac{\partial\Phi}{\partial\theta} + k\frac{\partial\Phi}{\partial\varphi}\right),$$
$$p = p_0 - \frac{\nu^2}{R}|\nabla_s\Phi|^2, \quad k = \pm 1,$$

where Δ_s and ∇_s are the Laplace–Beltrami and Hamilton operators on a unit sphere, or

$$\Delta_s\Phi = \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2\Phi}{\partial\theta^2}, \quad |\nabla_s\Phi|^2 = \frac{1}{R^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2 + \frac{1}{R^2\sin^2\theta}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2,$$

and the function $\Phi = \Phi(\varphi, \theta)$ is determined by the differential equation

$$\Delta_s\Phi = A\exp(-\Phi),$$

where A is an arbitrary constant. The last equation admits, for example, solutions of the form $\Phi = \Phi(\varphi)$.

⊙ *References*: S. N. Aristov (1995, 1998).

2°. Solutions in the spherical coordinates:

$$u_R = \frac{\nu}{R}\Delta_s\Phi, \quad u_\varphi = -\frac{\nu}{R}\frac{\partial\Phi}{\partial\varphi}, \quad u_\theta = -\frac{\nu}{R\sin\varphi}\frac{\partial\Phi}{\partial\theta}, \quad p = p_0 + \frac{\nu^2}{R}P, \tag{1}$$

where $\Delta_s = \Delta|_{R=1}$ is the Laplace–Beltrami operator on a unit sphere:

$$\Delta_s = \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2},$$

and the function $\Phi = \Phi(\varphi, \theta)$ is determined by the differential equation

$$\nabla_s(\Delta_s\Phi + 2)\cdot\nabla_s[\ln(\Delta_s\Phi + 2) + \Phi] = A, \tag{2}$$
$$\nabla_s F\cdot\nabla_s G \equiv \frac{1}{R^2}\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial\theta} + \frac{1}{R^2\sin^2\theta}\frac{\partial F}{\partial\varphi}\frac{\partial G}{\partial\varphi},$$

where A is an arbitrary constant.

For $A = 0$, from (2) we obtain the equation

$$\Delta_s\Phi = -2 + Be^{-\Phi}, \tag{3}$$

where B is an arbitrary constant. The transformation

$$x = \varphi, \quad y = \ln\left|\tan\frac{\theta}{2}\right|, \quad w = 2\ln|\sin\theta| + U(x, y)$$

leads to the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = Be^{-U},$$

which admits an exact linearization and is reduced to the Laplace equation; see equation 9.2.1.1.

Special case. Consider the following solution of equation (3):

$$\Phi = 2\ln\left[\frac{\alpha(1-\cos\theta)^m + \beta(1+\cos\theta)^m + \gamma\sin^m\theta\cos(m\varphi)}{\sin^{m-1}\theta}\right], \quad B = 2m^2(2\alpha\beta - \gamma^2), \tag{4}$$

where α, β, and γ are arbitrary constants, and m is a positive integer determining the number of jets. When $m = 0$, expression (4) describes a potential flow corresponding to a linear sink (Paull and Pillow, 1985). With $m = 1$, the flow is transformed into a submerged Landau jet (Landau, 1944) with an arbitrary orientation in space. This is a unique solution corresponding to a point impulse source located at the origin. For all other m, expression (4) determines a vortex flow with an arbitrary number of jets generated by a linear mass source whose specific flow rate is equal to $2(m-1)$.

With $A = 0$, the general solution of equation (2) satisfying the 2π-periodicity condition in θ admits a representation in terms of harmonic functions in the plane.

With $A \neq 0$, there exists a family of exact solutions to equation (2) in the form

$$\Phi = C\ln\sin\varphi + f(\theta),$$

where C is an arbitrary constant and the function $f = f(\theta)$ is described by a fourth-order ordinary differential equation, which admits double integration.

⊙ *References*: S. N. Aristov (1995, 1998).

Unsteady solutions.

3°. *Solutions with spiral symmetry* in the cylindrical coordinates:

$$\begin{aligned} &u_r = U(r,\xi,t), \quad u_\varphi = V(r,\xi,t), \quad u_z = W(r,\xi,t),\\ &p = P(r,\xi,t) + b(t)z, \quad \xi = \theta + az, \end{aligned} \tag{5}$$

where a is an arbitrary constant, $b(t)$ is an arbitrary function, and the functions $U = U(r,\xi,t)$, $V = V(r,\xi,t)$, $W = W(r,\xi,t)$, and $P = P(r,\xi,t)$ are described by an appropriate system of differential equations, which is not written out here.

In order to analyze this class of solutions, one can introduce a generalized stream function Ψ such that

$$U = \frac{1}{r}\frac{\partial\Psi}{\partial z}, \quad V + aW = -\frac{1}{r}\frac{\partial\Psi}{\partial r}. \tag{6}$$

In this case, the continuity equation is satisfied automatically.

Using (5) and (6) and eliminating the pressure from the Navier–Stokes equations, one can obtain two equations for two functions, Ψ and V or Ψ and W.

⊙ *References*: V. O. Bytev (1972), J. F. Sparenberg (1991), V. V. Pukhnachov (2006).

4°. Solution:

$$\begin{aligned} u &= \exp(-a^2\nu t)[A\sin(az + b_3) + C\cos(ay + b_2)],\\ v &= \exp(-a^2\nu t)(B\sin(ax + b_1) + A\cos(az + b_3)],\\ w &= \exp(-a^2\nu t)(C\sin(ay + b_2) + B\cos(ax + b_1)],\\ p &= p_0(t) - \exp(-a^2\nu t)[AC\sin(az + b_3)\cos(ay + b_2)\\ &\quad + AB\sin(ax + b_1)\cos(az + b_3) + BC\sin(ay + b_2)\cos(ax + b_1)], \end{aligned}$$

where a, b_1, b_2, b_3, A, B, and C are arbitrary constants and $p_0(t)$ is an arbitrary function.

⊙ *Reference*: V. Trkal (1919).

5°. Solution:

$$\mathbf{u}(\mathbf{x},t)=\exp(-\alpha^2\nu t)\sum_{i=1}^{n}\big[\sin(\alpha\mathbf{k}_i\cdot\mathbf{x})\,\mathbf{F}(\mathbf{k}_i)+\cos(\alpha\mathbf{k}_i\cdot\mathbf{x})\,\mathbf{k}_i\times\mathbf{F}(\mathbf{k}_i)\big],\quad p(\mathbf{x},t)=p_0(t)-\frac{1}{2}|\mathbf{u}|^2,$$

where α is an arbitrary constant, $\mathbf{k}_i\cdot\mathbf{k}_i=1$, and $\mathbf{F}(\mathbf{k})$ is any vector function that satisfies the conditions $\mathbf{F}(\mathbf{k}_i)\cdot\mathbf{k}_i=0$.

⊙ *References*: O I. Bogoyavlenskij (2002, 2003).

6°. Solution (generalizes the solution from Item 5°):

$$\mathbf{u}(\mathbf{x},t)=\exp(-\alpha^2\nu t)\iint_S\big[\sin(\alpha\mathbf{k}\cdot\mathbf{x})\,\mathbf{F}(\mathbf{k})+\cos(\alpha\mathbf{k}\cdot\mathbf{x})\,\mathbf{k}\times\mathbf{F}(\mathbf{k})\big]\,d\sigma,\quad p(\mathbf{x},t)=p_0(t)-\frac{1}{2}|\mathbf{u}|^2,$$

where $\alpha\neq0$ is an arbitrary constant; the integral is taken over any measure $d\sigma$ on the two-dimensional unit sphere S, $\mathbf{k}\cdot\mathbf{k}=1$, and $\mathbf{F}(\mathbf{k})$ is an arbitrary smooth vector field tangential to the unit sphere, with $\mathbf{F}(\mathbf{k})\cdot\mathbf{k}=0$.

⊙ *References*: O I. Bogoyavlenskij (2002, 2003).

21.6. Convective Fluid Motions

21.6.1. Equations for Convective Fluid Motions

Two-dimensional plane motions of a viscous incompressible fluid with inhomogeneous heat supply to its boundaries are described by the equations

$$\begin{aligned}
u_t+uu_x+vu_y&=-p_x+\nu(u_{xx}+u_{yy}),\\
v_t+uv_x+vv_y&=-p_y+\nu(v_{xx}+v_{yy})+\lambda\theta,\\
\theta_t+u\theta_x+v\theta_y&=\sigma(\theta_{xx}+\theta_{yy}),\\
u_x+v_y&=0,
\end{aligned}$$

where u and v are the fluid velocity components, p is the ratio of the pressure to the fluid density, θ is the temperature, ν is the kinematic viscosity of the fluid, λ is the Grashov number, and σ is the Prandtl number (e.g., see Landau and Lifshitz (1954)). For simplicity, the value of the kinematic viscosity is taken to be equal to one.

The above equations are often called *the Navier–Stokes equations in the Boussinesq approximation*.

21.6.2. Steady-State Solutions

1°. Solution:

$$u=-\frac{A}{2\nu}y^2+C_1y+C_2,\quad v=0,\quad \theta=C_3y+C_4,\quad p=-Ax+\frac{C_3\lambda}{2}y^2+C_4\lambda y+C_5,$$

where A and $C_1,\ldots,C_5$ are arbitrary constants. This solution describes the *generalized Couette–Poiseuille plane flows* arising under the action of a fixed pressure gradient between two moving parallel walls having different temperatures.

2°. Solution:

$$u=C_1\exp\Big(\frac{v_0}{\nu}y\Big)+C_2,\quad v=v_0,\quad \theta=C_3\exp\Big(\frac{v_0}{\sigma}y\Big)+C_4,\quad p=\frac{C_3\lambda\sigma}{v_0}\exp\Big(\frac{v_0}{\sigma}y\Big)+C_4\lambda y+C_5,$$

where C_1, …, C_5, and v_0 are arbitrary constants. This solution describes convection between two porous walls, when the fluid is injected through one of the walls and is sucked away through the other. There is no pressure gradient and the tilt of the velocity vector is different at the walls.

⊙ *Reference*: V. L. Katkov (1968).

3°. Solution:

$$u = C_1\exp\Big(\frac{v_0}{\nu}y\Big)+\frac{A}{v_0}y+C_2,\quad v=v_0,$$
$$\theta = C_3\exp\Big(\frac{v_0}{\sigma}y\Big)+C_4,\quad p=-Ax+\frac{C_3\lambda\sigma}{v_0}\exp\Big(\frac{v_0}{\sigma}y\Big)+C_4\lambda y+C_5,$$

where $A, C_1, \ldots, C_5$, and v_0 are arbitrary constants. This solution describes the generalized Poiseuille flow arising under the action of a pressure fixed gradient and a flow between two moving parallel porous walls having different temperatures.

⊙ *Reference*: V. L. Katkov (1968).

4°. Solution:

$$u = \frac{k\lambda}{6\nu}y^3+C_1y+C_2,\quad v=0,$$
$$\theta = kx+\frac{k}{\sigma}\left(\frac{k\lambda}{120\nu}y^5+\frac{C_1}{6}y^3+\frac{C_2}{2}y^2\right)+C_3y+C_4,$$
$$p = k\lambda xy+\frac{k\lambda}{\sigma}\left(\frac{k\lambda}{720\nu}y^6+\frac{C_1}{24}y^4+\frac{C_2}{6}y^3\right)+\frac{C_3\lambda}{2}y^2+C_4\lambda y+C_5,$$

where C_1, …, C_5, and k are arbitrary constants. This solution describes a flow inside a plane channel, when the temperature gradient k in the channel is constant.

⊙ *Reference*: V. L. Katkov (1968).

5°. Solution:

$$u=0,\quad v=-\frac{C_1\lambda}{6\nu}x^3+\frac{a-C_2\lambda}{2\nu}x^2+C_3x+C_4,\quad \theta=C_1x+C_2,\quad p=ay+C_5,$$

where a and C_1, …, C_5 are arbitrary constants. For $a=0$, the solution describes a fluid flow in the gap between two vertical walls having different temperatures.

6°. Solution:

$$u=u_0,\quad \theta=C_1\exp\Big(\frac{u_0}{\sigma}x\Big)+C_2,$$
$$v=C_3\exp\Big(\frac{u_0}{\nu}x\Big)+\frac{C_1\lambda\sigma^2}{u_0^2(\sigma-\nu)}\exp\Big(\frac{u_0}{\sigma}x\Big)+\frac{C_2\lambda-a}{u_0}x+C_4,\quad p=ay+C_5,$$

where a, u_0, and C_1, …, C_5 are arbitrary constants, and $\sigma\neq\nu$.

7°. Solution:

$$u = 0, \quad v = v(x), \quad \theta = \beta y - \frac{\nu}{\lambda} v''_{xx}(x), \quad p = \frac{1}{2}\beta\lambda y^2 + p_0,$$

where β and p_0 are arbitrary constants, and the function $v = v(x)$ is determined by the fourth-order linear ordinary differential equation

$$v''''_{xxxx} + av = 0, \qquad a = \frac{\beta\lambda}{\sigma\nu}.$$

The form of the general solution of this equation depends on the sign of a.

For $a = 4k^4 > 0$:

$$v = C_1 \cosh kx \cos kx + C_2 \cosh kx \sin kx + C_3 \sinh kx \cos kx + C_4 \sinh kx \sin kx.$$

For $a = -k^4 < 0$:

$$v = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx.$$

Here $C_1, \ldots, C_4$ are arbitrary constants.

21.6.3. Unsteady Solutions

1°. Solution:

$$u = u(y,t), \quad v = v(t), \quad \theta = \theta(y,t), \quad p = -a(t)x - v'_t(t)y + \lambda \int \theta(y,t)\,dy + p_0(t),$$

where $v(t)$, $a(t)$, and $p_0(t)$ are arbitrary functions, and the functions $u = u(y,t)$ and $\theta = \theta(y,t)$ are determined by the linear partial differential equations

$$\begin{aligned} u_t + v(t)u_y &= a(t) + \nu u_{yy}, \\ \theta_t + v(t)\theta_y &= \sigma\theta_{yy}. \end{aligned} \tag{1}$$

The transformation

$$u = U(z,t) + \int a(t)\,dt, \quad \theta = V(z,t), \quad z = y - \int v(t)\,dt$$

leads to two independent linear heat equations

$$U_t = \nu U_{zz}, \quad V_t = \sigma V_{zz}.$$

2°. Solution (generalizes the solution from Item 1°):

$$u = u(y,t), \quad v = v(t), \quad \theta = kx + f(y,t), \quad p = k\lambda xy - a(t)x - v'_t(t)y + \lambda \int f(y,t)\,dy + p_0(t),$$

where $v(t)$, $a(t)$, and $p_0(t)$ are arbitrary functions, k is an arbitrary constant, and the functions $u = u(y,t)$ and $f = f(y,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned} u_t + v(t)u_y &= -k\lambda y + a(t) + \nu u_{yy}, \\ f_t + v(t)f_y &= \sigma f_{yy} - ku. \end{aligned} \tag{2}$$

The transformation

$$u = U(z,t) - k\lambda t y + \int [a(t) + k\lambda t v(t)]\,dt, \quad z = y - \int v(t)\,dt,$$

reduces the first equation in (1) to the linear heat equation

$$U_t = \nu U_{zz}.$$

3°. Solution:

$$u = u(t), \quad v = v(x,t), \quad \theta = \theta(x,t), \quad p = -v_t'(t)x + a(t)y + p_0(t),$$

where $u(t)$, $a(t)$, and $p_0(t)$ are arbitrary functions, and the functions $v = v(x,t)$ and $\theta = \theta(x,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned} &v_t + u(t)v_x = -a(t) + \nu v_{xx} + \lambda\theta, \\ &\theta_t + u(t)\theta_x = \sigma\theta_{xx}. \end{aligned} \tag{3}$$

The solution should be started with the second equation; the transformation

$$\theta = \theta(z,t), \quad z = x - \int u(t)\,dt$$

leads to the linear heat equation $\theta_t = \sigma\theta_{zz}$.

4°. Solution (generalizes the solution from Item 3°):

$$u = u(t), \quad v = v(x,t), \quad \theta = ky + f(x,t), \quad p = -v_t'(t)x + \tfrac{1}{2}k\lambda y^2 + a(t)y + p_0(t),$$

where $u(t)$, $a(t)$, and $p_0(t)$ are arbitrary functions, k is an arbitrary constant, and the functions $v = v(x,t)$ and $f = f(x,t)$ are determined by the system of linear partial differential equations

$$\begin{aligned} &v_t + u(t)v_x = -a(t) + \nu v_{xx} + \lambda f, \\ &f_t + u(t)f_x = \sigma f_{xx} - kv. \end{aligned} \tag{4}$$

The transformation

$$v = V(z,t) + A(t), \quad f = F(z,t) + B(t), \quad z = x - \int u(t)\,dt,$$

where the functions $A = A(t)$ and $B = B(t)$ are determined by the system of ordinary differential equations

$$A_t' = \lambda B - a(t), \quad B_t' = kA, \tag{5}$$

leads to a simpler system for the parabolic constant-coefficient equations

$$\begin{aligned} V_t &= \nu V_{zz} + \lambda F, \\ F_t &= \sigma F_{zz} - kV. \end{aligned}$$

This system admits traveling-wave solutions, including simple exponential solutions

$$V = V_0 \exp(\alpha z - \beta t), \quad F = F_0 \exp(\alpha z - \beta t).$$

Note that system (5) is easy to integrate.

21.7. Boundary Layer Equations (Prandtl Equations)

21.7.1. Equations and Boundary Conditions. Stream Function

Boundary layer equations. The two-dimensional boundary layer equations for incompressible fluids are

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\end{gathered}\tag{1}$$

where u and v are the longitudinal and normal components of the fluid velocity, p is the pressure, ρ is the fluid density (ρ = const), and ν is the kinematic viscosity.

The right-hand side of the first equation in (1) involves the pressure, which is assumed to be known:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=f(x,t),\tag{2}$$

where $U=U(x,t)$ is the fluid velocity in the stream core* at the interface between the core and the boundary layer.

Boundary conditions. In the steady-state problems on fluid flows around bodies, to complete the statement of the problem, equations (1) must be supplemented with the no-slip boundary conditions for the fluid velocity at the surface of the body,

$$u=v=0 \qquad \text{at} \quad y=0,\tag{3}$$

and also with the condition

$$u\to U(x) \qquad \text{as} \quad y\to\infty\tag{4}$$

for the asymptotic matching of the longitudinal velocity component of the boundary layer with the fluid velocity in the flow core.

Equation for the stream function. System (1) is reduced to a single equation,

$$\frac{\partial^2 w}{\partial t\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=\nu\frac{\partial^3 w}{\partial y^3}+f(x,t),$$

by introducing a *stream function* w such that $u=\frac{\partial w}{\partial y}$ and $v=-\frac{\partial w}{\partial x}$.

⊙ *References for Subsection* 21.7.1: H. Schlichting (1981), L. Rosenhead (1988), L. G. Loitsyanskiy (1996), O. A. Oleinik and V. N. Samokhin (1999), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

21.7.2. Self-Similar Solutions of Some Boundary Layer Problems

Boundary layer on a flat plate (the Blasius problem). The boundary layer on a flat plate in a translational fluid flow directed along the plate is characterized by a constant pressure, p = const, and the fluid velocity components are expressed as

$$u=U_\infty f'_\eta(\eta),\quad v=\frac{1}{2}\sqrt{\frac{\nu U_{\rm i}}{x}}\,[\eta f'_\eta(\eta)-f(\eta)],\quad \eta=y\sqrt{\frac{U_{\rm i}}{\nu x}},\tag{1}$$

* The hydrodynamic problem on the flow of an ideal (inviscid) fluid about the body is solved in the stream core.

where x is the longitudinal coordinate measured from the front edge of the plate and y is the transverse coordinate, with $y = 0$ corresponding to the plate surface, U_i is the unperturbed incident fluid velocity far from the plate, and $f = f(\eta)$ is a function that satisfies the ordinary differential equation

$$f'''_{\eta\eta\eta} + \tfrac{1}{2} f f''_{\eta\eta} = 0 \tag{2}$$

and the boundary conditions

$$f(0) = f'_\eta(0) = 0, \qquad f'_\eta(\infty) = 1. \tag{3}$$

Detailed tables containing the numerical solution of the boundary value problem (2)–(3) can be found, for example, in Schlichting (1981). It should be noted that $f''_{\eta\eta}(0) \approx 0.332$.

⊙ *References*: H. Blasius (1908), H. Schlichting (1981), L. Rosenhead (1988), L. G. Loitsyanskiy (1996).

Reversed statement of the Blasius problem. In applications, one often deals with the "reversed" statement of the Blasius problem, in which a half-infinite plate moves in its plane at a velocity U_i. In this case, the fluid velocity components are determined by formulas (1), where the function $f = f(\eta)$ is described by the same ordinary differential equation (2) but with other boundary conditions

$$f(0) = 0, \qquad f'_\eta(0) = 1, \qquad f'_\eta(\infty) = 0. \tag{4}$$

Problem (2), (4) has also been solved numerically, and the function $f(\eta)$ is tabulated in Sakiadis (1961). We point out that in this case the solution differs from the corresponding Blasius solution. Thus, although physical consideration suggests that the "inversion" of flow is possible, the solution shows that it is impossible from the mathematical viewpoint. This is due to the fact that problems (2), (3) and (2), (4) are nonlinear.

⊙ *References*: B. C. Sakiadis (1961), A. D. Polyanin, A. M. Kutepov, A. V. Vyazmin, and D. A. Kazenin (2002).

Boundary layer on a V-shaped body (the Falkner–Skan problem). Consider a plane problem involving laminar flow past a V-shaped body (a wedge). In a potential flow of an ideal fluid past the front critical point of the V-shaped body with an angle ω of taper, the velocity close to the vertex is

$$U(x) = Ax^m. \tag{5}$$

Here the x-axis is directed along the wedge surface, A is a constant, and the exponent m and angle α are related by (see Schlichting, 1981)

$$m = \frac{\alpha}{2\pi - \alpha}.$$

The steady-state flow in the plane boundary layer near the surface of a V-shaped body is described by equations (1), (2) and boundary conditions (3), (4) from Subsection 21.7.1, where the y-axis is directed along the normal to the wedge surface (given by $y = 0$), ρ is the fluid density, u and v are, respectively, the longitudinal and transverse components of the fluid velocity, and $U = U(x)$ is defined by Eq. (5).

In the special case $m = 0$, this problem is reduced to the Blasius problem for the steady-state flow past a flat plate. The value $m = 1$, which corresponds to the angle $\alpha = \pi$, characterizes the plane flow near the critical point (Hiemenz, 1911).

The solution of problem (1)–(4) (from Subsection 21.7.1) with the velocity profile in the core flow (5) has the form

$$u = Ax^m \Phi'_\xi(\xi), \qquad \xi = \sqrt{\tfrac{1}{2}(m+1)A/\nu}\, x^{\frac{m-1}{2}} y;$$
$$v = -\sqrt{\tfrac{1}{2}(m+1)A\nu}\, x^{\frac{m-1}{2}} \left[\Phi(\xi) + \frac{m-1}{m+1}\xi\Phi'_\xi(\xi)\right],$$

where the function $\Phi = \Phi(\xi)$ is described by the following equation and boundary conditions:

$$\Phi'''_{\xi\xi\xi} + \Phi\Phi''_{\xi\xi} = \frac{2m}{m+1}\left[(\Phi'_\xi)^2 - 1\right],$$
$$\Phi(0) = \Phi'_\xi(0) = 0, \quad \Phi'_\xi(\infty) = 1.$$

To estimate the value of the second derivative, one can use the approximate formula (Polyanin and Dilman (1994))

$$\Phi''(0) = \sqrt{0.22 + 1.3\,b}, \quad b = 2m/(m+1).$$

The error of this formula does not exceed 0.9% for $0 \le b < \infty$.

⊙ *References*: V. M. Falkner and S. W. Skan (1931), H. Schlichting (1981), L. G. Loitsyanskiy (1996).

Plane jet (the Schlichting–Bickley problem). Consider a steady-state jet flow out of a long narrow slit into a quiescent fluid. We use the two-dimensional boundary layer equations (1) from Subsection 21.7.1 with constant pressure, $p = \text{const}$, and $u_t = 0$. The coordinate x is measured from the slit along the jet and the coordinate y is perpendicular to the jet, with $y = 0$ corresponding to the jet axis. The boundary layer equations are analyzed in conjunction with the boundary conditions

$$v = 0 \quad \text{at} \quad y = 0, \qquad \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0, \qquad u \to 0 \quad \text{as} \quad y \to \infty. \tag{6}$$

The first two conditions in (6) imply that the flow is symmetric about the jet axis and the last condition means that the fluid is quiescent far from the axis. Apart from conditions (6), the jet momentum flux is given:

$$J_0 = \rho \int_{-\infty}^{\infty} u^2\,dy = \text{const}. \tag{7}$$

The solution of problem (1), (6), (7) has the form

$$u = a\left(\frac{K^2}{\nu x}\right)^{1/3}(1 - \tanh^2\xi), \qquad v = b\left(\frac{K\nu}{x^2}\right)^{1/3}[2\xi(1 - \tanh^2\xi) - \tanh\xi],$$
$$\xi = c\left(\frac{K\nu}{x^2}\right)^{1/3}\frac{y}{x^{2/3}}, \quad K = \frac{J_0}{\rho}, \quad a \approx 0.4543, \quad b \approx 0.5503, \quad c = 0.2752,$$

where K is the kinematic momentum.

⊙ *References*: W. Bickley (1939), H. Schlichting (1981).

Axisymmetric jet. In Subsection 21.3.3 (Item 7°), there is an example of an exact solution to an axisymmetric jet-source problem for the Navier–Stokes equations. In the boundary layer approximation for the jet-source problem, the gradients of the normal stresses in the equations of motion are neglected. In the cylindrical coordinates (r, φ, z), in view of the axial symmetry ($u_\varphi = 0$), the system of boundary layer equations has the form

$$u_z\frac{\partial u_z}{\partial z} + u_r\frac{\partial u_z}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right),$$
$$\frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} = 0 \tag{8}$$

with the boundary conditions

$$u_r = 0 \quad \text{at} \quad r = 0, \qquad \frac{\partial u_z}{\partial r} = 0 \quad \text{at} \quad r = 0, \qquad u_z \to 0 \quad \text{as} \quad r \to \infty. \tag{9}$$

Apart from conditions (9), the jet momentum flux J_0 is assumed to be given by a relation similar to (7).

Problem (8)–(9) has an exact closed-form solution of the form

$$\begin{aligned} u_z &= \frac{3}{8\pi}\,\frac{J_0}{\rho\nu}\,\frac{1}{z}\left(1+\frac{\eta^2}{4}\right)^{-2}, \quad \eta = \frac{1}{4\nu}\sqrt{\frac{3J_0}{\pi\rho}}\,\frac{r}{z}, \\ u_r &= \frac{1}{4}\sqrt{\frac{3}{\pi}\,\frac{J_0}{\rho}}\,\frac{1}{z}\left(\eta-\frac{\eta^3}{4}\right)\left(1+\frac{\eta^2}{4}\right)^{-2}. \end{aligned} \tag{10}$$

Obviously, solution (10) differs from the Landau solution (see Eq. (15) in Item 7° of Subsection 21.3.3), but many characteristics of the flow remain qualitatively similar. For example, the dependence between the flow field and the jet momentum is the same, and the velocity at the jet axis decreases inversely proportionally to the distance from the source.

⊙ *Reference*: H. Schlichting (1981).

21.7.3. Other Solutions of the Boundary Layer Equations

1°. Solution:

$$u = -\frac{6\nu x + C_1}{(y+C_2)^2} - \frac{C_3}{(y+C_2)^3}, \quad v = -\frac{6\nu}{y+C_2}, \quad p = \text{const},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solution:

$$u = -C\varphi(x)\exp(-Cy), \quad v = -\varphi'_x(x)\exp(-Cy) - \nu C, \quad p = \text{const},$$

where C is an arbitrary constant and $\varphi(x)$ is an arbitrary function.

3°. Solution:

$$u = C_2y^2 + C_1y + C_0, \quad v = 0, \quad p = 2C_2\rho\nu x + C_3,$$

where C_0, C_1, C_2, and C_3 are arbitrary constants.

4°. Solution:

$$u = C_1\exp(ky) + C_2y + C_3, \quad v = k\nu, \quad p = -C_2k\nu\rho x + C_4,$$

where C_1, C_2, C_3, C_4, and k are arbitrary constants.

5°. Solution:

$$u = f'(y)x + g'(y), \quad v = -f(y), \quad p = -\rho(\tfrac{1}{2}Ax^2 + Bx + C),$$

where A, B, and C are arbitrary constants, and the functions $f = f(y)$ and $g = g(y)$ are determined by the system of ordinary differential equations

$$\begin{aligned} (f')^2 - ff'' &= \nu f''' + A, \\ f'g' - fg'' &= \nu g''' + B. \end{aligned}$$

⊙ *References for Subsection* 21.7.3: Yu. N. Pavlovskii (1961), A. D. Polyanin and V. F. Zaitsev (2004).

▷ *For other exact solutions of steady-state and unsteady boundary layer equations expressed in terms of the stream function, see Subsections 13.3.1 and 13.3.3.*

Chapter 22
Systems of General Form

22.1. Nonlinear Systems of Two Equations Involving the First Derivatives with Respect to t

1. $\dfrac{\partial u}{\partial t} = L[u] + uf(t, au - bw) + g(t, au - bw),$

$\dfrac{\partial w}{\partial t} = L[w] + wf(t, au - bw) + h(t, au - bw).$

Here, L is an arbitrary linear differential operator (of any order) with respect to the spatial variables $x_1, \ldots, x_n$.

1°. Solution:

$$u = \varphi(t) + b\exp\left[\int f(t, a\varphi - b\psi)\,dt\right]\theta(x,t), \quad w = \psi(t) + a\exp\left[\int f(t, a\varphi - b\psi)\,dt\right]\theta(x,t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = \varphi f(t, a\varphi - b\psi) + g(t, a\varphi - b\psi),$$
$$\psi'_t = \psi f(t, a\varphi - b\psi) + h(t, a\varphi - b\psi),$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

Remark 1. The coefficients of the linear differential operator L can be dependent on $x_1, \ldots, x_n, t$.

2°. Let us multiply the first equation by a and the second one by $-b$ and add the results together to obtain

$$\frac{\partial \zeta}{\partial t} = L[\zeta] + \zeta f(t,\zeta) + ag(t,\zeta) - bh(t,\zeta), \qquad \zeta = au - bw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial u}{\partial t} = L[u] + uf(t,\zeta) + g(t,\zeta). \tag{2}$$

Equation (1) can be treated separately. Given a solution of this equation, $\zeta = \zeta(x_1, \ldots, x_n, t)$, the function $u = u(x_1, \ldots, x_n, t)$ can be determined by solving the linear equation (2) and the function $w = w(x_1, \ldots, x_n, t)$ is found as $w = (au - \zeta)/b$.

Remark 2. Let L be a constant-coefficient differential operator with respect to the independent variable $x = x_1$ and let the condition

$$\frac{\partial}{\partial t}\Big[\zeta f(t,\zeta) + ag(t,\zeta) - bh(t,\zeta)\Big] = 0$$

hold true (for example, it is valid if the functions f, g, h are not implicitly dependent on t). Then equation (1) admits an exact, traveling-wave solution $\zeta = \zeta(z)$, where $z = kx - \lambda t$ with arbitrary constants k and λ.

⊙ *Reference for system* 22.1.1: A. D. Polyanin (2004 *b*).

2. $$\frac{\partial u}{\partial t} = L[u] + b_2 f(a_1u + b_1w) + b_1 g(a_2u + b_2w),$$
$$\frac{\partial w}{\partial t} = L[w] - a_2 f(a_1u + b_1w) - a_1 g(a_2u + b_2w).$$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can be dependent on x and t. It is assumed that $L[\text{const}] = 0$ and $a_1b_2 - a_2b_1 \neq 0$.

Multiplying the equations by suitable constants and adding together, one obtains two independent equations

$$\begin{aligned} \frac{\partial U}{\partial t} &= L[U] + (a_1b_2 - a_2b_1)f(U), \qquad U = a_1u + b_1w; \\ \frac{\partial W}{\partial t} &= L[W] - (a_1b_2 - a_2b_1)g(W), \qquad W = a_2u + b_2w. \end{aligned} \tag{1}$$

Special case 1. If L is an arbitrary linear differential operator with constant coefficients, then equations (1) admits traveling-wave solutions

$$U = U(k_1x - \lambda_1t), \quad W = W(k_2x - \lambda_2t),$$

where k_m and λ_m are arbitrary constants. The corresponding solution of the original system will be a superposition (linear combination) of two nonlinear traveling waves.

Special case 2. If the coefficients of the linear operator L depend on x alone, then equations (1) have simple solutions of the form

$$U = U(t), \quad W = W(x); \qquad U = U(x), \quad W = W(t).$$

Remark. The case where the linear differential operator L depends on several spatial coordinates $x_1, \ldots, x_n$ is treated likewise.

3. $$\frac{\partial u}{\partial t} = L_1[u] + uf\left(\frac{u}{w}\right), \qquad \frac{\partial w}{\partial t} = L_2[w] + wg\left(\frac{u}{w}\right).$$

Here, L_1 and L_2 are arbitrary constant-coefficient linear differential operators (of any order) with respect to x.

1°. Solution:

$$u = e^{kx-\lambda t}y(\xi), \qquad w = e^{kx-\lambda t}z(\xi), \qquad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$\begin{aligned} &M_1[y] + \lambda y + yf(y/z) = 0, \quad M_2[z] + \lambda z + zg(y/z) = 0, \\ &M_1[y] = e^{-kx}L_1[e^{kx}y(\xi)], \qquad M_2[z] = e^{-kx}L_2[e^{kx}z(\xi)]. \end{aligned}$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution.

2°. If the operators L_1 and L_2 contain only even derivatives, there are solutions of the form

$$\begin{aligned} &u = [C_1\sin(kx) + C_2\cos(kx)]\varphi(t), && w = [C_1\sin(kx) + C_2\cos(kx)]\psi(t); \\ &u = [C_1\exp(kx) + C_2\exp(-kx)]\varphi(t), && w = [C_1\exp(kx) + C_2\exp(-kx)]\psi(t); \\ &u = (C_1x + C_2)\varphi(t), && w = (C_1x + C_2)\psi(t), \end{aligned}$$

where C_1, C_2, and k are arbitrary constants. Note that the third solution is degenerate.

⊙ *References for system* 22.1.3: A. D. Polyanin (2004 *b*).

4. $\dfrac{\partial u}{\partial t} = L[u] + uf\left(t, \dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wg\left(t, \dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator with respect to the coordinates $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \ldots, x_n, t$:

$$L[u] = \sum A_{k_1 \ldots k_n}(x_1, \ldots, x_n, t) \frac{\partial^{k_1+\cdots+k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}. \tag{1}$$

1°. Solution:

$$\begin{aligned} u &= \varphi(t) \exp\left[\int g(t, \varphi(t))\, dt\right] \theta(x_1, \ldots, x_n, t), \\ w &= \exp\left[\int g(t, \varphi(t))\, dt\right] \theta(x_1, \ldots, x_n, t), \end{aligned} \tag{2}$$

where the function $\varphi = \varphi(t)$ is described by the first-order nonlinear ordinary differential equation

$$\varphi'_t = [f(t, \varphi) - g(t, \varphi)]\varphi, \tag{3}$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

2°. The transformation

$$u = a_1(t)U + b_1(t)W, \quad w = a_2(t)U + b_2(t)W,$$

where $a_n(t)$ and $b_n(t)$ are arbitrary functions ($n = 1, 2$), leads to an equation of the similar form for U and W.

Remark. The coefficients of the operator (1) can also depend on the ratio of the unknowns u/w, $A_{k_1 \ldots k_n} = A_{k_1 \ldots k_n}(x_1, \ldots, x_n, t, u/w)$ (in this case, L will be a quasilinear operator). Then there also exists a solution of the form (2), where $\varphi = \varphi(t)$ is described by the ordinary differential equation (3) and $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L^\circ[\theta], \qquad L^\circ = L_{u/w=\varphi}.$$

⊙ *References for system* 22.1.4: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5203.pdf).

5. $\dfrac{\partial u}{\partial t} = L[u] + uf\left(\dfrac{u}{w}\right) + g\left(\dfrac{u}{w}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wf\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator with respect to $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \ldots, x_n, t$:

$$L[u] = \sum A_{k_1 \ldots k_n}(x_1, \ldots, x_n, t) \frac{\partial^{k_1+\cdots+k_n} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}},$$

where $k_1 + \cdots + k_n \geq 1$.

Let λ be a root of the algebraic (transcendental) equation

$$g(\lambda) = \lambda h(\lambda). \tag{1}$$

1°. Solution if $f(\lambda) \neq 0$:

$$u(x,t) = \lambda\left(\exp[f(\lambda)t]\theta(x,t) - \frac{h(\lambda)}{f(\lambda)}\right), \quad w(x,t) = \exp[f(\lambda)t]\theta(x,t) - \frac{h(\lambda)}{f(\lambda)},$$

where the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta]. \tag{2}$$

2°. Solution if $f(\lambda) = 0$:

$$u(x,t) = \lambda[\theta(x,t) + h(\lambda)t], \quad w(x,t) = \theta(x,t) + h(\lambda)t,$$

where the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation (2).

6. $$\frac{\partial u}{\partial t} = L[u] + uf\left(\frac{w}{u}\right) - wg\left(\frac{w}{u}\right) + \frac{u}{\sqrt{u^2+w^2}}h\left(\frac{w}{u}\right),$$
$$\frac{\partial w}{\partial t} = L[w] + wf\left(\frac{w}{u}\right) + ug\left(\frac{w}{u}\right) + \frac{w}{\sqrt{u^2+w^2}}h\left(\frac{w}{u}\right).$$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u = r(x,t)\cos\varphi(t), \quad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined from the separable first-order ordinary differential equation

$$\varphi'_t = g(\tan\varphi),$$

and the function $r = r(x,t)$ satisfies the linear equation

$$\frac{\partial r}{\partial t} = L[r] + rf(\tan\varphi) + h(\tan\varphi). \tag{1}$$

The change of variable

$$r = F(t)\left[Z(x,t) + \int \frac{h(\tan\varphi)\,dt}{F(t)}\right], \quad F(t) = \exp\left[\int f(\tan\varphi)\,dt\right]$$

reduces (1) to the simpler equation

$$\frac{\partial Z}{\partial t} = L[Z].$$

Remark. The linear differential operator L can depend on several spatial coordinates $x_1, \ldots, x_n$.

7. $$\frac{\partial u}{\partial t} = L[u] + uf\Big(\frac{w}{u}\Big) + wg\Big(\frac{w}{u}\Big) + \frac{u}{\sqrt{u^2 - w^2}}h\Big(\frac{w}{u}\Big),$$
$$\frac{\partial w}{\partial t} = L[w] + wf\Big(\frac{w}{u}\Big) + ug\Big(\frac{w}{u}\Big) + \frac{w}{\sqrt{u^2 - w^2}}h\Big(\frac{w}{u}\Big).$$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u = r(x,t)\cosh\varphi(t), \qquad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined from the separable first-order ordinary differential equation

$$\varphi_t' = g(\tanh\varphi),$$

and the function $r = r(x,t)$ satisfies the linear equation

$$\frac{\partial r}{\partial t} = L[r] + rf(\tanh\varphi) + h(\tanh\varphi). \tag{1}$$

The change of variable

$$r = F(t)\Big[Z(x,t) + \int \frac{h(\tanh\varphi)\,dt}{F(t)}\Big], \qquad F(t) = \exp\Big[\int f(\tanh\varphi)\,dt\Big]$$

reduces (1) to the linear heat equation

$$\frac{\partial Z}{\partial t} = L[Z].$$

Remark. There are also solutions of the form

$$u = r(x,t)\sinh\varphi(t), \qquad w = r(x,t)\cosh\varphi(t).$$

8. $$\frac{\partial u}{\partial t} = L[u] + uf\left(t, \frac{u}{w}\right) + \frac{u}{w}h\left(t, \frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = L[w] + wg\left(t, \frac{u}{w}\right) + h\left(t, \frac{u}{w}\right).$$

Solution:

$$u = \varphi(t)G(t)\left[\theta(x_1,\ldots,x_n,t) + \int \frac{h(t,\varphi)}{G(t)}\,dt\right], \quad G(t) = \exp\left[\int g(t,\varphi)\,dt\right],$$
$$w = G(t)\left[\theta(x_1,\ldots,x_n,t) + \int \frac{h(t,\varphi)}{G(t)}\,dt\right],$$

where the function $\varphi = \varphi(t)$ is described by the first-order nonlinear ordinary differential equation

$$\varphi_t' = [f(t,\varphi) - g(t,\varphi)]\varphi,$$

and the function $\theta = \theta(x_1,\ldots,x_n,t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

⊙ *Reference*: A. D. Polyanin (2004 *b*).

9. $\dfrac{\partial u}{\partial t} = L[u] + uf\left(t, \dfrac{u}{w}\right)\ln u + ug\left(t, \dfrac{u}{w}\right),$

$\dfrac{\partial w}{\partial t} = L[w] + wf\left(t, \dfrac{u}{w}\right)\ln w + wh\left(t, \dfrac{u}{w}\right).$

Solution:

$$u(x,t) = \varphi(t)\psi(t)\theta(x_1,\ldots,x_n,t), \quad w(x,t) = \psi(t)\theta(x_1,\ldots,x_n,t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by solving the ordinary differential equations

$$\begin{aligned} \varphi_t' &= \varphi[g(t,\varphi) - h(t,\varphi) + f(t,\varphi)\ln\varphi], \\ \psi_t' &= \psi[h(t,\varphi) + f(t,\varphi)\ln\psi], \end{aligned} \tag{1}$$

and the function $\theta = \theta(x_1,\ldots,x_n,t)$ is determined by the differential equation

$$\frac{\partial\theta}{\partial t} = L[\theta] + f(t,\varphi)\theta\ln\theta. \tag{2}$$

Given a solution to the first equation in (1), the second equation can be solved with the change of variable $\psi = e^{\zeta}$ by reducing it to a linear equation for ζ. If L is a constant-coefficient one-dimensional operator ($n = 1$) and $f = \text{const}$, then equation (2) has a traveling-wave solution $\theta = \theta(kx - \lambda t)$.

⊙ *Reference*: A. D. Polyanin (2004 *b*).

10. $\dfrac{\partial u}{\partial t} = L[u] + uf(au+bw) - bwg\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wf(au+bw) + awg\left(\dfrac{w}{u}\right).$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x.

1°. Solution:

$$u = br(x,t)\cos^2\varphi(t), \quad w = ar(x,t)\sin^2\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined from the separable first-order ordinary differential equation

$$\varphi_t' = \frac{1}{2}a\tan\varphi\, g\left(\frac{a}{b}\tan^2\varphi\right), \tag{1}$$

and the function $r = r(x,t)$ satisfies the equation

$$\frac{\partial r}{\partial t} = L[r] + rf(abr). \tag{2}$$

The general solution to equation (1) can be written out in implicit form:

$$2\int \frac{d\varphi}{a\tan\varphi\, g(ab^{-1}\tan^2\varphi)} = t + C.$$

Equation (2) admits time-invariant solutions $r = r(x)$. If L is a linear differential operator with constant coefficients, then equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

2°. Solution:

$$u = br(x,t)\cosh^2\psi(t), \quad w = -ar(x,t)\sinh^2\psi(t),$$

where the function $\psi = \psi(t)$ is determined by the separable first-order ordinary differential equation

$$\psi_t' = \frac{1}{2}a\tanh\psi\, g\left(-\frac{a}{b}\tanh^2\psi\right),$$

and the function $r = r(x,t)$ satisfies equation (2).

3°. Multiplying the first equation by a and the second one by b adding together, one obtains the equation

$$\frac{\partial z}{\partial t} = L[z] + zf(z), \quad z = au + bw.$$

Remark. The solution presented in Items 1° and 2° are easy to generalize to the case where the linear differential operator L depends on n spatial coordinates $x_1, \ldots, x_n$.

11. $\dfrac{\partial u}{\partial t} = L[u] + uf(u^2+w^2) - wg\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wf(u^2+w^2) + ug\left(\dfrac{w}{u}\right).$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x.

Solution:

$$u = r(x,t)\cos\varphi(t), \quad w = r(x,t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the separable first-order ordinary differential equation

$$\varphi_t' = g(\tan\varphi), \tag{1}$$

and $r = r(x,t)$ satisfies the equation

$$\frac{\partial r}{\partial t} = L[r] + rf(r^2). \tag{2}$$

The general solution to equation (1) can be written out in implicit form:

$$\int \frac{d\varphi}{g(\tan\varphi)} = t + C.$$

It is noteworthy that equation (2) admits time-invariant solutions $r = r(x)$. If the coefficients of the linear differential operator L are constant, then equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

Remark. The above solution is easy to generalize to the case where the linear differential operator L depends on n spatial coordinates $x_1, \ldots, x_n$.

12. $\dfrac{\partial u}{\partial t} = L[u] + uf(u^2-w^2) + wg\left(\dfrac{w}{u}\right), \quad \dfrac{\partial w}{\partial t} = L[w] + wf(u^2-w^2) + ug\left(\dfrac{w}{u}\right).$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x.

Solution:

$$u = r(x,t)\cosh\varphi(t), \quad w = r(x,t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order ordinary differential equation

$$\varphi_t' = g(\tanh\varphi), \tag{1}$$

and $r = r(x,t)$ satisfies

$$\frac{\partial r}{\partial t} = L[r] + rf(r^2). \tag{2}$$

The general solution to equation (1) can be represented in implicit form:

$$\int \frac{d\varphi}{g(\tanh\varphi)} = t + C.$$

Note that equation (2) admits time-invariant solutions $r = r(x)$. If the coefficients of the linear differential operator L are constant, then equation (2) admits a traveling-wave solution $r = r(kx - \lambda t)$, where k and λ are arbitrary constants.

Remark. The above solution is easy to generalize to the case where the linear differential operator L depends on n spatial coordinates $x_1, \ldots, x_n$.

22.2. Nonlinear Systems of Two Equations Involving the Second Derivatives with Respect to t

1. $$\frac{\partial^2 u}{\partial t^2} = L[u] + uf(t, au - bw) + g(t, au - bw),$$
$$\frac{\partial^2 w}{\partial t^2} = L[w] + wf(t, au - bw) + h(t, au - bw).$$

Here, L is an arbitrary linear differential operator (of any order) with respect to the spatial variables $x_1, \ldots, x_n$, whose coefficients can be dependent on $x_1, \ldots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

1°. Solution:

$$u = \varphi(t) + a\theta(x_1, \ldots, x_n, t), \qquad w = \psi(t) + b\theta(x_1, \ldots, x_n, t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = \varphi f(t, a\varphi - b\psi) + g(t, a\varphi - b\psi),$$
$$\psi''_{tt} = \psi f(t, a\varphi - b\psi) + h(t, a\varphi - b\psi),$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies linear equation

$$\frac{\partial^2 \theta}{\partial t^2} = L[\theta] + f(t, a\varphi - b\psi)\theta.$$

2°. Let us multiply the first equation by a and the second one by $-b$ and add the results together to obtain

$$\frac{\partial^2 \zeta}{\partial t^2} = L[\zeta] + \zeta f(t,\zeta) + ag(t,\zeta) - bh(t,\zeta), \qquad \zeta = au - bw. \tag{1}$$

This equation will be considered in conjunction with the first equation of the original system

$$\frac{\partial^2 u}{\partial t^2} = L[u] + uf(t,\zeta) + g(t,\zeta). \tag{2}$$

Equation (1) can be treated separately. Given a solution $\zeta = \zeta(x,t)$ to equation (1), the function $u = u(x_1,\dots,x_n,t)$ can be determined by solving the linear equation (2) and the function $w = w(x_1,\dots,x_n,t)$ is found as $w = (au-\zeta)/b$.

Note three important cases where equation (1) admits exact solutions:

(i) Equation (1) admits a spatially homogeneous solution $\zeta = \zeta(t)$.

(ii) Suppose the coefficients of L and the functions f, g, h are not implicitly dependent on t. Then equation (1) admits a steady-state solution $\zeta = \zeta(x_1,\dots,x_n)$.

(iii) If the condition $\zeta f(t,\zeta) + bg(t,\zeta) - ch(t,\zeta) = k_1\zeta + k_0$ holds, equation (1) is linear. If the operator L is constant-coefficient, the method of separation of variables can be used to obtain solutions.

⊙ *Reference for system* 22.2.1: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5301.pdf).

2. $$\frac{\partial^2 u}{\partial t^2} = L[u] + b_2 f(a_1u + b_1w) + b_1 g(a_2u + b_2w),$$
$$\frac{\partial^2 w}{\partial t^2} = L[w] - a_2 f(a_1u + b_1w) - a_1 g(a_2u + b_2w).$$

Here L is an arbitrary linear differential operator in x (of any order in the derivatives) whose coefficients can depend on x and t. It is assumed that $L[\text{const}] = 0$ and $a_1b_2 - a_2b_1 \neq 0$.

Multiplying the equations by suitable constants and adding together, one arrives at two independent equations

$$\begin{aligned} &\frac{\partial^2 U}{\partial t^2} = L[U] + (a_1b_2 - a_2b_1)f(U), \qquad U = a_1u + b_1w; \\ &\frac{\partial^2 W}{\partial t^2} = L[W] - (a_1b_2 - a_2b_1)g(W), \qquad W = a_2u + b_2w. \end{aligned} \tag{1}$$

Special case 1. If the coefficients of the linear differential operator L are constant, then equations (1) admit traveling-wave solutions of the form

$$U = U(k_1x - \lambda_1 t), \quad W = W(k_2x - \lambda_2 t),$$

where k_m and λ_m are arbitrary constants. The corresponding solution of the original system will be a superposition (linear combination) of two traveling waves.

Special case 2. If the coefficients of the operator L depend on x alone, then equations (1) have simple solutions of the form

$$U = U(t), \quad W = W(x); \qquad U = U(x), \quad W = W(t).$$

Remark. The case where the linear differential operator L depends on several spatial coordinates $x_1,\dots,x_n$ can be treated likewise.

3. $$\frac{\partial^2 u}{\partial t^2} = L_1[u] + uf\left(\frac{u}{w}\right), \quad \frac{\partial^2 w}{\partial t^2} = L_2[w] + wg\left(\frac{u}{w}\right).$$

Here, L_1 and L_2 are arbitrary constant-coefficient linear differential operators (of any order) with respect to x. It is assumed that $L_1[\text{const}] = 0$ and $L_2[\text{const}] = 0$.

1°. Solution in the form of the product of two waves traveling at different speeds:

$$u = e^{kx-\lambda t}y(\xi), \quad w = e^{kx-\lambda t}z(\xi), \quad \xi = \beta x - \gamma t,$$

where k, λ, β, and γ are arbitrary constants, and the functions $y = y(\xi)$ and $z = z(\xi)$ are determined by the system of ordinary differential equations

$$\gamma^2 y''_{\xi\xi} + 2\lambda\gamma y'_\xi + \lambda^2 y = M_1[y] + yf(y/z), \quad \gamma^2 z''_{\xi\xi} + 2\lambda\gamma z'_\xi + \lambda^2 z = M_2[z] + zg(y/z),$$
$$M_1[y] = e^{-kx}L_1[e^{kx}y(\xi)], \qquad M_2[z] = e^{-kx}L_2[e^{kx}z(\xi)].$$

To the special case $k = \lambda = 0$ there corresponds a traveling-wave solution.

2°. Periodic multiplicative separable solution:

$$u = [C_1 \sin(kt) + C_2 \cos(kt)]\varphi(x), \quad w = [C_1 \sin(kt) + C_2 \cos(kt)]\psi(x),$$

where C_1, C_2, and k are arbitrary constants and the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$L_1[\varphi] + k^2\varphi + \varphi f(\varphi/\psi) = 0,$$
$$L_2[\psi] + k^2\psi + \psi g(\varphi/\psi) = 0.$$

3°. Multiplicative separable solution:

$$u = [C_1 \sinh(kt) + C_2 \cosh(kt)]\varphi(x), \quad w = [C_1 \sinh(kt) + C_2 \cosh(kt)]\psi(x),$$

where C_1, C_2, and k are arbitrary constants and the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$L_1[\varphi] - k^2\varphi + \varphi f(\varphi/\psi) = 0,$$
$$L_2[\psi] - k^2\psi + \psi g(\varphi/\psi) = 0.$$

4°. Degenerate multiplicative separable solution:

$$u = (C_1 t + C_2)\varphi(x), \quad w = (C_1 t + C_2)\psi(x),$$

where C_1 and C_2 are arbitrary constants and the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of ordinary differential equations

$$L_1[\varphi] + \varphi f(\varphi/\psi) = 0, \quad L_2[\psi] + \psi g(\varphi/\psi) = 0.$$

Remark 1. The coefficients of L_1, L_2 and the functions f and g in Items 2°–4° can be dependent on x.

Remark 2. If L_1 and L_2 contain only even derivatives, there are solutions of the form

$u = [C_1 \sin(kx) + C_2 \cos(kx)]U(t)$, $w = [C_1 \sin(kx) + C_2 \cos(kx)]W(t)$;
$u = [C_1 \exp(kx) + C_2 \exp(-kx)]U(t)$, $w = [C_1 \exp(kx) + C_2 \exp(-kx)]W(t)$;
$u = (C_1 x + C_2)U(t)$, $w = (C_1 x + C_2)W(t)$,

where C_1, C_2, and k are arbitrary constants. Note that the third solution is degenerate.

⊙ *Reference for system* 22.2.3: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5302.pdf).

4. $\dfrac{\partial^2 u}{\partial t^2} = L[u] + uf\left(t, \dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial t^2} = L[w] + wg\left(t, \dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator with respect to the coordinates $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on the coordinates.

Solution:

$$u = \varphi(t)\theta(x_1, \ldots, x_n),$$
$$w = \psi(t)\theta(x_1, \ldots, x_n),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the nonlinear system of second-order ordinary differential equations

$$\varphi''_{tt} = a\varphi + \varphi f(t, \varphi/\psi),$$
$$\psi''_{tt} = a\psi + \psi g(t, \varphi/\psi),$$

a is an arbitrary constant, and the function $\theta = \theta(x_1, \ldots, x_n)$ satisfies the linear steady-state equation

$$L[\theta] = a\theta.$$

⊙ *Reference*: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5303.pdf).

5. $\dfrac{\partial^2 u}{\partial t^2} = L[u] + uf\left(\dfrac{u}{w}\right) + g\left(\dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial t^2} = L[w] + wf\left(\dfrac{u}{w}\right) + h\left(\dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator with respect to the coordinates $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \ldots, x_n, t$.

Solution:

$$u = k\theta(x_1, \ldots, x_n, t), \quad w = \theta(x_1, \ldots, x_n, t),$$

where k is a root of the algebraic (transcendental) equation $g(k) = kh(k)$ and the function $\theta = \theta(x, t)$ satisfies the linear equation

$$\frac{\partial^2 \theta}{\partial t^2} = L[\theta] + f(k)w + h(k).$$

⊙ *Reference*: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5304.pdf).

6. $\dfrac{\partial^2 u}{\partial t^2} = L[u] + au \ln u + uf\left(t, \dfrac{u}{w}\right), \quad \dfrac{\partial^2 w}{\partial t^2} = L[w] + aw \ln w + wg\left(t, \dfrac{u}{w}\right).$

Here, L is an arbitrary linear differential operator with respect to the coordinates $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on the coordinates.

Solution:

$$u = \varphi(t)\theta(x_1, \ldots, x_n),$$
$$w = \psi(t)\theta(x_1, \ldots, x_n),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are described by the nonlinear system of second-order ordinary differential equations

$$\varphi''_{tt} = a\varphi \ln \varphi + b\varphi + \varphi f(t, \varphi/\psi),$$
$$\psi''_{tt} = a\psi \ln \psi + b\psi + \psi g(t, \varphi/\psi),$$

b is an arbitrary constant, and the function $\theta = \theta(x_1, \ldots, x_n)$ satisfies the steady-state equation

$$L[\theta] + a\theta \ln \theta - b\theta = 0.$$

In the special case $a = 0$, this equation is linear.

⊙ *Reference*: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5305.pdf).

22.3. Other Nonlinear Systems of Two Equations

1. $F_1\left(w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^m w}{\partial x^m}, \dfrac{1}{u^k}\dfrac{\partial w}{\partial t}, \dfrac{1}{u}\dfrac{\partial u}{\partial x}, \ldots, \dfrac{1}{u}\dfrac{\partial^n u}{\partial x^n}\right) = 0,$

$F_2\left(w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^m w}{\partial x^m}, \dfrac{1}{u^k}\dfrac{\partial w}{\partial t}, \dfrac{1}{u}\dfrac{\partial u}{\partial x}, \ldots, \dfrac{1}{u}\dfrac{\partial^n u}{\partial x^n}\right) = 0.$

Solution:

$$w = W(z), \quad u = [\varphi'(t)]^{1/k} U(z), \quad z = x + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function, and the functions $W(z)$ and $U(z)$ are determined by the autonomous system of ordinary differential equations

$$F_1\big(W, W'_z, \ldots, W^{(m)}_z, W'_z/U^k, U'_z/U, \ldots, U^{(n)}_z/U\big) = 0,$$
$$F_2\big(W, W'_z, \ldots, W^{(m)}_z, W'_z/U^k, U'_z/U, \ldots, U^{(n)}_z/U\big) = 0.$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

2. $F_1\left(au^k + bw, \dfrac{1}{u}\dfrac{\partial u}{\partial x}, \ldots, \dfrac{1}{u}\dfrac{\partial^m u}{\partial x^m}, \dfrac{1}{u^k}\dfrac{\partial w}{\partial t}, \ldots, \dfrac{1}{u^k}\dfrac{\partial^n w}{\partial t^n}\right) = 0,$

$F_2\left(au^k + bw, \dfrac{1}{u}\dfrac{\partial u}{\partial x}, \ldots, \dfrac{1}{u}\dfrac{\partial^m u}{\partial x^m}, \dfrac{1}{u^k}\dfrac{\partial w}{\partial t}, \ldots, \dfrac{1}{u^k}\dfrac{\partial^n w}{\partial t^n}\right) = 0.$

Solution:

$$u = e^{\lambda t}\theta(x), \qquad w = \varphi(x) - \frac{a}{b}e^{k\lambda t}[\theta(x)]^k,$$

where $\theta(x)$ and $\varphi(x)$ are described by the system of ordinary differential equations:

$$F_1\left(b\varphi, \frac{\theta_x}{\theta}, \ldots, \frac{\theta_x^{(m)}}{\theta}, -\frac{a}{b}k\lambda, \ldots, -\frac{a}{b}(k\lambda)^n\right) = 0,$$

$$F_2\left(b\varphi, \frac{\theta_x}{\theta}, \ldots, \frac{\theta_x^{(m)}}{\theta}, -\frac{a}{b}k\lambda, \ldots, -\frac{a}{b}(k\lambda)^n\right) = 0.$$

3. $ax\dfrac{\partial u}{\partial x} + ay\dfrac{\partial u}{\partial y} = F_1\left(u, w, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial y^2}\right),$

$ax\dfrac{\partial w}{\partial x} + ay\dfrac{\partial w}{\partial y} = F_2\left(u, w, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial y^2}\right).$

Traveling-wave solution:

$$u(x, y) = U(z), \qquad w(x, y) = W(z), \qquad z = k_1 x + k_2 y,$$

where k_1 and k_2 are arbitrary constants, and the functions $U = U(z)$ and $W = W(z)$ are determined by the system of ordinary differential equations

$$azU_z' = F_1\big(u, w, k_1U_z', k_2U_z', k_1W_z', k_2W_z', k_1^2U_{zz}'', k_2^2U_{zz}'', k_1^2W_{zz}'', k_2^2W_{zz}''\big),$$
$$azW_z' = F_2\big(u, w, k_1U_z', k_2U_z', k_1W_z', k_2W_z', k_1^2U_{zz}'', k_2^2U_{zz}'', k_1^2W_{zz}'', k_2^2W_{zz}''\big).$$

22.4. Nonlinear Systems of Many Equations Involving the First Derivatives with Respect to t

1. $\dfrac{\partial}{\partial t}\left[u + G(w_1, \ldots, w_n)\right] + \dfrac{\partial u}{\partial x} = 0, \qquad \dfrac{\partial w_k}{\partial t} = F_k(w_1, \ldots, w_n)u,$

$k = 1, \ldots, n.$

This system of $n + 1$ equations describes deep seepage of a multi-component particulate suspension into a porous medium with variable permeability (due to capture of particles by the porous medium).

The last n equations imply

$$\frac{1}{F_1(w_1, \ldots, w_n)}\frac{\partial w_1}{\partial t} = \cdots = \frac{1}{F_n(w_1, \ldots, w_n)}\frac{\partial w_n}{\partial t} = u. \tag{1}$$

Exact solutions to the system are sought in the form

$$w_1 = w_1(w_n), \quad \ldots, \quad w_{n-1} = w_{n-1}(w_n). \tag{2}$$

So it is assumed that the functions $w_1, \ldots, w_{n-1}$ can be expressed through w_n. From (1) and (2), one obtains the autonomous system of ordinary differential equations

$$\frac{dw_k}{dw_n} = \frac{F_k(w_1, \ldots, w_n)}{F_n(w_1, \ldots, w_n)}, \qquad k = 1, \ldots, n-1. \tag{3}$$

Further, by assuming that a solution to system (3) has been obtained and the dependences (2) are known, we substitute them into the original system to arrive at the system of two equations

$$\begin{aligned} &\frac{\partial}{\partial t}\big[u + g(w_n)\big] + \frac{\partial u}{\partial x} = 0, \\ &\frac{\partial w_n}{\partial t} = f_n(w_n)u, \end{aligned} \tag{4}$$

where

$$g(w_n) = G\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big), \qquad f_n(w_n) = F_n\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big).$$

By introducing, in (4), a new dependent variable

$$w = g(w_n) \equiv G\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big),$$

we obtain system 16.2.2.1 in which the function $f = f(w)$ is defined parametrically by

$$f = g'(w_n)f_n(w_n), \qquad w = g(w_n),$$

where w_n is treated as the parameter.

Remark. Solutions of the above form arise, for example, in problems with the initial and boundary conditions

$$\begin{aligned} w_1 = \cdots = w_n = u = 0 \quad &\text{at} \quad t = 0, \\ u = 1 \quad &\text{at} \quad x = 0. \end{aligned}$$

In this case, the system of ordinary differential equations (3) must be supplemented with the initial conditions

$$w_1 = \cdots = w_{n-1} = 0 \quad \text{at} \quad w_n = 0.$$

⊙ *Reference*: P. Bedrikovetsky (2006).

2. $$\frac{\partial u_m}{\partial t} = L[u_m] + u_m f(t, u_1 - b_1u_n, \ldots, u_{n-1} - b_{n-1}u_n) + g_m(t, u_1 - b_1u_n, \ldots, u_{n-1} - b_{n-1}u_n), \qquad m = 1, \ldots, n.$$

The system involves $n + 1$ arbitrary functions $f, g_1, \ldots, g_n$ that depend on n arguments; L is an arbitrary linear differential operator with respect to the spatial variables $x_1, \ldots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \ldots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u_m = \varphi_m(t) + \exp\Big[\int f(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n)\,dt\Big]\theta(x_1, \ldots, x_n, t).$$

Here, the functions $\varphi_m = \varphi_m(t)$ are determined by the system of ordinary differential equations

$$\varphi_m' = \varphi_m f(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n) + g_m(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n),$$

where $m = 1, \ldots, n$, the prime denotes the derivative with respect to t, and the function $\theta = \theta(x_1, \ldots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

⊙ *Reference*: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5401.pdf).

3. $$\frac{\partial u_m}{\partial t} = L[u_m] + u_m f_m\Big(t, \frac{u_1}{u_n}, \dots, \frac{u_{n-1}}{u_n}\Big) + \frac{u_m}{u_n} g\Big(t, \frac{u_1}{u_n}, \dots, \frac{u_{n-1}}{u_n}\Big),$$
$$\frac{\partial u_n}{\partial t} = L[u_n] + u_n f_n\Big(t, \frac{u_1}{u_n}, \dots, \frac{u_{n-1}}{u_n}\Big) + g\Big(t, \frac{u_1}{u_n}, \dots, \frac{u_{n-1}}{u_n}\Big).$$

Here, $m = 1, \dots, n-1$ and the system involves $n+1$ arbitrary functions $f_1, \dots, f_n$, g that depend on n arguments; L is an arbitrary linear differential operator with respect to the spatial variables $x_1, \dots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \dots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u_m = \varphi_m(t) F_n(t)\Big[\theta(x_1, \dots, x_n, t) + \int \frac{g(t, \varphi_1, \dots, \varphi_{n-1})}{F_n(t)}\, dt\Big], \qquad m = 1, \dots, n-1,$$
$$u_n = F_n(t)\Big[\theta(x_1, \dots, x_n, t) + \int \frac{g(t, \varphi_1, \dots, \varphi_{n-1})}{F_n(t)}\, dt\Big],$$
$$F_n(t) = \exp\Big[\int f_n(t, \varphi_1, \dots, \varphi_{n-1})\, dt\Big],$$

where the functions $\varphi_m = \varphi_m(t)$ are described by the nonlinear system of first-order ordinary differential equations

$$\varphi_m' = \varphi_m[f_m(t, \varphi_1, \dots, \varphi_{n-1}) - f_n(t, \varphi_1, \dots, \varphi_{n-1})], \qquad m = 1, \dots, n-1,$$

and the function $\theta = \theta(x_1, \dots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

⊙ *Reference*: A. D. Polyanin (2004, http://eqworld.ipmnet.ru/en/solutions/syspde/spde5402.pdf).

4. $$\frac{\partial u_m}{\partial t} = L[u_m] + \sum_{k=1}^{n} u_k f_{mk}\Big(t, \frac{u_1}{u_n}, \dots, \frac{u_{n-1}}{u_n}\Big), \qquad m = 1, \dots, n.$$

Here, the system involves n^2 arbitrary functions $f_{mk} = f_{mk}(t, z_1, \dots, z_{n-1})$ that depend on n arguments; L is an arbitrary linear differential operator with respect to the spatial variables $x_1, \dots, x_n$ (of any order in derivatives), whose coefficients can be dependent on $x_1, \dots, x_n, t$. It is assumed that $L[\text{const}] = 0$.

Solution:

$$u_m(x_1, \dots, x_n, t) = \varphi_m(t) F(t) \theta(x_1, \dots, x_n, t), \qquad m = 1, \dots, n,$$
$$F(t) = \exp\Big[\int \sum_{k=1}^{n} \varphi_k(t) f_{nk}(t, \varphi_1, \dots, \varphi_{n-1})\, dt\Big], \qquad \varphi_n(t) = 1,$$

where the functions $\varphi_m = \varphi_m(t)$ are described by the nonlinear system of first-order ordinary differential equations

$$\varphi_m' = \sum_{k=1}^{n} \varphi_k f_{mk}(t, \varphi_1, \dots, \varphi_{n-1}) - \varphi_m \sum_{k=1}^{n} \varphi_k f_{nk}(t, \varphi_1, \dots, \varphi_{n-1}), \qquad m = 1, \dots, n-1,$$

and the function $\theta = \theta(x_1, \dots, x_n, t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta].$$

⊙ *Reference*: A. D. Polyanin and E. A. Vyazmina (2006).

5. $\dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}\left[f\left(\sum\limits_{k=1}^{m} a_k u_k\right)\dfrac{\partial u_n}{\partial x}\right] + u_n g\left(\sum\limits_{k=1}^{m} a_k u_k\right) + u_1 h_n\left(\dfrac{u_2}{u_1}, \ldots, \dfrac{u_m}{u_1}\right),$
$n = 1, \ldots, m.$

Suppose that the condition

$$\sum_{n=1}^{m} a_n h_n(z_2, \ldots, z_m) \equiv 0$$

holds. Then the system admits the solution

$$u_n = \frac{1}{a_n} r(x,t)\varphi_n(t), \quad n = 1, \ldots, m,$$

where the function $r(x,t)$ satisfies the equation

$$\frac{\partial r}{\partial t} = \frac{\partial}{\partial x}\left[f(r)\frac{\partial r}{\partial x}\right] + r g(r),$$

and the functions $\varphi_n(t)$ satisfy the system of $(m-1)$ ordinary differential equations and the additional normalization condition

$$\varphi_n' = \frac{a_n}{a_1} h_n\left(\frac{a_1}{a_2}\frac{\varphi_2}{\varphi_1}, \ldots, \frac{a_1}{a_m}\frac{\varphi_m}{\varphi_1}\right), \quad n = 1, \ldots, m-1,$$
$$\sum_{k=1}^{m} \varphi_k = 1.$$

6. $\dfrac{\partial u_n}{\partial t} = L[u_n] + u_n f\left(\sum\limits_{k=1}^{m} u_k^2\right) + u_1 g_n\left(\dfrac{u_2}{u_1}, \ldots, \dfrac{u_m}{u_1}\right), \quad n = 1, \ldots, m.$

Suppose that the following condition holds:

$$\sum_{n=1}^{m} z_n g_n(z_2, \ldots, z_m) \equiv 0, \qquad z_1 \equiv 1, \ \text{ the other } z_m \text{ are arbitrary.}$$

Then the system admits the solution

$$u_n = r(x,t)\varphi_n(t), \quad n = 1, \ldots, m,$$

where the function $r(x,t)$ satisfies the equation

$$\frac{\partial r}{\partial t} = L[r] + r f(r^2),$$

and the functions $\varphi_n(t)$ satisfy the system of $(m-1)$ ordinary differential equations and the additional normalization condition

$$\varphi_n' = \varphi_1 g_n\left(\frac{\varphi_2}{\varphi_1}, \ldots, \frac{\varphi_m}{\varphi_1}\right), \quad n = 1, \ldots, m-1,$$
$$\sum_{k=1}^{m} \varphi_k^2 = 1.$$

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Part II

Exact Methods for Nonlinear Partial Differential Equations

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Chapter 23
Methods for Solving First-Order Quasilinear Equations

23.1. Characteristic System. General Solution

23.1.1. Equations with Two Independent Variables. General Solution

A *first-order quasilinear partial differential equation with two independent variables* has the general form

$$f(x,y,w)\frac{\partial w}{\partial x} + g(x,y,w)\frac{\partial w}{\partial y} = h(x,y,w). \tag{23.1.1.1}$$

Such equations are encountered in various applications (continuum mechanics, gas dynamics, hydrodynamics, heat and mass transfer, wave theory, acoustics, multiphase flows, chemical engineering, etc.).

If two independent integrals,

$$u_1(x,y,w) = C_1, \qquad u_2(x,y,w) = C_2, \tag{23.1.1.2}$$

of the *characteristic system*

$$\frac{dx}{f(x,y,w)} = \frac{dy}{g(x,y,w)} = \frac{dw}{h(x,y,w)} \tag{23.1.1.3}$$

are known, then the *general solution* of equation (23.1.1.1) is given by

$$\Phi(u_1,u_2) = 0, \tag{23.1.1.4}$$

where $\Phi(u,v)$ is an arbitrary function of two variables. With equation (23.1.1.4) solved for u_1 or u_2, we often specify the general solution in the form

$$u_k = \Psi(u_{3-k}),$$

where $k = 1, 2$ and $\Psi(u)$ is an arbitrary function of one variable.

This approach to solving first-order quasilinear partial differential equations is often called the *method of characteristics*.

Example 23.1. Consider the quasilinear equation

$$\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = be^{\lambda w}, \qquad \lambda \neq 0.$$

The characteristic system for this equation is

$$\frac{dx}{1} = \frac{dy}{a} = \frac{dw}{be^{\lambda w}}.$$

It has two independent integrals:

$$y - ax = C_1, \quad b\lambda x + e^{-\lambda w} = C_2.$$

Hence, the general solution of the original equation is given by $\Phi(y - ax,\, b\lambda x + e^{-\lambda w}) = 0$. On solving this equation for w, one obtains the general solution in explicit form

$$w = -\frac{1}{\lambda}\ln\bigl[\Psi(y - ax) - b\lambda x\bigr],$$

where $\Psi(u)$ is an arbitrary function.

TABLE 23.1
General solutions (in implicit form) to some special types of quasilinear first-order partial differential equations; $\Psi(u)$ is an arbitrary function. The subscripts x and y indicate the corresponding partial derivatives

No.	Equation	General solution
1	$aw_x + bw_y = f(x)g(w)$	$\int \frac{dw}{g(w)} = \frac{1}{a}\int f(x)\,dx + \Psi(bx - ay)$
2	$f(x)w_x + g(y)w_y = h(w)$	$\int \frac{dw}{h(w)} = \int \frac{dx}{f(x)} + \Psi(u)$, where $u = \int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}$
3	$w_x + f(w)w_y = 0$	$y = xf(w) + \Psi(w)$
4	$w_x + [f(w) + ay]w_y = 0$	$x = \frac{1}{a}\ln\lvert ay + f(w)\rvert + \Psi(w)$, $a \neq 0$
5	$w_x + [f(w) + g(x)]w_y = 0$	$y = xf(w) + \int g(x)\,dx + \Psi(w)$

Remark. If $h(x, y, w) \equiv 0$, then $w = C_1$ one of the integrals (23.1.1.2) of system (23.1.1.3) and another integral is determined by (23.1.1.3).

Example 23.2. Consider the quasilinear equation

$$\frac{\partial w}{\partial x} + \sin w \frac{\partial w}{\partial y} = 0.$$

The characteristic system

$$\frac{dx}{1} = \frac{dy}{\sin w} = \frac{dw}{0}$$

has two independent integrals:

$$w = C_1, \quad y - x\sin w = C_2.$$

Hence, the general solution of the original equation is given by

$$\Phi(w,\ y - x\sin w) = 0.$$

In the general case, this solution cannot be represented in explicit form as a function $w(x, y)$.

Table 23.1 lists general solutions to some linear and quasilinear first-order partial differential equations in two independent variables.

▷ *In Chapter 1, many more first-order quasilinear partial differential equations in two independent variables are considered than in Table 23.1.*

23.1.2. Quasilinear Equations with n Independent Variables

1°. *General solution.* A first-order quasilinear partial differential equation with n independent variables has the general form

$$f_1(x_1, \ldots, x_n, w)\frac{\partial w}{\partial x_1} + \cdots + f_n(x_1, \ldots, x_n, w)\frac{\partial w}{\partial x_n} = g(x_1, \ldots, x_n, w). \qquad (23.1.2.1)$$

Let n independent integrals,

$$u_1(x_1, \ldots, x_n, w) = C_1, \quad \ldots, \quad u_n(x_1, \ldots, x_n, w) = C_n,$$

of the characteristic system

$$\frac{dx_1}{f_1(x_1, \ldots, x_n, w)} = \cdots = \frac{dx_n}{f_n(x_1, \ldots, x_n, w)} = \frac{dw}{g(x_1, \ldots, x_n, w)}$$

be known. Then the general solution of equation (23.1.2.1) is given by

$$\Phi(u_1, \ldots, u_n) = 0,$$

where Φ is an arbitrary function of n variables.

2°. *Quasilinear equations of special form.* For constructing the general solutions of quasilinear equations with three or more independent variables, the following method may be used. Suppose the general solutions of the following two quasilinear equations with two independent variables are known:

$$\text{Equation:}\quad \frac{\partial w}{\partial x} + f_1(x,y,w)\frac{\partial w}{\partial y} = 0 \quad\Longrightarrow\quad \text{General solution:}\quad \Phi\big(u_1(x,y,w),w\big) = 0.$$

$$\text{Equation:}\quad \frac{\partial w}{\partial x} + f_2(x,y,w)\frac{\partial w}{\partial y} = 0 \quad\Longrightarrow\quad \text{General solution:}\quad \Phi\big(u_2(x,y,w),w\big) = 0.$$

Here $\Phi(u,w)$ is an arbitrary function of two variables. Then the general solution of the "composite" quasilinear equation with three variables

$$\frac{\partial w}{\partial x} + f_1(x,y,w)\frac{\partial w}{\partial y} + f_2(x,z,w)\frac{\partial w}{\partial z} = 0$$

is given by

$$\Phi\big(u_1(x,y,w),u_2(x,z,w),w\big) = 0,$$

where $\Phi(u,v,w)$ is an arbitrary function of three variables. The preceding can be easily extended to the case of the "composite" quasilinear equation with an arbitrary number of variables.

⊙ *References for Section* 23.1: E. Kamke (1965), I. G. Petrovskii (1970), E. Zauderer (1983), R. Courant and D. Hilbert (1989), D. Zwillinger (1998), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23.2. Cauchy Problem. Existence and Uniqueness Theorem

23.2.1. Cauchy Problem

Consider two formulations of the Cauchy problem.

1°. *Generalized Cauchy problem.* Find a solution $w = w(x,y)$ of equation (23.1.1.1) satisfying the initial conditions

$$x = h_1(\xi), \quad y = h_2(\xi), \quad w = h_3(\xi), \tag{23.2.1.1}$$

where ξ is a parameter ($\alpha \le \xi \le \beta$) and the $h_k(\xi)$ are given functions.

Geometric interpretation: find an integral surface of equation (23.1.1.1) passing through the line defined parametrically by equation (23.2.1.1).

2°. *Classical Cauchy problem.* Find a solution $w = w(x,y)$ of equation (23.1.1.1) satisfying the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0, \tag{23.2.1.2}$$

where $\varphi(y)$ is a given function.

It is convenient to represent the classical Cauchy problem as a generalized Cauchy problem by rewriting condition (23.2.1.2) in the parametric form

$$x = 0, \quad y = \xi, \quad w = \varphi(\xi). \tag{23.2.1.3}$$

23.2.2. Procedure of Solving the Cauchy Problem

The procedure of solving the Cauchy problem (23.1.1.1), (23.2.1.1) involves several steps. First, two independent integrals (23.1.1.2) of the characteristic system (23.1.1.3) are determined. Then, to find the constants of integration C_1 and C_2, the initial data (23.2.1.1) must be substituted into the integrals (23.1.1.2) to obtain

$$u_1\big(h_1(\xi), h_2(\xi), h_3(\xi)\big) = C_1, \qquad u_2\big(h_1(\xi), h_2(\xi), h_3(\xi)\big) = C_2. \tag{23.2.2.1}$$

Eliminating C_1 and C_2 from (23.1.1.2) and (23.2.2.1) yields

$$\begin{aligned} u_1(x, y, w) &= u_1\big(h_1(\xi), h_2(\xi), h_3(\xi)\big), \\ u_2(x, y, w) &= u_2\big(h_1(\xi), h_2(\xi), h_3(\xi)\big). \end{aligned} \tag{23.2.2.2}$$

Formulas (23.2.2.2) are a parametric form of the solution of the Cauchy problem (23.1.1.1), (23.2.1.1). In some cases, one may succeed in eliminating the parameter ξ from relations (23.2.2.2), thus obtaining the solution in an explicit form.

Example 23.3. Consider the Cauchy problem for Hopf's equation

$$\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} = 0 \tag{23.2.2.3}$$

subject to the initial condition (23.2.1.2).

First, we rewrite the initial condition (23.2.1.2) in the parametric form (23.2.1.3). Solving the characteristic system

$$\frac{dx}{1} = \frac{dy}{w} = \frac{dw}{0}, \tag{23.2.2.4}$$

we find two independent integrals,

$$w = C_1, \quad y - wx = C_2. \tag{23.2.2.5}$$

Using the initial conditions (23.2.1.3), we find that $C_1 = \varphi(\xi)$ and $C_2 = \xi$. Substituting these expressions into (23.2.2.5) yields the solution of the Cauchy problem (23.2.2.3), (23.2.1.2) in the parametric form

$$w = \varphi(\xi), \tag{23.2.2.6}$$

$$y = \xi + \varphi(\xi)x. \tag{23.2.2.7}$$

The characteristics (23.2.2.7) are straight lines in the xy-plane with slope $\varphi(\xi)$ that intersect the y-axis at the points ξ. On each characteristic, the function w has the same value equal to $\varphi(\xi)$ (generally, w takes different values on different characteristics).

For $\varphi'(\xi) > 0$, different characteristics do not intersect and, hence, formulas (23.2.2.6) and (23.2.2.7) define a unique solution. As an example, we consider the initial profile

$$\varphi(\xi) = \begin{cases} w_1 & \text{for } \xi \le 0, \\ \dfrac{w_2\xi^2 + \varepsilon w_1}{\xi^2 + \varepsilon} & \text{for } \xi > 0, \end{cases} \tag{23.2.2.8}$$

where $w_1 < w_2$ and $\varepsilon > 0$. Formulas (23.2.2.6)–(23.2.2.8) give a unique smooth solution in the entire half-plane $x > 0$. In the domain filled by the characteristics $y = \xi + w_1 x$ (for $\xi \le 0$), the solution is constant, i.e.,

$$w = w_1 \quad \text{for} \quad y/x \le w_1. \tag{23.2.2.9}$$

For $\xi > 0$, the solution is determined by relations (23.2.2.6)–(23.2.2.8).

Let us look how this solution is transformed in the limit case $\varepsilon \to 0$, which corresponds to the piecewise-continuous initial profile

$$\varphi(\xi) = \begin{cases} w_1 & \text{for } \xi \le 0, \\ w_2 & \text{for } \xi > 0, \end{cases} \qquad \text{where} \quad w_1 < w_2. \tag{23.2.2.10}$$

We further assume that $\xi > 0$ [for $\xi \le 0$, formula (23.2.2.9) is valid]. If $\xi = \text{const} \neq 0$ and $\varepsilon \to 0$, it follows from (23.2.2.8) that $\varphi(\xi) = w_2$. Hence, in the domain filled by the characteristics $y = \xi + w_2 x$ (for $\xi > 0$), the solution is constant, i.e., we have

$$w = w_2 \quad \text{for} \quad y/x \ge w_2 \qquad (\text{as } \varepsilon \to 0). \tag{23.2.2.11}$$

For $\xi \to 0$, the function φ can assume any value between w_1 and w_2 depending on the ratio of the small parameters ε and ξ; the first term on the right-hand side of equation (23.2.2.7) can be neglected. As a result, we find from equations (23.2.2.6) and (23.2.2.7) that the solution has the following asymptotic behavior in explicit form:

$$w = y/x \quad \text{for} \quad w_1 \le y/x \le w_2 \qquad (\text{as } \varepsilon \to 0). \tag{23.2.2.12}$$

By combining relations (23.2.2.9), (23.2.2.11), and (23.2.2.12) together, we obtain the solution of the Cauchy problem for equation (23.2.2.3) subject to the initial conditions (23.2.2.10) in the form

$$w(x,y) = \begin{cases} w_1 & \text{for } y \le w_1 x, \\ y/x & \text{for } w_1 x \le y \le w_2 x, \\ w_2 & \text{for } y \ge w_2 x. \end{cases} \tag{23.2.2.13}$$

Figure 23.1 shows characteristics of equation (23.2.2.3) that satisfy condition (23.2.2.10) with $w_1 = \frac{1}{2}$ and $w_2 = 2$. This figure also depicts the dependence of w on y (for $x = x_0 = 1$). In applications, such a solution is referred to as a centered rarefaction wave (see also Subsection 23.1.3).

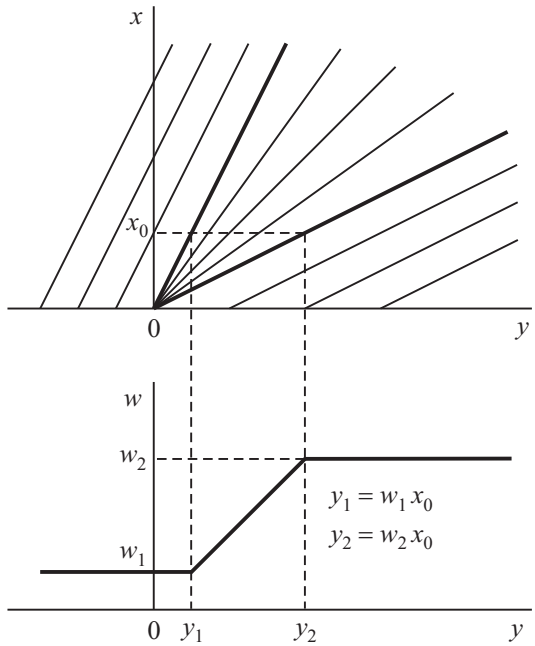

Figure 23.1. Characteristics of the Cauchy problem (23.2.2.3), (23.2.1.2) with the initial profile (23.2.2.10) and the dependence of the unknown w on the coordinate y for $w_1 = \frac{1}{2}$, $w_2 = 2$, and $x_0 = 1$.

Remark. If there is an interval where $\varphi'(\xi) < 0$, then the characteristics intersect in some domain. According to equation (23.2.2.6), at the point of intersection of two characteristics defined by two distinct values ξ_1 and ξ_2 of the parameter, the function w takes two distinct values equal to $\varphi(\xi_1)$ and $\varphi(\xi_2)$, respectively. Therefore, the solution is not unique in the domain of intersecting characteristics. This example illustrates the local character of the existence and uniqueness theorem. These issues are discussed in Sections 23.3 and 23.4 in more detail.

23.2.3. Existence and Uniqueness Theorem

Let Ω_0 be a domain in the xy-plane and let Ω be a cylindrical domain of the xyw-space obtained from Ω_0 by adding the coordinate w, with the condition $|w| < A_1$ being satisfied. Let the coefficients f, g, and h of equation (23.1.1.1) be continuously differentiable functions of x, y, and w in Ω and let $x = h_1(\xi)$, $y = h_2(\xi)$, and $w = h_3(\xi)$ be continuously differentiable functions of ξ for $|\xi| < A_2$ defining a curve C in Ω with a simple projection C_0 onto Ω_0. Suppose that $(h_1')^2 + (h_2')^2 \ne 0$ (the prime stands for the derivative with respect to ξ) and $fh_2' - gh_1' \ne 0$ on C. Then there exists a subdomain $\overline{\Omega}_0 \subset \Omega_0$ containing C_0 where

there exists a continuously differentiable function $w = w(x, y)$ satisfying the differential equation (23.1.1.1) in Ω_0 and the initial condition (23.2.1.1) on C_0. This function is unique.

It is important to note that this theorem has a local character, i.e., the existence of a solution is guaranteed in some "sufficiently narrow," unknown neighborhood of the line C (see the remark at the end of Example 23.3.

⊙ *References for Section* 23.2: E. Kamke (1965), I. G. Petrovskii (1970), S. J. Farlow (1982), E. Zauderer (1983), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. Courant and D. Hilbert (1989), D. Zwillinger (1998), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23.3. Qualitative Features and Discontinuous Solutions of Quasilinear Equations

23.3.1. Model Equation of Gas Dynamics

Consider a quasilinear equation of the special form*

$$\frac{\partial w}{\partial x} + f(w)\frac{\partial w}{\partial y} = 0, \tag{23.3.1.1}$$

which represents a conservation law of mass (or another quantity) and is often encountered in continuum mechanics, gas dynamics, hydrodynamics, wave theory, acoustics, multiphase flows, and chemical engineering. This equation is a model for numerous processes of mass transfer: sorption and chromatography, two-phase flows in porous media, flow of water in river, street traffic development, flow of liquid films along inclined surfaces, etc. The independent variables x and y in equation (23.3.1.1) usually play the role of time and spatial coordinate, respectively, w is the density of the quantity being transferred, and $f(w)$ is the rate of w.

23.3.2. Solution of the Cauchy Problem. Rarefaction Wave. Wave "Overturn"

1°. The solution $w = w(x, y)$ of the Cauchy problem for equation (23.3.1.1) subject to the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0 \qquad (-\infty < y < \infty) \tag{23.3.2.1}$$

can be represented in the parametric form

$$y = \xi + \mathcal{F}(\xi)x, \quad w = \varphi(\xi), \tag{23.3.2.2}$$

where $\mathcal{F}(\xi) = f\big(\varphi(\xi)\big)$.

Consider the characteristics $y = \xi + \mathcal{F}(\xi)x$ in the yx-plane for various values of the parameter ξ. These are straight lines with slope $\mathcal{F}(\xi)$. Along each of these lines, the unknown function is constant, $w = \varphi(\xi)$. In the special case $f(w) = a = \text{const}$, the equation in question is linear; solution (23.3.2.2) can be written explicitly as $w = \varphi(y - ax)$, thus representing a traveling wave with a fixed profile. The dependence of f on w leads to a typical nonlinear effect: distortion of the profile of the traveling wave.

We further consider the domain $x \geq 0$ and assume** that $f > 0$ for $w > 0$ and $f'_w > 0$. In this case, the greater values of w propagate faster than the smaller values. If the initial profile satisfies the condition $\varphi'(y) > 0$ for all y, then the characteristics in the yx-plane that come from the y-axis inside the domain $x > 0$ are divergent lines, and hence there exists a unique solution for all $x > 0$. In physics, such solutions are referred to as rarefaction waves.

* Equations of the general form are discussed in Section 23.4.

** By the change $x = -\widetilde{x}$ the consideration of the domain $x \leq 0$ can be reduced to that of the domain $\widetilde{x} \geq 0$. The case $f < 0$ can be reduced to the case $f > 0$ by the change $y = -\widetilde{y}$.

Example 23.4. Figures 23.2 and 23.3 illustrate characteristics and the evolution of a rarefaction wave for Hopf's equation [for $f(w) = w$ in (23.3.1.1)] with the initial profile

$$\varphi(y) = \frac{4}{\pi}\arctan(y-2) + 2. \tag{23.3.2.3}$$

It is apparent that the solution is smooth for all $x > 0$.

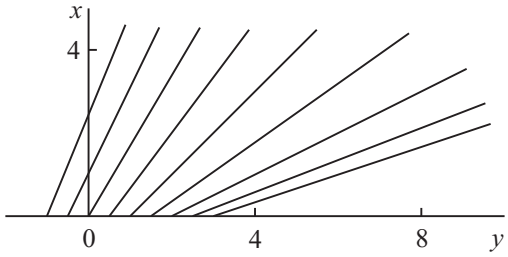

Figure 23.2. Characteristics for the Hopf's equation (23.2.2.3) with the initial profile (23.3.2.3).

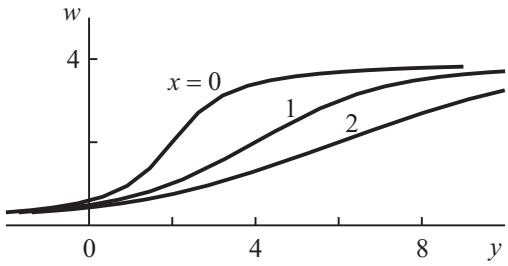

Figure 23.3. The evolution of a rarefaction wave for the Hopf's equation (23.2.2.3) with the initial profile (23.3.2.3).

2°. Let us now look at what happens if $\varphi'(y) < 0$ on some interval of the y-axis. Let y_1 and y_2 be points of this interval such that $y_1 < y_2$. Then $f(y_1) > f(y_2)$. It follows from the first relation in (23.3.2.2) that the characteristics issuing from the points y_1 and y_2 intersect at the "time instant"

$$x_* = \frac{y_2 - y_1}{f(w_1) - f(w_2)}, \qquad \text{where} \quad w_1 = \varphi(y_1), \quad w_2 = \varphi(y_2).$$

Since w has different values on these characteristics, the solution cannot be continuously extended to $x > x_*$. If $\varphi'(y) < 0$ on a bounded interval, then there exists $x_{\min} = \min\limits_{y_1,y_2} x_*$ such that the characteristics intersect in the domain $x > x_{\min}$ (see Fig. 23.4). Therefore, the front part of the wave where its profile is a decreasing function of y will "overturn" with time. The time $x_{\min}$ when the overturning begins is defined by

$$x_{\min} = -\frac{1}{\mathcal{F}'(\xi_0)},$$

where ξ_0 is determined by the condition $|\mathcal{F}'(\xi_0)| = \max|\mathcal{F}'(\xi)|$ for $\mathcal{F}'(\xi) < 0$, and the wave is also said to *break*. A formal extension of the solution to the domain $x > x_{\min}$ makes this solution nonunique. The boundary of the uniqueness domain in the yx-plane is the envelope of the characteristics. This boundary can be represented in parametric form as

$$y = \xi + \mathcal{F}(\xi)x, \quad 0 = 1 + \mathcal{F}'(\xi)x.$$

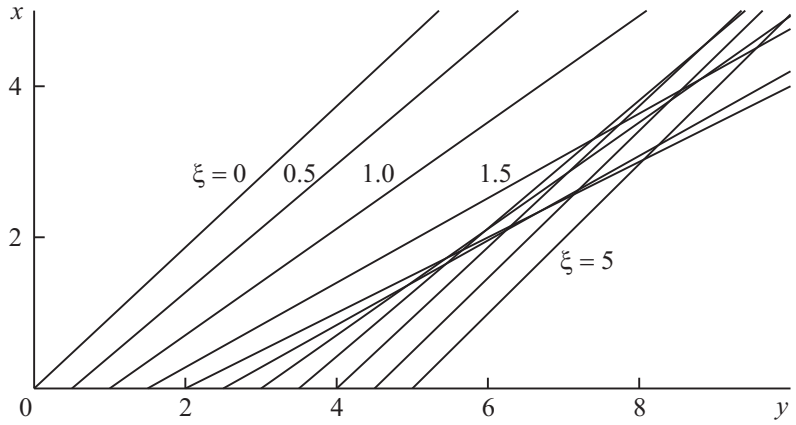

Figure 23.4. Characteristics for the Hopf's equation (23.2.2.3) with the initial profile (23.3.2.4).

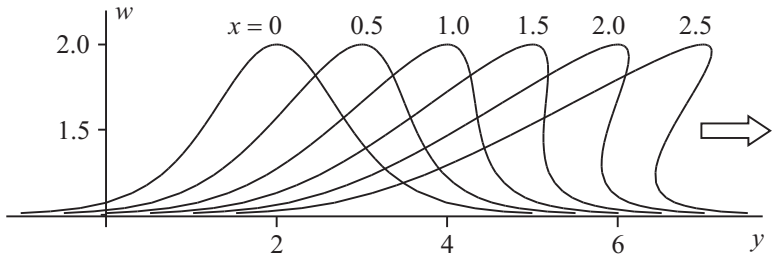

Figure 23.5. The evolution of a solitary wave for the Hopf's equation (23.2.2.3) with the initial profile (23.3.2.4).

Example 23.5. Figure 23.5 illustrates the evolution of a solitary wave with the initial profile

$$\varphi(y) = \cosh^{-2}(y-2) + 1 \tag{23.3.2.4}$$

for equation (23.3.1.1) with $f(w) = w$. It is apparent that for $x > x_{\min}$, where $x_{\min} = \frac{3}{4}\sqrt{3} \approx 1.3$, the wave "overturns" (the wave profile becomes triple-valued).

23.3.3. Shock Waves. Jump Conditions

In most applications where the equation under consideration is encountered, the unknown function $w(x, y)$ is the density of a medium and must be unique for its nature. In these cases, one has to deal with a generalized (nonsmooth) solution describing a step-shaped shock wave rather than a continuous smooth solution. The many-valued part of the wave profile is replaced by an appropriate discontinuity, as shown in Fig. 23.6. It should be emphasized that a discontinuity can occur for arbitrarily smooth functions $f(w)$ and $\varphi(y)$ entering equation (23.3.1.1) and the initial condition (23.3.2.1).

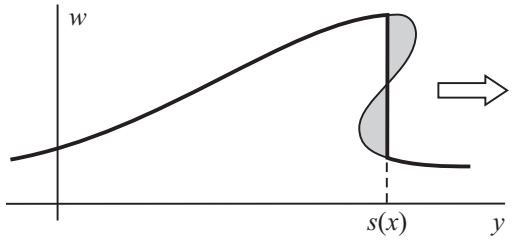

Figure 23.6. Replacement of the many-valued part of the wave profile by a discontinuity that cuts off domains with equal areas (shaded) from the profile of a breaking wave.

In what follows, we assume that $w(x, y)$ experiences a jump discontinuity at the line $y = s(x)$ in the yx-plane. On both sides of the discontinuity the function $w(x, y)$ is smooth and single-valued; as before, it is described by equations (23.3.2.2). The speed of propagation of the discontinuity, V, is expressed as $V = s'$ (the prime stands for the derivative) and must satisfy the condition

$$V = \frac{F(w_2) - F(w_1)}{w_2 - w_1}, \qquad F(w) = \int f(w)\, dw, \tag{23.3.3.1}$$

where the subscript 1 refers to values before the discontinuity and the subscript 2 to those after the discontinuity. In applications, relation (23.3.3.1), expressing a conservation law at discontinuity, is conventionally referred to as the Rankine–Hugoniot jump condition (this condition is derived below in Subsection 23.3.4).

The continuous wave "overturns" (breaks), thus resulting in a discontinuity if and only if the propagation velocity $f(w)$ decreases as y increases, i.e., the inequalities

$$f(w_2) < V < f(w_1) \tag{23.3.3.2}$$

are satisfied. Conditions (23.3.3.2) have the geometric meaning that the characteristics issuing from the x-axis (these characteristics "carry" information about the initial data) must intersect the line of discontinuity (see Fig. 23.7). In this case, the discontinuous solution is stable with respect to small perturbations of the initial profile (i.e., the corresponding solution varies only slightly).

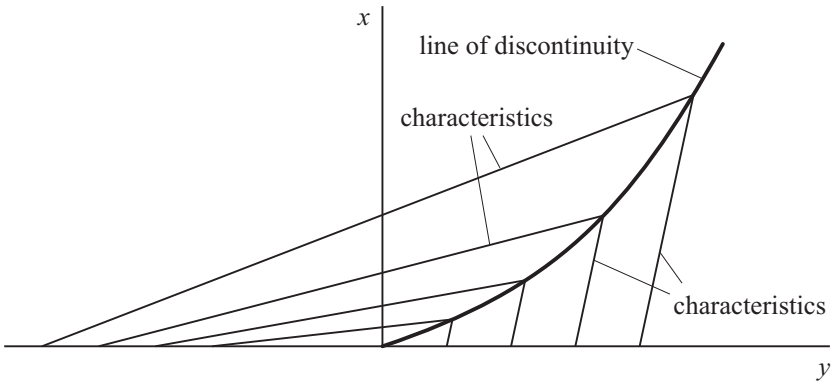

Figure 23.7. Mutual arrangement of characteristics and lines of discontinuity in the case of a stable shock wave.

The position of the point of discontinuity in the yw-plane may be determined geometrically by following Whitham's rule: the discontinuity must cut off domains with equal areas from the overturning wave profile (these domains are shaded in Fig. 23.6). Mathematically, the position of the point of discontinuity can be determined from the equations

$$\begin{aligned} s(x) &= \xi_1 + \mathcal{F}_1 x, \\ s(x) &= \xi_2 + \mathcal{F}_2 x, \\ w_2\mathcal{F}_2 - w_1\mathcal{F}_1 &= F(w_2) - F(w_1) + \frac{\mathcal{F}_2 - \mathcal{F}_1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} w\, d\xi. \end{aligned} \tag{23.3.3.3}$$

Here, w and $\mathcal{F}$ are defined as functions of ξ by $w = \varphi(\xi)$ and $\mathcal{F} = f(w)$, the function $F(w)$ is introduced in equation (23.3.3.1), and the subscripts 1 and 2 refer to the values of the corresponding quantities at $\xi = \xi_1$ and ξ_2. Equations (23.3.3.3) permit one to determine the dependences $s = s(x)$, $\xi_1 = \xi_1(x)$, and $\xi_2 = \xi_2(x)$. It is possible to show that the jump condition (23.3.3.1) follows from the last equation in (23.3.3.3).

Example 23.6. For Hopf's equation, which corresponds to $f(w) = w$ in equation (23.3.1.1), the jump condition (23.3.3.1) can be represented as

$$V = \frac{w_1 + w_2}{2}.$$

Here, we take into account the relation $F(w) = \frac{1}{2}w^2$. System (23.3.3.3), which determines the position of the point of discontinuity, becomes

$$s(x) = \xi_1 + \varphi(\xi_1)x,$$
$$s(x) = \xi_2 + \varphi(\xi_2)x,$$
$$\frac{\varphi(\xi_1) + \varphi(\xi_2)}{2} = \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \varphi(\xi)\, d\xi,$$

where the function $\varphi(\xi)$ specifies the initial wave profile.

Figure 23.8 illustrates the formation of the shock wave described by the generalized solution of Hopf's equation with $f(w) = w$ and generated from a solitary wave with the smooth initial profile (23.3.2.4).

The nonsmooth "step" curves depicted in Fig. 23.8 (for $x = 0.15$, 0.20, and 0.25) are obtained from the smooth (but many-valued) curves shown in Fig. 23.5 by means of Whitham's rule of equal areas.

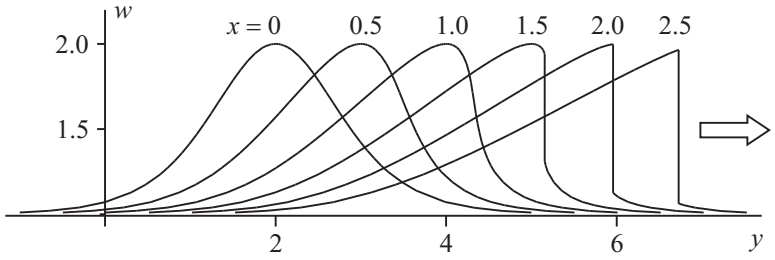

Figure 23.8. The formation of a shock wave generated from a solitary wave with the smooth initial profile.

23.3.4. Utilization of Integral Relations for Determining Generalized Solutions

Generalized solutions which are described by piecewise-smooth (piecewise-continuous) functions may formally be introduced by considering the following equation written in an integral form:

$$-\iint_D \left[w\frac{\partial \psi}{\partial x} + F(w)\frac{\partial \psi}{\partial y} \right] dy\, dx = 0. \tag{23.3.4.1}$$

Here, D is an arbitrary rectangle in the yx-plane, $\psi = \psi(x, y)$ is any "test" function with continuous first derivatives in D that is zero at the boundary of D, and the function $F(w)$ is defined in equation (23.3.3.1). If w and $F(w)$ are continuously differentiable, then equation (23.3.4.1) is equivalent to the original differential equation (23.3.1.1). Indeed, multiplying equation (23.3.1.1) by ψ, integrating over the domain D, and then integrating by parts, we obtain equation (23.3.4.1). Conversely, integrating (23.3.4.1) by parts yields

$$\iint_D \left[\frac{\partial w}{\partial x} + \frac{\partial F(w)}{\partial y} \right] \psi\, dy\, dx = 0.$$

Since this equation must be satisfied for any test function ψ and since $F'(w) = f(w)$, we obtain the original equation (23.3.1.1). However, equation (23.3.4.1) has a wider class of solutions since the admissible functions $w(x, y)$ need not necessarily be differentiable. The functions $w(x, y)$ satisfying the integral relation (23.3.4.1) for all test functions ψ are referred to as generalized (or weak) solutions of equation (23.3.1.1).

The use of generalized solutions is convenient for the description of discontinuities, since it permits one to obtain jump conditions automatically. Consider a solution of equation (23.3.4.1) continuously differentiable in two portions D_1 and D_2 of the rectangle D, which has a jump discontinuity at the interface Γ between D_1 and D_2. Integrating equation (23.3.4.1) by parts in each of the subdomains D_1 and D_2 yields

$$\iint_{D_1}\left[\frac{\partial w}{\partial x}+\frac{\partial F(w)}{\partial y}\right]\psi\,dy\,dx+\iint_{D_2}\left[\frac{\partial w}{\partial x}+\frac{\partial F(w)}{\partial y}\right]\psi\,dy\,dx+\int_\Gamma\big\{[w]\,dy-[F(w)]\,dx\big\}\psi=0,$$

where $[w]=w_2-w_1$ and $[F(w)]=F(w_2)-F(w_1)$ are jumps of w and $F(w)$ across Γ. The curvilinear integral over Γ is formed by the boundary terms of the integrals over D_1 and D_2 that result from the integration by parts. Since the relation obtained must be valid for all test functions ψ, it follows that equation (23.3.1.1) is valid inside each of the subdomains D_1 and D_2 and, moreover, the relation

$$[w]\,dy-[F(w)]\,dx=0\qquad(\text{on }\Gamma)$$

must hold. Assuming as before that the line of discontinuity is defined by the equation $y=s(x)$, we arrive at the jump condition (23.3.3.1).

It is worth noting that condition (23.3.3.2) does not follow from the integral relation (23.3.4.1) but is deduced from the additional condition of stability of the solution.

23.3.5. Conservation Laws. Viscosity Solutions

Point out also other ways of introducing generalized solutions.

1°. Generalized solution may be introduced using the conservation law

$$\frac{d}{dx}\int_{y_1}^{y_2} w\,dy+F(w_2)-F(w_1)=0, \tag{23.3.5.1}$$

where $w=w(x,y)$ and $w_n=w(x,y_n)$ $(n=1,2)$. Just as in equation (23.3.3.1), the function $F(w)$ is defined as $F(w)=\int f(w)\,dw$. Relation (23.3.5.1) is assumed to hold for any y_1 and y_2. It has a simple physical interpretation: the rate of change of the total value of w distributed over the interval $(y_1,\,y_2)$ is compensated for by the "flux" of the function $F(w)$ through the endpoints of the interval.

Let w be a continuously differentiable solution of the conservation law. Then, differentiating equation (23.3.5.1) with respect to y_2 and setting $y_2=y$, we arrive at equation (23.3.1.1). The conservation law (23.3.5.1) is convenient for the reason that it admits discontinuous solutions. It is not difficult to show that in this case the jump condition (23.3.3.1) must hold. For this reason, conservation laws like (23.3.5.1) are sometimes used as the basis for determining generalized solutions.

2°. An alternative approach to determining generalized solutions involves the consideration of an auxiliary equation of the parabolic type of the form

$$\frac{\partial w}{\partial x}+f(w)\frac{\partial w}{\partial y}=\varepsilon\frac{\partial^2 w}{\partial x^2},\qquad \varepsilon>0. \tag{23.3.5.2}$$

The generalized solution of the Cauchy problem (23.3.1.1), (23.3.2.1) (for a finite initial profile) is defined as the limit of the solution of equation (23.3.5.2) with the same initial

condition (23.3.2.1) as $\varepsilon \to 0$. It is shown by Oleinik (1957) and Gelfand (1959) that the above definitions of the generalized solution lead to the same results.

The parameter ε plays the role of "viscosity" (by analogy with the viscosity of a fluid), which "smooths out" the jump, thus making the profile of the unknown w continuous. Therefore, the above construction, based on proceeding to the limit as $\varepsilon \to 0$, is called the method of vanishing viscosity and the limit function obtained is called the viscosity solution. Equation (23.3.5.2) with small ε is not infrequently used as a basis for numerical simulation of discontinuous solutions of equation (23.3.1.1); in this case, one need not specially separate in the numerical scheme a domain of discontinuity.

Remark. In specific problems, first-order quasilinear equations are often a consequence of integral conservation laws, having clear physical interpretation. In such cases, one should introduce generalized solutions on the basis of these conservation laws; for example, see Whitham (1974) and Rozhdestvenskii and Yanenko (1983). The thus obtained nonsmooth generalized solutions may differ from those described above.

23.3.6. Hopf's Formula for the Generalized Solution

Below we give a general formula for a generalized solution of the Cauchy problem (23.3.1.1), (23.3.2.1), describing discontinuous solutions that satisfy the stability condition (23.3.3.2). As above, we assume that $x \ge 0$ and $f > 0$ for $w > 0$; $f'_w > 0$.

Consider the function

$$Z(s) = \min_w \bigl\{ws - F(w)\bigr\}, \quad \text{where} \quad F(w) = \int f(w)\,dw. \tag{23.3.6.1}$$

We set

$$H(x, y, \eta) = \int_0^\eta \varphi(\bar\eta)\,d\bar\eta + xZ\left(\frac{y-\eta}{x}\right). \tag{23.3.6.2}$$

This is a continuous function of η for fixed x and y. It can be shown that for fixed x and with the exception of a countable set of values of y, function (23.3.6.2) has a unique minimum with respect to η. Denote the position of this minimum by $\eta = \xi$, where $\xi = \xi(x, y)$. The stable generalized solution of equation (23.3.1.1) subject to the initial condition (23.3.2.1) is given by

$$w(x, y) = \mathcal{Z}\left(\frac{y-\xi}{x}\right), \quad \text{where} \quad \mathcal{Z}(s) = \frac{dZ}{ds}. \tag{23.3.6.3}$$

The function $Z = Z(s)$ defined by relation (23.3.6.1) can be represented in the parametric form

$$s = f(w), \qquad Z = ws - \int f(w)\,dw. \tag{23.3.6.4}$$

Hence follows the parametric representation for its derivative $\mathcal{Z} = \mathcal{Z}(s)$:

$$s = f(w), \qquad \mathcal{Z} = w. \tag{23.3.6.5}$$

The position of the minimum $\eta = \xi(x, y)$ of function (23.3.6.2) is determined by the condition $H_\eta = 0$, which results in the following equation for ξ:

$$\varphi(\xi) - \mathcal{Z}\left(\frac{y-\xi}{x}\right) = 0. \tag{23.3.6.6}$$

To illustrate the utilization of the above formulas, we consider two cases.

1°. Let the algebraic (or transcendental) equation (23.3.6.6) have a unique solution $\xi = \xi(x, y)$ in some domain of the xy-plane. We set $s = (y-\xi)/x$ in (23.3.6.5) and consider these relations in conjunction with equation (23.3.6.6). Eliminating the functions $f(w)$ and $\mathcal{Z}$ from these equations yields a solution of the problem in the parametric form (23.3.2.2). In this case, we obtain a smooth (classical) solution describing a rarefaction wave.

2°. Let the algebraic (or transcendental) equation (23.3.6.6) have two different solutions, ξ_1 and ξ_2, that are functions of x and y. For both cases, solution (23.3.2.2) is valid, where either $\xi = \xi_1$ or $\xi = \xi_2$. At each point (x, y), we choose that solution ξ_n $(n = 1, 2)$ which minimizes the function $H(x, y, \xi_n)$ defined by equation (23.3.6.2). In this case, we obtain a discontinuous (generalized) solution describing a shock wave.

23.3.7. Problem of Propagation of a Signal

In the problem of propagation of a signal and other physical applications, one seeks a solution of equation (23.3.1.1) subject to the conditions

$$\begin{aligned} &w = w_0 \quad \text{at} \quad x = 0 \quad \text{(initial condition)},\\ &w = g(x) \quad \text{at} \quad y = 0 \quad \text{(boundary condition)}, \end{aligned} \tag{23.3.7.1}$$

where w_0 is some constant and $g(x)$ is a prescribed function. One considers the domain $x > 0$, $y > 0$, where x plays the role of time and y the role of the spatial coordinate. It is assumed that $f(w) > 0$.

The characteristics of this problem issue from the positive semiaxis y and the positive semiaxis x (see Fig. 23.9). We have $w = w_0$ at the characteristics issuing from the y-axis. Hence, these characteristics are straight lines defined by $y - a_0 x = \text{const}$, where $a_0 = f(w_0)$. It follows that

$$w = w_0 \quad \text{for} \quad y > a_0 x. \tag{23.3.7.2}$$

As far as the characteristics issuing from the x-axis are concerned, we assume that one of the characteristics starts from a point $x = \tau$. The solution of equation (23.3.1.1) subject to conditions (23.3.7.1) can be represented in the parametric form

$$y = \mathcal{G}(\tau)(x - \tau), \quad w = g(\tau), \tag{23.3.7.3}$$

where $\mathcal{G}(\tau) = f\big(g(\tau)\big)$. This solution can be related to solution (23.3.2.2) of the Cauchy problem (23.3.1.1), (23.3.2.1) by setting

$$\xi = -\tau\mathcal{G}(\tau), \quad \varphi(\xi) = g(\tau), \quad \mathcal{F}(\xi) = \mathcal{G}(\tau). \tag{23.3.7.4}$$

This corresponds to the continuation of characteristics through the points $y = 0$, $x = \tau$ to the y-axis and to the designation of the points of intersection by $y = \xi$. In this case, the problem of propagation of a signal is formulated as a Cauchy problem.

Each domain of nonuniqueness in solution (23.3.7.3) should be replaced by a jump discontinuity. If

$$\mathcal{G}(+0) > a_0, \quad \text{where} \quad a_0 = f(w_0),$$

such a domain arises instantaneously, since the first characteristic $y = \mathcal{G}(+0)x$ is ahead of the last characteristic $y = a_0 x$ of the unperturbed domain. In this case, the discontinuity appears at the origin of coordinates and the relation

$$G - G_0 = (w - w_0)\mathcal{G} - \frac{1}{x - \tau}\int_0^\tau \big[G(\bar{\tau}) - G_0\big]\, d\bar{\tau} \tag{23.3.7.5}$$

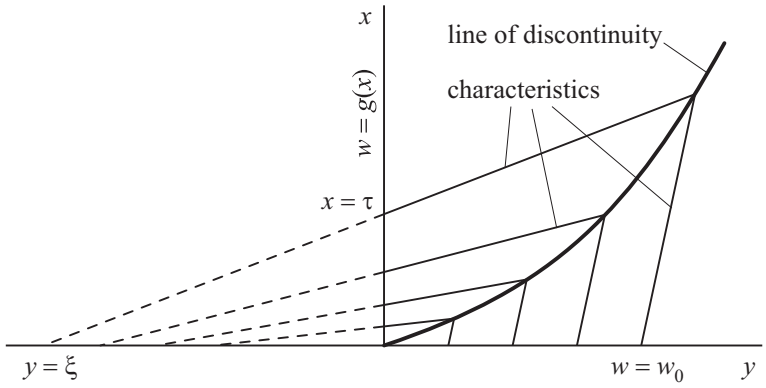

Figure 23.9. Characteristics of the problem of propagation of a signal.

holds. The quantities w, $\mathcal{G}$, and G are functions of τ in the domain behind the discontinuity and are given by

$$w = g(\tau), \quad \mathcal{G} = f\big(g(\tau)\big), \quad G = F\big(g(\tau)\big).$$

The subscript 0 refers to the values of these variables ahead of the discontinuity, $w = w_0$, $\mathcal{G}_0 = f(w_0)$, and $G_0 = F(w_0)$.

Relations (23.3.7.3) describe the solution in the perturbed domain behind the discontinuity. Equation (23.3.7.5) serves to determine $\tau(x)$ at the point of discontinuity; by setting this value into relations (23.3.7.3), we find both the position of the discontinuity and the value of w immediately behind it.

If $g(x)$ remains constant and equal to w_c, then for $a_c > a_0$, where $a_c = f(w_c)$, the solution has a jump discontinuity propagating at a constant velocity and separating two homogeneous domains with $w = w_c$ and $w = w_0$.

⊙ *References for Section* 23.3: E. Hopf (1950), P. D. Lax (1954), O. A. Oleinik (1957, 1959), I. M. Gelfand (1959), G. B. Whitham (1974), F. John (1982), B. L. Rozhdestvenskii and N. N. Yanenko (1983), C. M. Dafermos (1983), H. Rhee, R. Aris, and N. R. Amundson (1986, 1989), A. I. Subbotin (1991, 1995), R. J. LeVeque (1992), P. Bedrikovetsky (1993), J. Smoller (1994), E. Godlewski and P.-A. Raviart (1996), D. Serre (1996), M. Taylor (1996), A. A. Melikyan (1998), B. P. Andreyanov (1999), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

23.4. Quasilinear Equations of General Form

23.4.1. Quasilinear Equations in Conservative Form

In the general case, the quasilinear equation

$$\frac{\partial w}{\partial x} + f(x, y, w)\frac{\partial w}{\partial y} = g(x, y, w) \tag{23.4.1.1}$$

can be represented in an equivalent, conservative form as

$$\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}F(x, y, w) = G(x, y, w), \tag{23.4.1.2}$$

where

$$F(x, y, w) = \int_{w_0}^{w} f(x, y, t)\, dt, \quad G(x, y, w) = g(x, y, w) + \int_{w_0}^{w} \frac{\partial}{\partial y}\big[f(x, y, t)\big]\, dt, \tag{23.4.1.3}$$

and w_0 is an arbitrary number. In what follows we assume that the functions f and g are continuous and have continuous first derivatives.

As was shown by examples in Sections 23.2 and 23.3, characteristics of equation (23.4.1.1) can intersect in some domain (see Fig. 23.4), which results in the nonuniqueness of the solution and the absence of a physical interpretation of this solution. For this reason, one has to make use of a generalized solution, described by a discontinuous function instead of a classical smooth solution.

We consider the class of functions $w(x, y) \in \mathcal{K}$ satisfying the following conditions:

1°. In any bounded portion of the half-plane $x \geq 0$, there exists a finite number of lines and points of discontinuity; outside these lines and points, the function $w(x, y)$ is continuous and has continuous first derivatives.

2°. At the lines of discontinuity, $y = y(x)$, the left and right limit values of w exist: $w(x, y-0)$ and $w(x, y+0)$.

23.4.2. Generalized Solution. Jump Condition and Stability Condition

A generalized solution may be introduced in the following manner. Let $\psi(x, y) \in \mathcal{C}_1$ be a continuous finite function (which vanishes outside a finite portion of the xy-plane) having continuous first derivatives.

Multiply equation (23.4.1.1) by $\psi(x, y)$ and integrate the resulting relation over the half-plane $S = \{0 \leq x < \infty,\ -\infty < y < \infty\}$. On integrating by parts, we obtain

$$\iint_S \left[w\frac{\partial \psi}{\partial x} + F(x, y, w)\frac{\partial \psi}{\partial y} + G(x, y, w)\psi(x, y) \right] dy\, dx + \int_{-\infty}^{\infty} w(0, y)\psi(0, y)\, dy = 0. \tag{23.4.2.1}$$

The function $F(x, y, w)$ is defined in equation (23.4.1.3). The integral relation (23.4.2.1) does not contain derivatives of the unknown function and does not lose its meaning for discontinuous $w(x, y)$. The function $w(x, y) \in \mathcal{K}$ will be called the generalized solution of equation (23.4.1.1) if inequalities (23.4.2.1) hold for any finite $\psi(x, y) \in \mathcal{C}_1$.

Basic properties of the stable generalized solution:

1°. In the domain where the solution w is continuously differentiable, equations (23.4.1.1) and (23.4.2.1) are equivalent.

2°. Let $y = y(x)$ be the equation of a discontinuity line of $w(x, y)$. Then the Rankine–Hugoniot jump condition must hold. It expresses the speed of motion of the discontinuity line via the solution parameters ahead of and behind the discontinuity as

$$V = \frac{[F(x, y, w)]}{[w]} \equiv \frac{F\big(x, y(x), w_2(x)\big) - F\big(x, y(x), w_1(x)\big)}{w_2(x) - w_1(x)}, \tag{23.4.2.2}$$

where

$$V = y'(x), \quad w_1(x) = w\big(x, y(x) - 0\big), \quad w_2(x) = w\big(x, y(x) + 0\big).$$

3°. For $f'_w(x, y, w) \neq 0$, the generalized solution stable with respect to small perturbations of the initial profile (it is stable solutions that are physically realizable) must satisfy the condition

$$f\big(x, y(x), w_2(x)\big) \leq V \leq f\big(x, y(x), w_1(x)\big). \tag{23.4.2.3}$$

The stability condition (23.4.2.3) has the geometrical meaning that the characteristics issuing from the x-axis (these characteristics "carry" information about the initial data) must

intersect the discontinuity line (see Fig. 23.7). This condition is very important since it allows for the existence of a stable generalized solution and provides its uniqueness.

The properties of Items 1° and 2° follow from the integral relation (23.4.2.1), and the condition of Item 3° is additional [it cannot be deduced from the integral relation (23.4.2.1)]. If the stability condition of Item 3° is not imposed, then various generalized solutions satisfying Items 1° and 2° may be constructed.

Example 23.7. Consider the Cauchy problem for equation (23.2.2.3) with the initial condition (23.2.2.10). We set

$$\overline{w}(x,y)=\begin{cases} w_1 & \text{for } y<Vx,\\ w_2 & \text{for } y>Vx,\end{cases}\qquad \text{where}\quad V=\frac{w_1+w_2}{2}. \tag{23.4.2.4}$$

This function is constant from the left and right of the discontinuity line $y = Vx$, where the jump condition (23.4.2.2) is met [since $F(x,y,w)=\frac{1}{2}w^2$], and satisfies the initial condition (23.2.2.10). Hence, $\overline{w}$ is a generalized solution.

Figure 23.10 shows the discontinuity line and characteristics corresponding to solution (23.4.2.4). One can see that the characteristics "issue" from the discontinuity line and do not intersect the x-axis. Therefore, solution (23.4.2.4) is unstable, does not satisfy condition (23.4.2.3), and is not physically realizable. A stable solution of this problem was constructed earlier; see relation (23.2.2.13).

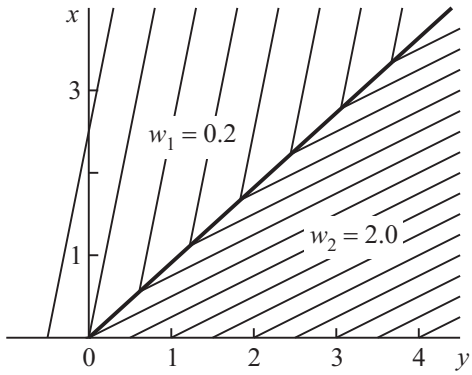

Figure 23.10. Characteristics and the discontinuity line for an unstable discontinuous solution (23.4.2.4).

If $f'_w(x,y,w)$ is not a function of fixed sign, the stability condition for the generalized solution becomes more complicated:

$$\frac{F(x,y,w_*)-F(x,y,w_2)}{w_*-w_2}\le V\le\frac{F(x,y,w_*)-F(x,y,w_1)}{w_*-w_1},$$
$$y=y(x),\quad w_1<w_*<w_2,$$

where w_* is any value from the interval (w_1, w_2).

Remark. Point out also other ways of defining generalized solutions (using conservation laws and viscosity solutions).

23.4.3. Method for Constructing Stable Generalized Solutions

Consider the Cauchy problem for the quasilinear equation

$$\frac{\partial w}{\partial x}+\frac{\partial}{\partial y}F(x,y,w)=0 \tag{23.4.3.1}$$

subject to the initial condition

$$w=\varphi(y)\quad\text{at}\quad x=0. \tag{23.4.3.2}$$

It is assumed that the function $F(x, y, w)$ is continuously differentiable with respect to all its arguments for $x \geq 0$, $-\infty < y < \infty$ and any bounded w. We also assume that the second derivative F_{ww} is positive. Let the functions $\varphi(y)$ and $\varphi'(y)$ be piecewise-continuous for any finite y.

The characteristic system for equation (23.4.3.1) has the form

$$y'_x = F_w(x, y, w), \quad w'_x = -F_y(x, y, w), \tag{23.4.3.3}$$

where F_w and F_y are the partial derivatives of the function F with respect to w and y.

Suppose the functions

$$y(x) = Y(x, \tau, \xi, \eta), \quad w(x) = W(x, \tau, \xi, \eta) \tag{23.4.3.4}$$

are solutions of system (23.4.3.3) satisfying the boundary conditions

$$y(0) = \eta, \quad y(\tau) = \xi. \tag{23.4.3.5}$$

Here, η and ξ are arbitrary numbers and $\tau > 0$. We assume that problem (23.4.3.3), (23.4.3.5) has a unique bounded solution.

The stable generalized solution of the Cauchy problem (23.4.3.1), (23.4.3.2) is given by

$$\begin{aligned} w(x, y-0) &= W\big(x, x, y, \xi_-(x, y)\big), \\ w(x, y+0) &= W\big(x, x, y, \xi_+(x, y)\big), \end{aligned} \tag{23.4.3.6}$$

where $\xi_-(x, y)$ and $\xi_+(x, y)$ denote, respectively, the greatest lower bound and the least upper bound of the set of values $\{\xi = \xi_n\}$ for which the function

$$I(x, y, \xi) = \int_0^{\xi} \big[\varphi(\eta) - W(0, x, y, \eta)\big]\, d\eta \tag{23.4.3.7}$$

takes the minimum value for fixed x and y ($x > 0$). If function (23.4.3.7) takes the minimum value for a single $\xi = \xi_1$, then $\xi_- = \xi_+$ and relation (23.4.3.7) describes the classical smooth solution.

⊙ *References for Section* 23.4: O. A. Oleinik (1954, 1957, 1959), I. M. Gelfand (1959), A. L. Hopf (1965), S. N. Kruzhkov (1966), B. L. Rozhdestvenskii and N. N. Yanenko (1983) F. John (1982), P.-L. Lions (1982), M. G. Crandall and P.-L. Lions (1983), H. Rhee, R. Aris, and N. R. Amundson (1986, 1989), A. I. Subbotin (1991, 1995), R. J. LeVeque (1992), J. Smoller (1994), E. Godlewski and P.-A. Raviart (1996), D. Serre (1996), M. Taylor (1996), A. A. Melikyan (1996, 1998), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

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Chapter 24
Methods for Solving First-Order Nonlinear Equations

24.1. Solution Methods

24.1.1. Complete, General, and Singular Integrals

A *nonlinear first-order partial differential equation* with two independent variables has the general form

$$F(x, y, w, p, q) = 0, \qquad \text{where} \quad p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}. \tag{24.1.1.1}$$

Such equations are encountered in analytical mechanics, calculus of variations, optimal control, differential games, dynamic programming, geometric optics, differential geometry, and other fields.

In this subsection, we consider only smooth solutions $w = w(x, y)$ of equation (24.1.1.1), which are continuously differentiable with respect to both arguments (Subsection 24.3 deals with nonsmooth solutions).

1°. Let a particular solution of equation (24.1.1.1),

$$w = \Xi(x, y, C_1, C_2), \tag{24.1.1.2}$$

depending on two parameters C_1 and C_2, be known. The two-parameter family of solutions (24.1.1.2) is called a *complete integral* of equation (24.1.1.1) if the rank of the matrix

$$M = \begin{pmatrix} \Xi_1 & \Xi_{x1} & \Xi_{y1} \\ \Xi_2 & \Xi_{x2} & \Xi_{y2} \end{pmatrix} \tag{24.1.1.3}$$

is equal to two in the domain being considered (for example, this is valid if $\Xi_{x1}\Xi_{y2} - \Xi_{x2}\Xi_{y1} \neq 0$). In equation (24.1.1.3), Ξ_n denotes the partial derivative of Ξ with respect to C_n ($n = 1, 2$), Ξ_{xn} is the second partial derivative with respect to x and C_n, and Ξ_{yn} is the second partial derivative with respect to y and C_n.

In some cases, a complete integral can be found using the method of undetermined coefficients by presetting an appropriate structure of the particular solution sought. (The complete integral is determined by the differential equation nonuniquely.)

Example 24.1. Consider the equation

$$\frac{\partial w}{\partial x} = a\left(\frac{\partial w}{\partial y}\right)^n + b.$$

We seek a particular solution as the sum $w = C_1 y + C_2 + C_3 x$. Substituting this expression into the equation yields the relation $C_3 = aC_1^n + b$ for the coefficients C_1 and C_3. With this relation, we find a complete integral in the form $w = C_1 y + \big(aC_1^n + b\big)x + C_2$.

A complete integral of equation (24.1.1.1) is often written in implicit form:*

$$\Xi(x, y, w, C_1, C_2) = 0. \tag{24.1.1.4}$$

2°. The *general integral* of equation (24.1.1.1) can be represented in parametric form by using the complete integral (24.1.1.2) [or (24.1.1.4)] and the two equations

$$\begin{aligned} &C_2 = f(C_1), \\ &\frac{\partial \Xi}{\partial C_1} + \frac{\partial \Xi}{\partial C_2} f'(C_1) = 0, \end{aligned} \tag{24.1.1.5}$$

where f is an arbitrary function and the prime stands for the derivative. In a sense, the general integral plays the role of the general solution depending on an arbitrary function (the question whether it describes all solutions calls for further analysis).

Example 24.2. For the equation considered in Example 24.1, the general integral can be written in parametric form by using the relations

$$w = C_1 y + \big(aC_1^n + b\big)x + C_2, \quad C_2 = f(C_1), \quad y + anC_1^{n-1}x + f'(C_1) = 0.$$

Eliminating C_2 from these relations and renaming C_1 by C, one can represent the general integral in a more graphic manner in the form

$$\begin{aligned} w &= Cy + \big(aC^n + b\big)x + f(C), \\ y &= -anC^{n-1}x + f'(C). \end{aligned}$$

3°. *Singular integrals* of equation (24.1.1.1) can be found without invoking a complete integral by eliminating p and q from the following system of three algebraic (or transcendental) equations:

$$F = 0, \quad F_p = 0, \quad F_q = 0,$$

where the first equation coincides with equation (24.1.1.1).

24.1.2. Method of Separation of Variables. Equations of Special Form

The method of separation of variables implies searching for a complete integral as the sum or product of functions of various arguments. Such solutions are called *additive separable* and *multiplicative separable*, respectively. Presented below are structures of complete integrals for some classes of nonlinear equations admitting separation of variables.

1°. If the equation does not depend explicitly on y and w, i.e.,

$$F(x, w_x, w_y) = 0,$$

then one can seek a complete integral in the form of the sum of two functions with different arguments

$$w = C_1 y + C_2 + u(x).$$

The new unknown function u is determined by solving the following ordinary differential equation:

$$F(x, u'_x, C_1) = 0.$$

Expressing u'_x from this equation in terms of x, one arrives at a separable differential equation for $u = u(x)$.

* In equations (24.1.1.2) and (24.1.1.4), the symbol Ξ denotes different functions.

2°. Consider an equation with separated variables

$$F_1(x, w_x) = F_2(y, w_y).$$

Then one can seek a complete integral as the sum of two functions with different arguments,

$$w = u(x) + v(y) + C_1,$$

which are determined by the following two ordinary differential equations:

$$F_1(x, u'_x) = C_2,$$
$$F_2(y, v'_y) = C_2.$$

3°. Let the equation have the form (generalizes the equation of Item 2°)

$$F_1(x, w_x) + F_2(y, w_y) = aw.$$

Then one can seek a complete integral as the sum of two functions with different arguments,

$$w = u(x) + v(y) + C_1,$$

which are determined by the following two ordinary differential equations:

$$F_1(x, u'_x) - au = aC_1 + C_2,$$
$$F_2(y, v'_y) - av = -C_2,$$

where C_1 is an arbitrary constant.

4°. Suppose the equation can be rewritten in the form

$$F\big(\varphi(x, w_x),\ y, w_y\big) = 0.$$

Then one can seek a complete integral as the sum of two functions with different arguments,

$$w(x, y) = u(x) + v(y) + C_1,$$

which are determined by the following two ordinary differential equations:

$$\varphi(x, u'_x) = C_2,$$
$$F(C_2, y, v'_y) = 0,$$

where C_2 is an arbitrary constant.

5°. Let the equation have the form

$$F_1(x, w_x/w) = w^k F_2(y, w_y/w).$$

Then one can seek a complete integral in the form of the product of two functions with different arguments,

$$w = u(x)v(y),$$

which are determined by the following two ordinary differential equations:

$$F_1(x, u'_x/u) = C_1 u^k,$$
$$F_2(y, v'_y/v) = C_1 v^{-k},$$

where C_1 is an arbitrary constant.

6°. Table 24.1 lists complete integrals of the above and some other nonlinear equations of general form involving arbitrary functions with several arguments.

▷ *Chapter 3 presents complete integrals for many more nonlinear first-order partial differential equations with two independent variables than in Table 24.1.*

TABLE 24.1
Complete integrals for some special types of nonlinear first-order partial differential equations; C_1 and C_2 are arbitrary constants

No.	Equations and comments	Complete integrals	Auxiliary equations
1	$F(w_x, w_y) = 0$, does not depend on x, y, and w implicitly	$w = C_1 + C_2x + C_3y$	$F(C_2, C_3) = 0$
2	$F(x, w_x, w_y) = 0$, does not depend on y and w implicitly	$w = C_1y + C_2 + u(x)$	$F(x, u'_x, C_1) = 0$
3	$F(w, w_x, w_y) = 0$, does not depend on x and y implicitly	$w = u(z),\ z = C_1x + C_2y$	$F(u, C_1u'_z, C_2u'_z) = 0$
4	$F_1(x, w_x) = F_2(y, w_y)$, separated equation	$w = u(x) + v(y) + C_1$	$F_1(x, u'_x) = C_2$, $F_2(y, v'_y) = C_2$
5	$F_1(x, w_x) + F_2(y, w_y) = aw$, generalizes equation 4	$w = u(x) + v(y)$	$F_1(x, u'_x) - au = C_1$, $F_2(y, v'_y) - av = -C_1$
6	$F_1(x, w_x) = e^{aw}F_2(y, w_y)$, generalizes equation 4	$w = u(x) + v(y)$	$F_1(x, u'_x) = C_1e^{au}$, $F_2(y, v'_y) = C_1e^{-av}$
7	$F_1(x, w_x/w) = w^kF_2(y, w_y/w)$, can be reduced to equation 6 by the change of variable $w = e^z$	$w = u(x)v(y)$	$F_1(x, u'_x/u) = C_1u^k$, $F_2(y, v'_y/v) = C_1v^{-k}$
8	$w = xw_x + yw_y + F(w_x, w_y)$, Clairaut equation	$w = C_1x + C_2y + F(C_1, C_2)$	—
9	$F(x, w_x, w_y, w - yw_y) = 0$, generalizes equation 2	$w = C_1y + u(x)$	$F\big(x, u'_x, C_1, u\big) = 0$
10	$F(w, w_x, w_y, xw_x + yw_y) = 0$, generalizes equations 3 and 8	$w = u(z),\ z = C_1x + C_2y$	$F(u, C_1u'_z, C_2u'_z, zu'_z) = 0$
11	$F\big(\varphi(x, w_x),\ y, w_y\big) = 0$, generalizes equation 4	$w = u(x) + v(y) + C_1$	$\varphi(x, u'_x) = C_2$, $F(C_2, y, v'_y) = 0$

24.1.3. Lagrange–Charpit Method

Suppose that a first integral,

$$\Phi(x, y, w, p, q) = C_1, \tag{24.1.3.1}$$

of the characteristic system of ordinary differential equations

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dw}{pF_p + qF_q} = -\frac{dp}{F_x + pF_w} = -\frac{dq}{F_y + qF_w} \tag{24.1.3.2}$$

is known. Here,

$$p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}, \quad F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_w = \frac{\partial F}{\partial w}, \quad F_p = \frac{\partial F}{\partial p}, \quad F_q = \frac{\partial F}{\partial q}.$$

We assume that solution (24.1.3.1) and equation (24.1.1.1) can be solved for the derivatives p and q, i.e.,

$$p = \varphi_1(x, y, w, C_1), \quad q = \varphi_2(x, y, w, C_1). \tag{24.1.3.3}$$

The first equation of this system can be treated as an ordinary differential equation with independent variable x and parameter y. On finding the solution of this equation depending on an arbitrary function $\psi(y)$, one substitutes this solution into the second equation to arrive at an ordinary differential equation for ψ. On determining $\psi(y)$ and on substituting it into the general solution of the first equation of (24.1.3.3), one finds a complete integral of equation (24.1.1.1). In a similar manner, one can start solving system (24.1.3.3) with the second equation, treating it as an ordinary differential equation with independent variable y and parameter x.

Example 24.3. Consider the equation

$$ywp^2 - q = 0, \qquad \text{where} \quad p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}.$$

In this case, the characteristic system (24.1.3.2) has the form

$$\frac{dx}{2ywp} = -\frac{dy}{1} = \frac{dw}{2ywp^2 - q} = -\frac{dp}{yp^3} = -\frac{dq}{wp^2 + yp^2 q}.$$

By making use of the original equation, we simplify the denominator of the third ratio to obtain an integrable combination: $dw/(ywp^2) = -dp/(yp^3)$. This yields the first integral $p = C_1/w$. Solving the original equation for q, we obtain the system

$$p = \frac{C_1}{w}, \quad q = \frac{C_1^2 y}{w}.$$

The general solution of the first equation has the form $w^2 = 2C_1 x + \psi(y)$, where $\psi(y)$ is an arbitrary function. With this solution, it follows from the second equation of the system that $\psi'(y) = 2C_1^2 y$. Thus, $\psi(y) = C_1^2 y^2 + C_2$. Finally, we arrive at a complete integral of the form

$$w^2 = 2C_1 x + C_1^2 y^2 + C_2.$$

Note that the general solution of the completely integrable Pfaffian equation (e.g., see A. D. Polyanin and A. V. Manzhirov 2007)

$$dw = \varphi_1(x, y, w, C_1)\, dx + \varphi_2(x, y, w, C_1)\, dy \tag{24.1.3.4}$$

is a complete integral of equation (24.1.1.1). Here, the functions φ_1 and φ_2 are the same as in system (24.1.3.3).

Remark. The relation $F(x, y, w, p, q) = C$ is an obvious first integral of the characteristic system (24.1.3.2). Hence, the function Φ determining the integral (24.1.3.1) must differ from F. However, the use of relation (24.1.1.1) makes it possible to reduce the order of system (24.1.3.2) by one.

24.1.4. Construction of a Complete Integral with the Aid of Two First Integrals

Suppose two independent first integrals,

$$\Phi(x, y, w, p, q) = C_1, \quad \Psi(x, y, w, p, q) = C_2, \tag{24.1.4.1}$$

of the characteristic system of ordinary differential equations (24.1.3.2) are known. Assume that the functions F, Φ, and Ψ determining equation (24.1.1.1) and the integrals (24.1.4.1) satisfy the two conditions

$$\begin{aligned} &\text{(a)} \quad J \equiv \frac{\partial(F, \Phi, \Psi)}{\partial(w, p, q)} \not\equiv 0, \\ &\text{(b)} \quad [\Phi, \Psi] \equiv \begin{vmatrix} \Phi_p & \Phi_x + p\Phi_w \\ \Psi_p & \Psi_x + p\Psi_w \end{vmatrix} + \begin{vmatrix} \Phi_q & \Phi_y + q\Phi_w \\ \Psi_q & \Psi_y + q\Psi_w \end{vmatrix} \equiv 0, \end{aligned} \tag{24.1.4.2}$$

where J is the Jacobian of F, Φ, and Ψ with respect to w, p, and q, and $[\Phi, \Psi]$ is the *Jacobi–Mayer bracket*. In this case, relations (24.1.1.1) and (24.1.4.1) form a parametric representation of the complete integral of equation (24.1.1.1) (p and q are considered to be parameters). Eliminating p and q from equations (24.1.1.1) and (24.1.4.1) followed by solving the obtained relation for w yields a complete integral in an explicit form $w = w(x, y, C_1, C_2)$.

Example 24.4. Consider the equation

$$pq - aw = 0, \qquad \text{where} \quad p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}.$$

The characteristic system (24.1.3.2) has the form

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dw}{2pq} = \frac{dp}{ap} = \frac{dq}{aq}.$$

Equating the first ratio with the fifth one and the second ratio with the fourth one, we obtain the first integrals

$$q - ax = C_1, \qquad p - ay = C_2.$$

Thus, $F = pq - aw$, $\Phi = q - ax$, and $\Psi = p - ax$. These functions satisfy conditions (24.1.4.2). Solving the equation and the first integrals for w yields a complete integral of the form

$$w = \frac{1}{a}(ax + C_1)(ay + C_2).$$

24.1.5. Case where the Equation Does Not Depend on w Explicitly

Suppose the original equation does not contain the unknown first explicitly, i.e., it has the form

$$F(x, y, p, q) = 0. \tag{24.1.5.1}$$

1°. Given a one-parameter family of solutions $w = \overline{\overline{\Xi}}(x, y, C_1)$ such that $\overline{\overline{\Xi}}'_1 \not\equiv \text{const}$, a complete integral is given by $w = \overline{\overline{\Xi}}(x, y, C_1) + C_2$.

2°. The first integral may be sought in the form

$$\Phi(x, y, p, q) = C_1$$

similar to that of equation (24.1.5.1). In this case, the characteristic system (24.1.3.2) is represented as

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dp}{F_x} = -\frac{dq}{F_y}.$$

The corresponding Pfaffian equation (24.1.3.4) becomes

$$dw = \varphi_1(x, y, C_1)\,dx + \varphi_2(x, y, C_1)\,dy.$$

One may integrate this equation in quadrature, thus arriving at the following expression for the complete integral:

$$w = \int_{x_0}^{x} \varphi_1(t, y, C_1)\,dt + \int_{y_0}^{y} \varphi_2(x_0, s, C_1)\,ds + C_2, \tag{24.1.5.2}$$

where x_0 and y_0 are arbitrary numbers.

3°. Suppose that equation (24.1.5.1) can be solved for p or q, for example,

$$p = -\mathcal{H}(x, y, q).$$

Then, by differentiating this relation with respect to y, we obtain a quasilinear equation for the derivative q in the form

$$\frac{\partial q}{\partial x} + \frac{\partial}{\partial y}\mathcal{H}(x, y, q) = 0, \qquad q = \frac{\partial w}{\partial y}.$$

This equation is simpler than the original one; qualitative features of it and solution methods can be found in Subsection 24.1.1.

24.1.6. Hamilton–Jacobi Equation

Equation (24.1.1.1) solved for one of the derivatives, e.g.,

$$p + \mathcal{H}(x, y, w, q) = 0, \qquad \text{where} \quad p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}, \tag{24.1.6.1}$$

is commonly referred to as the *Hamilton–Jacobi equation** and the function $\mathcal{H}$ as the *Hamiltonian*. Equations of the form (24.1.6.1) are frequently encountered in various fields of mechanics, control theory, and differential games, where the variable x usually plays the role of time and the variable y the role of the spatial coordinate. To the Hamilton–Jacobi equation (24.1.6.1) there corresponds the function $F(x, y, w, p, q) = p + \mathcal{H}(x, y, w, q)$ in equation (24.1.1.1).

The characteristic system (24.1.3.2) for equation (24.1.6.1) can be reduced, by taking into account the relation $p = -\mathcal{H}$, to a simpler system consisting of three differential equations,

$$y'_x = \mathcal{H}_q, \quad w'_x = q\mathcal{H}_q - \mathcal{H}, \quad q'_x = -q\mathcal{H}_w - \mathcal{H}_y, \tag{24.1.6.2}$$

which are independent of p; the left-hand sides of these equations are derivatives with respect to x.

⊙ *References for Section* 24.1: R. Bellman (1957), G. A. Korn and T. M. Korn (1961), L. S. Pontryagin, V. G. Boltyansky, R. V. Gamkrelidze, and E. F. Mishchenko (1962), E. Kamke (1965), I. G. Petrovskii (1970), V. I. Arnold (1980), E. Zauderer (1983), R. Courant and D. Hilbert (1989), J. Lewin (1994), V. F. Zaitsev and A. D. Polyanin (1996), A. A. Melikyan (1998), D. Zwillinger (1998).

24.2. Cauchy Problem. Existence and Uniqueness Theorem

24.2.1. Statement of the Problem. Solution Procedure

Consider the Cauchy problem for equation (24.1.1.1) subject to the initial conditions

$$x = h_1(\xi), \quad y = h_2(\xi), \quad w = h_3(\xi), \tag{24.2.1.1}$$

where ξ is a parameter ($\alpha \le \xi \le \beta$) and the $h_k(\xi)$ are given functions.

The solution of this problem is carried out in several steps:

1°. First, one determines additional initial conditions for the derivatives,

$$p = p_0(\xi), \quad q = q_0(\xi). \tag{24.2.1.2}$$

To this end, one must solve the algebraic (or transcendental) system of equations

$$F\big(h_1(\xi), h_2(\xi), h_3(\xi), p_0, q_0\big) = 0, \tag{24.2.1.3}$$

$$p_0 h'_1(\xi) + q_0 h'_2(\xi) - h'_3(\xi) = 0 \tag{24.2.1.4}$$

for p_0 and q_0. Equation (24.2.1.3) results from substituting the initial data (24.2.1.1) into the original equation (24.1.1.1). Equation (24.2.1.4) is a consequence of the dependence of w on x and y and the relation $dw = p\,dx + q\,dy$, where dx, dy, and dw are calculated in accordance with the initial data (24.2.1.1).

* The Hamilton–Jacobi equation is often the one in which the function $\mathcal{H}$ is independent of w explicitly, i.e., the equation $p + \mathcal{H}(x, y, q) = 0$.

2°. One solves the autonomous system

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dw}{pF_p + qF_q} = -\frac{dp}{F_x + pF_w} = -\frac{dq}{F_y + qF_w} = d\tau, \tag{24.2.1.5}$$

which is obtained from (24.1.3.2) by introducing the additional variable τ (playing the role of time).

3°. Finally, one determines the constant of integration from the initial conditions

$$x = h_1(\xi), \quad y = h_2(\xi), \quad w = h_3(\xi), \quad p = p_0(\xi), \quad q = q_0(\xi) \quad \text{at} \quad \tau = 0, \tag{24.2.1.6}$$

obtained by combining conditions (24.2.1.1) and (24.2.1.2). This results in the three functions

$$x = x(\tau, \xi), \quad y = y(\tau, \xi), \quad w = w(\tau, \xi), \tag{24.2.1.7}$$

which give the solution of the stated Cauchy problem in parametric form (τ and ξ are parameters).

24.2.2. Existence and Uniqueness Theorem

Let the function $F = F(x, y, w, p, q)$, determining equation (24.1.1.1), be twice continuously differentiable with respect to all five arguments (in the domain being considered), with $F_p^2 + F_q^2 \neq 0$. Let the functions $h_1(\xi)$, $h_2(\xi)$, and $h_3(\xi)$, determining the initial conditions (24.2.1.1), be twice differentiable with respect to ξ, with $(h_1')^2 + (h_2')^2 \neq 0$. Assume that the functions $p_0(\xi)$ and $q_0(\xi)$, determining the additional initial condition (24.2.1.2), satisfy system (24.2.1.3), (24.2.1.4). Moreover, we adopt the *transversality condition*

$$\Delta \equiv F_p h_2' - F_q h_1' \neq 0,$$

where the functions h_1, h_2, p, and q are defined by equations (24.2.1.1) and (24.2.1.2), and the prime denotes the derivative with respect to ξ. Under the adopted assumptions, there exists a unique, twice continuously differentiable solution of equation (24.1.1.1) satisfying the initial conditions (24.2.1.1) and (24.2.1.2).

Remark 1. This theorem has a local character, i.e., the existence of a unique solution of the Cauchy problem is merely guaranteed in some neighborhood of the line defined by the initial data (24.2.1.1) together with the additional conditions (24.2.1.2).

Remark 2. The algebraic (or transcendental) system (24.2.1.3), (24.2.1.4) can have several solutions (see Example 24.6 at the end of this subsection) that result in distinct additional conditions (24.2.1.2) for the derivatives p and q. Each pair of these additional conditions will generate a solution of the Cauchy problem (24.1.1.1), (24.2.1.1).

Remark 3. For nonlinear equations, the global solution of the Cauchy problem (24.1.1.1), (24.2.1.1) may be nonunique for another reason—because the characteristics in the xy-plane may intersect (see Example 24.7 in Section 24.3). Such a situation is discussed in Sections 23.3 and 23.4, where quasilinear equations are considered.

24.2.3. Cauchy Problem for the Hamilton–Jacobi Equation

The initial condition for the Hamilton–Jacobi equation (24.1.6.1) is usually stated in the form

$$w = \varphi(y) \quad \text{at} \quad x = L. \tag{24.2.3.1}$$

In this case, the solution of the Cauchy problem is reduced to the solution of the characteristic system (24.1.6.2) subject to the initial conditions

$$y = \xi, \quad w = \varphi(\xi), \quad q = \varphi'(\xi) \quad \text{at} \quad x = L, \tag{24.3.2.2}$$

where the prime stands for the derivative with respect to the parameter ξ.

24.2.4. Examples of Solving the Cauchy Problem

Consider a few specific examples.

Example 24.5. Find a solution of the equation

$$aw = pq, \qquad \text{where} \quad p = \frac{\partial w}{\partial x}, \quad q = \frac{\partial w}{\partial y}, \tag{24.2.4.1}$$

satisfying the initial condition

$$w = f(y) \quad \text{at} \quad x = 0. \tag{24.2.4.2}$$

Rewrite the initial condition (24.2.4.2) in parametric form:

$$x = 0, \quad y = \xi, \quad w = f(\xi). \tag{24.2.4.3}$$

The system of equations (24.2.1.3), (24.2.1.4) for $p_0(\xi)$ and $q_0(\xi)$ is written as

$$af(\xi) = p_0 q_0, \quad q_0 - f'(\xi) = 0.$$

It follows that

$$p_0 = a\frac{f(\xi)}{f'(\xi)}, \quad q_0 = f'(\xi). \tag{24.2.4.4}$$

If $F = pq - aw$, the characteristic system (24.2.1.5) is written as

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dw}{2pq} = \frac{dp}{ap} = \frac{dq}{aq} = d\tau. \tag{24.2.4.5}$$

Its solution is given by the relations

$$p = C_1 e^{a\tau}, \quad q = C_2 e^{a\tau}, \quad x = \frac{C_2}{a}e^{a\tau} + C_3, \quad y = \frac{C_1}{a}e^{a\tau} + C_4, \quad w = \frac{C_1 C_2}{a}e^{2a\tau} + C_5; \tag{24.2.4.6}$$

it is easy to integrate the last two equations first.

By using the initial conditions (24.2.4.3), (24.2.4.4), which must be satisfied at $\tau = 0$, we find the constants of integration in the form

$$C_1 = a\frac{f(\xi)}{f'(\xi)}, \quad C_2 = f'(\xi), \quad C_3 = -\frac{f'(\xi)}{a}, \quad C_4 = \xi - \frac{f(\xi)}{f'(\xi)}, \quad C_5 = 0.$$

Substituting theses values into (24.2.4.6) yields a solution of the Cauchy problem (24.2.4.1), (24.2.4.2) in the parametric form

$$x = \frac{1}{a}f'(\xi)\big(e^{a\tau} - 1\big), \quad y = \frac{f(\xi)}{f'(\xi)}\big(e^{a\tau} - 1\big) + \xi, \quad w = f(\xi)e^{2a\tau}.$$

Example 24.6. Find a solution of the equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = a^2, \tag{24.2.4.7}$$

passing through the circle

$$x^2 + y^2 = b^2, \quad w = 0. \tag{24.2.4.8}$$

By introducing a parameter ξ, we rewrite the equation of the circle in the form

$$x = b\sin\xi, \quad y = b\cos\xi, \quad w = 0. \tag{24.2.4.9}$$

In this case, the equations (24.2.1.3), (24.2.1.4) for determining additional initial conditions are

$$p_0^2 + q_0^2 = a^2, \quad p_0\cos\xi - \sin\xi q_0 = 0.$$

Whence,

$$p_0 = \varepsilon a\sin\xi, \quad q_0 = \varepsilon a\cos\xi, \qquad \text{where} \quad \varepsilon = \pm 1. \tag{24.2.4.10}$$

For $F = p^2 + q^2 - a^2$, system (24.2.1.5) is represented as

$$\frac{dx}{2p} = \frac{dy}{2q} = \frac{dw}{2(p^2+q^2)} = -\frac{dp}{0} = -\frac{dq}{0} = d\tau. \tag{24.2.4.11}$$

The general solution is given by (the last two equations are integrated first)

$$p = C_1, \quad q = C_2, \quad x = 2C_1\tau + C_3, \quad y = 2C_2\tau + C_4, \quad w = 2(C_1^2 + C_2^2)\tau + C_5. \tag{24.2.4.12}$$

Using the initial conditions (24.2.4.9), (24.2.4.10), which must be satisfied at $\tau = 0$, we find the constants of integration

$$C_1 = \varepsilon a \sin\xi, \quad C_2 = \varepsilon a \cos\xi, \quad C_3 = b \sin\xi, \quad C_4 = b \cos\xi, \quad C_5 = 0, \qquad \text{where} \quad \varepsilon = \pm 1.$$

Substituting these values into (24.2.4.12) yields a solution of the Cauchy problem (24.2.4.7), (24.2.4.8) in the parametric form

$$x = (2\varepsilon a\tau + b)\sin\xi, \quad y = (2\varepsilon a\tau + b)\cos\xi, \quad w = 2a^2\tau.$$

On eliminating the parameters ξ and τ, we can rewrite this solution in a more graphic form,

$$a^2(x^2 + y^2) = (ab \pm w)^2. \tag{24.2.4.13}$$

Relation (24.2.4.13) describes two circular coaxial cones in the space (x, y, w). The circle (24.2.4.8) is a base of the cones. The common axis coincides with the w-axis. The vertices of the cones have the coordinates $w = \pm ab$.

It is significant that solution (24.2.4.13) is a many-valued function.

⊙ *References for Section* 24.2: E. Kamke (1965), I. G. Petrovskii (1970), R. Courant and D. Hilbert (1989), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

24.3. Generalized Viscosity Solutions and Their Applications

24.3.1. Preliminary Remarks

Subsections 24.1 and 24.2 dealt with classical smooth solutions $w = w(x, y)$, having continuous derivatives with respect to both arguments. However, in optimal control, differential games, and some other applications, problems arise whose solutions are continuous but nonsmooth functions. To describe and construct generalized solutions of this sort, approaches other than those outlined above are required. It should be noted that for determining generalized solutions to nonlinear equations of the general forms (24.1.1.1) and (24.1.6.1), it turns out to be ineffective to use graphic constructs such as integral relations and conservation laws, which are frequently encountered in the theory of quasilinear equations (see Sections 23.3 and 23.4).

Note that nonsmoothness of a solution can be caused by (i) the intersection of characteristics in the xy-plane (see Example 24.7 below), (ii) nonsmoothness of initial conditions, and/or (iii) nonsmoothness of the functions F and $\mathcal{H}$ determining equations (24.1.1.1) and (24.1.6.1).

24.3.2. Viscosity Solutions Based on the Use of a Parabolic Equation

The solution of the Cauchy problem for equation (24.1.6.1) subject to the initial condition

$$w = \varphi(y) \quad \text{at} \quad x = 0 \tag{24.3.2.1}$$

may be approximated by the solution of the following second-order partial differential equation of the parabolic type:

$$\frac{\partial u}{\partial x} + \mathcal{H}\left(x, y, u, \frac{\partial u}{\partial y}\right) = \varepsilon \frac{\partial^2 u}{\partial y^2} \qquad (\varepsilon > 0). \tag{24.3.2.2}$$

This equation is subject to the same initial condition (24.3.2.1). For a fairly wide class of functions $\mathcal{H}$ and φ, the Cauchy problem for equation (24.3.2.2) is known to have a unique solution. In the theory of Hamilton–Jacobi equations, this fact is used to determine the solution of the Cauchy problem (24.1.6.1), (24.3.2.1) as a limit of the solution of problem (24.3.2.2), (24.3.2.1): $w(x,y) = \lim\limits_{\varepsilon\to 0} u(x,y)$. Just as in the theory of quasilinear equations (see Subsection 23.3.5), this structure, based on proceeding to the limit as $\varepsilon \to 0$, is referred to as the method of vanishing viscosity, and the limit function as the viscosity solution of the Hamilton–Jacobi equation.

The vanishing viscosity method can be implemented, for example, by a numerical solution of problem (24.3.2.2), (24.3.2.1) for sufficiently small ε (in this case, one need not search for singular points, at which the smoothness of the solution is violated). However, this method is difficult to use for constructing analytical solutions, since one has to treat a more complex, second-order partial differential equation.

24.3.3. Viscosity Solutions Based on Test Functions and Differential Inequalities

A continuous function $w = w(x,y)$ is called the viscosity solution of the initial value problem (24.1.1.1), (24.3.2.1) in a layer $0 \le x \le L$ if the following two conditions are satisfied:

1°. The function $w = w(x,y)$ satisfies the initial condition (24.3.2.1).

2°. Let $\psi(x,y)$ be any continuously differentiable test function. If (x°, y°) is a local extremum point of the difference

$$w(x,y) - \psi(x,y), \tag{24.3.3.1}$$

then the following relations hold at this point:

$$\begin{aligned} &F\big(x^\circ, y^\circ, w^\circ, \psi_x^\circ, \psi_y^\circ\big) \ge 0 \qquad \text{if } (x^\circ, y^\circ) \text{ is a local minimum point,} \\ &F\big(x^\circ, y^\circ, w^\circ, \psi_x^\circ, \psi_y^\circ\big) \le 0 \qquad \text{if } (x^\circ, y^\circ) \text{ is a local maximum point.} \end{aligned} \tag{24.3.3.2}$$

Only those local extremum points that lie within the layer in question ($0 < x^\circ < L$) are to be examined.

Note that it is not necessary that a test function $\psi(x,y)$ exists for which the difference (24.3.3.1) has a local extremum. But if such a function does exist, then condition (24.3.3.2) must hold.

If the Cauchy problem has a smooth classical solution, then it coincides with the viscosity generalized solution.

In optimal control and differential games, terminal value problems are encountered apart from initial value problems. In terminal value problems, the solution of equations (24.1.1.1) and (24.1.6.1) is sought in the layer $0 \le x \le L$ and the "initial" condition is set at the right endpoint $x = L$. For these problems, the inequalities in (24.3.3.2) must be changed for the opposites. The terminal value problems can be reduced to initial value problems by introducing a new independent variable $z = L - x$ instead of x.

24.3.4. Local Structure of Generalized Viscosity Solutions

A generalized solution, $w(x,y)$, consists of regular and singular points. In a neighborhood of a regular point, the function $w(x,y)$ is a solution in the classical sense (such twice

continuously differentiable solutions are discussed in the existence and uniqueness theorem in Section 24.2). All points that are not regular are called singular points.

Let D be a sufficiently small neighborhood of a singular point (x_*, y_*). Usually, it happens that the singular points form a smooth curve Γ, which passes through (x_*, y_*) and divides the domain D into two subdomains D_1 and D_2 (see Fig. 24.1). On both sides of Γ, the generalized solution w is defined by different classical solutions u_1 and u_2, i.e.,

$$w(x,y) = \begin{cases} u_1(x,y) & \text{if } (x,y) \in D_1, \\ u_2(x,y) & \text{if } (x,y) \in D_2. \end{cases} \tag{24.3.4.1}$$

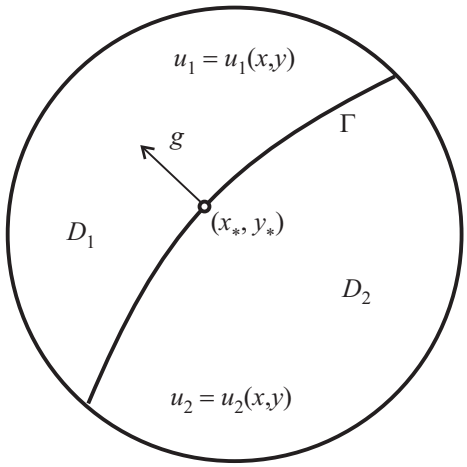

Figure 24.1. Generalized viscosity solution undergoes a discontinuity along the singular curve Γ; the solution is smooth in the subdomains D_1 and D_2.

The solutions u_1 and u_2 join together along the interface Γ in a continuous but nonsmooth manner. When crossing Γ, the derivatives of the generalized solution, $\partial w/\partial x$ and $\partial w/\partial y$, undergo a discontinuity. We assume that the smooth components u_1 and u_2 of the generalized solution are smoothly expended to the entire domain D. Then the equation of the curve Γ, formed by the singular points, can be represented as

$$g(x,y) = 0, \qquad \text{where} \quad g(x,y) = u_2(x,y) - u_1(x,y). \tag{24.3.4.2}$$

The gradient of g, directed along the normal to Γ, is given by

$$\nabla g = (p_2 - p_1)\mathbf{e}_x + (q_2 - q_1)\mathbf{e}_y,$$

with

$$p_n = \frac{\partial u_n}{\partial x}, \quad q_n = \frac{\partial u_n}{\partial y},$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the direction cosines along the x- and y-axes, respectively. Two situations are possible.

1°. The vector ∇g is directed from D_2 toward D_1. In this case, the following statements are valid:

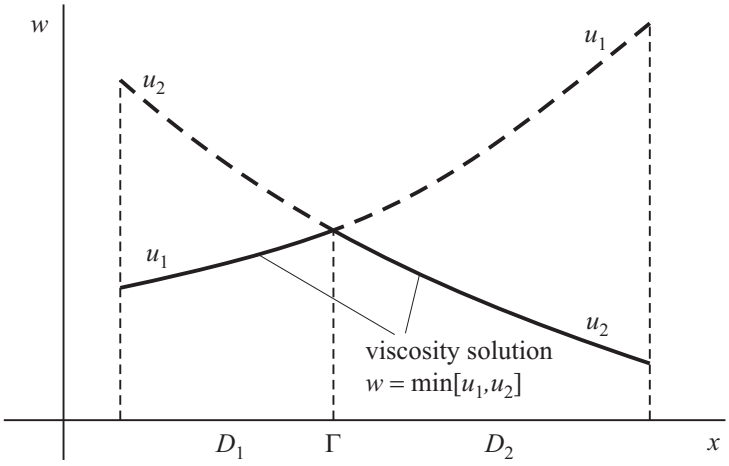

Figure 24.2. Graphic construction of a generalized viscosity solution by joining together smooth solutions.

(A) The generalized solution in D can be represented as $w = \min[u_1, u_2]$; see Fig. 24.2.

(B) There is no smooth test function $\psi(x, y)$ such that a local minimum of the difference (24.3.3.1) is attained at singular points comprising Γ.

(C) For the one-parameter family of test functions

$$\psi(x, y) = \lambda u_1(x, y) + (1 - \lambda)u_2(x, y), \qquad 0 \le \lambda \le 1, \tag{24.3.4.3}$$

the maximum

$$\max_{(x,y)\in D} \big[w(x, y) - \psi(x, y)\big] \tag{24.3.4.4}$$

is attained at $(x, y) \in \Gamma$.

Remark. For the generalized solution of the form $w = \min[u_1, u_2]$, one need not check the first inequality of (24.3.3.2); as far as the second inequality of (24.3.3.2) is concerned, it suffices to check this inequality against the one-parameter family of test functions defined by equation (24.3.4.3).

2°. The vector ∇g is directed from D_1 toward D_2. In this case, the generalized solution can be represented in the form $w = \max[u_1, u_2]$ and one needs to check only the first inequality of (24.3.3.2) against the one-parameter family of test functions of equation (24.3.4.3).

24.3.5. Generalized Classical Method of Characteristics

Consider the Cauchy problem for the Hamilton–Jacobi equation of the form

$$\begin{aligned} &\frac{\partial w}{\partial x} + \mathcal{H}\left(x, y, \frac{\partial w}{\partial y}\right) = 0, \\ &w = \varphi(y) \quad \text{at} \quad x = L. \end{aligned} \tag{24.3.5.1}$$

It is assumed here that (i) the function $\mathcal{H}(x, y, q)$ is convex with respect to q for all $x \in (0, L]$, $y \in \mathbb{R}$; (ii) the function $\mathcal{H}(x, y, q)$ is continuously differentiable with respect to x, y, and q; and (iii) the second derivatives $\mathcal{H}_{xy}$ and $\mathcal{H}_{xq}$ exist.

By

$$y = Y(x, \xi), \quad w = W(x, \xi), \quad q = Q(x, \xi) \tag{24.3.5.2}$$

we denote a solution of the characteristic system (24.1.6.2) satisfying condition (24.3.2.2).

Let $\{\xi_n = \xi_n(x, y)\}$ be the set of functions obtained by solving the first equation of (24.3.5.2) for the parameter ξ. The subscript n indicates the number of such functions.

The classical method of characteristics can be used to construct a generalized viscosity solution with the help of the relation

$$w(x, y) = \max_{\xi \in \{\xi_n\}} W(x, \xi) \tag{24.3.5.3}$$

for all $x \in (0, L]$, $y \in \mathbb{R}$. To the value $n = 1$ there corresponds the classical smooth solution.

24.3.6. Examples of Viscosity (Nonsmooth) Solutions

Below we solve two problems that are encountered in the theory of differential games.

Example 24.7. Consider the terminal value problem for the Hamilton–Jacobi equation

$$\frac{\partial w}{\partial x} + \sqrt{1 + \Big(\frac{\partial w}{\partial y}\Big)^2} = 0 \tag{24.3.6.1}$$

with the initial condition

$$w = \tfrac{1}{2}y^2 \quad \text{at} \quad x = L. \tag{24.3.6.2}$$

The solution is sought in the domain $0 \le x \le L$.

The characteristic system (24.1.6.2) for equation (24.3.6.1) with the Hamiltonian $\mathcal{H}(x, y, w, q) = \sqrt{1 + q^2}$ has the form

$$y'_x = \frac{q}{\sqrt{1 + q^2}}, \quad w'_x = -\frac{1}{\sqrt{1 + q^2}}, \quad q'_x = 0. \tag{24.3.6.3}$$

The initial conditions are obtained from (24.3.2.2) for $\varphi(\xi) = \frac{1}{2}\xi^2$:

$$y = \xi, \quad w = \tfrac{1}{2}\xi^2, \quad q = \xi \quad \text{at} \quad x = L. \tag{24.3.6.4}$$

Integrating equation (24.3.6.3) and applying conditions (24.3.6.4), we obtain a solution of the Cauchy problem (24.3.6.1), (24.3.6.2) of the form

$$y = \frac{\xi(x - L)}{\sqrt{1 + \xi^2}} + \xi, \quad w = \frac{L - x}{\sqrt{1 + \xi^2}} + \tfrac{1}{2}\xi^2. \tag{24.3.6.5}$$

Figure 24.3 displays the characteristics $y(x, \xi)$ in the xy-plane for $L = 2$ and $\xi = 0, \pm 0.2, \pm 0.4, \ldots, \pm 1.0$. It is apparent that characteristics intersect. In this example, it is possible to construct a local classical solution of problem (24.3.6.1), (24.3.6.2). But this solution cannot be extended to the entire layer $0 \le x \le L$, that is, there is no global classical solution. Pay attention to the fact that the Hamiltonian $\mathcal{H} = \sqrt{1 + q^2}$ of equation (24.3.6.1) and the function determining the initial condition (24.3.6.2) are infinitely differentiable functions.

The viscosity solution of the Cauchy problem (24.3.6.1), (24.3.6.2) has the form

$$w(x, y) = \max_{q \in \mathbb{R}} \big[qy + (L - x)\sqrt{1 + q^2} - \tfrac{1}{2}q^2\big], \tag{24.3.6.6}$$

where $0 \le x \le L$ and y is any number. Level curves of this function are depicted in Fig. 24.4. The heavy line indicates the set of singular points, at which the solution is nondifferentiable.

Example 24.8. Consider the terminal value problems for the more general Hamilton–Jacobi equation

$$\frac{\partial w}{\partial x} + \mathcal{H}\Big(\frac{\partial w}{\partial y}\Big) = 0 \qquad (0 \le x \le L) \tag{24.3.6.7}$$

with an arbitrary initial condition

$$w = \varphi(y) \quad \text{at} \quad x = L. \tag{24.3.6.8}$$

The following two statements hold:

1°. Let the Hamiltonian satisfy the Lipschitz condition

$$|\mathcal{H}(q_2) - \mathcal{H}(q_1)| \le \beta|q_2 - q_1| \qquad \text{for any} \quad q_1, q_2 \in \mathbb{R}, \tag{24.3.6.9}$$

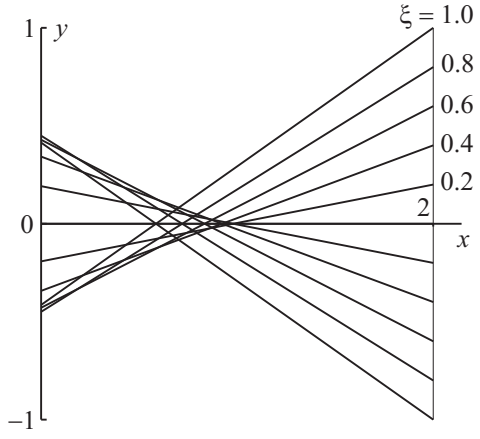

Figure 24.3. Characteristics of the terminal value problem (24.3.6.1)–(24.3.6.2) for the Hamilton–Jacobi equation.

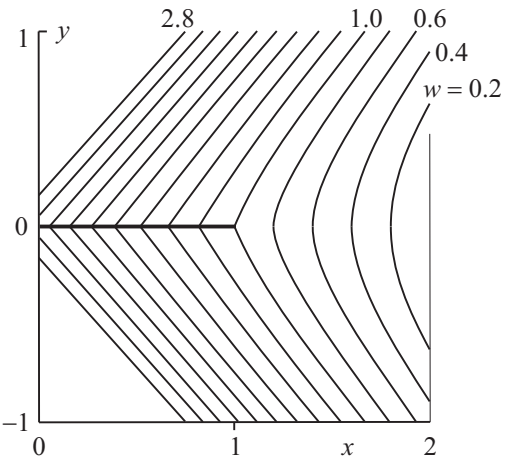

Figure 24.4. Level lines of the terminal value problem (24.3.6.1)–(24.3.6.2) for the Hamilton–Jacobi equation.

and let the function $\varphi(y)$ be convex. Then the function

$$w(x,y) = \sup_{q\in\mathbb{R}} \big[qy + (L-x)\mathcal{H}(q) - \varphi^*(q)\big]$$

is the viscosity solution of problem (24.3.6.7), (24.3.6.8). The function φ^* is the conjugate of φ, i.e.,

$$\varphi^*(q) = \sup_{x\in\mathbb{R}} \big[qx - \varphi(x)\big].$$

2°. Let the Hamiltonian $\mathcal{H}$ be convex and satisfy the Lipschitz condition (24.3.6.9). Let the function $\varphi(y)$ be continuous. Then the function

$$w(x,y) = \sup_{t\in\mathbb{R}} \big[\varphi(y + (L-x)t) - (L-x)\mathcal{H}^*(t)\big]$$

is the viscosity solution of problem (24.3.6.7), (24.3.6.8). The function

$$\mathcal{H}^*(t) = \sup_{q\in\mathbb{R}} \big[qt - \mathcal{H}(q)\big]$$

is the conjugate of the Hamiltonian.

⊙ *References for Section* 24.3: E. Hopf (1965), S. N. Kruzhkov (1966, 1975), P.-L. Lions (1982), M. G. Crandall and P.-L. Lions (1983), M. G. Crandall, L. C. Evans, and P.-L. Lions (1984), P.-L. Lions and P. E. Souganidis (1985), E. N. Barron and R. Jensen (1987), H. Ishii (1988), A. I. Subbotin (1991, 1995), M. G. Crandall, H. Ishii, and P.-L. Lions (1992), W. H. Fleming and H. M. Soner (1993), M. Bardi and I. C. Dolcetta (1998), A. A. Melikyan (1996, 1998), R. Courant and D. Hilbert (1989), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

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Chapter 25
Classification of Second-Order Nonlinear Equations

25.1. Semilinear Equations in Two Independent Variables

25.1.1. Types of Equations. Characteristic Equation

Consider a second-order semilinear partial differential equation in two independent variables of the form

$$a(x,y)\frac{\partial^2 w}{\partial x^2} + 2b(x,y)\frac{\partial^2 w}{\partial x \partial y} + c(x,y)\frac{\partial^2 w}{\partial y^2} = F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right), \qquad (25.1.1.1)$$

where a, b, and c are some functions of x and y that have continuous derivatives up to the second-order inclusive.

This equation is classified according to the sign of the discriminant

$$\delta = b^2 - ac, \qquad (25.1.1.2)$$

where the arguments of the equation coefficients are omitted for brevity. Given a point (x, y), equation (25.1.1.1) is said to be

$$\begin{aligned} &\textit{parabolic} && \text{if } \ b^2 - ac = 0, \\ &\textit{hyperbolic} && \text{if } \ b^2 - ac > 0, \\ &\textit{elliptic} && \text{if } \ b^2 - ac < 0 \end{aligned}$$

at this point.

In order to reduce equation (25.1.1.1) to a canonical form, one should first write out the characteristic equation

$$a\,dy^2 - 2b\,dx\,dy + c\,dx^2 = 0,$$

which splits into two equations

$$a\,dy - \left(b + \sqrt{b^2 - ac}\right)dx = 0, \qquad (25.1.1.3)$$

and

$$a\,dy - \left(b - \sqrt{b^2 - ac}\right)dx = 0, \qquad (25.1.1.4)$$

and find their general integrals.

Remark 1. The equations of characteristics (25.1.1.3)–(25.1.1.4) can be used if $a \not\equiv 0$. If $a \equiv 0$, one should use the simpler equations

$$dx = 0,$$
$$2b\,dy - c\,dx = 0.$$

The former equation has the obvious general solution $x = C$.

Remark 2. The classification of semilinear equations of the form (25.1.1.1) does not depend on their solutions—it is determined solely by the coefficients of the highest derivatives on the left-hand side.

25.1.2. Canonical Form of Parabolic Equations (Case $b^2 - ac = 0$)

In this case, equations (25.1.1.3) and (25.1.1.4) coincide and have a common general integral,

$$\varphi(x, y) = C.$$

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \eta(x, y),$$

where $\eta = \eta(x, y)$ is any twice differentiable function that satisfies the condition of nondegeneracy of the Jacobian $\frac{D(\xi,\eta)}{D(x,y)}$ in a given domain, we reduce equation (25.1.1.1) to the canonical form

$$\frac{\partial^2 w}{\partial \eta^2} = F_1\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right). \tag{25.1.2.1}$$

As η, one can take $\eta = x$ or $\eta = y$. Often $\eta = \eta(x, y)$ is selected so as to simplify the right-hand side of equation (25.1.2.1) as much as possible. In the special case $F_1 = \partial_\xi w$, we have the classical linear heat equation.

It is apparent that the transformed equation (25.1.2.1) has only one highest-derivative term.

Remark. In the degenerate case where the function F_1 is independent of the derivative $\partial_\xi w$, equation (25.1.2.1) is an ordinary differential equation for η, in which ξ serves as a parameter. A similar situation arises when, for example, the right-hand side of equation (25.1.2.1) is only dependent on x, y, and w (independent of the derivatives).

Example 25.1. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} = Ae^{-\lambda w}.$$

It is a special case of equation (25.1.1.1) with $a = c = 1$, $b = -1$, and $F = Ae^{\lambda w}$. In this case, characteristic equations (25.1.1.3) and (25.1.1.4) coincide and have a common integral,

$$x + y = C.$$

By changing to the new variables

$$\xi = x + y, \qquad \eta = x,$$

we arrive at the equation

$$\frac{\partial^2 w}{\partial \eta^2} = Ae^{-\lambda w}.$$

It can be treated as an ordinary differential equation. Its general solution involves two constants of integration, which are arbitrary functions of the argument ξ:

$$w = \frac{1}{\lambda} \ln\left[\frac{A\lambda}{2C_1^2} \cosh^2(C_1\eta + C_2)\right] \qquad (A\lambda > 0)$$

where $C_1 = C_1(\xi)$, $C_2 = C_2(\xi)$ are arbitrary functions, with $\xi = x + y$, and $\eta = x$.

25.1.3. Canonical Form of Hyperbolic Equations (Case $b^2 - ac > 0$)

The general integrals

$$\varphi(x, y) = C_1, \qquad \psi(x, y) = C_2$$

of equations (25.1.1.3) and (25.1.1.4) are real and different. These integrals determine two different families of real characteristics.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y),$$

we reduce equation (25.1.1.1) to

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = F_2\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right). \tag{25.1.3.1}$$

This is the so-called *first canonical form of a hyperbolic equation*.

If the right-hand side of equation (25.1.3.1) is independent of w and w_ξ (or w and w_η), the equation admits an order reduction with the substitution $u = w_\eta$ (or $u = w_\xi$).

Example 25.2. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = \alpha\left(\frac{\partial w}{\partial x}\right)^2 + \beta\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \gamma\left(\frac{\partial w}{\partial y}\right)^2. \tag{25.1.3.2}$$

It is a special case of equation (25.1.1.1) with $a = 1$, $b = 0$, $c = -1$, and $F = \alpha w_x^2 + \beta w_x w_y + \gamma w_y^2$. Let us find the values of α, β, and γ that would allow us to reduce the order of equation (25.1.3.2).

In this case, characteristic equations (25.1.1.3) and (25.1.1.4) have integrals $\varphi = y + x$ and $\psi = y - x$, so

$$\xi = y + x, \quad \eta = y - x \tag{25.1.3.3}$$

can be taken as the new variables. In the variables (25.1.3.3), equation (25.1.3.2) becomes

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = -\frac{1}{4}(\alpha + \beta + \gamma)\left(\frac{\partial w}{\partial \xi}\right)^2 - \frac{1}{4}(\alpha - \beta + \gamma)\left(\frac{\partial w}{\partial \eta}\right)^2 + \frac{1}{2}(\alpha - \gamma)\frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta}. \tag{25.1.3.4}$$

Consider three cases.

1°. The right-hand side of the equation is independent of w and w_ξ if the coefficients α, β, and γ satisfy the conditions $\alpha + \beta + \gamma = 0$ and $\alpha - \gamma = 0$ so that

$$\beta = -2\alpha, \quad \gamma = \alpha, \quad \alpha \text{ is any number.}$$

Then the original equation (25.1.3.2) can be rewritten as

$$w_{xx} - w_{yy} = \alpha(w_x - w_y)^2$$

and the related equation (25.1.3.4) admits an order reduction with the substitution $u = w_\eta$.

2°. The right-hand side of the equation is independent of w and w_η if α, β, and γ satisfy $\alpha - \beta + \gamma = 0$ and $\alpha - \gamma = 0$ so that

$$\beta = 2\alpha, \quad \gamma = \alpha, \quad \alpha \text{ is any number.}$$

Then the original equation (25.1.3.2) can be rewritten as

$$w_{xx} - w_{yy} = \alpha(w_x + w_y)^2$$

and the related equation (25.1.3.4) admits an order reduction with the substitution $u = w_\xi$.

3°. If α, β, and γ satisfy the conditions $\alpha + \beta + \gamma = 0$ and $\alpha - \beta + \gamma = 0$, which means that

$$\beta = 0, \quad \gamma = -\alpha, \quad \alpha \text{ is any number,}$$

then the original equation (25.1.3.2) becomes

$$w_{xx} - w_{yy} = \alpha(w_x^2 - w_y^2) \tag{25.1.3.5}$$

and the related equation (25.1.3.4) will coincide, up to obvious renaming, with the integrable equation 7.5.2.2. Hence, the general solution to equation (25.1.3.4) is given by

$$w = -\frac{1}{\alpha}\ln[f(y + x) + g(y - x)],$$

where $f(z_1)$ and $g(z_2)$ are arbitrary functions.

Remark. Equation (25.1.3.5) is also integrable by quadrature for arbitrary $\alpha = \alpha(w)$, since the associated equation (25.1.3.4) coincides, up to obvious renaming, with equation 7.5.3.8.

The transformation

$$\xi = t + z, \qquad \eta = t - z$$

brings equation (25.1.3.1) to another canonical form,

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = F_3\left(t, z, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial z}\right), \tag{25.1.3.6}$$

where $F_3 = 4F_2$. This is the so-called *second canonical form of a hyperbolic equation*.

In the special case $F_3 = 0$, equation (25.1.3.6) is the classical linear wave equation.

25.1.4. Canonical Form of Elliptic Equations (Case $b^2 - ac < 0$)

In this case, the general integrals of equations (25.1.1.3) and (25.1.1.4) are complex conjugate; these determine two families of complex characteristics.

Let the general integral of equation (25.1.1.3) have the form

$$\varphi(x, y) + i\psi(x, y) = C, \qquad i^2 = -1,$$

where $\varphi(x, y)$ and $\psi(x, y)$ are real-valued functions.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y),$$

we reduce equation (25.1.1.1) to the canonical form

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = F_4\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right).$$

In the special case $F_4 = 0$, we have the linear Laplace equation.

25.2. Nonlinear Equations in Two Independent Variables

25.2.1. Nonlinear Equations of General Form

In general, a *second-order nonlinear partial differential equation* in two independent variables has the form

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{25.2.1.1}$$

Denote

$$a = \frac{\partial F}{\partial p}, \quad b = \frac{1}{2}\frac{\partial F}{\partial q}, \quad c = \frac{\partial F}{\partial r}, \quad \text{where } p = \frac{\partial^2 w}{\partial x^2}, \quad q = \frac{\partial^2 w}{\partial x \partial y}, \quad r = \frac{\partial^2 w}{\partial y^2}. \tag{25.2.1.2}$$

Let us select a specific solution $w = w(x, y)$ of equation (25.2.1.1) and calculate a, b, and c by formulas (25.2.1.2) at some point (x, y), and substitute the resulting expressions into (25.1.1.2). Depending on the sign of the discriminant δ, the type of nonlinear equation (25.2.1.1) at the point (x, y) is determined according to (25.1.1.3): if $\delta = 0$, the equation is

parabolic, if $\delta > 0$, it is hyperbolic, and if $\delta < 0$, it is elliptic. In general, the coefficients a, b, and c of the nonlinear equation (25.2.1.1) depend not only on the selection of the point (x, y), but also on the selection of the specific solution. Therefore, it is impossible to determine the sign of δ without knowing the solution $w(x, y)$. To put it differently, the type of a nonlinear equation can be different for different solutions at the same point (x, y).

A line $\varphi(x, y) = C$ is called a *characteristic* of the nonlinear equation (25.2.1.1) if it is an integral curve of the *characteristic equation*

$$a\,(dy)^2 - 2b\,dx\,dy + c\,(dx)^2 = 0. \tag{25.2.1.3}$$

The form of characteristics depends on the selection of a specific solution.

In individual special cases, the type of a nonlinear equation [other than the semilinear equation (25.1.1)] may be independent of the selection of solutions.

Example 25.3. Consider the nonhomogeneous *Monge–Ampère equation*

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x, y).$$

It is a special case of equation (25.2.1.1) with

$$F(x, y, p, q, r) \equiv q^2 - pr - f(x, y) = 0, \qquad p = \frac{\partial^2 w}{\partial x^2}, \quad q = \frac{\partial^2 w}{\partial x \partial y}, \quad r = \frac{\partial^2 w}{\partial y^2}. \tag{25.2.1.4}$$

Using formulas (25.2.1.2) and (25.2.1.4), we find the discriminant (25.1.1.2):

$$\delta = q^2 - pr = f(x, y). \tag{25.2.1.5}$$

Here, the relation between the highest derivatives and $f(x, y)$ defined by equation (25.2.1.4) has been taken into account.

From (25.2.1.5) and (25.1.1.3) it follows that the type of the nonhomogeneous equation Monge–Ampère at a point (x, y) depends solely on the sign of $f(x, y)$ and is independent of the selection of a particular solution. At the points where $f(x, y) = 0$, the equation is of parabolic type; at the points where $f(x, y) > 0$, the equation is of hyperbolic type; and at the points where $f(x, y) < 0$, it is elliptic.

Example 25.4. For the nonlinear equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial}{\partial y}\left[f(w)\frac{\partial w}{\partial y}\right] = 0$$

we have $F = p - f(w)r - f'_w(w)w_y^2 = 0$. Using formulas (25.2.1.2), we find $a = 1$, $b = 0$, and $c = -f(w)$. Calculating the discriminant (25.1.1.2) gives

$$\delta = f(w).$$

It is apparent that the discriminant sign at the point (x, y) is determined by the sign of $f(w(x, y))$ at this point (i.e., is generally dependent on the function f and solution $w(x, y)$). If $f(w) > 0$ for any values of w, the nonlinear equation in question is hyperbolic (a wave equation). If $f(w) < 0$ for any values of w, the equation is elliptic.

25.2.2. Quasilinear Equations

A *second-order quasilinear partial differential equation* in two independent variables has the form

$$a(x, y, w, \xi, \eta)p + 2b(x, y, w, \xi, \eta)q + c(x, y, w, \xi, \eta)r = f(x, y, w, \xi, \eta), \tag{25.2.2.1}$$

with the short notation

$$\xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}, \quad p = \frac{\partial^2 w}{\partial x^2}, \quad q = \frac{\partial^2 w}{\partial x \partial y}, \quad r = \frac{\partial^2 w}{\partial y^2}.$$

Consider a curve C_0 defined in the x, y plane parametrically as

$$x = x(\tau), \qquad y = y(\tau). \tag{25.2.2.2}$$

Let us fix a set of boundary conditions on this curve, thus defining the initial values of the unknown function and its first derivatives:

$$w = w(\tau), \qquad \xi = \xi(\tau), \qquad \eta = \eta(\tau) \qquad (w'_\tau = \xi x'_\tau + \eta y'_\tau). \tag{25.2.2.3}$$

The derivative with respect to τ is obtained by the chain rule, since $w = w(x, y)$. It can be shown that the given set of functions (25.2.2.3) uniquely determines the values of the second derivatives p, q, and r (and also higher derivatives) at each point of the curve (25.2.2.2), satisfying the condition

$$a(y'_x)^2 - 2by'_x + c \neq 0 \qquad (y'_x = y'_\tau / x'_\tau). \tag{25.2.2.4}$$

Here and henceforth, the arguments of the functions a, b, and c are omitted.

Indeed, bearing in mind that $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, let us differentiate the second and the third equation in (25.2.2.3) with respect to the parameter τ:

$$\xi'_\tau = px'_\tau + qy'_\tau, \qquad \eta'_\tau = qx'_\tau + ry'_\tau. \tag{25.2.2.5}$$

On solving relations (25.2.2.1) and (25.2.2.5) for p, q, and r, we obtain formulas for the second derivatives at the points of the curve (25.2.2.2):

$$\begin{aligned} p &= \frac{c(x'_\tau \xi'_\tau - y'_\tau \eta'_\tau) - 2by'_\tau \xi'_\tau + f(y'_\tau)^2}{a(y'_\tau)^2 - 2bx'_\tau y'_\tau + c(x'_\tau)^2}, \\ q &= \frac{ay'_\tau \xi'_\tau + cx'_\tau \eta'_\tau - fx'_\tau y'_\tau}{a(y'_\tau)^2 - 2bx'_\tau y'_\tau + c(x'_\tau)^2}, \\ r &= \frac{a(y'_\tau \eta'_\tau - x'_\tau \xi'_\tau) - 2bx'_\tau \eta'_\tau + f(x'_\tau)^2}{a(y'_\tau)^2 - 2bx'_\tau y'_\tau + c(x'_\tau)^2}. \end{aligned} \tag{25.2.2.6}$$

The third derivatives at the points of the curve (25.2.2.2) can be calculated in a similar way. To this end, one differentiates (25.2.2.1) and (25.2.2.6) with respect to τ and then expresses the third derivatives from the resulting relations. This procedure can also be extended to higher derivatives. Consequently, the solution to equation (25.2.2.1) can be represented as a Taylor series about the points of the curve (25.2.2.2) that satisfy condition (25.2.2.4).

The singular points at which the denominators in the formulas for the second derivatives (25.2.2.6) vanish satisfy the characteristic equation (25.2.1.3). Conditions of the form (25.2.2.3) cannot be arbitrarily set on the characteristic curves, which are described by equation (25.2.1.3). The additional conditions of vanishing of the numerators in formulas (25.2.2.6) must be used; in this case, the second derivatives will be finite.

⊙ *References for Chapter* 25: V. M. Babich, M. B. Kapilevich, S. G. Mikhlin et al. (1964), R. Courant and D. Hilbert (1989), A. N. Tikhonov and A. A. Samarskii (1990), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 26
Transformations of Equations of Mathematical Physics

26.1. Point Transformations: Overview and Examples

26.1.1. General Form of Point Transformations

Let $w = w(x, y)$ be a function of independent variables x and y. In general, a *point transformation* is defined by the formulas

$$x = X(\xi, \eta, u), \qquad y = Y(\xi, \eta, u), \qquad w = W(\xi, \eta, u), \tag{26.1.1.1}$$

where ξ and η are new independent variables, $u = u(\xi, \eta)$ is a new dependent variable, and the functions X, Y, W may be either given or unknown (have to be found).

A point transformation not only preserves the order or the equation to which it is applied, but also mostly preserves the structure of the equation, since the highest-order derivatives of the new variables are linearly dependent on the highest-order derivatives of the original variables.

Transformation (26.1.1.1) is invertible if

$$\begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial w} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial w} \end{vmatrix} \neq 0.$$

In the general case, a point transformation (26.1.1.1) reduces a second-order equation with two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{26.1.1.2}$$

to a second-order equation

$$G\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial^2 u}{\partial \xi^2}, \frac{\partial^2 u}{\partial \xi \partial \eta}, \frac{\partial^2 u}{\partial \eta^2}\right) = 0. \tag{26.1.1.3}$$

If $u = u(\xi, \eta)$ is a solution of equation (26.1.1.3), then formulas (26.1.1.1) define the corresponding solution of equation (26.1.1.2) in parametric form.

Point transformations are employed to simplify equations and their reduction to known equations.

26.1.2. Linear Transformations

Linear point transformations (or simply *linear transformations*),

$$x = X(\xi,\eta), \quad y = Y(\xi,\eta), \quad w = f(\xi,\eta)u + g(\xi,\eta), \tag{26.1.2.1}$$

are most commonly used.

The simplest linear transformations of the independent variables are

$$\begin{aligned}
&x = \xi + x_0, && y = \eta + y_0 && \text{(\textit{translation transformation})},\\
&x = k_1\xi, && y = k_2\eta && \text{(\textit{scaling transformation})},\\
&x = \xi\cos\alpha - \eta\sin\alpha, && y = \xi\sin\alpha + \eta\cos\alpha && \text{(\textit{rotation transformation})}.
\end{aligned}$$

These transformations correspond to the translation of the origin of coordinates to the point (x_0, y_0), scaling (extension or contraction) along the x- and y-axes, and the rotation of the coordinate system through the angle α, respectively. These transformations do not affect the dependent variable ($w = u$).

Linear transformations (26.1.2.1) are employed to simplify linear and nonlinear equations and to reduce equations to the canonical forms (see Section 25.1).

Example 26.1. The nonlinear equation

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m\frac{\partial w}{\partial x}\right) + \bigl[xf(t) + g(t)\bigr]\frac{\partial w}{\partial x} + h(t)w$$

can be simplified to obtain

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right)$$

with the help of the linear transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right].$$

26.1.3. Simple Nonlinear Point Transformations

Point transformations can be used for the reduction of some nonlinear equations to linear ones. The simplest nonlinear transformations have the form

$$w = W(u) \tag{26.1.3.1}$$

and do not affect the independent variables ($x = \xi$ and $y = \eta$). Combinations of transformations (26.1.2.1) and (26.1.3.1) are also used quite often.

Example 26.2. Consider the nonlinear equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2. \tag{26.1.3.2}$$

With the substitution (26.1.3.1), this equation becomes

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{1}{W'_u}\bigl[aW''_{uu} + b(W'_u)^2\bigr]\left(\frac{\partial u}{\partial x}\right)^2. \tag{26.1.3.3}$$

The function $W(u)$ is chosen so that the expression in square brackets vanishes:

$$aW''_{uu} + b(W'_u)^2 = 0. \tag{26.1.3.4}$$

Integrating (26.1.3.2) yields the solution

$$W(u) = \frac{a}{b}\ln(C_1 u) + C_2, \tag{26.1.3.5}$$

where C_1 and C_2 are arbitrary constants. Hence, the transformation determined by (26.1.3.1) and (26.1.3.5) reduces the original equation (26.1.3.2) to the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

⊙ *References for Section* 26.1: A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin and A. V. Manzhirov (2007).

26.2. Hodograph Transformations (Special Point Transformations)

In some cases, nonlinear equations and systems of partial differential equations can be simplified by means of the *hodograph transformations*, which are special cases of point transformations.

26.2.1. One PDE: One of the Independent Variables Is Taken to Be the Dependent One

For a partial differential equation with two independent variables x, t and an unknown function $w = w(x, t)$, the hodograph transformation consists of representing the solution in implicit form

$$x = x(w, t) \tag{26.2.1.1}$$

[or $t = t(x, w)$]. Thus, t and w are treated as independent variables, while x is taken to be the dependent variable. The hodograph transformation (26.2.1.1) does not change the order of the equation and belongs to the class of point transformations (equivalently, it can be represented as $x = \widetilde{w}$, $t = \widetilde{t}$, $w = \widetilde{x}$).

Differentiating relation (26.2.1.1) with respect to both variables as an implicit function and taking into account that $w = w(x, t)$, we get

$$\begin{aligned}
&1 = w_x x_w && \text{(differentiation in } x\text{)},\\
&0 = x_t + w_t x_w && \text{(differentiation in } t\text{)},\\
&0 = w_x^2 x_{ww} + w_{xx} x_w && \text{(double differentiation in } x\text{)},\\
&0 = w_t w_x x_{ww} + w_x x_{tw} + w_{tx} x_w && \text{(differentiation in } x \text{ and } t\text{)},\\
&0 = x_{tt} + 2w_t x_{tw} + w_t^2 x_{ww} + w_{tt} x_w && \text{(double differentiation in } t\text{)},
\end{aligned}$$

where the subscripts indicate the corresponding partial derivatives. We solve these relations to express the "old" derivatives through the "new" ones,

$$\begin{aligned}
&w_x = \frac{1}{x_w}, \quad w_t = -\frac{x_t}{x_w}, \quad w_{xx} = -\frac{w_x^2 x_{ww}}{x_w} = -\frac{x_{ww}}{x_w^3},\\
&w_{tx} = -\frac{x_w x_{tw} - x_t x_{ww}}{x_w^3}, \quad w_{tt} = \frac{-x_w^2 x_{tt} + 2x_t x_w x_{tw} - x_t^2 x_{ww}}{x_w^3},\\
&\frac{\partial}{\partial x} = w_x\frac{\partial}{\partial w} = \frac{1}{x_w}\frac{\partial}{\partial w}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + w_t\frac{\partial}{\partial w} = \frac{\partial}{\partial t} - \frac{x_t}{x_w}\frac{\partial}{\partial w}.
\end{aligned} \tag{26.2.1.2}$$

Example 26.3. The hodograph transformation (26.2.1.1) brings the first-order quasilinear equation

$$\frac{\partial w}{\partial t} = f(t,w)\frac{\partial w}{\partial x} + g(t,w) \tag{26.2.1.3}$$

to a linear equation,

$$\frac{\partial x}{\partial t} + g(t,w)\frac{\partial x}{\partial w} + f(t,w) = 0.$$

Remark. For $f = f(w)$ and $g = 0$, equation (26.2.1.3) is the model equation of gas dynamics (see Subsection 23.3.1).

Example 26.4. Consider the nonlinear second-order equation

$$\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2 = f(t,w)\frac{\partial^2 w}{\partial x^2}. \tag{26.2.1.4}$$

Let us seek its solution in implicit form (26.2.1.1). Substituting expressions (26.2.1.2) into (26.2.1.4), we obtain the following second-order linear equation:

$$\frac{\partial x}{\partial t} = f(t,w)\frac{\partial^2 x}{\partial w^2}.$$

Example 26.5. The equation of a steady-state boundary layer of a Newtonian fluid at a flat plate has the form

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \nu\frac{\partial^3 w}{\partial y^3}, \tag{26.2.1.5}$$

where w is the stream function, x and y are the coordinates along and across the flow, and ν is the kinematic viscosity of the fluid. In equation (26.2.1.5), let us perform the hodograph transformation; by doing so, we assume that $y = y(x,w)$. Using the formulas (26.2.1.2) with the renaming $x \Rightarrow y$ and $t \Rightarrow x$, we get

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = -\frac{y_{xw}}{y_w^3}, \qquad \frac{\partial^3 w}{\partial y^3} = -\frac{1}{y_w}\frac{\partial}{\partial w}\left(\frac{y_{ww}}{y_w^3}\right).$$

As a result, equation (26.2.1.5) becomes

$$\frac{y_{xw}}{y_w^2} = \nu\frac{\partial}{\partial w}\left(\frac{y_{ww}}{y_w^3}\right). \tag{26.2.1.6}$$

This equation is simpler than the original one, since it admits an order reduction with the obvious substitution $y_w = f(x,w)$:

$$\frac{1}{f^2}\frac{\partial f}{\partial x} = \nu\frac{\partial}{\partial w}\left(\frac{1}{f^3}\frac{\partial f}{\partial w}\right).$$

The substitution $f = 1/U$ leads to the nonlinear heat equation

$$\frac{\partial U}{\partial t} = \nu\frac{\partial}{\partial w}\left(U\frac{\partial U}{\partial w}\right).$$

26.2.2. One PDE: Method of Conversion to an Equivalent System of Equations

In order to investigate equations with the unknown function $w = w(x,t)$, it may be useful to convert the original equation to an equivalent system of equations for two functions

$$w = w(x,t), \qquad v = v(x,t) \tag{26.2.2.1}$$

(the elimination of v from the system results in the original equation). Now there are two possibilities, which are discussed below.

1°. *The first hodograph transformation: one of the independent variables is taken to be the dependent one.* The hodograph transformation consists of representing the solution in implicit form

$$x = x(w, t), \qquad v = v(w, t). \tag{26.2.2.2}$$

Thus, w and t are treated as independent variables, while x and v are taken to be the dependent variables. The hodograph transformation (26.2.2.2) does not change the order of the equation and belongs to the class of point transformations (equivalently, it can be represented as $x = \widetilde{w}$, $t = \widetilde{t}$, $w = \widetilde{x}$, $v = \widetilde{v}$).

From (26.2.2.1) and (26.2.2.2) we get

$$dw = w_x\,dx + w_t\,dt, \qquad dv = v_x\,dx + v_t\,dt; \tag{26.2.2.3}$$

$$dx = x_w\,dw + x_t\,dt, \qquad dv = v_w\,dw + v_t\,dt. \tag{26.2.2.4}$$

Solving (26.2.2.4) for dw and dv, we have

$$dw = \frac{1}{x_w}\,dx - \frac{x_t}{x_w}dt, \qquad dv = \frac{v_w}{x_w}\,dx + \left(v_t - \frac{x_t}{x_w}v_w\right)dt. \tag{26.2.2.5}$$

The comparison between (26.2.2.3) and (26.2.2.5) allows us to express the "old" derivatives in terms of the "new" ones

$$w_x = \frac{1}{x_w}, \quad w_t = -\frac{x_t}{x_w}, \quad v_x = \frac{v_w}{x_w}, \quad v_t = v_t - \frac{x_t}{x_w}v_w. \tag{26.2.2.6}$$

For the hodograph transformation (26.2.2.2), the highest derivatives of $w = w(x, t)$ are given in (26.2.1.2) and are independent of the auxiliary function v.

Remark 1. In formulas (26.2.2.2) and (26.2.2.6), the variables x and t can be swapped.

Remark 2. The first transformation (26.2.2.2) will also be called a *nonclassical hodograph transformation*.

Let us illustrate the hodograph transformation (26.2.2.2) by some examples.

Example 26.6. Consider the second-order nonlinear equation

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{f(w)}{w_x}\right] = 0, \qquad w_x = \frac{\partial w}{\partial x}. \tag{26.2.2.7}$$

Assuming that $v = f(w)/w_x$, let us represent equation (26.2.2.7) as an equivalent system of two equations

$$\frac{\partial w}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad v\frac{\partial w}{\partial x} = f(w). \tag{26.2.2.8}$$

Using the hodograph transformation (26.2.2.2) and taking into account (26.2.2.6), we get the linear system of equations

$$\frac{\partial x}{\partial t} - \frac{\partial v}{\partial w} = 0, \quad f(w)\frac{\partial x}{\partial w} = v.$$

Eliminating v yields the linear heat equation

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial w}\left[f(w)\frac{\partial x}{\partial w}\right]. \tag{26.2.2.9}$$

Remark. In equations (26.2.2.7)–(26.2.2.9), the function $f(w)$ can be replaced by an arbitrary function of two variables, $f(t, w)$.

Example 26.7. Consider the *Calogero type-equation*

$$\frac{\partial^2 v}{\partial x \partial t} = v f\left(\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x^2} + g\left(\frac{\partial v}{\partial x}\right). \tag{26.2.2.10}$$

By letting $w = v_x$, we rewrite equation (26.2.2.10) as the system

$$w = \frac{\partial v}{\partial x}, \qquad \frac{\partial w}{\partial t} = v f(w)\frac{\partial w}{\partial x} + g(w).$$

Using the hodograph transformation (26.2.2.2), we obtain the linear system of equations

$$w\frac{\partial x}{\partial w} = \frac{\partial v}{\partial w}, \qquad \frac{\partial x}{\partial t} + g(w)\frac{\partial x}{\partial w} + f(w)v = 0.$$

Remark. In equation (26.2.2.10), the functions $f(w)$ and $g(w)$ can be replaced by arbitrary functions of two variables, $f(t, w)$ and $g(t, w)$.

2°. *The second hodograph transformation: both independent variables are taken to be the dependent ones.* First, one rewrites the original equation with the unknown function $w = w(x, t)$ as an equivalent system of equations for two functions (26.2.2.1). Then one seeks a solution in implicit form using the hodograph transformation

$$x = x(w, v), \qquad t = t(w, v), \tag{26.2.2.11}$$

where w, v are treated as the independent variables and x, t as the dependent variables. The hodograph transformation (26.2.2.11) does not change the order of the equation and belongs to the class of point transformations (equivalently, it can be represented as $x = \widetilde{w}$, $t = \widetilde{v}$, $w = \widetilde{x}$, $v = \widetilde{t}$).

From (26.2.2.1) we get the relations (26.2.2.3) and from (26.2.2.11) we have

$$dx = x_w\, dw + x_v\, dv, \qquad dt = t_w\, dw + t_v\, dv. \tag{26.2.2.12}$$

Solving (26.2.2.12) for dw and dv, we get

$$dw = \frac{t_v}{J}\, dx - \frac{x_v}{J}\, dt, \qquad dv = -\frac{t_w}{J}\, dx + \frac{x_w}{J}\, dt, \qquad J = x_w t_v - x_v t_w. \tag{26.2.2.13}$$

The comparison between (26.2.2.3) and (26.2.2.13) allows us to express the "old" derivatives in terms of the "new" ones

$$\begin{gathered} w_x = \frac{t_v}{J}, \quad w_t = -\frac{x_v}{J}, \quad v_x = -\frac{t_w}{J}, \quad v_t = \frac{x_w}{J}, \\ J = x_w t_v - x_v t_w = \frac{1}{w_x v_t - w_t v_x}. \end{gathered} \tag{26.2.2.14}$$

Remark 3. The second transformation in (26.2.2.11) will also be called the *classical hodograph transformation* (since it is used more frequently and was suggested long before the nonclassical hodograph transformation).

Let us illustrate the classical hodograph transformation (26.2.2.11) by examples of specific equations of mathematical physics.

Example 26.8. Rewrite the *nonlinear wave equation*

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\Big(w\frac{\partial w}{\partial x}\Big) \tag{26.2.2.15}$$

as the system of equations

$$\frac{\partial w}{\partial t} = \frac{\partial v}{\partial x}, \quad aw\frac{\partial w}{\partial x} = \frac{\partial v}{\partial t}. \tag{26.2.2.16}$$

We now take advantage of the hodograph transformation (26.2.2.11), which amounts to taking w, v as the independent variables and x, t as the dependent variables. Using (26.2.2.14) to eliminate the derivatives w_x, w_t, v_x, v_t from (26.2.2.16), we arrive at the system

$$\frac{\partial x}{\partial v} = \frac{\partial t}{\partial w}, \quad aw\frac{\partial t}{\partial v} = \frac{\partial x}{\partial w}. \tag{26.2.2.17}$$

Let us differentiate the first equation in w and the second in v, and then eliminate the mixed derivative x_{wv}. As a result, we obtain the following linear equation for the function $t = t(w, v)$:

$$\frac{\partial^2 t}{\partial w^2} - aw\frac{\partial^2 t}{\partial v^2} = 0. \tag{26.2.2.18}$$

Similarly, from system (26.2.2.17), we obtain another linear equation for the function $x = x(w, v)$,

$$\frac{\partial^2 x}{\partial v^2} - \frac{\partial}{\partial w}\left(\frac{1}{aw}\frac{\partial x}{\partial w}\right) = 0. \tag{26.2.2.19}$$

Given a particular solution $t = t(w, v)$ of equation (26.2.2.18), we substitute this solution into system (26.2.2.17) and find $x = x(w, v)$ by straightforward integration. Eliminating v from (26.2.2.11), we obtain an exact solution $w = w(x, t)$ of the nonlinear equation (26.2.2.15).

Remark. Equation (26.2.2.18) with an arbitrary a admits a simple particular solution, namely,

$$t = C_1 wv + C_2 w + C_3 v + C_4, \tag{26.2.2.20}$$

where $C_1, \ldots, C_4$ are arbitrary constants. Substituting this solution into system (26.2.2.17), we obtain

$$\frac{\partial x}{\partial v} = C_1 v + C_2, \quad \frac{\partial x}{\partial w} = a(C_1 w + C_3)w. \tag{26.2.2.21}$$

Integrating the first equation in (26.2.2.21) yields $x = \frac{1}{2}C_1 v^2 + C_2 v + \varphi(w)$. Substituting this solution into the second equation in (26.2.2.21), we find the function $\varphi(w)$, and consequently

$$x = \tfrac{1}{2}C_1 v^2 + C_2 v + \tfrac{1}{3}aC_1 w^3 + \tfrac{1}{2}aC_3 w^2 + C_5. \tag{26.2.2.22}$$

Formulas (26.2.2.20) and (26.2.2.22) define an exact solution of equation (26.2.2.15) in parametric form (v is the parameter).

In a similar way, one can construct more complex solutions of equation (26.2.2.15) in parametric form. See also equation 7.4.4.6 with $f(w) = aw$.

Example 26.9. Consider the *Born–Infeld equation*

$$\left[1 - \left(\frac{\partial w}{\partial t}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x \partial t} - \left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial t^2} = 0, \tag{26.2.2.23}$$

which is used in nonlinear electrodynamics (field theory).

By introducing the new variables

$$\xi = x - t, \quad \eta = x + t, \quad u = \frac{\partial w}{\partial \xi}, \quad v = \frac{\partial w}{\partial \eta},$$

equation (26.2.2.23) can be rewritten as the equivalent system

$$\begin{aligned} \frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \xi} &= 0, \\ v^2\frac{\partial u}{\partial \xi} - (1 + 2uv)\frac{\partial u}{\partial \eta} + u^2\frac{\partial v}{\partial \eta} &= 0. \end{aligned}$$

The hodograph transformation, where u, v are taken to be the independent variables and ξ, η the dependent ones, leads to the linear system

$$\begin{aligned} \frac{\partial \xi}{\partial v} - \frac{\partial \eta}{\partial u} &= 0, \\ v^2\frac{\partial \eta}{\partial v} + (1 + 2uv)\frac{\partial \xi}{\partial v} + u^2\frac{\partial \xi}{\partial u} &= 0. \end{aligned} \tag{26.2.2.24}$$

Eliminating η yields the linear second-order equation

$$u^2\frac{\partial^2 \xi}{\partial u^2} + (1 + 2uv)\frac{\partial^2 \xi}{\partial u \partial v} + v^2\frac{\partial^2 \xi}{\partial v^2} + 2u\frac{\partial \xi}{\partial u} + 2v\frac{\partial \xi}{\partial v} = 0.$$

Assuming that the solution of interest is in the domain of hyperbolicity, we write out the equation of characteristics (see Section 25.1):

$$u^2\,dv^2 - (1+2uv)\,du\,dv + v^2\,du^2 = 0.$$

This equation has the integrals $r = C_1$ and $s = C_2$, where

$$r = \frac{\sqrt{1+4uv}-1}{2v}, \qquad s = \frac{\sqrt{1+4uv}-1}{2u}. \tag{26.2.2.25}$$

Passing in (26.2.2.24) to the new variables (26.2.2.25), we obtain

$$\begin{aligned} r^2\frac{\partial\xi}{\partial r} + \frac{\partial\eta}{\partial r} &= 0,\\ \frac{\partial\xi}{\partial s} + s^2\frac{\partial\eta}{\partial s} &= 0. \end{aligned} \tag{26.2.2.26}$$

Eliminating η yields the simple equation

$$\frac{\partial^2\xi}{\partial r\partial s} = 0,$$

whose general solution is the sum of two arbitrary functions with different arguments: $\xi = f(r) + g(s)$. The function η is determined from system (26.2.2.26).

26.2.3. System of Two PDEs: One of the Independent Variables Is Taken to Be the Dependent One

Consider systems consisting of two equations with the unknown functions $w = w(x,t)$ and $v = v(x,t)$. We use the first hodograph transformation (26.2.2.2):

$$x = x(w,t), \qquad v = v(w,t)$$

where w, t are treated as the independent variables and x, v as the dependent variables. The expressions of the partial derivatives are given in (26.2.2.6).

Example 26.10. The quasilinear system

$$\begin{aligned} f_1(t,w)\frac{\partial w}{\partial t} + [f_2(t,w) + f_3(t,w)x + f_4(t,w)v]\frac{\partial w}{\partial x} + f_5(t,w)\frac{\partial v}{\partial x} &= f_6(t,w),\\ g_1(t,w)\frac{\partial w}{\partial t} + [g_2(t,w) + g_3(t,w)x + g_4(t,w)v]\frac{\partial w}{\partial x} + g_5(t,w)\frac{\partial v}{\partial x} &= g_6(t,w) \end{aligned}$$

is reduced by the hodograph transformation (26.2.2.2) to the linear system

$$\begin{aligned} f_1(t,w)\frac{\partial x}{\partial t} + f_6(t,w)\frac{\partial x}{\partial w} - f_5(t,w)\frac{\partial v}{\partial w} &= f_2(t,w) + f_3(t,w)x + f_4(t,w)v,\\ g_1(t,w)\frac{\partial x}{\partial t} + g_6(t,w)\frac{\partial x}{\partial w} - g_5(t,w)\frac{\partial v}{\partial w} &= g_2(t,w) + g_3(t,w)x + g_4(t,w)v. \end{aligned}$$

26.2.4. System of Two PDEs: Both of the Independent Variables Are Taken to Be the Dependent Ones

Consider systems consisting of two equations with the unknown functions $w = w(x,t)$ and $v = v(x,t)$. We use the second hodograph transformation (26.2.2.11):

$$x = x(w,v), \qquad t = t(w,v),$$

where w, v are treated as the independent variables and x, t as the dependent variables. The expressions of the partial derivatives are given in (26.2.2.14).

Example 26.11. Consider the *system of gas dynamic type equations*

$$f_1(w,v)\frac{\partial w}{\partial t} + f_2(w,v)\frac{\partial w}{\partial x} + f_3(w,v)\frac{\partial v}{\partial t} + f_4(w,v)\frac{\partial v}{\partial x} = 0,$$
$$g_1(w,v)\frac{\partial w}{\partial t} + g_2(w,v)\frac{\partial w}{\partial x} + g_3(w,v)\frac{\partial v}{\partial t} + g_4(w,v)\frac{\partial v}{\partial x} = 0.$$

Treating w, v as the independent variables and x, t as the dependent ones, we arrive at the following system of linear equations (the calculations are similar to those of Example 26.8):

$$-f_1(w,v)\frac{\partial x}{\partial v} + f_2(w,v)\frac{\partial t}{\partial v} + f_3(w,v)\frac{\partial x}{\partial w} - f_4(w,v)\frac{\partial t}{\partial w} = 0,$$
$$-g_1(w,v)\frac{\partial x}{\partial v} + g_2(w,v)\frac{\partial t}{\partial v} + g_3(w,v)\frac{\partial x}{\partial w} - g_4(w,v)\frac{\partial t}{\partial w} = 0.$$

⊙ *References for Section* 26.2: N. E. Kochin, I. A. Kibel', and N. V. Roze (1963), B. L. Rozhdestvenskii and N. N. Yanenko (1983), A. M. Siddiqui, P. N. Kaloni, and O. P. Chandna (1985), G. G. Chernyi (1988), R. Courant and D. Hilbert (1989), P. A. Clarkson, A. S. Fokas, and M. J. Ablowitz (1989), H. C. Lei and H. W. Chang (1996), H. C. Lei (1998, 2002), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin and A. V. Manzhirov (2007).

26.3. Contact Transformations. Legendre and Euler Transformations

26.3.1. Preliminary Remarks. Contact Transformations for Ordinary Differential Equations

Prior to describing the contact transformations for partial differential equations, let us first consider their application to ordinary differential equations.

1°. Consider functions of a one variable, $y = y(x)$. In general, a contact transformation has the form

$$\begin{aligned} x &= F(X, Y, Y'_X), \\ y &= G(X, Y, Y'_X), \end{aligned} \tag{26.3.1.1}$$

where the functions $F(X,Y,U)$ and $G(X,Y,U)$ are chosen so that the derivative y'_x does not depend on Y''_{XX}:

$$y'_x = \frac{y'_X}{x'_X} = \frac{G_X + G_Y Y'_X + G_U Y''_{XX}}{F_X + F_Y Y'_X + F_U Y''_{XX}} = H(X, Y, Y'_X). \tag{26.3.1.2}$$

The subscripts X, Y, and U after F and G denote the respective partial derivatives (it is assumed that $F_U \not\equiv 0$ and $G_U \not\equiv 0$).

It follows from (26.3.1.2) that the relation

$$\frac{\partial G}{\partial U}\Big(\frac{\partial F}{\partial X} + U\frac{\partial F}{\partial Y}\Big) - \frac{\partial F}{\partial U}\Big(\frac{\partial G}{\partial X} + U\frac{\partial G}{\partial Y}\Big) = 0 \tag{26.3.1.3}$$

holds; the derivative is calculated by

$$y'_x = \frac{G_U}{F_U}, \tag{26.3.1.4}$$

where $G_U/F_U \not\equiv \text{const}$.

2°. Suppose the function $F = F(X, Y, U)$ in the contact transformation (26.3.1.1) is specified. Then relation (26.3.1.3) can be viewed as a linear partial differential equation for the second function G. The corresponding characteristic system of ordinary differential equations (see Subsection 23.1.1),

$$\frac{dX}{1} = \frac{dY}{U} = -\frac{F_U\,dU}{F_X + UF_Y},$$

admits the obvious first integral:

$$F(X, Y, U) = C_1, \tag{26.3.1.5}$$

where C_1 is an arbitrary constant. It follows that, to obtain the general representation of the function $G = G(X, Y, U)$, one has to deal with the ordinary differential equation

$$Y'_X = U, \tag{26.3.1.6}$$

whose right-hand side is defined in implicit form by (26.3.1.5). Let the first integral of equation (26.3.1.6) have the form

$$\Phi(X, Y, C_1) = C_2.$$

Then the general representation of $G = G(X, Y, U)$ in transformation (26.3.1.1) is given by:

$$G = \Psi(F, \widetilde{\Phi}),$$

where $\Psi(F, \widetilde{\Phi})$ is an arbitrary function of two variables, $F = F(X, Y, U)$, and $\widetilde{\Phi} = \Phi(X, Y, F)$.

The application of contact transformations preserves the order of differential equations. The inverse of a contact transformation can be obtained by solving system (26.3.1.1) and (26.3.1.4) for X, Y, and Y'_X.

3°. Below we consider some examples of contact transformations.

Example 26.12. Legendre transformation:

$$\begin{aligned} &x = Y'_X, \quad y = XY'_X - Y, \quad y'_x = X \quad \text{(direct transformation);} \\ &X = y'_x, \quad Y = xy'_x - y, \quad Y'_X = x \quad \text{(inverse transformation).} \end{aligned} \tag{26.3.1.7}$$

The second derivative is calculated as follows:

$$y''_{xx} = (y'_x)'_x = \frac{(y'_x)'_X}{x'_X} = \frac{1}{Y''_{XX}}.$$

The Legendre transformation (26.3.1.7) allows us to rewrite a general second-order ordinary differential equation

$$F(x,\, y,\, y'_x,\, y''_{xx}) = 0 \tag{26.3.1.8}$$

in the form

$$F\Big(Y'_X,\, XY'_X - Y,\, X,\, \frac{1}{Y''_{XX}}\Big) = 0. \tag{26.3.1.9}$$

Sometimes equation (26.3.1.9) may be simpler than (26.3.1.8). In particular, the nonlinear equation

$$y''_{xx} = \big[xf(y'_x) + yg(y'_x) + h(y'_x)\big]^{-1}$$

is reduced by the Legendre transformation (26.3.1.7), leading to the second-order linear equation

$$Y''_{XX} = [f(X) + Xg(X)]Y'_X - g(X)Y + h(X).$$

Example 26.13. Contact transformation ($a \neq 0$):

$$x = Y'_X + aY, \quad y = be^{aX}Y'_X, \qquad y'_x = be^{aX} \quad \text{(direct transformation);}$$
$$X = \frac{1}{a}\ln\frac{y'_x}{b}, \quad Y = \frac{1}{a}\left(x - \frac{y}{y'_x}\right), \quad Y'_X = \frac{y}{y'_x} \quad \text{(inverse transformation).}$$

Example 26.14. Contact transformation:

$$x = Y'_X + aX, \qquad y = \tfrac{1}{2}(Y'_X)^2 + aY, \qquad y'_x = Y'_X \quad \text{(direct transformation);}$$
$$X = \frac{1}{a}\left(x - y'_x\right), \quad Y = \frac{1}{2a}\left[2y - (y'_x)^2\right], \quad Y'_X = y'_x \quad \text{(inverse transformation).}$$

Example 26.15. Contact transformation:

$$x = (Y'_X)^2 + Y^2, \quad y = Y'_X\cos X + Y\sin X, \quad y'_x = \frac{\cos X}{2Y'_X}.$$

26.3.2. General Form of Contact Transformations for Partial Differential Equations

Consider functions of two variables $w = w(x, y)$. A common property of contact transformations is the dependence of the original variables on the new variables and their first derivatives:

$$x = X\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \quad y = Y\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \quad w = W\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{26.3.2.1}$$

The functions X, Y, and W cannot be arbitrary and are selected so as to ensure that the first derivatives of the original variables depend only on the transformed variables and, possibly, their first derivatives,

$$\frac{\partial w}{\partial x} = U\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \qquad \frac{\partial w}{\partial y} = V\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{26.3.2.2}$$

Contact transformations (26.3.2.1)–(26.3.2.2) do not increase the order of the equations to which they are applied.

We now outline the procedure for finding the functions U and V in (26.3.2.2) and the relations that must hold for the functions X, Y, and W in (26.3.2.1).

Let us differentiate the first and second expressions in (26.3.2.1) with respect to x and y as composite functions taking into account that $u = u(\xi, \eta)$. Thus, we obtain the following four relations:

$$\begin{aligned}
&\left(\frac{\partial X}{\partial \xi} + \frac{\partial X}{\partial u}p + \frac{\partial X}{\partial p}p_\xi + \frac{\partial X}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial x} + \left(\frac{\partial X}{\partial \eta} + \frac{\partial X}{\partial u}q + \frac{\partial X}{\partial p}q_\xi + \frac{\partial X}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial x} = 1,\\
&\left(\frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial u}p + \frac{\partial Y}{\partial p}p_\xi + \frac{\partial Y}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial x} + \left(\frac{\partial Y}{\partial \eta} + \frac{\partial Y}{\partial u}q + \frac{\partial Y}{\partial p}q_\xi + \frac{\partial Y}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial x} = 0,\\
&\left(\frac{\partial X}{\partial \xi} + \frac{\partial X}{\partial u}p + \frac{\partial X}{\partial p}p_\xi + \frac{\partial X}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial y} + \left(\frac{\partial X}{\partial \eta} + \frac{\partial X}{\partial u}q + \frac{\partial X}{\partial p}q_\xi + \frac{\partial X}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial y} = 0,\\
&\left(\frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial u}p + \frac{\partial Y}{\partial p}p_\xi + \frac{\partial Y}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial y} + \left(\frac{\partial Y}{\partial \eta} + \frac{\partial Y}{\partial u}q + \frac{\partial Y}{\partial p}q_\xi + \frac{\partial Y}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial y} = 1,
\end{aligned} \tag{26.3.2.3}$$

where $p = \frac{\partial u}{\partial \xi}$, $q = \frac{\partial u}{\partial \eta}$, and $p_\eta = q_\xi$; the subscripts ξ and η denote the corresponding partial derivatives. The first two relations in (26.3.2.3) constitute a system of linear algebraic equations for $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$, and the other two relations form a system of linear algebraic equations for $\frac{\partial \xi}{\partial y}$ and $\frac{\partial \eta}{\partial y}$. Having solved these systems, we find the derivatives: $\frac{\partial \xi}{\partial x} = A$, $\frac{\partial \eta}{\partial x} = B$, $\frac{\partial \xi}{\partial y} = C$, $\frac{\partial \eta}{\partial y} = D$. Then, differentiating the third relation in (26.3.2.1) with respect to x and y, we express $U = \frac{\partial w}{\partial x}$ and $V = \frac{\partial w}{\partial y}$ in terms of the new variables to obtain

$$U = A\left(\frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial u}p + \frac{\partial W}{\partial p}p_\xi + \frac{\partial W}{\partial q}p_\eta\right) + B\left(\frac{\partial W}{\partial \eta} + \frac{\partial W}{\partial u}q + \frac{\partial W}{\partial p}q_\xi + \frac{\partial W}{\partial q}q_\eta\right),$$
$$V = C\left(\frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial u}p + \frac{\partial W}{\partial p}p_\xi + \frac{\partial W}{\partial q}p_\eta\right) + D\left(\frac{\partial W}{\partial \eta} + \frac{\partial W}{\partial u}q + \frac{\partial W}{\partial p}q_\xi + \frac{\partial W}{\partial q}q_\eta\right).$$

Relations (26.3.2.2) require that U and W should be independent of the second derivatives, i.e.,

$$\frac{\partial U}{\partial p_\xi} = \frac{\partial V}{\partial p_\xi} = \frac{\partial U}{\partial p_\eta} = \frac{\partial V}{\partial p_\eta} = \frac{\partial U}{\partial q_\eta} = \frac{\partial V}{\partial q_\eta} = 0 \quad (p_\eta \equiv q_\xi),$$

which results in additional relations for the functions X, Y, W.

In general, a contact transformation (26.3.2.1)–(26.3.2.2) reduces a second-order equation in two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{26.3.2.4}$$

to an equation of the form

$$G\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial^2 u}{\partial \xi^2}, \frac{\partial^2 u}{\partial \xi \partial \eta}, \frac{\partial^2 u}{\partial \eta^2}\right) = 0. \tag{26.3.2.5}$$

In some cases, equation (26.3.2.5) turns out to be more simple than (26.3.2.4). If $u = u(\xi, \eta)$ is a solution of equation (26.3.2.5), then formulas (26.3.2.1) define the corresponding solution of equation (26.3.2.4) in parametric form.

Example 26.16. Consider the equation

$$a\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} = f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}, \qquad a \neq 0.$$

The contact transformation

$$\bar{t} = t + a\ln|w_x|, \quad \bar{x} = w + aw_t, \quad \bar{w} = x + aw_t/w_x, \quad \bar{w}_{\bar{x}} = 1/w_x, \quad \bar{w}_{\bar{t}} = -w_t/w_x \tag{26.3.2.6}$$

leads to an equation of the similar form

$$a\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial \bar{w}}{\partial \bar{t}} = F\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}, \qquad \text{where} \qquad F(u) = \frac{1}{u^2} f\left(\frac{1}{u}\right).$$

Transformation (26.3.2.6) has an inverse; it is given by

$$t = \bar{t} + a\ln|\bar{w}_{\bar{x}}|, \quad x = \bar{w} + a\bar{w}_{\bar{t}}, \quad w = \bar{x} + a\bar{w}_{\bar{t}}/\bar{w}_{\bar{x}}, \quad w_x = 1/\bar{w}_{\bar{x}}, \quad w_t = -\bar{w}_{\bar{t}}/\bar{w}_{\bar{x}}. \tag{26.3.2.7}$$

The formulas of (26.3.2.7) can be used if the Jacobian function $J = \left[(\bar{w}_{\bar{x}} + a\bar{w}_{\bar{x}\bar{t}})^2 - a\bar{w}_{\bar{x}\bar{x}}(\bar{w}_{\bar{t}} + a\bar{w}_{\bar{t}\bar{t}})\right]$ is nonzero.

For the special case $f(w_x) = b(w_x)^{-2}$, transformation (26.3.2.6) leads to the linear telegraph equation

$$a\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + \frac{\partial \bar{w}}{\partial \bar{t}} = b\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}.$$

26.3.3. Legendre Transformation

An important special case of contact transformations is the *Legendre transformation* defined by the relations

$$x = \frac{\partial u}{\partial \xi}, \quad y = \frac{\partial u}{\partial \eta}, \quad w = x\xi + y\eta - u, \tag{26.3.3.1}$$

where u is the new dependent variable and ξ, η are the new independent variables.

Differentiating the last relation in (26.3.2.6) with respect to x and y and taking into account the other two relations, we obtain the first derivatives:

$$\frac{\partial w}{\partial x} = \xi, \quad \frac{\partial w}{\partial y} = \eta. \tag{26.3.3.2}$$

With (26.3.2.6)–(26.3.2.7), we find the second derivatives

$$\frac{\partial^2 w}{\partial x^2} = J\frac{\partial^2 u}{\partial \eta^2}, \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = -J\frac{\partial^2 u}{\partial \xi \partial \eta}, \quad \frac{\partial^2 w}{\partial y^2} = J\frac{\partial^2 u}{\partial \xi^2},$$

where

$$J = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2, \quad \frac{1}{J} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial^2 u}{\partial \eta^2} - \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)^2.$$

The Legendre transformation (26.3.2.6), with $J \neq 0$, allows us to rewrite a general second-order equation with two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{26.3.3.3}$$

in the form

$$F\left(\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta} - u, \xi, \eta, J\frac{\partial^2 u}{\partial \eta^2}, -J\frac{\partial^2 u}{\partial \xi \partial \eta}, J\frac{\partial^2 u}{\partial \xi^2}\right) = 0. \tag{26.3.3.4}$$

Sometimes equation (26.3.3.4) may be simpler than (26.3.3.3).

Let $u = u(\xi, \eta)$ be a solution of equation (26.3.3.4). Then the formulas (26.3.3.1) define the corresponding solution of equation (26.3.3.3) in parametric form.

Remark. The Legendre transformation may result in the loss of solutions for which $J = 0$.

Example 26.17. The equation of steady-state transonic gas flow

$$a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

is reduced by the Legendre transformation (26.3.3.1) to the linear equation with variable coefficients

$$a\xi\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2} = 0.$$

Example 26.18. The Legendre transformation (26.3.3.1) reduces the nonlinear equation

$$f\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x^2} + g\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y} + h\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial y^2} = 0$$

to the following linear equation with variable coefficients:

$$f(\xi, \eta)\frac{\partial^2 u}{\partial \eta^2} - g(\xi, \eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + h(\xi, \eta)\frac{\partial^2 u}{\partial \xi^2} = 0.$$

26.3.4. Euler Transformation

The *Euler transformation* belongs to the class of contact transformations and is defined by the relations

$$x = \frac{\partial u}{\partial \xi}, \quad y = \eta, \quad w = x\xi - u. \tag{26.3.4.1}$$

Differentiating the last relation in (26.3.4.1) with respect to x and y and taking into account the other two relations, we find that

$$\frac{\partial w}{\partial x} = \xi, \quad \frac{\partial w}{\partial y} = -\frac{\partial u}{\partial \eta}. \tag{26.3.4.2}$$

Differentiating these expressions in x and y, we find the second derivatives:

$$w_{xx} = \frac{1}{u_{\xi\xi}}, \quad w_{xy} = -\frac{u_{\xi\eta}}{u_{\xi\xi}}, \quad w_{yy} = \frac{u_{\xi\eta}^2 - u_{\xi\xi}u_{\eta\eta}}{u_{\xi\xi}}. \tag{26.3.4.3}$$

The subscripts indicate the corresponding partial derivatives.

The Euler transformation (26.3.4.1)–(26.3.4.3) is employed in finding solutions and linearization of certain nonlinear partial differential equations.

The Euler transformation (26.3.4.1) allows us to reduce a general second-order equation with two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{26.3.4.4}$$

to the equation

$$F\left(u_\xi, \eta, \xi u_\xi - u, \xi, -u_\eta, \frac{1}{u_{\xi\xi}}, -\frac{u_{\xi\eta}}{u_{\xi\xi}}, \frac{u_{\xi\eta}^2 - u_{\xi\xi}u_{\eta\eta}}{u_{\xi\xi}}\right) = 0. \tag{26.3.4.5}$$

In some cases, equation (26.3.4.5) may become simpler than equation (26.3.4.4).

Let $u = u(\xi, \eta)$ be a solution of equation (26.3.4.5). Then formulas (26.3.4.1) define the corresponding solution of equation (26.3.4.4) in parametric form.

Example 26.19. The equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} = f\left(y, \frac{\partial w}{\partial x}\right) \tag{26.3.4.6}$$

can be linearized with the help of the Euler transformation (26.3.4.1)–(26.3.4.3) to obtain

$$\frac{\partial u}{\partial \eta} = -f(\eta, \xi)\frac{\partial^2 u}{\partial \xi^2}.$$

Example 26.20. The equation

$$\frac{\partial^2 w}{\partial x \partial y} = f\left(y, \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}$$

can be linearized by the Euler transformation (26.3.4.1)–(26.3.4.3) to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = f(\eta, \xi)\frac{\partial u}{\partial \eta}.$$

This equation is reducible to a first-order linear equation with the substitution $v = \frac{\partial u}{\partial \eta}$.

26.3.5. Legendre Transformation with Many Variables

For a function of many variables $w = w(x_1, \ldots, x_n)$, the Legendre transformation and its inverse are defined as

Legendre transformation	*Inverse Legendre transformation*
$x_1 = X_1, \quad \ldots, \quad x_{k-1} = X_{k-1},$	$X_1 = x_1, \quad \ldots, \quad X_{k-1} = x_{k-1},$
$x_k = \dfrac{\partial W}{\partial X_k}, \quad \ldots, \quad x_n = \dfrac{\partial W}{\partial X_n},$	$X_k = \dfrac{\partial w}{\partial x_k}, \quad \ldots, \quad X_n = \dfrac{\partial w}{\partial x_n},$
$w(\mathbf{x}) = \sum_{i=k}^{n} X_i \dfrac{\partial W}{\partial X_i} - W(\mathbf{X}),$	$W(\mathbf{X}) = \sum_{i=k}^{n} x_i \dfrac{\partial w}{\partial x_i} - w(\mathbf{x}),$

where $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{X} = \{X_1, \ldots, X_n\}$, and the derivatives are related by

$$\frac{\partial w}{\partial x_1} = -\frac{\partial W}{\partial X_1}, \quad \ldots, \quad \frac{\partial w}{\partial x_{k-1}} = -\frac{\partial W}{\partial X_{k-1}}.$$

⊙ *References for Section* 26.3: M. G. Kurenskii (1934), E. Kamke (1977), N. H. Ibragimov (1985, 1994), S. R. Svirshchevskii (1986, 1988), D. Zwillinger (1989), E. Pucci and G. Saccomandi (1994), B. J. Cantwell (2002), A. D. Polyanin and V. F. Zaitsev (2003, 2004), S. V. Meleshko (2005), A. D. Polyanin and A. V. Manzhirov (2007).

26.4. Differential Substitutions. Von Mises Transformation

26.4.1. Differential Substitutions

In mathematical physics, apart from point transformations, contact transformations, and the Bäcklund transformations, one sometimes resorts to the so-called *differential substitutions*. Usually differential substitutions have the form

$$w = \Psi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

A differential substitution increases the order of an equation (if it is inserted into an equation for w) and allows us to obtain solutions of one equation from those of another. The relationship between the solutions of the two equations is generally not invertible and is, in a sense, unilateral. A differential substitution may decrease the order of an equation (if it is inserted into an equation for u).

Example 26.21. Consider the *Burgers equation*

$$\frac{\partial w}{\partial t} = w\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}. \tag{26.4.1.1}$$

Let us make the differential substitution (the *Hopf–Cole transformation*)

$$w = \frac{2u_x}{u}. \tag{26.4.1.2}$$

Substituting (26.4.1.2) into (26.4.1.1), we obtain the third-order equation

$$\frac{2u_{tx}}{u} - \frac{2u_t u_x}{u^2} = \frac{2u_{xxx}}{u} - \frac{2u_x u_{xx}}{u^2},$$

which can be converted to

$$\frac{\partial}{\partial x}\left[\frac{1}{u}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)\right] = 0. \tag{26.4.1.3}$$

Thus, using formula (26.4.1.2), one can transform each solution of the linear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{26.4.1.4}$$

into a solution of the Burgers equation (26.4.1.1). The converse is not generally true. Indeed, it follows from (26.4.1.3) that a solution of equation (26.4.1.1) generates a solution of the more general equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t)u,$$

where $f(t)$ is a function of t.

Example 26.22. Consider the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + 6w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0. \tag{26.4.1.5}$$

Let us make the *Miura transformation* (a differential substitution)

$$w = u_x - u^2. \tag{26.4.1.6}$$

This results in the fourth-order equation

$$\begin{gathered}\frac{\partial}{\partial x}M[u] - 2uM[u] = 0,\\ M[u] \equiv \frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}.\end{gathered} \tag{26.4.1.7}$$

It follows from (26.4.1.6)–(26.4.1.7) that any solution $u = u(x, t)$ to the modified Korteweg–de Vries equation

$$M[u] \equiv \frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{26.4.1.8}$$

generates, with the use of (26.4.1.6), an exact solution to the Korteweg–de Vries equation (26.4.1.5).

In the above examples, the differential substitutions (26.4.1.2) and (26.4.1.6) increased the order of equations (cf. the orders of (26.4.1.1) and (26.4.1.3) and those of (26.4.1.5) and (26.4.1.7)) on the one hand and, on the other hand, allowed us to obtain exact solutions to some equations using solutions to other equations of the same order (cf. the orders of (26.4.1.1) and (26.4.1.4), and those of (26.4.1.5) and (26.4.1.8)).

In Subsection 26.4.2, a differential substitution will be considered that reduces the order of some nonlinear equations of mathematical physics.

Remark 1. A differential substitution may be obtained as a consequence of a Bäcklund transformation (although this is not always the case); see Examples 26.29 and 26.32 in Subsection 26.5.1.

Remark 2. In general, differential substitutions are defined by formulas (26.3.2.1), where the function X, Y, and W can be defined arbitrarily.

26.4.2. Von Mises Transformation

An important special case of differential substitutions is the *von Mises transformation*, which is used in fluid dynamics to reduce the order of boundary layer equations. The von Mises transformation is the one in which the unknown function is treated as a new independent variable and a suitable partial derivative as the new dependent variable.

1°. Prior to describing the von Mises transformation for partial differential equations, let us first consider a simple analogue of this transformation that frequently arises in the theory of ordinary differential equations.

Let $w = w(x)$ be a function of one variable. The nth-order autonomous ordinary differential equation

$$F(w, w'_x, w''_{xx}, \ldots, w^{(n)}_x) = 0, \tag{26.4.2.1}$$

which does not depend on x explicitly, can be reduced, with the differential substitution

$$\eta(w) = w'_x \qquad (\textit{analogue of the von Mises transformation}), \tag{26.4.2.2}$$

to an $(n-1)$st-order equation for $\eta = \eta(w)$ where the derivatives with respect to x must be substituted by derivatives with respect to w using the formula $\eta^{(k)}_x = \eta(\eta^{(k-1)}_x)_w$.

Remark 1. Equation (26.4.2.1) has the characteristic property that it is invariant under translation in the independent variable $x = \bar{x} + C$, where C is an arbitrary constant.

2°. Now consider a nonlinear nth-order partial differential equation of the special form

$$\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial t\partial x}-\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x^2}=F\left(t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2},\dots,\frac{\partial^n w}{\partial x^n}\right). \tag{26.4.2.3}$$

If w and F are independent of t, then equation (26.4.2.3) becomes (26.4.2.1).

In equation (26.4.2.3), let us change from the old variables to the von Mises variables (analogue of transformation (26.4.2.2)):

$$t,\ x,\ w=w(t,x)\quad\Longrightarrow\quad t,\ w,\ \eta=\eta(t,w),\quad\text{where}\quad \eta=\frac{\partial w}{\partial x}. \tag{26.4.2.4}$$

The derivatives are transformed as follows:

$$\begin{gathered}\frac{\partial}{\partial x}=\frac{\partial w}{\partial x}\frac{\partial}{\partial w}=\eta\frac{\partial}{\partial w},\quad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\frac{\partial w}{\partial t}\frac{\partial}{\partial w},\\ \frac{\partial^2 w}{\partial x^2}=\eta\frac{\partial\eta}{\partial w},\ \frac{\partial^3 w}{\partial x^3}=\eta\frac{\partial}{\partial w}\left(\eta\frac{\partial\eta}{\partial w}\right),\ \dots,\ \frac{\partial^2 w}{\partial t\partial x}=\frac{\partial\eta}{\partial t}+\frac{\partial w}{\partial t}\frac{\partial\eta}{\partial w}.\end{gathered} \tag{26.4.2.5}$$

Hence, the von Mises transformation (26.4.2.4) reduces the nth-order equation (26.4.2.3) to the $(n-1)$st-order equation

$$\eta\frac{\partial\eta}{\partial t}=F\left(t,w,\eta,\eta\frac{\partial\eta}{\partial w},\dots,\frac{\partial^{n-1}\eta}{\partial x^{n-1}}\right), \tag{26.4.2.6}$$

where the third- and higher-order derivatives are calculated by the formulas

$$\frac{\partial^k w}{\partial x^k}=\frac{\partial^{k-1}\eta}{\partial x^{k-1}}=\eta\frac{\partial}{\partial w}\frac{\partial^{k-2}\eta}{\partial x^{k-2}},\quad \frac{\partial}{\partial x}=\eta\frac{\partial}{\partial w},\quad k=3,\dots,n.$$

Remark 2. Equation (26.4.2.3) has the following characteristic property. Suppose $\widetilde{w}(t,x)$ is a solution of the equation in question. Then the function

$$w(t,x)=\widetilde{w}\big(t,x+\varphi(t)\big),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

Example 26.23. The nonlinear second-order equation (26.4.2.3) with $F=f(t,w)w_{xx}+g(t,w)w_x^2+h(t,w)w_x$ is transformed into an equation (26.4.2.6) of special form. By canceling by η, one can reduce the latter to a linear first-order equation:

$$\eta_t-f(t,w)\eta_w=g(t,w)\eta+h(t,w).$$

Example 26.24. The nonlinear second-order equation (26.4.2.3) with $F=f(t,w)w_{xx}+g(t,w)w_x^2+h(t,w)$ is transformed into an equation (26.4.2.6) of special form. With the substitution $Z=\eta^2$, the latter can be reduced to a linear first-order equation.

Example 26.25. The equation of a steady-state laminar hydrodynamic boundary layer on a flat plate has the form (see Schlichting, 1981)

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=a\frac{\partial^3 w}{\partial y^3}, \tag{26.4.2.7}$$

where w is the stream function, x and y are the coordinates along and across the flow, and a is the kinematic viscosity of the fluid. Performing the renaming $x\Rightarrow t$ and $y\Rightarrow x$ in (26.4.2.7), one obtains a special case of equation (26.4.2.3).

The von Mises transformation

$$x,\ y,\ w=w(x,y)\quad\Longrightarrow\quad x,\ w,\ \eta=\eta(x,w),\quad\text{where}\quad \eta=\frac{\partial w}{\partial y},$$

decreases the order of equation (26.4.2.7) and brings it to the simpler nonlinear heat equation

$$\frac{\partial \eta}{\partial x} = a\frac{\partial}{\partial w}\Big(\eta\frac{\partial \eta}{\partial w}\Big).$$

Example 26.26. For $F = f(t,w)w_{xxx}/w_x$, equation (26.4.2.6) is reducible, with the substitution $Z = \eta^2$, to a linear second-order equation. For $F = aw_x^3 w_{xxx}$, the equation is reducible, with the same substitution, to the equation $Z_t = aZ^2 Z_{ww}$, which can reduced to the linear heat equation; see equation 5.1.9.13.

Any solution $w = g(t,x)$ to equation (26.4.2.3) determines a solution to equation (26.4.2.6), which can be written in the parametric form $w = g(t,x)$, $\eta = g_x(t,x)$.

Suppose $\eta = \eta(t,w)$ is a solution to equation (26.4.2.6). Then the function $w = w(t,x)$ is determined from the first-order equation $w_x = \eta(t,w)$, the integration of which gives a solution in implicit form:

$$\int_{w_0}^{w} \frac{du}{\eta(t,u)} = x + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function and w_0 is an arbitrary constant. Thus, the inverse von Mises transformation involves the nonlocal operation of integration of a function that depends on the solution. Such transformations are called *nonlocal*. In the general case, nonlocal transformations can be defined by differential equations.

3°. Equation (26.4.2.3) can be generalized by adding $I_x, \dots, I_x^{(m)}$, with $I = u_x u_{tx} - u_t u_{xx}$, as extra arguments of the function F.

4°. The von Mises transformation (26.4.2.4) reduces the order of the equation

$$w_x G_t - w_t G_x = F(t,w,w_x,\dots,w_x^{(n)}), \qquad G = G(t,w,w_x,\dots,w_x^{(m)}), \tag{26.4.2.8}$$

by taking it to the form (26.4.2.6), where η_t must be replaced by G_t. For $G = w_x$, equation (26.4.2.8) coincides with (26.4.2.3).

Example 26.27. The nonlinear third-order equation

$$w_x w_{txx} - w_t w_{xxx} = f(t,w)w_x$$

is a special case of equation (26.4.2.8) with $G = w_{xx}$ and $F = f(t,w)w_x$. The von Mises transformation (26.4.2.4) reduces it to the second-order equation $(\eta\eta_w)_t = f(t,w)$, whose general solution is given by

$$\eta^2 = 2\int_{t_0}^{t}\int_{w_0}^{w} f(\tau,\xi)\,d\xi\,d\tau + \varphi(t) + \psi(w),$$

where $\varphi(t)$, $\psi(w)$ are arbitrary functions and t_0, w_0 are arbitrary constants.

Remark 3. A more complex expression $H_x G_t - H_t G_x$ with $H = H(t,w,w_x,\dots,w_x^{(k)})$ can replace $w_x G_t - w_t G_x$ on the left-hand side of equation (26.4.2.8).

5°. The von Mises transformation (26.4.2.4) can be generalized to the case of greater number of independent variables.

Consider the nonlinear nth-order equation n

$$F(t,x,w,w_y,w_{yy},\dots,w_y^{(n)},w_y w_{ty} - w_t w_{yy}, w_y w_{xy} - w_x w_{yy}) = 0, \tag{26.4.2.9}$$

which involves three independent variables t, x, and y. The von Mises transformation

$$t,\ x,\ y,\ w = w(t,x,y) \quad \Longrightarrow \quad t,\ x,\ w,\ \eta = \eta(t,x,w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial y}, \tag{26.4.2.10}$$

reduces the order of equation (26.4.2.9) and takes it to the form

$$F(t,x,w,\eta,\eta\eta_w,\dots,\eta_y^{(n-1)},\eta\eta_t,\eta\eta_x) = 0, \tag{26.4.2.11}$$

where the derivatives with respect to y are calculated successively by the formula $\eta_y^{(k)} = \eta\big(\eta_y^{(k-1)}\big)_w$.

Example 26.28. Consider the Prandtl system

$$\begin{aligned} u_t + uu_x + vu_y &= \nu u_{yy} + f(t,x), \\ u_x + v_y &= 0, \end{aligned} \tag{26.4.2.12}$$

which describes a flat unsteady boundary layer with pressure gradient, with u and v being the fluid velocity components. Dividing the equation by u_y, differentiating with respect to y, and eliminating v_y using the second equation in (26.4.2.12), we arrive at a third-order equation of the form (26.4.2.9):

$$u_y u_{ty} - u_t u_{yy} + u(u_y u_{xy} - u_x u_{yy}) = \nu(u_y u_{yyy} - u_{yy}^2) - f(t,x)u_{yy}. \tag{26.4.2.13}$$

Applying the von Mises transformation (26.4.2.10) with $w \equiv u$, we obtain the second-order equation

$$\eta_t + u\eta_x + f(t,x)\eta_u = \nu\eta^2\eta_{uu}. \tag{26.4.2.14}$$

Once a solution of this equation has been found, the second velocity component v can be obtained directly from the first equation in (26.4.2.12) without integration.

It will be shown in Subsection 26.5.2 that the von Mises transformation (26.4.2.4) can successfully be used for constructing RF pairs and Bäcklund transformations for evolution equations of general form.

⊙ *References for Section* 26.4: H. Schlichting (1981), L. G. Loitsyanskiy (1996), A. D. Polyanin and V. F. Zaitsev (2003, 2004), G. Saccomandi (2004), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin and A. I. Zhurov (2009 *a*), A. D. Polyanin (2010).

26.5. Bäcklund Transformations. RF Pairs

26.5.1. Bäcklund Transformations for Second-Order Equations

Let $w = w(x,y)$ be a solution of the equation

$$F_1\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{26.5.1.1}$$

and let $u = u(x,y)$ be a solution of another equation

$$F_2\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0. \tag{26.5.1.2}$$

Equations (26.5.1.1) and (26.5.1.2) are said to be related by the *Bäcklund transformation*

$$\begin{aligned} \Phi_1\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) &= 0, \\ \Phi_2\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) &= 0 \end{aligned} \tag{26.5.1.3}$$

if the compatibility of the pair (26.5.1.1), (26.5.1.3) implies equation (26.5.1.2), and the compatibility of the pair (26.5.1.2), (26.5.1.3) implies (26.5.1.1). If, for some specific solution $u = u(x,y)$ of equation (26.5.1.2), one succeeds in solving equations (26.5.1.3) for $w = w(x,y)$, then this function $w = w(x,y)$ will be a solution of equation (26.5.1.1).

Bäcklund transformations may preserve the form of equations* (such transformations are used for obtaining new solutions) or establish relations between solutions of different equations (such transformations are used for obtaining solutions of one equation from solutions of another equation).

* In such cases, these are referred to as *auto-Bäcklund transformations*.

Example 26.29. The *Burgers equation*

$$\frac{\partial w}{\partial t} = w\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \tag{26.5.1.4}$$

is related to the linear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{26.5.1.5}$$

by the Bäcklund transformation

$$\begin{aligned} &\frac{\partial u}{\partial x} - \frac{1}{2}uw = 0, \\ &\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial (uw)}{\partial x} = 0. \end{aligned} \tag{26.5.1.6}$$

Eliminating w from (26.5.1.6), we obtain equation (26.5.1.5).

Conversely, let $u(x,t)$ be a nonzero solution of the heat equation (26.5.1.5). Dividing (26.5.1.5) by u, differentiating the resulting equation with respect to x, and taking into account that $(u_t/u)_x = (u_x/u)_t$, we obtain

$$\left(\frac{u_x}{u}\right)_t = \left(\frac{u_{xx}}{u}\right)_x. \tag{26.5.1.7}$$

From the first equation in (26.5.1.6) we have

$$\frac{u_x}{u} = \frac{w}{2} \quad \Longrightarrow \quad \frac{u_{xx}}{u} - \left(\frac{u_x}{u}\right)^2 = \frac{w_x}{2} \quad \Longrightarrow \quad \frac{u_{xx}}{u} = \frac{w_x}{2} + \frac{1}{4}w^2. \tag{26.5.1.8}$$

Replacing the expressions in parentheses in (26.5.1.7) with the right-hand sides of the first and the last relation (26.5.1.8), we obtain the Burgers equation (26.5.1.4).

The first relation in (26.5.1.6) is the Hopf–Cole transformation. It can be rewritten as a differential substitution by solving (26.5.1.6) for w, which leads to formula (26.4.1.2).

Example 26.30. Let us demonstrate that *Liouville's equation*

$$\frac{\partial^2 w}{\partial x \partial y} = e^{\lambda w} \tag{26.5.1.9}$$

is connected with the linear wave equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \tag{26.5.1.10}$$

by the Bäcklund transformation

$$\begin{aligned} &\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} = \frac{2k}{\lambda}\exp\Big[\frac{1}{2}\lambda(w+u)\Big], \\ &\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y} = -\frac{1}{k}\exp\Big[\frac{1}{2}\lambda(w-u)\Big], \end{aligned} \tag{26.5.1.11}$$

where $k \neq 0$ is an arbitrary constant.

Indeed, let us differentiate the first relation of (26.5.1.11) with respect to y and the second equation with respect to x. Then, taking into account that $u_{yx} = u_{xy}$ and $w_{yx} = w_{xy}$ and eliminating the combinations of the first derivatives using (26.5.1.11), we obtain

$$\begin{aligned} &\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial y} = k\exp\Big[\frac{1}{2}\lambda(w+u)\Big]\Big(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial y}\Big) = -\exp(\lambda w), \\ &\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} = \frac{\lambda}{2k}\exp\Big[\frac{1}{2}\lambda(w-u)\Big]\Big(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial x}\Big) = \exp(\lambda w). \end{aligned} \tag{26.5.1.12}$$

Adding relations (26.5.1.12) together, we get the linear equation (26.5.1.10). Subtracting the latter equation from the former gives the nonlinear equation (26.5.1.9).

Example 26.31. The nonlinear heat equation with a exponential source

$$w_{xx} + w_{yy} = ae^{\beta w}$$

is connected with the Laplace equation

$$u_{xx} + u_{yy} = 0$$

by the Bäcklund transformation

$$\begin{aligned} u_x + \tfrac{1}{2}\beta w_y &= \big(\tfrac{1}{2}a\beta\big)^{1/2}\exp\big(\tfrac{1}{2}\beta w\big)\sin u, \\ u_y - \tfrac{1}{2}\beta w_x &= \big(\tfrac{1}{2}a\beta\big)^{1/2}\exp\big(\tfrac{1}{2}\beta w\big)\cos u. \end{aligned}$$

This fact can be proved in a similar way as in Example 26.30.

Example 26.32. The Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + 6w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0$$

and the modified Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

are related by the Bäcklund transformation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \varepsilon(w + u^2), \quad \varepsilon = \pm 1, \\ \frac{\partial u}{\partial t} &= \varepsilon\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial}{\partial x}(uw). \end{aligned} \tag{26.5.1.13}$$

The first relation in (26.5.1.13) with $\varepsilon = 1$ is the Miura transformation. It can be rewritten as a differential substitution by solving (26.5.1.13) for w, which leads to formula (26.4.1.6).

Remark. It is significant that unlike the contact transformations, the Bäcklund transformations are determined by the specific equations (a Bäcklund transformation that relates given equations does not always exist).

26.5.2. RF Pairs and Their Use for Constructing Bäcklund Transformations

Outlined below is the method for constructing Bäcklund transformations for partial differential equations based on using RF pairs.

1°. Prior to describing RF pairs for partial differential equations, we describe them for simpler, ordinary differential equations. This order of presentation is chosen for better understanding of the subsequent material.

An RF pair is an operation of consecutive raising and lowering of the order of an equation (RF stands for "Rise" and "Fall"). The nth-order ordinary differential equation of general form

$$F(x, w, w'_x, w''_{xx}, \ldots, w^{(n)}_x) = 0 \tag{26.5.2.1}$$

admits two basic RF pairs, which are considered below.

The first RF pair. On solving equation (26.5.2.1) for the dependent variable (which is assumed to be feasible), we get

$$\Phi(x, w'_x, w''_{xx}, \ldots, w^{(n)}_x) = w. \tag{26.5.2.2}$$

The differentiation with respect to x takes (26.5.2.2) to an $(n+1)$st-order equation, which does not depend on w explicitly. By lowering the order of the resulting equation using the standard substitution

$$\xi(x) = w'_x, \tag{26.5.2.3}$$

we arrive at the nth-order equation

$$\frac{d}{dx}\Phi(x, \xi, \xi'_x, \ldots, \xi^{(n-1)}_x) = \xi. \tag{26.5.2.4}$$

Equations (26.5.2.2) and (26.5.2.4) are connected by the Bäcklund transformation

$$\begin{aligned} \Phi(x, \xi, \xi'_x, \ldots, \xi^{(n-1)}_x) &= w, \\ w'_x &= \xi. \end{aligned} \tag{26.5.2.5}$$

If we know a solution $w = w(x)$ to equation (26.5.2.2) (or (26.5.2.1)), then we can find a solution $\xi = \xi(x)$ to equation (26.5.2.4) using the second relation in (26.5.2.5). If we know a solution $\xi = \xi(x)$ to equation (26.5.2.4), then we obtain a solution $w = w(x)$ to equation (26.5.2.2) using the first relation in (26.5.2.5).

The second RF pair. On solving equation (26.5.2.1) for the independent variable (which is assumed to be feasible), we get

$$\Psi(w, w'_x, w''_{xx}, \ldots, w_x^{(n)}) = x. \tag{26.5.2.6}$$

Differentiating (26.5.2.6) with respect to x yields an $(n+1)$st-order autonomous equation, whose order is further reduced using substitution (26.4.2.2):

$$\eta(w) = w'_x.$$

As a result, we arrive at the nth-order equation

$$\eta \frac{d}{dw}\Psi(w, \eta, \eta\eta'_w, \ldots, \eta_x^{(n-1)}) = 1. \tag{26.5.2.7}$$

The derivatives with respect to x are found successively using $\eta_x^{(k)} = \eta(\eta_x^{(k-1)})_w$. The derivative with respect to w on the left-hand side of (26.5.2.7) is calculated by the chain rule taking into account that F depends on w, η, η_w, ...

Equations (26.5.2.6) and (26.5.2.7) are connected by the Bäcklund transformation

$$\begin{aligned} \Psi(w, \eta, \eta'_x, \ldots, \eta_x^{(n-1)}) &= x, \\ w'_x &= \eta. \end{aligned} \tag{26.5.2.8}$$

A considerable number of new integrable ordinary differential equations have been found in a systematic manner using RF pair operations and associated Bäcklund transformations (Zaitsev & Polyanin 1994, Polyanin & Zaitsev 2003).

The above ideas and results for ordinary differential equations turn out to be transferrable, when appropriately modified, to nonlinear partial differential equations.

2°. Consider the nth-order evolution equation

$$\frac{\partial w}{\partial t} = s(t)w + F\left(t, x, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^n w}{\partial x^n}\right), \tag{26.5.2.9}$$

which does not depend on w explicitly if $s(t) \equiv 0$. If the unknown function w as well as the functions s and F on the right-hand side are independent of t, then equation (26.5.2.9) becomes (26.5.2.2) (after F has been renamed $-s\Phi$).

Equation (26.5.2.9) has the following general property: if $\widetilde{w}(t, x)$ is a solution to this equation, then the function

$$w = \widetilde{w}(t, x) + C\exp\Big(\int s(t)\,dt\Big),$$

where C is an arbitrary constant, is also a solution to (26.5.2.9).

By analogy with ordinary differential equations, let us apply the first RF pair to equation (26.5.2.9).

Differentiating equation (26.5.2.9) with respect to x yields the $(n+1)$st-order equation

$$\frac{\partial^2 w}{\partial x \partial t} = s(t)\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}F\left(t, x, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^n w}{\partial x^n}\right), \tag{26.5.2.10}$$

which does not depend on w explicitly. In equation (26.5.2.10), let us change from the old variables to the new ones (analogue of transformation (26.5.2.3)):

$$t,\ x,\ w = w(t,x) \quad \Longrightarrow \quad t,\ x,\ \xi = \frac{\partial w}{\partial x}, \quad \text{where} \quad \xi = \xi(t,x). \tag{26.5.2.11}$$

The evolution nth-order equation for $\xi = \xi(x,t)$ has the form

$$\frac{\partial \xi}{\partial t} = s(t)\xi + \frac{\partial}{\partial x}F\left(t, x, \xi, \frac{\partial \xi}{\partial x}, \ldots, \frac{\partial^{n-1}\xi}{\partial x^{n-1}}\right). \tag{26.5.2.12}$$

The derivative with respect to x on the right-hand side of (26.5.2.12) is obtained by the chain rule taking into account that F depends on $x, \xi, \xi_x, \ldots$.

Equations (26.5.2.9) and (26.5.2.12) are connected by the Bäcklund transformation

$$\begin{aligned} w_t &= s(t)w + F(t, x, \xi, \xi_x, \ldots, \xi_x^{(n-1)}), \\ w_x &= \xi. \end{aligned} \tag{26.5.2.13}$$

If a solution to equation (26.5.2.9) is known, one can find a solution to equation (26.5.2.12) using the second relation in (26.5.2.13). Given a solution to equation (26.5.2.12), one obtains a solution to equation (26.5.2.9) by solving the linear first-order system (26.5.2.13) for w (by virtue of (26.5.2.12) the right-hand sides of the system satisfy the solvability condition).

Remark 1. By specifying an integrable equation of the form (26.5.2.9) using transformation (26.5.2.13), one can obtain an integrable equation of the form (26.5.2.12).

Remark 2. If the original equation (26.5.2.9) is linear, then the resulting equation (26.5.2.12) is also linear.

Example 26.33. Consider the nonlinear equation

$$w_t = aw_{xx} + bw_x^2 + f(x,t)w_x + g(x,t). \tag{26.5.2.14}$$

With the transformation

$$Z = \exp\left(\frac{b}{a}w\right) \tag{26.5.2.15}$$

it can be reduced to the linear equation

$$Z_t = aZ_{xx} + f(x,t)Z_x + \frac{b}{a}g(x,t)Z. \tag{26.5.2.16}$$

Note that if $f(x,t) = f_1(t)x + f_0(t)$ and $g(x,t) = g_1(t)x + g_0(t)$, equation (26.5.2.16) can be reduced to the classical heat equation. [see Polyanin (2002, p. 147)].

The Bäcklund transformation (26.5.2.13) corresponding to equation (26.5.2.14) has the form

$$\begin{aligned} w_t &= a\xi_x + b\xi^2 + f(x,t)\xi + g(x,t), \\ w_x &= \xi. \end{aligned} \tag{26.5.2.17}$$

Eliminating w from (26.5.2.17), we arrive at the nonlinear equation

$$\xi_t = a\xi_{xx} + 2b\xi\xi_x + [f(x,t)\xi]_x + g_x(x,t), \tag{26.5.2.18}$$

which generalizes the Burgers equation (corresponding to the special case $f = g = 0$). Replacing w in (26.5.2.17) by Z using (26.5.2.15), we obtain the Bäcklund transformation

$$\begin{aligned} Z_t &= \frac{b}{a}Z[a\xi_x + b\xi^2 + f(x,t)\xi + g(x,t)], \\ Z_x &= \frac{b}{a}Z\xi \end{aligned}$$

that connects the nonlinear equation (26.5.2.18) with the linear equation (26.5.2.16).

Example 26.34. The nonlinear third-order equation

$$w_t + w_{xxx} - 3w_x^2 + \frac{1}{2t}w = 0$$

can be reduced, with a Bäcklund transformation of the form (26.5.2.13),

$$w_t = -\xi_{xx} + 3\xi^2 - \frac{1}{2t}w,$$
$$w_x = \xi$$

to the cylindrical Korteweg–de Vries equation 13.1.2.1:

$$\xi_t + \xi_{xxx} - 6\xi\xi_x + \frac{1}{2t}\xi = 0.$$

Example 26.35. Consider the nonlinear equation

$$\frac{\partial w}{\partial t} = f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}.$$

It arises in the nonlinear theory of flows in porous media and also governs the motion of a nonlinear viscoplastic medium.

The Bäcklund transformation of the form (26.5.2.13)

$$w_t = f(\xi)\xi_x,$$
$$w_x = \xi,$$

reduces the original equation to the nonlinear hear equation

$$\xi_t = [f(\xi)\xi_x]_x.$$

3°. Consider the following nth-order evolution equation of a quite general form:

$$\frac{\partial w}{\partial t} = s(t)x\frac{\partial w}{\partial x} + F\left(t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^n w}{\partial x^n}\right). \qquad (26.5.2.19)$$

Equation (26.5.2.19) includes as special cases the Burgers, Korteweg–de Vries, and Harry Dym equations as well as nonlinear heat equations and other nonlinear equations of mathematical physics. If the unknown w and the right-hand side functions s and F are independent of t, then equation (26.5.2.19) reduces to (26.5.2.6) (with F renamed $-sw_x\Psi$).

Equation (26.5.2.19) has the following general property: if $\widetilde{w}(t, x)$ is a solution of the equation, then

$$w = \widetilde{w}(t, x + \psi(t)), \qquad \psi(t) = C\exp\left(-\int s(t)\,dt\right),$$

where C is an arbitrary constant, is also a solution.

By analogy with ordinary differential equations, let us apply the second RF pair to equation (26.5.2.19). The formula

$$\frac{\partial}{\partial x}\left(\frac{w_t}{w_x}\right) \equiv \frac{w_x w_{tx} - w_t w_{xx}}{w_x^2} = \frac{\eta_t}{\eta} \qquad (26.5.2.20)$$

will be required later; it is obtained by differentiating w_t/w_x with respect to x followed by changing from t, x, and w to the von Mises variables (26.4.2.4):

$$t,\ x,\ w = w(t, x) \quad \Longrightarrow \quad t,\ w,\ \eta = \eta(t, w), \quad \text{where} \quad \eta = \frac{\partial w}{\partial x}.$$

Dividing equation (26.5.2.19) by w_x and differentiating with respect to x, we obtain an $(n+1)$st-order equation whose left-hand side coincides with (26.5.2.20). Changing to the von Mises variables followed by elementary rearrangements leads to the nth-order equation

$$\eta_t = s(t)\eta + \eta^2 \frac{\partial}{\partial w}\Big(\frac{F}{\eta}\Big), \qquad F = F(t, w, \eta, \eta\eta_w, \ldots, \eta_x^{(n-1)}), \tag{26.5.2.21}$$

where the partial derivatives with respect to x are calculated successively with the formula $\eta_x^{(k)} = \eta(\eta_x^{(k-1)})_w$. On the right-hand side of (26.5.2.21), the derivative with respect to w is found by the chain rule taking into account the fact that F depends on w, η, η_w, ...

Equations (26.5.2.19) and (26.5.2.21) are connected by the Bäcklund transformation

$$\begin{aligned} w_t &= s(t)x\eta + F(t, w, \eta, \eta_x, \ldots, \eta_x^{(n-1)}), \\ w_x &= \eta. \end{aligned} \tag{26.5.2.22}$$

Any solution $w = g(t, x)$ to equation (26.5.2.19) determines a solution to equation (26.5.2.21), which can be written in parametric form as $w = g(t, x)$, $\eta = g_x(t, x)$.

Suppose $\eta = \eta(t, w)$ is a solution to equation (26.5.2.21). Let us calculate the spatial derivatives $\eta_x = \eta\eta_w$, $\eta_{xx} = \eta(\eta\eta_w)_w$, ... and substitute them into the right-hand side of equation (26.5.2.22). As a result, the first equation in (26.5.2.22) becomes $w_t = G(t, x, w)$, where $G(t, x, w)$ is a known function of three variables. This means that system (26.5.2.22) has been transformed into a system of first-order equations for $w = w(t, x)$.

Remark. Instead of solving the above system of first-order equations, one can proceed in a simpler way. Substituting the solution $\eta = \eta(t, w)$ of equation (26.5.2.21) into the second equation in (26.5.2.22) and integrating, one obtains

$$\int \frac{dw}{\eta(t, w)} = x + \varphi(t), \tag{26.5.2.23}$$

where $\varphi(t)$ is an arbitrary function. Further, on solving (26.5.2.23) for $w = w(t, x)$, one should substitute the resulting expression into the original equation (26.5.2.19) and find the function $\varphi(t)$.

New integrable equations can be obtained as follows. Starting from an integrable equation of the form (26.5.2.19), one can use the Bäcklund transformation (26.5.2.22) to arrive at another integrable equation (26.5.2.21). Conversely: if equation (26.5.2.21) is integrable for some function F, then the original equation (26.5.2.19) is also integrable.

Remark. Sometimes, it may be helpful to use additional point transformations of equation (26.5.2.21). In particular, it may be convenient to represent equation (26.5.2.21) in the form

$$\zeta_t = -s(t)\zeta - \frac{\partial}{\partial w}(\zeta F), \qquad \zeta = \frac{1}{\eta}.$$

Example 26.36. Consider the generalized Burgers equation

$$w_t + f(w)w_x = aw_{xx}. \tag{26.5.2.24}$$

The Bäcklund transformation (26.5.2.22) with $F = aw_{xx} - f(w)w_x = a\eta\eta_w - f(w)\eta$ and $s(t) \equiv 0$ takes (26.5.2.24) to the equation $\eta_t = a\eta^2\eta_{ww} - f'_w(w)\eta^2$, which can be reduced, with the substitution $\eta = 1/\zeta$, to the nonlinear heat equation with a source

$$\zeta_t = a(\zeta^{-2}\zeta_w)_w + f'_w(w). \tag{26.5.2.25}$$

Assuming that $f(w) = bw$ in (26.5.2.25), we have an integrable equation (it is reduced to the classical Burgers equation (26.5.2.24)), which was derived by Dorodnitsyn and Svirshchevskii (1983) using a different method.

Example 26.37. The linear third-order equation

$$w_t = aw_{xxx} + bw_{xx} + sxw_x + cw,$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, and $s = s(t)$ are arbitrary functions, is reduced, by the Bäcklund transformation (26.5.2.22) with $F = aw_{xxx} + bw_{xx} + cw = a\eta(\eta\eta_w)_w + b\eta\eta_w + cw$ followed by the substitution $\eta = 1/\zeta$, to the nonlinear equation

$$\zeta_t + c(w\zeta)_w + s\zeta = a(\zeta^{-3}\zeta_w)_{ww} + b(\zeta^{-2}\zeta_w)_w. \tag{26.5.2.26}$$

Example 26.38. Consider the nonlinear third-order equation

$$w_t + f(w)w_x = aw_{xxx}. \tag{26.5.2.27}$$

The Bäcklund transformation (26.5.2.22) with $F = aw_{xxx} - f(w)w_x = a\eta(\eta\eta_w)_w - f(w)\eta$ and $s(t) \equiv 0$ reduces (26.5.2.27) to the equation $\eta_t = a\eta^2(\eta\eta_w)_{ww} - f'_w(w)\eta^2$. With the substitution $\eta = 1/\zeta$, this equation can be rewritten as

$$\zeta_t = a(\zeta^{-3}\zeta_w)_{ww} + f'_w(w). \tag{26.5.2.28}$$

For $f(w) \equiv 0$, the original equation (26.5.2.27) is linear. For $f(w) = bw$ and $f(w) = bw^2$, we have the Korteweg–de Vries and modified Korteweg–de Vries equations, respectively. By substituting these functions $f(w)$ into (26.5.2.28), we obtain new integrable equations.

Remark. By applying again a Bäcklund transformation (26.5.2.22) based on the von Mises transformation to equations (26.5.2.25) with $f(w) = bw$, (26.5.2.26) with $c = 0$, and (26.5.2.28) with $f(w) = bw$, one obtains other integrable equations.

Table 26.1 lists integrable equations obtained by using the Bäcklund transformation (26.5.2.22) (in some cases, followed by an additional point transformation, specified in the last column). In the fifth row, a special case of the linearizable equation (26.3.4.6), with t substituted for y, is treated as the original equation; the function $g(\xi)$ is defined parametrically.

Table 26.2 lists nonlinear equations obtained by using the Bäcklund transformation (26.5.2.22) from some common classes of equations of mathematical physics. Whenever solutions to the original equations are known, one can construct exact solutions to the transformed equations.

4°. Transformations of the forms (26.5.2.13) and (26.5.2.22) are said to be *canonic Bäcklund transformations of evolution equations*.

One of the methods for finding other Bäcklund transformations involves the following. First, one looks for a point transformation that reduces the equation in question to an equation of the form (26.5.2.9) or (26.5.2.19). Then a canonical Bäcklund transformation of the form (26.5.2.13) or (26.5.2.22) is rewritten in terms of the new variables. The composition of these transformations determines the Bäcklund transformation of the original equation.

Example 26.39. Consider the following evolution equation, which is homogeneous in the unknown function w:

$$w_t = wF\Big(t, x, \frac{w_x}{w}, \frac{w_{xx}}{w}, \dots, \frac{w_x^{(n)}}{w}\Big). \tag{26.5.2.29}$$

This equation has the characteristic property of remaining unchanged under the change of variable $w = C\bar{w}$, where $C \neq 0$ is an arbitrary constant.

It is not difficult to verify that equation (26.5.2.29) is reduced to an equation of the form (26.5.2.9) with the substitution $w = e^u$. The second equation in (26.5.2.13), rewritten in terms of u and ξ, has the form $u_x = \xi$.

Returning back to the original variable $u = \ln w$, we obtain $w_x = w\xi$. Therefore the desired Bäcklund transformation of equation (26.5.2.29) has the form

$$\begin{aligned} w_t &= wF(t, x, \xi, \xi_x + \xi^2, \dots), \\ w_x &= w\xi. \end{aligned} \tag{26.5.2.30}$$

The arguments omitted in (26.5.2.30) are calculated using the recurrence relations

$$\frac{w_x}{w} = \xi, \quad \frac{w_x^{(k)}}{w} = \frac{\partial}{\partial x}\Big[\frac{w_x^{(k-1)}}{w}\Big] + \xi\frac{w_x^{(k-1)}}{w}, \quad k = 2, \dots, n,$$

which follow from the second equation in (26.5.2.30). In particular, for $n = 3$ we have $w_{xxx}/w = \xi_{xx} + 3\xi\xi_x + \xi^3$.

Dividing (26.5.2.30) by w and eliminating w by cross-differentiation with the aid of the relation

$$\frac{\partial}{\partial t}\Big(\frac{w_x}{w}\Big) = \frac{\partial}{\partial x}\Big(\frac{w_t}{w}\Big),$$

we arrive at the nth-order equation

$$\xi_t = \frac{\partial}{\partial x}F(t, x, \xi, \xi_x + \xi^2, \dots).$$

TABLE 26.1
Some integrable nonlinear equations obtained using the Bäcklund transformation (26.5.2.22)

No.	Original integrable equation (26.5.2.19)	Resulting integrable equation (26.5.2.21)	Additional notation and remarks
1	$w_t = aw_{xx} + bw$	$\zeta_t + b(w\zeta)_w = a(\zeta^{-2}\zeta_w)_w$	$\zeta = 1/\eta$, original equation is linear
2	$w_t + bww_x = aw_{xx}$	$\zeta_t = a(\zeta^{-2}\zeta_w)_w + b$	$\zeta = 1/\eta$, original equation is the Burgers equation
3	$w_t = w_{xx} + f(w)w_x^2$	$\eta_t = \eta^2\eta_{ww} + \eta^2[f(w)\eta]_w$	original equation is reduced to a linear one (see Eq. 5.6.6.8)
4	$w_t = aw_{xx} - aw_x^2 + be^w$	$\eta_t = a\eta^2\eta_{ww} - (a\eta^2 + be^w)\eta_w + be^w\eta$	original equation reduces to a linear one with the change $v = e^{-w}$
5	$w_t = \dfrac{f(w_x)}{w_{xx}}$	$\xi_t = g(\xi)\xi_w^{-2}\xi_{ww}$	$\xi = \displaystyle\int \frac{\eta^2 d\eta}{f(\eta)}$, $g = \dfrac{\eta^4}{f(\eta)}$
6	$w_t = aw_{xxx} + bw$	$\zeta_t + b(w\zeta)_w = a(\zeta^{-3}\zeta_w)_{ww}$	$\zeta = 1/\eta$, original equation is linear
7	$w_t = aw_{xxx} + bw_{xx}$	$\zeta_t = a(\zeta^{-3}\zeta_w)_{ww} + b(\zeta^{-2}\zeta_w)_w$	$\zeta = 1/\eta$, original equation is linear
8	$w_t + bww_x = aw_{xxx}$	$\zeta_t = a(\zeta^{-3}\zeta_w)_{ww} + b$	$\zeta = 1/\eta$, original equation is the KdV equation
9	$w_t + bw^2w_x = aw_{xxx}$	$\zeta_t = a(\zeta^{-3}\zeta_w)_{ww} + 2bw$	$\zeta = 1/\eta$, original equation is the modified KdV equation
10	$w_t = aw_{xxx} + bw_x^2$	$\theta_t = a\theta^{3/2}\theta_{www} + b\theta\theta_w$	$\theta = \eta^2$, original equation is the potential KdV equation
11	$w_t = aw_{xxx} + bw_x^3$	$\theta_t = a\theta^{3/2}\theta_{www} + 2b\theta^{3/2}\theta_w$	$\theta = \eta^2$, original equation is the potential modified KdV equation
12	$w_t = u_{xxx} - \frac{1}{8}w_x^3 + ae^w w_x$	$\theta_t = \theta^{3/2}\theta_{www} + \theta^{3/2}(-\frac{1}{4}\theta_w + 2ae^w)$	$\theta = \eta^2$, original equation is a special case of Eq. 13.1.4.7
13	$w_t = aw^3w_{xxx}$	$\theta_t = a\theta^{3/2}(w^3\theta_{ww})_w$	$\theta = \eta^2$, original equation is the Harry Dym equation (see Eq. 13.1.4.8)
14	$w_t = aw^{3/2}w_{xxx}$	$\theta_t = a\theta^{3/2}(w^{3/2}\theta_{ww})_w$	$\theta = \eta^2$, original equation is transformed equation 10 with $b = 0$
15	$w_t = aw_{xxxx}$	$\theta_t = a\theta^{3/2}(\theta^{1/2}\theta_{ww})_{ww}$	$\theta = \eta^2$, original equation is linear

⊙ *References for Section* 26.5: G. L. Lamb (1974), R. M. Miura (1976), R. L. Anderson and N. H. Ibragimov (1979), A. S. Fokas and R. L. Anderson (1979), A. S. Fokas and B. Fuchssteiner (1981), M. J. Ablowitz and H. Segur (1981), J. J. C. Nimmo and D. J. Crighton (1982), N. H. Ibragimov (1985, 1994), H. Stephani (1989), J. R. King (1990 *b*), V. F. Zaitsev and A. D. Polyanin (1993, 1994), B. J. Cantwell (2002), A. D. Polyanin and V. F. Zaitsev (2003, 2004), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin and A. I. Zhurov (2009 *a*), A. D. Polyanin (2010).

TABLE 26.2
Some nonlinear equations obtained using the Bäcklund transformation (26.5.2.22) and related equations. Notation for auxiliary variables: $\zeta = 1/\eta$ and $\theta = \eta^{1/2}$

Name of original equation	Original equation of the form (26.5.2.19)	Resulting equation of the form (26.5.2.21) and some other equations
Model equation of gas dynamics (with $g \equiv 0$)	$w_t + f(w)w_x = g(w)$	$\zeta_t + [g(w)\zeta]_w = f'_w(w)$ (linear equation)
Nonlinear heat equation with source	$w_t = [f(w)w_x]_x + g(w)$	$\eta_t = \eta^2[f(w)\eta]_{ww} + \eta^2[g(w)/\eta]_w$; $\zeta_t = [f(w)/\zeta]_{ww} - [g(w)\zeta]_w$
Generalized Burgers equation (case 1)	$w_t + f(w)w_x = g(w)w_{xx}$	$\eta_t = \eta^2[g(w)\eta_w]_w - \eta^2 f'_w(w)$; $\zeta_t = [g(w)\zeta^{-2}\zeta_w]_w + f'_w(w)$
Generalized Burgers equation (case 2)	$w_t + f(w)w_x = [g(w)w_x]_x$	$\eta_t = \eta^2[g(w)\eta]_{ww} - \eta^2 f'_w(w)$; $\zeta_t = [g(w)/\zeta]_{ww} + f'_w(w)$
Generalized equation of nonlinear theory of fluid flows through porous media	$w_t = f(w, w_x)w_{xx}$	$\eta_t = \eta^2[f(w, \eta)\eta_w]_w$
Heat equation for rheologically complex media	$w_t = [f(w, w_x)]_x$	$\eta_t = \eta^2[f(w, \eta)]_{ww}$; $\zeta_t = -[f(w, 1/\zeta)]_{ww}$
Korteweg–de Vries–Harry Dym type equation	$w_t + f(w)w_x = g(w)w_{xxx}$	$\eta_t = \eta^2[g(w)(\eta\eta_w)_w]_w - \eta^2 f'_w(w)$; $\theta_t = \theta^{3/2}[g(w)\theta_{ww}]_w - 2\theta^{3/2} f'_w(w)$
Generalized modified Korteweg–de Vries equation	$w_t + f(w_x) = aw_{xxx}$	$\zeta_t = a(\zeta^{-3}\zeta_w)_{ww} + [\zeta f(1/\zeta)]_w$
Generalized modified Harry Dym equation	$w_t = [f(w, w_x)]_{xx}$	$\eta_t = \eta^2\{\eta[f(w, \eta)]_w\}_{ww}$

26.6. Some Other Transformations

26.6.1. Crocco Transformation. Order Reduction of Hydrodynamic Type Equations

Crocco transformations are used in fluid dynamics to reduce the order of boundary layer equations. In a Crocco transformation, a suitable first-order partial derivative is taken as a new independent variable and a second-order derivative is taken as the new dependent one.

1°. Prior to describing Crocco transformations for partial differential equations, let us consider their simple analogue for ordinary differential equations.

Let $w = w(x)$ be a function of a one variable. On solving the autonomous nth-order ordinary differential equation of general form (26.4.2.1) for w, we rewrite the equation as

$$G(w'_x, w''_{xx}, \ldots, w^{(n)}_x) = w. \tag{26.6.1.1}$$

Differentiating (26.6.1.1) with respect to x and then changing from the old variables x and w to

$$\eta = w'_x, \quad \Phi = w''_{xx} \qquad (\textit{analogue of the Crocco transformation}), \tag{26.6.1.2}$$

we arrive at an $(n-1)$st-order equation.

2°. Consider the nonlinear nth-order partial differential equation with a mixed derivative

$$\frac{\partial^2 w}{\partial t\partial x}+[a(t)w+b(t)x]\frac{\partial^2 w}{\partial x^2}=F\Big(t,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2},\frac{\partial^3 w}{\partial x^3},\dots,\frac{\partial^n w}{\partial x^n}\Big). \tag{26.6.1.3}$$

Introduce the notation

$$\eta=\frac{\partial w}{\partial x},\qquad \Phi=\frac{\partial^2 w}{\partial x^2}. \tag{26.6.1.4}$$

Dividing equation (26.6.1.3) by $w_{xx}=\Phi$, differentiating with respect to x, and taking into account (26.6.1.4), we obtain

$$\frac{\Phi_t}{\Phi}-\frac{w_{tx}\Phi_x}{\Phi^2}+a(t)\eta+b(t)=\frac{\partial}{\partial x}\frac{F\big(t,\eta,\Phi,\Phi_x,\dots,\Phi_x^{(n-2)}\big)}{\Phi}. \tag{26.6.1.5}$$

In (26.6.1.5), we change from the old variables to the Crocco variables (analogue of transformation (26.6.1.2)):

$$t,\ x,\ w=w(t,x)\quad\Longrightarrow\quad t,\ \eta,\ \Phi=\Phi(t,\eta), \tag{26.6.1.6}$$

where η and Φ are given by (26.6.1.4). Then the derivatives are transformed as follows:

$$\frac{\partial}{\partial x}=\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=w_{xx}\frac{\partial}{\partial\eta}=\Phi\frac{\partial}{\partial\eta},\qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial t}+w_{tx}\frac{\partial}{\partial\eta}.$$

As a result, equation (26.6.1.5) and, hence, the original equation (26.6.1.3) are reduced to the $(n-1)$st-order equation

$$\frac{a(t)\eta+b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}=\frac{\partial}{\partial\eta}\left[\frac{1}{\Phi}F\Big(t,\eta,\Phi,\Phi\frac{\partial\Phi}{\partial\eta},\dots,\frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\Big)\right], \tag{26.6.1.7}$$

where the third- and higher-order derivatives are calculated by the formulas

$$\frac{\partial^k w}{\partial x^k}=\frac{\partial^{k-2}\Phi}{\partial x^{k-2}}=\Phi\frac{\partial}{\partial\eta}\frac{\partial^{k-3}\Phi}{\partial x^{k-3}},\qquad \frac{\partial}{\partial x}=\Phi\frac{\partial}{\partial\eta},\qquad k=3,\dots,n.$$

Remark. Equation (26.6.1.3) has the following characteristic property: if $\widetilde{w}(t,x)$ is a solution to the equation, then

$$w=\widetilde{w}(t,x+\varphi(t))+\frac{1}{a(t)}[b(t)\varphi(t)-\varphi_t'(t)],\qquad a(t)\not\equiv 0,$$

where $\varphi(t)$ is an arbitrary function, is also a solution to the equation. For $a(t)\equiv 0$, the function $w=\widetilde{w}(t,x)+\varphi(t)$ is a solution to equation (26.6.1.3).

Whenever a solution to the original equation (26.6.1.3) is known, formulas (26.6.1.4) define a solution to equation (26.6.1.7) in parametric form.

Suppose $\Phi=\Phi(t,\eta)$ is a solution to equation (26.6.1.7). Then the corresponding solution to the original equation (26.6.1.3) can represented in parametric form as

$$x=\int_{\eta_0}^{\eta}\frac{ds}{\Phi(t,s)}+\varphi(t),\qquad w=\int_{\eta_0}^{\eta}\frac{s\,ds}{\Phi(t,s)}+\frac{1}{a(t)}[b(t)\varphi(t)-\varphi_t'(t)], \tag{26.6.1.8}$$

where $\varphi(t)$ is an arbitrary function and η_0 is an arbitrary constant.

Example 26.40 (generalized Calogero equation). For $F = f(t, w_x)w_{xx} + g(t, w_x)$, which corresponds to the nonlinear second-order equation

$$\frac{\partial^2 w}{\partial t \partial x} = \left[f\left(t, \frac{\partial w}{\partial x}\right) - a(t)w - b(t)x \right] \frac{\partial^2 w}{\partial x^2} + g\left(t, \frac{\partial w}{\partial x}\right),$$

changing to the Crocco variables (26.6.1.6) with (26.6.1.4) leads to the first-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - \frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[f(t, \eta) + \frac{g(t, \eta)}{\Phi} \right].$$

This equation becomes linear after the substitution $\Phi = 1/\Psi$.

Example 26.41. Consider the nonlinear third-order equation

$$\frac{\partial^2 w}{\partial t \partial x} + w\frac{\partial^2 w}{\partial x^2} - m\left(\frac{\partial w}{\partial x}\right)^2 = \nu\frac{\partial^3 w}{\partial x^3} + q(t)\frac{\partial w}{\partial x} + p(t), \tag{26.6.1.9}$$

which arises in fluid dynamics and is derived from the Navier–Stokes equations, with w being one of the fluid velocity components and ν being the kinematic viscosity. The Crocco transformation (26.6.1.6) reduces (26.6.1.9) to a second-order equation, which can be written in the form

$$\frac{\partial \Phi}{\partial t} + [m\eta^2 + q(t)\eta + p(t)]\frac{\partial \Phi}{\partial \eta} = [(2m-1)\eta + q(t)]\Phi + \nu\Phi^2\frac{\partial^2 \Phi}{\partial \eta^2}. \tag{26.6.1.10}$$

Note that in the degenerate case of inviscid fluid ($\nu = 0$), the original nonlinear equation (26.6.1.9) is reduced to the linear first-order equation (26.6.1.10).

3°. Consider the nonlinear nth-order equation

$$c(t)w_{tx} + [a(t)w + b(t)x]w_{xx} + d(t)(w_x w_{tx} - w_t w_{xx}) = F(t, w_x, w_{xx}, \ldots, w_x^{(n)}), \tag{26.6.1.11}$$

which becomes (26.6.1.3) if $c(t) = 1$ and $d(t) = 0$.

Dividing equation (26.6.1.11) by w_{xx}, differentiating with respect to x, and changing to the Crocco variables (26.6.1.6) with (26.6.1.4), we arrive at the $(n-1)$st-order equation

$$\frac{a(t)\eta + b(t)}{\Phi} - [d(t)\eta + c(t)]\frac{\partial}{\partial t}\frac{1}{\Phi} = \frac{\partial}{\partial \eta}\left[\frac{1}{\Phi} F\left(t, \eta, \Phi, \Phi\frac{\partial \Phi}{\partial \eta}, \ldots, \frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right) \right].$$

Example 26.42. The equation of a steady-state boundary layer of a non-Newtonian fluid on a flat plate has the form

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 w}{\partial y^2}\right) \right], \tag{26.6.1.12}$$

where w is the stream function, x and y are the coordinates along and across the flow, and $f(u)$ is a function determined by the rheological properties of the fluid (for a power-law fluid, we have $f(u) = ku^m$). With the renaming $x \Rightarrow t$ and $y \Rightarrow x$ in equation (26.6.1.12), we have a special case of equation (26.6.1.11) with $n = 3$, $a(t) = b(t) = c(t) = 0$, $d(t) = 1$, and $F = [f(w_{xx})]_x$.

The Crocco transformation

$$x,\ y,\ w = w(x, y) \quad \Longrightarrow \quad x,\ \eta,\ \Phi = \Phi(x, \eta), \quad \text{where} \quad \eta = \frac{\partial w}{\partial y}, \quad \Phi = \frac{\partial^2 w}{\partial y^2},$$

decreases the order of equation (26.4.2.7) and brings it to the second-order equation

$$\eta\frac{\partial \Phi}{\partial x} = \Phi^2\frac{\partial^2}{\partial \eta^2}[f(\Phi)].$$

For $f(\Phi) = k/\Phi$ with $k = \text{const}$, this equation is linearized with the substitution $\Psi = 1/\Phi$.

Remark. Equation (26.6.1.11) can be generalized by adding $J_x, \ldots, J_x^{(m)}$ with $J = w_{xx}w_{txx} - w_{tx}w_{xxx}$ as extra arguments of the function F.

4°. Consider the evolution nth-order equation of a quite general form

$$\frac{\partial w}{\partial t}+[a(t)w+b(t)x]\frac{\partial w}{\partial x}=F\left(t,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2},\frac{\partial^3 w}{\partial x^3},\dots,\frac{\partial^n w}{\partial x^n}\right),\tag{26.6.1.13}$$

which includes as special cases the Burgers and Korteweg–de Vries equations as well as equations of the nonlinear theory of fluid flows through porous media and some other nonlinear equations of mathematical physics.

On differentiating (26.6.1.13) with respect to x, we obtain an $(n+1)$st-order equation with a mixed derivative of the form (26.6.1.3):

$$w_{tx}+[a(t)w+b(t)x]w_{xx}=-a(t)w_x^2-b(t)w_x+\frac{\partial}{\partial x}F(t,w_x,w_{xx},w_{xxx},\dots,w_x^{(n)}).\tag{26.6.1.14}$$

In (26.6.1.14), by changing from t, x, and w to the Crocco variables (26.6.1.6), we arrive at the nth-order equation

$$\frac{3a(t)\eta+2b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}+[a(t)\eta^2+b(t)\eta]\frac{\partial}{\partial\eta}\frac{1}{\Phi}=\frac{\partial^2}{\partial\eta^2}F\left(t,\eta,\Phi,\Phi\frac{\partial\Phi}{\partial\eta},\dots,\frac{\partial^{n-2}\Phi}{\partial x^{n-2}}\right).\tag{26.6.1.15}$$

Equations (26.6.1.13) and (26.6.1.15) are connected by a Bäcklund-type transformation:

$$\begin{aligned}w_t+[a(t)w+b(t)x]\eta&=F(t,\eta,\Phi,\Phi_x,\dots,\Phi_x^{(n-2)}),\\ w_x=\eta,\quad w_{xx}&=\Phi.\end{aligned}\tag{26.6.1.16}$$

Example 26.43. Consider the nonlinear second-order equation

$$w_t+[a(t)w+b(t)x]w_x=\frac{f(t,w_x)}{w_{xx}}+g(t,w_x),\tag{26.6.1.17}$$

which is a special case of equation (26.6.1.13). The Bäcklund transformation (26.6.1.16) reduces (26.6.1.17) to the equation

$$\frac{3a(t)\eta+2b(t)}{\Phi}-\frac{\partial}{\partial t}\frac{1}{\Phi}+[a(t)\eta^2+b(t)\eta]\frac{\partial}{\partial\eta}\frac{1}{\Phi}=\frac{\partial^2}{\partial\eta^2}\Big[\frac{f(t,\eta)}{\Phi}+g(t,\eta)\Big],$$

which becomes linear after the substitution $\Phi=1/\Psi$.

Example 26.44. The nonlinear third-order equation

$$w_t+aww_x=\frac{f(t,w_x)}{w_{xx}^3}w_{xxx}$$

can be reduced, using the Bäcklund transformation (26.6.1.16) with $b(t)\equiv 0$ and $F=f(t,w_x)w_{xx}^{-3}w_{xxx}=f(t,\eta)\Phi^{-2}\Phi_\eta$ followed by the substitution $\Phi=1/\Psi$, to the linear equation

$$\Psi_t-a\eta^2\Psi_\eta-3a\eta\Psi=[f(t,\eta)\Psi]_{\eta\eta\eta}.$$

Remark. The sum $p(t)w+q(t)x$ can be added to the right-hand side of equation (26.6.1.13) and the first equation in transformation (26.6.1.16). Equation (26.6.1.15) will then contain additional terms.

26.6.2. Transformations Based on Conservation Laws

1°. Consider a differential equation written as a conservation law,

$$\frac{\partial}{\partial t}\left[G\left(t,x,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t},\dots\right)\right]+\frac{\partial}{\partial x}\left[F\left(t,x,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t},\dots\right)\right]=0.\tag{26.6.2.1}$$

In (26.6.2.1), let us change from the original variables to the new variables

$$t,\ x,\ w = w(t,x) \quad \Longrightarrow \quad t,\ z,\ w = w(t,z), \tag{26.6.2.2}$$

where the function $z = z(t,x)$ is determined by solving the system of first-order partial differential equations

$$\frac{\partial z}{\partial t} = F, \qquad \frac{\partial z}{\partial x} = -G. \tag{26.6.2.3}$$

Here and henceforth, the arguments of F and G are omitted for brevity. By virtue of equation (26.6.2.1), system (26.6.2.3) is consistent.

The transformation (26.6.2.2), (26.6.2.3) determines the passage from the variables t and x to the new independent variables t and z according to the rule

$$\frac{\partial}{\partial x} = -G\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + F\frac{\partial}{\partial z}.$$

Here, F and G are the same as in (26.6.2.1).

Whenever a solution $w = w(t,x)$ of equation (26.6.2.1) is known, the corresponding function $z = z(t,x)$ can be found by integrating the relation

$$dz = F(t,x,w,w_x,w_t,\ldots)\,dt - G(t,x,w,w_x,w_t,\ldots)\,dx, \tag{26.6.2.4}$$

which is obtained from equations (26.6.2.3) taking into account the expression of the total differential $dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt$. Relation (26.6.2.4) is convenient to treat as a short representation of the transformation (26.6.2.2), (26.6.2.3).

Remark 1. The transformation (26.6.2.2), (26.6.2.3) preserves the order of the equation.

Remark 2. Often one may encounter transformations (26.6.2.4) that are supplemented with a transformation of the unknown function in the form $u = \varphi(w)$.

Example 26.45. Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right], \tag{26.6.2.5}$$

which represents a special case of equation (26.6.2.1) for $G = -w$ and $F = f(w)w_x$.

In this case, transformation (26.6.2.4) has the form

$$dz = w\,dx + [f(w)w_x]\,dt \tag{26.6.2.6}$$

and determines a transformation from the variables x and t to the new independent variables z and t according to the rule

$$\frac{\partial}{\partial x} = w\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + [f(w)w_x]\frac{\partial}{\partial z}.$$

Applying transformation (26.6.2.6) to equation (26.6.2.5), we obtain

$$\frac{\partial w}{\partial t} = w^2\frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right]. \tag{26.6.2.7}$$

The substitution $w = 1/u$ reduces (26.6.2.7) to an equation of the form (26.6.2.5),

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left[\frac{1}{u^2}f\Big(\frac{1}{u}\Big)\frac{\partial u}{\partial z}\right]. \tag{26.6.2.8}$$

Suppose $w = w(t,x)$ is a solution of equation (26.6.2.1). The corresponding function $z = z(t,x)$ is found by integrating relation (26.6.2.6):

$$z = \int_{x_0}^{x} w(t,y)\,dy + \int_{t_0}^{t} f\big(w(\tau,x_0)\big)\left[\frac{\partial w}{\partial x}(\tau,x)\right]_{x=x_0} d\tau,$$

where x_0 and t_0 are arbitrary constants (for which the integrals are meaningful).

In the special case of $f(w) = aw^{-2}$, the nonlinear equation (26.6.2.5) is reduced to the linear equation $u_t = au_{zz}$ by the transformation (26.6.2.6).

Example 26.46. Consider the third-order nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2}{\partial x^2}\left[f(w)\frac{\partial w}{\partial x}\right], \tag{26.6.2.9}$$

which represents a special case of equation (26.6.2.1) for $G = -w$ and $F = [f(w)w_x]_x$.

In this case, transformation (26.6.2.4) has the form

$$dz = w\,dx + [f(w)w_x]_x\,dt, \tag{26.6.2.10}$$

and determines a transformation from the variables x and t to the new independent variables z and t according to the rule

$$\frac{\partial}{\partial x} = w\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + [f(w)w_x]_x\frac{\partial}{\partial z}.$$

Applying transformation (26.6.2.10) to equation (26.6.2.9), we obtain

$$\frac{\partial w}{\partial t} = w^2\frac{\partial^2}{\partial z^2}\left[wf(w)\frac{\partial w}{\partial z}\right]. \tag{26.6.2.11}$$

The substitution $w = 1/u$ reduces (26.6.2.11) to an equation of the form (26.6.2.9),

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial z^2}\left[\frac{1}{u^3}f\left(\frac{1}{u}\right)\frac{\partial u}{\partial z}\right].$$

In the special case of $f(w) = aw^{-3}$, the nonlinear equation (26.6.2.9) is reduced to the linear equation $u_t = au_{zzz}$ by the transformation (26.6.2.10).

2°. Evolution equations of the form

$$\frac{\partial w}{\partial t} = F\left(t, x, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}, \ldots, \frac{\partial^n w}{\partial x^n}\right)$$

are reduced, by the differentiation with respect to x followed by the substitution

$$u = \frac{\partial w}{\partial x},$$

to the equation of the form (26.6.2.1):

$$-\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}F\left(t, x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1}u}{\partial x^{n-1}}\right) = 0, \qquad \text{where } G = -u.$$

⊙ *References for Section* 26.6: J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin and S. N. Aristov (2009), A. D. Polyanin and A. I. Zhurov (2009 *b*), R. Z. Zhdanov (2009), A. D. Polyanin (2010).

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Chapter 27
Traveling-Wave Solutions and Self-Similar Solutions

27.1. Preliminary Remarks

There are a number of methods for the construction of exact solutions to equations of mathematical physics that are based on the reduction of the original equations to equations in fewer dependent and/or independent variables. The main idea is to find such variables and, by passing to them, to obtain simpler equations. In particular, in this way, finding exact solutions of some partial differential equations in two independent variables may be reduced to finding solutions of appropriate ordinary differential equations (or systems of ordinary differential equations). Naturally, the ordinary differential equations thus obtained do not give all solutions of the original partial differential equation, but provide only a class of solutions with some specific properties.

The simplest classes of exact solutions described by ordinary differential equations involve traveling-wave solutions and self-similar solutions. The existence of such solutions is usually due to the invariance of the equations in question under translations and scaling transformations.

Traveling-wave solutions and self-similar solutions often occur in various applications. Below we consider some characteristic features of such solutions.

It is assumed that the unknown w depends on two variables, x and t, where t plays the role of time and x is a spatial coordinate.

27.2. Traveling-Wave Solutions. Invariance of Equations under Translations

27.2.1. General Form of Traveling-Wave Solutions

Traveling-wave solutions, by definition, are of the form

$$w(x,t) = W(z), \qquad z = kx - \lambda t, \tag{27.2.1.1}$$

where λ/k plays the role of the wave propagation velocity (the sign of λ can be arbitrary, the value $\lambda = 0$ corresponds to a stationary solution, and the value $k = 0$ corresponds to a space-homogeneous solution). Traveling-wave solutions are characterized by the fact that the profiles of these solutions at different time* instants are obtained from one another by appropriate shifts (translations) along the x-axis. Consequently, a Cartesian coordinate system moving with a constant speed can be introduced in which the profile of the desired quantity is stationary. For $k > 0$ and $\lambda > 0$, the wave (27.2.1.1) travels along the x-axis to the right (in the direction of increasing x).

* We also use the term *traveling-wave solution* in the cases where the variable t plays the role of a spatial coordinate.

A traveling-wave solution is found by directly substituting the representation (27.2.1.1) into the original equation and taking into account the relations $w_x = kW'$, $w_t = -\lambda W'$, etc. (the prime denotes a derivative with respect to z).

Traveling-wave solutions occur for equations that do not explicitly involve independent variables,

$$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial t}, \frac{\partial^2 w}{\partial t^2}, \ldots\right) = 0. \tag{27.2.1.2}$$

Substituting (27.2.1.1) into (27.2.1.2), we obtain an autonomous ordinary differential equation for the function $W(z)$:

$$F(W, kW', -\lambda W', k^2 W'', -k\lambda W'', \lambda^2 W'', \ldots) = 0,$$

where k and λ are arbitrary constants.

Example 27.1. The nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] \tag{27.2.1.3}$$

admits a traveling-wave solution. Substituting (27.2.1.1) into (27.2.1.3), we arrive at the ordinary differential equation

$$k^2[f(W)W']' + \lambda W' = 0.$$

Integrating this equation twice yields its solution in implicit form:

$$k^2 \int \frac{f(W)\,dW}{\lambda W + C_1} = -z + C_2,$$

where C_1 and C_2 are arbitrary constants.

Example 27.2. Consider the homogeneous Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t^2} = 0. \tag{27.2.1.4}$$

Inserting (27.2.1.1) into this equation, we obtain an identity. Therefore, equation (27.2.1.4) admits solutions of the form

$$w = W(kx - \lambda t),$$

where $W(z)$ is an arbitrary function and k and λ are arbitrary constants.

Remark. There are isolated (degenerate) cases of equations of the form (27.2.1.2) that do not admit traveling-wave solutions (27.2.1.1). As an example, consider the equation

$$\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t^2} = 1, \tag{27.2.1.5}$$

which differs from (27.2.1.4) in the right-hand side. Indeed, substituting (27.2.1.1) into the left-hand side of equation (27.2.1.5) gives zero ($\neq 1$).

27.2.2. Invariance of Solutions and Equations under Translation Transformations

Traveling-wave solutions (27.2.1.1) are invariant under the translation transformations

$$x = \bar{x} + C\lambda, \quad t = \bar{t} + Ck, \tag{27.2.2.1}$$

where C is an arbitrary constant.

It should be observed that equations of the form (27.2.1.2) are invariant (i.e., preserve their form) under transformation (27.2.2.1); furthermore, these equations are also invariant under general translations in both independent variables:

$$x = \bar{x} + C_1, \quad t = \bar{t} + C_2, \tag{27.2.2.2}$$

where C_1 and C_2 are arbitrary constants. The property of the invariance of specific equations under translation transformations (27.2.2.1) or (27.2.2.2) is inseparably linked with the existence of traveling-wave solutions to such equations (the former implies the latter).

Remark 1. Traveling-wave solutions that stem from the invariance of equations under translations will be called *classical traveling-wave solutions*. Thses solutions are simplest *invariant solutions*.

Remark 2. The condition of invariance of equations under translations is not a necessary condition for the existence of traveling-wave solutions. It can be verified directly that the second-order equation

$$F\big(w, w_x, w_t, xw_x + tw_t, w_{xx}, w_{xt}, w_{tt}\big) = 0$$

does not admit transformations of the form (27.2.2.1) and (27.2.2.2) but has an exact traveling-wave solution (27.2.1.1) described by the ordinary differential equation

$$F\big(W, kW', -\lambda W', zW', k^2W'', -k\lambda W'', \lambda^2 W''\big) = 0.$$

A traveling-wave solution (27.2.1.1) will be called nonclassical (noninvariant) if the equation does not admit the transformation of translation (27.2.2.1) or (27.2.2.2).

27.2.3. Functional Equation Describing Traveling-Wave Solutions

Let us demonstrate that traveling-wave solutions can be defined as solutions of the functional equation

$$w(x,t) = w(x + C\lambda, t + Ck), \tag{27.2.3.1}$$

where k and λ are some constants and C is an arbitrary constant. Equation (27.2.3.1) states that the unknown function does not change upon increasing both arguments by proportional quantities, with C being the coefficient of proportionality.

For $C = 0$, equation (27.2.3.1) turns into an identity. Let us expand (27.2.3.1) into a series in powers of C about $C = 0$, then divide the resulting expression by C, and proceed to the limit as $C \to 0$ to obtain the linear first-order partial differential equation

$$\lambda\frac{\partial w}{\partial x} + k\frac{\partial w}{\partial t} = 0.$$

The general solution to this equation is constructed by the method of characteristics (see Subsection 24.3.5) and has the form (27.2.1.1), which was to be proved.

27.3. Self-Similar Solutions. Invariance of Equations Under Scaling Transformations

27.3.1. General Form of Self-Similar Solutions. Similarity Method

By definition, a *self-similar solution* is a solution of the form

$$w(x,t) = t^\alpha U(\zeta), \qquad \zeta = xt^\beta. \tag{27.3.1.1}$$

The profiles of these solutions at different time instants are obtained from one another by a similarity transformation (like scaling).

Self-similar solutions exist if the scaling of the independent and dependent variables,

$$t = C\bar{t}, \qquad x = C^k\bar{x}, \qquad w = C^m\bar{w}, \qquad \text{where } C \neq 0 \text{ is an arbitrary constant,} \tag{27.3.1.2}$$

for some k and m $(|k| + |m| \neq 0)$, is equivalent to the identical transformation. This means that the original equation

$$F(x, t, w, w_x, w_t, w_{xx}, w_{xt}, w_{tt}, \ldots) = 0, \tag{27.3.1.3}$$

when subjected to transformation (27.3.1.2), turns into the same equation in the new variables,

$$F(\bar{x}, \bar{t}, \bar{w}, \bar{w}_{\bar{x}}, \bar{w}_{\bar{t}}, \bar{w}_{\bar{x}\bar{x}}, \bar{w}_{\bar{x}\bar{t}}, \bar{w}_{\bar{t}\bar{t}}, \ldots) = 0. \tag{27.3.1.4}$$

Here, the function F is the same as in the original equation (27.3.1.3); it is assumed that equation (27.3.1.3) is independent of the parameter C.

Let us find the connection between the parameters α, β in solution (27.3.1.1) and the parameters k, m in the scaling transformation (27.3.1.2). Suppose

$$w = \Phi(x, t) \tag{27.3.1.5}$$

is a solution of equation (27.3.1.3). Then the function

$$\bar{w} = \Phi(\bar{x}, \bar{t}) \tag{27.3.1.6}$$

is a solution of equation (27.3.1.4).

In view of the explicit form of solution (27.3.1.1), it follows from (27.3.1.6) that

$$\bar{w} = \bar{t}^{\alpha} U(\bar{x}\bar{t}^{\beta}). \tag{27.3.1.7}$$

Using (27.3.1.2) to return to the new variables in (27.3.1.7), we get

$$w = C^{m-\alpha} t^{\alpha} U\left(C^{-k-\beta} x t^{\beta}\right). \tag{27.3.1.8}$$

By construction, this function satisfies equation (27.3.1.3) and hence is its solution. Let us require that solution (27.3.1.8) coincide with (27.3.1.1), so that the condition for the uniqueness of the solution holds for any $C \neq 0$. To this end, we must set

$$\alpha = m, \qquad \beta = -k. \tag{27.3.1.9}$$

In practice, the above existence criterion is checked: if a pair of k and m in (27.3.1.2) has been found, then a self-similar solution is defined by formulas (27.3.1.1) with parameters (27.3.1.9).

The method for constructing self-similar solutions on the basis of scaling transformations (27.3.1.2) is called the *similarity method*. It is significant that these transformations involve the arbitrary constant C as a parameter.

To make easier to understand, Fig. 27.1 depicts the basic stages for constructing self-similar solutions.

27.3.2. Examples of Self-Similar Solutions to Mathematical Physics Equations

Example 27.3. Consider the heat equation with a nonlinear power-law source term

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw^n. \tag{27.3.2.1}$$

The scaling transformation (27.3.1.2) converts equation (27.3.2.1) into

$$C^{m-1}\frac{\partial \bar{w}}{\partial \bar{t}} = aC^{m-2k}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + bC^{mn}\bar{w}^n.$$

Equating the powers of C yields the following system of linear algebraic equations for the constants k and m:

$$m - 1 = m - 2k = mn.$$

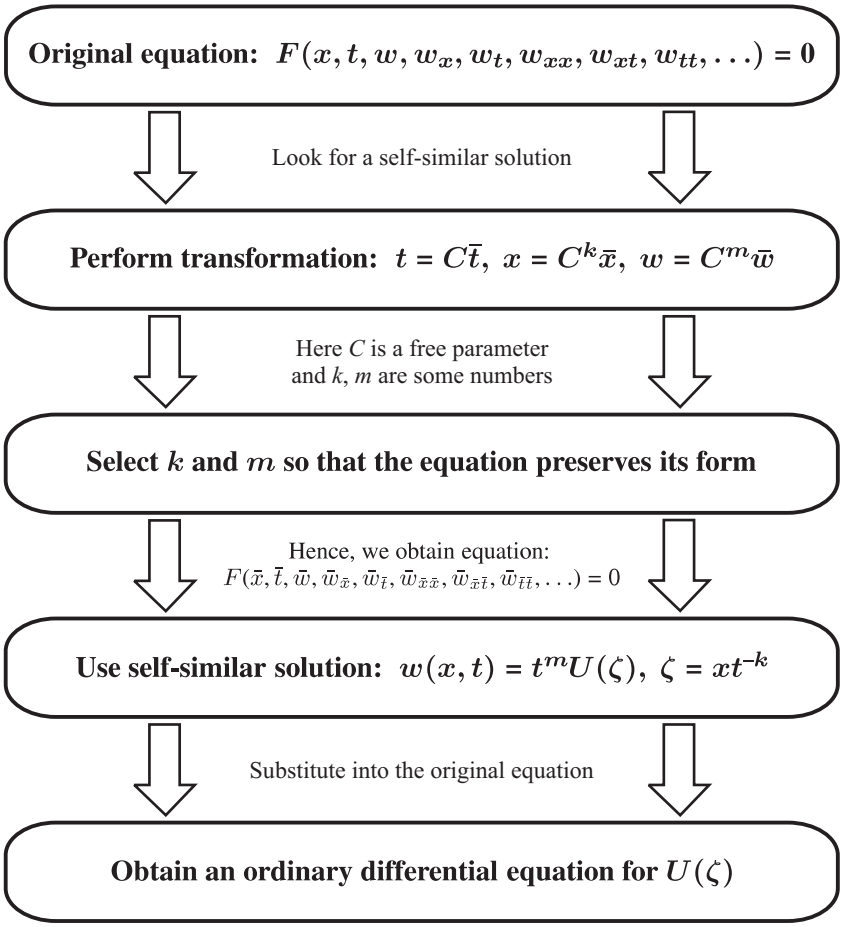

Figure 27.1. A simple scheme that is often used in practice for constructing self-similar solutions.

This system admits a unique solution: $k = \frac{1}{2}$, $m = \frac{1}{1-n}$. Using this solution together with relations (27.3.1.1) and (27.3.1.9), we obtain self-similar variables in the form

$$w = t^{1/(1-n)}U(\zeta), \quad \zeta = xt^{-1/2}.$$

Inserting these into (27.3.2.1), we arrive at the following ordinary differential equation for the function $U(\zeta)$:

$$aU''_{\zeta\zeta} + \frac{1}{2}\zeta U'_\zeta + \frac{1}{n-1}U + bU^n = 0.$$

Example 27.4. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right), \tag{27.3.2.2}$$

which occurs in problems of wave and gas dynamics. Inserting (27.3.1.2) into (27.3.2.2) yields

$$C^{m-2}\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = aC^{mn+m-2k}\frac{\partial}{\partial \bar{x}}\left(\bar{w}^n\frac{\partial \bar{w}}{\partial \bar{x}}\right).$$

Equating the powers of C results in a single linear equation, $m - 2 = mn + m - 2k$. Hence, we obtain $k = \frac{1}{2}mn + 1$, where m is arbitrary. Further, using (27.3.1.1) and (27.3.1.9), we find self-similar variables:

$$w = t^m U(\zeta), \quad \zeta = xt^{-\frac{1}{2}mn-1} \quad (m \text{ is arbitrary}).$$

Substituting these into (27.3.2.2), one obtains an ordinary differential equation for the function $U(\zeta)$.

TABLE 27.1
Some nonlinear equations of mathematical physics that admit self-similar solutions

Equation	Equation name	Form of solutions	Determining equation
$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$	Unsteady heat equation	$w=w(z),\ z=xt^{-1/2}$	$[f(w)w']'+\frac{1}{2}zw'=0$
$\frac{\partial w}{\partial t}=a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right)+bw^k$	Heat equation with source	$w=t^pu(z),\ z=xt^q,$ $p=\frac{1}{1-k},\ q=\frac{k-n-1}{2(1-k)}$	$a(u^nu')'-qzu'$ $+bu^k-pu=0$
$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+bw\frac{\partial w}{\partial x}$	Burgers equation	$w=t^{-1/2}u(z),\ z=xt^{-1/2}$	$au''+buu'+\frac{1}{2}zu'+\frac{1}{2}u=0$
$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+b\left(\frac{\partial w}{\partial x}\right)^2$	Potential Burgers equation	$w=w(z),\ z=xt^{-1/2}$	$aw''+b(w')^2+\frac{1}{2}zw'=0$
$\frac{\partial w}{\partial t}=a\left(\frac{\partial w}{\partial x}\right)^k\frac{\partial^2 w}{\partial x^2}$	Filtration equation	$w=t^pu(z),\ z=xt^q,$ $p=-\frac{(k+2)q+1}{k},\ q$ is any	$a(u')^ku''=qzu'+pu$
$\frac{\partial w}{\partial t}=f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}$	Filtration equation	$w=t^{1/2}u(z),\ z=xt^{-1/2}$	$2f(u')u''+zu'-u=0$
$\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$	Wave equation	$w=w(z),\ z=x/t$	$(z^2w')'=[f(w)w']'$
$\frac{\partial^2 w}{\partial t^2}=a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right)$	Wave equation	$w=t^{2k}u(z),\ z=xt^{-nk-1},$ k is any	$\frac{2k(2k-1)}{(nk+1)^2}u+\frac{nk-4k+2}{nk+1}zu'$ $+z^2u''=\frac{a}{(nk+1)^2}(u^nu')'$
$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=aw^n$	Heat equation with source	$w=x^{\frac{2}{1-n}}u(z),\ z=y/x$	$(1+z^2)u''-\frac{2(1+n)}{1-n}zu'$ $+\frac{2(1+n)}{(1-n)^2}u-au^n=0$
$\frac{\partial^2 w}{\partial x^2}+a\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}=0$	Equation of steady transonic gas flow	$w=x^{-3k-2}u(z),\ z=x^ky,$ k is any	$\frac{a}{k+1}u'u''+\frac{k^2}{k+1}z^2u''$ $-5kzu'+3(3k+2)u=0$
$\frac{\partial w}{\partial t}=a\frac{\partial^3 w}{\partial x^3}+bw\frac{\partial w}{\partial x}$	Korteweg–de Vries equation	$w=t^{-2/3}u(z),\ z=xt^{-1/3}$	$au'''+buu'+\frac{1}{3}zu'+\frac{2}{3}u=0$
$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=a\frac{\partial^3 w}{\partial y^3}$	Boundary-layer equation	$w=x^{\lambda+1}u(z),\ z=x^\lambda y,$ λ is any	$(2\lambda+1)(u')^2-(\lambda+1)uu''$ $=au'''$

Table 27.1 gives examples of self-similar solutions to some other nonlinear equations of mathematical physics.

The above method for constructing self-similar solutions is also applicable to systems of partial differential equations (see Example 28.18 in Section 28.4).

27.3.3. More General Approach Based on Solving a Functional Equation. Some Remarks

The algorithm for the construction of a self-similar solution, presented in Subsection 27.3.1, relies on representing this solution in the form (27.3.1.1) explicitly. However, there is a more general approach that allows the derivation of relation (27.3.1.1) directly from the condition of the invariance of equation (27.3.1.3) under transformations (27.3.1.2).

Indeed, let us assume that transformations (27.3.1.2) convert equation (27.3.1.3) into the same equation (27.3.1.4). Let (27.3.1.5) be a solution of equation (27.3.1.3). Then (27.3.1.6) will be a solution of equation (27.3.1.4). Switching back to the original variables (27.3.1.2) in (27.3.1.6),we obtain

$$w=C^m\Phi\left(C^{-k}x,C^{-1}t\right). \tag{27.3.3.1}$$

By construction, this function satisfies equation (27.3.1.3) and hence is its solution. Let us require that solution (27.3.3.1) coincide with (27.3.1.5), so that the uniqueness condition for the solution is met for any $C \neq 0$. This results in the functional equation

$$\Phi(x,t) = C^m \Phi\big(C^{-k}x, C^{-1}t\big). \tag{27.3.3.2}$$

Solution methods for such functional equations are outlined in Chapter 43. Further, we follow the method presented in Section 43.1. For $C = 1$, equation (27.3.3.2) is satisfied identically. Let us expand (27.3.3.2) in a power series in C about $C = 1$, then divide the resulting expression by $(C - 1)$, and proceed to the limit as $C \to 1$. This results in a linear first-order partial differential equation for Φ:

$$kx\frac{\partial \Phi}{\partial x} + t\frac{\partial \Phi}{\partial t} - m\Phi = 0. \tag{27.3.3.3}$$

The associated characteristic system of ordinary differential equations (see Subsection 23.1.1) has the form

$$\frac{dx}{kx} = \frac{dt}{t} = \frac{d\Phi}{m\Phi}.$$

Its first integrals are

$$xt^{-k} = A_1, \qquad t^{-m}\Phi = A_2,$$

where A_1 and A_2 are arbitrary constants. The general solution of the partial differential equation (27.3.3.3) is sought in the form $A_2 = U(A_1)$, where $U(A)$ is an arbitrary function (see Subsection 23.1.1). As a result, one obtains a solution of the functional equation (27.3.3.2) in the form

$$\Phi(x,t) = t^m U(\zeta), \qquad \zeta = xt^{-k}. \tag{27.3.3.4}$$

Substituting (27.3.3.4) into (27.3.1.5) yields the self-similar solution (27.3.1.1) with parameters (27.3.1.9).

Remark 1. Self-similar solutions (27.3.1.1) with $\alpha = 0$ arise in problems with simple initial and boundary conditions of the form

$$w = w_1 \quad \text{at} \quad t = 0 \quad (x > 0), \qquad w = w_2 \quad \text{at} \quad x = 0 \quad (t > 0),$$

where w_1 and w_2 are some constants.

Remark 2. Self-similar solutions that stem from the invariance of equations under scaling transformations will be called *classical self-similar solutions*. These solutions are simplest *invariant solutions*.

The condition for the existence of a transformation (27.3.1.2) preserving the form of the given equation is sufficient for the existence of a self-similar solution. However, this condition is not necessary: there are equations that do not admit transformations of the form (27.3.1.2) but have self-similar solutions.

For example, the equation

$$a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial t^2} = (bx^2 + at^2)f(w)$$

has a self-similar solution

$$w = w(z), \quad z = xt \quad \Longrightarrow \quad w'' - f(w) = 0,$$

but does not admit transformations of the form (27.3.1.2). In this equation, a and b can be arbitrary functions of the arguments x, t, w, w_x, w_t, w_{xx}, ...

A self-similar solution of the form (27.3.1.1) will be called nonclassical (noninvariant) if the equation does not admit scaling transformations (27.3.1.2).

Remark 3. Traveling-wave solutions are closely related to self-similar solutions. Indeed, setting

$$t = \ln\tau, \quad x = \ln y$$

in (27.2.1.1), we obtain a self-similar representation of a traveling wave:

$$w = W(kx - \lambda t) = W\bigl(k\ln(y\tau^{-\lambda/k})\bigr) = U(y\tau^{-\lambda/k}),$$

where $U(z) = W(k\ln z)$.

The examples considered in Sections 27.2 and 27.3 demonstrate that the existence of exact solutions is due to the fact the partial differential equations concerned are invariant under some transformations or, what is the same, possess some symmetries. The next Chapter 28 outlines a different, more advanced approach to finding self-similar and some other simple invariant solutions. In Chapter 32, a general method for the investigation of symmetries of differential equations (the group-theoretic method) will be described that allows finding similar and more complicated invariant solutions on a routine basis.

27.3.4. Generalized Self-Similar Solutions

A *generalized self-similar solution* has the form

$$w(x,t) = \varphi(t)u(z), \qquad z = \psi(t)x. \tag{27.3.4.1}$$

Formula (27.3.4.1) comprises the above self-similar and exponential self-similar solutions (27.3.1.1) and (27.4.1.1) as special cases.

The procedure of finding generalized self-similar solutions is briefly as follows: after substituting (27.3.4.1) into the given equation, one chooses the functions $\varphi(t)$ and $\psi(t)$ so that $u(z)$ satisfies a single ordinary differential equation.

Example 27.5. A solution of the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(w^m\frac{\partial w}{\partial x}\right) \tag{27.3.4.2}$$

will be sought in the form (27.3.4.1). Taking into account that $x = z/\psi(t)$, we find the derivatives

$$w_t = \varphi_t' u + \varphi\psi_t' x u_z' = \varphi_t' u + \frac{\varphi\psi_t'}{\psi} z u_z', \quad w_x = \varphi\psi u_z', \quad (w^n w_x)_x = \psi^2\varphi^{n+1}(u^n u_z')_z'.$$

Substituting them into (27.3.4.2) and dividing by φ_t', we have

$$u + \frac{\varphi\psi_t'}{\varphi_t'\psi} z u_z' = \frac{\psi^2\varphi^{n+1}}{\varphi_t'}(u^n u_z')_z'. \tag{27.3.4.3}$$

For this relation to be an ordinary differential equation for $u(z)$, the functional coefficients of zu_z' and $(u^n u_z')_z'$ must be constant:

$$\frac{\varphi\psi_t'}{\varphi_t'\psi} = a, \qquad \frac{\psi^2\varphi^{n+1}}{\varphi_t'} = b. \tag{27.3.4.4}$$

The function $u(z)$ will satisfy the equation

$$u + azu_z' = b(u^n u_z')_z'.$$

From the first equation in (27.3.4.4) it follows that

$$\psi = C_1\varphi^a, \tag{27.3.4.5}$$

where C_1 is an arbitrary constant. Substituting the resulting expression into the second equation in (27.3.4.4) and integrating, we obtain

$$\begin{aligned} &\frac{C_1^2}{b}t + C_2 = -\frac{1}{2a+n}\varphi^{-2a-n} \quad \text{for} \quad a \neq -\frac{n}{2}, \\ &\frac{C_1^2}{b}t + C_2 = \ln|\varphi|, \qquad\qquad\quad \text{for} \quad a = -\frac{n}{2}, \end{aligned} \tag{27.3.4.6}$$

where C_2 is an arbitrary constant. From (27.3.4.5)–(27.3.4.6) we have, in particular,

$$\varphi(t) = t^{-\frac{1}{2a+n}}, \quad \psi(t) = t^{-\frac{a}{2a+n}} \quad \text{at} \quad C_1 = 1, \quad C_2 = 0, \quad b = -\frac{1}{2a+n};$$

$$\varphi(t) = e^{2t}, \qquad \psi(t) = e^{-nt} \qquad \text{at} \quad C_1 = 1, \quad C_2 = 0, \quad b = \frac{1}{2}.$$

The first pair of functions $\varphi(t)$ and $\psi(t)$ corresponds to a self-similar solution (with any $a \neq -n/2$), and the second pair, to an exponential self-similar solution.

⊙ *References for Chapter* 27: P. W. Bridgman (1931), W. F. Ames (1972), G. W. Bluman and J. D. Cole (1974), G. I. Barenblatt and Ya. B. Zel'dovich (1972), W. F. Ames, R. J. Lohner, and E. Adams (1981), L. Dresner (1983), G. I. Barenblatt (1989), L. I. Sedov (1993), A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin (2004, *a*).

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Chapter 28
Elementary Theory of Using Invariants for Solving Equations

This chapter describes a simple scheme for the analysis of mathematical equations which relies on using invariants and makes it possible to simplify algebraic equations, reduce the order of ordinary differential equations (or integrate them), and find exact solutions of nonlinear partial differential equations. Invariants are constructed by searching for transformations that preserve the form of the equations; the notions and complex techniques of symmetry analysis (see Chapter 32) are not used here. Numerous examples of solving specific differential equations are given. It is significant that even with the simplest linear transformations of translation and scaling, as well as their compositions, the number of solvable ordinary differential equations (or those admitting order reduction) that can be described in a unified way is more than those discussed in the overwhelming majority of available textbooks. For nonlinear equations of mathematical physics, this approach makes it possible to find all most common invariant solutions. To use this simple method, one does not have to have a strong mathematical background—what is required is to be able to solve simple algebraic equations (and system of equations) and differentiate. To distinguish it from the classical group analysis method, the approach presented in this chapter will be called the *method of invariants*.

28.1. Introduction. Symmetries. General Scheme of Using Invariants for Solving Mathematical Equations

28.1.1. Symmetries. Transformations Preserving the Form of Equations. Invariants

Symmetries of mathematical equations are understood as transformations that preserve the form of equations. Given below are examples of specific equations that remain unchanged under some simple transformations.

Example 28.1. Consider the biquadratic equation

$$x^4 + ax^2 + 1 = 0. \tag{28.1.1.1}$$

The change of variable

$$x = -\bar{x}$$

results in exactly the same equation

$$\bar{x}^4 + a\bar{x}^2 + 1 = 0.$$

This means that equation (28.1.1.1) preserves its form under the transformation $x = -\bar{x}$.

Two other transformations

$$x = \pm\frac{1}{\bar{x}}$$

also preserve the form of equation (28.1.1.1), since multiplying by $\widetilde{x}^4$ gives

$$\widetilde{x}^4 + a\widetilde{x}^2 + 1 = 0.$$

Example 28.2. The form of the differential equation

$$y''_{xx} - y'_x = 0 \tag{28.1.1.2}$$

will not change if we make any of the transformations

$$\begin{aligned} &x = \bar{x} + a, \quad y = \bar{y} && (a \text{ is any number});\\ &x = \bar{x}, \qquad y = \bar{y} + b && (b \text{ is any number});\\ &x = \bar{x}, \qquad y = c\bar{y} && (c \text{ is any nonzero number}), \end{aligned} \tag{28.1.1.3}$$

since we obtain exactly the same equation

$$\bar{y}''_{\bar{x}\bar{x}} - \bar{y}'_{\bar{x}} = 0$$

for each of the three transformations.

It will be shown below that transformations preserving the form of equations allow us to "multiply" solutions.

An *invariant of a transformation* is a nonconstant function that remains unchanged under the action of the transformation. Invariants of transformations can depend on the independent and dependent variables and their derivatives (when we deal with differential equations). To clarify the concept of an invariant that preserves its form under a transformation, we consider a few simple examples.

Example 28.3. The transformation of simultaneous translation in two coordinate axes

$$x = \bar{x} + a, \quad y = \bar{y} + a,$$

where a is any number, has the invariant

$$I = y - x = \bar{y} - \bar{x}.$$

If x is the independent variable and y is the dependent one, then the derivatives

$$I_2 = y'_x = \bar{y}'_{\bar{x}}, \quad I_3 = y''_{xx} = \bar{y}''_{\bar{x}\bar{x}}, \quad \ldots$$

are also invariants of the transformation.

Example 28.4. The transformation of uniform scaling in two coordinate axes

$$x = a\bar{x}, \quad y = a\bar{y},$$

where a is any nonzero number, has the invariant

$$I_1 = \frac{y}{x} = \frac{\bar{y}}{\bar{x}}.$$

If x is the independent variable and y is the dependent one, then there are also more complicated invariants that depend on derivatives and remain unchanged under the transformation. Examples are

$$I_2 = y'_x = \bar{y}'_{\bar{x}}, \quad I_3 = xy''_{xx} = \bar{x}\bar{y}''_{\bar{x}\bar{x}}, \quad \ldots.$$

28.1.2. General Scheme of Using Invariants for Solving Mathematical Equations

Displayed below is a schematic diagram for the analysis of mathematical equations which is based on searching for transformations that preserve the form of equations followed by changing, in the equations, from the original variable to new ones—invariants of the transformations.

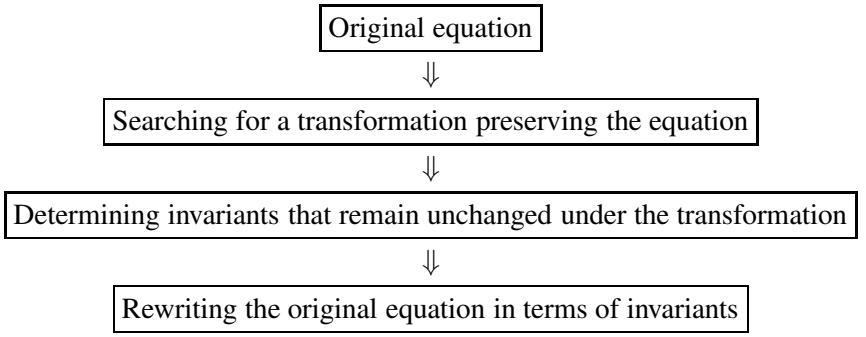

Figure 28.1. General scheme of using invariants for solving mathematical equations.

Once the above steps have been completed, the equation is often simplified and reduced to a solvable form. It is important to note that the above scheme can successfully be applied to various types of mathematical equations (see Sections 28.2–28.4 below).

For better understanding and learning of the ideas of how to use invariants or solving mathematical equations, we follow the approach "from simple to complex," first parenting results for algebraic equations, then for ordinary differential equations, and finally for nonlinear partial differential equations.

28.2. Algebraic Equations and Systems of Equations

28.2.1. Algebraic Equations with Even Powers

Consider the algebraic equation

$$a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + a_{2n-4}x^{2n-4} + \cdots + a_4x^4 + a_2x^2 + a_0 = 0, \tag{28.2.1.1}$$

which only contains even powers of x. A biquadratic equation is a special case of equation (28.2.1.1) with $n = 2$.

The change of variable

$$x = -\bar{x} \tag{28.2.1.2}$$

leads to exactly the same equation for $\bar{x}$; equation (28.2.1.1) is said to be invariant under transformation (28.2.1.2). It follows that if $x = x_1$ is a solution of equation (28.2.1.1), then $x = -x_1$ is also a solution of this equation.

By squaring (28.2.1.2), we get a simple algebraic function that is left unchanged by transformation (28.2.1.2):

$$x^2 = \bar{x}^2. \tag{28.2.1.3}$$

This function is an invariant of transformation (28.2.1.2). By taking the invariant (28.2.1.3) as the new variable, $z = x^2$, we can represent equation (28.2.1.1) of degree $2n$ as an equation of degree n:

$$a_{2n}z^n + a_{2n-2}z^{n-1} + a_{2n-4}z^{n-2} + \cdots + a_4z^2 + a_2z + a_0 = 0.$$

Thus, in this case, the change from the original variable x to the invariant $z = x^2$ of transformation (28.2.1.3) allows us to simplify the original equation—its degree has been halved.

28.2.2. Reciprocal Equations

1°. A *reciprocal* (*palindromic*) *polynomial equation* of even degree has the form

$$a_0x^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0 \qquad (a_0 \neq 0). \tag{28.2.2.1}$$

The left-hand side of this equation is called a *reciprocal polynomial* or *palindromic polynomial*.

The change of variable

$$x = \frac{1}{\bar{x}} \tag{28.2.2.2}$$

transforms (28.2.2.1) into exactly the same equation (after multiplication by $\bar{x}^{2n}$). It follows that if $x = x_1$ is a root of equation (28.2.2.1), then $x = 1/x_1$ is also a root of the equation.

The simplest reciprocal equation is a quadratic equation

$$a_0x^2 + a_1x + a_0 = 0.$$

Dividing it by x and grouping the first and last terms together, we get

$$a_0\left(x + \frac{1}{x}\right) + a_1 = 0.$$

The result is convenient to rewrite as a first-degree equation

$$a_0z + a_1 = 0,$$

where

$$z = x + \frac{1}{x} = \bar{x} + \frac{1}{\bar{x}} \tag{28.2.2.3}$$

is the simplest *invariant of transformation* (28.2.2.2).

THEOREM 1. *In the general case, the reciprocal equation (28.2.2.1) of even degree $2n$ can be simplified with substitution (28.2.2.3), resulting in an algebraic equation of degree n.*

Example 28.5. Consider the quartic reciprocal equation

$$ax^4 + bx^3 + cx^2 + bx + a = 0.$$

Dividing it by x^2 and grouping terms, we get

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0. \tag{28.2.2.4}$$

Taking into account that

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} \quad \Longrightarrow \quad x^2 + \frac{1}{x^2} = z^2 - 2$$

and using the change of variable (28.2.2.3), which is an invariant of transformation (28.2.2.2), we reduce (28.2.2.4) to the quadratic equation

$$az^2 + bz + c - 2a = 0.$$

Theorem 2. *In the general case, a reciprocal equation of odd order*

$$P_{2n+1}(x) = 0, \quad \text{where} \quad P_{2n+1}(x) \equiv a_0x^{2n+1} + a_1x^{2n} + a_2x^{2n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

has a root $x = -1$, and the left-hand side can be represented as

$$P_{2n+1}(x) = (x + 1)Q_{2n}(x),$$

where $Q_{2n}(x)$ is a reciprocal polynomial of degree $2n$.

Example 28.6. The cubic reciprocal equation

$$ax^3 + bx^2 + bx + a = 0$$

can be represented in the form

$$(x + 1)[ax^2 + (b - a)x + a] = 0.$$

It follows from Theorem 2 that a reciprocal equation of degree $2n + 1$ can be reduced, by dividing by $(x + 1)$ and introducing the new variable (28.2.2.3), to an algebraic equation of degree n.

2°. A *generalized reciprocal polynomial equation* of even degree has the form

$$\begin{aligned} &a_0x^{2n} + a_1x^{2n-1} + \cdots + a_{n-1}x^{n+1} + a_nx^n \\ &\quad + \lambda a_{n-1}x^{n-1} + \lambda^2 a_{n-2}x^{n-2} + \cdots + \lambda^{n-1}a_1x + \lambda^n a_0 = 0 \qquad (a_0 \neq 0). \end{aligned} \tag{28.2.2.5}$$

The first $n + 1$ terms (written in the first row) coincide with the respective terms of the reciprocal equation (28.2.2.1) and the remaining terms (in the second row) differ from the respective terms of equation (28.2.2.1) by factors λ^m. In the special case $\lambda = 1$, equation (28.2.2.5) coincides with (28.2.2.1).

It is not difficult to verify that the transformation

$$x = \frac{\lambda}{\bar{x}} \tag{28.2.2.6}$$

leaves equation (28.2.2.5) unchanged, and the simplest invariant of transformation (28.2.2.6) is written as

$$z = x + \frac{\lambda}{x} = \bar{x} + \frac{\lambda}{\bar{x}}. \tag{28.2.2.7}$$

The introduction of the new variable (28.2.2.7) reduces (28.2.2.5) to an equation of degree n.

Example 28.7. Consider the quartic equation

$$ax^4 + bx^3 + cx^2 - bx + a = 0,$$

which is a special case of equation (28.2.2.5) with $n = 2$ and $\lambda = -1$. The change of variable

$$z = x - \frac{1}{x}$$

leads to the quadratic equation

$$az^2 + bz + 2a + c = 0.$$

28.2.3. Systems of Algebraic Equations Symmetric with Respect to Permutation of Arguments

A bivariate polynomial $P(x, y)$ is called *symmetric* if it does not change after the permutation of its arguments: $P(x, y) = P(y, x)$.

Remark. In terms of transformations, a symmetric polynomial is defined as a polynomial that is left unchanged by the transformation $x = \bar{y}$, $y = \bar{x}$.

The simplest symmetric polynomials

$$u = x + y, \quad w = xy \tag{28.2.3.1}$$

are called *elementary*. These polynomials are the simplest algebraic invariants to the permutation of arguments. Any symmetric bivariate polynomial can be uniquely expressed in terms of the elementary polynomials.

For the solution of systems of two algebraic equations

$$P(x, y) = 0, \quad Q(x, y) = 0,$$

where P and Q are symmetric polynomials, it is helpful to use the elementary symmetric polynomials (28.2.3.1) as the new variables. Such systems possess the following property: if $x = x_0$, $y = y_0$ is a solution to the system, then $x = y_0$, $y = x_0$ is also a solution.

Example 28.8. Consider the nonlinear system of algebraic equations

$$\begin{aligned} x^2 + axy + y^2 &= b, \\ x^4 + cx^2y^2 + y^4 &= d. \end{aligned} \tag{28.2.3.2}$$

It remains unchanged under the permutation of the variables.

In (28.2.3.2), by changing from x and y to the variables (28.2.3.1) and taking into account the formulas

$$\begin{aligned} &x^2 + y^2 = (x + y)^2 - 2xy = u^2 - 2w, \\ &x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (u^2 - 2w)^2 - 2w^2 = u^4 - 4u^2w + 2w^2, \end{aligned}$$

we obtain

$$\begin{aligned} u^2 - (a - 2)w &= b, \\ u^4 - 4u^2w + (c + 2)w^2 &= d. \end{aligned} \tag{28.2.3.3}$$

Eliminating u, we arrive at the quadratic equation

$$(a^2 + c - 2)w^2 - 2abw + b^2 - d = 0.$$

The further procedure of finding solutions is straightforward and omitted here.

Example 28.9. Consider the nonlinear system of algebraic equations

$$\begin{aligned} x^2 + y^2 &= a, \\ x^3 + y^3 &= b. \end{aligned} \tag{28.2.3.4}$$

Changing to the variables (28.2.3.1) and taking into account that $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$, we get

$$\begin{aligned} u^2 - 2w &= a, \\ u^3 - 3uw &= b. \end{aligned}$$

Eliminating w yields the cubic equation

$$u^3 - 3au + 2b = 0. \tag{28.2.3.5}$$

Note that the straightforward elimination of y from system (28.2.3.4) results in a much more complex equation of degree 6:

$$(a - x^2)^3 = (b - x^3)^2 \quad \Longrightarrow \quad 2x^6 - 3ax^4 - 2bx^3 + 3a^2x^2 + b^2 - a^3 = 0.$$

28.3. Ordinary Differential Equations

28.3.1. Transformations Preserving the Form of Equations. Invariants

An ordinary differential equation

$$F(x, y, y'_x, \ldots, y^{(n)}_x) = 0 \tag{28.3.1.1}$$

is said to be *invariant under an invertible transformation*

$$x = \varphi(\bar{x}, \bar{y}), \quad y = \psi(\bar{x}, \bar{y}) \tag{28.3.1.2}$$

if the substitution of (28.3.1.2) into (28.3.1.1) leads to exactly the same equation

$$F(\bar{x}, \bar{y}, \bar{y}'_{\bar{x}}, \ldots, \bar{y}^{(n)}_{\bar{x}}) = 0. \tag{28.3.1.3}$$

The function F is the same in both equations (28.3.1.1) and (28.3.1.3).

Transformations that preserve the form on an equation can be used to "multiply" its solutions. Indeed, suppose

$$y = g(x) \tag{28.3.1.4}$$

is a particular solution to equation (28.3.1.1). Since equation (28.3.1.1) is left the same by the change of variables (28.3.1.2), then

$$\bar{y} = g(\bar{x}) \tag{28.3.1.5}$$

is a solution to the transformed equation (28.3.1.3). In (28.3.1.5), changing back to the old variables using (28.3.1.2) (the relations must be solved for $\bar{x}$ and $\bar{y}$), we obtain a solution to equation (28.3.1.1) that differs, in general, from the original solution (28.3.1.4).

Example 28.10. The third-order equation

$$y'''_{xxx} - y'_x = 0 \tag{28.3.1.6}$$

has a particular solution

$$y = e^x.$$

The transformation

$$x = \bar{x} + a, \quad y = \bar{y} + b \tag{28.3.1.7}$$

leaves the equation unchanged, and therefore the transformed equation $\bar{y}'''_{\bar{x}\bar{x}\bar{x}} - \bar{y}'_{\bar{x}} = 0$ has a solution $\bar{y} = e^{\bar{x}}$. Inserting the old variables, by inverting the formulas (28.3.1.7), we obtain a new solution to equation (28.3.1.6):

$$y = Ae^x + b, \quad A = e^{-a},$$

which involves two arbitrary constants A and b.

A function $I(x, y, y'_x)$, other than a constant, is called an *invariant of transformation* (28.3.1.2) if it remains unchanged under the transformation:

$$I(x, y, y'_x) = I(\bar{x}, \bar{y}, \bar{y}'_{\bar{x}}).$$

Remark. If $I = I(x, y, y'_x)$ is an invariant of transformation (28.3.1.2), then $\Psi(I)$, where Ψ is an arbitrary function, is also an invariant of the transformation.

28.3.2. Order Reduction Procedure for Equations with $n \geq 2$ (Reduction to Solvable Form with $n = 1$)

Let us now consider in more detail the scheme outlined in Section 28.1 for using invariants in the analysis of mathematical equations as applied to ordinary differential equations.

Given an nth-order equation (28.3.1.1), one should seek, at the first stage, a transformation

$$x = \varphi(\bar{x}, \bar{y}; a), \quad y = \psi(\bar{x}, \bar{y}; a) \tag{28.3.2.1}$$

that preserves the form of the equation. Transformation (28.3.2.1) must depend on a *single free parameter* $a \in [a_1, a_2]$; the original equation (28.3.1.1) is independent of this parameter.

At the second stage, for second- and higher-order equations ($n \geq 2$), one constructs two functionally independent invariants of transformation (28.3.2.1):

$$I_1 = I_1(x, y), \quad I_2 = I_2(x, y, y'_x). \tag{28.3.2.2}$$

At the third stage, the invariants (28.3.2.2) are taken as the new variable for equation (28.3.1.1) and the transformation

$$u = I_2, \quad z = I_1, \quad u = u(z)$$

is performed. This results in an $(n-1)$st-order equation, so that the order of the original equation is reduced by one.

For first-order equations ($n = 1$), one should make the change of variable

$$z = I_1, \quad z = z(x),$$

in (28.3.1.1) at the third stage. This results in a solvable (separable) equation.

28.3.3. Simple Transformations. Invariant Determination Procedure

In what follows, we will only use the simplest transformations

$$\begin{aligned} &x = \bar{x} + A, \quad y = \bar{y} + B \qquad (\textit{translation}); \\ &x = A\bar{x}, \qquad y = B\bar{y} \qquad\quad (\textit{scaling}) \end{aligned}$$

and their compositions

$$x = A_1\bar{x} + B_1, \quad y = A_2\bar{y} + B_2. \tag{28.3.3.1}$$

Then the derivatives satisfy linear relations:

$$y'_x = \frac{A_2}{A_1}\bar{y}'_{\bar{x}}, \quad y''_{xx} = \frac{A_2}{A_1^2}\bar{y}''_{\bar{x}\bar{x}}, \quad y_x^{(n)} = \frac{A_2}{A_1^n}\bar{y}_{\bar{x}}^{(n)}. \tag{28.3.3.2}$$

The transformation coefficients A_1, A_2, B_1, and B_2 are determined from the invariance condition for the equation; these coefficients must depend on a single free parameter a.

The following statement is true. Suppose transformation (28.3.3.1) preserves the form of an equation that has a particular solution (28.3.1.4). Then

$$y = B_2 + A_2 g\left(\frac{x - B_1}{A_1}\right)$$

is also a solution of the equation.

The first invariant I_1 is obtained by eliminating a from (28.3.3.1). The second invariant I_2 is obtained by eliminating a from one of the relations in (28.3.3.1) and the first relation in (28.3.3.2).

28.3.4. Analysis of Some Ordinary Differential Equations. Useful Remarks

Example 28.11. The second-order equation

$$y''_{xx} = F(x, y'_x), \tag{28.3.4.1}$$

which does not involve y explicitly, remains unchanged under an arbitrary translation in the dependent variable: $y \Longrightarrow y + a$ (which corresponds to $y = \bar{y} + a$), where a is a free parameter. Moreover, out of the three variables x, y, and y'_x, two remain unchanged:

$$x, \quad y'_x.$$

These are invariants of equation (28.3.4.1); hence, $I_1 = x$ and $I_2 = y'_x$. These can be taken as the new variables:

$$u = y'_x, \quad z = x, \quad u = u(z).$$

As a result, we arrive at a first-order equation: $u'_x = F(x, u)$.

Example 28.12. The autonomous second-order equation

$$y''_{xx} = F(y, y'_x), \tag{28.3.4.2}$$

which does not involve x explicitly, remains unchanged under an arbitrary translation in the independent variable: $x \Longrightarrow x + a$, where a is a free parameter. Out of the three variables x, y, and y'_x, two remain unchanged:

$$y, \quad y'_x.$$

These are invariants of equation (28.3.4.2); hence, $I_1 = y$ and $I_2 = y'_x$. We choose them as the new variables:

$$u = y'_x, \quad z = y, \quad u = u(z).$$

This results in a first-order equation: $uu'_y = F(y, u)$.

Example 28.13. The nonlinear second-order equation

$$y''_{xx} = yF\left(x, \frac{y'_x}{y}\right) \tag{28.3.4.3}$$

does not change if the dependent variable is scaled: $y \Longrightarrow ay$. Two combinations out of the three variables x, y, and y'_x do not change:

$$x, \quad \frac{y'_x}{y}.$$

These are invariants of equation (28.3.4.3) and can be taken as the new variables:

$$u = \frac{y'_x}{y}, \quad z = x, \quad u = u(z).$$

Differentiating u linear x yields

$$u'_x = \frac{y''_{xx}}{y} - \left(\frac{y'_x}{y}\right)^2 = \frac{y''_{xx}}{y} - u^2.$$

Using this relation to eliminate y''_{xx} in (28.3.4.3), one arrives at the first-order equation

$$u'_x = F(x, u) - u^2.$$

Remark. For $F(x, u) = g(x) + f(x)u$, the original equation (28.3.4.3) is a general linear homogeneous second-order equation. With the above transformation, it is reduced to a first-order equation with a quadratic nonlinearity.

Example 28.14. Consider the nonlinear second-order equation

$$yy''_{xx} - (y'_x)^2 = ky^3 e^{\lambda x}. \tag{28.3.4.4}$$

We look for an invariant transformation of the form

$$x = \bar{x} + b, \quad y = a\bar{y}. \tag{28.3.4.5}$$

Substituting (28.3.4.5) into (28.3.4.4) and canceling by a, we obtain

$$\bar{y}\bar{y}''_{\bar{x}\bar{x}} - (\bar{y}'_{\bar{x}})^2 = ae^{\lambda b} k\bar{y}^3 e^{\lambda \bar{x}}.$$

Requiring that this equation coincide with (28.3.4.4), we get the relation for determining the parameter b:

$$ae^{\lambda b} = 1 \quad \Longrightarrow \quad b = -\frac{1}{\lambda}\ln a. \tag{28.3.4.6}$$

The parameter a remains free.

Substituting (28.3.4.6) into (28.3.4.5) and eliminating a from the second relation with the aid of the first relation, we obtain

$$y = \bar{y}e^{\lambda(\bar{x}-x)} \quad \Longrightarrow \quad ye^{\lambda x} = \bar{y}e^{\lambda\bar{x}}.$$

Hence,

$$I_1 = ye^{\lambda x} \tag{28.3.4.7}$$

is an invariant of transformation (28.3.4.5)–(28.3.4.6). Another invariant can be found by calculating the derivative

$$y'_x = a\bar{y}'_{\bar{x}}.$$

Eliminating a with the aid of the second relation in (28.3.4.5) yields

$$\frac{y'_x}{y} = \frac{\bar{y}'_{\bar{x}}}{\bar{y}} = I_2. \tag{28.3.4.8}$$

To reduce the order of the original equation, one should take the invariants (28.3.4.7)–(28.3.4.8) as the new variables:

$$z = e^{\lambda x}y, \quad u = \frac{y'_x}{y}, \quad u = u(z). \tag{28.3.4.9}$$

On the one hand,

$$u'_x = \frac{y''_{xx}}{y} - \left(\frac{y'_x}{y}\right)^2 = \frac{y''_{xx}}{y} - u^2; \tag{28.3.4.10}$$

on the other hand,

$$u'_x = u_z z'_x = (\lambda e^{\lambda x}y + e^{\lambda x}y'_x)u'_z = \left(\lambda z + e^{\lambda x}y\frac{y'_x}{y}\right)u'_z = (\lambda z + zu)u'_z. \tag{28.3.4.11}$$

Equating (28.3.4.10) with (28.3.4.11), we get

$$\frac{y''_{xx}}{y} - u^2 = (\lambda z + zu)u'_z \quad \Longrightarrow \quad \frac{y''_{xx}}{y} = u^2 + (\lambda z + zu)u'_z.$$

Inserting this into (28.3.4.4) and performing elementary rearrangements, we arrive at a separable first-order equation:

$$(\lambda + u)u'_z = k.$$

Remark. The more general, nonlinear second-order equation

$$y''_{xx} = yF\left(e^{\lambda x}y, \frac{y'_x}{y}\right)$$

has similar properties. Transformation (28.3.4.9) reduces it to the first-order equation

$$u^2 + (\lambda z + zu)u'_z = F(z, u).$$

Example 28.15. Now consider the nonlinear first-order equation

$$y'_x = f\left(\frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2}\right). \tag{28.3.4.12}$$

Its left-hand side remains unchanged under transformations of the form

$$x = a\bar{x} + b, \quad y = a\bar{y} + c, \tag{28.3.4.13}$$

where a, b, and c are arbitrary constants. Substituting (28.3.4.13) into the argument of the right-hand side function of (28.3.4.12) gives

$$\frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2} = \frac{a(\alpha_1\bar{x} + \beta_1\bar{y}) + \alpha_1 b + \beta_1 c + \gamma_1}{a(\alpha_2\bar{x} + \beta_2\bar{y}) + \alpha_2 b + \beta_2 c + \gamma_2}. \tag{28.3.4.14}$$

TABLE 28.1
Some second-order ordinary differential equations that admit order reduction, and invariants.

Equation	Invariant transformation	First invariant	Second invariant
$F(x, y'_x, y''_{xx}) = 0$	$y = \bar{y} + a$	$I_1 = x$	$I_2 = y'_x$
$F(y, y'_x, y''_{xx}) = 0$	$x = \bar{x} + a$	$I_1 = y$	$I_2 = y'_x$
$F(\alpha x + \beta y, y'_x, y''_{xx}) = 0$	$x = \bar{x} + a\beta$, $y = \bar{y} - a\alpha$	$I_1 = \alpha x + \beta y$	$I_2 = y'_x$
$F\left(x, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\right) = 0$	$y = a\bar{y}$	$I_1 = x$	$I_2 = \frac{y'_x}{y}$
$F(y, xy'_x, x^2 y''_{xx}) = 0$	$x = a\bar{x}$	$I_1 = y$	$I_2 = xy'_x$
$F\left(e^{\lambda x} y, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\right) = 0$	$x = \bar{x} - \frac{1}{\lambda} \ln a$, $y = a\bar{y}$	$I_1 = e^{\lambda x} y$	$I_2 = \frac{y'_x}{y}$
$F(e^{\lambda x} y, e^{\lambda x} y'_x, e^{\lambda x} y''_{xx}) = 0$	$x = \bar{x} - \frac{1}{\lambda} \ln a$, $y = a\bar{y}$	$I_1 = e^{\lambda x} y$	$I_2 = e^{\lambda x} y'_x$
$F(xe^{\lambda y}, xy'_x, x^2 y''_{xx}) = 0$	$x = a\bar{x}$, $y = \bar{y} - \frac{1}{\lambda} \ln a$	$I_1 = xe^{\lambda y}$	$I_2 = xy'_x$
$F(x^k y, x^{k+1} y'_x, x^{k+2} y''_{xx}) = 0$	$x = a\bar{x}$, $y = a^{-k}\bar{y}$	$I_1 = x^k y$	$I_2 = x^{k+1} y'_x$
$F\left(x^n y^m, \frac{xy'_x}{y}, \frac{x^2 y''_{xx}}{y}\right) = 0$	$x = a^m \bar{x}$, $y = a^{-n}\bar{y}$	$I_1 = x^n y^m$	$I_2 = \frac{xy'_x}{y}$

For equation (28.3.4.12) to be invariant under transformation (28.3.4.13), one must set

$$\alpha_1 b + \beta_1 c + \gamma_1 = a\gamma_1, \quad \alpha_2 b + \beta_2 c + \gamma_2 = a\gamma_2 \tag{28.3.4.15}$$

in (28.3.4.14). These relations can be viewed as a system of two linear algebraic equations for the coefficients b and c; the coefficient a remains arbitrary. Thus, equation (28.3.4.12) is invariant under transformation (28.3.4.13) subject to conditions (28.3.4.15). The argument of the right-hand side function of (28.3.4.12) is an invariant of the transformation. Therefore, the change of variable

$$z = \frac{\alpha_1 x + \beta_1 y + \gamma_1}{\alpha_2 x + \beta_2 y + \gamma_2}, \quad \text{where} \quad z = z(x), \tag{28.3.4.16}$$

should be made in equation (28.3.4.12). Solving (28.3.4.16) for y, differentiating with respect to x, substituting y'_x by $f(z)$, which follows from (28.3.4.12) and (28.3.4.16), and performing elementary rearrangements, one arrives at the separable equation

$$\frac{(\alpha_2\beta_1 - \alpha_1\beta_2)x + \beta_1\gamma_2 - \beta_2\gamma_1}{(\beta_2 z - \beta_1)^2} z'_x = f(z) + \frac{\alpha_2 z - \alpha_1}{\beta_2 z - \beta_1}.$$

Table 28.1 lists some second-order ordinary differential equations that admit order reduction by using the simplest invariant transformations. For first-order equations, where $F(u, v, w)$ is independent of the third argument, the equations listed in Table 28.1 can be solved by changing from y to the new dependent variable $z = I_1(x, y)$, where I_1 is the first invariant.

The results presented in Table 28.1 are easy to generalize to nonlinear equations of arbitrary order.

Remark. The above method of invariants for the analysis of ordinary differential equations takes advantage of the ideas of the group analysis method but is much simpler. To learn how to apply the former method, one should only be able to solve simple algebraic equations (and systems) and differentiate, whereas the application of the group analysis method requires, at intermediate stages, the solution of partial differential equations (which leads beyond the standard courses of ordinary differential equations). Other advantages of the simple method of invariants described include the fact that there is no need to introduce new concepts, which are abundant in the group analysis method, and that the number of solvable ordinary differential equations (or those admitting order reduction) describable in a unified way is more than those discussed in the overwhelming majority of available textbooks.

28.4. Partial Differential Equations

28.4.1. Transformations Preserving the Form of Equations. Invariants

We consider (invertible) point transformations of the form

$$x = \varphi(\bar{x}, \bar{t}, \bar{w}), \quad t = \psi(\bar{x}, \bar{t}, \bar{w}), \quad w = \chi(\bar{x}, \bar{t}, \bar{w}). \tag{28.4.1.1}$$

A partial differential equation

$$F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial t}, \frac{\partial^2 w}{\partial t^2}, \ldots\right) = 0 \tag{28.4.1.2}$$

is said to be *invariant under transformation* (28.4.1.1) if the substitution of (28.4.1.1) in (28.4.1.2) results in exactly the same equation

$$F\left(\bar{x}, \bar{t}, \bar{w}, \frac{\partial \bar{w}}{\partial \bar{x}}, \frac{\partial \bar{w}}{\partial \bar{t}}, \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}, \frac{\partial^2 \bar{w}}{\partial \bar{x} \partial \bar{t}}, \frac{\partial^2 \bar{w}}{\partial \bar{t}^2}, \ldots\right) = 0. \tag{28.4.1.3}$$

The function F in equations (28.4.1.1) and (28.4.1.2) is the same.

Transformations that preserve the form of equations allow us to "multiply" solutions. Indeed, suppose

$$w = g(x, y) \tag{28.4.1.4}$$

is a particular solution to equation (28.4.1.2). Since equation (28.4.1.2) is left unchanged by the change of variables (28.4.1.1), the transformed equation (28.4.1.3) has a solution

$$\bar{w} = g(\bar{x}, \bar{y}). \tag{28.4.1.5}$$

In (28.4.1.5), changing back to the old variables by formulas (28.4.1.1), solved for $\bar{x}$, $\bar{y}$, and $\bar{w}$, one arrives at a solution to equation (28.4.1.2) that, in general, is different from the original solution (28.4.1.4).

A function $I(x, t, w)$ is called an *invariant of transformation* (28.4.1.1) if it is left unchanged by the transformation:

$$I(x, t, w) = I(\bar{x}, \bar{t}, \bar{w}).$$

Remark. If $I = I(x, t, w)$ is an invariant of transformation (28.4.1.1), then an arbitrary function $\Psi(I)$ is also an invariant of the transformation.

28.4.2. Procedure for Constructing Exact Solutions

Consider in more detail the scheme outlined in Section 28.1 for using invariants in the analysis of mathematical equations as applied to partial differential equations.

First, one searches for a transformation

$$x = \varphi(\bar{x}, \bar{t}, \bar{w}; C), \quad t = \psi(\bar{x}, \bar{t}, \bar{w}; C), \quad w = \chi(\bar{x}, \bar{t}, \bar{w}; C) \tag{28.4.2.1}$$

that leaves the form of equation (28.4.1.2) unchanged. Transformation (28.4.2.1) must depend on a *single free parameter* $C \in [C_1, C_2]$; equation (28.4.1.2) itself does involve this parameter.

Secondly, one constructs two functionally independent invariants of transformation (28.4.2.1):

$$I_1 = I_1(x, y, w), \qquad I_2 = I_2(x, y, w). \tag{28.4.2.2}$$

Thirdly, the invariants (28.4.2.2) are taken as the new variables and one looks for an exact solution to equation (28.4.1.2) in the form

$$I_2 = \Phi(I_1), \tag{28.4.2.3}$$

where $\Phi(z)$ is the function that solves the ordinary differential equation resulting from the substitution of (28.4.2.3) into (28.4.1.2); the equation obtained is called an *invariant solution*.

As before, we will only use the simplest transformations

$$x = \bar{x} + a, \quad t = \bar{t} + b, \quad w = \bar{w} + c \qquad (\textit{translation}), \tag{28.4.2.4}$$

$$x = a\bar{x}, \quad t = b\bar{t}, \quad w = c\bar{w} \qquad (\textit{scaling}) \tag{28.4.2.5}$$

and their compositions as transformations (28.4.2.1).

It is significant that the overwhelming majority of linear and nonlinear equations of mathematical physics admit translation transformations in independent variables and many equations admit scaling transformations.

28.4.3. Examples of Constructing Invariant Solutions to Nonlinear Partial Differential Equations

Below are a few examples of constructing invariant solutions to nonlinear equations of mathematical physics; the technique is based on transformations (28.4.2.4) and (28.4.2.5) and their composition.

Example 28.16. Consider the nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]. \tag{28.4.3.1}$$

It remains unchanged under an arbitrary translation in the independent variables:

$$t = \bar{t} + Ck, \quad x = \bar{x} + C\lambda, \quad w = \bar{w}, \tag{28.4.3.2}$$

where C is a free parameter, and k and λ are some given numbers (which can be chosen arbitrarily). Eliminating C from the first two relations in (28.4.3.2), we find one of the invariants

$$\frac{x - \bar{x}}{\lambda} = \frac{t - \bar{t}}{k} \quad \Longrightarrow \quad kx - \lambda t = k\bar{x} - \lambda\bar{t} = I_1.$$

Another invariant is $I_2 = w = \bar{w}$. By virtue of (28.4.2.3), equation (28.4.3.1) admits a traveling-wave solution

$$w = \Phi(z), \qquad z = kx - \lambda t. \tag{28.4.3.3}$$

Substituting (28.4.3.3) into (28.4.3.1) yields the ordinary differential equation

$$\lambda^2 \Phi''_{zz} = k^2[f(\Phi)\Phi'_z]'_z. \tag{28.4.3.4}$$

Integrating it twice, one obtains the general solution in the implicit form

$$k^2 \int f(\Phi)\, d\Phi - \lambda^2 \Phi = C_1 z + C_2,$$

where C_1 and C_2 are arbitrary constants.

Example 28.17. Consider the equation

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial^2 w}{\partial x^2} + \beta w^n, \tag{28.4.3.5}$$

which describes unsteady heat conduction with a nonlinear power-law source.

We look for an invariant scaling transformation. Substituting (28.4.2.5) into (28.4.3.5) and multiplying all terms by b/c, we have

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{b}{a^2}\alpha \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + bc^{n-1}\beta \bar{w}^n.$$

This equation must coincide with (28.4.3.5), which leads to the relations

$$\frac{b}{a^2} = 1, \quad bc^{n-1} = 1.$$

Expressing a and c in terms of b,

$$a = \sqrt{b}, \quad c = b^{\frac{1}{1-n}},$$

and substituting these into (28.4.2.5), we arrive at an invariant transformation

$$x = \sqrt{b}\,\bar{x}, \quad t = b\bar{t}, \quad w = b^{\frac{1}{1-n}}\bar{w}, \tag{28.4.3.6}$$

which involves a free parameter $C = b$.

We now proceed to the determination of invariants. Eliminating b from the first two relations in (28.4.3.6), we get

$$x = \Big(\frac{t}{\bar{t}}\Big)^{1/2}\bar{x} \quad \Longrightarrow \quad xt^{-1/2} = \bar{x}\bar{t}^{-1/2} \quad \Longrightarrow \quad I_1 = xt^{-1/2}. \tag{28.4.3.7}$$

Eliminating b from the second and last relations in (28.4.3.6) yields

$$w = \Big(\frac{t}{\bar{t}}\Big)^{\frac{1}{1-n}}\bar{w} \quad \Longrightarrow \quad wt^{\frac{1}{n-1}} = \bar{w}\bar{t}^{\frac{1}{n-1}} \quad \Longrightarrow \quad I_2 = wt^{\frac{1}{n-1}}. \tag{28.4.3.8}$$

Relation (28.4.2.3) together with (28.4.3.7)–(28.4.3.8) determines the form of a self-similar solution

$$w = t^{\frac{1}{1-n}}\Phi(z), \quad z = xt^{-1/2}. \tag{28.4.3.9}$$

By substituting (28.4.3.9) into (28.4.3.5) and rearranging, we arrive at the ordinary differential equation

$$\frac{1}{1-n}\Phi - \frac{1}{2}z\Phi'_z = a\Phi''_{zz} + b\Phi^n.$$

Remark. Transformation (28.4.2.5) is often more convenient to write as

$$t = C\bar{t}, \quad x = C^k\bar{x}, \quad w = C^m\bar{w} \qquad (C > 0), \tag{28.4.3.10}$$

and then seek the values of k and m for which the form of the original equation does not change; C is a free parameter. The exact solutions corresponding to such transformations are called *self-similar* (see Section 27.3).

More examples of equations admitting self-similar solutions can be found in Table 27.1.

A similar approach can be successfully used for constructing exact solutions to systems of partial differential equations.

Example 28.18. Consider the system of equations of a steady laminar hydrodynamic boundary layer on a flat plate (see Schlichting, 1981)

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{28.4.3.11}$$

Let us scale the independent and dependent variables in (28.4.3.11) according to

$$x = C\bar{x}, \quad y = C^k\bar{y}, \quad u = C^m\bar{u}, \quad v = C^n\bar{v}. \tag{28.4.3.12}$$

Multiplying these relations by appropriate constant factors, we have

$$\begin{aligned} \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + C^{n-m-k+1}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} &= C^{-m-2k+1}\nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \\ \frac{\partial \bar{u}}{\partial \bar{x}} + C^{n-m-k+1}\frac{\partial \bar{v}}{\partial \bar{y}} &= 0. \end{aligned} \tag{28.4.3.13}$$

Let us require that the form of the equations of the transformed system (28.4.3.13) coincide with that of the original system (28.4.3.11). This condition results in two linear algebraic equations, $n - m - k + 1 = 0$ and $-2k - m + 1 = 0$. On solving them for m and n, we obtain

$$m = 1 - 2k, \quad n = -k, \tag{28.4.3.14}$$

where the exponent k can be chosen arbitrarily. Substituting (28.4.3.14) into (28.4.3.12) yields

$$x = C\bar{x}, \quad y = C^k\bar{y}, \quad u = C^{1-2k}\bar{u}, \quad v = C^{-k}\bar{v}.$$

By eliminating C, we arrive at the following invariants:

$$\begin{aligned} y = \left(\frac{x}{\bar{x}}\right)^k \bar{y} \quad &\Longrightarrow \quad yx^{-k} = \bar{y}\bar{x}^{-k} = I_1 = \zeta; \\ u = \left(\frac{x}{\bar{x}}\right)^{1-2k} \bar{u} \quad &\Longrightarrow \quad ux^{2k-1} = \bar{u}\bar{x}^{2k-1} = I_2; \\ v = \left(\frac{x}{\bar{x}}\right)^k \bar{v} \quad &\Longrightarrow \quad vx^k = \bar{v}\bar{x}^k = I_3. \end{aligned}$$

Solutions are sought in the form $I_2 = U(I_1)$ and $I_3 = V(I_1)$. We have

$$ux^{2k-1} = U(\zeta), \quad vx^k = V(\zeta)$$

or

$$u(x,y) = x^{1-2k}U(\zeta), \quad v(x,y) = x^{-k}V(\zeta), \quad \zeta = yx^{-k}, \tag{28.4.3.15}$$

where k is an arbitrary constant. Inserting (28.4.3.15) into the original system (28.4.3.11), we arrive at a system of ordinary differential equations for $U = U(\zeta)$ and $V = V(\zeta)$:

$$\begin{aligned} U\left[(1-2k)U - k\zeta U'_\zeta\right] + VU'_\zeta &= \nu U''_{\zeta\zeta}, \\ (1-2k)U - k\zeta U'_\zeta + V'_\zeta &= 0. \end{aligned}$$

Example 28.19. Consider the Klein–Gordon equation with an exponential nonlinearity

$$\frac{\partial^2 w}{\partial t^2} = \alpha\frac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w}. \tag{28.4.3.16}$$

We look for an invariant transformation as a composition of translation and scaling:

$$x = a\bar{x}, \quad t = b\bar{t}, \quad w = \bar{w} + c. \tag{28.4.3.17}$$

Substituting (28.4.3.17) into (28.4.3.16) and multiplying by b^2, we get

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = \frac{b^2}{a^2}\alpha\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + b^2 e^{\lambda c}\beta e^{\lambda \bar{w}}.$$

For this equation to coincide with (28.4.3.16), we must set

$$\frac{b^2}{a^2} = 1, \quad b^2 e^{\lambda c} = 1.$$

It follows that

$$a = b, \quad c = -\frac{2}{\lambda}\ln b.$$

Then transformation (28.4.3.17) becomes

$$x = b\bar{x}, \quad t = b\bar{t}, \quad w = \bar{w} - \frac{2}{\lambda}\ln b, \tag{28.4.3.18}$$

where $b > 0$ is an arbitrary constant.

Eliminating b from (28.4.3.17) yields the following invariants:

$$I_1 = \frac{x}{t} = \frac{\bar{x}}{\bar{t}}, \quad I_2 = w + \frac{2}{\lambda}\ln t = \bar{w} + \frac{2}{\lambda}\ln \bar{t}. \tag{28.4.3.19}$$

Relation (28.4.2.3) together with (28.4.3.19) determines the form of the solution

$$w = \Phi(z) - \frac{2}{\lambda}\ln t, \quad z = \frac{x}{t}. \tag{28.4.3.20}$$

Substituting (28.4.3.20) into (28.4.3.16), we arrive at the ordinary differential equation

$$(z^2\Phi'_z)'_z + \frac{2}{\lambda} = \alpha\Phi''_{zz} + \beta e^{\lambda\Phi}.$$

Remark. An exact solution to the steady-state equation of combustion theory with exponential heat release

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \beta e^{\lambda w}$$

can be found in much the same way.

Example 28.20. Consider the nonlinear equation of unsteady heat conduction

$$\frac{\partial w}{\partial t} = \alpha\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right). \tag{28.4.3.21}$$

We look for an invariant transformation as a composition of translation and scaling:

$$x = a\bar{x}, \quad t = \bar{t} + b, \quad w = c\bar{w}. \tag{28.4.3.22}$$

Substituting (28.4.3.22) into (28.4.3.21) and dividing by c, we get

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{c^n}{a^2}\alpha\frac{\partial}{\partial \bar{x}}\left(\bar{w}^n\frac{\partial \bar{w}}{\partial \bar{x}}\right).$$

The requirement that this equation must coincide with (28.4.3.21) yields the constraint

$$\frac{c^n}{a^2} = 1 \quad \Longrightarrow \quad a = c^{n/2},$$

where c is an arbitrary constant. The other parameter, b, is also arbitrary, so the resulting invariant transformation

$$x = c^{n/2}\bar{x}, \quad t = \bar{t} + b, \quad w = c\bar{w} \tag{28.4.3.23}$$

is two-parameter.

Since the method of invariants requires a one-parameter invariant transformation, we assume that

$$b = f(c), \tag{28.4.3.24}$$

and will determine the function f later on. Eliminating the parameter c from the last two relations in (28.4.3.23) by using (28.4.3.24), we obtain

$$t = \bar{t} + f\left(\frac{w}{\bar{w}}\right). \tag{28.4.3.25}$$

This equation determines an invariant $I(t, w) = I(\bar{t}, \bar{w})$ if the condition

$$f\left(\frac{w}{\bar{w}}\right) = \varphi(w) - \varphi(\bar{w}) \tag{28.4.3.26}$$

holds. This condition can be viewed as a functional equation. It is not difficult to verify that equation (28.4.3.26) has the following solution:

$$f(c) = A\ln c, \tag{28.4.3.27}$$

where A is an arbitrary constant. Substituting (28.4.3.24) into (28.4.3.23) and taking into account (28.4.3.27), we obtain

$$x = c^{n/2}\bar{x}, \quad t = \bar{t} + A\ln c, \quad w = c\bar{w}. \tag{28.4.3.28}$$

Assuming that $A \neq 0$, we express c through t using the second relation in (28.4.3.28) and substitute into the other two relations to obtain the invariants

$$I_1 = xe^{knt} = \bar{x}e^{kn\bar{t}}, \quad I_2 = we^{2kt} = \bar{w}e^{2k\bar{t}}, \quad \text{where} \quad k = -\frac{1}{2A}.$$

Using formula (28.4.2.3), we determine the form of exact solutions to (28.4.3.21):

$$w = e^{-2kt}\Phi(z), \quad z = xe^{knt}. \tag{28.4.3.29}$$

Substituting (28.4.3.29) into (28.4.3.21), we arrive at an ordinary differential equation for $\Phi(z)$:

$$-2k\Phi + knz\Phi'_z = \alpha(w^n w'_z)'_z.$$

Remark 1. Sometimes solutions of the form (28.4.3.29) are called *limiting self-similar solutions*.

Remark 2. Transformation (28.4.3.23) with $b = c$ leaves equation (28.4.3.21) unchanged but does not have invariants. Consequently, it is not necessary that every transformation that preserves the form of an equation must have invariants; for conditions transformations with invariants must satisfy, see Section 32.4.1.

Table 28.2 lists invariant solutions that may be obtained using combinations of translation and scaling transformations in the independent and dependent variables. The transformations are assumed to preserve the form of equations (the given equation is converted into the same equation). Apart from traveling-wave solutions, self-similar solutions, and exponential self-similar solutions, considered above, another five invariant solutions are presented. The right column of Table 28.2 gives examples of equations that admit the solutions specified.

TABLE 28.2
Invariant solutions that may be obtained using combinations of translation and scaling transformations preserving the form of equations (C is an arbitrary constant, $C > 0$)

No.	Invariant transformations	Form of invariant solutions	Example of equation
1	$t = \bar{t} + Ck,\ x = \bar{x} + C\lambda$	$w = U(z),\ z = kx - \lambda t$	$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$
2	$t = C\bar{t},\ x = C^k\bar{x},\ w = C^m\bar{w}$	$w = t^m U(z),\ z = xt^{-k}$	see Table 27.1
3	$t = \bar{t} + \ln C,\ x = C^k\bar{x},\ w = C^m\bar{w}$	$w = e^{mt}U(z),\ z = xe^{-kt}$	$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right)$ ($k = \frac{1}{2}mn$, m is any)
4	$t = C\bar{t},\ x = \bar{x} + k\ln C,\ w = C^m\bar{w}$	$w = t^m U(z),\ z = x - k\ln t$	$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right)$ ($m = -1/n$, k is any)
5	$t = C\bar{t},\ x = C^\beta\bar{x},\ w = \bar{w} + \alpha\ln C$	$w = U(z) + \alpha\ln t,\ z = xt^{-\beta}$	$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(e^w\frac{\partial w}{\partial x}\right)$ ($\alpha = 2\beta - 1$, β is any)
6	$t = C\bar{t},\ x = \bar{x} + \beta\ln C,\ w = \bar{w} + \alpha\ln C$	$w = U(z) + \alpha\ln t,\ z = x - \beta\ln t$	$\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t^2} = 0$ (α and β are any)
7	$t = \bar{t} + C,\ x = \bar{x} + C\lambda,\ w = \bar{w} + Ck$	$w = U(z) + kt,\ z = x - \lambda t$	$\frac{\partial w}{\partial t} = f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}$ (k and λ are any)
8	$t = \bar{t} + \ln C,\ x = \bar{x} + k\ln C,\ w = C^m\bar{w}$	$w = e^{mt}U(z),\ z = x - kt$	$\left(\frac{\partial^2 w}{\partial x\partial t}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t^2} = 0$ (k and m are any)

28.4.4. Simple Inverse Problems (Determination of the Form of Equations from Their Properties)

So far we have looked for exact solutions to given nonlinear partial differential equations. However, sometimes it is required to determine the form of an equation that possesses given properties. Such problems are called *inverse problems*. In some cases, solving inverse problems is reduced to solving functional equations, solution methods for which are described in Chapter 43. Below are a few examples that illustrate the statement of such inverse problems and methods for solving them.

Example 28.21. Consider the nonlinear parabolic equation of a quite general form

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + F\left(w, \frac{\partial w}{\partial x}\right). \tag{28.4.4.1}$$

Find the (general) form of the function $F = F(w, u)$ for which equation (28.4.4.1) admits self-similar solutions.

To have a self-similar solution, equation (28.4.4.1) must be invariant under the scaling transformation (28.4.3.10). Substituting (28.4.3.10) into (28.4.4.1) and rearranging the resulting equation, we get

$$\frac{\partial \bar{w}}{\partial \bar{t}} = C^{1-2k} a \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + C^{1-m} F\left(C^m \bar{w}, C^{m-k} \frac{\partial \bar{w}}{\partial \bar{x}}\right). \tag{28.4.4.2}$$

The requirement that equation (28.4.4.2) must coincide, up to renaming of the variables, with (28.4.4.1) results in $k = \frac{1}{2}$ and yields a functional equation for $F = F(w, u)$:

$$F(w, u) = C^{1-m} F(C^m w, C^{m-1/2} u). \tag{28.4.4.3}$$

Solution methods for similar functional equations are described in Sections 43.1 and 43.3. The general solution to equation (28.4.4.3) is expressed as

$$F(w, u) = w^{\frac{m-1}{m}} f\left(w^{\frac{1-2m}{2m}} u\right), \tag{28.4.4.4}$$

where $f(z)$ is an arbitrary function.

Substituting (28.4.4.4) into (28.4.4.1) yields the equations

$$w_t = a w_{xx} + w^{\frac{m-1}{m}} f\left(w^{\frac{1-2m}{2m}} w_x\right), \tag{28.4.4.5}$$

which is invariant under transformation (28.4.3.10) with $k = \frac{1}{2}$ (m is a free parameter). Equation (28.4.4.4) has self-similar solutions of the form

$$w = t^m \varphi(z), \quad z = x t^{-1/2},$$

where the function $\varphi = \varphi(z)$ is described by the ordinary differential equation

$$m\varphi - \tfrac{1}{2} z \varphi'_z = a \varphi''_{zz} + \varphi^{\frac{m-1}{m}} f\left(\varphi^{\frac{1-2m}{2m}} \varphi'_z\right).$$

Example 28.22. Consider another nonlinear equation

$$\frac{\partial^2 w}{\partial t^2} = F\left(w, \frac{\partial w}{\partial x}\right) \frac{\partial^2 w}{\partial x^2}. \tag{28.4.4.6}$$

Find the (general) form of the function $F = F(w, u)$ for which equation (28.4.4.6) admits limiting self-similar solutions (an example of a limiting self-similar solution is given by formulas (28.4.3.29)).

To have a limiting self-similar solution, equation (28.4.4.6) must be invariant under a composition of scaling and translation of the form

$$t = \bar{t} + A \ln C, \quad x = C\bar{x}, \quad w = C^m \bar{w}, \tag{28.4.4.7}$$

where C is an arbitrary constant ($C > 0$), and A and m are some numbers.

Substituting (28.4.4.7) into (28.4.4.6) and canceling by C^m, we get

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = C^{-2} F\left(C^m \bar{w}, C^{m-1} \frac{\partial \bar{w}}{\partial \bar{x}}\right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}. \tag{28.4.4.8}$$

The requirement that equation (28.4.4.8) must coincide, up to renaming, with (28.4.4.6) results in a functional equation for $F = F(w, u)$:

$$F(w, u) = C^{-2} F(C^m w, C^{m-1} u). \tag{28.4.4.9}$$

The general solution of the functional equation (28.4.4.9) is given by (see Sections 43.1 or 43.3):

$$F(w, u) = w^{\frac{2}{m}} f\left(w^{\frac{1-m}{m}} u\right), \tag{28.4.4.10}$$

where $f(z)$ is an arbitrary function.

Equation (28.4.4.6) with the function (28.4.4.10) admits limiting self-similar solutions of the form

$$w = e^{kmt} \psi(y), \quad y = x e^{-kt},$$

where $k = 1/A$ is an arbitrary constant (m is a free parameter).

Example 28.23. Now let us find the form of $F = F(w, u)$ for which equation (28.4.4.6) is invariant under another composition of scaling and translation:

$$t = C\bar{t}, \quad x = C^k\bar{x}, \quad w = \bar{w} + A\ln C, \tag{28.4.4.11}$$

where C is an arbitrary constant ($C > 0$), and A and k are some numbers.

Substituting (28.4.4.11) into (28.4.4.6), we have

$$\frac{\partial^2\bar{w}}{\partial\bar{t}^2} = C^{2(1-k)}F\left(\bar{w} + A\ln C,\ C^{-k}\frac{\partial\bar{w}}{\partial\bar{x}}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}. \tag{28.4.4.12}$$

Comparing (28.4.4.12) and (28.4.4.6) leads to the functional equation

$$F(w, u) = C^{2(1-k)}F(w + A\ln C, C^{-k}u), \tag{28.4.4.13}$$

whose general solution is given by

$$F(w, u) = e^{2(k-1)\lambda w}f(e^{k\lambda w}u), \quad \lambda = 1/A, \tag{28.4.4.14}$$

where $f(z)$ is an arbitrary function.

Equation (28.4.4.6) with the function (28.4.4.14) admits invariant solutions of the form

$$w = \theta(r) + \frac{1}{\lambda}\ln t, \quad r = xt^{-k}.$$

28.5. General Conclusions and Remarks

The method of invariants described above is effective for solving various types of mathematical equations. As applied to ordinary differential equations and equations of mathematical physics, the method is much easier to understand than the classical group analysis method (see Chapter 32) and can be treated as its special case. The usage of the method only requires the ability to solve simple algebraic equations (and systems of equations) and to differentiate. It is significant that, as applied to ordinary differential equations, the above method is virtually not inferior to the classical group analysis method (see Olver 1995, Ibragimov 1999, Hydon 2000, and Polyanin and Zaitsev 2003). For nonlinear equations of mathematical physics, the method of invariants allows finding virtually all most common solutions, which make up, by the most conservative estimates, more than half of solutions that can be obtained using the group analysis method; this estimate follows from the comparison of the exact solutions presented in the handbooks by Ibragimov (1994, 1995) and the current handbook. The above suggests that it is reasonable to include the method of invariants in standard lecture courses on ordinary differential equations and equations of mathematical physics for students specializing in applied mathematics, physics, and engineering; it may be useful to include the method in optional courses for engineering and education students. Furthermore, this simple method is useful to study at a preliminary (introductory) stage prior to giving the main material within much more difficult courses on the methods of group (symmetry) analysis of differential equations.

As applied to equations of mathematical physics, the main limitations of the method of invariants include:

1. A nonidentity transformation based on formulas (28.4.2.4) and (28.4.2.5) or their compositions and leaving the equation unchanged may not exist.

2. A transformation that preserves the form of the equation may not always be effectively used (the equation must depend on a free parameter).

3. An invariant transformation may not always have an invariant.

⊙ *References for Chapter* 28: G. W. Bluman and J. D. Cole (1974), P. J. Olver (1995), N. A. Kudryashov (1998), N. H. Ibragimov (1994, 1999), P. E. Hydon (2000), V. G. Boltyanskii and N. Ya. Vilenkin (2002), A. D. Polyanin and V. F. Zaitsev (2003, 2004), A. D. Polyanin (2008).

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Chapter 29
Method of Generalized Separation of Variables

29.1. Exact Solutions with Simple Separation of Variables

29.1.1. Multiplicative and Additive Separable Solutions

Separation of variables is the most common approach to solve linear equations of mathematical physics [see, for example, Tikhonov and Samarskii (1990), Zwillinger (1998), and Polyanin (2002)]. For equations in two independent variables x, t and a dependent variable w, this approach involves searching for exact solutions in the form of the product of functions depending on different arguments:

$$w(x,t) = \varphi(x)\psi(t). \tag{29.1.1.1}$$

The integration of a few classes of first-order nonlinear partial differential equations is based on searching for exact solutions in the form of the sum of functions depending on different arguments (see Subsection 24.1.2):

$$w(x,t) = \varphi(x) + \psi(t). \tag{29.1.1.2}$$

Some second- and higher-order nonlinear equations of mathematical physics also have exact solutions of the form (29.1.1.1) or (29.1.1.2). Such solutions are called *multiplicative separable* and *additive separable*, respectively.

29.1.2. Simple Separation of Variables in Nonlinear Partial Differential Equations

1°. In isolated cases, the separation of variables in nonlinear equations is carried out following the same technique as in linear equations. Specifically, an exact solution is sought in the form of the product or sum of functions depending on different arguments. On substituting it into the equation and performing elementary algebraic manipulations, one obtains an equation with the two sides dependent on different variables (for equations with two variables). Then one concludes that the expressions on each side must be equal to the same constant quantity, called a *separation constant*.

Example 29.1. The heat equation with a power nonlinearity

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^k\frac{\partial w}{\partial x}\right) \tag{29.1.2.1}$$

has a multiplicative separable solution. Substituting (29.1.1.1) into (29.1.2.1) yields

$$\varphi\psi'_t = a\psi^{k+1}(\varphi^k\varphi'_x)'_x.$$

Separating the variables by dividing both sides by $\varphi\psi^{k+1}$, we obtain

$$\frac{\psi_t'}{\psi^{k+1}} = \frac{a(\varphi^k\varphi_x')_x'}{\varphi}.$$

The left-hand side depends on t alone and the right-hand side on x alone. This is possible only if

$$\frac{\psi_t'}{\psi^{k+1}} = C, \quad \frac{a(\varphi^k\varphi_x')_x'}{\varphi} = C, \tag{29.1.2.2}$$

where C is an arbitrary constant (separation constant). On solving the ordinary differential equations (29.1.2.2), we obtain a solution of equation (29.1.2.1) with the form (29.1.1.1).

The procedure for constructing a separable solution (29.1.1.1) of the nonlinear equation (29.1.2.1) is identical to that used in solving linear equations [in particular, equation (29.1.2.1) with $k = 0$]. We refer to similar cases as *simple separation of variables*.

Example 29.2. The wave equation with an exponential nonlinearity

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) \tag{29.1.2.3}$$

has an additive separable solution. On substituting (29.1.1.2) into (29.1.2.3) and dividing by $e^{\lambda\psi}$, we arrive at the equation

$$e^{-\lambda\psi}\psi_{tt}'' = a(e^{\lambda\varphi}\varphi_x')_x',$$

whose left-hand side depends on t alone and the right-hand side on x alone. This is possible only if

$$e^{-\lambda\psi}\psi_{tt}'' = C, \quad a(e^{\lambda\varphi}\varphi_x')_x' = C, \tag{29.1.2.4}$$

where C is an arbitrary constant. Solving the ordinary differential equations (29.1.2.4) yields a solution of equation (29.1.2.3) with the form (29.1.1.2).

Example 29.3. The steady-state heat equation in an anisotropic medium with a logarithmic source

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] = aw\ln w \tag{29.1.2.5}$$

has a multiplicative separable solution

$$w = \varphi(x)\psi(y). \tag{29.1.2.6}$$

On substituting (29.1.2.6) into (29.1.2.5), dividing by $\varphi\psi$, and rearranging individual terms of the resulting equation, we obtain

$$\frac{1}{\varphi}[f(x)\varphi_x']_x' - a\ln\varphi = -\frac{1}{\psi}[g(y)\psi_y']_y' + a\ln\psi.$$

The left-hand side of this equation depends only on x and the right-hand only on y. By equating both sides to a constant quantity, one obtains ordinary differential equations for $\varphi(x)$ and $\psi(y)$.

Table 29.1 gives other examples of simple, additive or multiplicative, separable solutions for some nonlinear equations.

2°. Below are two assertions that allow one to generalize some solutions obtained by simple separation of variables.

Assertion 1. Suppose that an equation

$$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x\partial t}, \frac{\partial^2 w}{\partial t^2}, \ldots\right) = 0, \tag{29.1.2.7}$$

which does not involve independent variables explicitly, can be solved by simple separation of variables to give a solution of the special form

$$w = t^a\varphi(x), \quad a \neq 0. \tag{29.1.2.8}$$

Then equation (29.1.2.7) also admits a more complex solution

$$w = (t + C_1)^a\psi(z), \quad z = x + C_2\ln|t + C_1| + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

TABLE 29.1
Some nonlinear equations of mathematical physics that admit additive or multiplicative separable solutions (C, C_1, and C_2 are arbitrary constants)

Equation	Equation name	Form of solutions	Determining equations
$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\ln w$	Heat equation with source	$w = \varphi(x)\psi(t)$	$a\varphi''_{xx}/\varphi - b\ln\varphi =$ $-\psi'_t/\psi + b\ln\psi = C$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^k\frac{\partial w}{\partial x}\right) + bw$	Heat equation with source	$w = \varphi(x)\psi(t)$	$(\psi'_t - b\psi)/\psi^{k+1} =$ $a(\varphi^k\varphi'_x)'_x/\varphi = C$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^k\frac{\partial w}{\partial x}\right) + bw^{k+1}$	Heat equation with source	$w = \varphi(x)\psi(t)$	$\psi'_t/\psi^{k+1} =$ $a(\varphi^k\varphi'_x)'_x/\varphi + b\varphi^k = C$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + b$	Heat equation with source	$w = \varphi(x) + \psi(t)$	$e^{-\lambda\psi}(\psi'_t - b) = a(e^{\lambda\varphi}\varphi'_x)'_x = C$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{w}\frac{\partial w}{\partial x}\right) + be^{w}$	Heat equation with source	$w = \varphi(x) + \psi(t)$	$e^{-\psi}\psi'_t = a(e^{\varphi}\varphi'_x)'_x + be^{\varphi} = C$
$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2$	Potential Burgers equation	$w = \varphi(x) + \psi(t)$	$\psi'_t = a\varphi''_{xx} + b(\varphi'_x)^2 = C$
$\frac{\partial w}{\partial t} = a\left(\frac{\partial w}{\partial x}\right)^k\frac{\partial^2 w}{\partial x^2}$	Filtration equation	$w = \varphi(x) + \psi(t)$, $w = f(x)g(t)$	$\psi'_t = a(\varphi'_x)^k\varphi''_{xx} = C_1$, $g'_t/g^{k+1} = a(f'_x)^k f''_{xx}/f = C_2$
$\frac{\partial w}{\partial t} = F\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}$	Filtration equation	$w = \varphi(x) + \psi(t)$	$\psi'_t = F(\varphi'_x)\varphi''_{xx} = C$
$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^k\frac{\partial w}{\partial x}\right)$	Wave equation	$w = \varphi(x)\psi(t)$	$\psi''_{tt}/\psi^{k+1} = a(\varphi^k\varphi'_x)'_x/\varphi = C$
$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right)$	Wave equation	$w = \varphi(x) + \psi(t)$	$e^{-\lambda\psi}\psi''_{tt} = a(e^{\lambda\varphi}\varphi'_x)'_x = C$
$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + bw\ln w$	Wave equation with source	$w = \varphi(x)\psi(t)$	$\psi''_{tt}/\psi - b\ln\psi =$ $a\varphi''_{xx}/\varphi + b\ln\varphi = C$
$\frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\left(w^k\frac{\partial w}{\partial y}\right) = 0$	Anisotropic steady heat equation	$w = \varphi(x)\psi(y)$	$\varphi''_{xx}/\varphi^{k+1} = -a(\psi^k\psi'_y)'_y/\psi = C$
$\frac{\partial^2 w}{\partial x^2} + a\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} = 0$	Equation of steady transonic gas flow	$w = \varphi(x) + \psi(y)$, $w = f(x)g(y)$	$\varphi''_{xx} = -a\psi'_y\psi''_{yy} = C_1$, $f''_{xx}/f = -ag'_y g''_{yy}/g = C_2$
$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$	Monge–Ampère equation	$w = \varphi(x) + \psi(y)$, $w = f(x)g(y)$	$\varphi''_{xx} = 0$ or $\psi''_{yy} = 0$, $(f'_x)^2/(ff''_{xx}) = gg''_{yy}/(g'_y)^2 = C$
$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} + b\left(\frac{\partial w}{\partial x}\right)^2$	Potential Korteweg-de Vries equation	$w = \varphi(x) + \psi(t)$	$\psi'_t = a\varphi'''_{xxx} + b(\varphi'_x)^2 = C$
$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = a\frac{\partial^3 w}{\partial y^3}$	Boundary-layer equation	$w = \varphi(x) + \psi(y)$, $w = f(x)g(y)$	$-\varphi'_x = a\psi'''_{yyy}/\psi''_{yy} = C_1$, $f'_x = ag'''_{yyy}[(g'_y)^2 - gg''_{yy}]^{-1} = C_2$

Example 29.4. The nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^k\frac{\partial w}{\partial x}\right) \tag{29.1.2.9}$$

has a multiplicative separable solution of the form (29.1.2.8) with $a = -2/k$. Therefore, equation (29.1.2.9) has a more complex solution

$$w = (t + C_1)^{-2/k}\psi(z), \qquad z = x + C_2\ln|t + C_1| + C_3.$$

The function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\frac{2(k+2)}{k^2}\psi - \frac{k+4}{k}C_2\psi'_z + C_2^2\psi''_{zz} = a(\psi^k\psi'_z)'_z.$$

Assertion 2. Suppose that simple separation of variables leads to the following solution of equation (29.1.2.7):

$$w = e^{\lambda_0 t}\varphi(x), \tag{29.1.2.10}$$

where $\lambda_0 \neq 0$ is some number. Then equation (29.1.2.7) also admits a more complex solution

$$w = C_1 e^{\lambda t}\psi(z), \qquad z = x + C_2 t + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants.

Example 29.5. The nonlinear equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) \tag{29.1.2.11}$$

has a multiplicative separable solution of the form (29.1.2.10) with $\lambda_0 = 1$. Therefore, equation (29.1.2.11) has a more complex solution

$$w = C_1 e^{\lambda t}\psi(z), \qquad z = x + C_2 t + C_3.$$

The function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\lambda\psi + C_2\psi'_z = a\psi''_{zz} + \psi f(\psi'_z/\psi).$$

29.1.3. Complex Separation of Variables in Nonlinear Partial Differential Equations

The variables in nonlinear equations often separate more complexly than in linear equations. We exemplify this below.

Example 29.6. Consider the equation with a cubic nonlinearity

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + w\left(\frac{\partial w}{\partial x}\right)^2 - bw^3, \tag{29.1.3.1}$$

where $b > 0$. We look for exact solutions in the product form. We substitute (29.1.1.1) into (29.1.3.1) and divide the resulting equation by $\varphi(x)\psi(t)$ to obtain

$$\frac{\psi'_t}{\psi} = a\frac{\varphi''_{xx}}{\varphi} + \psi^2[(\varphi'_x)^2 - b\varphi^2]. \tag{29.1.3.2}$$

In the general case, this expression cannot be represented as the sum of two functions depending on different arguments. This, however, does not mean that equation (29.1.3.1) has no solutions of the form (29.1.1.1).

1°. One can make sure by direct check that the functional differential equation (29.1.3.2) has solutions

$$\varphi(x) = Ce^{\pm x\sqrt{b}}, \qquad \psi(t) = e^{abt}, \tag{29.1.3.3}$$

where C is an arbitrary constant. Solutions (29.1.3.3) for φ make the expression in square brackets in (29.1.3.2) vanish, which allows the separation of variables.

2°. There is a more general solution of the functional differential equation (29.1.3.2):

$$\varphi(x) = C_1 e^{x\sqrt{b}} + C_2 e^{-x\sqrt{b}}, \qquad \psi(t) = e^{abt}\left(C_3 + 4C_1C_2e^{2abt}\right)^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants. The function $\varphi = \varphi(x)$ is such that both x-dependent expressions in (29.1.3.2) are constant simultaneously:

$$\varphi''_{xx}/\varphi = \text{const}, \qquad (\varphi'_x)^2 - b\varphi^2 = \text{const}.$$

It is this circumstance that makes it possible to separate the variables.

Example 29.7. Consider the second-order equation with a quadratic nonlinearity

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}+a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=b\frac{\partial^3 w}{\partial x^3}+c\frac{\partial^3 w}{\partial y^3}. \tag{29.1.3.4}$$

We look for additive separable solutions

$$w=f(x)+g(y). \tag{29.1.3.5}$$

Substituting (29.1.3.5) into (29.1.3.4) yields

$$g'_y f'''_{xx}+af'_x g'''_{yy}=bf''''_{xxx}+cg''''_{yyy}. \tag{29.1.3.6}$$

This expression cannot be rewritten as the equality of two functions depending on different arguments.

It can be shown that equation (29.1.3.4) has a solution of the form (29.1.3.5):

$$w=C_1e^{-a\lambda x}+\frac{c\lambda}{a}x+C_2e^{\lambda y}-ab\lambda y+C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants. The mechanism of separation of variables is different here: both nonlinear terms on the left-hand side in (29.1.3.6) contain terms that cannot be rewritten in additive form but are equal in magnitude and have unlike signs. In adding, the two terms cancel out, thus resulting in separation of variables:

$$\begin{array}{rll} & g'_y f'''_{xx} & = C_1C_2a^2\lambda^3e^{\lambda y-a\lambda x}-C_1b(a\lambda)^3e^{-a\lambda x} \\ + & af'_x g'''_{yy} & = -C_1C_2a^2\lambda^3e^{\lambda y-a\lambda x}+C_2c\lambda^3e^{\lambda y} \\ \hline & g'_y f'''_{xx}+af'_x g'''_{yy} & = -C_1b(a\lambda)^3e^{-a\lambda x}+C_2c\lambda^3e^{\lambda y}=bf''''_{xxx}+cg''''_{yyy} \end{array}.$$

Example 29.8. Consider the second-order equation with a cubic nonlinearity

$$(1+w^2)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-2w\left(\frac{\partial w}{\partial x}\right)^2-2w\left(\frac{\partial w}{\partial y}\right)^2=aw(1-w^2). \tag{29.1.3.7}$$

We seek an exact solution of this equation in the product form

$$w=f(x)g(y). \tag{29.1.3.8}$$

Substituting (29.1.3.8) into (29.1.3.7) yields

$$(1+f^2g^2)(gf''_{xx}+fg''_{yy})-2fg[g^2(f'_x)^2+f^2(g'_y)^2]=afg(1-f^2g^2). \tag{29.1.3.9}$$

This expression cannot be rewritten as the equality of two functions with different arguments. Nevertheless, equation (29.1.3.7) has solutions of the form (29.1.3.8). One can make sure by direct check that the functions $f=f(x)$ and $g=g(y)$ satisfying the nonlinear ordinary differential equations

$$\begin{aligned} (f'_x)^2&=Af^4+Bf^2+C, \\ (g'_y)^2&=Cg^4+(a-B)g^2+A, \end{aligned} \tag{29.1.3.10}$$

where A, B, and C are arbitrary constants, reduce equation (29.1.3.9) to an identity; to verify this, one should use the relations $f''_{xx}=2Af^3+Bf$ and $g''_{yy}=2Cg^3+(a-B)g$ that follow from (29.1.3.10).

Remark. By the change of variable $u=4\arctan w$ equation (29.1.3.7) can be reduced to a nonlinear heat equation with a sinusoidal source, $\Delta u=a\sin u$.

The examples considered above illustrate some specific features of separable solutions to nonlinear equations. Sections 29.2, 24.4–24.6 outlines fairly general methods for constructing similar and more complicated solutions to nonlinear partial differential equations.

⊙ *References for Section* 29.1: R. Steuerwald (1936), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2004).

29.2. Structure of Generalized Separable Solutions

29.2.1. General Form of Solutions. Classes of Nonlinear Equations Considered

To simplify the presentation, we confine ourselves to the case of mathematical physics equations in two independent variables x, y and a dependent variable w (one of the independent variables can play the role of time).

Linear separable equations of mathematical physics admit exact solutions in the form

$$w(x,y)=\varphi_1(x)\psi_1(y)+\varphi_2(x)\psi_2(y)+\cdots+\varphi_n(x)\psi_n(y), \tag{29.2.1.1}$$

where $w_i=\varphi_i(x)\psi_i(y)$ are particular solutions; the functions $\varphi_i(x)$, as well as the functions $\psi_i(y)$, with different numbers i, are not related to one another.

Many nonlinear partial differential equations with quadratic or power nonlinearities,

$$f_1(x)g_1(y)\Pi_1[w]+f_2(x)g_2(y)\Pi_2[w]+\cdots+f_m(x)g_m(y)\Pi_m[w]=0, \tag{29.2.1.2}$$

also have exact solutions of the form (29.2.1.1). In (29.2.1.2), the $\Pi_i[w]$ are differential forms that are the products of nonnegative integer powers of the function w and its partial derivatives $\partial_x w, \partial_y w, \partial_{xx} w, \partial_{xy} w, \partial_{yy} w, \partial_{xxx} w$, etc. We will refer to solutions (29.2.1.1) of nonlinear equations (29.2.1.2) as *generalized separable solutions*. Unlike linear equations, in nonlinear equations the functions $\varphi_i(x)$ with different subscripts i are usually related to one another [and to functions $\psi_j(y)$]. In general, the functions $\varphi_i(x)$ and $\psi_j(y)$ in (29.2.1.1) are not known in advance and are to be identified. Section 29.1 gives simple examples of exact solutions of the form (29.2.1.1) with $n=1$ and $n=2$ (for $\psi_1=\varphi_2=1$) to some nonlinear equations.

Note that most common of the generalized separable solutions are solutions of the special form

$$w(x,y)=\varphi(x)\psi(y)+\chi(x);$$

the independent variables on the right-hand side can be swapped. In the special case of $\chi(x)=0$, this is a multiplicative separable solution, and if $\varphi(x)=1$, this is an additive separable solution.

Remark 1. Expressions of the form (29.2.1.1) are often used in applied and computational mathematics for constructing approximate solutions to differential equations by the Galerkin method (and its modifications).

Remark 2. Solutions of the form (29.2.1.1) can also be admitted by equations that have nonlinearities other than (29.2.1.2); see Example 29.18, Section 29.5.

29.2.2. General Form of Functional Differential Equations

In general, on substituting expression (29.2.1.1) into the differential equation (29.2.1.2), one arrives at a functional differential equation

$$\Phi_1(X)\Psi_1(Y)+\Phi_2(X)\Psi_2(Y)+\cdots+\Phi_k(X)\Psi_k(Y)=0 \tag{29.2.2.1}$$

for the $\varphi_i(x)$ and $\psi_i(y)$. The functionals $\Phi_j(X)$ and $\Psi_j(Y)$ depend only on x and y, respectively,

$$\begin{aligned}\Phi_j(X)&\equiv\Phi_j\big(x,\varphi_1,\varphi_1',\varphi_1'',\ldots,\varphi_n,\varphi_n',\varphi_n''\big),\\ \Psi_j(Y)&\equiv\Psi_j\big(y,\psi_1,\psi_1',\psi_1'',\ldots,\psi_n,\psi_n',\psi_n''\big).\end{aligned} \tag{29.2.2.2}$$

Here, for simplicity, the formulas are written out for the case of a second-order equation (29.2.1.2); for higher-order equations, the right-hand sides of relations (29.2.2.2) will contain higher-order derivatives of φ_i and ψ_j.

Sections 29.4 and 29.5 outline two different methods for solving functional differential equations of the form (29.2.2.1)–(29.2.2.2).

Remark. Unlike ordinary differential equations, equation (29.2.2.1)–(29.2.2.2) involves several functions (and their derivatives) with different arguments.

⊙ *References for Section* 29.2: S. S. Titov (1988), V. A. Galaktionov and S. A. Posashkov (1989, 1994), V. A. Galaktionov (1995), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2004), V. A. Galaktionov and S. R. Svirshchevskii (2006).

29.3. Simplified Scheme for Constructing Generalized Separable Solutions

29.3.1. Description of the Simplified Scheme for Constructing Solutions Based on Presetting One System of Coordinate Functions

To construct exact solutions of equations (29.2.1.2) with quadratic or power nonlinearities that do not depend explicitly on x (all f_i constant), it is reasonable to use the following simplified approach. As before, we seek solutions in the form of finite sums (29.2.1.1). We assume that the system of coordinate functions $\{\varphi_i(x)\}$ is governed by linear differential equations with constant coefficients. The most common solutions of such equations are of the forms

$$\varphi_i(x) = x^i, \quad \varphi_i(x) = e^{\lambda_i x}, \quad \varphi_i(x) = \sin(\alpha_i x), \quad \varphi_i(x) = \cos(\beta_i x). \tag{29.3.1.1}$$

Finite chains of these functions (in various combinations) can be used to search for separable solutions (29.2.1.1), where the quantities λ_i, α_i, and β_i are regarded as free parameters. The other system of functions $\{\psi_i(y)\}$ is determined by solving the nonlinear equations resulting from substituting (29.2.1.1) into the equation under consideration [or into equation (29.2.2.1)–(29.2.2.2)].

This simplified approach lacks the generality of the methods outlined in Sections 29.4–29.6. However, specifying one of the systems of coordinate functions, $\{\varphi_i(x)\}$, simplifies the procedure of finding exact solutions substantially. The drawback of this approach is that some solutions of the form (29.2.1.1) can be overlooked. It is significant that the overwhelming majority of generalized separable solutions known to date, for partial differential equations with quadratic nonlinearities, are determined by coordinate functions (29.3.1.1) (usually with $n = 2$).

29.3.2. Examples of Finding Exact Solutions of Second- and Third-Order Equations

Below we consider specific examples that illustrate the application of the above simplified scheme to the construction of generalized separable solutions of second- and third-order nonlinear equations.

Example 29.9. The equations of a laminar boundary layer on a flat plate are reduced to a single third-order nonlinear equation for the stream function (see Schlichting, 1981, and Loitsyanskiy, 1996):

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = a\frac{\partial^3 w}{\partial y^3}. \tag{29.3.2.1}$$

We look for generalized separable solutions with the form

$$w(x, y) = x\psi(y) + \theta(y), \tag{29.3.2.2}$$

which corresponds to the simplest set of functions $\varphi_1(x) = x$, $\varphi_2(x) = 1$ with $n = 2$ in formula (29.2.1.1). On substituting (29.3.2.2) into (29.3.2.1) and collecting terms, we obtain

$$x[(\psi'_y)^2 - \psi\psi''_{yy} - a\psi'''_{yyy}] + [\psi'_y\theta'_y - \psi\theta''_{yy} - a\theta'''_{yyy}] = 0.$$

To meet this equation for any x, one should equate both expressions in square brackets to zero. This results in a system of ordinary differential equations for $\psi = \psi(y)$ and $\theta = \theta(y)$:

$$(\psi'_y)^2 - \psi\psi''_{yy} - a\psi''_{yyy} = 0,$$
$$\psi'_y\theta'_y - \psi\theta''_{yy} - a\theta''_{yyy} = 0.$$

For example, this system has an exact solution

$$\psi = \frac{6a}{y + C_1}, \quad \theta = \frac{C_2}{y + C_1} + \frac{C_3}{(y + C_1)^2} + C_4,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Other generalized separable solutions of equation (29.3.2.1) can be found in Subsection 9.4.2 (see Example 29.12).

Example 29.10. Consider a nonhomogeneous Monge–Ampère equation of the form

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x). \tag{29.3.2.3}$$

We look for generalized separable solutions with the form

$$w(x, y) = \varphi(x)y^k + \psi(x), \qquad k \neq 0. \tag{29.3.2.4}$$

On substituting (29.3.2.4) into (29.3.2.3) and collecting terms, we obtain

$$[k^2(\varphi'_x)^2 - k(k-1)\varphi\varphi''_{xx}]y^{2k-2} - k(k-1)\varphi\psi''_{xx}y^{k-2} - f(x) = 0. \tag{29.3.2.5}$$

This equation can be satisfied only if $k = 1$ or $k = 2$.

First case. If $k = 1$, (29.3.2.5) reduces one equation

$$(\varphi'_x)^2 - f(x) = 0.$$

It has two solutions: $\varphi(x) = \pm\int \sqrt{f(x)}\,dx$. They generate two solutions of equation (29.3.2.3) in the form (29.3.2.4):

$$w(x, y) = \pm y \int \sqrt{f(x)}\,dx + \psi(x),$$

where $\psi(x)$ is an arbitrary function.

Second case. If $k = 2$, equating the functional coefficients of the different powers of y to zero, we obtain two equations:

$$2(\varphi'_x)^2 - \varphi\varphi''_{xx} = 0,$$
$$2\varphi\psi''_{xx} + f(x) = 0.$$

Their general solutions are given by

$$\varphi(x) = \frac{1}{C_1x + C_2}, \quad \psi(x) = -\frac{1}{2}\int_0^x (x - t)(C_1t + C_2)f(t)\,dt + C_3x + C_4.$$

Here, C_1, C_2, C_3, and C_4 are arbitrary constants.

Example 29.11. Consider the third-order nonlinear equation

$$\frac{\partial^2 w}{\partial x \partial t} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = a\frac{\partial^3 w}{\partial x^3}, \tag{29.3.2.7}$$

which is encountered in hydrodynamics.

We look for exact solutions of the form

$$w = \varphi(t)e^{\lambda x} + \psi(t), \qquad \lambda \neq 0. \tag{29.3.2.8}$$

On substituting (29.3.2.8) into (29.3.2.7), we have

$$\varphi'_t - \lambda\varphi\psi = a\lambda^2\varphi.$$

We now solve this equation for ψ and substitute the resulting expression into (29.3.2.8) to obtain a solution of equation (29.3.2.7) in the form

$$w = \varphi(t)e^{\lambda x} + \frac{1}{\lambda}\frac{\varphi'_t(t)}{\varphi(t)} - a\lambda,$$

where $\varphi(t)$ is an arbitrary function and λ is an arbitrary constant.

⊙ *References for Section* 29.3: A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2004).

29.4. Solution of Functional Differential Equations by Differentiation

29.4.1. Description of the Method

Below we describe a procedure for constructing solutions to functional differential equations of the form (29.2.2.1)–(29.2.2.2). It involves three successive stages.

1°. Assume that $\Psi_k \not\equiv 0$. We divide equation (29.2.2.1) by Ψ_k and differentiate with respect to y. This results in a similar equation but with fewer terms:

$$\widetilde{\Phi}_1(X)\widetilde{\Psi}_1(Y) + \widetilde{\Phi}_2(X)\widetilde{\Psi}_2(Y) + \cdots + \widetilde{\Phi}_{k-1}(X)\widetilde{\Psi}_{k-1}(Y) = 0,$$
$$\widetilde{\Phi}_j(X) = \Phi_j(X), \quad \widetilde{\Psi}_j(Y) = [\Psi_j(Y)/\Psi_k(Y)]'_y.$$

We repeat the above procedure $(k-3)$ times more to obtain the separable two-term equation

$$\widehat{\Phi}_1(X)\widehat{\Psi}_1(Y) + \widehat{\Phi}_2(X)\widehat{\Psi}_2(Y) = 0. \tag{29.4.1.1}$$

Two cases must be considered.

Nondegenerate case: $|\widehat{\Phi}_1(X)\widehat{\Psi}_1(Y)| + |\widehat{\Phi}_2(X)\widehat{\Psi}_2(Y)| \not\equiv 0$. Then the solutions of equation (29.4.1.1) are determined by the ordinary differential equations

$$\widehat{\Phi}_1(X) + C\widehat{\Phi}_2(X) = 0, \quad C\widehat{\Psi}_1(Y) - \widehat{\Psi}_2(Y) = 0,$$

where C is an arbitrary constant. The equations $\widehat{\Phi}_2 = 0$ and $\widehat{\Psi}_1 = 0$ correspond to the limit case $C = \infty$.

Degenerate case: $|\widehat{\Phi}_1(X)\widehat{\Psi}_1(Y)| + |\widehat{\Phi}_2(X)\widehat{\Psi}_2(Y)| \equiv 0$. Studying this is reduced to analyzing the following cases:

$$\widehat{\Phi}_1(X) \equiv 0, \quad \widehat{\Phi}_2(X) \equiv 0 \quad \Longrightarrow \quad \widehat{\Psi}_{1,2}(Y) \text{ are any functions;}$$
$$\widehat{\Psi}_1(Y) \equiv 0, \quad \widehat{\Psi}_2(Y) \equiv 0 \quad \Longrightarrow \quad \widehat{\Phi}_{1,2}(X) \text{ are any functions,}$$

with two similar cases omitted.

2°. The solutions of the two-term equation (29.4.1.1) should be substituted into the original functional differential equation (29.2.2.1) to "remove" redundant constants of integration [these arise because equation (29.4.1.1) is obtained from (29.2.2.1) by differentiation].

3°. The case $\Psi_k \equiv 0$ should be treated separately (since we divided the equation by Ψ_k at the first stage). Likewise, we have to study all other cases where the functionals by which the intermediate functional differential equations were divided vanish.

Remark 1. The functional differential equation (29.2.2.1) happens to have no solutions.

Remark 2. At each subsequent stage, the number of terms in the functional differential equation can be reduced by differentiation with respect to either y or x. For example, we can assume at the first stage that $\Phi_k \not\equiv 0$. On dividing equation (29.2.2.1) by Φ_k and differentiating with respect to x, we again obtain a similar equation that has fewer terms.

29.4.2. Examples of Constructing Exact Generalized Separable Solutions

Below we consider specific examples illustrating the application of the above method of constructing exact generalized separable solutions of nonlinear equations.

Example 29.12. Consider once again the equation for a laminar boundary layer on a flat plate

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=a\frac{\partial^3 w}{\partial y^3} \tag{29.4.2.1}$$

(see equation (29.3.2.1)).

We look for generalized separable solutions to equation (29.4.2.1) in the form

$$w(x,y)=\varphi(x)\psi(y)+\chi(x). \tag{29.4.2.2}$$

On substituting (29.4.2.2) into (29.4.2.1) and canceling by φ, we arrive at the functional differential equation

$$\varphi'_x[(\psi'_y)^2-\psi\psi''_{yy}]-\chi'_x\psi''_{yy}=a\psi'''_{yyy}. \tag{29.4.2.3}$$

We differentiate (29.4.2.3) with respect to x to obtain

$$\varphi''_{xx}[(\psi'_y)^2-\psi\psi''_{yy}]=\chi''_{xx}\psi''_{yy}. \tag{29.4.2.4}$$

Nondegenerate case. On separating the variables in (29.4.2.4), we get

$$\chi''_{xx}=C_1\varphi''_{xx},$$
$$(\psi'_y)^2-\psi\psi''_{yy}-C_1\psi''_{yy}=0.$$

Integrating yields

$$\psi(y)=C_4e^{\lambda y}-C_1,\quad \varphi(x)\text{ is any function},\quad \chi(x)=C_1\varphi(x)+C_2x+C_3, \tag{29.4.2.5}$$

where $C_1,\dots,C_4$, and λ are constants of integration. On substituting (29.4.2.5) into (29.4.2.3), we establish the relationship between constants to obtain $C_2=-a\lambda$. Ultimately, taking into account the aforesaid and formulas (29.4.2.2) and (29.4.2.5), we arrive at a solution of equation (29.4.2.1) of the form (29.4.2.2):

$$w(x,y)=\varphi(x)e^{\lambda y}-a\lambda x+C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants ($C=C_3$, $C_4=1$).

Degenerate case. It follows from (29.4.2.4) that

$$\varphi''_{xx}=0,\quad \chi''_{xx}=0,\quad \psi(y)\text{ is any function}. \tag{29.4.2.6}$$

Integrating the first two equations in (29.4.2.6) twice yields

$$\varphi(x)=C_1x+C_2,\quad \chi(x)=C_3x+C_4, \tag{29.4.2.7}$$

where $C_1,\dots,C_4$ are arbitrary constants.

Substituting (29.4.2.7) into (29.4.2.3), we arrive at an ordinary differential equation for $\psi=\psi(y)$:

$$C_1(\psi'_y)^2-(C_1\psi+C_3)\psi''_{yy}=a\psi'''_{yyy}. \tag{29.4.2.8}$$

Formulas (29.4.2.2) and (29.4.2.7) together with equation (29.4.2.8) determine an exact solution of equation (29.4.2.1).

Example 29.13. The two-dimensional stationary equations of motion of a viscous incompressible fluid are reduced to a single fourth-order nonlinear equation for the stream function (see Loitsyanskiy, 1996):

$$\frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w)-\frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w)=a\Delta\Delta w,\qquad \Delta w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}. \tag{29.4.2.9}$$

Here, a is the kinematic viscosity of the fluid and x, y are Cartesian coordinates.

We seek exact separable solutions of equation (29.4.2.9) in the form

$$w=f(x)+g(y). \tag{29.4.2.10}$$

Substituting (29.4.2.10) into (29.4.2.9) yields

$$g'_yf'''_{xxx}-f'_xg'''_{yyy}=af''''_{xxxx}+ag''''_{yyyy}. \tag{29.4.2.11}$$

Differentiating (29.4.2.11) with respect to x and y, we obtain

$$g''_{yy}f''''_{xxxx}-f''_{xx}g''''_{yyyy}=0. \tag{29.4.2.12}$$

Nondegenerate case. If $f''_{xx}\not\equiv 0$ and $g''_{yy}\not\equiv 0$, we separate the variables in (29.4.2.12) to obtain the ordinary differential equations

$$f''''_{xxxx}=Cf''_{xx}, \tag{29.4.2.13}$$
$$g''''_{yyyy}=Cg''_{yy}, \tag{29.4.2.14}$$

which have different solutions depending on the value of the integration constant C.

1°. Solutions of equations (29.4.2.13) and (29.4.2.14) for $C = 0$:

$$f(x) = A_1 + A_2x + A_3x^2 + A_4x^3,$$
$$g(y) = B_1 + B_2y + B_3y^2 + B_4y^3, \tag{29.4.2.15}$$

where A_k and B_k are arbitrary constants ($k = 1, 2, 3, 4$). On substituting (29.4.2.15) into (29.4.2.11), we evaluate the integration constants. Three cases are possible:

$$\begin{aligned} &A_4 = B_4 = 0, && A_n, B_n \text{ are any numbers} && (n = 1, 2, 3);\\ &A_k = 0, && B_k \text{ are any numbers} && (k = 1, 2, 3, 4);\\ &B_k = 0, && A_k \text{ are any numbers} && (k = 1, 2, 3, 4). \end{aligned}$$

The first two sets of constants determine two simple solutions (29.4.2.10) of equation (29.4.2.9):

$$w = C_1x^2 + C_2x + C_3y^2 + C_4y + C_5,$$
$$w = C_1y^3 + C_2y^2 + C_3y + C_4,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

2°. Solutions of equations (29.4.2.13) and (29.4.2.14) for $C = \lambda^2 > 0$:

$$f(x) = A_1 + A_2x + A_3e^{\lambda x} + A_4e^{-\lambda x},$$
$$g(y) = B_1 + B_2y + B_3e^{\lambda y} + B_4e^{-\lambda y}. \tag{29.4.2.16}$$

Substituting (29.4.2.16) into (29.4.2.11), dividing by λ^3, and collecting terms, we obtain

$$A_3(a\lambda - B_2)e^{\lambda x} + A_4(a\lambda + B_2)e^{-\lambda x} + B_3(a\lambda + A_2)e^{\lambda y} + B_4(a\lambda - A_2)e^{-\lambda y} = 0.$$

Equating the coefficients of the exponentials to zero, we find

$$\begin{aligned} &A_3 = A_4 = B_3 = 0, \quad A_2 = a\lambda && && \text{(case 1)},\\ &A_3 = B_3 = 0, \quad A_2 = a\lambda, \quad B_2 = -a\lambda && && \text{(case 2)},\\ &A_3 = B_4 = 0, \quad A_2 = -a\lambda, \quad B_2 = -a\lambda && && \text{(case 3)}. \end{aligned}$$

(The other constants are arbitrary.) These sets of constants determine three solutions of the form (29.4.2.10) for equation (29.4.2.9):

$$w = C_1e^{-\lambda y} + C_2y + C_3 + a\lambda x,$$
$$w = C_1e^{-\lambda x} + a\lambda x + C_2e^{-\lambda y} - a\lambda y + C_3,$$
$$w = C_1e^{-\lambda x} - a\lambda x + C_2e^{\lambda y} - a\lambda y + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants.

3°. Solution of equations (29.4.2.13) and (29.4.2.14) for $C = -\lambda^2 < 0$:

$$f(x) = A_1 + A_2x + A_3\cos(\lambda x) + A_4\sin(\lambda x),$$
$$g(y) = B_1 + B_2y + B_3\cos(\lambda y) + B_4\sin(\lambda y). \tag{29.4.2.17}$$

Substituting (29.4.2.17) into (29.4.2.11) does not yield new real solutions.

Degenerate cases. If $f''_{xx} \equiv 0$ or $g''_{yy} \equiv 0$, equation (29.4.2.12) becomes an identity for any $g = g(y)$ or $f = f(x)$, respectively. These cases should be treated separately from the nondegenerate case. For example, if $f''_{xx} \equiv 0$, we have $f(x) = Ax + B$, where A and B are arbitrary numbers. Substituting this f into (29.4.2.11), we arrive at the equation $-Ag'''_{yyy} = ag''''_{yyyy}$. Its general solution is given by $g(y) = C_1\exp(-Ay/a) + C_2y^2 + C_3y + C_4$. Thus, we obtain another solution of the form (29.4.2.10) for equation (29.4.2.9):

$$w = C_1e^{-\lambda y} + C_2y^2 + C_3y + C_4 + a\lambda x \qquad (A = a\lambda,\ B = 0).$$

Example 29.14. Consider the second-order nonlinear parabolic equation

$$\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + c. \tag{29.4.2.18}$$

We look for exact separable solutions of equation (29.4.2.18) in the form

$$w = \varphi(t) + \psi(t)\theta(x). \tag{29.4.2.19}$$

Substituting (29.4.2.19) into (29.4.2.18) and collecting terms yields

$$\varphi'_t - c + \psi'_t\theta = a\varphi\psi\theta''_{xx} + \psi^2\left[a\theta\theta''_{xx} + b(\theta'_x)^2\right]. \tag{29.4.2.20}$$

On dividing this relation by ψ^2 and differentiating with respect to t and x, we obtain

$$(\psi'_t/\psi^2)'_t\theta'_x = a(\varphi/\psi)'_t\theta'''_{xxx}.$$

Separating the variables, we arrive at the ordinary differential equations

$$\theta'''_{xxx} = K\theta'_x, \tag{29.4.2.21}$$

$$(\psi'_t/\psi^2)'_t = aK(\varphi/\psi)'_t, \tag{29.4.2.22}$$

where K is an arbitrary constant. The general solution of equation (29.4.2.21) is given by

$$\theta = \begin{cases} A_1x^2 + A_2x + A_3 & \text{if } K = 0, \\ A_1e^{\lambda x} + A_2e^{-\lambda x} + A_3 & \text{if } K = \lambda^2 > 0, \\ A_1\sin(\lambda x) + A_2\cos(\lambda x) + A_3 & \text{if } K = -\lambda^2 < 0, \end{cases} \tag{29.4.2.23}$$

where A_1, A_2, and A_3 are arbitrary constants. Integrating (29.4.2.22) yields

$$\begin{aligned} &\psi = \frac{B}{t + C_1}, \quad \varphi(t) \text{ is any} && \text{if } K = 0, \\ &\varphi = B\psi + \frac{1}{aK}\frac{\psi'_t}{\psi}, \quad \psi(t) \text{ is any} && \text{if } K \neq 0, \end{aligned} \tag{29.4.2.24}$$

where B is an arbitrary constant. On substituting solutions (29.4.2.23) and (29.4.2.24) into (29.4.2.20), one can "remove" the redundant constants and define the functions φ and ψ. Below we summarize the results.

1°. Solution for $a \neq -b$ and $a \neq -2b$:

$$w = \frac{c(a+2b)}{2(a+b)}(t + C_1) + C_2(t + C_1)^{-\frac{a}{a+2b}} - \frac{(x + C_3)^2}{2(a+2b)(t + C_1)} \qquad \text{(corresponds to } K = 0\text{)},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solution for $b = -a$:

$$w = \frac{1}{a\lambda^2}\frac{\psi'_t}{\psi} + \psi(A_1e^{\lambda x} + A_2e^{-\lambda x}) \qquad \text{(corresponds to } K = \lambda^2 > 0\text{)},$$

where the function $\psi = \psi(t)$ is determined from the autonomous ordinary differential equation

$$Z''_{tt} = ac\lambda^2 + 4a^2\lambda^4A_1A_2e^{2Z}, \qquad \psi = e^Z,$$

whose solution can be found in implicit form. In the special case $A_1 = 0$ or $A_2 = 0$, one easily obtains $\psi = C_1\exp\left(\frac{1}{2}ac\lambda^2t^2 + C_2t\right)$.

3°. Solution for $b = -a$:

$$w = -\frac{1}{a\lambda^2}\frac{\psi'_t}{\psi} + \psi[A_1\sin(\lambda x) + A_2\cos(\lambda x)] \qquad \text{(corresponds to } K = -\lambda^2 < 0\text{)},$$

where the function $\psi = \psi(t)$ is determined from the autonomous ordinary differential equation

$$Z''_{tt} = -ac\lambda^2 + a^2\lambda^4(A_1^2 + A_2^2)e^{2Z}, \qquad \psi = e^Z,$$

whose solution can be found in implicit form.

⊙ *References for Section* 29.4: Galaktionov (1995), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002, 2004).

29.5. Solution of Functional-Differential Equations by Splitting

29.5.1. Preliminary Remarks. Description of the Method

As one reduces the number of terms in the functional differential equation (29.2.2.1)–(29.2.2.2) by differentiation, redundant constants of integration arise. These constants must be "removed" at the final stage. Furthermore, the resulting equation can be of a higher order than the original equation. To avoid these difficulties, it is convenient to reduce the solution of the functional differential equation to the solution of a bilinear functional equation of a standard form and solution of a system of ordinary differential equations. Thus, the original problem splits into two simpler problems. Below we outline the basic stages of the splitting method.

1°. At the first stage, we treat equation (29.2.2.1) as a purely functional equation that depends on two variables X and Y, where $\Phi_n = \Phi_n(X)$ and $\Psi_n = \Psi_n(Y)$ are unknown quantities ($n = 1, \ldots, k$).

It can be shown* that the bilinear functional equation (29.2.2.1) has $k - 1$ different solutions:

$$\begin{aligned} \Phi_i(X) &= C_{i,1}\Phi_{m+1}(X) + C_{i,2}\Phi_{m+2}(X) + \cdots + C_{i,k-m}\Phi_k(X), \qquad i = 1, \ldots, m; \\ \Psi_{m+j}(Y) &= -C_{1,j}\Psi_1(Y) - C_{2,j}\Psi_2(Y) - \cdots - C_{m,j}\Psi_m(Y), \qquad j = 1, \ldots, k-m; \\ & m = 1, 2, \ldots, k-1, \end{aligned} \tag{29.5.1.1}$$

where $C_{i,j}$ are arbitrary constants. The functions $\Phi_{m+1}(X)$, …, $\Phi_k(X)$, $\Psi_1(Y)$, …, $\Psi_m(Y)$ on the right-hand sides of formulas (29.5.1.1) are defined arbitrarily. It is apparent that for fixed m, solution (29.5.1.1) contains $m(k-m)$ arbitrary constants.

2°. At the second stage, we successively substitute the $\Phi_i(X)$ and $\Psi_j(Y)$ of (29.2.2.2) into all solutions (29.5.1.1) to obtain systems of ordinary differential equations (such systems are usually overdetermined) for the unknown functions $\varphi_p(x)$ and $\psi_q(y)$. Solving these systems, we get generalized separable solutions of the form (29.2.1.1).

3°. The degenerate cases where either function $\Phi_i(X)$ or $\Psi_j(Y)$ vanishes must be treated separately.

Remark 1. It is important that, for fixed k, the bilinear functional equation (29.2.2.1) used in the splitting method is the same for different classes of original nonlinear mathematical physics equations.

Remark 2. For fixed m, solution (29.5.1.1) contains $m(k-m)$ arbitrary constants $C_{i,j}$. Given k, the solutions having the maximum number of arbitrary constants are defined by

Solution number	*Number of arbitrary constants*	*Conditions on k*
$m = \frac{1}{2}k$	$\frac{1}{4}k^2$	if k is even,
$m = \frac{1}{2}(k \pm 1)$	$\frac{1}{4}(k^2 - 1)$	if k is odd.

It is these solutions of the bilinear functional equation that most frequently result in nontrivial generalized separable solution in nonlinear partial differential equations.

Remark 3. The bilinear functional equation (29.2.2.1) and its solutions (29.5.1.1) play an important role in the method of functional separation of variables.

For visualization, the main stages of constructing generalized separable solutions by the splitting method are displayed in Fig. 29.1.

* These solutions can be obtained by differentiation following the procedure outlined in Section 29.4, and by induction. Another simple method for finding solutions is described in Subsection 29.5.2, Item 3°.

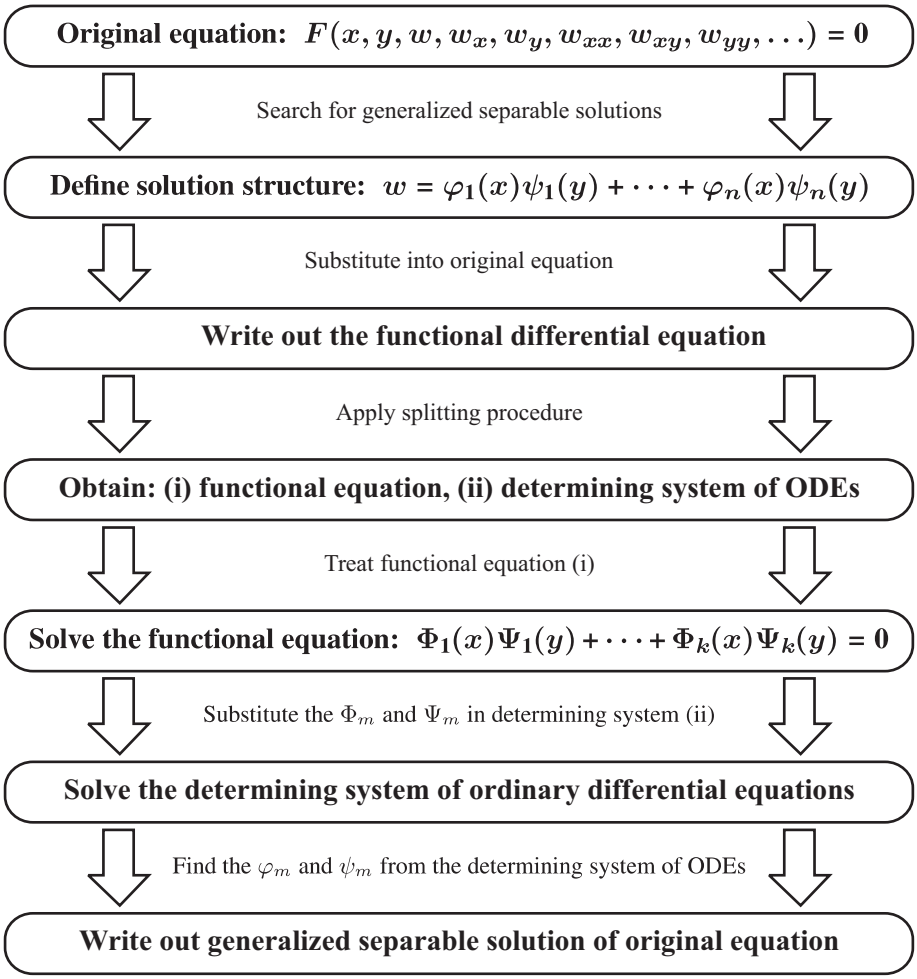

Figure 29.1. General scheme for constructing generalized separable solutions by the splitting method. Abbreviation: ODE stands for ordinary differential equation.

29.5.2. Solutions of Simple Functional Equations and Their Application

Below we give solutions to two simple bilinear functional equations of the form (29.2.2.1) that will be used subsequently for solving specific nonlinear partial differential equations.

1°. The functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 = 0, \tag{29.5.2.1}$$

where Φ_i are all functions of the same argument and Ψ_i are all functions of another argument, has two solutions:

$$\begin{aligned} &\Phi_1 = A_1\Phi_3, \quad \Phi_2 = A_2\Phi_3, \quad \Psi_3 = -A_1\Psi_1 - A_2\Psi_2; \\ &\Psi_1 = A_1\Psi_3, \quad \Psi_2 = A_2\Psi_3, \quad \Phi_3 = -A_1\Phi_1 - A_2\Phi_2. \end{aligned} \tag{29.5.2.2}$$

The arbitrary constants are renamed as follows: $A_1 = C_{1,1}$ and $A_2 = C_{2,1}$ in the first solution, and $A_1 = -1/C_{1,2}$ and $A_2 = C_{1,1}/C_{1,2}$ in the second solution. The functions on the right-hand sides of the formulas in (29.5.2.2) are assumed to be arbitrary.

2°. The functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 = 0, \tag{29.5.2.3}$$

where Φ_i are all functions of the same argument and Ψ_i are all functions of another argument, has a solution

$$\begin{aligned} &\Phi_1 = A_1\Phi_3 + A_2\Phi_4, \quad \Phi_2 = A_3\Phi_3 + A_4\Phi_4, \\ &\Psi_3 = -A_1\Psi_1 - A_3\Psi_2, \quad \Psi_4 = -A_2\Psi_1 - A_4\Psi_2 \end{aligned} \tag{29.5.2.4}$$

dependent on four arbitrary constants $A_1, \ldots, A_4$; see solution (29.5.1.1) with $k = 4$, $m = 2$, $C_{1,1} = A_1$, $C_{1,2} = A_2$, $C_{2,1} = A_3$, and $C_{2,2} = A_4$. The functions on the right-hand sides of the solutions in (29.5.2.2) are assumed to be arbitrary.

Equation (29.5.2.3) also has two other solutions:

$$\begin{aligned} &\Phi_1 = A_1\Phi_4, \quad \Phi_2 = A_2\Phi_4, \quad \Phi_3 = A_3\Phi_4, \quad \Psi_4 = -A_1\Psi_1 - A_2\Psi_2 - A_3\Psi_3; \\ &\Psi_1 = A_1\Psi_4, \quad \Psi_2 = A_2\Psi_4, \quad \Psi_3 = A_3\Psi_4, \quad \Phi_4 = -A_1\Phi_1 - A_2\Phi_2 - A_3\Phi_3 \end{aligned} \tag{29.5.2.5}$$

involving three arbitrary constants. In the first solution, $A_1 = C_{1,1}$, $A_2 = C_{2,1}$, and $A_3 = C_{3,1}$, and in the second solution, $A_1 = -1/C_{1,3}$, $A_2 = C_{1,1}/C_{1,3}$, and $A_3 = C_{1,2}/C_{1,3}$.

Solutions (29.5.2.5) will sometimes be called *degenerate*, to emphasize the fact that they contain fewer arbitrary constants than solution (29.5.2.4).

3°. Solutions of the functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 + \Phi_5\Psi_5 = 0 \tag{29.5.2.6}$$

can be found by formulas (29.5.1.1) with $k = 5$. Below is a simple technique for finding solutions, which is quite useful in practice, based on equation (29.5.2.6) itself. Let us assume that Φ_1, Φ_2, and Φ_3 are linear combinations of Φ_4 and Φ_5:

$$\Phi_1 = A_1\Phi_4 + B_1\Phi_5, \quad \Phi_2 = A_2\Phi_4 + B_2\Phi_5, \quad \Phi_3 = A_3\Phi_4 + B_3\Phi_5, \tag{29.5.2.7}$$

where A_n, B_n are arbitrary constants. Let us substitute (29.5.2.7) into (29.5.2.6) and collect the terms proportional to Φ_4 and Φ_5 to obtain

$$(A_1\Psi_1 + A_2\Psi_2 + A_3\Psi_3 + \Psi_4)\Phi_4 + (B_1\Psi_1 + B_2\Psi_2 + B_3\Psi_3 + \Psi_5)\Phi_5 = 0.$$

Equating the expressions in parentheses to zero, we have

$$\begin{aligned} \Psi_4 &= -A_1\Psi_1 - A_2\Psi_2 - A_3\Psi_3, \\ \Psi_5 &= -B_1\Psi_1 - B_2\Psi_2 - B_3\Psi_3. \end{aligned} \tag{29.5.2.8}$$

Formulas (29.5.2.7) and (29.5.2.8) give solutions to equation (29.5.2.6). Other solutions are found likewise.

Example 29.15. Consider the nonlinear hyperbolic equation

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + f(t)w + g(t), \tag{29.5.2.9}$$

where $f(t)$ and $g(t)$ are arbitrary functions. We look for generalized separable solutions of the form

$$w(x,t) = \varphi(x)\psi(t) + \chi(t). \tag{29.5.2.10}$$

Substituting (29.5.2.10) into (29.5.2.9) and collecting terms yield

$$a\psi^2(\varphi\varphi'_x)'_x + a\psi\chi\varphi''_{xx} + (f\psi - \psi''_{tt})\varphi + f\chi + g - \chi''_{tt} = 0.$$

This equation can be represented as a functional equation (29.5.2.3) in which

$$\begin{aligned}&\Phi_1 = (\varphi\varphi_x')_x', \quad \Phi_2 = \varphi_{xx}'', \quad \Phi_3 = \varphi, \qquad \Phi_4 = 1,\\ &\Psi_1 = a\psi^2, \qquad \Psi_2 = a\psi\chi, \quad \Psi_3 = f\psi - \psi_{tt}'', \quad \Psi_4 = f\chi + g - \chi_{tt}''.\end{aligned} \tag{29.5.2.11}$$

On substituting (29.5.2.11) into (29.5.2.4), we obtain the following overdetermined system of ordinary differential equations for the functions $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $\chi = \chi(t)$:

$$\begin{aligned}&(\varphi\varphi_x')_x' = A_1\varphi + A_2, \qquad \varphi_{xx}'' = A_3\varphi + A_4,\\ &f\psi - \psi_{tt}'' = -A_1a\psi^2 - A_3a\psi\chi, \quad f\chi + g - \chi_{tt}'' = -A_2a\psi^2 - A_4a\psi\chi.\end{aligned} \tag{29.5.2.12}$$

The first two equations in (29.5.2.12) are compatible only if

$$A_1 = 6B_2, \quad A_2 = B_1^2 - 4B_0B_2, \quad A_3 = 0, \quad A_4 = 2B_2, \tag{29.5.2.13}$$

where B_0, B_1, and B_2 are arbitrary constants, and the solution is given by

$$\varphi(x) = B_2x^2 + B_1x + B_0. \tag{29.5.2.14}$$

On substituting the expressions (29.5.2.13) into the last two equations in (29.5.2.12), we obtain the following system of equations for $\psi(t)$ and $\chi(t)$:

$$\begin{aligned}&\psi_{tt}'' = 6aB_2\psi^2 + f(t)\psi,\\ &\chi_{tt}'' = [2aB_2\psi + f(t)]\chi + a(B_1^2 - 4B_0B_2)\psi^2 + g(t).\end{aligned} \tag{29.5.2.15}$$

Relations (29.5.2.10), (29.5.2.14) and system (29.5.2.15) determine a generalized separable solution of equation (29.5.2.9). The first equation in (29.5.2.15) can be solved independently; it is linear if $B_2 = 0$ and is integrable by quadrature for $f(t) = \text{const}$. The second equation in (29.5.2.15) is linear in χ (for ψ known).

Equation (29.5.2.9) does not have other solutions with the form (29.5.2.10) if f and g are arbitrary functions and $\varphi \not\equiv 0$, $\psi \not\equiv 0$, and $\chi \not\equiv 0$.

Remark. It can be shown that equation (29.5.2.9) has a more general solution with the form (see Galaktionov, 1995)

$$w(x,y) = \varphi_1(x)\psi_1(t) + \varphi_2(x)\psi_2(t) + \psi_3(t), \qquad \varphi_1(x) = x^2, \quad \varphi_2(x) = x, \tag{29.5.2.16}$$

where the functions $\psi_i = \psi_i(t)$ are determined by the ordinary differential equations

$$\begin{aligned}&\psi_1'' = 6a\psi_1^2 + f(t)\psi_1,\\ &\psi_2'' = [6a\psi_1 + f(t)]\psi_2,\\ &\psi_3'' = [2a\psi_1 + f(t)]\psi_3 + a\psi_2^2 + g(t).\end{aligned} \tag{29.5.2.17}$$

(The prime denotes a derivative with respect to t.) The second equation in (29.5.2.17) has a particular solution $\psi_2 = \psi_1$. Hence, its general solution can be represented as (see Polyanin and Zaitsev, 2003)

$$\psi_2 = C_1\psi_1 + C_2\psi_1\int\frac{dt}{\psi_1^2}.$$

The solution obtained in Example 1 corresponds to the special case $C_2 = 0$.

Example 29.16. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x\partial t} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = a\frac{\partial^3 w}{\partial x^3}, \tag{29.5.2.18}$$

which arises in hydrodynamics (see Polyanin and Zaitsev, 2004).

We look for exact solutions of the form

$$w = \varphi(t)\theta(x) + \psi(t). \tag{29.5.2.19}$$

Substituting (29.5.2.19) into (29.5.2.18) yields

$$\varphi_t'\theta_x' - \varphi\psi\theta_{xx}'' + \varphi^2\left[(\theta_x')^2 - \theta\theta_{xx}''\right] - a\varphi\theta_{xxx}''' = 0.$$

This functional differential equation can be reduced to the functional equation (29.5.2.3) by setting

$$\begin{aligned}&\Phi_1 = \varphi_t', \quad \Phi_2 = \varphi\psi, \qquad \Phi_3 = \varphi^2, \qquad \Phi_4 = a\varphi,\\ &\Psi_1 = \theta_x', \quad \Psi_2 = -\theta_{xx}'', \quad \Psi_3 = (\theta_x')^2 - \theta\theta_{xx}'', \quad \Psi_4 = -\theta_{xxx}'''.\end{aligned} \tag{29.5.2.20}$$

On substituting these expressions into (29.5.2.4), we obtain the system of ordinary differential equations

$$\begin{aligned} &\varphi'_t = A_1\varphi^2 + A_2 a\varphi, && \varphi\psi = A_3\varphi^2 + A_4 a\varphi, \\ &(\theta'_x)^2 - \theta\theta''_{xx} = -A_1\theta'_x + A_3\theta''_{xx}, && \theta'''_{xxx} = A_2\theta'_x - A_4\theta''_{xx}. \end{aligned} \tag{29.5.2.21}$$

It can be shown that the last two equations in (29.5.2.21) are compatible only if the function θ and its derivative are linearly dependent,

$$\theta'_x = B_1\theta + B_2. \tag{29.5.2.22}$$

The six constants B_1, B_2, A_1, A_2, A_3, and A_4 must satisfy the three conditions

$$\begin{aligned} B_1(A_1 + B_2 - A_3B_1) &= 0, \\ B_2(A_1 + B_2 - A_3B_1) &= 0, \\ B_1^2 + A_4B_1 - A_2 &= 0. \end{aligned} \tag{29.5.2.23}$$

Integrating (29.5.2.22) yields

$$\theta = \begin{cases} B_3\exp(B_1x) - \dfrac{B_2}{B_1} & \text{if } B_1 \neq 0, \\ B_2x + B_3 & \text{if } B_1 = 0, \end{cases} \tag{29.5.2.24}$$

where B_3 is an arbitrary constant.

The first two equations in (29.5.2.21) lead to the following expressions for φ and ψ:

$$\varphi = \begin{cases} \dfrac{A_2a}{C\exp(-A_2at) - A_1} & \text{if } A_2 \neq 0, \\ -\dfrac{1}{A_1t + C} & \text{if } A_2 = 0, \end{cases} \qquad \psi = A_3\varphi + A_4a, \tag{29.5.2.25}$$

where C is an arbitrary constant.

Formulas (29.5.2.24), (29.5.2.25) and relations (29.5.2.23) allow us to find the following solutions of equation (29.5.2.18) with the form (29.5.2.19):

$$\begin{aligned} &w = \frac{x + C_1}{t + C_2} + C_3 && \text{if} \quad A_2 = B_1 = 0,\ B_2 = -A_1; \\ &w = \frac{C_1e^{-\lambda x} + 1}{\lambda t + C_2} + a\lambda && \text{if} \quad A_2 = 0,\ B_1 = -A_4,\ B_2 = -A_1 - A_3A_4; \\ &w = C_1e^{-\lambda(x+a\beta t)} + a(\lambda + \beta) && \text{if} \quad A_1 = A_3 = B_2 = 0,\ A_2 = B_1^2 + A_4B_1; \\ &w = \frac{a\beta + C_1e^{-\lambda x}}{1 + C_2e^{-a\lambda\beta t}} + a(\lambda - \beta) && \text{if} \quad A_1 = A_3B_1 - B_2,\ A_2 = B_1^2 + A_4B_1, \end{aligned}$$

where C_1, C_2, C_3, β, and λ are arbitrary constants (these can be expressed in terms of the A_k and B_k).

The analysis of the second solution (29.5.2.5) of the functional equation (29.5.2.3) in view of (29.5.2.20) leads to the following two more general solutions of the differential equation (29.5.2.18):

$$\begin{aligned} w &= \frac{x}{t + C_1} + \psi(t), \\ w &= \varphi(t)e^{-\lambda x} - \frac{\varphi'_t(t)}{\lambda\varphi(t)} + a\lambda, \end{aligned}$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, and C_1 and λ are arbitrary constants.

Example 29.17. Consider the nonlinear Monge–Ampère type equation

$$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + k\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x)g(y). \tag{29.5.2.26}$$

For $k = -1$, equations of this form are encountered in differential geometry, gas dynamics, and meteorology.

We look for generalized separable solutions with the form

$$w(x, y) = \varphi(x)\theta(y) + \psi(x). \tag{29.5.2.27}$$

On substituting (29.5.2.27) into (29.5.2.26) and collecting terms, we obtain

$$k\varphi\varphi''_{xx}\theta\theta''_{yy} + k\varphi\psi''_{xx}\theta''_{yy} + (\varphi'_x)^2(\theta'_y)^2 - f(x)g(y) = 0. \tag{29.5.2.28}$$

TABLE 29.2

Exact solutions of the Monge–Ampère type equation (29.5.2.26) of the form (29.5.2.27); a, b, n, and λ are some numbers (the sum $C_1x + C_2y + C_3$ can be added to all solutions; C_1, C_2, and C_3 are arbitrary constants).

No.	Function $f(x)$	Function $g(y)$	Generalized separable solution $w(x,y)$
1	Any	$ay^n + by^{2n+2}$ $(n \neq -1, -2)$	$w = \varphi(x)y^{n+2} + \dfrac{a}{k(n+1)(n+2)} \displaystyle\int_{x_0}^{x} (x-t)\frac{f(t)}{\varphi(t)}\,dt;$ $k(n+1)(n+2)\varphi\varphi''_{xx} + (n+2)^2(\varphi'_x)^2 = bf(x)$
2	Any	$\dfrac{a\ln y + b}{y^2}$	$w = \varphi(x)\ln y + \dfrac{1}{k}\displaystyle\int_{x_0}^{x} (x-t)\frac{[\varphi'_t(t)]^2 - bf(t)}{\varphi(t)}\,dt;$ $k\varphi\varphi''_{xx} + af(x) = 0$
3	Any	$ae^{\lambda y} + be^{2\lambda y}$	$w = \varphi(x)e^{\lambda y} + \dfrac{a}{k\lambda^2}\displaystyle\int_{x_0}^{x} (x-t)\frac{f(t)}{\varphi(t)}\,dt;$ $k\lambda^2\varphi\varphi''_{xx} + \lambda^2(\varphi'_x)^2 = bf(x)$
4	Any	Any	$w = \dfrac{C_1}{k}\displaystyle\int_{x_0}^{x} (x-t)f(t)\,dt + \frac{1}{C_1}\int_{y_0}^{y} (y-\xi)g(\xi)\,d\xi$
5	Any	1	$w = \pm y\displaystyle\int_{x_0}^{x} \sqrt{f(x)}\,dx + \psi(x);$ $\psi(x)$ is an arbitrary function
6	1	Any	$w = (ax+b)\theta(y) + c(ax+b)[\ln(ax+b) - 1];$ $a^2ck\theta''_{yy} + a^2(\theta'_y)^2 - g(y) = 0$

This functional differential equation can be reduced to the functional equation (29.5.2.3) by setting

$$\begin{aligned} &\Phi_1 = k\varphi\varphi''_{xx}, \quad \Phi_2 = k\varphi\psi''_{xx}, \quad \Phi_3 = (\varphi'_x)^2, \quad \Phi_4 = f(x), \\ &\Psi_1 = \theta\theta''_{yy}, \qquad \Psi_2 = \theta''_{yy}, \qquad \Psi_3 = (\theta'_y)^2, \quad \Psi_4 = -g(y). \end{aligned} \tag{29.5.2.29}$$

On substituting (29.5.2.29) into (29.5.2.4), we have

$$\begin{aligned} &k\varphi\varphi''_{xx} = A_1(\varphi'_x)^2 + A_2f(x), \quad k\varphi\psi''_{xx} = A_3(\varphi'_x)^2 + A_4f(x), \\ &(\theta'_y)^2 = -A_1\theta\theta''_{yy} - A_3\theta''_{yy}, \quad g(y) = A_2\theta\theta''_{yy} + A_4\theta''_{yy}. \end{aligned} \tag{29.5.2.30}$$

The function $\varphi(x)$ can be determined from the first equation. Then, by double simple integration of the second equation, we find $\psi(x)$:

$$\psi(x) = \frac{1}{k}\int_{x_0}^{x} (x-t)\frac{A_3[\varphi'_t(t)]^2 + A_4f(t)}{\varphi(t)}\,dt + B_1x + B_2,$$

where B_1 and B_2 are arbitrary constants. The third equation serves to determine $\theta(y)$ and the last equation serves to determine the admissible function $g(y)$.

For $A_1 = -k$, the first equation in (29.5.2.30) is integrable by quadrature for any $f(x)$:

$$\varphi^2 = \frac{2A_2}{k}\int_{x_0}^{x} (x-t)f(t)\,dt + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants and x_0 is any number for which the integral exists (if the integrand does not have singularities, we can set $x_0 = 0$).

The third equation in (29.5.2.30) is easy to integrate for $A_1 \neq 0$; without loss of generality, it can be assumed that $A_3 = 0$ (this is achieved by translation of θ by a constant, which leads to redefining ψ in (29.5.2.27)). For $A_1 \neq 0$ and $A_3 = 0$, power-law functions and exponentials are solutions to the equation. In the special case $A_1 = 0$, the equation is solved by a logarithmic function. The last relation in (29.5.2.30) serves to identify the admissible functions $g(y)$. These results are summarized in the first three rows of Table 29.2. The table also lists three degenerate solutions, which correspond to vanishing second derivatives of the determining functions in solution (29.5.2.27).

It is noteworthy that the determining equations (29.5.2.30) and the first five solutions remain valid if $k = k(x)$ is an arbitrary function; in these solutions, k can be included in the integral and replaced by $k(t)$.

Example 29.18. Consider the equation

$$w_t = f(x)\exp(aw_{xx}), \tag{29.5.2.31}$$

which has an exponential nonlinearity in the highest derivative.

We look for exact solutions of the form

$$w = \varphi(x) + \psi(x)\theta(t). \tag{29.5.2.32}$$

By substituting (29.5.2.32) into (29.5.2.31), dividing the resulting equation by $f(x)$, taking the logarithm of both sides while assuming $\psi/f > 0$, and rearranging, we obtain

$$a\varphi''_{xx} - \ln(\psi/f) + a\theta\psi''_{xx} - \ln\theta'_t = 0. \tag{29.5.2.33}$$

This functional differential equation can be rewritten in the form (29.5.2.1) with

$$\Phi_1 = a\varphi''_{xx} - \ln(\psi/f), \quad \Phi_2 = \psi''_{xx}, \quad \Phi_3 = 1, \quad \Psi_1 = 1, \quad \Psi_2 = a\theta, \quad \Psi_3 = -\ln\theta'_t.$$

Substituting these expressions into the first solution in (29.5.2.2) leads to the ordinary differential equations

$$\Phi_1 = a\varphi''_{xx} - \ln(\psi/f) = A_1, \quad \psi''_{xx} = A_2, \quad \ln\theta'_t = A_1 + A_2 a\theta.$$

Integrating yields

$$\begin{aligned}
\varphi(x) &= \frac{1}{2a}A_1x^2 + C_3x + C_4 + \frac{1}{a}\int_{x_0}^{x}(x-\xi)\ln\frac{\psi(\xi)}{f(\xi)}\,d\xi,\\
\psi(x) &= \frac{1}{2}A_2x^2 + C_1x + C_2,\\
\theta(t) &= -\frac{1}{A_2a}\ln\bigl(C_5 - A_2ae^{A_1}t\bigr).
\end{aligned} \tag{29.5.2.34}$$

Formula (29.5.2.32) with the functions of (29.5.2.34) define an exact solution to equation (29.5.2.31).

⊙ *References for Section* 29.5: E. R. Rozendorn (1984), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and A. I. Zhurov (2002), A. D. Polyanin and V. F. Zaitsev (2004).

29.6. Titov–Galaktionov Method

29.6.1. Method Description. Linear Subspaces Invariant under a Nonlinear Operator

Consider the nonlinear evolution equation

$$\frac{\partial w}{\partial t} = F[w], \tag{29.6.1.1}$$

where $F[w]$ is a nonlinear differential operator with respect to x,

$$F[w] \equiv F\left(x, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^m w}{\partial x^m}\right). \tag{29.6.1.2}$$

Definition. A finite-dimensional linear subspace

$$\mathcal{L}_n = \bigl\{\varphi_1(x), \ldots, \varphi_n(x)\bigr\} \tag{29.6.1.3}$$

formed by linear combinations of linearly independent functions $\varphi_1(x), \ldots, \varphi_n(x)$ is called invariant under the operator F if $F[\mathcal{L}_n] \subseteq \mathcal{L}_n$. This means that there exist functions $f_1, \ldots, f_n$ such that

$$F\left[\sum_{i=1}^{n} C_i\varphi_i(x)\right] = \sum_{i=1}^{n} f_i(C_1, \ldots, C_n)\varphi_i(x) \tag{29.6.1.4}$$

for arbitrary constants $C_1, \ldots, C_n$.

Let the linear subspace (29.6.1.3) be invariant under the operator F. Then equation (29.6.1.1) possesses generalized separable solutions of the form

$$w(x,t)=\sum_{i=1}^{n}\psi_i(t)\varphi_i(x). \tag{29.6.1.5}$$

Here, the functions $\psi_1(t),\dots,\psi_n(t)$ are described by the autonomous system of ordinary differential equations

$$\psi_i'=f_i(\psi_1,\dots,\psi_n),\qquad i=1,\dots,n, \tag{29.6.1.6}$$

where the prime denotes a derivative with respect to t.

The following example illustrates the scheme for constructing generalized separable solutions.

Example 29.19. Consider the nonlinear second-order parabolic equation

$$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+\left(\frac{\partial w}{\partial x}\right)^2+kw^2+bw+c. \tag{29.6.1.7}$$

Obviously, the nonlinear differential operator

$$F[w]=aw_{xx}+(w_x)^2+kw^2+bw+c \tag{29.6.1.8}$$

for $k>0$ has a two-dimensional invariant subspace $\mathcal{L}_2=\{1,\cos(x\sqrt{k}\,)\}$. Indeed, for arbitrary C_1 and C_2 we have

$$F\big[C_1+C_2\cos(x\sqrt{k}\,)\big]=k(C_1^2+C_2^2)+bC_1+c+C_2(2kC_1-ak+b)\cos(x\sqrt{k}\,).$$

Therefore, there is a generalized separable solution of the form

$$w(x,t)=\psi_1(t)+\psi_2(t)\cos(x\sqrt{k}\,), \tag{29.6.1.9}$$

where the functions $\psi_1(t)$ and $\psi_2(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}\psi_1'&=k(\psi_1^2+\psi_2^2)+b\psi_1+c,\\ \psi_2'&=\psi_2(2k\psi_1-ak+b).\end{aligned} \tag{29.6.1.10}$$

Remark 1. Example 29.22 below shows how one can find all two-dimensional linear subspaces invariant under the nonlinear differential operator (29.6.1.8).

Remark 2. For $k>0$, the nonlinear differential operator (29.6.1.8) has a three-dimensional invariant subspace $\mathcal{L}_3=\{1,\sin(x\sqrt{k}\,),\cos(x\sqrt{k}\,)\}$; see Example 29.22.

Remark 3. For $k<0$, the nonlinear differential operator (29.6.1.8) has a three-dimensional invariant subspace $\mathcal{L}_3=\{1,\sinh(x\sqrt{-k}\,),\cosh(x\sqrt{-k}\,)\}$; see Example 29.22.

Remark 4. A more general equation (29.6.1.7), with $a=a(t)$, $b=b(t)$, and $c=c(t)$ being arbitrary functions, and $k=\text{const}>0$, also admits a generalized separable solution of the form (29.6.1.9), where the functions $\psi_1(t)$ and $\psi_2(t)$ are determined by the system of ordinary differential equations (29.6.1.10).

Example 29.20. The nonlinear differential operator

$$F[w]=w_x^{(m)}w_x^{(n)} \tag{29.6.1.11}$$

has the $(m+n+1)$-dimensional invariant subspace

$$\mathcal{L}_{m+n}=\{1,x,x^2,\dots,x^{m+n}\}. \tag{29.6.1.12}$$

In particular, for $n=0$ and $m=2$ we have $F[w]=ww_{xx}$ and $\mathcal{L}_3=\{1,x,x^2\}$.

The nonlinear differential operator

$$F[w]=\sum_{i=0}^{k}a_iw_x^{(m+i)}w_x^{(n-i)},$$

which is more general than (29.6.1.11), also admits an $(n+m+1)$-dimensional invariant subspace (29.6.1.12).

Table 29.3 lists some nonlinear differential operators and linear subspaces invariant under these operators.

TABLE 29.3
Some nonlinear differential operators and linear subspaces invariant under these operators; adding the linear operator $L[w] = \alpha w_{xx} + \beta w_x + \gamma w + \delta$ to the first five nonlinear operators does not change the invariant subspaces.

No.	Nonlinear differential operator $F[w]$	Subspaces invariant under the operator $F[w]$
1	$aw_{xx}+(w_x)^2+bw^2$	$\mathcal{L}_3=\{1, \sin(x\sqrt{b}), \cos(x\sqrt{b})\}$ if $b>0$, $\mathcal{L}_3=\{1, \sinh(x\sqrt{\lvert b\rvert}), \cosh(x\sqrt{\lvert b\rvert})\}$ if $b<0$, $\mathcal{L}_3=\{1, x, x^2\}$ if $b=0$
2	$aww_{xx}+bw_x^2+cw^2$	$\mathcal{L}_3=\{1, \sin(\lambda x), \cos(\lambda x)\}$ if $c/(a+b)=\lambda^2>0$, $\mathcal{L}_3=\{1, \sinh(\lambda x), \cosh(\lambda x)\}$ if $c/(a+b)=-\lambda^2<0$, $\mathcal{L}_3=\{1, x, x^2\}$ if $c=0$, $\mathcal{L}_2=\{x^2, x^\beta\}$, $\beta=a/(a+b)$ if $c=0$, $a\neq -b$
3	$ww_{xx}-w_x^2$ (special case of the 2nd operator)	$\mathcal{L}_3=\{1, \sin(\lambda x), \cos(\lambda x)\}$, λ is an arbitrary constant, $\mathcal{L}_3=\{1, \sinh(\lambda x), \cosh(\lambda x)\}$, λ is an arbitrary constant, $\mathcal{L}_3=\{1, x, x^2\}$
4	$ww_{xx}-\frac{2}{3}w_x^2$ (special case of the 2nd operator)	$\mathcal{L}_4=\{1, x, x^2, x^3\}$
5	$ww_{xx}-\frac{3}{4}w_x^2+aw^2$ (special case of the 2nd operator)	$\mathcal{L}_5=\{1, \cos(kx), \sin(kx), \cos(2kx), \sin(2kx)\}$ if $a=k^2>0$, $\mathcal{L}_5=\{1, \cosh(kx), \sinh(kx), \cosh(2kx), \sinh(2kx)\}$ if $a=-k^2<0$, $\mathcal{L}_5=\{1, x, x^2, x^3, x^4\}$ if $a=0$
6	$w^2w_{xx}-\frac{1}{2}ww_x^2+aw^3$	$\mathcal{L}_3=\{1, \cos(\sqrt{2a}\,x), \sin(\sqrt{2a}\,x)\}$ if $a>0$, $\mathcal{L}_3=\{1, \cosh(\sqrt{2\lvert a\rvert}\,x), \sinh(\sqrt{2\lvert a\rvert}\,x)\}$ if $a<0$, $\mathcal{L}_3=\{1, x, x^2\}$ if $a=0$
7	w_xw_{xx}	$\mathcal{L}_4=\{1, x, x^2, x^3\}$, $\mathcal{L}_3=\{1, x^{3/2}, x^3\}$
8	$(w^2)_{xxxx}$	$\mathcal{L}_5=\{1, x, x^2, x^3, x^4\}$, $\mathcal{L}_3=\{x^{1/2}, x^{3/2}, x^4\}$
9	$(w^2)_x^{(n)}$	$\mathcal{L}_{n+1}=\{1, x, x^2, \ldots, x^n\}$, $\mathcal{L}_3=\{x^{k/2}, x^{m/2}, x^n\}$, where $k<n$ and $m<n$, with k, m, and $\frac{1}{2}(k+m)$ being nonnegative integers
10	$w_x^{(m)}w_x^{(n)}$	$\mathcal{L}_{m+n+1}=\{1, x, x^2, \ldots, x^{m+n}\}$, $\mathcal{L}_3=\{1, x^{(m+n)/2}, x^{n+m}\}$

29.6.2. Some Generalizations

Likewise, one can consider a more general equation of the form

$$L_1[w] = L_2[U], \qquad U = F[w], \tag{29.6.2.1}$$

where $L_1[w]$ and $L_2[U]$ are linear differential operators with respect to t,

$$L_1[w] \equiv \sum_{i=0}^{s_1} a_i(t)\frac{\partial^i w}{\partial t^i}, \qquad L_2[U] \equiv \sum_{j=0}^{s_2} b_j(t)\frac{\partial^j U}{\partial t^j}, \tag{29.6.2.2}$$

and $F[w]$ is a nonlinear differential operator with respect to x,

$$F[w] \equiv F\left(t, x, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^m w}{\partial x^m}\right), \tag{29.6.2.3}$$

and may depend on t as a parameter.

Let the linear subspace (29.6.1.3) be invariant under the operator F, i.e., for arbitrary constants $C_1, \ldots, C_n$ the following relation holds:

$$F\left[\sum_{i=1}^{n} C_i\varphi_i(x)\right] = \sum_{i=1}^{n} f_i(t, C_1, \ldots, C_n)\varphi_i(x). \tag{29.6.2.4}$$

Then equation (29.6.2.1) possesses generalized separable solutions of the form (29.6.1.5), where the functions $\psi_1(t), \ldots, \psi_n(t)$ are described by the system of ordinary differential equations

$$L_1\big[\psi_i(t)\big] = L_2\big[f_i(t, \psi_1, \ldots, \psi_n)\big], \qquad i = 1, \ldots, n. \tag{29.6.2.5}$$

Example 29.21. Consider the equation

$$a_2(t)\frac{\partial^2 w}{\partial t^2} + a_1(t)\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}, \tag{29.6.2.6}$$

which, in the special case of $a_2(t) = k_2$ and $a_1(t) = k_1/t$, is used for describing transonic gas flows (where t plays the role of a spatial variable).

Equation (29.6.2.6) is a special case of equation (29.6.2.1), where

$$L_1[w] = a_2(t)w_{tt} + a_1(t)w_t, \quad L_2[U] = U, \quad F[w] = w_x w_{xx}.$$

It can be shown that the nonlinear differential operator $F[w] = w_x w_{xx}$ admits the three-dimensional invariant subspace $\mathcal{L}_3 = \{1, x^{3/2}, x^3\}$. Therefore, equation (29.6.2.6) possesses generalized separable solutions of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)x^{3/2} + \psi_3(t)x^3,$$

where the functions $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are described by the system of ordinary differential equations

$$\begin{aligned} a_2(t)\psi_1'' + a_1(t)\psi_1' &= \tfrac{9}{8}\psi_2^2, \\ a_2(t)\psi_2'' + a_1(t)\psi_2' &= \tfrac{45}{4}\psi_2\psi_3, \\ a_2(t)\psi_3'' + a_1(t)\psi_3' &= 18\psi_3^2. \end{aligned}$$

Remark. The operator $F[w] = w_x w_{xx}$ is a special case of operator (29.6.1.11) with $m = 1$ and $n = 2$, so it has a four-dimensional invariant subspace $\mathcal{L}_4 = \{1, x, x^2, x^3\}$. Therefore, equation (29.6.2.6) has a generalized separable solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)x + \psi_3(t)x^2 + \psi_4(t)x^3.$$

29.6.3. Finding Linear Subspaces Invariant Under a Given Nonlinear Operator

The most difficult part in using the Titov–Galaktionov method for the construction of exact solutions to specific equations is to find linear subspaces invariant under a given nonlinear operator.

In order to determine basis functions $\varphi_i = \varphi_i(x)$, let us substitute the linear combination $\sum_{i=1}^{n} C_i\varphi_i(x)$ into the differential operator (29.6.1.2). This is assumed to result in an expression like

$$\begin{aligned} F\left[\sum_{i=1}^{n} C_i\varphi_i(x)\right] &= A_1(C)\Phi_1(X) + A_2(C)\Phi_2(X) + \cdots + A_k(C)\Phi_k(X) \\ &\quad + B_1(C)\varphi_1(x) + B_2(C)\varphi_2(x) + \cdots + B_n(C)\varphi_n(x), \end{aligned} \tag{29.6.3.1}$$

where $A_j(C)$ and $B_i(C)$ depend on $C_1, \ldots, C_n$ only, and the functionals $\Phi_j(X)$ depend on x and are independent of $C_1, \ldots, C_n$:

$$\begin{aligned} A_j(C) &\equiv A_j(C_1, \ldots, C_n), \quad j = 1, \ldots, k, \\ B_i(C) &\equiv B_i(C_1, \ldots, C_n), \quad i = 1, \ldots, n, \\ \Phi_j(X) &\equiv \Phi_j\big(x, \varphi_1, \varphi_1', \varphi_1'', \ldots, \varphi_n, \varphi_n', \varphi_n''\big). \end{aligned} \tag{29.6.3.2}$$

Here, for simplicity, the formulas are written out for the case of a second-order differential operator. For higher-order operators, the right-hand sides of relations (29.6.3.2) will contain higher-order derivatives of φ_i. The functionals and functions $\Phi_1(X)$, …, $\Phi_k(X)$, $\varphi_1(x)$, …, $\varphi_n(x)$ together are assumed to be linearly independent, and the $A_j(C)$ are linearly independent functions of $C_1, \ldots, C_n$.

The basis functions are determined by solving the (usually overdetermined) system of ordinary differential equations

$$\Phi_j\big(x, \varphi_1, \varphi_1', \varphi_1'', \ldots, \varphi_n, \varphi_n', \varphi_n''\big) = p_{j,1}\varphi_1 + p_{j,2}\varphi_2 + \cdots + p_{j,n}\varphi_n, \quad j = 1, \ldots, k, \tag{29.6.3.3}$$

where $p_{j,i}$ are some constants independent of the parameters $C_1, \ldots, C_n$. If for some collection of the constants $p_{i,j}$, system (29.6.3.3) is solvable (in practice, it suffices to find a particular solution), then the functions $\varphi_i = \varphi_i(x)$ define a linear subspace invariant under the nonlinear differential operator (29.6.1.2). In this case, the functions appearing on the right-hand side of (29.6.1.4) are given by

$$\begin{aligned} f_i(C_1, \ldots, C_n) = {} & p_{1,i}A_1(C_1, \ldots, C_n) + p_{2,i}A_2(C_1, \ldots, C_n) + \cdots \\ & + p_{k,i}A_k(C_1, \ldots, C_n) + B_i(C_1, \ldots, C_n). \end{aligned}$$

Remark. The analysis of nonlinear differential operators is useful to begin with searching for two-dimensional invariant subspaces of the form $\mathcal{L}_2 = \{1, \varphi(x)\}$.

Proposition 1. Let a nonlinear differential operator $F[w]$ admit a two-dimensional invariant subspace of the form $\mathcal{L}_2 = \{1, \varphi(x)\}$, where $\varphi(x) = p\varphi_1(x) + q\varphi_2(x)$, p and q are arbitrary constants, and the functions 1, $\varphi_1(x)$, $\varphi_2(x)$ are linearly independent. Then the operator $F[w]$ also admits a three-dimensional invariant subspace $\mathcal{L}_2 = \{1, \varphi_1(x), \varphi_2(x)\}$.

Proposition 2. Let two nonlinear differential operators $F_1[w]$ and $F_2[w]$ admit an invariant subspace $\mathcal{L}_n = \{\varphi_1(x), \ldots, \varphi_n(x)\}$. Then the nonlinear operator $pF_1[w] + qF_2[w]$, where p and q are arbitrary constants, also admits the same invariant subspace.

Example 29.22. Consider the nonlinear differential operator (29.6.1.8). We look for its invariant subspaces of the form $\mathcal{L}_2 = \{1, \varphi(x)\}$. We have

$$F[C_1 + C_2\varphi(x)] = C_2^2[(\varphi_x')^2 + k\varphi^2] + C_2 a\varphi_{xx}'' + kC_1^2 + bC_1 + c + (bC_2 + 2kC_1C_2)\varphi.$$

Here, $\Phi_1(X) = (\varphi_x')^2 + k\varphi^2$ and $\Phi_2(X) = a\varphi_{xx}''$. Hence, the basis function $\varphi(x)$ is determined by the overdetermined system of ordinary differential equations

$$\begin{aligned} (\varphi_x')^2 + k\varphi^2 &= p_1 + p_2\varphi, \\ \varphi_{xx}'' &= p_3 + p_4\varphi, \end{aligned} \tag{29.6.3.4}$$

where $p_1 = p_{1,1}$, $p_2 = p_{1,2}$, $p_3 = p_{2,1}/a$, and $p_4 = p_{2,2}/a$. Let us investigate system (29.6.3.4) for consistency. To this end, we differentiate the first equation with respect to x and then divide by φ_x' to obtain $\varphi_{xx}'' = -k\varphi + p_2/2$. Using this relation to eliminate the second derivative from the second equation in (29.6.3.4), we get $(p_4 + k)\varphi + p_3 - \frac{1}{2}p_2 = 0$. For this equation to be satisfied, the following identities must hold:

$$p_4 = -k, \quad p_3 = \tfrac{1}{2}p_2. \tag{29.6.3.5}$$

TABLE 29.4
Some admissible functions of equation (29.6.3.7) and the associated basis functions $\varphi(x)$; a, b, n, and λ are arbitrary constants ($n \neq 2$, $\lambda \neq 0$).

No.	Function $f(x)$	Function $\varphi(x)$	Parameters p_1, p_2, p_3, and p_4
1	a	$\dfrac{1}{2a}x^2 + bx$	$p_1 = ab^2$, $p_2 = 3$, $p_3 = 1$, $p_4 = 0$
2	ax^n	x^{2-n}	$p_1 = p_4 = 0$, $p_2 = a(2-n)(3-2n)$, $p_3 = a(1-n)(2-n)$
3	ax^2	$\ln x$	$p_4 = a$, $p_2 = p_3 = -a$, $p_4 = 0$
4	$ae^{\lambda x}$	$e^{-\lambda x}$	$p_1 = p_4 = 0$, $p_2 = 2a\lambda^2$, $p_3 = a\lambda^2$

The simultaneous solution of system (29.6.3.4) under condition (29.6.3.5) is given by

$$\begin{aligned}
&\varphi(x) = px^2 + qx && \text{if} \quad k = 0 \qquad (p_1 = q^2,\ p_2 = 4p),\\
&\varphi(x) = p\sin\bigl(x\sqrt{k}\,\bigr) + q\cos\bigl(x\sqrt{k}\,\bigr) && \text{if} \quad k > 0 \qquad (p_1 = kp^2 + kq^2,\ p_2 = 0),\\
&\varphi(x) = p\sinh\bigl(x\sqrt{-k}\,\bigr) + q\cosh\bigl(x\sqrt{-k}\,\bigr) && \text{if} \quad k < 0 \qquad (p_1 = -kp^2 + kq^2,\ p_2 = 0),
\end{aligned} \tag{29.6.3.6}$$

where p and q are arbitrary constants.

Since formulas (29.6.3.6) involve two arbitrary parameters p and q, it follows from Proposition 1 that the nonlinear differential operator (29.6.1.8) admits the following invariant subspaces:

$$\begin{aligned}
&\mathcal{L}_3 = \bigl\{1, x, x^2\bigr\} && \text{if} \quad k = 0,\\
&\mathcal{L}_3 = \bigl\{1, \sin(x\sqrt{k}\,), \cos(x\sqrt{k}\,)\bigr\} && \text{if} \quad k > 0,\\
&\mathcal{L}_3 = \bigl\{1, \sinh(x\sqrt{-k}\,), \cosh(x\sqrt{-k}\,)\bigr\} && \text{if} \quad k < 0.
\end{aligned}$$

Example 29.23. Consider the heat equation with a quadratic nonlinearity

$$w_t = f(x)(ww_x)_x, \tag{29.6.3.7}$$

where the nonlinear differential operator is dependent on x, $F[w] = f(x)(ww_x)_x$. We look for its invariant subspaces of the form $\mathcal{L}_2 = \{1, \varphi(x)\}$. We have

$$F[C_1 + C_2\varphi(x)] = C_2^2 f(x)[\varphi\varphi''_{xx} + (\varphi'_x)^2] + C_1C_2 f(x)\varphi''_{xx}.$$

Hence, $\Phi_1(X) = f(x)[\varphi\varphi''_{xx} + (\varphi'_x)^2]$ and $\Phi_2(X) = f(x)\varphi''_{xx}$ and the basis function $\varphi(x)$ is determined by the overdetermined system of ordinary differential equations

$$\begin{aligned}
f(x)[\varphi\varphi''_{xx} + (\varphi'_x)^2] &= p_1 + p_2\varphi,\\
f(x)\varphi''_{xx} &= p_3 + p_4\varphi.
\end{aligned} \tag{29.6.3.8}$$

Let us find the form of admissible functions $f(x)$ for which system (29.6.3.8) is consistent.

In the nondegenerate case with $\varphi''_{xx} \neq 0$, eliminating $f(x)$ gives an equation for φ:

$$(p_3 + p_4\varphi)[\varphi\varphi''_{xx} + (\varphi'_x)^2] = (p_1 + p_2\varphi)\varphi''_{xx}. \tag{29.6.3.9}$$

Any solution to this equation generates a function $f(x)$ such that

$$f(x) = \frac{p_3 + p_4\varphi}{\varphi''_{xx}}. \tag{29.6.3.10}$$

The substitution $(\varphi'_x)^2 = 2u(\varphi)$ reduces (29.6.3.9) to a separable linear first-order equation

$$[p_4\varphi^2 + (p_3 - p_2)\varphi - p_1]u'_\varphi = -2(p_3 + p_4\varphi)u,$$

which is easy to integrate. Some cases where $\varphi(x)$ can be expressed explicitly in terms of elementary functions are gathered in Table 29.4.

The corresponding exact solutions of equation (29.6.3.7) are given by $w = \psi_1(t) + \psi_2(t)\varphi(x)$, where the functions $\psi_m = \psi_m(t)$ are determined by solving the autonomous system of two ordinary differential equations

$$\begin{aligned}
\psi'_1 &= p_1\psi_2^2 + p_3\psi_1\psi_2,\\
\psi'_2 &= p_2\psi_2^2 + p_4\psi_1\psi_2,
\end{aligned}$$

where the prime denotes a derivative with respect to t. By dividing the former equation by the latter, one reduces the system to a single homogeneous first-order equation, which is easy to integrate.

Remark. Reducible to equation (29.6.3.7) with $f(x) = a/x$ is an equation of diffusion boundary layer where the diffusion coefficient is linearly dependent on concentration. Also reducible to equations of the form (29.6.3.7) are equations

$$w_t = a(z)[b(z)ww_z]_z, \tag{29.6.3.11}$$

which is done with the substitution $x = \int \frac{dz}{b(z)}$, where $f(x) = a(z)$. Equation (29.6.3.11) with $a(z) = z^{-n}$ and $b(z) = z^n$ describes nonlinear heat and mass transfer in the radial symmetric case ($n = 1$ corresponds to a plane problem and $n = 2$ to a spatial one).

29.6.4. Generalizations to Pseudo-Differential Equations

1°. Suppose $f = f(x)$ is a function that can be represented as a finite or infinite power series,

$$f(x) = \sum_{n=0}^{\infty} \beta_n x^n. \tag{29.6.4.1}$$

For simplicity, we assume that the radius of convergence of the series is infinitely large. Let

$$f(D_x) \equiv \sum_{n=0}^{\infty} \beta_n (D_x)^n, \qquad D_x = \frac{\partial}{\partial x}, \tag{29.6.4.2}$$

be a linear differential operator corresponding to the function (29.6.4.1).

Such operators are called *pseudo-differential* and possess the properties

$$f(D_x)(C_1 w_1 + C_2 w_2) = C_1 f(D_x)w_1 + C_2 f(D_x)w_2,$$
$$[f_1(D_x) + f_2(D_x)]w = f_1(D_x)w + f_2(D_x)w,$$

where w_1, w_2, and w are arbitrary functions, and C_1 and C_2 are arbitrary constants.

Note also the useful properties

$$(D_x)^n E = \lambda^n E, \quad f(D_x)E = f(\lambda)E, \quad \text{where} \quad E = e^{\lambda x},$$

which will be required later on.

Considered below are a few nonlinear differential equations that involve the operator (29.6.4.2).

2°. The nonlinear "parabolic" equation

$$\frac{\partial w}{\partial t} = f(D_x)(w^2) + aw^2 + bw + c \tag{29.6.4.3}$$

admits a generalized separable solution of the form

$$w = \varphi(t) + e^{\lambda x}\psi(t). \tag{29.6.4.4}$$

This can be shown as follows. With the representation (29.6.4.2) and formula (29.6.4.4), we have

$$\begin{aligned} w^2 &= \varphi^2 + 2\varphi\psi E + \psi^2 E^2, \quad E = e^{\lambda x}, \\ f(D_x)(w^2) &= \beta_0\varphi^2 + 2\varphi\psi E \sum_{n=0}^{\infty} \beta_n \lambda^n + \psi^2 E^2 \sum_{n=0}^{\infty} \beta_n (2\lambda)^n \\ &= \beta_0\varphi^2 + 2f(\lambda)\varphi\psi E + f(2\lambda)\psi^2 E^2, \quad \beta_0 = f(0). \end{aligned} \tag{29.6.4.5}$$

Substituting (29.6.4.4) into (29.6.4.3) and taking into account (29.6.4.5), we arrive at a system of ordinary differential equations for $\varphi = \varphi(t)$ and $\psi = \psi(t)$:

$$\begin{aligned} \varphi' &= [a + f(0)]\varphi^2 + b\varphi + c, \\ \psi' &= 2[a + f(\lambda)]\varphi\psi + b\psi. \end{aligned} \tag{29.6.4.6}$$

We also obtain the following transcendental equation for the constant λ:

$$f(2\lambda) + a = 0. \tag{29.6.4.7}$$

Remark 1. Equation (29.6.4.1) and similar equations considered below are called nonlinear *pseudo-differential equations*.

Remark 2. The transcendental equation (29.6.4.7) can have more than one root or no roots at all.

Example 29.24. Consider the equation

$$\frac{\partial w}{\partial t} = \cos(\sigma D_x)(w^2) + aw^2 + bw + c, \tag{29.6.4.8}$$

which has an exact solution of the form (29.6.4.6), where the functions φ_1 and φ_2 are determined by the system of ordinary differential equations (29.6.4.6) with $f(0) = 1$ and $f(\lambda) = \cos(\sigma\lambda)$, and the constant λ is determined by solving the transcendental equation

$$\cos(2\sigma\lambda) + a = 0. \tag{29.6.4.9}$$

For $-1 \le a \le 1$, equation (29.6.4.9) has infinitely many real roots (in particular, for $a = 0$, we have $\lambda = \frac{1}{2\sigma}(\frac{\pi}{2} + \pi m)$ with $m = 0, \pm1, \pm2, \ldots$), which generate infinitely many exact solutions of the form (29.6.4.6). For $|a| > 1$, equation (29.6.4.9) does not have real roots, and hence equation (29.6.4.8) does not admit exact solutions of the form (29.6.4.6).

Likewise, it can be shown that equation (29.6.4.5) admits a more complicated solution of the form

$$w = \varphi_1(t) + \varphi_2(t)e^{\lambda x} + \varphi_3(t)e^{-\lambda x}, \tag{29.6.4.10}$$

provided that the real constant $\lambda \ne 0$ satisfies both transcendental equations

$$f(2\lambda) + a = 0, \qquad f(-2\lambda) + a = 0 \tag{29.6.4.11}$$

simultaneously. In this case, the functions $\varphi_m = \varphi_m(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \varphi_1' &= [a + f(0)](\varphi_1^2 + 2\varphi_2\varphi_3) + b\varphi_1 + c, \\ \varphi_2' &= [a + f(\lambda)]\varphi_1\varphi_2 + b\varphi_2, \\ \varphi_3' &= [a + f(-\lambda)]\varphi_1\varphi_3 + b\varphi_3. \end{aligned} \tag{29.6.4.12}$$

The overdetermined system of equations (29.6.4.11) is reduced to a single equation in many cases, two of which include:

(i) a is any number and the function $f(x)$ is even, $f(x) = f(-x)$. In this case, the overdetermined system is reduced to a single equation (29.6.4.7) and it follows from the last two equations in (29.6.4.12) that $\varphi_2 = C\varphi_3$, where C is an arbitrary constant.

(ii) $a = 0$ and the function $f(x)$ is odd, $f(x) = -f(-x)$.

Example 29.25. Consider equation (29.6.4.8) again. It is determined by an even function, $f(x) = \cos(\sigma x)$ with $\sigma = \text{const}$, and, hence, corresponds to case (i). Therefore, the equation admits an exact solution of the form

$$w = \varphi_1(t) + \varphi_2(t)(C_1e^{\lambda x} + C_2e^{-\lambda x}),$$

where the constant λ is determined by solving the transcendental equation (29.6.4.9); it has been taken into account that $\varphi_3/\varphi_3 = \text{const}$. As in Example 29.24, for $-1 \le a \le 1$, there are infinitely many real roots λ, which generate infinitely many exact solutions of the above form.

Suppose that $f(x)$ is an even function, which corresponds to case (i), and equation (29.6.4.7) purely imaginary root, $\lambda = ik$, where $i^2 = -1$ and k is a real number. Then equation (29.6.4.5) admits a solution involving trigonometric functions:

$$w = \psi_1(t) + \psi_2(t)\cos(kx) + \psi_3(t)\sin(kx), \tag{29.6.4.13}$$

where the functions $\psi_m = \psi_m(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1' &= [a + f(0)][\psi_1^2 + \tfrac{1}{2}(\psi_2^2 + \psi_3^2)] + b\psi_1 + c,\\
\psi_2' &= [a + f(ik)]\psi_1\psi_2 + b\psi_2,\\
\psi_3' &= [a + f(ik)]\psi_1\psi_3 + b\psi_3.
\end{aligned} \tag{29.6.4.14}$$

This fact can be proved by substituting

$$e^{\lambda x} = e^{ikx} = \cos(kx) + i\sin(kx), \quad e^{-\lambda x} = e^{-ikx} = \cos(kx) - i\sin(kx),$$
$$\varphi_1 = \psi_1, \quad \varphi_2 = \frac{\psi_2 - i\psi_3}{2}, \quad \varphi_3 = \frac{\psi_2 + i\psi_3}{2}$$

into solution (29.6.4.10) and equations (29.6.4.12) followed by separating the real and imaginary parts. It should be noted that $f(ik) = f(-ik)$ is a real number and the last equation in (29.6.4.14) can be replaced by the simpler equation $\varphi_2 = C\varphi_3$, where C is an arbitrary constant.

Example 29.26. Consider equation (29.6.4.8), determined by the even function $f(x) = \cos(\sigma x)$ and corresponding to the case $a < -1$. The transcendental equation (29.6.4.9) has two purely imaginary roots, $\lambda = \pm ik$, where the real number k is determined from the equation $\cosh(k) = -a$. Hence, equation (29.6.4.8) has a single solution of the form (29.6.4.13), and the two roots $\lambda = \pm ik$ generate identical solutions.

Example 29.27. Likewise, it can be shown that the equation

$$\frac{\partial w}{\partial t} = f(D_x)(w^2) + g(D_x)w \tag{29.6.4.15}$$

has an exact solution of the form (29.6.4.4), where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi' &= f(0)\varphi^2 + g(0)\varphi,\\
\psi' &= 2f(\lambda)\varphi\psi + g(\lambda)\psi,
\end{aligned} \tag{29.6.4.16}$$

and the constant λ is determined from the transcendental equation

$$f(2\lambda) = 0.$$

3°. The more complex equation

$$\frac{\partial w}{\partial t} = f(D_x)w\, g(D_x)w + h(D_x)w, \tag{29.6.4.17}$$

where $f(D_x)w\, g(D_x)w$ stands for the product of the functions $f(D_x)w$ and $g(D_x)w$, also admits an exact solution of the form (29.6.4.4), where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi' &= f(0)g(0)\varphi^2 + h(0)\varphi,\\
\psi' &= [f(0)g(\lambda) + g(0)f(\lambda)]\varphi\psi + h(\lambda)\psi,
\end{aligned} \tag{29.6.4.18}$$

and the parameter λ is determined by solving either transcendental equation

$$f(\lambda) = 0 \quad \text{or} \quad g(\lambda) = 0.$$

4°. Instead of the "parabolic" equations (29.6.4.3), (29.6.4.15), and (29.6.4.18), one could treat the corresponding "hyperbolic" equations, in which the first derivative w_t is substituted for by the second derivative w_{tt}. As an example, let us consider the equation

$$\frac{\partial^2 w}{\partial t^2} = f(D_x)(w^2) + aw^2 + bw + c, \tag{29.6.4.19}$$

which corresponds to (29.6.4.3).

Equation (29.6.4.19) has a solution of the form (29.6.4.4), where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of ordinary differential equations (29.6.4.6) in which the first derivatives φ' and ψ' must be replaced by the second derivatives φ'' and ψ'', respectively, and the equation for the parameter λ remains the same, (29.6.4.7). This also applies to solutions (29.6.4.10) and (29.6.4.13); in systems (29.6.4.12) and (29.6.4.14), the first derivatives must be replaced by the respective second derivatives, and the equations for λ do not change.

5°. Nonlinear equations of the form

$$f(D_t)w = g(D_x)(w^p)$$

admit a multiplicative separable solution

$$w = \varphi(t)\psi^{1/p}(x),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations

$$f(D_t)\varphi = C\varphi^p, \quad g(D_x)\psi = C\psi^{1/p},$$

with C being an arbitrary constant.

⊙ *References for Section* 29.6: S. S. Titov (1988), V. A. Galaktionov and S. A. Posashkov (1994), V. A. Galaktionov (1995), V. A. Galaktionov, S. A. Posashkov, and S. R. Svirshchevskii (1995), S. R. Svirshchevskii (1995, 1996), Yu. A. Dubinskii (1991), A. D. Polyanin and V. F. Zaitsev (2004), V. A. Galaktionov and S. R. Svirshchevskii (2007), A. D. Polyanin and A. V. Manzhirov (2007).

Chapter 30
Method of Functional Separation of Variables

30.1. Structure of Functional Separable Solutions. Solution by Reduction to Equations with Quadratic Nonlinearities

30.1.1. Structure of Functional Separable Solutions

Suppose a nonlinear equation for $w = w(x, y)$ is obtained from a linear mathematical physics equation for $z = z(x, y)$ by a nonlinear change of variable $w = F(z)$. Then, if the linear equation for z admits separable solutions, the transformed nonlinear equation for w will have exact solutions of the form

$$w(x, y) = F(z), \quad \text{where} \quad z = \sum_{m=1}^{n} \varphi_m(x)\psi_m(y). \tag{30.1.1.1}$$

It is noteworthy that many nonlinear partial differential equations that are not reducible to linear equations have exact solutions of the form (30.1.1.1) as well. We will call such solutions *functional separable solutions*. In general, the functions $\varphi_m(x)$, $\psi_m(y)$, and $F(z)$ in (30.1.1.1) are not known in advance and are to be identified.

Main idea: The functional differential equation resulting from the substitution of (30.1.1.1) in the original partial differential equation should be reduced to the standard bilinear functional equation (29.2.2.1) or to a functional differential equation of the form (29.2.2.1)–(29.2.2.2), and then the results of Sections 29.4–29.6 should be used.

Remark. The function $F(z)$ can be determined by a single ordinary differential equation or by an overdetermined system of equations; both possibilities must be taken into account.

30.1.2. Solution by Reduction to Equations with Quadratic (or Power) Nonlinearities

In some cases, solutions of the form (30.1.1.1) can be searched for in two stages. First, one looks for a transformation that would reduce the original equation to an equation with a quadratic (or power) nonlinearity. Then the methods outlined in Subsections Sections 29.4–29.6 are used for finding solutions of the resulting equation.

Table 30.1 gives examples of nonlinear heat equations with power, exponential, and logarithmic nonlinearities reducible, by simple substitutions of the form $w = F(z)$, to quadratically nonlinear equations. For these equations, it can be assumed that the form of the function $F(z)$ in solution (30.1.1.1) is given a priori.

⊙ *References for Section* 30.1: V. A. Galaktionov and S. A. Posashkov (1989, 1994), V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2004), V. A. Galaktionov and S. R. Svirshchevskii (2006).

30.2. Special Functional Separable Solutions. Generalized Traveling-Wave Solutions

30.2.1. Special Functional Separable and Generalized Traveling-Wave Solutions

To simplify the analysis, some of the functions in (30.1.1.1) can be specified a priori and the other functions will be defined in the analysis. We call such solutions *special functional separable solutions*.

TABLE 30.1
Some nonlinear heat equations reducible to quadratically nonlinear equations by a transformation of the form $w = F(z)$; the constant σ is expressed in terms of the coefficients of the transformed equation

Original equation	Transformation	Transformed equation	Form of solutions
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + bw + cw^{1-n}$	$w = z^{1/n}$	$\frac{\partial z}{\partial t} = az\frac{\partial^2 z}{\partial x^2} + \frac{a}{n}\left(\frac{\partial z}{\partial x}\right)^2 + bnz + cn$	$z = \varphi(t)x^2 + \psi(t)x + \chi(t)$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + bw^{n+1} + cw$	$w = z^{1/n}$	$\frac{\partial z}{\partial t} = az\frac{\partial^2 z}{\partial x^2} + \frac{a}{n}\left(\frac{\partial z}{\partial x}\right)^2 + bnz^2 + cnz$	$z = \varphi(t)e^{\sigma x} + \psi(t)e^{-\sigma x} + \chi(t)$ $z = \varphi(t)\sin(\sigma x) + \psi(t)\cos(\sigma x) + \chi(t)$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}$	$w = \frac{1}{\lambda}\ln z$	$\frac{\partial z}{\partial t} = az\frac{\partial^2 z}{\partial x^2} + b\lambda z + c\lambda$	$z = \varphi(t)x^2 + \psi(t)x + \chi(t)$
$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + be^{\lambda w} + c$	$w = \frac{1}{\lambda}\ln z$	$\frac{\partial z}{\partial t} = az\frac{\partial^2 z}{\partial x^2} + bz^2 + c\lambda z$	$z = \varphi(t)e^{\sigma x} + \psi(t)e^{-\sigma x} + \chi(t)$ $z = \varphi(t)\sin(\sigma x) + \psi(t)\cos(\sigma x) + \chi(t)$
$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\ln w + cw$	$w = e^z$	$\frac{\partial z}{\partial t} = a\frac{\partial^2 z}{\partial x^2} + a\left(\frac{\partial z}{\partial x}\right)^2 + bz + c$	$z = \varphi(t)x^2 + \psi(t)x + \chi(t)$
$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\ln^2 w + cw$	$w = e^z$	$\frac{\partial z}{\partial t} = a\frac{\partial^2 z}{\partial x^2} + a\left(\frac{\partial z}{\partial x}\right)^2 + bz^2 + c$	$z = \varphi(t)e^{\sigma x} + \psi(t)e^{-\sigma x} + \chi(t)$ $z = \varphi(t)\sin(\sigma x) + \psi(t)\cos(\sigma x) + \chi(t)$

A generalized separable solution (see Section 29.2) is a functional separable solution of the special form corresponding to $F(z) = z$.

Below we consider two simplest functional separable solutions of special forms:

$$\begin{aligned} &w = F(z), \quad z = \varphi_1(x)y + \varphi_2(x);\\ &w = F(z), \quad z = \varphi(x) + \psi(y). \end{aligned} \tag{30.2.1.1}$$

The first solution (30.2.1.1) will be called a *generalized traveling-wave solution* (x and y can be swapped). After substituting this solution into the original equation, one should eliminate y with the help of the expression for z. This will result in a functional differential equation with two arguments, x and z. Its solution may be obtained with the methods outlined in Sections 29.4–29.6.

Remark 1. In functional separation of variables, searching for solutions in the forms $w = F\big(\varphi(x)+\psi(y)\big)$ [it is the second solution in (30.1.2.1)] and $w = F\big(\varphi(x)\psi(y)\big)$ leads to equivalent results because the two forms are functionally equivalent. Indeed, we have $F\big(\varphi(x)\psi(y)\big) = F_1\big(\varphi_1(x) + \psi_1(y)\big)$, where $F_1(z) = F(e^z)$, $\varphi_1(x) = \ln\varphi(x)$, and $\psi_1(y) = \ln\psi(y)$.

Remark 2. In constructing functional separable solutions with the form $w = F\big(\varphi(x) + \psi(y)\big)$ [it is the second solution in (30.1.2.1)], it is assumed that $\varphi \not\equiv \text{const}$ and $\psi \not\equiv \text{const}$.

Example 30.1. Consider the third-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = a\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3},$$

which describes a boundary layer of a power-law fluid on a flat plate; w is the stream function, x and y are coordinates along and normal to the plate, and n is a rheological parameter (the value $n = 1$ corresponds to a Newtonian fluid). Searching for solutions in the form

$$w = w(z), \qquad z = \varphi(x)y + \psi(x)$$

leads to the equation $\varphi'_x(w'_z)^2 = a\varphi^{2n}(w''_{zz})^{n-1}w'''_{zzz}$, which is independent of ψ. Separating the variables and integrating yields

$$\varphi(x) = (ax + C)^{1/(1-2n)}, \quad \psi(x) \text{ is arbitrary},$$

and $w = w(z)$ is determined by solving the ordinary differential equation $(w'_z)^2 = (1-2n)(w''_{zz})^{n-1}w'''_{zzz}$.

Remark. Below are two more functional separable solutions of special form:

$$\begin{aligned} &w = F(z), \quad z = \psi_1(y)x^2 + \psi_2(y) \quad (z \text{ is quadratic in } x);\\ &w = F(z), \quad z = \psi_1(y)e^{\lambda x} + \psi_2(y) \quad (z \text{ contains an exponential of } x). \end{aligned}$$

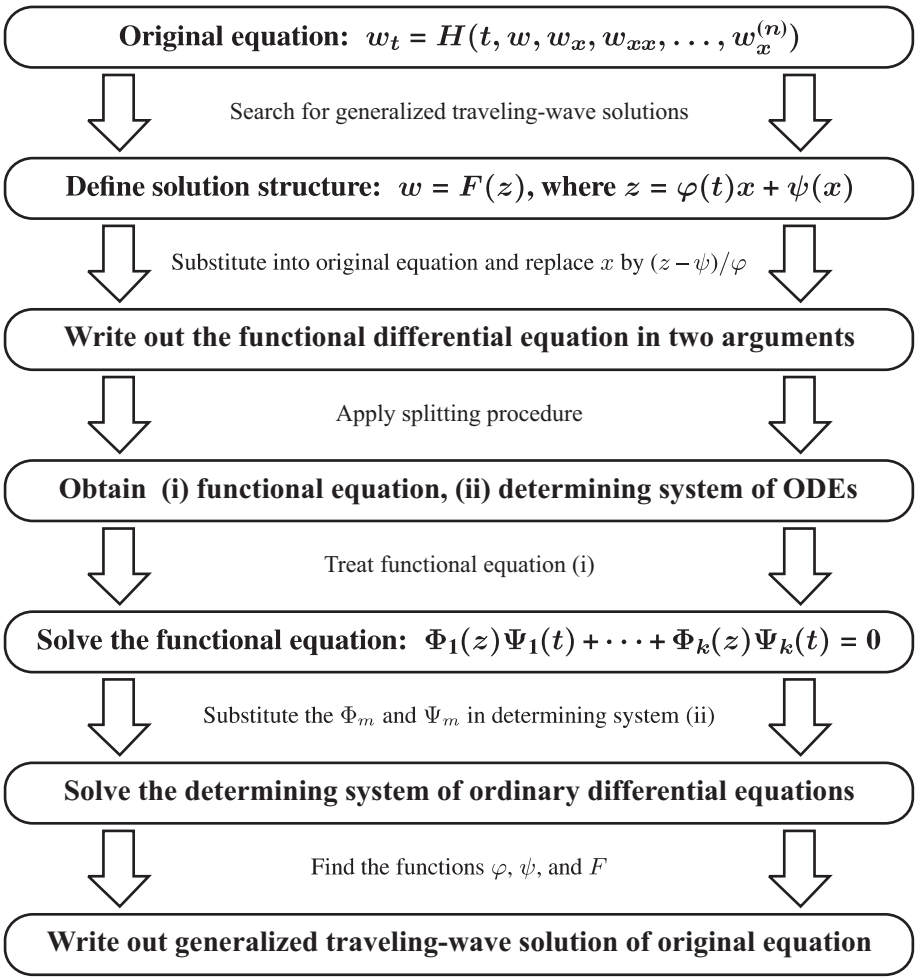

Figure 30.1. Algorithm for constructing generalized traveling-wave solutions for evolution equations. Abbreviation: ODE stands for ordinary differential equation.

30.2.2. General Scheme for Constructing Generalized Traveling-Wave Solutions

For visualization, the general scheme for constructing generalized traveling-wave solutions for evolution equations is displayed in Fig. 30.1.

Example 30.2. Consider the unsteady heat equation with a nonlinear source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \mathcal{F}(w). \tag{30.2.2.1}$$

We look for functional separable solutions of the special form

$$w = w(z), \quad z = \varphi(t)x + \psi(t). \tag{30.2.2.2}$$

The functions $w(z)$, $\varphi(t)$, $\psi(t)$, and $\mathcal{F}(w)$ are to be determined.

On substituting (30.2.2.2) into (30.2.2.1) and on dividing by w'_z, we have

$$\varphi'_t x + \psi'_t = \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z}. \tag{30.2.2.3}$$

We express x from (30.2.2.2) in terms of z and substitute into (30.2.2.3) to obtain a functional differential equation with two variables, t and z:

$$-\psi'_t + \frac{\psi}{\varphi}\varphi'_t - \frac{\varphi'_t}{\varphi}z + \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z} = 0,$$

which can be treated as the functional equation (29.5.2.3), where

$$\Phi_1 = -\psi_t' + \frac{\psi}{\varphi}\varphi_t', \quad \Phi_2 = -\frac{\varphi_t'}{\varphi}, \quad \Phi_3 = \varphi^2, \qquad \Phi_4 = 1,$$
$$\Psi_1 = 1, \qquad \Psi_2 = z, \qquad \Psi_3 = \frac{w_{zz}''}{w_z'}, \quad \Psi_4 = \frac{\mathcal{F}(w)}{w_z'}.$$

Substituting these expressions into relations (29.5.2.4) yields the system of ordinary differential equations

$$-\psi_t' + \frac{\psi}{\varphi}\varphi_t' = A_1\varphi^2 + A_2, \quad -\frac{\varphi_t'}{\varphi} = A_3\varphi^2 + A_4,$$
$$\frac{w_{zz}''}{w_z'} = -A_1 - A_3 z, \quad \frac{\mathcal{F}(w)}{w_z'} = -A_2 - A_4 z, \tag{30.2.2.4}$$

where $A_1, \ldots, A_4$ are arbitrary constants.

Case 1. For $A_4 \neq 0$, the solution of system (30.2.2.4) is given by

$$\varphi(t) = \pm\left(C_1 e^{2A_4 t} - \frac{A_3}{A_4}\right)^{-1/2},$$
$$\psi(t) = -\varphi(t)\left[A_1 \int \varphi(t)\,dt + A_2 \int \frac{dt}{\varphi(t)} + C_2\right],$$
$$w(z) = C_3 \int \exp\left(-\tfrac{1}{2}A_3 z^2 - A_1 z\right) dz + C_4,$$
$$\mathcal{F}(w) = -C_3(A_4 z + A_2)\exp\left(-\tfrac{1}{2}A_3 z^2 - A_1 z\right), \tag{30.2.2.5}$$

where $C_1, \ldots, C_4$ are arbitrary constants. The dependence $\mathcal{F} = \mathcal{F}(w)$ is defined by the last two relations in parametric form (z is considered the parameter). If $A_3 \neq 0$ in (30.2.2.5), the source function is expressed in terms of elementary functions and the inverse of the error function.

In the special case $A_3 = C_4 = 0$, $A_1 = -1$, and $C_3 = 1$, the source function can be represented in explicit form as

$$\mathcal{F}(w) = -w(A_4 \ln w + A_2). \tag{30.2.2.6}$$

Case 2. For $A_4 = 0$, the solution to the first two equations in (30.2.2.4) is given by

$$\varphi(t) = \pm\frac{1}{\sqrt{2A_3 t + C_1}}, \qquad \psi(t) = \frac{C_2}{\sqrt{2A_3 t + C_1}} - \frac{A_1}{A_3} - \frac{A_2}{3A_3}(2A_3 t + C_1),$$

and the solutions to the other equations are determined by the last two formulas in (30.2.2.5) where $A_4 = 0$.

Example 30.3. Consider the more general equation

$$\frac{\partial w}{\partial t} = a(t)\frac{\partial^2 w}{\partial x^2} + b(t)\frac{\partial w}{\partial x} + c(t)\mathcal{F}(w).$$

We look for solutions in the form (30.2.2.2). In this case, only the first two equations in system (30.2.2.4) will change, and the functions $w(z)$ and $\mathcal{F}(w)$ will be given by (30.2.2.5).

Example 30.4. The nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\mathcal{G}(w)\frac{\partial w}{\partial x}\right] + \mathcal{F}(w)$$

has also solutions of the form (30.2.2.2). The unknown quantities are governed by system (30.2.2.4) in which w_{zz}'' must be replaced by $[\mathcal{G}(w)w_z']_z'$. The functions $\varphi(t)$ and $\psi(t)$ are determined by the first two formulas in (30.2.2.5). One of the two functions $\mathcal{G}(w)$ and $\mathcal{F}(w)$ can be assumed arbitrary and the other is identified in the course of the solution. The special case $\mathcal{F}(w) = \text{const}$ yields $\mathcal{G}(w) = C_1 e^{2ke} + (C_2 w + C_3)e^{kw}$.

Functional separable solutions (30.2.2.2) of the given equation are discussed in more detail in 5.6.15.2, Items 4° and 5°; some other solutions are also specified there.

Example 30.5. We can treat the nth-order nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^n w}{\partial x^n} + \mathcal{F}(w)$$

likewise. As before, we look for solutions in the form (30.2.2.2). In this case, the quantities φ^2 and w''_{zz} in (30.2.2.4) must be replaced by φ^n and $w_z^{(n)}$, respectively. In particular, for $A_3 = 0$, apart from equations with logarithmic nonlinearities of the form (30.2.2.6), we obtain other equations.

Example 30.6. For the nth-order nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^n w}{\partial x^n} + \mathcal{F}(w)\frac{\partial w}{\partial x},$$

the search for exact solutions of the form (30.2.2.2) leads to the following system of equations for $\varphi(t)$, $\psi(t)$, $w(z)$, and $\mathcal{F}(w)$:

$$-\psi'_t + \frac{\psi}{\varphi}\varphi'_t = A_1\varphi^n + A_2\varphi, \quad -\frac{\varphi'_t}{\varphi} = A_3\varphi^n + A_4\varphi,$$
$$\frac{w_z^{(n)}}{w'_z} = -A_1 - A_3 z, \qquad \mathcal{F}(w) = -A_2 - A_4 z,$$

where $A_1, \ldots, A_4$ are arbitrary constants.

In the case $n = 3$, we assume $A_3 = 0$ and $A_1 > 0$ to find in particular that $\mathcal{F}(w) = -A_2 - A_4 \arcsin(kw)$.

Some functional separable solutions (30.2.2.2) of the given equation can be found in Subsection 15.1.3.

Example 30.7. Consider the equation

$$\frac{\partial^{n+1} w}{\partial x^n \partial y} = f(w). \tag{30.2.2.7}$$

We look for functional separable solutions of the special form

$$w = w(z), \quad z = \varphi(y)x + \psi(y). \tag{30.2.2.8}$$

We substitute (30.2.2.8) in (30.2.2.7), eliminate x with the expression for z, divide the resulting equation by $w_z^{(n+1)}$, and rearrange terms to obtain the functional-differential equation with two arguments

$$\varphi^n\psi'_y - \varphi^{n-1}\psi\varphi'_y + \varphi^{n-1}\varphi'_y\left(z + n\frac{w_z^{(n)}}{w_z^{(n+1)}}\right) - \frac{f(w)}{w_z^{(n+1)}} = 0. \tag{30.2.2.9}$$

It is reduced to a three-term bilinear functional equation, which has two solutions (see Subsection 29.5.2). Accordingly, we consider two cases.

1°. First, we set the expression in parentheses and the last fraction in (30.2.2.9) equal to constants. On rearranging terms, we obtain

$$(z - C_1)w_z^{(n+1)} + nw_z^{(n)} = 0,$$
$$C_2 w_z^{(n+1)} - f(w) = 0,$$
$$\varphi^n\psi'_y - \varphi^{n-1}\psi\varphi'_y + C_1\varphi^{n-1}\varphi'_y - C_2 = 0,$$

where C_1 and C_2 are arbitrary constants. Setting $C_1 = 0$, which corresponds to a translation in z and renaming ψ, and integrating yields

$$\begin{aligned} w &= A\ln|z| + B_{n-1}z^{n-1} + \cdots + B_1 z + B_0,\\ f(w) &= AC_2 n!\,(-1)^n z^{-n-1},\\ \psi(y) &= C_2\varphi(y)\int \frac{dy}{[\varphi(y)]^{n+1}} + C_3\varphi(y), \end{aligned} \tag{30.2.2.10}$$

where A, the B_m, and C_3 are arbitrary constants and $\varphi(y)$ is an arbitrary function.

The first two formulas in (30.2.2.10) give a parametric representation of $f(w)$. In the special case of $B_{n-1} = \cdots = B_0 = 0$, on eliminating z, we arrive at the exponential dependence

$$f(w) = \alpha e^{\beta w}, \qquad \alpha = AC_2 n!\,(-1)^n, \quad \beta = -(n+1)/A.$$

By virtue of (30.2.2.10), the corresponding solution of equation (30.2.2.7) will have functional arbitrariness.

2°. In the second case, (30.2.2.9) splits into three ordinary differential equations:

$$\begin{aligned} \varphi^{n-1}\varphi'_y &= C_1,\\ \varphi^n\psi'_y - \varphi^{n-1}\psi\varphi'_y &= C_2,\\ (C_1 z + C_2)w_z^{(n+1)} + C_1 n w_z^{(n)} - f(w) &= 0, \end{aligned} \tag{30.2.2.11}$$

where C_1 and C_2 are arbitrary constants. The solutions of the first two equations are given by

$$\varphi = (C_1 nt + C_3)^{1/n}, \quad \psi = C_4(C_1 nt + C_3)^{1/n} - \frac{C_2}{C_1}.$$

Together with the last equation in (30.2.2.11), these formulas define a self-similar solution of the form (30.2.2.8).

Example 30.8. In addition, searching for solutions of equation (30.2.2.1) with z quadratically dependent on x,

$$w = w(z), \quad z = \varphi(t)x^2 + \psi(t), \tag{30.2.2.12}$$

also makes sense here. Indeed, on substituting (30.2.2.12) into (30.2.2.1), we arrive at an equation that contains terms with x^2 and does not contain terms linear in x. Eliminating x^2 from the resulting equation with the aid of (30.2.2.12), we obtain

$$-\psi'_t + \frac{\psi}{\varphi}\varphi'_t + 2\varphi - \frac{\varphi'_t}{\varphi}z + 4\varphi z\frac{w''_{zz}}{w'_z} - 4\varphi\psi\frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z} = 0.$$

To solve this functional-differential equation with two arguments, we apply the splitting method outlined in Subsection 29.5.2. It can be shown that equation (30.2.2.1) with a logarithmic nonlinearity of the form (30.2.2.6), has a solution of the form (30.2.2.12).

Remark. The algorithm presented in Fig. 30.1 can also be used for finding exact solutions of the more general form $w = \sigma(t)F(z) + \varphi_1(t)x + \psi_1(t)$, where $z = \varphi_2(t)x + \psi_2(t)$. For an example of this sort of solution, see Subsection 31.2.1 (Example 31.7).

⊙ *References for Section* 30.2: A. D. Polyanin (2002, Supplement B), A. D. Polyanin and A. I. Zhurov (2002), A. D. Polyanin and V. F. Zaitsev (2002, 2004).

30.3. Differentiation Method

30.3.1. Basic Ideas of the Method. Reduction to a Standard Equation

In general, the substitution of expression (30.1.1.1) into the nonlinear partial differential equation under study leads to a functional differential equation with three arguments — two usual arguments, x and y, and one composite argument, z. In some cases, the resulting equation can be reduced by differentiation to a standard functional differential equation with two arguments (either x or y is eliminated). To solve the two-argument equation, one can use the methods outlined in Sections 29.4–29.6.

30.3.2. Examples of Constructing Functional Separable Solutions

Below we consider specific examples illustrating the application of the differentiation method for constructing functional separable solutions of nonlinear equations.

Example 30.9. Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\mathcal{F}(w)\frac{\partial w}{\partial x}\right]. \tag{30.3.2.1}$$

We look for exact solutions with the form

$$w = w(z), \quad z = \varphi(x) + \psi(t). \tag{30.3.2.2}$$

On substituting (30.3.2.2) into (30.3.2.1) and dividing by w'_z, we obtain the functional differential equation with three variables

$$\psi'_t = \varphi''_{xx}\mathcal{F}(w) + (\varphi'_x)^2 H(z), \tag{30.3.2.3}$$

where

$$H(z) = \mathcal{F}(w)\frac{w''_{zz}}{w'_z} + \mathcal{F}'_z(w), \qquad w = w(z). \tag{30.3.2.4}$$

Differentiating (30.3.2.3) with respect to x yields

$$\varphi'''_{xxx}\mathcal{F}(w) + \varphi'_x\varphi''_{xx}[\mathcal{F}'_z(w) + 2H(z)] + (\varphi'_x)^3 H'_z = 0. \tag{30.3.2.5}$$

This functional differential equation with two variables can be treated as the functional equation (29.5.2.1). This three-term functional equation has two different solutions. Accordingly, we consider two cases.

Case 1. The solutions of the functional differential equation (30.3.2.5) are determined from the system of ordinary differential equations

$$\begin{aligned} &\mathcal{F}'_z + 2H = 2A_1\mathcal{F}, \quad H'_z = A_2\mathcal{F}, \\ &\varphi'''_{xxx} + 2A_1\varphi'_x\varphi''_{xx} + A_2(\varphi'_x)^3 = 0, \end{aligned} \tag{30.3.2.6}$$

where A_1 and A_2 are arbitrary constants.

The first two equations (30.3.2.6) are linear and independent of the third equation. Their general solution is given by

$$\mathcal{F} = \begin{cases} e^{A_1 z}(B_1 e^{kz} + B_2 e^{-kz}) & \text{if } A_1^2 > 2A_2, \\ e^{A_1 z}(B_1 + B_2 z) & \text{if } A_1^2 = 2A_2, \\ e^{A_1 z}[B_1 \sin(kz) + B_2 \cos(kz)] & \text{if } A_1^2 < 2A_2, \end{cases} \qquad H = A_1\mathcal{F} - \tfrac{1}{2}\mathcal{F}'_z, \qquad k = \sqrt{|A_1^2 - 2A_2|}. \tag{30.3.2.7}$$

Substituting H of (30.3.2.7) into (30.3.2.4) yields an ordinary differential equation for $w = w(z)$. On integrating this equation, we obtain

$$w = C_1 \int e^{A_1 z} |\mathcal{F}(z)|^{-3/2} dz + C_2, \tag{30.3.2.8}$$

where C_1 and C_2 are arbitrary constants. The expression of $\mathcal{F}$ in (30.3.2.7) together with expression (30.3.2.8) defines the function $\mathcal{F} = \mathcal{F}(w)$ in parametric form.

Without full analysis, we will study the case $A_2 = 0$ ($k = A_1$) and $A_1 \neq 0$ in more detail. It follows from (30.3.2.7) and (30.3.2.8) that

$$\mathcal{F}(z) = B_1 e^{2A_1 z} + B_2, \quad H = A_1 B_2, \quad w(z) = C_3(B_1 + B_2 e^{-2A_1 z})^{-1/2} + C_2 \quad (C_1 = A_1 B_2 C_3). \tag{30.3.2.9}$$

Eliminating z yields

$$\mathcal{F}(w) = \frac{B_2 C_3^2}{C_3^2 - B_1 w^2}. \tag{30.3.2.10}$$

The last equation in (30.3.2.6) with $A_2 = 0$ has the first integral $\varphi''_{xx} + A_1(\varphi'_x)^2 = \text{const}$. The corresponding general solution is given by

$$\begin{aligned} \varphi(x) &= -\frac{1}{2A_1} \ln\left[\frac{D_2}{D_1} \frac{1}{\sinh^2\left(A_1\sqrt{D_2}\, x + D_3\right)}\right] && \text{for} \quad D_1 > 0 \ \text{ and } \ D_2 > 0, \\ \varphi(x) &= -\frac{1}{2A_1} \ln\left[-\frac{D_2}{D_1} \frac{1}{\cos^2\left(A_1\sqrt{-D_2}\, x + D_3\right)}\right] && \text{for} \quad D_1 > 0 \ \text{ and } \ D_2 < 0, \\ \varphi(x) &= -\frac{1}{2A_1} \ln\left[-\frac{D_2}{D_1} \frac{1}{\cosh^2\left(A_1\sqrt{D_2}\, x + D_3\right)}\right] && \text{for} \quad D_1 < 0 \ \text{ and } \ D_2 > 0, \end{aligned} \tag{30.3.2.11}$$

where D_1, D_2, and D_3 are constants of integration. In all three cases, the following relations hold:

$$(\varphi'_x)^2 = D_1 e^{-2A_1\varphi} + D_2, \quad \varphi''_{xx} = -A_1 D_1 e^{-2A_1\varphi}. \tag{30.3.2.12}$$

We substitute (30.3.2.9) and (30.3.2.12) into the original functional differential equation (30.3.2.3). With reference to the expression of z in (30.3.2.2), we obtain the following equation for $\psi = \psi(t)$:

$$\psi'_t = -A_1 B_1 D_1 e^{2A_1\psi} + A_1 B_2 D_2.$$

Its general solution is given by

$$\psi(t) = \frac{1}{2A_1} \ln \frac{B_2 D_2}{D_4 \exp(-2A_1^2 B_2 D_2 t) + B_1 D_1}, \tag{30.3.2.13}$$

where D_4 is an arbitrary constant.

Formulas (30.3.2.2), (30.3.2.9) for w, (30.3.2.11), and (30.3.2.13) define three solutions of the nonlinear equation (30.3.2.1) with $\mathcal{F}(w)$ of the form (30.3.2.10) [recall that these solutions correspond to the special case $A_2 = 0$ in (30.3.2.7) and (30.3.2.8)].

Case 2. The solutions of the functional differential equation (30.3.2.5) are determined from the system of ordinary differential equations

$$\begin{aligned}&\varphi'''_{xxx} = A_1(\varphi'_x)^3, \quad \varphi'_x\varphi''_{xx} = A_2(\varphi'_x)^3,\\ &A_1\mathcal{F} + A_2(\mathcal{F}'_z + 2H) + H'_z = 0.\end{aligned} \tag{30.3.2.14}$$

The first two equations in (30.3.2.14) are compatible in the two cases

$$\begin{aligned}&A_1 = A_2 = 0 \quad \Longrightarrow \quad \varphi(x) = B_1x + B_2,\\ &A_1 = 2A_2^2 \quad \Longrightarrow \quad \varphi(x) = -\frac{1}{A_2}\ln|B_1x + B_2|.\end{aligned} \tag{30.3.2.15}$$

The first solution in (30.3.2.15) eventually leads to the traveling-wave solution $w = w(B_1x + B_2t)$ of equation (30.3.2.1) and the second solution to the self-similar solution of the form $w = \widetilde{w}(x^2/t)$. In both cases, the function $\mathcal{F}(w)$ in (30.3.2.1) is arbitrary.

Remark. The more general nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\mathcal{F}(w)\frac{\partial w}{\partial x}\right] + \mathcal{G}(w)$$

has also solutions of the form (30.3.2.2). For the unknown functions $\varphi(x)$ and $\psi(t)$, we have the functional-differential equation in three variables

$$\psi'_t = \varphi''_{xx}\mathcal{F}(w) + (\varphi'_x)^2H(z) + \mathcal{G}(w)/w'_z,$$

where $w = w(z)$ and $H(z)$ is defined by (30.3.2.4). Differentiating with respect to x yields

$$\varphi'''_{xxx}\mathcal{F}(w) + \varphi'_x\varphi''_{xx}[\mathcal{F}'_z(w) + 2H(z)] + (\varphi'_x)^3H'_z + \varphi'_x[\mathcal{G}(w)/w'_z]'_z = 0.$$

This functional-differential equation in two variables can be treated as the bilinear functional equation (29.5.2.3) with $\Phi_1 = \varphi'''_{xxx}$, $\Phi_2 = \varphi'_x\varphi''_{xx}$, $\Phi_3 = (\varphi'_x)^3$, and $\Phi_4 = \varphi'_x$.

See also Estévez, Qu, and Zhang (2002), where a more general equation was considered.

Example 30.10. Consider the *nonlinear Klein–Gordon equation*

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = \mathcal{F}(w). \tag{30.3.2.16}$$

We look for functional separable solutions in additive form:

$$w = w(z), \qquad z = \varphi(x) + \psi(t). \tag{30.3.2.17}$$

Substituting (30.3.2.17) into (30.3.2.16) yields

$$\psi''_{tt} - \varphi''_{xx} + \left[(\psi'_t)^2 - (\varphi'_x)^2\right]g(z) = h(z), \tag{30.3.2.18}$$

where

$$g(z) = w''_{zz}/w'_z, \quad h(z) = \mathcal{F}\big(w(z)\big)/w'_z. \tag{30.3.2.19}$$

On differentiating (30.3.2.18) first with respect to t and then with respect to x and on dividing by $\psi'_t\varphi'_x$, we have

$$2(\psi''_{tt} - \varphi''_{xx})\,g'_z + \left[(\psi'_t)^2 - (\varphi'_x)^2\right]g''_{zz} = h''_{zz}.$$

Eliminating $\psi''_{tt} - \varphi''_{xx}$ from this equation with the aid of (30.3.2.18), we obtain

$$\left[(\psi'_t)^2 - (\varphi'_x)^2\right](g''_{zz} - 2gg'_z) = h''_{zz} - 2g'_zh. \tag{30.3.2.20}$$

This relation holds in the following cases:

$$\begin{aligned}&g''_{zz} - 2gg'_z = 0, \quad h''_{zz} - 2g'_zh = 0 && \text{(case 1)},\\ &(\psi'_t)^2 = A\psi + B, \quad (\varphi'_x)^2 = -A\varphi + B - C, \quad h''_{zz} - 2g'_zh = (Az + C)(g''_{zz} - 2gg'_z) && \text{(case 2)},\end{aligned} \tag{30.3.2.21}$$

where A, B, and C are arbitrary constants. We consider both cases.

Case 1. The first two equations in (30.3.2.21) enable one to determine $g(z)$ and $h(z)$. Integrating the first equation once yields $g'_z = g^2 + \text{const}$. Further, the following cases are possible:

$$g = k, \tag{30.3.2.22a}$$
$$g = -1/(z + C_1), \tag{30.3.2.22b}$$
$$g = -k\tanh(kz + C_1), \tag{30.3.2.22c}$$
$$g = -k\coth(kz + C_1), \tag{30.3.2.22d}$$
$$g = k\tan(kz + C_1), \tag{30.3.2.22e}$$

where C_1 and k are arbitrary constants.

The second equation in (30.3.2.21) has a particular solution $h = g(z)$. Hence, its general solution is expressed by [e.g., see Polyanin and Zaitsev (2003)]:

$$h = C_2 g(z) + C_3 g(z) \int \frac{dz}{g^2(z)}, \tag{30.3.2.23}$$

where C_2 and C_3 are arbitrary constants.

The functions $w(z)$ and $\mathcal{F}(w)$ are found from (30.3.2.19) as

$$w(z) = B_1 \int G(z)\,dz + B_2, \quad \mathcal{F}(w) = B_1 h(z) G(z), \quad \text{where} \quad G(z) = \exp\left[\int g(z)\,dz\right], \tag{30.3.2.24}$$

and B_1 and B_2 are arbitrary constants ($\mathcal{F}$ is defined parametrically).

Let us dwell on the case (30.3.2.22b). According to (30.3.2.23),

$$h = A_1(z + C_1)^2 + \frac{A_2}{z + C_1}, \tag{30.3.2.25}$$

where $A_1 = -C_3/3$ and $A_2 = -C_2$ are any numbers. Substituting (30.3.2.22b) and (30.3.2.25) into (30.3.2.24) yields

$$w = B_1 \ln|z + C_1| + B_2, \quad \mathcal{F} = A_1 B_1 (z + C_1) + \frac{A_2 B_1}{(z + C_1)^2}.$$

Eliminating z, we arrive at the explicit form of the right-hand side of equation (30.3.2.16):

$$\mathcal{F}(w) = A_1 B_1 e^u + A_2 B_1 e^{-2u}, \quad \text{where} \quad u = \frac{w - B_2}{B_1}. \tag{30.3.2.26}$$

For simplicity, we set $C_1 = 0$, $B_1 = 1$, and $B_2 = 0$ and denote $A_1 = a$ and $A_2 = b$. Thus, we have

$$w(z) = \ln|z|, \quad \mathcal{F}(w) = ae^w + be^{-2w}, \quad g(z) = -1/z, \quad h(z) = az^2 + b/z. \tag{30.3.2.27}$$

It remains to determine $\psi(t)$ and $\varphi(x)$. We substitute (30.3.2.27) into the functional differential equation (30.3.2.18). Taking into account (30.3.2.17), we find

$$[\psi\psi''_{tt} - (\psi'_t)^2 - a\psi^3 - b] - [\varphi\varphi''_{xx} - (\varphi'_x)^2 + a\varphi^3] + (\psi''_{tt} - 3a\psi^2)\varphi - \psi(\varphi''_{xx} + 3a\varphi^2) = 0. \tag{30.3.2.28}$$

Differentiating (30.3.2.28) with respect to t and x yields the separable equation*

$$(\psi'''_{ttt} - 6a\psi\psi'_t)\varphi'_x - (\varphi'''_{xxx} + 6a\varphi\varphi'_x)\psi'_t = 0,$$

whose solution is determined by the ordinary differential equations

$$\psi'''_{ttt} - 6a\psi\psi'_t = A\psi'_t,$$
$$\varphi'''_{xxx} + 6a\varphi\varphi'_x = A\varphi'_x,$$

where A is the separation constant. Each equation can be integrated twice, thus resulting in

$$\begin{aligned}(\psi'_t)^2 &= 2a\psi^3 + A\psi^2 + C_1\psi + C_2,\\ (\varphi'_x)^2 &= -2a\varphi^3 + A\varphi^2 + C_3\varphi + C_4,\end{aligned} \tag{30.3.2.29}$$

where $C_1, \ldots, C_4$ are arbitrary constants. Eliminating the derivatives from (30.3.2.28) using (30.3.2.29), we find that the arbitrary constants are related by $C_3 = -C_1$ and $C_4 = C_2 + b$. So, the functions $\psi(t)$ and $\varphi(x)$ are determined by the first-order nonlinear autonomous equations

$$(\psi'_t)^2 = 2a\psi^3 + A\psi^2 + C_1\psi + C_2,$$
$$(\varphi'_x)^2 = -2a\varphi^3 + A\varphi^2 - C_1\varphi + C_2 + b.$$

* To solve equation (30.3.2.28), one can use the solution of functional equation (29.5.2.3) [see (29.5.2.4)].

TABLE 30.2
Nonlinear Klein–Gordon equations $\partial_{tt}w - \partial_{xx}w = \mathcal{F}(w)$ admitting functional separable solutions of the form $w = w(z)$, $z = \varphi(x) + \psi(t)$. Notation: A, C_1, and C_2 are arbitrary constants; $\sigma = 1$ for $z > 0$ and $\sigma = -1$ for $z < 0$

No.	Right-hand side $\mathcal{F}(w)$	Solution $w(z)$	Equations for $\psi(t)$ and $\varphi(x)$
1	$aw\ln w + bw$	e^z	$(\psi_t')^2 = C_1e^{-2\psi} + a\psi - \frac{1}{2}a + b + A,$ $(\varphi_x')^2 = C_2e^{-2\varphi} - a\varphi + \frac{1}{2}a + A$
2	$ae^w + be^{-2w}$	$\ln\lvert z\rvert$	$(\psi_t')^2 = 2a\psi^3 + A\psi^2 + C_1\psi + C_2,$ $(\varphi_x')^2 = -2a\varphi^3 + A\varphi^2 - C_1\varphi + C_2 + b$
3	$a\sin w + b\Big(\sin w\ln\tan\frac{w}{4} + 2\sin\frac{w}{4}\Big)$	$4\arctan e^z$	$(\psi_t')^2 = C_1e^{2\psi} + C_2e^{-2\psi} + b\psi + a + A,$ $(\varphi_x')^2 = -C_2e^{2\varphi} - C_1e^{-2\varphi} - b\varphi + A$
4	$a\sinh w + b\Big(\sinh w\ln\tanh\frac{w}{4} + 2\sinh\frac{w}{2}\Big)$	$2\ln\Big\lvert\coth\frac{z}{2}\Big\rvert$	$(\psi_t')^2 = C_1e^{2\psi} + C_2e^{-2\psi} - \sigma b\psi + a + A,$ $(\varphi_x')^2 = C_2e^{2\varphi} + C_1e^{-2\varphi} + \sigma b\varphi + A$
5	$a\sinh w + 2b\Big(\sinh w\arctan e^{w/2} + \cosh\frac{w}{2}\Big)$	$2\ln\Big\lvert\tan\frac{z}{2}\Big\rvert$	$(\psi_t')^2 = C_1\sin 2\psi + C_2\cos 2\psi + \sigma b\psi + a + A,$ $(\varphi_x')^2 = -C_1\sin 2\varphi + C_2\cos 2\varphi - \sigma b\varphi + A$

The solutions of these equations are expressed in terms of elliptic functions.

For the other cases in (30.3.2.22), the analysis is performed in a similar way. Table 30.2 presents the final results for the cases (30.3.2.22a)–(30.3.2.22e).

Case 2. Integrating the third and fourth equations in (30.3.2.21) yields

$$\begin{aligned} &\psi = \pm\sqrt{B}\,t + D_1, && \varphi = \pm\sqrt{B - C}\,t + D_2 && \text{if} \quad A = 0;\\ &\psi = \frac{1}{4A}(At + D_1)^2 - \frac{B}{A}, && \varphi = -\frac{1}{4A}(Ax + D_2)^2 + \frac{B - C}{A} && \text{if} \quad A \neq 0, \end{aligned} \tag{30.3.2.30}$$

where D_1 and D_2 are arbitrary constants. In both cases, the function $\mathcal{F}(w)$ in equation (30.3.2.16) is arbitrary. The first row in (30.3.2.30) corresponds to the traveling-wave solution $w = w(kx + \lambda t)$. The second row leads to a solution of the form $w = w(x^2 - t^2)$.

Example 30.11. The nonlinear stationary heat (diffusion) equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \mathcal{F}(w)$$

is analyzed in much the same way as the nonlinear Klein–Gordon equation considered in Example 30.10. The final results are listed in Table 30.3; the traveling wave solutions $w = w(kx + \lambda t)$ and solutions of the form $w = w(x^2 + y^2)$, existing for any $\mathcal{F}(w)$, are omitted.

⊙ *References for Section* 30.3: A. M. Grundland and E. Infeld (1992), W. Miller and L. A. Rubel (1993), R. Z. Zhdanov (1994), P. W. Doyle and P. J. Vassiliou (1998), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002, 2004).

30.4. Splitting Method. Solutions of Some Nonlinear Functional Equations and Their Applications

30.4.1. Splitting Method. Reduction to a Standard Functional Equation

The procedure for constructing functional separable solutions, which is based on the splitting method, involves several stages outlined below.

1°. Substitute expression (30.1.1.1) into the nonlinear partial differential equation under study. This results in a functional-differential equation with three arguments—two usual arguments, x and y, and one composite argument, z.

TABLE 30.3
Nonlinear equations $\partial_{xx}w + \partial_{yy}w = \mathcal{F}(w)$ admitting functional separable solutions of the form $w = w(z)$, $z = \varphi(x) + \psi(y)$. Notation: A, C_1, and C_2 are arbitrary constants; $\sigma = 1$ for $z > 0$, $\sigma = -1$ for $z < 0$

No.	Right-hand side $\mathcal{F}(w)$	Solution $w(z)$	Equations for $\varphi(x)$ and $\psi(y)$
1	$aw\ln w+bw$	e^z	$(\varphi'_x)^2=C_1e^{-2\varphi}+a\varphi-\frac{1}{2}a+b+A,$ $(\psi'_y)^2=C_2e^{-2\psi}+a\psi-\frac{1}{2}a-A$
2	ae^w+be^{-2w}	$\ln\lvert z\rvert$	$(\varphi'_x)^2=2a\varphi^3+A\varphi^2+C_1\varphi+C_2,$ $(\psi'_y)^2=2a\psi^3-A\psi^2+C_1\psi-C_2-b$
3	$a\sin w+b\left(\sin w\ln\tan\frac{w}{4}+2\sin\frac{w}{4}\right)$	$4\arctan e^z$	$(\varphi'_x)^2=C_1e^{2\varphi}+C_2e^{-2\varphi}+b\varphi+a+A,$ $(\psi'_y)^2=C_2e^{2\psi}+C_1e^{-2\psi}+b\psi-A$
4	$a\sinh w+b\left(\sinh w\ln\tanh\frac{w}{4}+2\sinh\frac{w}{2}\right)$	$2\ln\left\lvert\coth\frac{z}{2}\right\rvert$	$(\varphi'_x)^2=C_1e^{2\varphi}+C_2e^{-2\varphi}-\sigma b\varphi+a+A,$ $(\psi'_y)^2=-C_2e^{2\psi}-C_1e^{-2\psi}-\sigma b\psi-A$
5	$a\sinh w+2b\left(\sinh w\arctan e^{w/2}+\cosh\frac{w}{2}\right)$	$2\ln\left\lvert\tan\frac{z}{2}\right\rvert$	$(\varphi'_x)^2=C_1\sin 2\varphi+C_2\cos 2\varphi+\sigma b\varphi+a+A,$ $(\psi'_y)^2=C_1\sin 2\psi-C_2\cos 2\psi+\sigma b\psi-A$

2°. Reduce the functional-differential equation to a purely functional equation with three arguments x, y, and z with the aid of elementary differential substitutions (by selecting and renaming terms with derivatives).

3°. Reduce the three-argument functional-differential equation by the differentiation method to the bilinear functional equation of a standard form with two arguments (either x or y is eliminated).

4°. Construct the solutions of the two-argument functional equation using the formulas given in Section 29.5.

5°. Solve the (overdetermined) systems formed by the solutions of Item 4° and the differential substitutions of Item 2°.

6°. Substitute the solutions of Item 5° into the original functional-differential equation of Item 1° to establish the relations for the constants of integration and determine all unknown quantities.

7°. Consider all degenerate cases possibly arising due to the violation of assumptions adopted in the previous analysis.

Remark. Stage 3° is the most difficult here; it may not always be realizable.

The splitting method reduces solving the three-argument functional-differential equation to (i) solving a purely functional equation with three arguments (by reducing it to a standard functional equation with two arguments) and (ii) solving systems of ordinary differential equations. Thus, the initial problem splits into several simpler problems. Examples of constructing functional separable solutions by the splitting method are given in Subsection 30.4.4 and 30.4.5.

30.4.2. Three-Argument Functional Equations of Special Form

The substitution of the expression

$$w = F(z), \qquad z = \varphi(x) + \psi(t) \tag{30.4.2.1}$$

into a nonlinear partial differential equation sometimes leads to functional differential equations of the form

$$f(t) + \Phi_1(x)\Psi_1(z) + \cdots + \Phi_k(x)\Psi_k(z) = 0, \tag{30.4.2.2}$$

where $\Phi_j(x)$ and $\Psi_j(z)$ are functionals dependent on the variables x and z, respectively,

$$\begin{aligned}\Phi_j(x) &\equiv \Phi_j\left(x, \varphi, \varphi'_x, \ldots, \varphi_x^{(n)}\right),\\ \Psi_j(z) &\equiv \Psi_j\left(F, F'_z, \ldots, F_z^{(n)}\right).\end{aligned} \tag{30.4.2.3}$$

It is reasonable to solve equation (30.4.2.2) by the splitting method. At the first stage, we treat (30.4.2.2) as a purely functional equation, thus disregarding (30.4.2.3). Differentiating (30.4.2.2) with respect to x yields the standard bilinear functional differential equation in two independent variables x and z:

$$\Phi'_1(x)\Psi_1(z) + \cdots + \Phi'_k(x)\Psi_k(z) + \Phi_1(x)\varphi'(x)\Psi'_1(z) + \cdots + \varphi'(x)\Phi_k(x)\Psi'_k(z) = 0, \tag{30.4.2.4}$$

which can be solved using the results of Sections 29.4–29.6. Then, substituting the solutions $\Phi_m(x)$ and $\Psi_m(z)$ into (30.4.2.2) and taking into account the second relation in (30.4.2.1), we find the function $f(t)$. Further, substituting the functionals (30.4.2.3) into the solutions of the functional equation (30.4.2.2), we obtain determining systems of ordinary differential equations for $F(z)$, $\varphi(x)$, and $\psi(t)$.

Below, we discuss several types of three-argument functional equations of the form (30.4.2.2) that arise most frequently in the functional separation of variables in nonlinear equations of mathematical physics. The results are used for constructing exact solutions for some classes of nonlinear heat and wave equations.

30.4.3. Functional Equation $f(t) + g(x) = Q(z)$, with $z = \varphi(x) + \psi(t)$

Here, one of the two functions $f(t)$ and $\psi(t)$ is prescribed and the other is assumed unknown; also one of the functions $g(x)$ and $\psi(x)$ is prescribed and the other is unknown, and the function $Q(z)$ is assumed unknown.*

Differentiating the equation with respect to x and t yields $Q''_{zz} = 0$. Consequently, the solution is given by

$$f(t) = A\psi(x) + B, \quad g(x) = A\varphi(x) - B + C, \quad Q(z) = Az + C, \tag{30.4.3.1}$$

where A, B, and C are arbitrary constants.

30.4.4. Functional Equation $f(t) + g(x) + h(x)Q(z) + R(z) = 0$, with $z = \varphi(x) + \psi(t)$

Differentiating the equation with respect to x yields the two-argument equation

$$g'_x + h'_x Q + h\varphi'_x Q'_z + \varphi'_x R'_z = 0. \tag{30.4.4.1}$$

Such equations were discussed in Sections 29.4 and 29.5. Hence, the following relations hold [see formulas (29.5.2.3) and (29.5.2.4)]:

$$\begin{aligned} g'_x &= A_1 h\varphi'_x + A_2\varphi'_x, \\ h'_x &= A_3 h\varphi'_x + A_4\varphi'_x, \\ Q'_z &= -A_1 - A_3 Q, \\ R'_z &= -A_2 - A_4 Q, \end{aligned} \tag{30.4.4.2}$$

where $A_1, \ldots, A_4$ are arbitrary constants. By integrating system (30.4.4.2) and substituting the resulting solutions into the original functional equation, one obtains the results given below.

Case 1. If $A_3 = 0$ in (30.4.4.2), the corresponding solution of the functional equation is given by

$$\begin{aligned} f &= -\tfrac{1}{2}A_1A_4\psi^2 + (A_1B_1 + A_2 + A_4B_3)\psi - B_2 - B_1B_3 - B_4, \\ g &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2, \\ h &= A_4\varphi + B_1, \\ Q &= -A_1 z + B_3, \\ R &= \tfrac{1}{2}A_1A_4 z^2 - (A_2 + A_4B_3)z + B_4, \end{aligned} \tag{30.4.4.3}$$

where A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 2. If $A_3 \neq 0$ in (30.4.4.2), the corresponding solution of the functional equation is

$$\begin{aligned} f &= -B_1B_3e^{-A_3\psi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\psi - B_2 - B_4 - \frac{A_1A_4}{A_3^2}, \\ g &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2, \\ h &= B_1e^{A_3\varphi} - \frac{A_4}{A_3}, \\ Q &= B_3e^{-A_3 z} - \frac{A_1}{A_3}, \\ R &= \frac{A_4B_3}{A_3}e^{-A_3 z} + \left(\frac{A_1A_4}{A_3} - A_2\right)z + B_4, \end{aligned} \tag{30.4.4.4}$$

where A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

* In similar equations with a composite argument, it is assumed that $\varphi(x) \not\equiv \text{const}$ and $\psi(t) \not\equiv \text{const}$.

Case 3. In addition, the functional equation has two degenerate solutions:

$$f = A_1\psi + B_1, \quad g = A_1\varphi + B_2, \quad h = A_2, \quad R = -A_1 z - A_2 Q - B_1 - B_2, \tag{30.4.4.5a}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions; A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = A_1\psi + B_1, \quad g = A_1\varphi + A_2 h + B_2, \quad Q = -A_2, \quad R = -A_1 z - B_1 - B_2, \tag{30.4.4.5b}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $h = h(x)$ are arbitrary functions; and A_1, A_2, B_1, and B_2 are arbitrary constants. The degenerate solutions (30.4.4.5a) and (30.4.4.5b) can be obtained directly from the original equation and its consequence (30.4.4.1) using formulas (29.5.2.5).

Example 30.12. Consider the unsteady heat equation with a nonlinear source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \mathcal{F}(w). \tag{30.4.4.6}$$

We look for exact solutions of the form

$$w = w(z), \quad z = \varphi(x) + \psi(t). \tag{30.4.4.7}$$

Substituting (30.4.4.7) into (30.4.4.6) and dividing by w'_z yields the functional differential equation

$$\psi'_t = \varphi''_{xx} + (\varphi'_x)^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w(z))}{w'_z}.$$

Let us solve it by the splitting method. To this end, we represent this equation as the functional equation $f(t) + g(x) + h(x)Q(z) + R(z) = 0$, where

$$f(t) = -\psi'_t, \quad g(x) = \varphi''_{xx}, \quad h(x) = (\varphi'_x)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z. \tag{30.4.4.8}$$

On substituting the expressions of g and h of (30.4.4.8) into (30.4.4.3)–(30.4.4.5), we arrive at overdetermined systems of equations for $\varphi = \varphi(x)$.

Case 1. The system

$$\begin{aligned}
\varphi''_{xx} &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2,\\
(\varphi'_x)^2 &= A_4\varphi + B_1
\end{aligned}$$

following from (30.4.4.3) and corresponding to $A_3 = 0$ in (30.4.4.2) is consistent in the cases

$$\begin{aligned}
&\varphi = C_1 x + C_2 \quad \text{for} \quad A_2 = -A_1C_1^2,\ A_4 = B_2 = 0,\ B_1 = C_1^2,\\
&\varphi = \tfrac{1}{4}A_4x^2 + C_1x + C_2 \quad \text{for} \quad A_1 = A_2 = 0,\ B_1 = C_1^2 - A_4C_2,\ B_2 = \tfrac{1}{2}A_4,
\end{aligned} \tag{30.4.4.9}$$

where C_1 and C_2 are arbitrary constants.

The first solution in (30.4.4.9) with $A_1 \neq 0$ leads to a right-hand side of equation (30.4.4.6) containing the inverse of the error function [the form of the right-hand side is identified from the last two relations in (30.4.4.3) and (30.4.4.8)]. The second solution in (30.4.4.9) corresponds to the right-hand side of $\mathcal{F}(w) = k_1 w \ln w + k_2 w$ in (30.4.4.6). In both cases, the first relation in (30.4.4.3) is, taking into account that $f = -\psi'_t$, a first-order linear solution with constant coefficients, whose solution is an exponential plus a constant.

Case 2. The system

$$\begin{aligned}
\varphi''_{xx} &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2,\\
(\varphi'_x)^2 &= B_1e^{A_3\varphi} - \frac{A_4}{A_3},
\end{aligned}$$

following from (30.4.4.4) and corresponding to $A_3 \neq 0$ in (30.4.4.2), is consistent in the following cases:

$$\begin{aligned}
&\varphi = \pm\sqrt{-A_4/A_3}\,x + C_1 && \text{for } A_2 = A_1A_4/A_3,\ B_1 = B_2 = 0,\\
&\varphi = -\frac{2}{A_3}\ln|x| + C_1 && \text{for } A_1 = \tfrac{1}{2}A_3^2,\ A_2 = A_4 = B_2 = 0,\ B_1 = 4A_3^{-2}e^{-A_3C_1},\\
&\varphi = -\frac{2}{A_3}\ln\left|\cos\left(\tfrac{1}{2}\sqrt{A_3A_4}\,x + C_1\right)\right| + C_2 && \text{for } A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3A_4,\ B_2 = 0,\ A_3A_4 > 0,\\
&\varphi = -\frac{2}{A_3}\ln\left|\sinh\left(\tfrac{1}{2}\sqrt{-A_3A_4}\,x + C_1\right)\right| + C_2 && \text{for } A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3A_4,\ B_2 = 0,\ A_3A_4 < 0,\\
&\varphi = -\frac{2}{A_3}\ln\left|\cosh\left(\tfrac{1}{2}\sqrt{-A_3A_4}\,x + C_1\right)\right| + C_2 && \text{for } A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3A_4,\ B_2 = 0,\ A_3A_4 < 0,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The right-hand sides of equation (30.4.4.6) corresponding to these solutions are represented in parametric form.

Case 3. Traveling-wave solutions of the nonlinear heat equation (30.4.4.6) and solutions of the linear equation (30.4.4.6) with $\mathcal{F}'_w = \text{const}$ correspond to the degenerate solutions of the functional equation (30.4.4.5).

Example 30.13. Likewise, one can analyze the more general equation

$$\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + \mathcal{F}(w). \tag{30.4.4.10}$$

It arises in convective heat/mass exchange problems (a = const and b = const), problems of heat transfer in inhomogeneous media ($b = a'_x \neq \text{const}$), and spatial heat transfer problems with axial or central symmetry (a = const and $b = \text{const}/x$).

Searching for exact solutions of equation (30.4.4.10) in the form (30.4.4.7) leads to the functional equation $f(t) + g(x) + h(x)Q(z) + R(z) = 0$, where

$$f(t) = -\psi'_t, \quad g(x) = a(x)\varphi''_{xx} + b(x)\varphi'(x), \quad h(x) = a(x)(\varphi'_x)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z.$$

Substituting these expressions into (30.4.4.3)–(30.4.4.5) yields a system of ordinary differential equations for the unknowns.

Remark. In Examples 30.12 and 30.13, different equations were all reduced to the same functional equation. This demonstrates the utility of the isolation and independent analysis of individual types of functional equations, as well as the expedience of developing methods for solving functional equations with a composite argument.

30.4.5. Functional Equation $f(t) + g(x)Q(z) + h(x)R(z) = 0$, with $z = \varphi(x) + \psi(t)$

Differentiating with respect to x yields the two-argument functional differential equation

$$g'_x Q + g\varphi'_x Q'_z + h'_x R + h\varphi'_x R'_z = 0, \tag{30.4.5.1}$$

which coincides with equation (29.5.2.3), up to notation.

Nondegenerate case. Equation (30.4.5.1) can be solved using formulas (29.5.2.4). In this way, we arrive at the system of ordinary differential equations

$$\begin{aligned} g'_x &= (A_1 g + A_2 h)\varphi'_x, \\ h'_x &= (A_3 g + A_4 h)\varphi'_x, \\ Q'_z &= -A_1 Q - A_3 R, \\ R'_z &= -A_2 Q - A_4 R, \end{aligned} \tag{30.4.5.2}$$

where $A_1, \ldots, A_4$ are arbitrary constants.

The solution of equation (30.4.5.2) is given by

$$\begin{aligned} g(x) &= A_2 B_1 e^{k_1\varphi} + A_2 B_2 e^{k_2\varphi}, \\ h(x) &= (k_1 - A_1)B_1 e^{k_1\varphi} + (k_2 - A_1)B_2 e^{k_2\varphi}, \\ Q(z) &= A_3 B_3 e^{-k_1 z} + A_3 B_4 e^{-k_2 z}, \\ R(z) &= (k_1 - A_1)B_3 e^{-k_1 z} + (k_2 - A_1)B_4 e^{-k_2 z}, \end{aligned} \tag{30.4.5.3}$$

where $B_1, \ldots, B_4$ are arbitrary constants and k_1 and k_2 are roots of the quadratic equation

$$(k - A_1)(k - A_4) - A_2 A_3 = 0. \tag{30.4.5.4}$$

In the degenerate case $k_1 = k_2$, the terms $e^{k_2\varphi}$ and $e^{-k_2 z}$ in (30.4.5.3) must be replaced by $\varphi e^{k_1\varphi}$ and $z e^{-k_1 z}$, respectively. In the case of purely imaginary or complex roots, one should extract the real (or imaginary) part of the roots in solution (30.4.5.3).

On substituting (30.4.5.3) into the original functional equation, one obtains conditions that must be met by the free coefficients and identifies the function $f(t)$, specifically,

$$\begin{aligned} B_2 = B_4 = 0 \quad &\Longrightarrow \quad f(t) = [A_2 A_3 + (k_1 - A_1)^2]B_1 B_3 e^{-k_1\psi}, \\ B_1 = B_3 = 0 \quad &\Longrightarrow \quad f(t) = [A_2 A_3 + (k_2 - A_1)^2]B_2 B_4 e^{-k_2\psi}, \\ A_1 = 0 \quad &\Longrightarrow \quad f(t) = (A_2 A_3 + k_1^2)B_1 B_3 e^{-k_1\psi} + (A_2 A_3 + k_2^2)B_2 B_4 e^{-k_2\psi}. \end{aligned} \tag{30.4.5.5}$$

Solution (30.4.5.3), (30.4.5.5) involves arbitrary functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$.

Degenerate case. In addition, the functional equation has two degenerate solutions,

$$f = B_1B_2e^{A_1\psi}, \quad g = A_2B_1e^{-A_1\varphi}, \quad h = B_1e^{-A_1\varphi}, \quad R = -B_2e^{A_1z} - A_2Q,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions; A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = B_1B_2e^{A_1\psi}, \quad h = -B_1e^{-A_1\varphi} - A_2g, \quad Q = A_2B_2e^{A_1z}, \quad R = B_2e^{A_1z},$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $g = g(x)$ are arbitrary functions; and A_1, A_2, B_1, and B_2 are arbitrary constants. The degenerate solutions can be obtained immediately from the original equation or its consequence (30.4.5.1) using formulas (29.5.2.5).

Example 30.14. For the first-order nonlinear equation

$$\frac{\partial w}{\partial t} = \mathcal{F}(w)\left(\frac{\partial w}{\partial x}\right)^2 + \mathcal{G}(x),$$

the search for exact solutions in the form (30.4.4.7) leads to the functional equation that appears in the title of Subsection 30.4.5, where

$$f(t) = -\psi_t', \quad g(x) = (\varphi_x')^2, \quad h(x) = \mathcal{G}(x), \quad Q(z) = \mathcal{F}(w)w_z', \quad R(z) = 1/w_z', \quad w = w(z).$$

Example 30.15. For the nonlinear heat equation (30.3.2.1), searching for exact solutions in the form $w = w(z)$, with $z = \varphi(x) + \psi(t)$, leads to the functional equation (30.3.2.3), which coincides with the equation $f(t) + g(x)Q(z) + h(x)R(z) = 0$ if

$$f(t) = -\psi_t', \quad g(x) = \varphi_{xx}'', \quad h(x) = (\varphi_x')^2, \quad Q(z) = \mathcal{F}(w), \quad R(z) = \frac{[\mathcal{F}(w)w_z']_z'}{w_z'}, \quad w = w(z).$$

30.4.6. Equation $f_1(x) + f_2(y) + g_1(x)P(z) + g_2(y)Q(z) + R(z) = 0$, $z = \varphi(x) + \psi(y)$

Differentiating with respect to y and dividing the resulting relation by $\psi_y'P_z'$ and differentiating with respect to y again, one arrives at the bilinear functional equation with two arguments, y and z, that is discussed in Sections 29.2, 29.4, and 29.5 [see equation (29.2.2.1) and its solutions (29.5.1.1)].

Example 30.16. Consider the following equation of steady-state heat transfer in an anisotropic inhomogeneous medium with a nonlinear source:

$$\frac{\partial}{\partial x}\left[a(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[b(y)\frac{\partial w}{\partial y}\right] = \mathcal{F}(w). \tag{30.4.6.1}$$

The search for exact solutions in the form $w = w(z)$, $z = \varphi(x) + \psi(y)$, leads to the functional equation $f_1(x) + f_2(y) + g_1(x)P(z) + g_2(y)Q(z) + R(z) = 0$ in which

$$f_1(x) = a(x)\varphi_{xx}'' + a_x'(x)\varphi_x', \quad f_2(y) = b(y)\psi_{yy}'' + b_y'(y)\psi_y', \quad g_1(x) = a(x)(\varphi_x')^2, \quad g_2(y) = b(y)(\psi_y')^2,$$
$$P(z) = Q(z) = w_{zz}''/w_z', \quad R(z) = -\mathcal{F}(w)/w_z', \quad w = w(z).$$

Here we confine ourselves to studying functional separable solutions existing for arbitrary right-hand side $\mathcal{F}(w)$.

With the change of variable $z = \zeta^2$, we look for solutions of equation (30.4.6.1) in the form

$$w = w(\zeta), \quad \zeta^2 = \varphi(x) + \psi(y). \tag{30.4.6.2}$$

Taking into account that $\frac{\partial\zeta}{\partial x} = \frac{\varphi_x'}{2\zeta}$ and $\frac{\partial\zeta}{\partial y} = \frac{\psi_y'}{2\zeta}$, we find from (30.4.6.1)

$$\left[(a\varphi_x')_x' + (b\psi_y')_y'\right]\frac{w_\zeta'}{2\zeta} + \left[a(\varphi_x')^2 + b(\psi_y')^2\right]\frac{\zeta w_{\zeta\zeta}'' - w_\zeta'}{4\zeta^3} = \mathcal{F}(w), \qquad \mathcal{F}(w) = \mathcal{F}\big(w(\zeta)\big). \tag{30.4.6.3}$$

For this functional-differential equation to be solvable we require that the expressions in square brackets be functions of ζ:

$$(a\varphi_x')_x' + (b\psi_y')_y' = M(\zeta), \quad a(\varphi_x')^2 + b(\psi_y')^2 = N(\zeta).$$

Differentiating the first relation with respect to x and y yields the equation $(M'_\zeta/\zeta)'_\zeta = 0$, whose general solution is $M(\zeta) = C_1\zeta^2 + C_2$. Likewise, we find $N(\zeta) = C_3\zeta^2 + C_4$. Here, $C_1, \ldots, C_4$ are arbitrary constants. Consequently, we have

$$(a\varphi'_x)'_x + (b\psi'_y)'_y = C_1(\varphi + \psi) + C_2, \quad a(\varphi'_x)^2 + b(\psi'_y)^2 = C_3(\varphi + \psi) + C_4.$$

The separation of variables results in a system of ordinary differential equations for $\varphi(x)$, $a(x)$, $\psi(y)$, and $b(y)$:

$$\begin{aligned} &(a\varphi'_x)'_x - C_1\varphi - C_2 = k_1, \qquad (b\psi'_y)'_y - C_1\psi = -k_1, \\ &a(\varphi'_x)^2 - C_3\varphi - C_4 = k_2, \qquad b(\psi'_y)^2 - C_3\psi = -k_2. \end{aligned}$$

This system is always integrable by quadrature and can be rewritten as

$$\begin{aligned} &(C_3\varphi + C_4 + k_2)\varphi''_{xx} + (C_1\varphi + C_2 + k_1 - C_3)(\varphi'_x)^2 = 0, \quad a = (C_3\varphi + C_4 + k_2)(\varphi'_x)^{-2}; \\ &(C_3\psi - k_2)\psi''_{yy} + (C_1\psi - k_1 - C_3)(\psi'_y)^2 = 0, \qquad\qquad b = (C_3\psi - k_2)(\psi'_y)^{-2}. \end{aligned} \tag{30.4.6.4}$$

Here, the equations for φ and ψ do not involve a and b and, hence, can be solved independently. Without full analysis of system (30.4.6.4), we note a special case where the system can be solved in explicit form.

For $C_1 = C_2 = C_4 = k_1 = k_2 = 0$ and $C_3 = C \neq 0$, we find

$$a(x) = \alpha e^{\mu x}, \quad b(y) = \beta e^{\nu y}, \quad \varphi(x) = \frac{Ce^{-\mu x}}{\alpha\mu^2}, \quad \psi(y) = \frac{Ce^{-\nu y}}{\beta\nu^2},$$

where α, β, μ, and ν are arbitrary constants. Substituting these expressions into (30.4.6.3) and taking into account (30.4.6.2), we obtain the ordinary differential equation for $w(\zeta)$

$$w''_{\zeta\zeta} - \frac{1}{\zeta}w'_\zeta = \frac{4}{C}\mathcal{F}(w).$$

System (30.4.6.4) has other solutions as well; these lead to various expressions of $a(x)$ and $b(y)$. Table 30.4 lists the cases where these functions can be written in explicit form (the traveling-wave solution, which corresponds to a = const and b = const, is omitted). In general, the solution of system (30.4.6.4) enables one to represent $a(x)$ and $b(y)$ in parametric form.

TABLE 30.4
Functional separable solutions of the form $w = w(\zeta)$, $\zeta^2 = \varphi(x) + \psi(y)$, for heat equations in an anisotropic inhomogeneous medium with an arbitrary nonlinear source.
Notation: C, α, β, μ, ν, n, and k are free parameters ($C \neq 0$, $\mu \neq 0$, $\nu \neq 0$, $n \neq 2$, and $k \neq 2$)

Heat equation	Functions $\varphi(x)$ and $\psi(y)$	Equation for $w = w(\zeta)$
$\dfrac{\partial}{\partial x}\left(\alpha x^m\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta y^n\dfrac{\partial w}{\partial y}\right)=\mathcal{F}(w)$	$\varphi=\dfrac{Cx^{2-m}}{\alpha(2-m)^2}$, $\psi=\dfrac{Cy^{2-n}}{\beta(2-n)^2}$	$w''_{\zeta\zeta}+\dfrac{4-mn}{(2-m)(2-n)}\dfrac{1}{\zeta}w'_\zeta=\dfrac{4}{C}\mathcal{F}(w)$
$\dfrac{\partial}{\partial x}\left(\alpha e^{\mu x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta e^{\nu y}\dfrac{\partial w}{\partial y}\right)=\mathcal{F}(w)$	$\varphi=\dfrac{C}{\alpha\mu^2}e^{-\mu x}$, $\psi=\dfrac{C}{\beta\nu^2}e^{-\nu y}$	$w''_{\zeta\zeta}-\dfrac{1}{\zeta}w'_\zeta=\dfrac{4}{C}\mathcal{F}(w)$
$\dfrac{\partial}{\partial x}\left(\alpha e^{\mu x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta y^n\dfrac{\partial w}{\partial y}\right)=\mathcal{F}(w)$	$\varphi=\dfrac{C}{\alpha\mu^2}e^{-\mu x}$, $\psi=\dfrac{Cy^{2-n}}{\beta(2-n)^2}$	$w''_{\zeta\zeta}+\dfrac{n}{2-n}\dfrac{1}{\zeta}w'_\zeta=\dfrac{4}{C}\mathcal{F}(w)$
$\dfrac{\partial}{\partial x}\left(\alpha x^2\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(\beta y^2\dfrac{\partial w}{\partial y}\right)=\mathcal{F}(w)$	$\varphi=\mu\ln\lvert x\rvert$, $\psi=\nu\ln\lvert y\rvert$	Equation (30.4.6.3); expressions in square brackets are both constant
$\alpha\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(\beta y^2\dfrac{\partial w}{\partial y}\right)=\mathcal{F}(w)$	$\varphi=\mu x$, $\psi=\nu\ln\lvert y\rvert$	Equation (30.4.6.3); expressions in square brackets are both constant

⊙ *References for Section* 30.4: V. F. Zaitsev and A. D. Polyanin (1996), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002, 2004).

Chapter 31
Direct Method of Symmetry Reductions of Nonlinear Equations

31.1. Clarkson–Kruskal Direct Method

31.1.1. Simplified Scheme. Examples of Constructing Exact Solutions

The basic idea of the simplified scheme is as follows: for an equation with the unknown function $w = w(x,t)$, an exact solution is sought in the form

$$w = f(t)u(z) + g(x,t), \qquad z = \varphi(t)x + \psi(t). \tag{31.1.1.1}$$

The functions $f(t)$, $g(x,t)$, $\varphi(t)$, and $\psi(t)$ are found in the subsequent analysis and are chosen in such a way that, ultimately, the function $u(z)$ would satisfy a *single ordinary differential equation*.

Below we consider some cases in which it is possible to construct exact solutions of nonlinear equations of the form (31.1.1.1).

Example 31.1. Consider the *generalized Burgers–Korteweg–de Vries equation*

$$\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + bw\frac{\partial w}{\partial x}. \tag{31.1.1.2}$$

We seek its exact solution in the form (31.1.1.1). Inserting (31.1.1.1) into (31.1.1.2), we obtain

$$af\varphi^n u_z^{(n)} + bf^2\varphi uu_z' + f(bg\varphi - \varphi_t' x - \psi_t')u_z' + (bfg_x - f_t')u + ag_x^{(n)} + bgg_x - g_t = 0. \tag{31.1.1.3}$$

Equating the functional coefficients of $u_z^{(n)}$ and uu_z' in (31.1.1.3), we get

$$f = \varphi^{n-1}. \tag{31.1.1.4}$$

Further, equating the coefficient of u_z' to zero, we obtain

$$g = \frac{1}{b\varphi}(\varphi_t' x + \psi_t'). \tag{31.1.1.5}$$

Inserting the expressions (31.1.1.4) and (31.1.1.5) into (31.1.1.3), we arrive at the relation

$$\varphi^{2n-1}(au_z^{(n)} + buu_z') + (2-n)\varphi^{n-2}\varphi_t' u + \frac{1}{b\varphi^2}\left[(2\varphi_t^2 - \varphi\varphi_{tt})x + 2\varphi_t\psi_t - \varphi\psi_{tt}\right] = 0.$$

Dividing each term by φ^{2n-1} and then eliminating x with the help of the relation $x = (z-\psi)/\varphi$, we obtain

$$\begin{aligned} au_z^{(n)} + buu_z' + (2-n)\varphi^{-n-1}\varphi_t' u + \frac{1}{b}\varphi^{-2n-2}(2\varphi_t^2 - \varphi\varphi_{tt})z \\ + \frac{1}{b}\varphi^{-2n-2}(\varphi\psi\varphi_{tt} - \varphi^2\psi_{tt} + 2\varphi\varphi_t\psi_t - 2\psi\varphi_t^2) = 0. \end{aligned} \tag{31.1.1.6}$$

Let us require that the functional coefficient of u and the last term be constant,

$$\varphi^{-n-1}\varphi_t' = -A, \qquad \varphi^{-2n-2}(\varphi\psi\varphi_{tt} - \varphi^2\psi_{tt} + 2\varphi\varphi_t\psi_t - 2\psi\varphi_t^2) = B,$$

where A and B are arbitrary. As a result, we arrive at the following system of ordinary differential equations for φ and ψ:

$$\begin{aligned} \varphi_t &= -A\varphi^{n+1}, \\ \psi_{tt} + 2A\varphi^n\psi_t + A^2(1-n)\varphi^{2n}\psi &= -B\varphi^{2n}. \end{aligned} \tag{31.1.1.7}$$

Using (31.1.1.6) and (31.1.1.7), we obtain an equation for $u(z)$,

$$au_z^{(n)} + buu_z' + A(n-2)u + \frac{A^2}{b}(1-n)z + \frac{B}{b} = 0. \tag{31.1.1.8}$$

For $A \neq 0$, the general solution of equations (31.1.1.7) has the form

$$\begin{aligned} \varphi(t) &= (Ant + C_1)^{-\frac{1}{n}}, \\ \psi(t) &= C_2(Ant + C_1)^{\frac{n-1}{n}} + C_3(Ant + C_1)^{-\frac{1}{n}} + \frac{B}{A^2(n-1)}, \end{aligned} \tag{31.1.1.9}$$

where C_1, C_2, and C_3 are arbitrary constants.

Formulas (31.1.1.1), (31.1.1.4), (31.1.1.5), and (31.1.1.9), together with equation (31.1.1.8), describe an exact solution of the generalized Burgers–Korteweg–de Vries equation (31.1.1.2).

Example 31.2. Consider the *Boussinesq equation*

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + a\frac{\partial^4 w}{\partial x^4} = 0. \tag{31.1.1.10}$$

Just as in Example 31.1, we seek its solutions in the form (31.1.1.1), where the functions $f(t)$, $g(x,t)$, $\varphi(t)$, and $\psi(t)$ are found in the subsequent analysis. Substituting (31.1.1.1) into (31.1.1.10) yields

$$\begin{aligned} &af\varphi^4 u'''' + f^2\varphi^2 uu'' + f(z_t^2 + g\varphi^2)u'' + f^2\varphi^2(u')^2 + (fz_{tt} + 2fg_x\varphi + 2f_t z_t)u' \\ &\quad + (fg_{xx} + f_{tt})u + g_{tt} + gg_{xx} + g_x^2 + ag_x^{(4)} = 0. \end{aligned} \tag{31.1.1.11}$$

Equating the functional coefficients of u'''' and uu'', we get

$$f = \varphi^2. \tag{31.1.1.12}$$

Equating the functional coefficient of u'' to zero and taking into account (31.1.1.12), we obtain

$$g = -\frac{1}{\varphi^2}(\varphi_t' x + \psi_t')^2. \tag{31.1.1.13}$$

Substituting the expressions (31.1.1.12) and (31.1.1.13) into (31.1.1.11), we arrive at the relation

$$\varphi^6(au'''' + uu'' + u'^2) + \varphi^2(x\varphi_{tt} + \psi_{tt})u' + 2\varphi\varphi_{tt}u - \left[\varphi^{-2}(\varphi_t x + \psi_t)^2\right]_{tt} + 6\varphi^{-4}\varphi_t^2(\varphi_t x + \psi_t)^2 = 0.$$

Let us perform the double differentiation of the expression in square brackets and then divide all terms by φ^6. Excluding x with the help of the relation $x = (z - \psi)/\varphi$, we get

$$au'''' + uu'' + (u')^2 + \varphi^{-5}(\varphi_{tt}z + \varphi\psi_{tt} - \psi\varphi_{tt})u' + 2\varphi^{-5}\varphi_{tt}u + \cdots = 0. \tag{31.1.1.14}$$

Let us require that the functional coefficient of u' be a function of only one variable, z, i.e.,

$$\varphi^{-5}(\varphi_{tt}z + \varphi\psi_{tt} - \psi\varphi_{tt}) = \varphi^{-5}\varphi_{tt}z + \varphi^{-5}(\varphi\psi_{tt} - \psi\varphi_{tt}) \equiv Az + B,$$

where A and B are arbitrary constants. Hence, we obtain the following system of ordinary differential equations for the functions φ and ψ:

$$\begin{aligned} \varphi_{tt} &= A\varphi^5, \\ \psi_{tt} &= (A\psi + B)\varphi^4. \end{aligned} \tag{31.1.1.15}$$

Let us eliminate the second and the third derivatives of the functions φ and ψ from (31.1.1.14). As a result, we arrive at the following ordinary differential equation for the function $u(z)$:

$$au'''' + uu'' + (u')^2 + (Az + B)u' + 2Au - 2(Az + B)^2 = 0. \tag{31.1.1.16}$$

Formulas (31.1.1.1), (31.1.1.12), and (31.1.1.13), together with equations (31.1.1.15)–(31.1.1.16), describe an exact solution of the Boussinesq equation (31.1.1.10).

Example 31.3. Consider the nonlinear third-order equation

$$\frac{\partial^2 w}{\partial t\partial x} + w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2 = \nu\frac{\partial^3 w}{\partial x^3} + q(t)\frac{\partial w}{\partial x} + p(t), \tag{31.1.1.17}$$

which describes a broad class of exact solutions to the three-dimensional Navier–Stokes equations (Aristov and Polyanin, 2009). The functions $p = p(t)$ and $q = q(t)$ that appear in equation (31.1.1.17) can be chosen arbitrarily.

We look for exact solutions in the form

$$w = a(t)u(z) + b(t)x + c(t), \quad z = \lambda(t)x + \sigma(t), \quad \tau = \int \lambda^2(t)\,dt + C_0, \tag{31.1.1.18}$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, $\lambda = \lambda(t)$, and $\sigma = \sigma(t)$ are unknown functions. Substituting (31.1.1.18) into (31.1.1.17) yields

$$[\widetilde{a}(\tau)u + \widetilde{b}(\tau)z + \widetilde{c}(\tau)]u''_{zz} - \widetilde{a}(\tau)(u'_z)^2 = \nu u'''_{zzz} + \widetilde{q}(\tau)u'_z + \widetilde{p}(\tau), \tag{31.1.1.19}$$

where

$$\begin{aligned}
\widetilde{a} &= \frac{a}{\lambda}, \quad \widetilde{b} = \frac{1}{\lambda^3}(b\lambda + \lambda'_t),\\
\widetilde{c} &= \frac{1}{\lambda^3}(c\lambda^2 - b\lambda\sigma + \lambda\sigma'_t - \sigma\lambda'_t),\\
\widetilde{q} &= \frac{1}{a\lambda^3}[aq\lambda + 2ab\lambda - (a\lambda)'_t],\\
\widetilde{p} &= \frac{1}{a\lambda^3}(p + bq + b^2 - b'_t).
\end{aligned} \tag{31.1.1.20}$$

The left-hand side functions have the argument τ and the functions appearing on the right-hand sides depend on t; the variable τ and t are connected by the last relation in (31.1.1.18).

Now, by setting

$$\widetilde{a} = C_1, \quad \widetilde{b} = C_2, \quad \widetilde{c} = C_3, \quad \widetilde{q} = C_4, \quad \widetilde{p} = C_5, \tag{31.1.1.21}$$

where C_n are arbitrary constants, we obtain from (31.1.1.19) an ordinary differential equation for $u = u(z)$. In this case, relations (31.1.1.20) subject to the conditions (31.1.1.21) make up a system of algebraic and ordinary differential equations for the functional coefficients of transformation (31.1.1.18) and the functional parameters $p = p(t)$ and $q = q(t)$ of equation (31.1.1.17). The functions $a = a(t)$ and $\sigma = \sigma(t)$ appearing in this system can be regarded as arbitrary, and the other quantities, $\lambda = \lambda(t)$, $b = b(t)$, $c = c(t)$, $p = p(t)$, and $q = q(t)$, are expressed in terms of $a(t)$ and $\sigma(t)$ without quadrature.

31.1.2. Description of the Method: A Special Form for Symmetry Reduction

1°. The basic idea of the method is the following: for an equation with the unknown function $w = w(x, t)$, an exact solution is sought in the form

$$w(x, t) = f(x, t)u(z) + g(x, t), \quad z = z(x, t). \tag{31.1.2.1}$$

The functions $f(x, t)$, $g(x, t)$, and $z(x, t)$ are determined in the subsequent analysis, so that ultimately one obtains a *single ordinary differential equation* for the function $u(z)$.

2°. Inserting (31.1.2.1) into a nonlinear partial differential equation with a quadratic or a power nonlinearity, we obtain

$$\Phi_1(x, t)\Pi_1[u] + \Phi_2(x, t)\Pi_2[u] + \cdots + \Phi_m(x, t)\Pi_m[u] = 0. \tag{31.1.2.2}$$

Here, $\Pi_k[u]$ are differential forms that are the products of nonnegative integer powers of the function u and its derivatives u'_z, u''_{zz}, etc., and $\Phi_k(x, t)$ depend on the functions $f(x, t)$, $g(x, t)$, and $z(x, t)$ and their partial derivatives with respect to x and t. Suppose that the differential form $\Pi_1[u]$ contains the highest-order derivative with respect to z. Then the function $\Phi_1(x, t)$ is used as a normalizing factor. This means that the following relations should hold:

$$\Phi_k(x, t) = \Gamma_k(z)\Phi_1(x, t), \qquad k = 1, \ldots, m, \tag{31.1.2.3}$$

where $\Gamma_k(z)$ are functions to be determined.

3°. The representation of a solution in the form (31.1.2.1) has "redundant" generality and the functions f, g, u, and z are ambiguously determined. In order to remove the ambiguity, we use the following three degrees of freedom in the determination of the above functions:

(*a*) if $f = f(x, t)$ has the form $f = f_0(x, t)\Omega(z)$, then we can take $\Omega \equiv 1$, which corresponds to the replacement $u(z) \to u(z)/\Omega(z)$;

(*b*) if $g = g(x, t)$ has the form $g = g_0(x, t) + f(x, t)\Omega(z)$, then we can take $\Omega \equiv 0$, which corresponds to the replacement $u(z) \to u(z) - \Omega(z)$;

(*c*) if $z = z(x, t)$ is determined by an equation of the form $\Omega(z) = h(x, y)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$, which corresponds to the replacement $z \to \Omega^{-1}(z)$.

4°. Having determined the functions $\Gamma_k(z)$, we substitute (31.1.2.3) into (31.1.2.2) to obtain an ordinary differential equation for $u(z)$,

$$\Pi_1[u] + \Gamma_2(z)\Pi_2[u] + \cdots + \Gamma_m(z)\Pi_m[u] = 0. \tag{31.1.2.4}$$

Below we illustrate the main points of the Clarkson–Kruskal direct method by an example.

Example 31.4. We seek a solution of the Boussinesq equation (31.1.1.10) in the form (31.1.2.1). We have

$$af z_x^4 u'''' + a(6f z_x^2 z_{xx} + 4f_x z_x^3)u''' + f^2 z_x^2 uu'' + \cdots = 0. \tag{31.1.2.5}$$

Here, we have written out only the first three terms and have omitted the arguments of the functions f and z. The functional coefficients of u'''' and uu'' should satisfy the condition [see (31.1.2.3)]:

$$f^2 z_x^2 = af z_x^4 \Gamma_3(z),$$

where $\Gamma_3(z)$ is a function to be determined. Hence, using the degree of freedom mentioned in Item $3^\circ(a)$, we choose

$$f = z_x^2, \quad \Gamma_3(z) = 1/a. \tag{31.1.2.6}$$

Similarly, the functional coefficients of u'''' and u''' must satisfy the condition

$$6f z_x^2 z_{xx} + 4f_x z_x^3 = f z_x^4 \Gamma_2(z), \tag{31.1.2.7}$$

where $\Gamma_2(z)$ is another function to be determined. Hence, with (31.1.2.6), we find

$$14\, z_{xx}/z_x = \Gamma_2(z) z_x.$$

Integrating with respect to x yields

$$\ln z_x = I(z) + \ln \widetilde{\varphi}(t), \quad I(z) = \frac{1}{14}\int \Gamma_2(z)\, dz,$$

where $\widetilde{\varphi}(t)$ is an arbitrary function. Integrate again to obtain

$$\int e^{-I(z)}\, dz = \widetilde{\varphi}(t)x + \widetilde{\psi}(t),$$

where $\widetilde{\psi}(t)$ is another arbitrary function. We have a function of z on the left and, therefore, using the degree of freedom mentioned in Item $3^\circ(c)$, we obtain

$$z = x\varphi(t) + \psi(t), \tag{31.1.2.8}$$

where $\varphi(t)$ and $\psi(t)$ are to be determined.

From formulas (31.1.2.6)–(31.1.2.8) it follows that

$$f = \varphi^2(t), \quad \Gamma_2(z) = 0. \tag{31.1.2.9}$$

Substituting (31.1.2.8) and (31.1.2.9) into (31.1.2.1), we obtain a solution of the form (31.1.2.1) with the function f defined by (31.1.2.12). Thus, the general approach based on the representation of a solution in the form (31.1.2.1) ultimately leads us to the same result as the approach based on the more simple formula (31.1.2.1).

Remark 1. In a similar way, it can be shown that formulas (31.1.2.1) and (31.1.2.1) used for the construction of an exact solution of the generalized Burgers–Korteweg–de Vries equation (31.1.2.2) lead us to the same result.

Remark 2. The above examples clearly show that it is more reasonable to perform the initial analysis of specific equations on the basis of the simpler formula (31.1.2.1) rather than the general formula (31.1.2.1).

31.1.3. Description of the Method: the General Form for Symmetry Reduction

The basic idea of the method is the following: for an equation with the unknown function $w = w(x,t)$, an exact solution is sought in the form

$$w(x,t) = F\big(x,t,u(z)\big), \quad z = z(x,t). \tag{31.1.3.1}$$

The functions $F(x,t,u)$ and $z(x,t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $u(z)$. Unlike formulas (31.1.1.1) and (31.1.2.1), the relationship between the functions w and u in (31.1.3.1) can be nonlinear.

Below we illustrate the main features of the Clarkson–Kruskal direct method by examples.

Example 31.5. Consider once again the Boussinesq equation (31.1.1.10). Substituting (31.1.3.1) into (31.1.1.10), we get

$$aF_u z_x^4 u'''' + 4aF_{uu} z_x^4 u' u''' + a(4F_{xu} z_x^3 + 6F_u z_x^2 z_{xx}) u''' + \cdots = 0. \tag{31.1.3.2}$$

Here, we have written out only the first three principal terms and omitted the arguments of the functions F and z. In order to ensure that (31.1.3.2) is reducible to an ordinary differential equation for $u = u(z)$, the ratios of the functional coefficients of $u'u'''$, u''', ... to the coefficient of the highest-order derivative u'''' must be functions of z and u, i.e.,

$$\frac{4aF_{uu} z_x^4}{aF_u z_x^4} = \Gamma_2(z,u), \qquad \frac{a(4F_{xu} z_x^3 + 6F_u z_x^2 z_{xx})}{aF_u z_x^4} = \Gamma_3(z,u), \qquad \ldots .$$

From the first relation we have

$$F_{uu}/F_u = \Gamma_2(z,u).$$

Integrating twice with respect to u yields

$$F(x,t,u) = f(x,t)\Theta(z,u) + g(x,t), \tag{31.1.3.3}$$

where $f(x,t)$ and $g(x,t)$ are arbitrary functions of two arguments, $\Theta = \int \exp\bigl(\int \Gamma_2\, du\bigr)\, du$.

Denoting $\Theta(z,u(z)) = U(z)$ in (31.1.3.3) and using the representation (31.1.3.1), we arrive at a solution which, up to notation, coincides with (31.1.2.1). Therefore, if we seek a similarity reduction of the Boussinesq equation (31.1.1.10) in the general form (31.1.3.1), we are naturally led to the special form (31.1.2.1).

Example 31.6. Consider the Harry Dym equation

$$\frac{\partial w}{\partial t} + 2\frac{\partial^3}{\partial x^3}\frac{1}{\sqrt{w}} = 0. \tag{31.1.3.4}$$

Let us seek a similarity reduction in the form (31.1.3.1). Inserting the expression (31.1.3.1) into (31.1.3.4), we arrive at the relation

$$-F^{-3/2} F_u z_x^3 u''' + \bigl(-3F^{-3/2} F_{uu} + \tfrac{9}{2} F^{-5/2} F_u^2\bigr) z_x^3 u' u'' + \cdots = 0.$$

The ratio of the functional coefficients of $u'u''$ and u''' must be a function of z and u, i.e.,

$$3\frac{F_{uu}}{F_u} - \frac{9}{2}\frac{F_u}{F} = \Gamma(z,u).$$

The double integration yields

$$F^{-1/2}(x,t,u) = f(x,t)\Theta(z,u) + g(x,t), \tag{31.1.3.5}$$

where $f(x,t)$ and $g(x,t)$ are arbitrary functions of two arguments, $\Theta = -\int \exp\bigl(\frac{1}{3}\int \Gamma\, du\bigr)\, du$. From (31.1.3.1) and (31.1.3.5) it follows that one can seek similarity reductions of the Harry Dym equation (31.1.3.4) in the form

$$w^{-1/2}(x,t) = f(x,t)U(z) + g(x,t), \qquad z = z(x,t).$$

⊙ *References for Section* 31.1: P. A. Clarkson and M. D. Kruskal (1989), D. Arrigo, P. Broadbridge, and J. M. Hill (1993), D. Levi and P. Winternitz (1989), P. Olver (1994), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999, 2000), A. D. Polyanin and V. F. Zaitsev (2004), S. N. Aristov and A. D. Polyanin (2009), A. D. Polyanin and S. N. Aristov (2009).

31.2. Some Modifications and Generalizations

31.2.1. Symmetry Reductions Based on the Generalized Separation of Variables

1°. The Clarkson–Kruskal direct method based on the representation of solutions in the forms (31.1.2.1) and (31.1.2.10) attaches particular significance to the function $u = u(z)$, because the choice of the other functions is meant to ensure a *single ordinary differential equation* for $u(z)$. However, in some cases it is reasonable to combine these methods with the ideas of the generalized and functional separation of variables, with all determining functions being regarded as equally important. Then the function $u(z)$ is described by an *overdetermined system of equations*.

2°. Exact solutions of nonlinear partial differential equations with quadratic or power nonlinearities may be sought in the form (31.1.2.1) with $g(x,t) = g_1(t)x + g_0(t)$. Substituting (31.1.2.1) into an equation under consideration, we replace x by $x = [z - \psi(t)]/\varphi(t)$. As a result, we obtain a functional differential equation with two arguments, t and z. Its solution can sometimes be obtained by the differentiation and splitting methods outlined in Sections 29.4 and 29.5.

Example 31.7. Consider the *equation of an axisymmetric steady hydrodynamic boundary layer*

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(y\frac{\partial^2 w}{\partial y^2}\right) + \mathcal{F}(x), \tag{31.2.1.1}$$

where w is the stream function, $y = \frac{1}{4}r^2$, and x and r are axial and radial coordinates.

Its solution is sought in the form

$$w(x,y) = f(x)u(z) + g(x), \qquad z = \varphi(x)y + \psi(x). \tag{31.2.1.2}$$

Let us substitute this expression into equation (31.2.1.1) and eliminate y using the relation $y = [z - \psi(x)]/\varphi(x)$. After the division by $\varphi^2 f$, we arrive at the functional differential equation

$$(zu''_{zz})'_z - \psi u'''_{zzz} + f'_x u u''_{zz} + g'_x u''_{zz} - \frac{(f\varphi)'_x}{\varphi}(u'_z)^2 + \frac{\mathcal{F}}{f\varphi^2} = 0. \tag{31.2.1.3}$$

General methods for solving such equations are outlined in Section 29.5. Here we use a simplified scheme for the construction of exact solutions. Assume that the functional coefficients of uu''_{zz}, u''_{zz}, $(u'_z)^2$, and 1 are linear combinations of the coefficients 1 and ψ of the highest-order terms $(zu''_{zz})'_z$ and u'''_{zzz}, respectively. We have

$$\begin{aligned} f'_x &= A_1 + B_1\psi, \\ g'_x &= A_2 + B_2\psi, \\ -(f\varphi)'_x/\varphi &= A_3 + B_3\psi, \\ \mathcal{F}/(f\varphi^2) &= A_4 + B_4\psi, \end{aligned} \tag{31.2.1.4}$$

where A_k and B_k are arbitrary constants. Let us substitute the expressions of (31.2.1.4) into (31.2.1.3) and sum up the terms proportional to ψ (it is assumed that $\psi \neq \text{const}$). Equating the functional coefficient of ψ to zero, we obtain the following overdetermined system:

$$(zu''_{zz})'_z + A_1 u u''_{zz} + A_2 u''_{zz} + A_3(u'_z)^2 + A_4 = 0, \tag{31.2.1.5}$$

$$-u'''_{zzz} + B_1 u u''_{zz} + B_2 u''_{zz} + B_3(u'_z)^2 + B_4 = 0. \tag{31.2.1.6}$$

Case 1. Let

$$A_1 = A_3 = A_4 = 0, \qquad A_2 = -n. \tag{31.2.1.7}$$

Then the solution of equation (31.2.1.5) has the form

$$u(z) = \frac{C_1}{n(n+1)}z^{n+1} + C_2 z + C_3, \tag{31.2.1.8}$$

where C_1, C_2, and C_3 are integration constants. The solution (31.2.1.8) of equation (31.2.1.5) can be a solution of equation (31.2.1.6) only if the following conditions are satisfied:

$$n = -2, \quad B_1 = B_3, \quad C_1 = -4/B_1, \quad C_2^2 = -B_4/B_1, \quad C_3 = -B_2/B_1. \tag{31.2.1.9}$$

Let us insert the coefficients (31.2.1.7), (31.2.1.9) into system (31.2.1.4). Integrating yields

$$g(x) = 2x - C_3 f, \quad \varphi = \frac{C_4}{f^2}, \quad \psi = -\frac{C_1}{4}f'_x, \quad \mathcal{F} = -(C_2C_4)^2\frac{f'_x}{f^3}, \tag{31.2.1.10}$$

where $f = f(x)$ is an arbitrary function.

Formulas (31.2.1.2), (31.2.1.8), (31.2.1.10) define an exact solution of the axisymmetric boundary layer equation (31.2.1.1).

Case 2. For

$$B_1 = B_3 = B_4 = 0, \quad B_2 = -\lambda, \quad A_2 = 0, \quad A_3 = -A_1, \quad A_4 = \lambda^2/A_1 \tag{31.2.1.11}$$

a common solution of system (31.2.1.5), (31.2.1.6) can be written in the form

$$u(z) = \frac{1}{A_1}(C_1 e^{-\lambda z} + \lambda z - 3). \tag{31.2.1.12}$$

A solution of system (31.2.1.4) with coefficients (31.2.1.11) is described by the formulas

$$f = A_1x + C_2, \quad \varphi = C_3, \quad \psi = -\frac{1}{\lambda}g'_x, \quad \mathcal{F} = \frac{(C_3\lambda)^2}{A_1}(A_1x + C_2), \tag{31.2.1.13}$$

where C_1, C_2, and C_3 are arbitrary constants and $g = g(x)$ is an arbitrary function.

Formulas (31.2.1.2), (31.2.1.12), (31.2.1.13) define an exact solution of the axisymmetric boundary layer equation (31.2.1.1).

Case 3. System (31.2.1.5)–(31.2.1.6) also admits solutions of the form

$$u(z) = C_1z^2 + C_2z + C_3,$$

with constants C_1, C_2, and C_3 related to A_n and B_n. The corresponding solution is easier to obtain directly from the original equation (31.2.1.1) by substituting $w = \varphi_2(x)y^2 + \varphi_1(x)y + \varphi_0(x)$ into it, which corresponds to the method of generalized separation of variables. This results in the solution

$$w(x, y) = C_1y^2 + \varphi(x)y + \frac{1}{4C_1}\varphi^2(x) - \frac{1}{2C_1}\int \mathcal{F}(x)\,dx - x + C_3,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_3 are arbitrary constants.

Example 31.8. Consider the equation with a cubic nonlinearity

$$\frac{\partial w}{\partial t} + \sigma w\frac{\partial w}{\partial x} = a\frac{\partial^2 w}{\partial x^2} + b_3w^3 + b_2w^2 + b_1w + b_0. \tag{31.2.1.14}$$

Let us seek its solution in the form

$$w(x, t) = f(x, t)u(z) + \lambda, \quad z = z(x, t), \tag{31.2.1.15}$$

where the functions $f = f(x, t)$, $z = z(x, t)$, and $u = u(z)$, as well as the constant λ, are to be determined. Substituting (31.2.1.15) into equation (31.2.1.14), we obtain

$$\begin{aligned} &af z_x^2u'' - \sigma f^2z_xuu' + (af z_{xx} + 2af_xz_x - \sigma\lambda f z_x - f z_t)u' + b_3f^3u^3 \\ &\quad + (3b_3\lambda f^2 + b_2f^2 - \sigma f f_x)u^2 + (3b_3\lambda^2 f + 2b_2\lambda f + b_1f + af_{xx} - \sigma\lambda f_x - f_t)u \\ &\quad + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0. \end{aligned} \tag{31.2.1.16}$$

From the overdetermined system of ordinary differential equations resulting from the condition of proportionality of the three functions u'', uu', and u^3 and that of the two functions u' and u^2, it follows that

$$u(z) = 1/z, \tag{31.2.1.17}$$

where the constant factor is taken equal to unity [this factor can be included in f, since formula (31.2.1.15) contains the product of u and f]. Let us substitute (31.2.1.17) into (31.2.1.16) and represent the resulting expression as a finite expansion in negative powers of z. Equating the functional coefficient of z^{-3} to zero, we obtain

$$f = \beta z_x, \tag{31.2.1.18}$$

where β is a root of the quadratic equation

$$b_3\beta^2 + \sigma\beta + 2a = 0. \tag{31.2.1.19}$$

Equating the functional coefficients of the other powers of z to zero and taking into account (31.2.1.18), we find that

$$\begin{aligned} z_t - (3a + \beta\sigma)z_{xx} + (\sigma\lambda + \beta b_2 + 3\beta b_3\lambda)z_x = 0 \qquad &\text{(coefficient of } z^{-2}), \\ z_{xt} - az_{xxx} + \sigma\lambda z_{xx} - (b_1 + 2\lambda b_2 + 3b_3\lambda^2)z_x = 0 \qquad &\text{(coefficient of } z^{-1}), \\ b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0 \qquad &\text{(coefficient of } z^0). \end{aligned} \tag{31.2.1.20}$$

Here, the first two linear partial differential equations form an overdetermined system for the function $z(x, t)$, while the last cubic equation serves for the determination of the constant λ.

Using (31.2.1.15), (31.2.1.17), and (31.2.1.18), we can write out a solution of equation (31.2.1.14) in the form

$$w(x, t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda. \tag{31.2.1.21}$$

Let β be a root of the quadratic equation (31.2.1.19), and λ be a root of the last (cubic) equation in (31.2.1.20). According to the value of the constant b_3, one should consider two cases.

1°. *Case* $b_3 \neq 0$. From the first two equations in (31.2.1.20), one obtains

$$z_t + p_1 z_{xx} + p_2 z_x = 0,$$
$$z_{xxx} + q_1 z_{xx} + q_2 z_x = 0,$$

where

$$p_1 = -\beta\sigma - 3a, \quad p_2 = \lambda\sigma + \beta b_2 + 3\beta\lambda b_3, \quad q_1 = -\frac{\beta b_2 + 3\beta\lambda b_3}{\beta\sigma + 2a}, \quad q_2 = -\frac{3b_3\lambda^2 + 2b_2\lambda + b_1}{\beta\sigma + 2a}.$$

Four situations are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$z(x,t) = C_1 \exp(k_1 x + s_1 t) + C_2 \exp(k_2 x + s_2 t) + C_3,$$
$$k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2 p_1 - k_n p_2,$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1, 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$,

$$z(x,t) = C_1 \exp(kx + s_1 t) + C_2(kx + s_2 t)\exp(kx + s_1 t) + C_3,$$
$$k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1, \quad s_2 = -\tfrac{1}{2}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1.$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$z(x,t) = C_1(x - p_2 t) + C_2 \exp[-q_1 x + q_1(p_2 - p_1 q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$z(x,t) = C_1(x - p_2 t)^2 + C_2(x - p_2 t) - 2C_1 p_1 t + C_3.$$

2°. *Case* $b_3 = 0$, $b_2 \neq 0$. The solutions are determined by (31.2.1.21), where

$$\beta = -\frac{2a}{\sigma}, \quad z(x,t) = C_1 + C_2 \exp\left[Ax + A\left(\frac{b_1\sigma}{2b_2} + \frac{2ab_2}{\sigma}\right)t\right], \quad A = \frac{\sigma(b_1 + 2b_2\lambda)}{2ab_2},$$

and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $b_2\lambda^2 + b_1\lambda + b_0 = 0$.

31.2.2. Similarity Reductions in Equations with Three or More Independent Variables

The procedure of the construction of exact solutions to nonlinear equations with three or more independent variables sometimes involves (at intermediate stages) the solution of functional differential equations considered in Sections 29.4 and 29.5.

Example 31.9. Consider the nonlinear wave equation anisotropic in one of the directions

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[(bw + c)\frac{\partial w}{\partial y}\right]. \tag{31.2.2.1}$$

Let us seek its solution in the form

$$w = U(z) + f(x,t), \quad z = y + g(x,t). \tag{31.2.2.2}$$

Substituting (31.2.2.2) into equation (31.2.2.1), we get

$$[(bU + ag_x^2 - g_t^2 + bf + c)U_z']_z' + (ag_{xx} - g_{tt})U_z' + af_{xx} - f_{tt} = 0.$$

Suppose that the functions f and g satisfy the following overdetermined system of equations:

$$af_{xx} - f_{tt} = C_1, \tag{31.2.2.3}$$
$$ag_{xx} - g_{tt} = C_2, \tag{31.2.2.4}$$
$$ag_x^2 - g_t^2 + bf = C_3, \tag{31.2.2.5}$$

where C_1, C_2, and C_3 are arbitrary constants. Then the function $U(z)$ is determined by the autonomous ordinary differential equation

$$[(bU + c + C_3)U_z']_z' + C_2 U_z' + C_1 = 0. \tag{31.2.2.6}$$

The general solutions of equations (31.2.2.3)–(31.2.2.4) are expressed as

$$f = \varphi_1(\xi) + \psi_1(\eta) - \tfrac{1}{2}C_1t^2,$$
$$g = \varphi_2(\xi) + \psi_2(\eta) - \tfrac{1}{2}C_2t^2,$$
$$\xi = x + t\sqrt{a}, \quad \eta = x - t\sqrt{a}.$$

Let us insert these expressions into equation (31.2.2.5) and then eliminate t with the help of the formula $t = \dfrac{\xi-\eta}{2\sqrt{a}}$. After simple transformations, we obtain a functional differential equation with two arguments,

$$b\varphi_1(\xi)+C_2\xi\varphi_2'(\xi)-k\xi^2-C_3+b\psi_1(\eta)+C_2\eta\psi_2'(\eta)-k\eta^2+\psi_2'(\eta)[4a\varphi_2'(\xi)-C_2\xi]+\eta[2k\xi-C_2\varphi_2'(\xi)]=0, \quad (31.2.2.7)$$

where

$$k = \frac{1}{8a}(bC_1 + 2C_2^2).$$

Equation (31.2.2.7) can be solved by the splitting method described in Section 29.5. According to the simplified scheme, set

$$\begin{aligned} b\varphi_1(\xi) + C_2\xi\varphi_2'(\xi) - k\xi^2 - C_3 &= A_1,\\ 4a\varphi_2'(\xi) - C_2\xi &= A_2,\\ 2k\xi - C_2\varphi_2'(\xi) &= A_3, \end{aligned} \quad (31.2.2.8)$$

where A_1, A_2, and A_3 are constants. The common solution of system (31.2.2.8) has the form

$$\varphi_1(\xi) = -\frac{C_2^2}{8ab}\xi^2 - \frac{BC_2}{b}\xi + \frac{A_1 + C_3}{b}, \quad \varphi_2(\xi) = \frac{C_2}{8a}\xi^2 + B\xi \quad (31.2.2.9)$$

and corresponds to the following values of the constants:

$$\begin{aligned} &A_1 \text{ is arbitrary}, \quad A_2 = 4aB, \quad A_3 = -BC_2, \quad B \text{ is arbitrary},\\ &C_1 = -\frac{C_2^2}{b}, \quad C_2 \text{ and } C_3 \text{ are arbitrary}, \quad k = \frac{C_2^2}{8a}. \end{aligned} \quad (31.2.2.10)$$

From (31.2.2.7) and (31.2.2.8) we obtain an equation that establishes a relation between the functions ψ_1 and ψ_2,

$$A_1 + b\psi_1(\eta) + C_2\eta\psi_2'(\eta) - k\eta^2 + A_2\psi_2'(\eta) + A_3\eta = 0.$$

Hence, taking into account (31.2.2.10), we get

$$\psi_1(\eta) = -\frac{1}{b}(C_2\eta + 4aB)\psi_2'(\eta) + \frac{1}{b}\left(\frac{C_2^2}{8a}\eta^2 + BC_2\eta - A_1\right),$$

$\psi_2(\eta)$ is an arbitrary function.

Ultimately, we find the functions that determine solution (31.2.2.2):

$$f(x,t) = -\frac{C_2^2}{2\sqrt{a}\,b}xt + \frac{C_2^2}{2b}t^2 - \frac{2\sqrt{a}\,BC_2}{b}t + \frac{C_3}{b} - \frac{1}{b}(C_2\eta + 4aB)\psi_2'(\eta),$$
$$g(x,t) = \frac{C_2}{8a}\left(x^2 + 2\sqrt{a}\,xt - 3at^2\right) + B(x + \sqrt{a}\,t) + \psi_2(\eta), \quad \eta = x - t\sqrt{a}.$$

Remark. In the special case of $a = 1$, $b < 0$, and $c > 0$, equation (31.2.2.1) describes spatial transonic flows of an ideal polytropic gas (Pokhozhaev, 1989).

⊙ *References for Section* 31.2: M. C. Nucci and P. A. Clarkson (1992), N. A. Kudryashov (1993), G. I. Burde (1994, 1995), A. D. Polyanin and V. F. Zaitsev (2002, 2004), A. D. Polyanin and A. V. Manzhirov (2007).

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Chapter 32
Classical Method of Symmetry Reductions

Preliminary remarks. The classical method of symmetry reduction of partial differential equations* presents a routine procedure that allows obtaining the following:

(i) transformations under which differential equations are invariant (such transformations bring the given equation to itself);

(ii) new variables (dependent and independent) in which differential equations become considerably simpler.

The transformations of (i) convert a solution of an equation to the same or another solution of this equation. In the former case, we have an *invariant solution*, which can be found by symmetry reduction, rewriting the equation in new, fewer variables. In the latter case, we have noninvariant solutions, which may be "multiplied" to a family of solutions.

Remark 1. The classical method of symmetry analysis of differential equations may be treated as a significant extension of the method outlined in Section 28.4.

Remark 2. Sections 32.1–32.3 give a description of the classical method of symmetry reduction in a nontraditional way, with minimal use of the special (group) terminology, for the reader's easier understanding. Section 32.4 will explain the origin of the term "Lie group analysis."

32.1. One-Parameter Transformations and Their Local Properties

32.1.1. One-Parameter Transformations. Infinitesimal Operator

We will consider invertible transformations of the form

$$\begin{aligned} \bar{x} &= \varphi_1(x,y,w,\varepsilon), \quad \bar{x}|_{\varepsilon=0} = x, \\ \bar{y} &= \varphi_2(x,y,w,\varepsilon), \quad \bar{y}|_{\varepsilon=0} = y, \\ \bar{w} &= \psi(x,y,w,\varepsilon), \quad \bar{w}|_{\varepsilon=0} = w, \end{aligned} \tag{32.1.1.1}$$

where φ_1, φ_2, and ψ are sufficiently smooth functions of their arguments, and ε is a real parameter. It is assumed that the successive application (composition) of two transformations of the form (32.1.1.1) with parameters ε and $\bar{\varepsilon}$ is equivalent to a single transformation of the form with parameter $\varepsilon + \bar{\varepsilon}$ (this means that such transformations have the group property).

Remark. In the special case of transformations in the plane, the functions φ_1 and φ_2 in (32.1.1.1) are independent of w, and $\psi = w$ (i.e., $\bar{w} = w$).

The expansion of (32.1.1.1) into truncated Taylor series in ε about $\varepsilon = 0$ to linear terms gives

$$\bar{x} \simeq x + \varepsilon\xi(x,y,w), \quad \bar{y} \simeq y + \varepsilon\eta(x,y,w), \quad \bar{w} \simeq w + \varepsilon\zeta(x,y,w), \tag{32.1.1.2}$$

where

$$\xi(x,y,w) = \frac{\partial\varphi_1}{\partial\varepsilon}\Big|_{\varepsilon=0}, \quad \eta(x,y,w) = \frac{\partial\varphi_2}{\partial\varepsilon}\Big|_{\varepsilon=0}, \quad \zeta(x,y,w) = \frac{\partial\psi}{\partial\varepsilon}\Big|_{\varepsilon=0}.$$

The linear first-order differential operator

$$\mathrm{X} = \xi(x,y,w)\frac{\partial}{\partial x} + \eta(x,y,w)\frac{\partial}{\partial y} + \zeta(x,y,w)\frac{\partial}{\partial w}, \tag{32.1.1.3}$$

which corresponds to the infinitesimal transformation (32.1.1.2), is called an *infinitesimal operator*.*

* It is also known as the *Lie group analysis of differential equations.*

* In the literature, it is also known as an *infinitesimal generator or a group generator. In this book, we call an operator (32.1.1.3) an infinitesimal operator, a group generator, or, for short, an operator.*

TABLE 32.1
One-parameter transformations in the plane

Name	Transformation	Transformation	Invariant
Translation in the x-axis	$\bar{x} = x + \varepsilon,\ \bar{y} = y$	$\mathrm{X} = \frac{\partial}{\partial x}$	$I_1 = y$
Translation along the straight line $ax + by = 0$	$\bar{x} = x + b\varepsilon,\ \bar{y} = y - a\varepsilon$	$\mathrm{X} = b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}$	$I_1 = ax + by$
Rotation	$\bar{x} = x\cos\varepsilon + y\sin\varepsilon,$ $\bar{y} = y\cos\varepsilon - x\sin\varepsilon$	$\mathrm{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$I_1 = x^2 + y^2$
Galileo transformation	$\bar{x} = x + \varepsilon y,\ \bar{y} = y$	$\mathrm{X} = y\frac{\partial}{\partial x}$	$I_1 = y$
Lorentz transformation	$\bar{x} = x\cosh\varepsilon + y\sinh\varepsilon,$ $\bar{y} = y\cosh\varepsilon + x\sinh\varepsilon$	$\mathrm{X} = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$I_1 = y^2 - x^2$
Uniform extension	$\bar{x} = xe^{\varepsilon},\ \bar{y} = ye^{\varepsilon}$	$\mathrm{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$I_1 = y/x$
Nonuniform extension	$\bar{x} = xe^{a\varepsilon},\ \bar{y} = ye^{b\varepsilon}$	$\mathrm{X} = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$	$I_1 = \lvert y\rvert^a\lvert x\rvert^{-b}$

THEOREM (LIE). *Suppose the coordinates $\xi(x, y, w)$, $\eta(x, y, w)$, $\zeta(x, y, z)$ of the infinitesimal operator (32.1.1.3) are known. Then the transformation (32.1.1.1), having the group property, can be completely recovered by solving the Lie equations*

$$\frac{d\varphi_1}{d\varepsilon} = \xi(\varphi_1, \varphi_2, \psi), \quad \frac{d\varphi_2}{d\varepsilon} = \eta(\varphi_1, \varphi_2, \psi), \quad \frac{d\psi}{d\varepsilon} = \zeta(\varphi_1, \varphi_2, \psi)$$

with the initial conditions

$$\varphi_1\big|_{\varepsilon=0} = x, \quad \varphi_2\big|_{\varepsilon=0} = y, \quad \psi\big|_{\varepsilon=0} = w.$$

32.1.2. Invariant of an Infinitesimal Operator. Transformations in the Plane

An *invariant of the operator* (32.1.1.3) is a function $I(x, y, w)$ that satisfies the condition

$$I(\bar{x}, \bar{y}, \bar{w}) = I(x, y, w).$$

Let us expand this equation in Taylor series in the small parameter ε, divide the resulting relation by ε, and then proceed to the limit as $\varepsilon \to 0$ to obtain a linear partial differential equation for I:

$$\mathrm{X}I = \xi(x, y, w)\frac{\partial I}{\partial x} + \eta(x, y, w)\frac{\partial I}{\partial y} + \zeta(x, y, w)\frac{\partial I}{\partial w} = 0. \tag{32.1.2.1}$$

Let the associated characteristic system of ordinary differential equations (see Subsection 23.1.2)

$$\frac{dx}{\xi(x, y, w)} = \frac{dy}{\eta(x, y, w)} = \frac{dw}{\zeta(x, y, w)} \tag{32.1.2.2}$$

have the functionally independent integrals

$$I_1(x, y, w) = C_1, \quad I_2(x, y, w) = C_2, \tag{32.1.2.3}$$

where C_1 and C_2 are arbitrary constants. Then the general solution of equation (32.1.2.1) has the form

$$I = \Psi(I_1, I_2), \tag{32.1.2.4}$$

where $\Psi(I_1, I_2)$ is an arbitrary function of two arguments, $I_1 = I_1(x, y, w)$ and $I_2 = I_2(x, y, w)$.

This means that the operator (32.1.1.3) has two functionally independent invariants, I_1 and I_2, and any function $\Phi(x, y, w)$ invariant under the operator (32.1.1.3) can be represented as a function of the two invariants.

Table 32.1 lists the most common transformations in the plane and the corresponding operator (32.1.1.3) and invariants; only one invariant is specified, with the other being the same: $I_2 = w$.

32.1.3. Formulas for Derivatives. Coordinates of the First and Second Prolongations

In the new variables (32.1.1.1), the first derivatives become

$$\frac{\partial\bar{w}}{\partial\bar{x}} \simeq \frac{\partial w}{\partial x} + \varepsilon\zeta_1, \quad \frac{\partial\bar{w}}{\partial\bar{y}} \simeq \frac{\partial w}{\partial y} + \varepsilon\zeta_2. \tag{32.1.3.1}$$

Here, ζ_1 and ζ_2 are the coordinates of the first prolongation, which are found as

$$\begin{aligned}
\zeta_1 &= D_x(\zeta) - w_x D_x(\xi) - w_y D_x(\eta)\\
&= \zeta_x + (\zeta_w - \xi_x)w_x - \eta_x w_y - \xi_w w_x^2 - \eta_w w_x w_y,\\
\zeta_2 &= D_y(\zeta) - w_x D_y(\xi) - w_y D_y(\eta)\\
&= \zeta_y - \xi_y w_x + (\zeta_w - \eta_y)w_y - \xi_w w_x w_y - \eta_w w_y^2,
\end{aligned} \tag{32.1.3.2}$$

where D_x and D_y are the total differential operators with respect to x and y:

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{xy}\frac{\partial}{\partial w_y} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + w_y\frac{\partial}{\partial w} + w_{xy}\frac{\partial}{\partial w_x} + w_{yy}\frac{\partial}{\partial w_y} + \cdots.
\end{aligned} \tag{32.1.3.3}$$

Let us verify that the first formula in (32.1.3.1) holds. Obviously,

$$\bar{w}_x = \bar{w}_{\bar{x}}\bar{x}_x + \bar{w}_{\bar{y}}\bar{y}_x, \quad \bar{w}_y = \bar{w}_{\bar{x}}\bar{x}_y + \bar{w}_{\bar{y}}\bar{y}_y. \tag{32.1.3.4}$$

Differentiating (32.1.1.2) with respect to x and y and retaining terms to the first order of ε, we have

$$\begin{aligned}
\bar{x}_x &= 1 + \varepsilon D_x\xi, \qquad \bar{x}_y = \varepsilon D_y\xi,\\
\bar{y}_x &= \varepsilon D_x\eta, \qquad \bar{y}_y = 1 + \varepsilon D_y\eta,\\
\bar{w}_x &= w_x + \varepsilon D_x\zeta, \quad \bar{w}_y = w_y + \varepsilon D_y\zeta.
\end{aligned} \tag{32.1.3.5}$$

In order to calculate $\bar{w}_{\bar{x}}$, let us eliminate $\bar{w}_{\bar{y}}$ from (32.1.3.4) and then substitute the derivatives $\bar{x}_x$, $\bar{x}_y$, $\bar{y}_x$, $\bar{y}_y$, $\bar{w}_x$, $\bar{w}_y$ for their respective expression from (32.1.3.5) to obtain

$$\bar{w}_{\bar{x}} = \frac{w_x + \varepsilon(D_x\zeta + w_x D_y\eta - w_y D_x\eta) + \varepsilon^2(D_x\zeta D_y\eta - D_x\eta D_y\zeta)}{1 + \varepsilon(D_x\xi + D_y\eta) + \varepsilon^2(D_x\xi D_y\eta - D_x\eta D_y\xi)}.$$

Expanding into a series in ε, we have

$$\bar{w}_{\bar{x}} \simeq w_x + \varepsilon\zeta_1, \quad \zeta_1 = D_x\zeta - w_x D_x\xi - w_y D_x\eta,$$

which was to be proved. The coordinate ζ_2 is calculated likewise.

The second derivatives in the new variables (32.1.1.1) are calculated as

$$\frac{\partial^2\bar{w}}{\partial\bar{x}^2} \simeq \frac{\partial^2 w}{\partial x^2} + \varepsilon\zeta_{11}, \quad \frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{y}} \simeq \frac{\partial^2 w}{\partial x\partial y} + \varepsilon\zeta_{12}, \quad \frac{\partial^2\bar{w}}{\partial\bar{y}^2} \simeq \frac{\partial^2 w}{\partial y^2} + \varepsilon\zeta_{22}. \tag{32.1.3.6}$$

Here, the ζ_{ij} are the coordinates of the second prolongation and are found as

$$\begin{aligned}
\zeta_{11} &= D_x(\zeta_1) - w_{xx}D_x(\xi) - w_{xy}D_x(\eta),\\
\zeta_{12} &= D_y(\zeta_1) - w_{xx}D_y(\xi) - w_{xy}D_y(\eta),\\
\zeta_{22} &= D_y(\zeta_2) - w_{xy}D_y(\xi) - w_{yy}D_y(\eta),
\end{aligned}$$

or, in detailed form,

$$\begin{aligned}
\zeta_{11} &= \zeta_{xx} + (2\zeta_{wx} - \xi_{xx})w_x - \eta_{xx}w_y + (\zeta_{ww} - 2\xi_{wx})w_x^2 - 2\eta_{wx}w_x w_y -\\
&\quad - \xi_{ww}w_x^3 - \eta_{ww}w_x^2 w_y + (\zeta_w - 2\xi_x - 3\xi_w w_x - \eta_w w_y)w_{xx} - 2(\eta_x + \eta_w w_x)w_{xy},\\
\zeta_{12} &= \zeta_{xy} + (\zeta_{wy} - \xi_{xy})w_x + (\zeta_{wx} - \eta_{xy})w_y - \xi_{wy}w_x^2 -\\
&\quad - (\zeta_{ww} - \xi_{wx} - \eta_{wy})w_x w_y - \eta_{wx}w_y^2 - \xi_{ww}w_x^2 w_y - \eta_{ww}w_x w_y^2 -\\
&\quad - (\xi_y + \xi_w w_y)w_{xx} + (\zeta_w - \xi_x - \eta_y - 2\xi_w w_x - 2\eta_w w_y)w_{xy} - (\eta_x + \eta_w w_x)w_{yy},\\
\zeta_{22} &= \zeta_{yy} - \xi_{yy}w_x + (2\zeta_{wy} - \eta_{yy})w_y - 2\xi_{wy}w_x w_y + (\zeta_{ww} - 2\eta_{wy})w_y^2 -\\
&\quad - \xi_{ww}w_x w_y^2 - \eta_{ww}w_y^3 - 2(\xi_y + \xi_w w_y)w_{xy} + (\zeta_w - 2\eta_y - \xi_w w_x - 3\eta_w w_y)w_{yy}.
\end{aligned} \tag{32.1.3.7}$$

The above formulas for the coordinates of the first and second prolongation, (32.1.3.2) and (32.1.3.7), will be required later for the analysis of differential equations.

⊙ *References for Section* 32.1: L. V. Ovsiannikov (1962, 1982), N. H. Ibragimov (1985, 1994), P. J. Olver (1986, 1995).

32.2. Symmetries of Nonlinear Second-Order Equations. Invariance Condition

32.2.1. Invariance Condition. Splitting in Derivatives

We will consider second-order partial differential equations in two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{32.2.1.1}$$

The procedure for finding symmetries* of equation (32.2.1.1) consists of several stages. At the first stage, one requires that equation (32.2.1.1) must be invariant (preserve its form) under transformations (32.1.1.1), so that

$$F\left(\bar{x}, \bar{y}, \bar{w}, \frac{\partial \bar{w}}{\partial \bar{x}}, \frac{\partial \bar{w}}{\partial \bar{y}}, \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}, \frac{\partial^2 \bar{w}}{\partial \bar{x} \partial \bar{y}}, \frac{\partial^2 \bar{w}}{\partial \bar{y}^2}\right) = 0. \tag{32.2.1.2}$$

Let us expand this equation into a series in ε about $\varepsilon = 0$, taking into account that the leading term vanishes, according to (32.2.1.1). Using formulas (32.1.1.2), (32.1.3.1), (32.1.3.6) and retaining the terms to the first-order of ε, we obtain

$$\underset{2}{\mathrm{X}} F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{F=0} = 0, \tag{32.2.1.3}$$

where

$$\underset{2}{\mathrm{X}} F = \xi \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} + \zeta \frac{\partial F}{\partial w} + \zeta_1 \frac{\partial F}{\partial w_x} + \zeta_2 \frac{\partial F}{\partial w_y} + \zeta_{11} \frac{\partial F}{\partial w_{xx}} + \zeta_{12} \frac{\partial F}{\partial w_{xy}} + \zeta_{22} \frac{\partial F}{\partial w_{yy}}. \tag{32.2.1.4}$$

The coordinates of the first, ζ_i, and the second, ζ_{ij}, prolongation are defined by formulas (32.1.3.2) and (32.1.3.7). Relation (32.2.1.3) is called the *invariance condition*, and the operator $\underset{2}{\mathrm{X}}$ is called the *second prolongation of the operator* (32.1.1.3).

At the second stage, the derivative $\frac{\partial^2 w}{\partial y^2}$ is eliminated from (32.2.1.3) using equation (32.2.1.1); instead of $\frac{\partial^2 w}{\partial y^2}$, one can eliminate either of the first derivatives or another second derivative. The resulting relation is then written as a polynomial in the "independent variables," the various combinations of the products of the derivatives (involving various powers of w_x, w_y, w_{xx}, and w_{xy}):

$$\sum A_{k_1 k_2 k_3 k_4} (w_x)^{k_1} (w_y)^{k_2} (w_{xx})^{k_3} (w_{xy})^{k_4} = 0, \tag{32.2.1.5}$$

where the functional coefficients $A_{k_1 k_2 k_3 k_4}$ are dependent on x, y, w, ξ, η, ζ and the derivatives of ξ, η, ζ only and are independent of the derivatives of w. Equation (32.2.1.5) is satisfied if all $A_{k_1 k_2 k_3 k_4}$ are zero. Thus, the invariance condition is split to an overdetermined determining system, resulting from equating all functional coefficients of the various products of the remaining derivatives to zero (recall that the unknowns ξ, η, and ζ are independent of w_x, w_y, w_{xx}, and w_{xy}).

At the third stage, one solves the determining system and finds admissible coordinates ξ, η, and ζ of the infinitesimal operator (32.1.1.3).

Remark 1. It should be noted that the functional coefficients $A_{k_1 k_2 k_3 k_4}$ and the determining system are linear in the unknowns ξ, η, and ζ.

Remark 2. An invariant I that is a solution of equation (32.1.2.1) is also a solution of the equation $\underset{2}{\mathrm{X}} I = 0$.

The procedure for finding symmetries of differential equations is illustrated below by specific examples.

32.2.2. Examples of Finding Symmetries of Nonlinear Equations

Example 32.1. Consider the two-dimensional stationary heat equation with a nonlinear source

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(w). \tag{32.2.2.1}$$

The corresponding left-hand side of equation (32.2.1.1) is $F = w_{xx} + w_{yy} - f(w)$.

* A symmetry of an equation is a transformation that preserves its form.

Infinitesimal operators X admitted by the equations are sought in the form (32.1.2.1), where the coordinates $\xi=\xi(x,y,w)$, $\eta=\eta(x,y,w)$, $\zeta=\zeta(x,y,w)$ are yet unknown and are to be determined in the subsequent analysis. In view of identity $F = w_{xx} + w_{yy} - f(w)$, the invariance condition (32.2.1.3)–(32.2.1.4) is written as

$$\big[\zeta_{22} + \zeta_{11} - \zeta f'(w)\big]_{F=0} = 0.$$

Substituting here the expressions of the coordinates of the second prolongation (32.1.3.7) and then replacing w_{yy} by $f(w) - w_{xx}$, which follows from equation (32.2.2.1), we obtain

$$\begin{aligned}
&-2\xi_w w_x w_{xx} + 2\eta_w w_y w_{xx} - 2\eta_w w_x w_{xy} - 2\xi_w w_y w_{xy} - 2(\xi_x - \eta_y)w_{xx} - 2(\xi_y + \eta_x)w_{xy} -\\
&-\xi_{ww} w_x^3 - \eta_{ww} w_x^2 w_y - \xi_{ww} w_x w_y^2 - \eta_{ww} w_y^3 + (\zeta_{ww} - 2\xi_{xw})w_x^2 - 2(\xi_{yw} + \eta_{xw})w_x w_y +\\
&+(\zeta_{ww} - 2\eta_{yw})w_y^2 + (2\zeta_{xw} - \xi_{xx} - \xi_{yy} - f\xi_w)w_x + (2\zeta_{yw} - \eta_{xx} - \eta_{yy} - 3f\eta_w)w_y +\\
&+\zeta_{xx} + \zeta_{yy} + f(\zeta_w - 2\eta_y) - \zeta f' = 0,
\end{aligned}$$

where $f = f(w)$ and $f' = df/dw$. Equating the coefficients of all combinations of the derivatives to zero, we have the system

$$\begin{aligned}
w_x w_{xx}&: \quad \xi_w = 0,\\
w_y w_{xx}&: \quad \eta_w = 0,\\
w_{xx}&: \quad \xi_x - \eta_y = 0,\\
w_{xy}&: \quad \xi_y + \eta_x = 0,\\
w_x^2&: \quad \zeta_{ww} - 2\xi_{wx} = 0,\\
w_x w_y&: \quad \eta_{wx} + \xi_{wy} = 0,\\
w_x&: \quad 2\zeta_{wx} - \xi_{xx} - \xi_{yy} - \xi_w f(w) = 0,\\
w_y^2&: \quad \zeta_{ww} - 2\eta_{wy} = 0,\\
w_y&: \quad 2\zeta_{wy} - \eta_{xx} - \eta_{yy} - 3\eta_w f(w) = 0,\\
1&: \quad \zeta_{xx} + \zeta_{yy} - f'(w)\zeta + f(w)(\zeta_w - 2\eta_y) = 0.
\end{aligned} \tag{32.2.2.2}$$

Here, the left column indicates a combination of the derivatives and the right column gives the associated coefficient. The coefficients of $w_y w_{xy}$, $w_x w_{xy}$, w_x^3, $w_x^2 w_y$, $w_x w_y^2$, w_y^3 either are among those already listed or are their differential consequences, and therefore they are omitted. It follows from the first, second, and fifth equations and their consequences that

$$\xi = \xi(x,y), \quad \eta = \eta(x,y), \quad \zeta = a(x,y)w + b(x,y). \tag{32.2.2.3}$$

The third and fourth equations of system (32.2.2.2) give

$$\xi_{xx} + \xi_{yy} = 0, \quad \eta_{xx} + \eta_{yy} = 0. \tag{32.2.2.4}$$

Substituting (32.2.2.3) into the seventh and ninth equations of (32.2.2.2) and using (32.2.2.4), we find that $a_x = a_y = 0$, whence

$$a(x,y) = a = \text{const}. \tag{32.2.2.5}$$

In view of (32.2.2.3) and (32.2.2.5), system (32.2.2.2) becomes

$$\begin{aligned}
&\xi_x - \eta_y = 0,\\
&\xi_y + \eta_x = 0,\\
&b_{xx} + b_{yy} - aw f'(w) - bf'(w) + f(w)(a - 2\eta_y) = 0.
\end{aligned} \tag{32.2.2.6}$$

For arbitrary f, it follows that $a = b = \eta_y = 0$, and then $\xi = C_1 y + C_2$, $\eta = -C_1 x + C_3$, and $\zeta = 0$. By setting one of the constants to unity and the others to zero, we establish that the original equation admits three different operators:

$$\begin{aligned}
&\mathrm{X}_1 = \partial_x && (C_2 = 1,\ C_1 = C_3 = 0);\\
&\mathrm{X}_2 = \partial_y && (C_3 = 1,\ C_1 = C_2 = 0);\\
&\mathrm{X}_3 = y\partial_x - x\partial_y && (C_1 = 1,\ C_2 = C_3 = 0).
\end{aligned} \tag{32.2.2.7}$$

The first two operators define a translation along the x- and y-axis. The third operator represents a rotational symmetry.

Let us dwell on the third equation of system (32.2.2.6). If the relation

$$(aw + b)f'(w) - f(w)(a - 2\eta_y) = 0 \tag{32.2.2.8}$$

holds, there may exist other solutions of system (32.2.2.6) that will result in operators other than (32.2.2.7). Let us study two cases: $a \neq 0$ and $a = 0$.

Case 1. If $a \neq 0$, the solution of equation (32.2.2.8) gives

$$f(w) = C(aw + b)^{1-\frac{2\gamma}{a}},$$

where $\gamma = \eta_y = \text{const}$ and $b = \text{const}$. Therefore, for $f(w) = w^k$, equation (32.2.2.1) admits another operator,

$$\mathrm{X}_4 = x\partial_x + y\partial_y + \frac{2}{1-k}w\partial_w,$$

that defines a nonuniform extension.

Case 2. If $a = 0$, we have

$$f(w) = Ce^{\lambda w},$$

where $\lambda = \text{const}$. Then $b = -2\eta_y/\lambda$, and the functions ξ and η satisfy the first two equations (32.2.2.6), which coincide with the Cauchy–Riemann conditions for analytic functions. The real and the imaginary part of any analytic function, $\Phi(z) = \xi(x, y) + i\eta(x, y)$, of the complex variable $z = x + iy$ satisfies the Cauchy–Riemann conditions. In particular, if $b = \text{const}$ and $f(w) = e^w$, the equation admits another operator,

$$\mathrm{X}_4 = x\partial_x + y\partial_y - 2\partial_w,$$

which corresponds to extension in x and y with simultaneous translation in w.

Example 32.2. Consider the nonlinear unsteady heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\Big[f(w)\frac{\partial w}{\partial x}\Big]. \tag{32.2.2.9}$$

In the invariance condition (32.2.1.3)–(32.2.1.4), one should set

$$y = t, \quad F = w_t - f(w)w_{xx} - f'(w)w_x^2 \qquad (\zeta_{12} = \zeta_{22} = 0).$$

The coordinates of the first and second prolongations, ζ_1, ζ_2, and ζ_{11}, are expressed by (32.1.3.2) and (32.1.3.7) with $y = t$. In the resulting equation, one should replace w_t with the right-hand side of equation (32.2.2.9) and equate the coefficients of the various combinations of the remaining derivatives to zero, thus arriving at the system of equations

$$\begin{aligned}
w_x w_{xx}:&\quad 2f(w)[\eta_{wx}f(w) + \xi_w] + f'(w)\eta_x = 0,\\
w_{xx}:&\quad \zeta f'(w) - f^2(w)\eta_{xx} - f(w)(2\xi_x - \eta_t) = 0,\\
w_x w_{xt}:&\quad f(w)\eta_w = 0,\\
w_{xt}:&\quad f(w)\eta_x = 0,\\
w_x^4:&\quad f'(w)\eta_w + f(w)\eta_{ww} = 0,\\
w_x^3:&\quad 2[f'(w)]^2\eta_x + f(w)\xi_{ww} + f'(w)\xi_w + 2f(w)f'(w)\eta_{wx} = 0,\\
w_x^2:&\quad f(w)\zeta_{ww} + f''(w)\zeta - 2f(w)\xi_{wx} - f'(w)(2\xi_x - \eta_t) + f'(w)\zeta_w - f(w)f'(w)\eta_{ww} = 0,\\
w_x:&\quad 2f(w)\zeta_{wx} + 2f'(w)\zeta_x - f(w)\xi_{xx} + \xi_t = 0,\\
1:&\quad \zeta_t - f(w)\zeta_{xx} = 0.
\end{aligned}$$

Here, the left column indicates a combination of the derivatives, and the right column gives the associated equation (to a constant factor); identities and differential consequences are omitted. Since $f(w) \not\equiv 0$, it follows from the third and fourth equations of the system that $\eta = \eta(t)$. Then from the first and second equations, we have

$$\xi = \xi(x, t), \quad \zeta = \frac{f(w)(2\xi_x - \eta_t)}{f'(w)}.$$

With these relations, the remaining equations of the system become

$$\begin{aligned}
&[ff'f''' - f(f'')^2 + (f')^2 f''](2\xi_x - \eta_t) = 0,\\
&f[4ff'' - 7(f')^2]\xi_{xx} - (f')^2\xi_t = 0,\\
&2f\xi_{xxx} - 2\xi_{xt} + \eta_{tt} = 0;
\end{aligned}$$

the equations have been canceled by nonzero factors. In the general case, for arbitrary function f, from the first equation it follows that $2\xi_x - \eta_t = 0$ and from the second it follows that $\xi_t = 0$. The third equation gives $\xi = C_1 + C_2 x$, and then $\eta = 2C_2 t + C_3$. Therefore, for arbitrary f, equation (32.2.2.9) admits three operators

$$\begin{aligned} &\mathrm{X}_1 = \partial_x && (C_1 = 1,\ C_2 = C_3 = 0);\\ &\mathrm{X}_2 = \partial_t && (C_3 = 1,\ C_1 = C_2 = 0);\\ &\mathrm{X}_3 = 2t\partial_t + x\partial_x && (C_2 = 1,\ C_1 = C_3 = 0). \end{aligned}$$

Similarly, it can be shown that the following special forms of f result in additional operators:

1. $f = e^w$: $\quad \mathrm{X}_4 = x\partial_x + 2\partial_w$;
2. $f = w^k$, $k \neq 0, -4/3$: $\quad \mathrm{X}_4 = kx\partial_x + 2w\partial_w$;
3. $f = w^{-4/3}$: $\quad \mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\quad \mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$.

Example 32.3. Consider now the *nonlinear wave equation*

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\Big[f(w)\frac{\partial w}{\partial x}\Big]. \tag{32.2.2.10}$$

In the invariance condition (32.2.1.3)–(32.2.1.4), one should set

$$y = t, \quad F = w_{tt} - f(w)w_{xx} - f'(w)w_x^2 \qquad (\zeta_2 = \zeta_{12} = 0),$$

and use the coordinates of the first and second prolongations, ζ_1, ζ_{11}, and ζ_{22}, expressed by (32.1.3.2) and (32.1.3.7) with $y = t$. In the resulting equation, one should replace w_{tt} with the right-hand side of equation (32.2.2.10) and equate the coefficients of the various combinations of the remaining derivatives to zero, thus arriving at the system of equations

$$\begin{aligned} w_x w_{xx}&: \quad f(w)\xi_w = 0,\\ w_t w_{xx}&: \quad f(w)\eta_w = 0,\\ w_{xx}&: \quad f'(w)\zeta + 2f(w)(\eta_t - \xi_x) = 0,\\ w_{xt}&: \quad f(w)\eta_x - \xi_t = 0,\\ w_x^3&: \quad f'(w)\xi_w + f(w)\xi_{ww} = 0,\\ w_x^2 w_t&: \quad f(w)\eta_{ww} - f'(w)\eta_w = 0,\\ w_x^2&: \quad f(w)\zeta_{ww} + f'(w)\zeta_w + f''(w)\zeta - 2f(w)\xi_{wx} - 2f'(w)(\xi_x - \eta_t) = 0,\\ w_x w_t&: \quad 2f'(w)\eta_x + 2f(w)\eta_{wx} - 2\xi_{wt} = 0,\\ w_x&: \quad 2f'(w)\zeta_x - f(w)\xi_{xx} + 2f(w)\zeta_{wx} + \xi_{tt} = 0,\\ w_t^2&: \quad \zeta_{ww} - 2\eta_{wt} = 0,\\ w_t&: \quad f(w)\eta_{xx} + 2\zeta_{wt} - \eta_{tt} = 0,\\ 1&: \quad \zeta_{tt} - f(w)\zeta_{xx} = 0. \end{aligned}$$

Identities and differential consequences have been omitted. Since $f(w) \neq \text{const}$, it follows from the first two equations that $\xi = \xi(x,t)$ and $\eta = \eta(x,t)$. The tenth equation of the system becomes $\zeta_{ww} = 0$, thus giving $\zeta = a(x,t)w + b(x,t)$. As a result, the system becomes

$$\begin{aligned} &wf'(w)a(x,y) + f'(w)b(x,y) + 2f(w)(\eta_t - \xi_x) = 0,\\ &f'(w)a(x,y) + wf''(w)a(x,y) + f''(w)b(x,y) - 2f'(w)(\xi_x - \eta_t) = 0,\\ &2f'(w)(a_x w + b_x) - f(w)\xi_{xx} + 2f(w)a_x = 0,\\ &2a_t - \eta_{tt} = 0,\\ &a_{tt}w + b_{tt} - f(w)(a_{xx}w + b_{xx}) = 0. \end{aligned}$$

For arbitrary function $f(w)$, we have $a = b = 0$, $\xi_{xx} = 0$, $\eta_{tt} = 0$, and $\xi_x - \eta_t = 0$. Integrating yields three operators:

$$\mathrm{X}_1 = \partial_x, \quad \mathrm{X}_2 = \partial_t, \quad \mathrm{X}_3 = x\partial_x + t\partial_t.$$

Similarly, it can be established that the following special forms of f result in additional operators:

1. $f = e^w$: $\quad \mathrm{X}_4 = x\partial_x + 2\partial_w$;
2. $f = w^k$, $k \neq 0, -4/3, -4$: $\quad \mathrm{X}_4 = kx\partial_x + 2w\partial_w$;
3. $f = w^{-4/3}$: $\quad \mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\quad \mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$;
4. $f = w^{-4}$: $\quad \mathrm{X}_4 = 2x\partial_x - w\partial_w$, $\quad \mathrm{X}_5 = t^2\partial_t + tw\partial_w$.

The symmetries obtained with the procedure presented can be used to find exact solutions of the differential equations considered (see below).

⊙ *References for Section* 32.2: L. V. Ovsiannikov (1962, 1982), G. W. Bluman and J. D. Cole (1974), N. H. Ibragimov (1985, 1994), P. J. Olver (1986, 1995), P. P. Kiryakov, S. I. Senashov, and A. N. Yakhno (2001), D. M. Klimov and V. F. Zhuravlev (2002).

32.3. Using Symmetries of Equations for Finding Exact Solutions. Invariant Solutions

32.3.1. Using Symmetries of Equations for Constructing One-Parameter Solutions

Suppose a particular solution,

$$w = g(x, y), \tag{32.3.1.1}$$

of a given equation is known. Let us show that any symmetry of the equation defined by a transformation of the form (32.1.1.1) generates a one-parameter family of solutions (except for the cases where the solution is not mapped into itself by the transformations; see Subsection 32.3.2).

Indeed, since equation (32.2.1.1) converted to the new variables (32.1.1.1) acquires the same form (32.2.1.2), then the transformed equation (32.2.1.2) has a solution

$$\bar{w} = g(\bar{x}, \bar{y}). \tag{32.3.1.2}$$

In (32.3.1.2), going back to the old variables by formulas (32.1.1.1), we obtain a one-parameter solution of the original equation (32.2.1.1).

Example 32.4. The two-dimensional heat equation with an exponential source

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^w \tag{32.3.1.3}$$

has a one-dimensional solution

$$w = \ln \frac{2}{x^2}. \tag{32.3.1.4}$$

Equation (32.3.1.3) admits the operator $X_3 = y\partial_x - x\partial_y$ (see Example 32.1 in Subsection 32.2.2), which defines rotation in the plane. The corresponding transformation is given in Table 32.1. Replacing x in (32.3.1.4) by $\bar{x}$ (from Table 32.1), we obtain a one-parameter solution of equation (32.3.1.3):

$$w = \ln \frac{2}{(x\cos\varepsilon + y\sin\varepsilon)^2},$$

where ε is a free parameter.

32.3.2. Procedure for Constructing Invariant Solutions

Solution (32.3.1.1) of equation (32.2.1.1) is called *invariant* under transformations (32.1.1.1) if it coincides with solution (32.3.1.2), which must be rewritten in terms of the old variables using formulas (32.1.1.1). This means that an invariant solution is converted to itself under the given transformation. The basic stages of constructing invariant solutions are outlined below.

First, one finds a one-parameter transformation with operator (32.1.1.3) whose coordinates are determined from the invariance condition (32.2.1.3) following the procedure described in Subsection 32.2.1. Then, one looks for two functionally independent integrals (32.1.2.3) of the characteristic system of ordinary differential equations (32.1.2.2). The invariant solution of the partial differential equation (32.1.2.1) is sought in the form

$$I_2 = \Phi(I_1), \tag{32.3.2.1}$$

where the functions $I_1 = I_1(x, y, w)$ and $I_2 = I_2(x, y, w)$ are known,* and the function Φ is to be determined. Relation (32.3.2.1) is the basis for the construction of invariant solutions: solving (32.3.2.1) for w and

* Usually, the invariant that is independent of w is taken to be I_1.

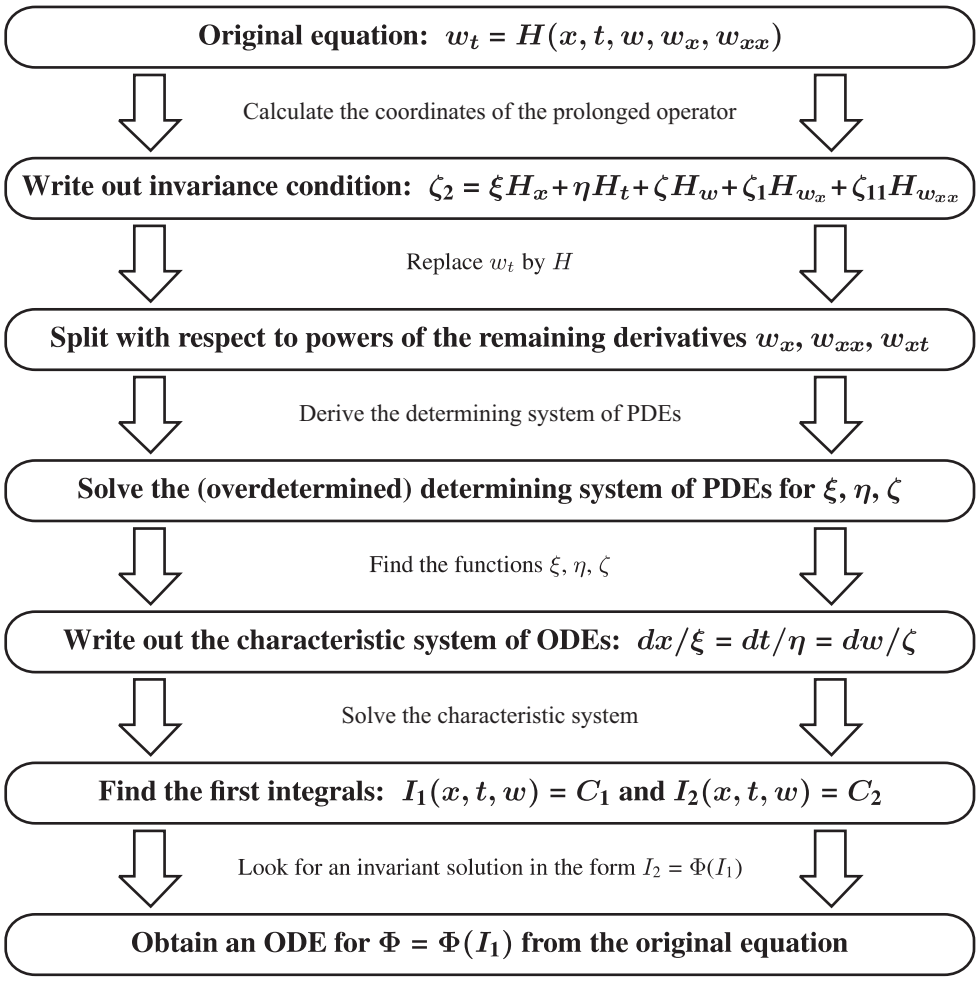

Figure 32.1. An algorithm for constructing invariant solutions for evolution second-order equations. Notation: ODE stands for ordinary differential equation and PDE stands for partial differential equation; $\xi = \xi(x, t, w)$, $\eta = \eta(x, t, w)$, $\zeta = \zeta(x, t, w)$; ζ_1, ζ_2, and ζ_{11} are the coordinates of the prolonged operator, which are defined by formulas (32.1.3.2) and (32.1.3.7) with $y = t$.

substituting the resulting expression into (32.2.1.1), we arrive at an ordinary differential equation for Φ.

Example 32.5. A well-known and very important special case of invariant solutions is the self-similar solutions (see Sections 27.3 and 28.4); they are based on invariants of scaling groups. The corresponding infinitesimal operator and its invariants are

$$X = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cw\frac{\partial}{\partial w}; \qquad I_1 = |y|^a|x|^{-b}, \quad I_2 = |w|^a|x|^{-c}.$$

Substituting the invariants into (32.3.2.1) gives $|w|^a|x|^{-c} = \Phi\big(|y|^a|x|^{-b}\big)$. On solving this equation for w, we obtain the form of the desired solution, $w = |x|^{c/a}\Psi\big(y|x|^{-b/a}\big)$, where $\Psi(z)$ is an unknown function.

To make it clearer, the general scheme for constructing invariant solutions for evolution second-order equations is depicted in Fig. 32.1. The first-order partial differential equation (32.1.2.1) for finding group invariants is omitted, since the corresponding characteristic system of ordinary differential equations (32.1.2.2) can be immediately used.

32.3.3. Examples of Constructing Invariant Solutions to Nonlinear Equations

Example 32.6. Consider once again the stationary heat equation with nonlinear source

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(w).$$

1°. Let us dwell on the case $f(w)=w^k$, where the equation admits an additional operator (see Example 32.1 from Subsection 32.2.2):

$$\mathrm{X}_4=x\partial_x+y\partial_y+\frac{2}{1-k}w\partial_w.$$

In order to find invariants of this operator, we have to consider the linear first-order partial differential equation $\mathrm{X}_4 I=0$, or, in detailed form,

$$x\frac{\partial I}{\partial x}+y\frac{\partial I}{\partial y}+\frac{2}{1-k}w\frac{\partial I}{\partial w}=0.$$

The corresponding characteristic system of ordinary differential equations,

$$\frac{dx}{x}=\frac{dy}{y}=\frac{1-k}{2}\frac{dw}{w},$$

has the first integrals

$$y/x=C_1,\qquad x^{2/(k-1)}w=C_2,$$

where C_1, C_2 are arbitrary constants. Therefore, the functions $I_1=y/x$ and $I_2=x^{2/(k-1)}w$ are invariants of the operator X_4.

Assuming that $I_2=\Phi(I_1)$ and expressing w, we find the form of the invariant (self-similar) solution:

$$w=x^{-2/(k-1)}\Phi(y/x). \tag{32.3.3.1}$$

Substituting (32.3.3.1) into the original equation (32.2.2.1) yields a second-order ordinary differential equation for $\Phi(z)$:

$$(k-1)^2(z^2+1)\Phi''_{zz}+2(k^2-1)z\Phi'_z+2(k+1)\Phi-(k-1)^2\Phi^k=0,$$

where $z=y/x$.

2°. The functions $u=x^2+y^2$ and w are invariants of the operator X_3 for the nonlinear heat equation concerned. The substitutions $w=w(u)$ and $u=x^2+y^2$ lead to an ordinary differential equation describing solutions of the original equation which are invariant under rotation:

$$uw''_{uu}+w'_u=\tfrac{1}{4}f(w).$$

Remark. In applications, the polar radius $r=\sqrt{x^2+y^2}$ is normally used as an invariant instead of $u=x^2+y^2$.

Example 32.7. Consider the nonlinear unsteady heat equation (32.2.2.9).

1°. For arbitrary $f(w)$, the equation admits the operator (see Example 32.2 from Subsection 32.2.2)

$$\mathrm{X}_3=2t\partial_t+x\partial_x.$$

Invariants of X_3 are found for the linear first-order partial differential equation $\mathrm{X}_3 I=0$, or

$$2t\frac{\partial I}{\partial t}+x\frac{\partial I}{\partial x}+0\frac{\partial I}{\partial w}=0.$$

The associated characteristic system of ordinary differential equations,

$$\frac{dx}{x}=\frac{dt}{2t}=\frac{dw}{0},$$

has the first integrals

$$xt^{-1/2}=C_1,\qquad w=C_2,$$

where C_1 and C_2 are arbitrary constants. Therefore, the functions $I_1=xt^{-1/2}$ and $I_2=w$ are invariants of the operator X_3.

Assuming $I_2=\Phi(I_1)$, we get

$$w=\Phi(z),\qquad z=xt^{-1/2}, \tag{32.3.3.2}$$

where $\Phi(z)$ is to be determined in the subsequent analysis. Substituting (32.3.3.2) in the original equation (32.2.2.9) yields the second-order ordinary differential equation

$$2[f(\Phi)\Phi'_z]'_z+z\Phi'_z=0,$$

which describes an invariant (self-similar) solution.

TABLE 32.2
Operators, invariants, and solution structures admitted by the nonlinear unsteady heat equation (32.2.2.9)

Function $f(w)$	Operators	Invariants	Solution structure
Arbitrary	$\mathrm{X}_1 = \partial_x$, $\mathrm{X}_2 = \partial_t$, $\mathrm{X}_3 = 2t\partial_t + x\partial_x$	$I_1 = t$, $I_2 = w$, $I_1 = x$, $I_2 = w$, $I_1 = x^2/t$, $I_2 = w$	$w = w(t) = \text{const}$, $w = w(x)$, $w = w(z)$, $z = x^2/t$
e^w	$\mathrm{X}_4 = x\partial_x + 2\partial_w$	$I_1 = t$, $I_2 = w - 2\ln\lvert x\rvert$	$w = 2\ln\lvert x\rvert + \theta(t)$
w^k $(k \neq 0, -\frac{4}{3})$	$\mathrm{X}_4 = kx\partial_x + 2w\partial_w$	$I_1 = t$, $I_2 = w\lvert x\rvert^{-k/2}$	$w = \lvert x\rvert^{k/2}\theta(t)$
$w^{-4/3}$	$\mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$	$I_1 = t$, $I_2 = wx^{2/3}$, $I_1 = t$, $I_2 = wx^3$	$w = x^{-2/3}\theta(t)$, $w = x^{-3}\theta(t)$

2°. Let us dwell on the case $f(w) = w^k$, where the equation admits the operator

$$\mathrm{X}_4 = kx\partial_x + 2w\partial_w.$$

The invariants are described by the first-order partial differential equation $\mathrm{X}_4 I = 0$, or

$$0\frac{\partial I}{\partial t} + kx\frac{\partial I}{\partial x} + 2w\frac{\partial I}{\partial w} = 0.$$

The associated characteristic system of ordinary differential equations,

$$\frac{dt}{0} = \frac{dx}{kx} = \frac{dw}{2w},$$

has the first integrals

$$t = C_1, \quad x^{-2/k}w = C_2,$$

where C_1, C_2 are arbitrary constants. Therefore, $I_1 = t$ and $I_2 = x^{-2/k}w$ are invariants of the operator X_4.

Assuming $I_2 = \theta(I_1)$ and expressing w, we get

$$w = x^{2/k}\theta(t), \tag{32.3.3.3}$$

where $\theta(t)$ is to be determined in the subsequent analysis. Substituting (32.3.3.3) in the original equation (32.2.2.9) with $f(w) = w^k$ gives the first-order ordinary differential equation

$$k^2\theta'_t = 2(k+2)\theta^{k+1}.$$

Integrating yields

$$\theta(t) = \left[A - \frac{2(k+2)}{k}t\right]^{-1/k},$$

where A is an arbitrary constant. Hence, the solution of equation (32.2.2.9) with $f(w) = w^k$, which is invariant under scaling, has the form

$$w(x,t) = x^{2/k}\left[A - \frac{2(k+2)}{k}t\right]^{-1/k}.$$

Table 32.2 summarizes the symmetries of equation (32.2.2.9) (see Example 32.2 from Subsection 32.2.2 and Example 32.7 from Subsection 32.3.3).

Example 32.8. Consider the nonlinear wave equation (32.2.2.10). For arbitrary $f(w)$, this equation admits the following operator (see Example 32.3 from Subsection 32.2.2):

$$\mathrm{X}_3 = t\partial_t + x\partial_x.$$

The invariants are found from the linear first-order partial differential equation $\mathrm{X}_3 I_1 = 0$, or

$$t\frac{\partial I}{\partial t} + x\frac{\partial I}{\partial x} + 0\frac{\partial I}{\partial w} = 0.$$

TABLE 32.3
Operators, invariants, and solution structures admitted by the nonlinear wave equation (32.2.2.10)

Functions $f(w)$	Operators	Invariants	Solution structure
Arbitrary	$\mathrm{X}_1 = \partial_x$, $\mathrm{X}_2 = \partial_t$, $\mathrm{X}_3 = t\partial_t + x\partial_x$	$I_1 = t$, $I_2 = w$, $I_1 = x$, $I_2 = w$, $I_1 = x/t$, $I_2 = w$	$w = w(t)$, $w = w(x)$, $w = w(z)$, $z = x/t$
e^w	$\mathrm{X}_4 = x\partial_x + 2\partial_w$	$I_1 = t$, $I_2 = w - 2\ln\lvert x\rvert$	$w = 2\ln\lvert x\rvert + \theta(t)$
w^k $(k \neq 0, -\frac{4}{3}, -4)$	$\mathrm{X}_4 = kx\partial_x + 2w\partial_w$	$I_1 = t$, $I_2 = w\lvert x\rvert^{-k/2}$	$w = \lvert x\rvert^{k/2}\theta(t)$
$w^{-4/3}$	$\mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$	$I_1 = t$, $I_2 = wx^{2/3}$, $I_1 = t$, $I_2 = wx^3$	$w = x^{-2/3}\theta(t)$, $w = x^{-3}\theta(t)$
w^{-4}	$\mathrm{X}_4 = 2x\partial_x - w\partial_w$, $\mathrm{X}_5 = t^2\partial_t + tw\partial_w$	$I_1 = t$, $I_2 = w\lvert x\rvert^{1/2}$, $I_1 = x$, $I_2 = w/t$	$w = \lvert x\rvert^{-1/2}\theta(t)$, $w = t\theta(x)$

The associated characteristic system of ordinary differential equations

$$\frac{dx}{x} = \frac{dt}{t} = \frac{dw}{0}$$

admits the first integrals

$$xt^{-1} = C_1, \quad w = C_2,$$

where C_1, C_2 are arbitrary constants. Therefore, $I_1 = xt^{-1}$ and $I_2 = w$ are invariants of the operator X_3.

Taking $I_2 = \Phi(I_1)$, we get

$$w = \Phi(y), \quad y = xt^{-1}. \tag{32.3.3.4}$$

The function $\Phi(y)$ is found by substituting (32.3.3.4) in the original equation (32.2.2.10). This results in the ordinary differential equation

$$[f(\Phi)\Phi'_y]'_y = (y^2\Phi'_y)'_y,$$

which defines an invariant (self-similar) solution. This equation has the obvious first integral $f(\Phi)\Phi'_y = y^2\Phi'_y + C$.

Table 32.3 summarizes the symmetries of equation (32.2.2.10) (see Example 32.3 from Subsection 32.2.2 and Example 32.8 from Subsection 32.3.3).

32.3.4. Solutions Induced by Linear Combinations of Admissible Operators

If a given equation admits N operators, then we have N associated different invariant solutions. However, when dealing with operators individually, one may overlook solutions that are invariant under a linear superposition of the operators; such solutions may have a significantly different form. In order to find all types of invariant solutions, one should study all possible linear combinations of the admissible operators.

Example 32.9. Consider once again the nonlinear unsteady heat equation (32.2.2.9).

1°. For arbitrary $f(w)$, this equation admits three operators (see Table 32.2):

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = 2t\partial_t + x\partial_x.$$

The respective invariant solutions are

$$w = \Phi(x), \quad w = \Phi(t), \quad w = \Phi(x^2/t).$$

However, various linear combinations give another operator,

$$X_{1,2} = X_1 + aX_2 = \partial_t + a\partial_x, \tag{32.3.4.1}$$

where $a \neq 0$ is an arbitrary constant. The solution invariant under this operator is written as

$$w = \Phi(x - at). \tag{32.3.4.2}$$

It is apparent that solutions of this type (traveling waves) are not contained in the invariant solutions associated with the individual operators X_1, X_2, and X_3.

2°. If $f(w) = e^w$, apart from the above three operators, there is another one, $X_4 = x\partial_x + 2\partial_w$ (see Table 32.2). In this case, the linear combination

$$X_{3,4} = X_3 + aX_4 = 2t\partial_t + (a+1)x\partial_x + 2a\partial_w$$

gives another invariant solution,

$$w = \Phi(\xi) + a\ln t, \quad \xi = xt^{-\frac{a+1}{2}},$$

where the function $\Phi = \Phi(\xi)$ satisfies the ordinary differential equation

$$(e^\Phi \Phi'_\xi)'_\xi + \tfrac{1}{2}(a+1)\xi\Phi'_\xi = a.$$

3°. If $f(w) = w^k$ ($k \neq 0, -4/3$), apart from the three operators from 1°, there is another one $X_4 = kx\partial_x + 2w\partial_w$. The linear combination

$$X_{3,4} = X_3 + aX_4 = 2t\partial_t + (ak+1)x\partial_x + 2aw\partial_w$$

generates the invariant (self-similar) solution

$$w = t^a\Phi(\zeta), \quad \zeta = xt^{-\frac{ak+1}{2}},$$

where the function $\Phi = \Phi(\zeta)$ satisfies the ordinary differential equation

$$(\Phi^k\Phi'_\zeta)'_\zeta + \tfrac{1}{2}(ak+1)\zeta\Phi'_\zeta = a\Phi.$$

The invariant solutions presented in Items 1°–3° are not listed in Table 32.2. It is clearly important to consider solutions induced by linear combinations of admissible operators.

Example 32.10. Now consider the nonlinear wave equation (32.2.2.10).

1°. For arbitrary $f(w)$, this equation admits three operators (see Table 32.3):

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t + x\partial_x.$$

The corresponding invariant solutions are

$$w = \Phi(x), \quad w = \Phi(t), \quad w = \Phi(x/t).$$

Going over the various linear combinations yields one more operator (32.3.4.1). There is a traveling-wave solution (32.3.4.2) which is invariant under this operator and not contained among the solutions corresponding to the "pure" operators X_1, X_2, and X_3.

2°. If $f(w) = e^w$, a fourth operator, $X_4 = x\partial_x + 2\partial_w$, arises in addition to the above three (see Table 32.3). In this case, the linear combination

$$X_{3,4} = X_3 + aX_4 = t\partial_t + (a+1)x\partial_x + 2a\partial_w$$

($a \neq 0$ is an arbitrary constant) gives an invariant solution

$$w = \Phi(\xi) + 2a\ln t, \quad \xi = xt^{-a-1},$$

where the function $\Phi = \Phi(\xi)$ satisfies the ordinary differential equation

$$(e^\Phi\Phi'_\xi)'_\xi = (a+1)^2\xi^2\Phi''_{\xi\xi} + (a+1)(a+2)\xi\Phi'_\xi - 2a.$$

3°. If $f(w) = w^k$ ($k \neq 0, -4/3, -4$), there arises a fourth operator $X_4 = kx\partial_x + 2w\partial_w$. The linear combination

$$X_{3,4} = X_3 + aX_4 = t\partial_t + (ak+1)x\partial_x + 2aw\partial_w$$

gives an invariant (self-similar) solution

$$w = t^{2a}\Phi(\zeta), \quad \zeta = xt^{-ak-1},$$

where the function $\Phi = \Phi(\zeta)$ satisfies the ordinary differential equation

$$(\Phi^k\Phi'_\zeta)'_\zeta = (ak+1)^2\zeta^2\Phi''_{\zeta\zeta} + (ak+1)(ak+2-4a)\zeta\Phi'_\zeta + 2a(2a-1)\Phi.$$

⊙ *References for Section* 32.3: L. V. Ovsiannikov (1962, 1982), G. W. Bluman and J. D. Cole (1974), J. M. Hill (1982, 1992), N. H. Ibragimov (1985, 1994), P. J. Olver (1986, 1995), D. H. Sattinger and O. L. Weaver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), W. I. Fushchich, V. M. Stelen, and N. I. Serov (1993), G. Gaeta (1994), A. M. Vinogradov and I. S. Krasilshchik (1997), G. Baumann (2000), P. A. Clarkson (2000), P. E. Hydon (2000), P. P. Kiryakov, S. I. Senashov, and A. N. Yakhno (2001), B. J. Cantwell (2002), D. M. Klimov and V. F. Zhuravlev (2002).

32.4. Some Generalizations. Higher-Order Equations

32.4.1. One-Parameter Lie Groups of Point Transformations. Group Generator

Here we will be considering functions dependent on $n+1$ variables, $x_1, \dots, x_n, w$. The brief notation $x = (x_1, \dots, x_n)$ will be used.

The set of invertible transformations of the form

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x}_i = \varphi_i(x, w, \varepsilon), & \bar{x}_i\big|_{\varepsilon=0} = x_i, \\ \bar{w} = \psi(x, w, \varepsilon), & \bar{w}\big|_{\varepsilon=0} = w, \end{cases} \tag{32.4.1.1}$$

where φ_i and ψ are sufficiently smooth functions of their arguments ($i = 1, \dots, n$) and ε is a real parameter, is called a *one-parameter continuous point group* of transformations G if for any ε_1 and ε_2 the relation $\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_2} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}$ holds, that is, the successive application (composition) of two transformations of the form (32.4.1.1) with parameters ε_1 and ε_2 is equivalent to a single transformation of the same form with parameter $\varepsilon_1 + \varepsilon_2$.

Further on, we consider local one-parameter continuous Lie groups of point transformations (or, for short, point groups), corresponding to the infinitesimal transformation (32.4.1.1) as $\varepsilon \to 0$. The expansion of (32.4.1.1) into Taylor series in the parameter ε about $\varepsilon = 0$ to the first order gives

$$\bar{x}_i \simeq x_i + \varepsilon\xi_i(x, w), \quad \bar{w} \simeq w + \varepsilon\zeta(x, w), \tag{32.4.1.2}$$

where

$$\xi_i(x, w) = \frac{\partial\varphi_i(x, w, \varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}, \quad \zeta(x, w) = \frac{\partial\psi(x, w, \varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}.$$

The linear first-order differential operator

$$\mathrm{X} = \xi_i(x, w)\frac{\partial}{\partial x_i} + \zeta(x, w)\frac{\partial}{\partial w} \tag{32.4.1.3}$$

corresponding to the infinitesimal transformation (32.4.1.2) is called a *group generator* (or an *infinitesimal operator*). In formula (32.4.1.3), summation is assumed over the repeated index i.

THEOREM (LIE). *Suppose the coordinates $\xi_i(x, w)$ and $\zeta(x, w)$ of the group generator (32.4.1.3) are known. Then the one-parameter group of transformations (32.4.1.1) can be completely recovered by solving the Lie equations*

$$\frac{d\varphi_i}{d\varepsilon} = \xi_i(\varphi, \psi), \quad \frac{d\psi}{d\varepsilon} = \zeta(\varphi, \psi) \qquad (i = 1, \dots, n)$$

with the initial conditions

$$\varphi_i\big|_{\varepsilon=0} = x_i, \quad \psi\big|_{\varepsilon=0} = w.$$

Here, the short notation $\varphi = (\varphi_1, \dots, \varphi_n)$ has been used.

Remark. The widely known terms "Lie group analysis of differential equations," "group-theoretic methods," and others are due to the prevailing concept of a local one-parameter Lie group of point transformations. However, in this book, we prefer the widely used term "classical method of symmetry reduction."

32.4.2. Group Invariants. Local Transformations of Derivatives

A *universal invariant* (or, for short, an *invariant*) of a group (32.4.1.1) and a operator (32.4.1.3) is a function $I(x, w)$ that satisfies the condition $I(\bar{x}, \bar{w}) = I(x, w)$. The expansion in a series in powers of the small parameter ε gives rise to the linear partial differential equation for I:

$$\mathrm{X}I = \xi_i(x, w)\frac{\partial I}{\partial x_i} + \zeta(x, w)\frac{\partial I}{\partial w} = 0. \tag{32.4.2.1}$$

Here summation is assumed over the repeated index i. From the theory of first-order partial differential equations it follows that the group (32.4.1.1) and the operator (32.4.1.3) have n functionally independent universal invariants. On the other side, this means that any function $F(x, w)$ that is invariant under the group (32.4.1.1) can be written as a function of n invariants.

In the new variable (32.4.1.1), the derivatives are transformed as follows:

$$\begin{gathered} \frac{\partial\bar{w}}{\partial\bar{x}_i} \simeq \frac{\partial w}{\partial x_i} + \varepsilon\zeta_i, \quad \frac{\partial^2\bar{w}}{\partial\bar{x}_i\partial\bar{x}_j} \simeq \frac{\partial^2 w}{\partial x_i\partial x_j} + \varepsilon\zeta_{ij}, \\ \frac{\partial^3\bar{w}}{\partial\bar{x}_i\partial\bar{x}_j\partial\bar{x}_k} \simeq \frac{\partial^3 w}{\partial x_i\partial x_j\partial x_k} + \varepsilon\zeta_{ijk}, \quad \dots . \end{gathered} \tag{32.4.2.2}$$

The coordinates ζ_i, ζ_{ij}, and ζ_{ijk} of the first three prolongations are expressed as

$$\begin{aligned}\zeta_i &= D_i(\zeta) - p_s D_i(\xi_s),\\ \zeta_{ij} &= D_j(\zeta_i) - q_{is} D_j(\xi_s),\\ \zeta_{ijk} &= D_k(\zeta_{ij}) - r_{ijs} D_k(\xi_s),\end{aligned} \tag{32.4.2.3}$$

where summation is assumed over the repeated index s and the following short notation partial derivatives are used:

$$p_i = \frac{\partial w}{\partial x_i}, \quad q_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad r_{ijk} = \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k},$$
$$D_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial w} + q_{ij} \frac{\partial}{\partial p_j} + r_{ijk} \frac{\partial}{\partial q_{jk}} + \cdots,$$

with D_i being the total differential operator with respect to x_i.

32.4.3. Invariant Condition. Splitting Procedure. Invariant Solutions

We will consider partial differential equations of order m in n independent variables

$$F\left(x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}, \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k}, \ldots\right) = 0, \tag{32.4.3.1}$$

where $i, j, k = 1, \ldots, n$.

The group analysis of equation (32.4.3.1) consists of several stages. At the first stage, let us require that equation (32.4.3.1) be invariant (must preserve its form) under transformations (32.4.1.1), so that

$$F\left(\bar{x}, \bar{w}, \frac{\partial \bar{w}}{\partial \bar{x}_i}, \frac{\partial^2 \bar{w}}{\partial x_i \partial \bar{x}_j}, \frac{\partial^3 \bar{w}}{\partial x_i \partial \bar{x}_j \partial \bar{x}_k}, \ldots\right) = 0. \tag{32.4.3.2}$$

Let us expand this expression into a series in powers of ε about $\varepsilon = 0$ taking into account that the leading term (32.4.3.1) vanishes. Using formulas (32.4.1.1) and (32.4.2.2) and retaining terms to the first order of ε, we get

$$\underset{m}{\mathrm{X}} F\left(x, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}, \frac{\partial^3 w}{\partial x_i \partial x_j \partial x_k}, \ldots\right)\Big|_{F=0} = 0, \tag{32.4.3.3}$$

where

$$\underset{m}{\mathrm{X}} F = \xi_i \frac{\partial F}{\partial x_i} + \zeta \frac{\partial F}{\partial w} + \zeta_i \frac{\partial F}{\partial w_{x_i}} + \zeta_{ij} \frac{\partial F}{\partial w_{x_i x_j}} + \zeta_{ijk} \frac{\partial F}{\partial w_{x_i x_j x_k}} + \cdots. \tag{32.4.3.4}$$

The coordinates ζ_i, ζ_{ij}, and ζ_{ijk} of the first three prolongations are defined by formulas (32.4.2.2)–(32.4.2.3). Summation is assumed over repeated indices. Relation (32.4.3.3) is called the *invariance condition* and the operator $\underset{m}{\mathrm{X}}$ is called the mth prolongation of the group generator; the partial derivatives of F with respect to all m derivatives of w appear last in (32.4.3.4).

At the second stage, one of the highest mth-order derivatives is eliminated from (32.4.3.3) using equation (32.4.3.1). The resulting relation is then represented as a polynomial in "independent variables," the various combinations of the remaining derivatives, which are the products of different powers of $w_x, w_y, w_{xx}, w_{xy}, \ldots$ All the coefficients of this polynomial — they depend on x, w, ξ_i, and ζ only and are independent of the derivatives of w — are further equated to zero. As a result, the invariance condition is split into an overdetermined linear determining system.

The third stage involves solving the determining system and finding admissible coordinates ξ_i and ζ of the group generator (32.4.1.3).

For mth-order equations in two independent variables, the invariant solutions are defined in a similar way as for second-order equations. In this case, the procedure of constructing invariant solutions (for known coordinates of the group generator) is identical to that described in Subsection 32.3.2.

32.5. Symmetries of Systems of Equations of Mathematical Physics

32.5.1. Basic Relations Used in Symmetry Analysis of Systems of Equations

Symmetry analysis (group analysis) of systems of partial differential equations differs from the analysis of individual equations by only the fact that all dependent and independent variables must be taken into account

in the operator (32.1.1.3) and its prolongations. Below are some formulas that are frequently used in practical symmetry analysis of simultaneous equations.

For system consisting of two equations with two independent variables, x and y, and two unknown functions, $u = u(x, y)$ and $v = v(x, y)$, admissible operators are sought in the form

$$\mathrm{X} = \xi(x,y,u,v)\frac{\partial}{\partial x} + \eta(x,y,u,v)\frac{\partial}{\partial y} + \zeta(x,y,u,v)\frac{\partial}{\partial u} + \chi(x,y,u,v)\frac{\partial}{\partial v}. \tag{32.5.1.1}$$

Generally, the second prolongation of (32.5.1.1) to the "independent variables" u_x, u_y, v_x, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, and v_{yy} is written as

$$\begin{aligned}\underset{2}{\mathrm{X}} = {}& \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial u} + \chi\frac{\partial}{\partial v} + \zeta_1\frac{\partial}{\partial u_x} + \zeta_2\frac{\partial}{\partial u_y} + \chi_1\frac{\partial}{\partial v_x} + \chi_2\frac{\partial}{\partial v_y} \\ &+ \zeta_{11}\frac{\partial}{\partial u_{xx}} + \zeta_{12}\frac{\partial}{\partial u_{xy}} + \zeta_{22}\frac{\partial}{\partial u_{yy}} + \chi_{11}\frac{\partial}{\partial v_{xx}} + \chi_{12}\frac{\partial}{\partial v_{xy}} + \chi_{22}\frac{\partial}{\partial v_{yy}}.\end{aligned} \tag{32.5.1.2}$$

The coordinates of the first prolongation are calculated by formulas similar to (32.1.3.2) from Subsection 32.1.3,

$$\begin{aligned}\zeta_1 &= D_x(\zeta) - u_x D_x(\xi) - u_y D_x(\eta), \qquad \chi_1 = D_x(\chi) - v_x D_x(\xi) - v_y D_x(\eta),\\ \zeta_2 &= D_y(\zeta) - u_x D_y(\xi) - u_y D_y(\eta), \qquad \chi_2 = D_y(\chi) - v_x D_y(\xi) - v_y D_y(\eta);\end{aligned} \tag{32.5.1.3}$$

the coordinates of the second prolongation are calculated by

$$\begin{aligned}\zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi) - u_{xy} D_x(\eta), \qquad \chi_{11} = D_x(\chi_1) - v_{xx} D_x(\xi) - v_{xy} D_x(\eta),\\ \zeta_{12} &= D_y(\zeta_1) - u_{xx} D_y(\xi) - u_{xy} D_y(\eta), \qquad \chi_{12} = D_y(\chi_1) - v_{xx} D_y(\xi) - v_{xy} D_y(\eta),\\ \zeta_{22} &= D_y(\zeta_2) - u_{xy} D_y(\xi) - u_{yy} D_y(\eta), \qquad \chi_{22} = D_y(\chi_2) - v_{xy} D_y(\xi) - v_{yy} D_y(\eta),\end{aligned} \tag{32.5.1.4}$$

where D_x and D_y are total differential operators in the independent variables:

$$\begin{aligned}D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + v_{xx}\frac{\partial}{\partial v_x} + v_{xy}\frac{\partial}{\partial v_y} + \cdots,\\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + v_{xy}\frac{\partial}{\partial v_x} + v_{yy}\frac{\partial}{\partial v_y} + \cdots.\end{aligned}$$

In the symmetry analysis of simultaneous partial differential equations, it is required that each equation must satisfy the invariance condition generated by the prolonged operator (32.5.1.2).

32.5.2. Symmetries of Equations of Steady Hydrodynamic Boundary Layer

The technique of symmetry analysis of nonlinear simultaneous partial differential equations is illustrated below by a specific example using formulas (32.5.1.1)–(32.5.1.4).

Example 32.11. Consider the system of equations of a steady hydrodynamic boundary layer

$$\begin{aligned}uu_x + vu_y + f(x) &= u_{yy},\\ u_x + v_y &= 0.\end{aligned} \tag{32.5.2.1}$$

Determining relations for the coordinates of the admissible operator. We look for an operator admitted by system (32.5.2.1) in the form (32.5.1.1). The second prolongation is given by formula (32.5.1.2) in which

$$\chi_1 = 0, \quad \zeta_{11} = \zeta_{12} = 0, \quad \chi_{11} = \chi_{12} = \chi_{22} = 0. \tag{32.5.2.2a}$$

(It has been taken into account that the only derivatives appearing in the system are u_x, u_y, v_y, and u_{yy}.) The nonzero coordinates of the prolonged operator (32.5.1.2) are calculated by formulas (32.5.1.3) and (32.5.1.4):

$$\begin{aligned}\zeta_1 &= \zeta_x + \zeta_u u_x + \zeta_v v_x - u_x(\xi_x + \xi_u u_x + \xi_v v_x) - u_y(\eta_x + \eta_u u_x + \eta_v v_x),\\ \zeta_2 &= \zeta_y + \zeta_u u_y + \zeta_v v_y - u_x(\xi_y + \xi_u u_y + \xi_v v_y) - u_y(\eta_y + \eta_u u_y + \eta_v v_y),\\ \chi_2 &= \chi_y + \chi_u u_y + \chi_v v_y - v_x(\xi_y + \xi_u u_y + \xi_v v_y) - v_y(\eta_y + \eta_u u_y + \eta_v v_y),\\ \zeta_{22} &= \zeta_{yy} + 2\zeta_{vy} v_y + \zeta_{vv}(v_y)^2 + [2\zeta_{uy} - \eta_{yy} + 2(\zeta_{uv} - \eta_{vy})v_y - \eta_{vv}(v_y)^2]u_y\\ &\quad - [\xi_{yy} + 2(\xi_{uy} + \xi_{uv} v_y)u_y + 2\xi_{vy} v_y + \xi_{uu}(u_y)^2 + \xi_{vv}(v_y)^2]u_x +\\ &\quad - (\zeta_{uu} - 2\eta_{uy} - 2\eta_{uv} v_y)(u_y)^2 - \eta_{uu}(u_y)^3 - 2(\xi_y + \xi_u u_y + \xi_v v_y)u_{xy}\\ &\quad - [2\eta_y - \zeta_u - \zeta_v + (\xi_u + \xi_v)u_x + (3\eta_u + \eta_v)u_y + 2\eta_v v_y]u_{yy}.\end{aligned} \tag{32.5.2.2b}$$

Taking into account (32.5.2.2a) and (32.5.2.2b), we apply the prolonged operator (32.5.1.2) to system (32.5.2.1). Eliminating u_{yy} and v_y from the resulting relations using the equations of system (32.5.2.1) yields two invariance conditions. By collecting similar terms and rearranging, we arrive at the determining equations

$$\begin{aligned}
&[-3\xi_v u_x + (2\xi_u - \eta_v)u_y + 2\xi_y + \zeta_v]u_{xy} + \xi_{vv}u_x^3 - (2\xi_{uv} - \eta_{vv})u_x^2 u_y \\
&\quad + (\xi_{uu} - 2\eta_{uv})u_x u_y^2 + \eta_{uu}u_y^3 + (v\xi_v - 2u\eta_v - \zeta_{vv} - 2\xi_{yv})u_x^2 \\
&\quad + (2\zeta_{uv} + 2u\eta_u - 2\eta_{yv} - v\eta_v + 2\xi_{yu})u_x u_y + (2v\eta_u + 2\eta_{yu} - \zeta_{uu})u_y^2 \\
&\quad - u\xi_v u_x v_x - u\eta_v u_y v_x + (-v\xi_y - v\zeta_v - u\xi_x - 2f\eta_v + 2u\eta_y + f\xi_u \\
&\quad + \zeta + 2\zeta_{yv} + \xi_{yy})u_x + (v\eta_y - u\eta_x + 3f\eta_u - 2\zeta_{yu} + \chi + \eta_{yy})u_y \\
&\quad + u\zeta_v v_x + f'_x\xi - \zeta_{yy} + u\zeta_x + v\zeta_y - f\zeta_u + 2f\eta_y = 0 \quad (\textit{first equation}), \\
&-(\xi_u + \eta_v)u_x^2 + (\zeta_u - \chi_v + \eta_y - \xi_x)u_x - (\eta_v + \xi_u)u_y v_x + \zeta_v v_x \\
&\quad + (\chi_u - \eta_x - \xi_y)u_y + \zeta_x + \chi_y = 0 \qquad (\textit{second equation}).
\end{aligned}$$

Splitting these equations into "independent" variables results in two overdetermined systems, which must hold simultaneously:

First system:

$$\begin{aligned}
u_x u_{xy}&: && 3\xi_v = 0, && (1)\\
u_y u_{xy}&: && 2\xi_u - \eta_v = 0, && (2)\\
u_{xy}&: && 2\xi_y + \zeta_v = 0, && (3)\\
u_x^3&: && \xi_{vv} = 0, && (4)\\
u_x^2 u_y&: && 2\xi_{uv} - \eta_{vv} = 0, && (5)\\
u_x u_y^2&: && \xi_{uu} - 2\eta_{uv} = 0, && (6)\\
u_y^3&: && \eta_{uu} = 0, && (7)\\
u_x^2&: && v\xi_v - 2u\eta_v - \zeta_{vv} - 2\xi_{yv} = 0, && (8)\\
u_x u_y&: && 2\zeta_{uv} + 2u\eta_u - 2\eta_{yv} - v\eta_v + 2\xi_{yu} = 0, && (9)\\
u_y^2&: && 2v\eta_u + 2\eta_{yu} - \zeta_{uu} = 0, && (10)\\
u_x v_x&: && u\xi_v = 0, && (11)\\
u_y v_x&: && u\eta_v = 0, && (12)\\
u_x&: && \xi_{yy} - v\xi_y - v\zeta_v - u\xi_x - 2f\eta_v + 2u\eta_y + f\xi_u + \zeta + 2\zeta_{yv} = 0, && (13)\\
u_y&: && \eta_{yy} + v\eta_y - u\eta_x + 3f\eta_u - 2\zeta_{yu} + \chi = 0, && (14)\\
v_x&: && u\zeta_v = 0, && (15)\\
1&: && f'_x\xi - \zeta_{yy} + u\zeta_x + v\zeta_y - f\zeta_u + 2f\eta_y = 0. && (16)
\end{aligned}$$

Second system:

$$\begin{aligned}
u_x^2&: && \xi_u + \eta_v = 0, && (1)\\
u_y v_x&: && \eta_v + \xi_u = 0, && (2)\\
u_x&: && \zeta_u - \chi_v + \eta_y - \xi_x = 0, && (3)\\
u_y&: && \chi_u - \eta_x - \xi_y = 0, && (4)\\
v_x&: && \zeta_v = 0, && (5)\\
1&: && \zeta_x + \chi_y = 0. && (6)
\end{aligned}$$

It follows from the 11th and 12th equations of the first system that $\xi = \xi(x, y, u)$ and $\eta = \eta(x, y, u)$; using equation 2, it can be concluded that $\xi = \xi(x, y)$. From the 15th equation it follows that $\zeta = \zeta(x, y, u)$ and equation 3 then gives $\xi = \xi(x)$. Equations 1 and 4–6 are satisfied identically. If follows from equation 9 that $\eta_u = 0$, and hence $\eta = \eta(x, y)$ (equation 7 is satisfied identically). So we have

$$\xi = \xi(x), \quad \eta = \eta(x, y), \quad \zeta = \zeta(x, y, u). \tag{32.5.2.3}$$

The remaining equations become

$$\begin{aligned}
u_y^2&: && \zeta_{uu} = 0,\\
u_x&: && -u\xi'_x + 2u\eta_y + \zeta = 0,\\
u_y&: && \eta_{yy} + v\eta_y - u\eta_x - 2\zeta_{yu} + \chi = 0,\\
1&: && f'_x\xi - \zeta_{yy} + u\zeta_x + v\zeta_y - f\zeta_u + 2f\eta_y = 0.
\end{aligned} \tag{32.5.2.4}$$

By virtue of (32.5.2.3), equations 1, 2, and 5 in the second system are satisfied identically and the remaining equations become

$$\begin{aligned} u_x\colon &\quad \zeta_u - \chi_v + \eta_y - \xi_x = 0,\\ u_y\colon &\quad \chi_u - \eta_x = 0,\\ 1\colon &\quad \zeta_x + \chi_y = 0. \end{aligned} \tag{32.5.2.5}$$

The first equation in (32.5.2.4) implies that $\zeta = \zeta_1(x,y)u + \zeta_0(x,y)$. By substituting this expression into the second equation in (32.5.2.4) and splitting in u, one finds that $\zeta_0 \equiv 0$. Likewise, from the fourth equation in (32.5.2.4), one finds that $\zeta_1 = C_1 = \text{const}$, or

$$\zeta = C_1 u. \tag{32.5.2.6}$$

By virtue of this, the second equation in (32.5.2.4) and the last equation in (32.5.2.5) lead to

$$\eta = \tfrac{1}{2}(\xi_x - C_1)y + h(x), \quad \chi = \chi(x,u,v), \tag{32.5.2.7}$$

where $h = h(x)$ is an arbitrary function. By substituting (32.5.2.6) and (32.5.2.7) into the third and fourth equations of system (32.5.2.4) and substituting the second equation of (32.5.2.5), one arrives at the following system of three equations:

$$\begin{aligned} &\chi = \tfrac{1}{2}\xi''_{xx}yu + h'_x u - \tfrac{1}{2}(\xi'_x - C_1)v,\\ &f'_x\xi + f(\xi'_x - 2C_1) = 0,\\ &\chi_u - \eta_x = 0. \end{aligned} \tag{32.5.2.8}$$

Note that the first equation of (32.5.2.5) coincides with the first equation of (32.5.2.8) differentiated with respect to v.

Since χ is independent of y, the first equation of (32.5.2.8) implies $\xi''_{xx} = 0$, or

$$\xi = C_2 x + C_3.$$

Substituting this expression into the first and last equations of system (32.5.2.8), taking into account (32.5.2.6) and (32.5.2.7), and renaming the constants of integration, one obtains the coordinates of the operator (32.5.1.1):

$$\xi = C_1 + (C_2 + 2C_3)x, \quad \eta = C_3 y + h(x), \quad \zeta = C_2 u, \quad \chi = -C_3 v + h'(x)u. \tag{32.5.2.9}$$

In addition, the condition

$$\big[C_1 + (C_2 + 2C_3)x\big]f'(x) = (C_2 - 2C_3)f(x) \tag{32.5.2.10}$$

must be satisfied, which follows from the second equation of (32.5.2.8) and contains the function $f(x)$ and the constants of integration C_1, C_2, and C_3.

Symmetries of the hydrodynamic boundary layer equations. Using (32.5.2.10), let us classify the symmetries of the boundary layer system (32.5.2.1) for all $f(x)$.

1°. In order that condition (32.5.2.10) is satisfied for arbitrary $f(x)$, one must set $C_1 = C_2 = C_3 = 0$. Then, from (32.5.2.9) it follows that $\xi = \zeta = 0$, $\eta = h(x)$, and $\chi = h'(x)u$, so the corresponding admissible operator is

$$X_0 = h(x)\partial_y + h'(x)u\partial_v. \tag{32.5.2.11}$$

2°. If $f(x) = 0$, condition (32.5.2.10) is satisfied identically for any C_1, C_2, and C_3. Taking into account (32.5.2.9), we obtain the following three additional operators:

$$\begin{aligned} &X_1 = \partial_x && (C_1 = 1,\ C_2 = C_3 = 0,\ h = 0);\\ &X_2 = x\partial_x + u\partial_u && (C_2 = 1,\ C_1 = C_3 = 0,\ h = 0);\\ &X_3 = 2x\partial_x + y\partial_y - v\partial_v && (C_3 = 1,\ C_1 = C_2 = 0,\ h = 0). \end{aligned} \tag{32.5.2.12}$$

The operator (32.5.2.11) exists in all cases.

3°. If $f(x) \equiv 1$, condition (32.5.2.10) implies $C_2 = 2C_3$. Then, along with X_0, there are two more admissible operators: X_1 and $2X_2 + X_3$, where the X_n are given by (32.5.2.12).

4°. If $f(x) = \pm e^x$, condition (32.5.2.10) implies $C_1 = -4C_3$ and $C_2 = -2C_3$, which corresponds to a single operator $4X_1 + 2X_2 - X_3$.

TABLE 32.4
Symmetries of the system of boundary layer equations (32.5.2.1); a = const

Function $f(x)$	Operator	Invariant 1	Invariant 2	Invariant 3
Arbitrary	$X_0 = h(x)\partial_y + h'(x)u\partial_v$	x	u	$v - (\ln h)'_x yu$
0	$X_1 = \partial_x$ $X_2 = x\partial_x + u\partial_u$ $X_3 = 2x\partial_x + y\partial_y - v\partial_v$	y y $yx^{-1/2}$	u u/x u	v v $x^{1/2}v$
a	$X_1 = \partial_x$ $X_2 = 4x\partial_x + y\partial_y + 2u\partial_u - v\partial_v$	y $yx^{-1/4}$	u $ux^{-1/2}$	v $x^{1/4}v$
ae^x	$X_1 = 2x\partial_x - y\partial_y + 2u\partial_u - v\partial_v$	$yx^{1/2}$	u/x	$x^{1/2}v$
ax^n, $n \neq 0$	$X_1 = 4x\partial_x - (n-1)y\partial_y$ $+ 2(n+1)u\partial_u + (n-1)v\partial_v$	$yx^{\frac{n-1}{4}}$	$x^{-\frac{n+1}{2}}u$	$x^{\frac{1-n}{4}}v$

5°. If $f(x) = \pm x^n$ ($n \neq 0$), we have $C_1 = 0$ and $C_2 = -\dfrac{2(1+n)}{(1-n)}$, which results in a single operator $2(n+1)X_2 - (n-1)X_3$.

These results and the corresponding invariants of the system of boundary layer equations (32.5.2.1) are gathered in Table 32.4. The invariants I_n are determined by the first integrals of the characteristic system of ordinary differential equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{du}{\zeta} = \frac{dv}{\chi},$$

which corresponds to the linear first-order partial differential equation $XI = 0$.

Remark 1. The first invariant was taken linear in y (wherever this was possible). This simplified the calculations, since the equations contain the highest-order derivative in y, while the derivatives with respect to x have a lower order. Recall that the invariants I_n can be replaced by $F_n(I_n)$, where the functions F_n are chosen for reasons of convenience.

Remark 2. The symmetry X_0 corresponds to the invariance of system (32.5.2.1) under the transformation

$$x = \bar{x}, \quad y = \bar{y} + h(x), \quad u = \bar{u}, \quad v = \bar{v} - h'_x u,$$

where $h = h(x)$ is an arbitrary function.

Exact solutions to the hydrodynamic boundary layer equations. We look for solutions in the form

$$I_2 = \Phi(I_1), \quad I_3 = \Psi(I_1), \tag{32.5.2.13}$$

where the I_n are invariants and the functions Φ and Ψ are determined from the system of ordinary differential equations resulting from the substitution of (32.5.2.13) into the original system of equations (32.5.2.1).

1°. For arbitrary $f(x)$, exact solutions are sought in the form (see the first row in Table 32.4)

$$u = \Phi(x), \quad v = (\ln h)'_x yu + \Psi(x). \tag{32.5.2.14}$$

Substituting (32.5.2.14) into (32.5.2.1) yields a system of ordinary differential equations, which is easy to integrate. We have

$$\Phi(x) = \pm\Big[C_1 - 2\int f(x)\,dx\Big]^{1/2}, \quad h(x) = \frac{C_1}{\Phi(x)}, \quad \Psi(x) \text{ is an arbitrary function}, \tag{32.5.2.15}$$

where C_1 and C_2 are arbitrary constants. Solution (32.5.2.14)–(32.5.2.15) corresponds to inviscid fluid flow, since $u_{yy} = 0$.

2°. For $f(x) = 0$, the invariants of the first operator in Table 32.4 give a one-dimensional solution independent of the coordinate x. Substituting $u = u(y)$ and $v = v(y)$ into system (32.5.2.1) and integrating we obtain

$$u = C_2 \exp(C_1 y) + C_3, \quad v = C_1 \qquad \text{if } C_1 \neq 0;$$
$$u = C_2 y + C_3, \qquad\qquad v = 0 \qquad \text{if } C_1 = 0.$$

The invariants of the second operator in Table 32.4 correspond to solutions of the form

$$u = x\varphi(y), \quad v = \psi(y), \tag{32.5.2.16}$$

where the functions $\varphi(y)$ and $\psi(y)$ are described by the system of ordinary differential equations

$$\varphi^2 + \psi\varphi'_y = \varphi''_{yy},$$
$$\varphi + \psi'_y = 0,$$

which results from substituting (32.5.2.16) into the original system (32.5.2.1).

The invariants of the third operator (see Table 32.4) lead to the self-similar solution

$$u = U(\theta), \quad v = x^{-1/2}V(\theta), \quad \theta = yx^{-1/2},$$

where the functions $U(\theta)$ and $V(\theta)$ are determined by the system of ordinary differential equations

$$-\tfrac{1}{2}\theta U U'_\theta + V U'_\theta = U''_{\theta\theta},$$
$$-\tfrac{1}{2}\theta U'_\theta + V'_\theta = 0.$$

3°. Exact solutions for the other $f(x)$ presented in Table 32.4 are sought in a similar manner.

⊙ *References for Section* 32.4: Yu. N. Pavlovskii (1961), L. V. Ovsiannikov (1962, 1982), G. W. Bluman and J. D. Cole (1974), N. H. Ibragimov (1985, 1994), P. J. Olver (1986, 1995), A. M. Vinogradov and I. S. Krasilshchik (1997).

Chapter 33
Nonclassical Method of Symmetry Reductions*

33.1. General Description of the Method

33.1.1. Description of the Method. Invariant Surface Condition

For simplicity, we consider a second-order equation in two independent variables of the general form

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{33.1.1.1}$$

The results of the classical group analysis (see Chapter 32) can be substantially extended if, instead of finding invariants of an admissible infinitesimal operator X by means of solving the characteristic system of equations

$$\frac{dx}{\xi(x, y, w)} = \frac{dy}{\eta(x, y, w)} = \frac{dw}{\zeta(x, y, w)},$$

one imposes the corresponding *invariant surface condition* (Bluman and Cole, 1969)

$$\xi(x, y, w)\frac{\partial w}{\partial x} + \eta(x, y, w)\frac{\partial w}{\partial y} = \zeta(x, y, w). \tag{33.1.1.2}$$

Equation (33.1.1.1) and condition (33.1.1.2) are supplemented by the invariance condition

$$\underset{2}{\mathrm{X}} F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{F=0} = 0, \tag{33.1.1.3}$$

which coincides with equation (32.2.1.3).

All three equations (33.1.1.1)–(33.1.1.3) are used for the construction of exact solutions of the original equation (33.1.1.1). It should be observed that in this case, the determining equations obtained for the unknown functions $\xi(x, y, w)$, $\eta(x, y, w)$, and $\zeta(x, y, w)$ by the splitting procedure are nonlinear. The symmetries determined by the invariant surface (33.1.1.2) are called *nonclassical*.

33.1.2. Scheme for Constructing Exact Solutions by the Nonclassical Method

Figure 33.1 is intended to clarify the general scheme for constructing of exact solutions of a second-order evolution equation by the nonclassical method on the basis of the invariant surface condition (33.1.1.2).

Remark. Apart from the algorithm shown in Fig. 33.1, its modification can also be used. Instead of solving the characteristic system of ordinary differential equations, the derivative w_t is eliminated from (33.1.1.1)–(33.1.1.2) after finding the coordinates ξ, η, and ζ. Then the resulting equation is solved, which can be treated as an ordinary differential equation for x with parameter t.

⊙ *References for Section* 33.1: G. W. Bluman and J. D. Cole (1969), P. J. Olver and Ph. Rosenau (1987), E. Pucci and G. Saccomandi (2000), A. D. Polyanin, V. F. Zaitsev (2004).

* Prior to reading this section, the reader may find it useful to get acquainted with Chapter 32.

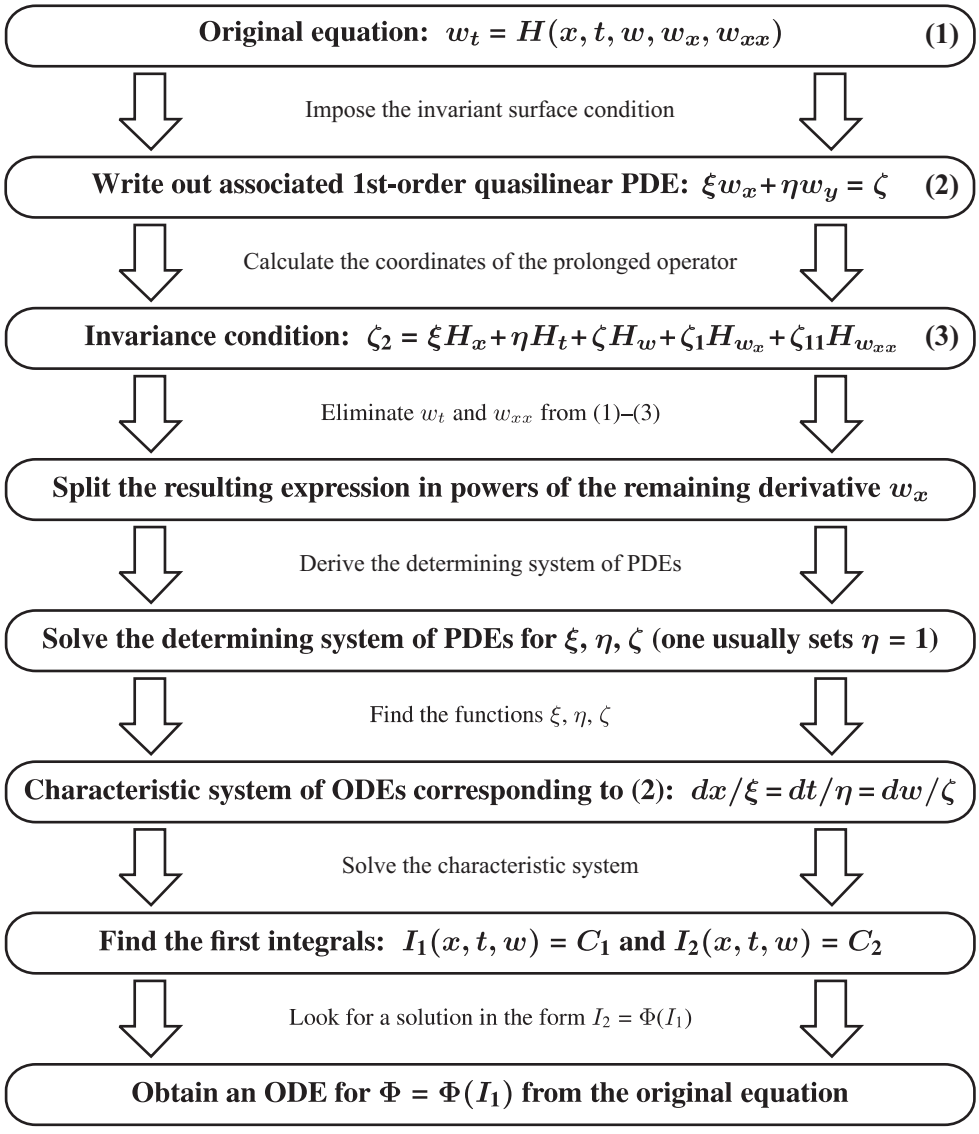

Figure 33.1. Algorithm for the construction of exact solutions by a nonclassical method for second-order evolution equations. Here, ODE stands for ordinary differential equation and PDE for partial differential equation.

33.2. Examples of Constructing Exact Solutions

33.2.1. Newell–Whitehead Equation

Consider the *Newell–Whitehead equation*

$$w_t = w_{xx} + w - w^3, \tag{33.2.1.1}$$

which corresponds to the left-hand side $F = -w_t + w_{xx} + w - w^3$ of equation (33.1.1.1) with $y = t$.

Without loss of generality, we set $\eta = 1$ in the invariant surface condition (33.1.1.2) with $y = t$, thus assuming that $\eta \neq 0$. We have

$$\frac{\partial w}{\partial t} + \xi(x, t, w)\frac{\partial w}{\partial x} = \zeta(x, t, w). \tag{33.2.1.2}$$

The invariance condition is obtained by a procedure similar to the classical algorithm (see Sections 32.1–32.2). Namely, we apply the operator

$$\underset{2}{\mathrm{X}} = \xi\partial_x + \eta\partial_t + \zeta\partial_w + \zeta_1\partial_{w_x} + \zeta_2\partial_{w_t} + \zeta_{11}\partial_{w_{xx}} + \zeta_{12}\partial_{w_{xt}} + \zeta_{22}\partial_{w_{tt}} \tag{33.2.1.3}$$

to equation (33.2.1.1). Taking into account that $\partial_x = \partial_t = \partial_{w_x} = \partial_{w_{xt}} = \partial_{w_{tt}} = 0$, since the equation is explicitly independent of x, t, w_x, w_{xt}, w_{tt}, we get the invariance condition in the form

$$\zeta_2 = \zeta(3w^2 - 1) + \zeta_{11}.$$

Substituting here the expressions (32.1.3.2) and (32.1.3.7) for the coordinates ζ_2 and ζ_{11} of the first and second prolongations, with $y = t$ and $\eta = 1$, we obtain

$$\zeta_t - \xi_t w_x + \zeta_w w_t - \xi_w w_x w_t = \zeta(-3w^2 + 1) + \zeta_{xx} + (2\zeta_{wx} - \xi_{xx})w_x + (\zeta_{ww} - 2\xi_{wx})w_x^2 - \xi_{ww} w_x^3 + (\zeta_w - 2\xi_x - 3\xi_w w_x)w_{xx}. \qquad (33.2.1.4)$$

Let us express the derivatives w_t and w_{xx} with the help of (33.2.1.1)–(33.2.1.2) via the other quantities:

$$w_t = \zeta - \xi w_x, \quad w_{xx} = \zeta - \xi w_x + w(w^2 - 1). \qquad (33.2.1.5)$$

Inserting these expressions into the invariance condition (33.2.1.4), we arrive at a cubic polynomial in the remaining "independent" derivative w_x. Equating all functional coefficients of the various powers of this polynomial to zero, we get the determining system

$$\begin{aligned} w_x^3&: \quad \xi_{ww} = 0,\\ w_x^2&: \quad \zeta_{ww} - 2(\xi_{wx} - \xi\xi_w) = 0,\\ w_x&: \quad 2\zeta_{wx} - 2\xi_w\zeta - 3w(w^2 - 1)\xi_w - \xi_{xx} + 2\xi\xi_x + \xi_t = 0,\\ 1&: \quad \zeta_t - \zeta_{xx} + 2\xi_x\zeta + (2\xi_x - \zeta_w)w(w^2 - 1) + (3w^2 - 1)\zeta = 0, \end{aligned} \qquad (33.2.1.6)$$

which consists of only four equations.

The analysis of the system (33.2.1.6) allows us to conclude that

$$\xi = \xi(x, t), \quad \zeta = -\xi_x w,$$

where the function $\xi(x, t)$ satisfies the system

$$\begin{aligned} \xi_t - 3\xi_{xx} + 2\xi\xi_x = 0,\\ \xi_{xt} - \xi_{xxx} + 2\xi_x^2 + 2\xi\xi_x = 0. \end{aligned} \qquad (33.2.1.7)$$

The associated invariant surface condition has the form (33.2.1.2):

$$w_t + \xi w_x = -\xi_x w. \qquad (33.2.1.8)$$

We look for a stationary particular solution to equation (33.2.1.7) in the form $\xi = \xi(x)$. Let us differentiate the first equation of (33.2.1.7) with respect to x and then eliminate the third derivative, using the second equation of (33.2.1.7), from the resulting expression. This will give a second-order equation. Eliminating from it the second derivative, using the first equation of (33.2.1.7), after simple rearrangements we obtain

$$(6\xi_x' - 2\xi^2 + 9)\xi_x' = 0. \qquad (33.2.1.9)$$

Equating the expression in parentheses in (33.2.1.9) to zero, we get a separable first-order equation. Its general solution can be written as

$$\xi = -\frac{3}{\sqrt{2}}\,\frac{A\exp\bigl(\sqrt{2}\,x\bigr) + B}{A\exp\bigl(\sqrt{2}\,x\bigr) - B} = -\frac{3}{\sqrt{2}}\,\frac{A\exp\bigl(\frac{\sqrt{2}}{2}\,x\bigr) + B\exp\bigl(-\frac{\sqrt{2}}{2}\,x\bigr)}{A\exp\bigl(\frac{\sqrt{2}}{2}\,x\bigr) - B\exp\bigl(-\frac{\sqrt{2}}{2}\,x\bigr)}, \qquad (33.2.1.10)$$

where A and B are arbitrary constants.

The characteristic system corresponding to (33.2.1.8),

$$\frac{dt}{1} = \frac{dx}{\xi} = -\frac{dw}{\xi_x' w}, \qquad (33.2.1.11)$$

admits the first integrals

$$t + \tfrac{2}{3}\ln\bigl|A\exp\bigl(\tfrac{\sqrt{2}}{2}\,x\bigr) + B\exp\bigl(-\tfrac{\sqrt{2}}{2}\,x\bigr)\bigr| = C_1, \quad \xi w = C_2, \qquad (33.2.1.12)$$

where C_1 and C_2 are arbitrary constants. For convenience, we introduce the new constant $\overline{C}_1 = \exp\bigl(\frac{3}{2}C_1\bigr)$ and look for a solution in the form $C_2 = -\frac{3}{\sqrt{2}}\overline{C}_1 F(\overline{C}_1)$. Inserting (33.2.1.12) here and taking into account (33.2.1.10), we obtain the solution structure

$$\begin{aligned} w(x, t) &= \bigl\{A\exp\bigl[\tfrac{1}{2}(\sqrt{2}x + 3t)\bigr] - B\exp\bigl[\tfrac{1}{2}(-\sqrt{2}x + 3t)\bigr]\bigr\}F(z),\\ z &= A\exp\bigl[\tfrac{1}{2}(\sqrt{2}x + 3t)\bigr] + B\exp\bigl[\tfrac{1}{2}(-\sqrt{2}x + 3t)\bigr]. \end{aligned} \qquad (33.2.1.13)$$

Substituting (33.2.1.13) in the original equation, we find the equation for $F = F(z)$:

$$F_{zz}'' = 2F^3. \qquad (33.2.1.14)$$

The general solution of equation (33.2.1.14) is expressed in terms of elliptic functions. This equation admits the following particular solutions: $F = \pm\dfrac{1}{z + C}$, where C is an arbitrary constant.

Remark 1. To the degenerate case $\xi'_x = 0$ in (33.2.1.9) there corresponds a traveling-wave solution.

Remark 2. It is apparent from this example that using the invariant surface condition (33.1.1.2) gives much more freedom in determining the coordinates ξ, η, ζ as compared with the classical scheme presented in Chapter 32. This stems from the fact that in the classical scheme, the invariance condition is split with respect to two derivatives, w_x and w_{xx}, which are considered to be independent (see Example 32.2 in Subsection 32.2.2). In the nonclassical scheme, the derivatives w_x and w_{xx} are related by the second equation of (33.2.1.5) and the invariance conditions are split in only one derivative, w_x. This is why in the classical scheme the determining system consists of a larger number of equations, which impose additional constraints on the unknown quantities, as compared with the nonclassical scheme. In particular, the classical scheme fails to find exact solutions to equation Newell–Whitehead (33.2.1.1) in the form (33.2.1.13).

33.2.2. Nonlinear Wave Equation

Consider the nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = w\frac{\partial^2 w}{\partial x^2}, \tag{33.2.2.1}$$

which corresponds to the left-hand side $F = w_{tt} - ww_{xx}$ of equation (33.1.1.1) with $y = t$.

Let us add the invariant surface condition (33.2.1.2). Applying the prolonged operator (33.2.1.3) to equation (33.2.2.1) gives the invariance condition. Taking into account that $\partial_x = \partial_t = \partial_{w_x} = \partial_{w_t} = \partial_{w_{xt}} = 0$ (since the equation is independent of x, t, w_x, w_t, w_{xt} explicitly) and $\eta = 1$, we get the invariance condition

$$\zeta_{22} = \zeta w_{xx} + w\zeta_{11}.$$

Substituting here the expressions (32.1.3.7) for the coordinates ζ_{11} and ζ_{22} of the second prolongation with $y = t$ and $\eta = 1$, we have

$$\begin{aligned}
&\zeta_{tt} - \xi_{tt}w_x + 2\zeta_{wt}w_t - 2\xi_{wt}w_xw_t + \zeta_{ww}w_t^2 \\
&\quad - \xi_{ww}w_xw_t^2 - 2(\xi_t + \xi_w w_t)w_{xt} + (\zeta_w - \xi_w w_x)w_{tt} \\
&\qquad = \zeta w_{xx} + w\big[\zeta_{xx} + (2\zeta_{wx} - \xi_{xx})w_x + (\zeta_{ww} - 2\xi_{wx})w_x^2 \\
&\qquad\quad - \xi_{ww}w_x^3 + (\zeta_w - 2\xi_x - 3\xi_w w_x)w_{xx}\big].
\end{aligned} \tag{33.2.2.2}$$

From the invariant surface condition (33.2.1.2) and equation (33.2.2.1) it follows that

$$w_t = \zeta - \xi w_x, \quad w_{tt} = ww_{xx}, \quad w_{xt} = \zeta_x - \xi_x w_x - \xi w_{xx}. \tag{33.2.2.3}$$

The last formula has been obtained by differentiating the first one with respect to x. Substituting w_t, w_{tt}, w_{xt} from (33.2.2.3) into (33.2.2.2) yields a polynomial in two "independent" derivatives, w_x and w_{xx}. Equating the functional coefficients of this polynomial to zero, we arrive at the determining system:

$$\begin{aligned}
w_xw_{xx}\colon&\quad (\xi^2 - w)\xi_w = 0,\\
w_{xx}\colon&\quad 2\xi\xi_t + 2w\xi_x + 2\xi\xi_w\zeta - \zeta = 0,\\
w_x^3\colon&\quad (\xi^2 - w)\xi_{ww} = 0,\\
w_x^2\colon&\quad (\xi^2 - w)\zeta_{ww} + 2\xi\xi_{wt} + 2\xi\xi_{ww}\zeta - 2\xi\xi_x\xi_w + 2w\xi_{wx} = 0,\\
w_x\colon&\quad \xi_{tt} + 2\xi\zeta_{wt} + 2\xi_{wt}\zeta + 2\xi\zeta\zeta_{ww} + \xi_{ww}\zeta^2 - 2\xi_t\xi_x - 2\xi_x\xi_w\zeta\\
&\qquad - 2\xi\xi_w\zeta_x + 2w\zeta_{wx} - w\xi_{xx} = 0,\\
1\colon&\quad \zeta_{tt} + 2\zeta\zeta_{wt} + \zeta^2\zeta_{ww} - 2\xi_t\zeta_x - 2\xi_w\zeta\zeta_x - w\zeta_{xx} = 0.
\end{aligned}$$

From the first equation it follows that

$$1)\quad \xi = \xi(x,t); \tag{33.2.2.4}$$

$$2)\quad \xi = \sqrt{w}. \tag{33.2.2.5}$$

Consider both cases.

1°. For $\xi = \xi(x,t)$, the third equation of the determining system (33.2.2.4) is satisfied identically. From the second equation it follows that

$$\zeta = 2w\xi_x + 2\xi\xi_t. \tag{33.2.2.6}$$

The fourth equation is also satisfied identically in view of (33.2.2.4) and (33.2.2.6). Substituting (33.2.2.4) and (33.2.2.6) into the fifth and sixth equations of the determining system yields two solutions:

$$\begin{aligned} &\xi = \alpha t + \beta, \quad \zeta = 2\alpha(\alpha t + \beta) \quad \text{(first solution);} \\ &\xi = \alpha x + \beta, \quad \zeta = 2\alpha w \qquad\quad \text{(second solution);} \end{aligned} \tag{33.2.2.7}$$

where α and β are arbitrary constants.

First solution. The characteristic system of ordinary differential equations associated with the first solution (33.2.2.7), with $\alpha = 2$ and $\beta = 0$, has the form

$$\frac{dt}{1} = \frac{dx}{2t} = \frac{dw}{8t}.$$

Find the first integrals: $C_1 = x - t^2$, $C_2 = w - 4t^2$. Following the scheme shown in Fig. 33.1, we look for a solution in the form $w - 4t^2 = \Phi(x - t^2)$. Substituting

$$w = \Phi(z) + 4t^2, \quad z = x - t^2 \tag{33.2.2.8}$$

into (33.2.2.1) yields the following autonomous ordinary differential equation for $\Phi = \Phi(z)$:

$$\Phi\Phi''_{zz} + 2\Phi'_z = 8.$$

It is easy to integrate. Reducing its order gives a separable equation. As a result, one can obtain an exact solution to equation (33.2.2.1) of the form (33.2.2.8).

Second solution. The characteristic system of ordinary differential equations associated with the second solution of (33.2.2.7), with $\alpha = 1$ and $\beta = 0$, has the form

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dw}{2w}.$$

Find the first integrals: $C_1 = \ln|x| - t$, $C_2 = w/x^2$. Following the scheme shown in Fig. 33.1, we look for a solution in the form $w/x^2 = \Phi(\ln|x| - t)$. Substituting

$$w = x^2\Phi(z), \quad z = \ln|x| - t$$

into (33.2.2.1) yields the autonomous ordinary differential equation

$$(\Phi - 1)\Phi''_{zz} + 3\Phi\Phi'_z + 2\Phi^2 = 0,$$

which admits order reduction via the substitution $U(\Phi) = \Phi'_z$.

2°. The second case of (33.2.2.5) gives rise to the trivial solution $\zeta = 0$ (as follows from the fourth equation of the determining system), which generates the obvious solution $w = \text{const}$.

⊙ *References for Section* 33.2: G. W. Bluman and J. D. Cole (1969), P. J. Olver and Ph. Rosenau (1987), D. Levi and P. Winternitz (1989), M. C. Nucci and P. A. Clarkson (1992), D. Arrigo, P. Broadbridge, and J. M. Hill (1993), P. A. Clarkson, E. L. Mansfield (1994), P. A. Clarkson (1995), P. A. Clarkson, D. K. Ludlow, T. J. Priestley (1997), A. D. Polyanin, V. F. Zaitsev (2004).

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Chapter 34
Method of Differential Constraints

34.1. Preliminary Remarks. Method of Differential Constraints for Ordinary Differential Equations

34.1.1. Description of the Method. First-Order Differential Constraints

Prior to describing the method of differential constraints for partial differential equations, let us first consider it as applied to simpler, ordinary differential equations.

The main idea of the method is that exact solutions to a complex (nonintegrable) equation are sought by jointly analyzing this equation and an auxiliary simpler (integrable) equation, called a *differential constraint*.

The order of a differential constraint is the order of the highest derivative involved. Usually, the order of the differential constraint is less than that of the equation; first-order differential constraints are simplest and most common. The equation and differential constraint must involve a set of free parameters (or even arbitrary functions) whose values are chosen by ensuring that the equation and the constraint are consistent. After the consistency analysis, all solutions obtained by integrating the differential constraint will be simultaneously solutions to the original equation. The method makes it possible to find particular solutions to the original equation for some values of the determining parameters.

For simplicity, we first consider autonomous ordinary differential equations of the form*

$$F(y, y'_x, \ldots, y_x^{(n)}; \mathbf{a}) = 0, \tag{34.1.1.1}$$

which do not involve the independent variable x explicitly and depend on a vector of free parameters $\mathbf{a} = \{a_1, \ldots, a_k\}$. For equations (34.1.1.1), one should take first-order differential constraints in the autonomous form

$$G(y, y'_x; \mathbf{b}) = 0, \tag{34.1.1.2}$$

dependent on a vector of free parameters $\mathbf{b} = \{b_1, \ldots, b_s\}$.

By differentiating relation (34.1.1.2) successively several times, one can express higher-order derivatives in terms of y and y'_x: $y_x^{(k)} = \varphi_k(y, y'_x; \mathbf{b})$. Substituting these expressions into the original equation (34.1.1.1), one arrives at a first-order equation

$$H(y, y'_x; \mathbf{a}, \mathbf{b}) = 0. \tag{34.1.1.3}$$

By eliminating the derivative y'_x from (34.1.1.2) and (34.1.1.3), one obtains an algebraic/transcendental equation

$$P(y; \mathbf{a}, \mathbf{b}) = 0. \tag{34.1.1.4}$$

Further, one looks for the values of $\mathbf{a}$ and $\mathbf{b}$ at which equation (34.1.1.4) is satisfied identically for any y (this may result in some restrictions on the components of the vector $\mathbf{a}$). After this, one expresses the vector $\mathbf{b}$ in terms of $\mathbf{a}$, so that $\mathbf{b} = \mathbf{b}(\mathbf{a})$, and substitutes it back into the differential constraint (34.1.1.2) to obtain a first-order ordinary differential equation

$$g(y, y'_x; \mathbf{a}) = 0 \qquad (g = G|_{\mathbf{b}=\mathbf{b}(\mathbf{a})}). \tag{34.1.1.5}$$

This equation is consistent with the original equation (34.1.1.1); in other words, the original equation is a consequence of equation (34.1.1.5) and, therefore, inherits all of its solutions. Finally, by solving for the derivative, equation (34.1.1.5) is reduced to a separable equation, which is integrated to obtain a general solution. The general solution of equation (34.1.1.5) is also an exact solution of the original equation (34.1.1.1).

* Similar equations often arise in mathematical physics when exact solutions to nonlinear partial differential equations are sought in the form of a traveling wave (see Section 27.2).

Remark 1. If a first-order differential constraint is defined in explicit form, $y'_x = h(y; \mathbf{b})$, the successive differentiation enables one to express the higher-order derivatives in terms of y, so that $y_x^{(k)} = \varphi_k(y; \mathbf{b})$. Using these expressions and the differential constraint to eliminate the derivatives from (34.1.1.1), one immediately arrives at an algebraic/transcendental equation of the form (34.1.1.4).

Remark 2. Instead of y'_x, one can eliminate the dependent variable y from (34.1.1.2) and (34.1.1.3) to obtain an algebraic/transcendental equation for the derivative: $Q(y'_x; \mathbf{a}, \mathbf{b}) = 0$.

The structure of the nonlinearity of the differential constraint (34.1.1.2) can often be taken to be similar to that of the original equation (34.1.1.1) so as to have different determining parameters. This will be illustrated below by specific examples of second-, third-, fourth-, and higher-order equations.

Example 34.1. Consider the second-order ordinary differential equation with a power-law nonlinearity

$$y''_{xx} - cy'_x = ay + by^n, \tag{34.1.1.6}$$

which arises in the theory of chemical reactors, combustion theory, and mathematical biology.*

Let us supplement equation (34.1.1.6) with the first-order differential constraint

$$y'_x = \alpha y + \beta y^m, \tag{34.1.1.7}$$

which is a separable equation and is easy to integrate. The form of the right-hand side of (34.1.1.7) has been chosen to be similar to that of the original equation (34.1.1.6).

The equation and differential constraint involve seven parameters: a, b, c, n, m, α, and β. The further analysis aims at determining the parameters α, β, and m of the differential constraint so as to express them in terms of a, b, c, and n. Simultaneously, restrictions on the equation parameters will be found.

Differentiating (34.1.1.7) and replacing the first derivative with the right-hand side of (34.1.1.7), we get

$$y''_{xx} = (\alpha + m\beta y^{m-1})y'_x = (\alpha + m\beta y^{m-1})(\alpha y + \beta y^m) = \alpha^2 y + \alpha\beta(m+1)y^m + m\beta^2 y^{2m-1}. \tag{34.1.1.8}$$

Eliminating the first and second derivatives from (34.1.1.6) using (34.1.1.7) and (34.1.1.8) and rearranging, we obtain

$$(\alpha^2 - \alpha c - a)y + \beta[\alpha(m+1) - c]y^m + m\beta^2 y^{2m-1} - by^n = 0.$$

For this equation to hold for all y, one must set

$$\begin{aligned} &\alpha^2 - \alpha c - a = 0, \\ &\alpha(m+1) - c = 0, \\ &2m - 1 = n, \\ &m\beta^2 - b = 0. \end{aligned} \tag{34.1.1.9}$$

If conditions (34.1.1.9) hold, then solutions to equation (34.1.1.7) are also solutions to the more complex equation (34.1.1.6). The determining system of four equations (34.1.1.9) contains seven parameters a, b, c, n, m, α, and β. The three parameters b, c, and n of the original equation can be regarded as arbitrary and the other parameters are expressed as follows:

$$a = -\frac{2c^2(n+1)}{(n+1)^2}, \quad m = \frac{n+1}{2}, \quad \alpha = \frac{2c}{n+3}, \quad \beta = \pm\sqrt{\frac{2b}{n+1}}. \tag{34.1.1.10}$$

It is apparent that for equations (34.1.1.6) and (34.1.1.7) to be consistent, the original equation parameter a must be connected with two other parameters, c and n. In this case, two families of parameters (34.1.1.10) of equation (34.1.1.7) can be identified that determine two different one-parameter solutions to equations (34.1.1.6) and (34.1.1.7); recall that equation (34.1.1.7) is separable and is easy to integrate.

Example 34.2. The second-order equation with an exponential nonlinearity

$$y''_{xx} - cy'_x = a + be^{\lambda y} \tag{34.1.1.11}$$

can be investigated in a similar manner. The equation will be considered in conjunction with the first-order differential constraint

$$y'_x = \alpha + \beta e^{\mu y}. \tag{34.1.1.12}$$

* Equations (34.1.1.6) and (34.1.1.11) describe traveling-wave solutions of Kolmogorov–Petrovskii–Piskunov equation, $w_t = w_{zz} - f(w)$, for some forms of the kinetic function $f(w)$. In this case, we have $w = y(x)$ with $x = z + ct$.

The analysis shows that three parameters of the original equation, b, c, and λ, can be regarded as arbitrary and the other parameters are expressed as

$$a = -\frac{2c^2}{\lambda}, \quad \alpha = \frac{2c}{\lambda}, \quad \beta = \pm\sqrt{\frac{2b}{\lambda}}, \quad \mu = \frac{\lambda}{2}. \tag{34.1.1.13}$$

It is apparent that for equations (34.1.1.11) and (34.1.1.12) to be consistent, the parameter a must be related in a certain way to two other parameters of the equation, c and λ. In this case, two families of parameters (34.1.1.13) of the differential constraint (34.1.1.12) can be identified which two different one-parameter solutions to equations (34.1.1.11) and (34.1.1.12). Equation (34.1.1.12) is separable and is easy to integrate.

Example 34.3. Consider the nonlinear third-order equation

$$y'''_{xxx} = ay^4 + by^2 + c \tag{34.1.1.14}$$

in conjunction with the first-order differential constraint

$$y'_x = \alpha y^2 + \beta. \tag{34.1.1.15}$$

Using (34.1.1.15), we find the derivatives

$$y''_{xx} = 2\alpha y y'_x = 2\alpha y(\alpha y^2 + \beta) = 2\alpha^3 y^3 + 2\alpha\beta y,$$
$$y'''_{xxx} = (6\alpha^2 y^2 + 2\alpha\beta)y'_x = (6\alpha^2 y^2 + 2\alpha\beta)(\alpha y^2 + \beta) = 6\alpha^3 y^4 + 8\alpha^2\beta y^2 + 2\alpha\beta^2.$$

For the last equation to coincide with (34.1.1.14), the relations

$$a = 6\alpha^3, \quad b = 8\alpha^2\beta, \quad c = 2\alpha\beta^2$$

must hold. On solving the first two equations for α and β and substituting the resulting expressions into the last equation, we obtain

$$\alpha = \left(\frac{a}{6}\right)^{1/3}, \quad \beta = \left(\frac{a}{6}\right)^{-2/3}\frac{b}{8}, \quad c = \frac{3b^2}{16a}. \tag{34.1.1.16}$$

It follows that with this c, the third-order equation (34.1.1.14) has a particular solution resulting from solving the first-order separable equation (34.1.1.15) whose parameters are connected with those of the original equation by the first two relations in (34.1.1.16).

Example 34.4. Consider the nonlinear fourth-order equation

$$y''''_{xxxx} = ay^n + by^{2n+3} \tag{34.1.1.17}$$

in conjunction with the first-order differential constraint

$$(y'_x)^2 = \alpha y^m + \beta. \tag{34.1.1.18}$$

Differentiating (34.1.1.18), we get the derivatives

$$\begin{aligned} y''_{xx} &= \tfrac{1}{2}\alpha m y^{m-1} \qquad \text{(after canceling by } y'_x\text{)},\\ y'''_{xxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y'_x,\\ y''''_{xxxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y''_{xx} + \tfrac{1}{2}\alpha m(m-1)(m-2)y^{m-3}(y'_x)^2\\ &= \tfrac{1}{2}\alpha\beta m(m-1)(m-2)y^{m-3} + \tfrac{1}{4}\alpha^2 m(m-1)(3m-4)y^{2m-3}. \end{aligned} \tag{34.1.1.19}$$

Comparing the right-hand side of (34.1.1.17) and that of the last equation in (34.1.1.19) allows us to draw the following conclusions about the consistency of (34.1.1.17) and (34.1.1.19).

1°. For any values of the parameters of the original equation (34.1.1.17) except for $n \neq -1, -2, -3, -\frac{5}{3}$ and $b \neq 0$, one can calculate the parameters of the differential constraint (34.1.1.18) by the formulas

$$m = n + 3, \quad \alpha = \pm 2\sqrt{\frac{b}{(n+2)(n+3)(3n+5)}}, \quad \beta = \frac{2a}{\alpha(n+1)(n+2)(n+3)}.$$

2°. For $b = 0$ and $n = -\frac{5}{3}$, we have

$$m = \frac{4}{3}, \quad \beta = -\frac{27a}{4\alpha}, \quad \alpha \neq 0 \text{ is an arbitrary constant.}$$

In this case, the solution to equation (34.1.1.18) will depend on two arbitrary constants (α plays the role of an additional constant of integration).

Remark. For $b = 0$ and $n = -\frac{5}{3}$, one can find the general solution of equation (34.1.1.17) (see Polyanin and Zaitsev, 2003, p. 659).

Example 34.5. The fourth-order equation with an exponential nonlinearity

$$y'''' _{xxxx} = ae^{\lambda y} + b^{2\lambda y}$$

can be analyzed using the differential constraint

$$(y'_x)^2 = \alpha e^{\lambda y} + \beta.$$

Example 34.6. The nonlinear nth-order equation

$$y_x^{(n)} = ae^{\lambda y}$$

admits the first-order differential constraint

$$y'_x = \alpha e^{\mu y}.$$

The successive differentiation of the differential constraint gives $y_x^{(n)} = \alpha^n \mu^{n-1}(n-1)!\, e^{n\mu y}$. Comparing this expression with the equation yields $\lambda = n\mu$ and $a = \alpha^n \mu^{n-1}(n-1)!$, or

$$\mu = \frac{\lambda}{n}, \quad \alpha = \left[\frac{an^{n-1}}{\lambda^{n-1}(n-1)!}\right]^{1/n}.$$

34.1.2. Differential Constraints of Arbitrary Order. General Consistency Method for Two Equations

In general, a differential constraint is an ordinary differential equation of arbitrary order. Therefore, it is necessary to be able to analyze overdetermined systems of two ordinary differential equations for consistency. Outlined below is the general algorithm for the analysis of such systems.

1°. First, let us consider two ordinary differential equations of the same order

$$F_1(x, y, y'_x, \ldots, y_x^{(n)}) = 0, \tag{34.1.2.1}$$

$$F_2(x, y, y'_x, \ldots, y_x^{(n)}) = 0; \tag{34.1.2.2}$$

here and henceforth, it is assumed that the equations depend on free parameters, which are omitted for brevity. We eliminate the highest derivative (by solving one of the equations for $y_x^{(n)}$ and substituting the resulting expression into the other equation) to obtain the $(n-1)$st-order equation

$$G_1(x, y, y'_x, \ldots, y_x^{(n-1)}) = 0. \tag{34.1.2.3}$$

Differentiating (34.1.2.3) with respect to x and eliminating the derivative $y_x^{(n)}$ from the resulting equation using either of the equations (34.1.2.1) and (34.1.2.2), one arrives at another $(n-1)$st-order equation

$$G_2(x, y, y'_x, \ldots, y_x^{(n-1)}) = 0. \tag{34.1.2.4}$$

Thus, the analysis of two nth-order equations (34.1.2.1) and (34.1.2.2) is reduced to the analysis of two $(n-1)$st-order equations (34.1.2.3) and (34.1.2.4). By reducing the order of equations in a similar manner further, one ultimately arrives at a single algebraic/transcendental equation (since two first-order differential equations are reducible to a single algebraic equation). The analysis of the resulting algebraic equation presents no fundamental difficulties and is performed in the same way as previously in Subsection 34.1.1 for the case of a first-order differential constraint.

2°. Suppose there are two ordinary differential equations having different orders:

$$F_1(x, y, y'_x, \ldots, y_x^{(n)}) = 0, \tag{34.1.2.5}$$

$$F_2(x, y, y'_x, \ldots, y_x^{(m)}) = 0, \tag{34.1.2.6}$$

with $m < n$. Then, by differentiating (34.1.2.6) $n - m$ times, one reduces system (34.1.2.5)–(34.1.2.6) to a system of the form (34.1.2.1)–(34.1.2.2), in which both equations have the same order n.

Example 34.7. Consider the fourth-order equation with a quadratic nonlinearity

$$y''''_{xxxx} = a(y''_{xx})^2 - by^2 + c \tag{34.1.2.7}$$

in conjunction with the second-order differential constraint

$$y''_{xx} = \alpha y + \beta. \tag{34.1.2.8}$$

Differentiating (34.1.2.8) twice gives $y''''_{xxxx} = \alpha^2 y + \alpha\beta$. Using this expression and the differential constraint (34.1.2.8) to eliminate the derivatives from (34.1.2.7), one arrives at a quadratic equation for y, which is satisfied identically if the conditions

$$a\alpha^2 - b = 0, \quad \alpha - 2a\beta = 0, \quad c = \alpha\beta - a\beta^2$$

hold. Two parameters of the original equation, a and b, can be regarded as arbitrary and the others are expressed in terms of them as follows:

$$c = \frac{b}{4a^2}, \quad \alpha = \pm\sqrt{\frac{b}{a}}, \quad \beta = \pm\frac{1}{2a}\sqrt{\frac{b}{a}}.$$

Example 34.8. The equation of order mn with a quadratic nonlinearity

$$y_x^{(mn)} = a[y_x^{(n)}]^2 + byy_x^{(n)} + cy_x^{(n)} + dy^2 + ky + p \qquad (m \text{ is positive integer}),$$

which generalizes equation (34.1.2.7), can be investigated using the nth-order differential constraint

$$y_x^{(n)} = \alpha y + \beta.$$

3°. The general autonomous second-order differential constraint

$$y''_{xx} = f(y)$$

is equivalent to the autonomous first-order differential constraint

$$(y'_x)^2 = F(y),$$

where $F(y) = 2\int f(y)\,dy + C$ and C is an arbitrary constant. This is proved by differentiating the latter relation and comparing with the original differential constraint.

With this in mind, the second-order differential constraint (34.1.2.8) in Example 34.7 could be replaced by the first-order constraint $(y'_x)^2 = \alpha y^2 + 2\beta y + \gamma$, where γ is an extra free parameter. However, the differential constraint (34.1.2.8) is linear and is easy to integrate.

4°. In principle, any differential constraint of arbitrary order (34.1.2.6) can be replaced by a suitable first-order differential constraint. Indeed, the above algorithm for successive order reduction of system (34.1.2.5)–(34.1.2.6) leads, in the nondegenerate case, to a system of first-order equations, one of which can be treated as a first-order differential constraint.

34.1.3. Some Generalizations. The Case of Several Differential Constraints

1°. In some situations, the equation under study is supplemented with several differential constraints containing an additional unknown function. To be specific, let us return to the nth-order autonomous equation (34.1.1.1) and supplement it with two first-order differential constraints

$$y = G(w, w'_x; \mathbf{b}), \tag{34.1.3.1}$$

$$H(w, w'_x; \mathbf{c}) = 0, \tag{34.1.3.2}$$

where $\mathbf{b}$ and $\mathbf{c}$ are vectors of free parameters. On substituting (34.1.3.1) into (34.1.1.1), one obtains an $(n+1)$st-order equation for $w = w(x)$:

$$F_1(w, w'_x, \ldots, w_x^{(n+1)}; \mathbf{a}, \mathbf{b}) = 0. \tag{34.1.3.3}$$

This equation in conjunction with the differential constraint (34.1.3.2) is analyzed with the method outlined in Subsection 34.1.1. There is an insignificant distinction that the order of equation (34.1.3.3) is higher than that of the original equation (34.1.1.1).

Remark. The differential constraints (34.1.3.1) and (34.1.3.2) can involve higher derivatives of w with respect to x.

Kudryashov (2004, 2005) chose the differential constraint (34.1.3.2) in one of the following three forms:

$$w_x + w^2 - c_1 w - c_2 = 0, \tag{34.1.3.4}$$

$$(w'_x)^2 - 4w^3 - c_1 w^2 - c_2 w - c_3 = 0, \tag{34.1.3.5}$$

$$(w'_x)^2 - w^4 - c_1 w^3 - c_2 w^2 - c_3 w - c_4 = 0. \tag{34.1.3.6}$$

The differential constraint (34.1.3.1) was chosen from the class of functions

$$y = \sum_{k=0}^{K} c_{1k} w^k + w'_x \sum_{l=0}^{L} c_{2l} w^l + \sum_{m=1}^{M} c_{3m} \left(\frac{w'_x}{w} \right)^m. \tag{34.1.3.7}$$

For the differential constraint (34.1.3.4), it was assumed that $K = M$ and $c_{2l} = 0$ ($l = 1, \ldots, L$) in (34.1.3.7). This resulted in a number of new exact solutions to nonlinear second-, third-, and fourth-order equations.

Remark. All equations (34.1.3.4)–(34.1.3.6) are reduced to separable equations, whose solutions are expressed in terms of elementary functions or/and integrals of elementary functions. A solution to (34.1.3.5) can be expressed through the Weierstrass function $\wp = \wp(z, g_2, g_3)$. A solution to equation (34.1.3.6) is expressed through the Jacobi elliptic function.

2°. In some cases, it is useful to reduce the equation under study, with a point transformation (or, sometimes, with a differential substitution), to another equation, which can be analyzed using a suitable differential constraint.

Example 34.9. Consider the second-order ordinary differential equation with a power-law nonlinearity of arbitrary degree

$$y''_{xx} - cy'_x = ay + by^n + dy^{2n-1}, \tag{34.1.3.8}$$

which generalizes equation (34.1.1.6) to the case of $d \neq 0$. The substitution $u = y^{1-n}$ reduces (34.1.3.8) to an equation with a quadratic nonlinearity,

$$uu''_{xx} + \frac{n}{1-n}(u'_x)^2 - cuu_x = a(1-n)u^2 + b(1-n)u + d(1-n), \tag{34.1.3.9}$$

which is easier to analyze.

1°. Let us supplement equation (34.1.3.9) with the linear differential constraint

$$u'_x = \alpha u + \beta. \tag{34.1.3.10}$$

We use this relation to eliminate the derivatives in (34.1.3.9) to obtain a quadratic equation for u, which is satisfied identically if the conditions (determining system of algebraic equations)

$$\frac{1}{1-n}\alpha^2 - c\alpha = a(1-n), \quad \frac{1+n}{1-n}\alpha\beta - c\beta = b(1-n), \quad n\beta^2 = d(1-n)^2$$

hold. The first and last equations give to pairs of solutions each,

$$\alpha_{1,2} = \frac{1}{2}(1-n)\Big(c \pm \sqrt{c^2 + 4a}\Big), \quad \beta_{1,2} = \pm\frac{1}{1-n}\sqrt{\frac{d}{n}}, \tag{34.1.3.11}$$

which are then substituted into the second equation. As a result, we obtain one restriction (not written out here) for each of the parameters a, b, c, d, and n per pair α_i, β_j.

It is apparent from (34.1.3.11) that for $a = 0$, equation (34.1.3.9) admits a simple, degenerate first-order differential constraint

$$u'_x = \beta = \text{const}.$$

2°. For $c = 0$, equation (34.1.3.9) can be supplemented with the differential constraint

$$(u'_x)^2 = \alpha u^2 + \beta u + \gamma. \tag{34.1.3.12}$$

A simple analysis shows that the constraint coefficients in (34.1.3.12) are expressed in terms of the equation coefficients in (34.1.3.8) as follows:

$$\alpha = \frac{a(1-n)^2}{2-n}, \quad \beta = b(1-n)^2, \quad \gamma = \frac{d(1-n)^2}{n}.$$

3°. For $b = 0$, differentiating (34.1.3.9) leads to a third-order differential equation with a quadratic nonlinearity,

$$uu'''_{xxx} + \frac{1+n}{1-n}u'_x u''_{xx} - cuu_{xx} - c(u'_x)^2 - 2a(1-n)uu_x = 0,$$

which is homogeneous in u and whose order can be reduced using the differential substitution $w = u'_x/u$. The resulting equivalent third- and second-order equations can be analyzed using the differential constraint method.

⊙ *References for Section* 34.1: N. A. Kudryashov (2004, 2005), A. D. Polyanin and V. F. Zaitsev (2003, 2004).

34.2. Description of the Method for Partial Differential Equations

34.2.1. Preliminary Remarks. A Simple Example

Basic idea: Try to find an exact solution to a complex partial differential equation by analyzing it in conjunction with a simpler, auxiliary equation, called a *differential constraint*.

In Section 29.1, we considered examples of additive separable solutions of nonlinear equations in the form

$$w(x,y)=\varphi(x)+\psi(y). \tag{34.2.1.1}$$

At the initial stage, the functions $\varphi(x)$ and $\psi(y)$ are assumed arbitrary and are to be determined in the subsequent analysis.

Differentiating the expression (34.2.1.1) with respect to y, we obtain

$$\frac{\partial w}{\partial y}=f(y)\qquad (f=\psi'_y). \tag{34.2.1.2}$$

Conversely, relation (34.2.1.2) implies a representation of the solution in the form (34.2.1.1).

Further, differentiating (34.2.1.2) with respect to x gives

$$\frac{\partial^2 w}{\partial x\partial y}=0. \tag{34.2.1.3}$$

Conversely, from (34.2.1.3) we obtain a representation of the solution in the form (34.2.1.1).

Thus, the problem of finding exact solutions of the form (34.2.1.1) for a specific partial differential equation may be replaced by an equivalent problem of finding exact solutions of the given equation supplemented with the condition (34.2.1.2) or (34.2.1.3). Such supplementary conditions in the form of one or several differential equations will be called *differential constraints*.

Prior to giving a general description of the differential constraints method, we demonstrate its features by a simple example.

Example 34.10. Consider the third-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=a\frac{\partial^3 w}{\partial y^3}, \tag{34.2.1.4}$$

which occurs in the theory of the hydrodynamic boundary layer. Let us seek a solution of equation (34.2.1.4) satisfying the linear first-order differential constraint

$$\frac{\partial w}{\partial x}=\varphi(y). \tag{34.2.1.5}$$

Here, the function $\varphi(y)$ cannot be arbitrary, in general, but must satisfy the condition of compatibility of equations (34.2.1.4) and (34.2.1.5). The compatibility condition is a differential equation for $\varphi(y)$ and is a consequence of equations (34.2.1.4), (34.2.1.5), and those obtained by their differentiation.

Successively differentiating (34.2.1.5) with respect to different variables, we calculate the derivatives

$$w_{xx}=0,\quad w_{xy}=\varphi'_y,\quad w_{xxy}=0,\quad w_{xyy}=\varphi''_{yy},\quad w_{xyyy}=\varphi'''_{yyy}. \tag{34.2.1.6}$$

Differentiating (34.2.1.4) with respect to x yields

$$w_{xy}^2+w_y w_{xxy}-w_{xx}w_{yy}-w_x w_{xyy}=aw_{xyyy}. \tag{34.2.1.7}$$

Substituting the derivatives of the function w from (34.2.1.5) and (34.2.1.6) into (34.2.1.7), we obtain the following third-order ordinary differential equation for φ:

$$(\varphi'_y)^2-\varphi\varphi''_{yy}=a\varphi'''_{yyy}, \tag{34.2.1.8}$$

which represents the compatibility condition for equations (34.2.1.4) and (34.2.1.5).

In order to construct an exact solution, we integrate equation (34.2.1.5) to obtain

$$w=\varphi(y)x+\psi(y). \tag{34.2.1.9}$$

The function $\psi(y)$ is found by substituting (34.2.1.9) into (34.2.1.4) and taking into account the condition (34.2.1.8). As a result, we arrive at the ordinary differential equation

$$\varphi'_y\psi'_y-\varphi\psi''_{yy}=a\psi'''_{yyy}. \tag{34.2.1.10}$$

Finally, we obtain an exact solution of the form (34.2.1.9), with the functions φ and ψ described by equations (34.2.1.8) and (34.2.1.10).

Remark. It is easier to obtain the above solution by directly substituting expression (34.2.1.9) into the original equation (34.2.1.4).

34.2.2. General Description of the Method of Differential Constraints

In the general case, the procedure of the construction of exact solutions for nonlinear partial equations of mathematical physics by the method of differential constraints is significantly more complex than that for ordinary differential equations and consists of several steps described below.

1°. Searching for exact solutions of the equation

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \ldots\right) = 0 \tag{34.2.2.1}$$

is performed by supplementing this equation with a differential constraint (other differential equation)

$$G\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \ldots\right) = 0. \tag{34.2.2.2}$$

The form of the differential constraint (34.2.2.2) may be prescribed on the basis of (i) a priori considerations (for instance, it may be required that the constraint should represent a solvable equation); (ii) certain properties of the equation under consideration (for instance, it may be required that the constraint should follow from symmetries of the equation or the corresponding conservation laws).

2°. In general, the thus obtained overdetermined system (34.2.2.1)–(34.2.2.2) requires a compatibility analysis. If the differential constraint (34.2.2.2) is specified on the basis of a priori considerations, it should allow for sufficient freedom in choosing functions (i.e., involve arbitrary determining functions). The compatibility analysis of system (34.2.2.1)–(34.2.2.2) should provide conditions that specify the structure of the determining functions. These conditions (*compatibility conditions*) are written as a system of ordinary differential equations (or a system of partial differential equations).

Usually, the compatibility analysis is performed by means of differentiating (possibly, several times) equations (34.2.2.1) and (34.2.2.2) with respect to x and y and eliminating the highest-order derivatives from the resulting differential relations and equations (34.2.2.1) and (34.2.2.2). As a result, one arrives at an equation involving powers of lower-order derivatives. Equating the coefficients of all powers of the derivatives to zero, one obtains compatibility conditions connecting the functional coefficients of equations (34.2.2.1) and (34.2.2.2).

3°. One solves the system of differential equations obtained in Item 2° for the determining functions. Then these functions are substituted into the differential constraint (34.2.2.2) to obtain an equation of the form

$$g\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \ldots\right) = 0. \tag{34.2.2.3}$$

A differential constraint (34.2.2.3) that is compatible with equation (34.2.2.1) under consideration is called an *invariant manifold* of equation (34.2.2.1).

4°. One should find the general solution of (i) equation (34.2.2.3) or (ii) some equation that follows from equations (34.2.2.1) and (34.2.2.3). The solution thus obtained will involve some arbitrary functions $\{\varphi_m\}$ (these may depend on x and y, as well as w). Note that in some cases, one can use, instead of the general solution, some particular solutions of equation (34.2.2.3) or equations that follow from (34.2.2.3).

5°. The solution obtained in Item 4° should be substituted into the original equation (34.2.2.1). As a result, one arrives at a functional differential equation from which the functions $\{\varphi_m\}$ should be found. Having found the $\{\varphi_m\}$, one should insert these functions into the solution from Item 4°. Thus, one obtains an exact solution of the original equation (34.2.2.1).

Remark 1. Should the choice of a differential constraint be inadequate, equations (34.2.2.1) and (34.2.2.2) may happen to be incompatible (having no common solutions).

Remark 2. There may be several differential constraints of the form (34.2.2.2).

Remark 3. At the last three steps of the method of differential constraints, one has to solve various equations (systems of equations). If no solution can be constructed at one of those steps, one fails to construct an exact solution of the original equation.

For the sake of clarity, the general scheme of the method of differential constraints is represented in Figure 34.1.

⊙ *References for Subsection* 34.2: N. N. Yanenko (1964), A. F. Sidorov, V. P. Shapeev, and N. N. Yanenko (1984), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2004).

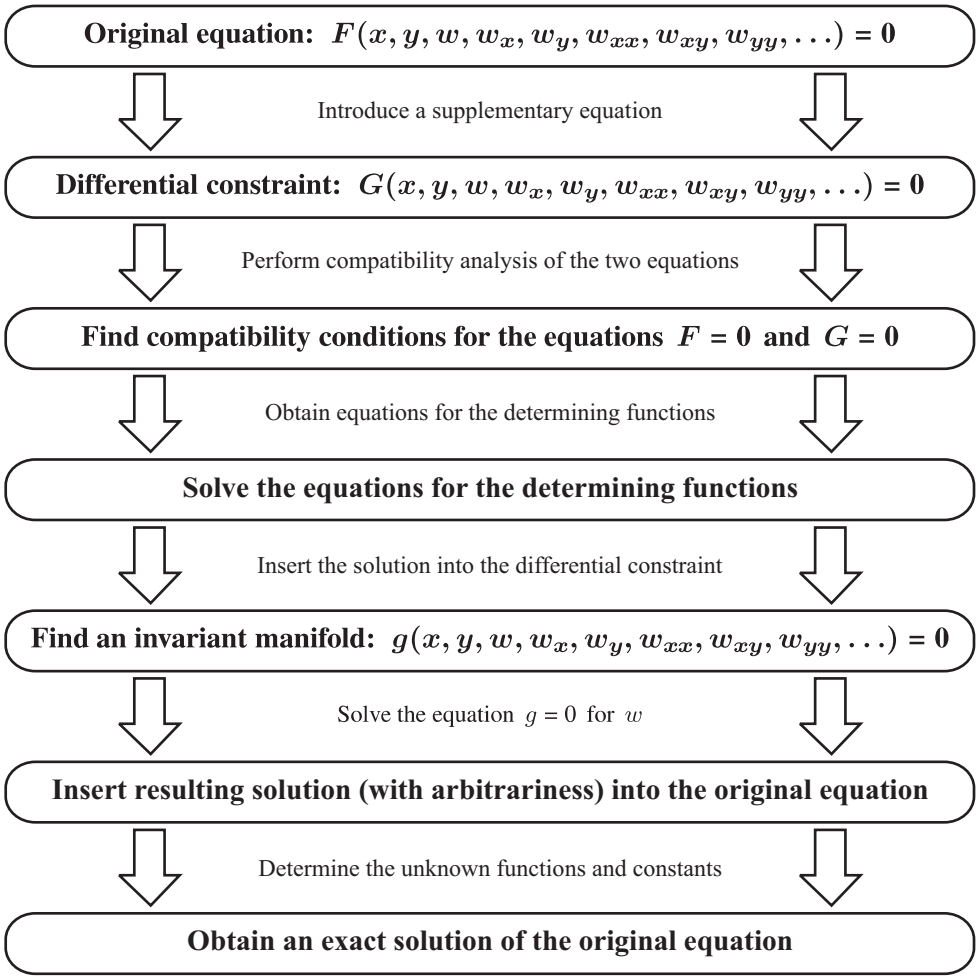

Figure 34.1. Algorithm for the construction of exact solutions by the method of differential constraints.

34.3. First-Order Differential Constraints for PDEs

34.3.1. Second-Order Evolution Equations

Consider a general second-order evolution equation solved for the highest-order derivative:

$$\frac{\partial^2 w}{\partial x^2} = \mathcal{F}\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right). \tag{34.3.1.1}$$

Let us supplement this equation with a first-order differential constraint

$$\frac{\partial w}{\partial t} = \mathcal{G}\left(x, t, w, \frac{\partial w}{\partial x}\right). \tag{34.3.1.2}$$

The condition of compatibility of these equations is obtained by differentiating (34.3.1.1) with respect to t once and differentiating (34.3.1.2) with respect to x twice, and then equating the two resulting expressions for the third derivatives w_{xxt} and w_{txx}:

$$\mathrm{D}_t\mathcal{F} = \mathrm{D}_x^2\mathcal{G}. \tag{34.3.1.3}$$

Here, D_t and D_x are the total differentiation operators with respect to t and x:

$$\mathrm{D}_t = \frac{\partial}{\partial t} + w_t\frac{\partial}{\partial w} + w_{xt}\frac{\partial}{\partial w_x} + w_{tt}\frac{\partial}{\partial w_t}, \quad \mathrm{D}_x = \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{xt}\frac{\partial}{\partial w_t}. \tag{34.3.1.4}$$

The partial derivatives w_t, w_{xx}, w_{xt}, and w_{tt} in (34.3.1.4) should be expressed in terms of x, t, w, and w_x by means of the relations (34.3.1.1), (34.3.1.2) and those obtained by differentiation of (34.3.1.1), (34.3.1.2). As a result, we get

$$\begin{aligned} &w_t = \mathcal{G}, \quad w_{xx} = \mathcal{F}, \quad w_{xt} = \mathrm{D}_x\mathcal{G} = \frac{\partial \mathcal{G}}{\partial x} + w_x\frac{\partial \mathcal{G}}{\partial w} + \mathcal{F}\frac{\partial \mathcal{G}}{\partial w_x}, \\ &w_{tt} = \mathrm{D}_t\mathcal{G} = \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G}\frac{\partial \mathcal{G}}{\partial w} + w_{xt}\frac{\partial \mathcal{G}}{\partial w_x} = \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G}\frac{\partial \mathcal{G}}{\partial w} + \left(\frac{\partial \mathcal{G}}{\partial x} + w_x\frac{\partial \mathcal{G}}{\partial w} + \mathcal{F}\frac{\partial \mathcal{G}}{\partial w_x}\right)\frac{\partial \mathcal{G}}{\partial w_x}. \end{aligned} \tag{34.3.1.5}$$

In the expression for $\mathcal{F}$, the derivative w_t should be replaced by $\mathcal{G}$ by virtue of (34.3.1.2).

By virtue of relations (34.3.1.2), (34.3.1.4), and (34.3.1.5), the compatibility condition (34.3.1.3) becomes $R(x, t, w, \frac{\partial w}{\partial x}) = 0$. We assume that the left-hand side of this relation can be represented as a polynomial in the derivative:

$$\sum_{m=1}^{M} R_m(x, t, w)\Big(\frac{\partial w}{\partial x}\Big)^m = 0.$$

The splitting procedure in the derivative leads to the system of determining equations

$$R_m(x, t, w) = 0, \qquad m = 1, \ldots, M.$$

Example 34.11. From the class of nonlinear heat equations with a source

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + g(w), \tag{34.3.1.6}$$

let us single out equations possessing invariant manifolds of the simplest form

$$\frac{\partial w}{\partial t} = \varphi(w). \tag{34.3.1.7}$$

Equations (34.3.1.6) and (34.3.1.7) are special cases of (34.3.1.1) and (34.3.1.2) with

$$\mathcal{F} = \frac{w_t - f'(w)w_x^2 - g(w)}{f(w)} = \frac{\varphi(w) - g(w) - f'(w)w_x^2}{f(w)}, \quad \mathcal{G} = \varphi(w).$$

The functions $f(w)$, $g(w)$, and $\varphi(w)$ are unknown in advance and are to be determined in the subsequent analysis.

Using (34.3.1.5) and (34.3.1.4), we find partial derivatives and the total differentiation operators:

$$\begin{aligned} &w_t = \varphi, \quad w_{xx} = \mathcal{F}, \quad w_{xt} = \varphi' w_x, \quad w_{tt} = \varphi\varphi', \\ &\mathrm{D}_t = \frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial w} + \varphi' w_x\frac{\partial}{\partial w_x} + \varphi\varphi'\frac{\partial}{\partial w_t}, \quad \mathrm{D}_x = \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + \mathcal{F}\frac{\partial}{\partial w_x} + \varphi' w_x\frac{\partial}{\partial w_t}. \end{aligned}$$

We insert the expressions of D_x and D_t into the compatibility conditions (34.3.1.3) and rearrange terms to obtain

$$\left[\frac{(f\varphi)'}{f}\right]' w_x^2 + \frac{\varphi - g}{f}\varphi' - \varphi\left(\frac{\varphi - g}{f}\right)' = 0.$$

In order to ensure that this equality holds true for any w_x, one should take

$$\left[\frac{(f\varphi)'}{f}\right]' = 0, \qquad \frac{\varphi - g}{f}\varphi' - \varphi\left(\frac{\varphi - g}{f}\right)' = 0. \tag{34.3.1.8}$$

Nondegenerate case. Assuming that the function $f = f(w)$ is given, we obtain a three-parameter solution of equations (34.3.1.8) for the functions $g(w)$ and $\varphi(w)$:

$$g(w) = \frac{a + cf}{f}\left(\int f\,dw + b\right), \quad \varphi(w) = \frac{a}{f}\left(\int f\,dw + b\right), \tag{34.3.1.9}$$

where a, b, and c are arbitrary constants.

We substitute $\varphi(w)$ of (34.3.1.9) into equation (34.3.1.7) and integrate to obtain

$$\int f\,dw = \theta(x)e^{at} - b. \tag{34.3.1.10}$$

Differentiating (34.3.1.10) with respect to x and t, we get $w_t = ae^{at}\theta/f$ and $w_x = e^{at}\theta'_x/f$. Substituting these expressions into (34.3.1.6) and taking into account (34.3.1.9), we arrive at the equation $\theta''_{xx} + c\theta = 0$, whose general solution is given by

$$\theta = \begin{cases} C_1 \sin\bigl(x\sqrt{c}\bigr) + C_2 \cos\bigl(x\sqrt{c}\bigr) & \text{if } c > 0, \\ C_1 \sinh\bigl(x\sqrt{-c}\bigr) + C_2 \cosh\bigl(x\sqrt{-c}\bigr) & \text{if } c < 0, \\ C_1 x + C_2 & \text{if } c = 0, \end{cases} \tag{34.3.1.11}$$

where C_1 and C_2 are arbitrary constants. Formulas (34.3.1.10)–(34.3.1.11) describe exact solutions (in implicit form) of equation (34.3.1.6) with $f(w)$ arbitrary and $g(w)$ given by (34.3.1.9).

Degenerate case. There also exists a two-parameter solution of equations (34.3.1.8) for the functions $g(w)$ and $\varphi(w)$ (as above, f is assumed arbitrary):

$$g(w) = \frac{b}{f} + c, \qquad \varphi(w) = \frac{b}{f},$$

where b and c are arbitrary constants. This solution can be obtained from (34.3.1.9) by renaming variables, $b \to b/a$ and $c \to ac/b$, and letting $a \to 0$. After simple calculations, we obtain the corresponding solution of equation (34.3.1.6) in implicit form:

$$\int f\,dw = bt - \frac{1}{2}cx^2 + C_1 x + C_2.$$

The example given below shows that calculations may be performed without the use of the general formulas (34.3.1.3)–(34.3.1.5).

Example 34.12. Consider the problem of finding second-order nonlinear equations

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f_1(w)\frac{\partial w}{\partial x} + f_0(w) \tag{34.3.1.12}$$

admitting first-order invariant manifolds of the form

$$\frac{\partial w}{\partial t} = g_1(w)\frac{\partial w}{\partial x} + g_0(w). \tag{34.3.1.13}$$

Equations (34.3.1.12) and (34.3.1.13) are special cases of (34.3.1.1) and (34.3.1.2) for $\mathcal{F} = w_t - f_1(w)w_x - f_0(w)$ and $\mathcal{G} = g_1(w)w_x + g_0(w)$. The functions $f_1(w)$, $f_0(w)$, $g_1(w)$, and $g_0(w)$ are unknown in advance and are to be determined in the subsequent analysis.

First, we calculate derivatives. Equating the right-hand sides of (34.3.1.12) and (34.3.1.13), we get

$$w_{xx} = h_1 w_x + h_0, \quad \text{where} \quad h_1 = g_1 - f_1, \quad h_0 = g_0 - f_0. \tag{34.3.1.14}$$

Here and in what follows, the argument of the functions f_1, f_0, g_1, g_0, h_1, and h_0 is omitted. Differentiating (34.3.1.13) with respect to x twice and using the expression (34.3.1.14) for w_{xx}, we find the mixed derivatives

$$\begin{aligned} w_{xt} &= g_1 w_{xx} + g_1' w_x^2 + g_0' w_x = g_1' w_x^2 + (g_1 h_1 + g_0')w_x + g_1 h_0, \\ w_{xxt} &= g_1'' w_x^3 + (g_1 h_1' + 3g_1' h_1 + g_0'')w_x^2 + (g_1 h_0' + 3g_1' h_0 + g_1 h_1^2 + g_0' h_1)w_x + (g_1 h_1 + g_0')h_0, \end{aligned} \tag{34.3.1.15}$$

where the prime denotes a derivative with respect to w. Differentiating (34.3.1.14) with respect to t and using the expressions (34.3.1.13) and (34.3.1.15) for w_t and w_{xt}, we obtain

$$w_{xxt} = h_1 w_{xt} + h_1' w_x w_t + h_0' w_t = (g_1 h_1' + g_1' h_1)w_x^2 + (g_1 h_1^2 + g_0' h_1 + g_0 h_1' + g_1 h_0')w_x + g_1 h_0 h_1 + g_0 h_0'. \tag{34.3.1.16}$$

We equate the expressions for the third derivative w_{xxt} from (34.3.1.15) and (34.3.1.16) and collect terms with the same power of w_x to obtain an invariance condition in the form

$$g_1'' w_x^3 + (2g_1' h_1 + g_0'')w_x^2 + (3g_1' h_0 - g_0 h_1')w_x + g_0' h_0 - g_0 h_0' = 0. \tag{34.3.1.17}$$

For condition (34.3.1.17) to hold we require that the coefficients of like powers of w_x be zero:

$$g_1'' = 0, \quad 2g_1' h_1 + g_0'' = 0, \quad 3g_1' h_0 - g_0 h_1' = 0, \quad g_0' h_0 - g_0 h_0' = 0.$$

The general solution of this system of ordinary differential equations is given by the following formulas:

$$g_1 = C_1 w + C_2, \quad g_0 = -C_1^2 C_3 w^3 - C_1 C_4 w^2 + C_5 w + C_6, \quad h_1 = 3C_1 C_3 w + C_4, \quad h_0 = C_3 g_0, \tag{34.3.1.18}$$

where $C_1, \ldots, C_6$ are arbitrary constants. Using formulas (34.3.1.14) for h_0 and h_1 together with (34.3.1.18), we find the unknown functions involved in equations (34.3.1.12) and (34.3.1.13):

$$\begin{aligned} &f_1(w) = C_1(1-3C_3)w + C_2 - C_4, \quad f_0(w) = (-C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6)(1-C_3),\\ &g_1(w) = C_1w + C_2, \quad g_0(w) = -C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6. \end{aligned} \tag{34.3.1.19}$$

Let us dwell on the special case of

$$C_1 = -k, \quad C_2 = C_4 = 0, \quad C_3 = -1/k, \quad C_5 = ak, \quad C_6 = bk$$

in (34.3.1.19), where a, b, and k are arbitrary constants ($k \neq 0$). The corresponding equation (34.3.1.12) and the invariant manifold (34.3.1.13) have the form

$$w_t = w_{xx} - (k+3)ww_x + (k+1)(w^3 + aw + b), \tag{34.3.1.20}$$

$$w_t = -kww_x + k(w^3 + aw + b). \tag{34.3.1.21}$$

The general solution of the first-order quasilinear equation (34.3.1.21) can be written out in implicit form; it involves the integral $I(w) = \int w(w^3 + aw + b)^{-1}\,dw$ and its inversion. Due to its complex structure, this solution is inconvenient for the construction of exact solutions of equation (34.3.1.20).

In this situation, instead of (34.3.1.21) one can use equations obtained from (34.3.1.20) and (34.3.1.21) by eliminating the derivative w_t:

$$w_{xx} = 3ww_x - w^3 - aw - b. \tag{34.3.1.22}$$

This ordinary differential equation coincides with (34.3.1.14), where h_1 and h_0 are expressed by (34.3.1.18). The substitution $w = -U_x/U$ transforms (34.3.1.22) into a third-order linear equation with constant coefficients,

$$U_{xxx} + aU_x - b = 0, \tag{34.3.1.23}$$

whose solutions are determined by the roots of the cubic equation $\lambda^3 + a\lambda - b = 0$. In particular, if all its roots λ_n are real, then the general solutions of equations (34.3.1.22) and (34.3.1.23) are given by

$$w = -U_x/U, \quad U = r_1(t)\exp(\lambda_1 x) + r_2(t)\exp(\lambda_2 x) + r_3(t)\exp(\lambda_3 x). \tag{34.3.1.24}$$

The functions $r_n(t)$ are found by substituting (34.3.1.24) into equation (34.3.1.20) or (34.3.1.21).

Note that equation (34.3.1.20) was studied in more detail by another method in Subsection 31.2.1 (see Example 31.8 with $a = 1$ and $b_2 = 0$).

Example 34.13. Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 + w^2 \tag{34.3.1.25}$$

in conjunction with the first-order differential constraint

$$\frac{\partial w}{\partial x} = \varphi(x,t), \tag{34.3.1.26}$$

where φ is some (yet unknown) function of two arguments.

The original equation (34.3.1.25) can be rewritten in the form

$$\frac{\partial w}{\partial t} = \frac{\partial \varphi}{\partial x} + \varphi^2 + w^2. \tag{34.3.1.27}$$

Let us find the consistency condition for equations (34.3.1.26) and (34.3.1.27). To this end, we differentiate (34.3.1.26) with respect to t and (34.3.1.27) with respect to x and eliminate the mixed derivative from the resulting relations bearing in mind that $w_{xt} = w_{tx}$. Using (34.3.1.26) to replace w_x with φ, we get

$$\varphi_t = \varphi_{xx} + 2\varphi\varphi_x + 2w\varphi.$$

By expressing w,

$$w = \frac{\varphi_t - \varphi_{xx} - 2\varphi\varphi_x}{2\varphi}, \tag{34.3.1.28}$$

and substituting (34.3.1.28) into (34.3.1.26) and (34.3.1.27), we arrive at the overdetermined system of equations

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{\varphi_t - \varphi_{xx} - 2\varphi\varphi_x}{2\varphi}\right) = \varphi,\\ &\frac{\partial}{\partial t}\left(\frac{\varphi_t - \varphi_{xx} - 2\varphi\varphi_x}{2\varphi}\right) = \varphi_x + \varphi^2 + \left(\frac{\varphi_t - \varphi_{xx} - 2\varphi\varphi_x}{2\varphi}\right)^2. \end{aligned} \tag{34.3.1.29}$$

Using the method of separation of variables, we look for a solution to the first equation in (34.3.1.29) in the product form $\varphi(x,t)=\psi(t)\theta(x)$. Elementary calculations show that the first equation in (34.3.1.29) is satisfied by the function

$$\varphi(x,t)=\psi(t)\sin(x+C), \tag{34.3.1.30}$$

where $\psi(t)$ is an arbitrary function and C is an arbitrary constant. Substituting (34.3.1.30) into the second equation in (34.3.1.29), we arrive at an equation for $\psi(t)$:

$$\frac{d}{dt}\left(\frac{\dot\psi+\psi}{2\psi}\right)=\left(\frac{\dot\psi+\psi}{2\psi}\right)^2+\psi^2; \tag{34.3.1.31}$$

a dot over a symbol denotes a derivative with respect to t. If there is a solution to the second-order ordinary differential equation (34.3.1.31), then a solution to the original equation (34.3.1.25) can be calculated as

$$w(x,t)=\frac{\dot\psi+\psi}{2\psi}-\psi\cos(x+C);$$

this formula results from substituting (34.3.1.30) into (34.3.1.28).

Remark 1. In the general case, for a given function $\mathcal{F}$, the compatibility condition (34.3.1.3) is a nonlinear partial differential equation for the function $\mathcal{G}$. This equation admits infinitely many solutions (by the theorem on the local existence of solutions). Therefore, the second-order partial differential equation (34.3.1.1) admits infinitely many compatible first-order differential constraints (34.3.1.2).

Remark 2. In the general case, the solution of the first-order partial differential equation (34.3.1.2) reduces to the solution of a system of ordinary differential equations; see Sections 23.1 and 24.2.

34.3.2. Second-Order Hyperbolic Equations

In a similar way, one can consider second-order hyperbolic equations of the form

$$\frac{\partial^2 w}{\partial x\partial t}=\mathcal{F}\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t}\right), \tag{34.3.2.1}$$

supplemented by a first-order differential constraint (34.3.1.2). Assume that $\mathcal{G}_{w_x}\neq 0$.

A compatibility condition for these equations is obtained by differentiating (34.3.2.1) with respect to t and (34.3.1.2) with respect to t and x, and then equating the resulting expressions of the third derivatives w_{xtt} and w_{ttx} to one another:

$$\mathrm{D}_t\mathcal{F}=\mathrm{D}_x[\mathrm{D}_t\mathcal{G}]. \tag{34.3.2.2}$$

Here, D_t and D_x are the total differential operators of (34.3.1.4) in which the partial derivatives w_t, w_{xx}, w_{xt}, and w_{tt} must be expressed in terms of x, t, w, and w_x with the help of relations (34.3.2.1) and (34.3.1.2) and those obtained by differentiating (34.3.2.1) and (34.3.1.2).

Let us show how the second derivatives can be calculated. We differentiate (34.3.1.2) with respect to x and replace the mixed derivative by the right-hand side of (34.3.2.1) to obtain the following expression for the second derivative with respect to x:

$$\frac{\partial\mathcal{G}}{\partial x}+w_x\frac{\partial\mathcal{G}}{\partial w}+w_{xx}\frac{\partial\mathcal{G}}{\partial w_x}=\mathcal{F}\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t}\right)\quad\Longrightarrow\quad\frac{\partial^2 w}{\partial x^2}=\mathcal{H}_1\left(x,t,w,\frac{\partial w}{\partial x}\right). \tag{34.3.2.3}$$

Here and in what follows, we have taken into account that (34.3.1.2) allows us to express the derivative with respect to t through the derivative with respect to x. Further, differentiating (34.3.1.2) with respect to t yields

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial\mathcal{G}}{\partial t}+w_t\frac{\partial\mathcal{G}}{\partial w}+w_{xt}\frac{\partial\mathcal{G}}{\partial w_x}=\frac{\partial\mathcal{G}}{\partial t}+\mathcal{G}\frac{\partial\mathcal{G}}{\partial w}+\mathcal{F}\frac{\partial\mathcal{G}}{\partial w_x}\quad\Longrightarrow\quad\frac{\partial^2 w}{\partial t^2}=\mathcal{H}_2\left(x,t,w,\frac{\partial w}{\partial x}\right). \tag{34.3.2.4}$$

Replacing the derivatives w_t, w_{xt}, w_{xx}, and w_{tt} in (34.3.1.4) by their expressions from (34.3.1.2), (34.3.2.1), (34.3.2.3), and (34.3.2.4), we find the total differential operators D_t and D_x, which are required for the compatibility condition (34.3.2.2).

Example 34.14. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x\partial t}=f(w). \tag{34.3.2.5}$$

Let us supplement (34.3.2.5) with a quasilinear differential constraint of the form

$$\frac{\partial w}{\partial x}=\varphi(t)g(w). \tag{34.3.2.6}$$

Differentiating (34.3.2.5) with respect to x and then replacing the first derivative with respect to x by the right-hand side of (34.3.2.6), we get

$$w_{xxt} = \varphi g f'_w. \tag{34.3.2.7}$$

Differentiating further (34.3.2.6) with respect to x and t, we obtain two relations

$$w_{xx} = \varphi g'_w w_x = \varphi^2 g g'_w, \tag{34.3.2.8}$$

$$w_{xt} = \varphi'_t g + \varphi g'_w w_t. \tag{34.3.2.9}$$

Eliminating the mixed derivative from (34.3.2.9) using equation (34.3.2.5), we find the first derivative with respect to t:

$$w_t = \frac{f - \varphi'_t g}{\varphi g'_w}. \tag{34.3.2.10}$$

Differentiating (34.3.2.8) with respect to t and replacing w_t by the right-hand side of (34.3.2.10), we have

$$w_{xxt} = 2\varphi\varphi'_t g g'_w + \varphi^2 (g g'_w)'_w w_t = 2\varphi\varphi'_t g g'_w + \varphi (g g'_w)'_w \frac{f - \varphi'_t g}{g'_w}. \tag{34.3.2.11}$$

Equating now the third derivatives (34.3.2.7) and (34.3.2.11), canceling them by φ, and performing simple rearrangements, we get the determining equation

$$\varphi'_t g[(g'_w)^2 - g g''_{ww}] = g g'_w f'_w - f(g g'_w)'_w, \tag{34.3.2.12}$$

which has two different solutions.

Solution 1. Equation (34.3.2.12) is satisfied identically for any $\varphi = \varphi(t)$ if

$$(g'_w)^2 - g g''_{ww} = 0,$$

$$g g'_w f'_w - f(g g'_w)'_w = 0.$$

The general solution of this system has the form

$$f(w) = a e^{\lambda w}, \quad g(w) = b e^{\lambda w/2}, \tag{34.3.2.13}$$

where a, b, and λ are arbitrary constants. For simplicity, we set

$$a = b = 1, \quad \lambda = -2. \tag{34.3.2.14}$$

Substitute $g(w)$ defined by (34.3.2.13)–(34.3.2.14) into the differential constraint (34.3.2.6) and integrate the resulting equation to obtain

$$w = \ln[\varphi(t)x + \psi(t)], \tag{34.3.2.15}$$

where $\psi(t)$ is an arbitrary function. Substituting (34.3.2.15) into the original equation (34.3.2.5) with the right-hand side (34.3.2.13)–(34.3.2.14), we arrive at a linear ordinary differential equation for $\psi(t)$:

$$\psi\varphi'_t - \varphi\psi'_t = 1.$$

The general solution of this equation is expressed as

$$\psi(t) = C\varphi(t) - \varphi(t) \int \frac{dt}{\varphi^2(t)}, \tag{34.3.2.16}$$

where C is an arbitrary constant.

Thus, formulas (34.3.2.15)–(34.3.2.16), where $\varphi(t)$ is an arbitrary function, define an exact solution to the nonlinear equation $w_{xt} = e^{-2w}$.

Solution 2. The second solution is determined by the linear relation

$$\varphi(t) = at + b, \tag{34.3.2.17}$$

where a and b are arbitrary constants. In this case, the functions $f(w)$ and $g(w)$ are related by (34.3.2.12), with $\varphi'_t = a$. Integrating (34.3.2.6) with constraint (34.3.2.17) yields the solution structure

$$w = w(z), \quad z = (at + b)x + \psi(t), \tag{34.3.2.18}$$

where $\psi(t)$ is an arbitrary function. Inserting it into the original equation (34.3.2.5) and changing x to z with the help of (34.3.2.18), we obtain

$$[az + (at + b)\psi'_t - a\psi]w''_{zz} + a w'_z = f(w). \tag{34.3.2.19}$$

In order for this relation to be an ordinary differential equation for $w = w(z)$, one should set

$$(at + b)\psi'_t - a\psi = \text{const}.$$

Integrating yields $\psi(t)$ in the form

$$\psi(t) = ct + d, \tag{34.3.2.20}$$

where c and d are arbitrary constants.

Formulas (34.3.2.18) and (34.3.2.20) define a solution to equation (34.3.2.5) for arbitrary $f(w)$. The function $w(z)$ is described by equation (34.3.2.19) with constraint (34.3.2.20). To the special case $a = d = 0$ there corresponds a traveling-wave solution and to the case $b = c = d = 0$, a self-similar solution.

34.3.3. Second-Order Equations of General Form

Consider a second-order equation of the general form

$$\mathcal{F}(x,t,w,w_x,w_t,w_{xx},w_{xt},w_{tt})=0 \tag{34.3.3.1}$$

with a first-order differential constraint

$$\mathcal{G}(x,t,w,w_x,w_t)=0. \tag{34.3.3.2}$$

Let us successively differentiate equations (34.3.3.1) and (34.3.3.2) with respect to both variables so as to obtain differential relations involving second and third derivatives. We get

$$\begin{aligned}&\mathrm{D}_x\mathcal{F}=0,\quad \mathrm{D}_t\mathcal{F}=0,\quad \mathrm{D}_x\mathcal{G}=0,\quad \mathrm{D}_t\mathcal{G}=0,\\ &\mathrm{D}_x[\mathrm{D}_x\mathcal{G}]=0,\quad \mathrm{D}_x[\mathrm{D}_t\mathcal{G}]=0,\quad \mathrm{D}_t[\mathrm{D}_t\mathcal{G}]=0.\end{aligned} \tag{34.3.3.3}$$

The compatibility condition for (34.3.3.1) and (34.3.3.2) can be found by eliminating the derivatives w_t, w_{xx}, w_{xt}, w_{tt}, w_{xxx}, w_{xxt}, w_{xtt}, and w_{ttt} from the nine equations of (34.3.3.1)–(34.3.3.3). In doing so, we obtain an expression of the form

$$\mathcal{H}(x,t,w,w_x)=0. \tag{34.3.3.4}$$

If the left-hand side of (34.3.3.4) is a polynomial in w_x, then the compatibility conditions result from equating the functional coefficients of the polynomial to zero.

⊙ *References for Section* 34.3: V. A. Galaktionov (1994), P. J. Olver and E. M. Vorob'ev (1996), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov (2005), A. D. Polyanin and A. V. Manzhirov (2007).

34.4. Second-Order Differential Constraints for PDEs. Some Generalized

34.4.1. Second-Order Differential Constraints

Constructing exact solutions of nonlinear partial differential equations with the help of second- and higher-order differential constraints requires finding exact solutions of these differential constraints. The latter is generally rather difficult or even impossible. For this reason, one employs some special differential constraints that involve derivatives with respect to only one variable. In practice, one considers second-order ordinary differential equations in, say, x and the other variable, t, is involved implicitly or is regarded as a parameter, so that integration constants depend on t.

The problem of compatibility of a second-order evolution equation

$$\frac{\partial w}{\partial t}=\mathcal{F}_1\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2}\right) \tag{34.4.1.1}$$

with a similar differential constraint

$$\frac{\partial w}{\partial t}=\mathcal{F}_2\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2}\right) \tag{34.4.1.2}$$

may be reduced to a problem with the first-order differential constraint considered in Subsection 34.2.1. To that end, one should first eliminate the second derivative w_{xx} from the equations. Then the resulting first-order equation is examined together with the original equation (or the original differential constraint).

By eliminating the derivative with respect to t from (34.4.1.1) and (34.4.1.2) one obtains an equation of the form

$$H\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2}\right)=0, \tag{34.4.1.3}$$

which can be treated as an ordinary differential equation for x involving t as a parameter. The constants of integration arising in solving the equation will be arbitrary functions of t: $C_1=C_1(t)$ and $C_2=C_2(t)$.

Thus, an evolution second-order differential constraint of the general form (34.4.1.1) is equivalent, on the one hand, to a suitable first-order constraint (34.3.1.2) and, on the other hand, to a second-order differential constraint in the form of the ordinary differential equation (34.4.1.3). These equivalent constraints are easier to use than the original differential constraint (34.4.1.3).

Remark. The left-hand sides of equation (34.4.1.1) and the differential constraint (34.4.1.2) can have the second derivative w_{tt} (or w_{tx}) instead of w_t. Then, the elimination of w_{tt} (or w_{tx}) also results in a second-order ordinary differential equation (34.4.1.3), which is more reasonable to use rather than the original differential constraint.

Example 34.15. From the class of nonlinear heat equations with a source

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right] + f_2(w) \tag{34.4.1.4}$$

one singles out equations that admit invariant manifolds of the form

$$\frac{\partial^2 w}{\partial x^2} = g_1(w)\left(\frac{\partial w}{\partial x}\right)^2 + g_2(w). \tag{34.4.1.5}$$

The functions $f_2(w)$, $f_1(w)$, $g_2(w)$, and $g_1(w)$ are to be determined in the further analysis.

Eliminating the second derivative from (34.4.1.4) and (34.4.1.5), we obtain

$$\frac{\partial w}{\partial t} = \varphi(w)\left(\frac{\partial w}{\partial x}\right)^2 + \psi(w), \tag{34.4.1.6}$$

where

$$\varphi(w) = f_1(w)g_1(w) + f_1'(w), \quad \psi(w) = f_1(w)g_2(w) + f_2(w). \tag{34.4.1.7}$$

The condition of invariance of the manifold (34.4.1.5) under equation (34.4.1.4) is obtained by differentiating (34.4.1.5) with respect to t:

$$w_{xxt} = 2g_1 w_x w_{xt} + g_1' w_x^2 w_t + g_2' w_t.$$

The derivatives w_{xxt}, w_{xt}, and w_t should be eliminated from this relation with the help of equations (34.4.1.5) and (34.4.1.6) and those obtained by their differentiation. As a result, we get

$$(2\varphi g_1^2 + 3\varphi' g_1 + \varphi g_1' + \varphi'')w_x^4 + (4\varphi g_1 g_2 + 5\varphi' g_2 + \varphi g_2' - g_1\psi' - \psi g_1' + \psi'')w_x^2 + 2\varphi g_2^2 + \psi' g_2 - \psi g_2' = 0.$$

Equating the coefficients of like powers of w_x to zero, one obtains three equations, which, for convenience, may be written in the form

$$\begin{gathered}(\varphi' + \varphi g_1)' + 2g_1(\varphi' + \varphi g_1) = 0,\\ 4g_2(\varphi' + \varphi g_1) + (\varphi g_2 - \psi g_1)' + \psi'' = 0,\\ \varphi = -\tfrac{1}{2}(\psi/g_2)'.\end{gathered} \tag{34.4.1.8}$$

The first equation can be satisfied by taking $\varphi' + \varphi g_1 = 0$. The corresponding particular solution of system (34.4.1.8) has the form

$$\varphi = -\frac{1}{2}\mu', \quad \psi = \mu g_2, \quad g_1 = -\frac{\mu''}{\mu'}, \quad g_2 = \left(2C_1 + \frac{C_2}{\sqrt{|\mu|}}\right)\frac{1}{\mu'}, \tag{34.4.1.9}$$

where $\mu = \mu(w)$ is an arbitrary function.

Taking into account (34.4.1.7), we find the functional coefficients of the original equation (34.4.1.4) and the invariant set (34.4.1.5):

$$f_1 = \left(C_3 - \frac{1}{2}w\right)\mu', \quad f_2 = (\mu - f_1)g_2, \quad g_1 = -\frac{\mu''}{\mu'}, \quad g_2 = \left(2C_1 + \frac{C_2}{\sqrt{|\mu|}}\right)\frac{1}{\mu'}. \tag{34.4.1.10}$$

Equation (34.4.1.5), together with (34.4.1.10), admits the first integral

$$w_x^2 = \left[4C_1\mu + 4C_2\sqrt{|\mu|} + 2\sigma_t'(t)\right]\frac{1}{(\mu')^2}, \tag{34.4.1.11}$$

where $\sigma(t)$ is an arbitrary function. Let us eliminate w_x^2 from (34.4.1.6) by means of (34.4.1.11) and substitute the functions φ and ψ from (34.4.1.9) to obtain the equation

$$\mu' w_t = -C_2\sqrt{|\mu|} - \sigma_t'(t). \tag{34.4.1.12}$$

Let us dwell on the special case $C_2 = C_3 = 0$. Integrating equation (34.4.1.12) and taking into account that $\mu_t = \mu' w_t$ yields

$$\mu = -\sigma(t) + \theta(x), \tag{34.4.1.13}$$

where $\theta(x)$ is an arbitrary function. Substituting (34.4.1.13) into (34.4.1.11) and taking into account the relation $\mu_x = \mu' w_x$, we obtain

$$\theta_x^2 - 4C_1\theta = 2\sigma_t - 4C_1\sigma.$$

Equating both sides of this equation to zero and integrating the resulting ordinary differential equations, we find the functions on the right-hand side of (34.4.1.13):

$$\sigma(t) = A\exp(2C_1t), \quad \theta(x) = C_1(x+B)^2, \tag{34.4.1.14}$$

where A and B are arbitrary constants. Thus, an exact solution of equation (34.4.1.4) with the functions f_1 and f_2 from (34.4.1.10) can be represented in implicit form for $C_2 = C_3 = 0$ as follows:

$$\mu(w) = -A\exp(2C_1t) + C_1(x+B)^2.$$

In the solution and the determining relations (34.4.1.10), the function $\mu(w)$ can be chosen arbitrarily.

Example 34.16. Consider the problem of finding nonlinear second-order equations

$$\frac{\partial w}{\partial t} = f_2(w)\frac{\partial^2 w}{\partial x^2} + f_1(w)\frac{\partial w}{\partial x} + f_0(w)$$

admitting invariant manifolds of the form

$$\frac{\partial^2 w}{\partial x^2} = g_1(w)\frac{\partial w}{\partial x} + g_0(w).$$

The compatibility analysis of these equations leads us to the following relations for the determining functions:

$$\begin{aligned}
&f_2(w) \text{ is an arbitrary function,}\\
&f_1(w) = C_1w + C_2 - (3C_1C_3w + C_4)f_2(w),\\
&f_0(w) = (-C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6)[1 - C_3f_2(w)],\\
&g_1(w) = 3C_1C_3w + C_4,\\
&g_0(w) = C_3(-C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6),
\end{aligned}$$

where $C_1, \ldots, C_6$ are arbitrary constants.

34.4.2. Higher-Order Differential Constraints. Determining Equations

1°. Third- and higher-order differential constraints involving derivatives in two or more independent variables are used very rarely, since they lead to cumbersome computations and rather complex equations (often, the original equations are simpler). A higher-order differential constraint represents an ordinary differential equation, only involving derivatives in a single independent variable, with the other independent variables appearing as free parameters; an example of such a differential constraint of the second order is given by equation (34.4.1.3).

For simplicity of the presentation, let us first consider the second-order evolution equation

$$w_t = f(t, x, w, w_x, w_{xx}) \tag{34.4.2.1}$$

and a n-order differential constraint of the form

$$h = w_x^{(n)} + g(t, x, w, w_x, \ldots, w_x^{(n-1)}) = 0. \tag{34.4.2.2}$$

We denote by $[f]$ equation (34.4.2.1) and its differential consequences with respect to x. Constraint (34.4.2.2) and its differential consequences with respect to x are denoted by $[h]$.

Equation (34.4.2.1) and differential constraint (34.4.2.2) satisfy the compatibility if and only if

$$D_t(h)\big|_{[f]\cap[h]} = 0. \tag{34.4.2.3}$$

If condition (34.4.2.3) holds, the differential constraint (34.4.2.2) represents an invariant manifold of equation (34.4.2.1).

It can be shown that with $n \geq 4$, condition (34.4.2.3) can be represented as an equivalent *determining equation* (Kaptsov, 1998):

$$\begin{aligned}
D_t(h)\big|_{[f]} &= f_2D_x^2(h) + [f_1 + nD_x(f_2)]D_x(h)\\
&\quad + [f_0 + nD_x(f_1) - h_{n-1}D_x(f_2) + \tfrac{1}{2}n(n-1)D_x^2(f_2) + f_2hh_{n-1,n-1} - 2f_2D_x(h_{n-1})]h.
\end{aligned} \tag{34.4.2.4}$$

Here and henceforth, D_t and D_x are the total derivatives with respect to t and x; also the following short notation is used:

$$f_0 = \frac{\partial f}{\partial w}, \quad f_1 = \frac{\partial f}{\partial w_x}, \quad f_2 = \frac{\partial f}{\partial w_{xx}}, \quad h_{n-1} = \frac{\partial h}{\partial u}, \quad h_{n-1,n-1} = \frac{\partial^2 h}{\partial u^2}, \quad u = w_x^{(n-1)}.$$

Equation (34.4.2.4) is a complicated nonlinear equation for h.

2°. For determining the invariant manifold of equation (34.4.2.1), one can use, instead of the nonlinear equation (34.4.2.4), a simpler *linear determining equation* of the form

$$D_t(h)\big|_{[f]} = f_2 D_x^2(h) + [c_1 f_1 + c_2 D_x(f_2)]D_x(h) + [c_3 f_0 + c_4 D_x(f_1) + c_5 D_x^2(f_2)]h, \tag{34.4.2.5}$$

where $c_1, \ldots, c_5$ are some constants to be determined. Equation (34.4.2.5) has been obtained by discarding the nonlinear terms in equation (34.4.2.4) and replacing the constants with undetermined coefficients.

If, for some values of the constants $c_1, \ldots, c_5$, the function h satisfies (34.4.2.5), then the equation $h = 0$ is an invariant manifold of equation (34.4.2.1).

Example 34.17. For the equation

$$w_t = w_{xx} + w_x^2 + w^2, \tag{34.4.2.6}$$

the linear determining equation (34.4.2.5) is written as

$$D_t(h)\big|_{[f]} = D_x^2(h) + 2c_1 w_x D_x(h) + 2(c_2 w + c_3 w_{xx})h. \tag{34.4.2.7}$$

A solution to equation (34.4.2.7) is sought in the form

$$h = w_{xxx} + a w_x, \tag{34.4.2.8}$$

where a is some constant.

Successively differentiating (34.4.2.6) with respect to x, we get

$$\begin{aligned} w_{xt} &= w_{xxx} + 2w_x w_{xx} + 2w w_x, \\ w_{xxt} &= w_x^{(4)} + 2w_x w_{xxx} + 2w_{xx}^2 + 2w w_{xx} + 2w_x^2, \\ w_{xxxt} &= w_x^{(5)} + 2w_x w_x^{(4)} + 6w_{xx} w_{xxx} + 2w w_{xxx} + 6w_x w_{xx}. \end{aligned} \tag{34.4.2.9}$$

Taking into account (34.4.2.8) and (34.4.2.9), we calculate the left- and right-hand sides of the determining equation (34.4.2.7) to obtain

$$\begin{aligned} D_t(h)\big|_{[f]} &= w_x^{(5)} + 2w_x w_x^{(4)} + 6w_{xx} w_{xxx} + 2w w_{xxx} + (6 + 2a) w_x w_{xx} + a w_{xxx} + 2a w w_x, \\ D_x^2(h) + 2c_1 w_x D_x(h) + 2(c_2 w + c_3 w_{xx})h &= w_x^{(5)} + 2c_1 w_x w_x^{(4)} + 2c_3 w_{xx} w_{xxx} \\ &\quad + 2c_2 w w_{xxx} + 2(c_1 + a c_3) w_x w_{xx} + a w_{xxx} + 2a c_2 w w_x. \end{aligned}$$

It follows that the linear determining equation (34.4.2.7) can be satisfied by setting

$$a = 1, \quad c_1 = c_2 = 1, \quad c_3 = 3$$

in (34.4.2.7) and (34.4.2.8). By setting $a = 1$ in (34.4.2.8) and solving the equation $h = 0$, we find that

$$w = \varphi_0(t) + \varphi_2(t)\cos x + \varphi_3(t)\sin x. \tag{34.4.2.10}$$

By substituting this expression into the original equation (34.4.2.6), one can obtain a system of ordinary differential equations for $\varphi_k(t)$.

Example 34.18. For the equation

$$w_t = w w_{xx} - \tfrac{2}{3} w_x^2, \tag{34.4.2.11}$$

the linear determining equation (34.4.2.5) becomes

$$D_t(h)\big|_{[f]} = w D_x^2(h) + b_1 w_x D_x(h) + b_2 w_{xx} h, \tag{34.4.2.12}$$

where the coefficients b_1 and b_2 can be expressed in terms of $c_1, \ldots, c_5$. It can be verified by direct substitution that for $n = 4$ and with a suitable selection of the coefficients b_1 and b_2, equation (34.4.2.12) can be satisfied by setting $h = u_{xxxx}$. It follows that the equation $w_{xxxx} = 0$ represents an invariant manifold. Hence, the original equation (34.4.2.11) admits a solution in the form of a cubic polynomial in the space variable x:

$$w = \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

By substituting this expression into (34.4.2.11), one can obtain a system of ordinary differential equations for $\varphi_k(t)$.

Remark. Examples 34.17 and 34.18 as well as some other examples of usage of linear determining equations can be found in Kaptsov (1995, 1998) and Kaptsov and Verevkin (2003). In the latter paper, nonlinear heat equations of the form $w_t = (w^k w_x)_x + f(w)$ were investigated using second- and third-order linear determining equations.

3°. Consider the mth-order evolution equation

$$w_t = f(t, x, w, w_x, \ldots, w_x^{(m)}) \tag{34.4.2.13}$$

in conjunction with the differential constraint (34.4.2.2).

To find the function h, one can use a *linear determining equation* of the form

$$D_t(h)\big|_{[f]} = \sum_{i=0}^{m} \sum_{j=0}^{i} c_{ij} D_x^{i-j}(f_{m-j}) D_x^{m-i}(h), \tag{34.4.2.14}$$

where c_{ij} are some constants to be determined. Equation (34.4.2.14) is a generalization of (34.4.2.5) to the case of $m > 2$.

Now consider equation (34.4.2.13) with the first time derivative replaced by the second one:

$$w_{tt} = f(t, x, w, w_x, \ldots, w_x^{(m)}). \tag{34.4.2.15}$$

The function h appearing in the differential constraint (34.4.2.2) can be found using a linear determining equation of the form

$$D_{tt}(h)\big|_{[f]} = \sum_{i=0}^{m} \sum_{j=0}^{i} c_{ij} D_x^{i-j}(f_{m-j}) D_x^{m-i}(h). \tag{34.4.2.16}$$

Remark 1. In some cases, the constants c_{ij} appearing in the linear determining equations (34.4.2.14) and (34.4.2.16) are expressed through binomial coefficients as

$$c_{ij} = C_n^{i-j}, \tag{34.4.2.17}$$

where n is the order of the highest derivative in the differential constraint (34.4.2.2). In particular, in Examples 34.17 and 34.18 considered above, the linear determining equation (34.4.2.5) with coefficients (34.4.2.17) holds true.

Remark 2. Section 34.5 contains examples of second- and third-order differential constraints that are essentially equivalent to most common structures of exact solutions.

34.4.3. Usage of Several Differential Constraints. Systems of Nonlinear Equations

As mentioned in Remark 2 from Subsection 34.2.2 (see also Subsection 34.5.3), more than one differential constraint of the form (34.2.2.2) can be used. In general, all differential constraints must be analyzed for consistency with the original equation.

The method of differential constraints is also quite effective in constructing exact solutions to nonlinear systems of equations.

This will be exemplified below.

Example 34.19. Consider the nonlinear heat equation with a source

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + g(w) \tag{34.4.3.1}$$

in conjunction with two first-order differential constraints

$$\begin{aligned} \frac{\partial w}{\partial t} &= \varphi(x, t, w), \\ \frac{\partial w}{\partial x} &= \psi(x, t, w), \end{aligned} \tag{34.4.3.2}$$

where φ and ψ are some (yet arbitrary) functions of three arguments.

First, let us find the consistency condition for the differential constraints (34.4.3.2). To this end, we differentiate the first relation in (34.4.3.2) with respect to x and the second relation with respect to t. In the resulting relations, we replace the first derivatives with the right-hand sides of (34.4.3.2) to obtain

$$\begin{aligned} w_{tx} &= \varphi_x + \varphi_w w_x = \varphi_x + \psi\varphi_w, \\ w_{xt} &= \psi_t + \psi_w w_t = \psi_t + \varphi\psi_w. \end{aligned}$$

Equating the mixed derivatives, $w_{tx} = w_{xt}$, we get the consistency condition

$$\varphi_x + \psi\varphi_w - \psi_t - \varphi\psi_w = 0. \tag{34.4.3.3}$$

Now substituting (34.4.3.2) into the original equation (34.4.3.1), we obtain

$$\varphi = (\psi_x + \psi\psi_w)f + \psi^2 f' + g. \tag{34.4.3.4}$$

Eliminating φ from the consistency condition (34.4.3.3) using (34.4.3.4) and taking into account (34.4.3.2), we arrive at the following equation for ψ:

$$\psi_t = (\psi_{xx} + 2\psi\psi_{xw} + \psi^2\psi_{ww})f + (3\psi\psi_x + 2\psi^2\psi_w)f' + \psi^3 f'' + \psi g' - g\psi_w. \tag{34.4.3.5}$$

Equation (34.4.3.5) has three independent variables x, t, and w and looks more complicated than the original equation (34.4.3.1), which contains only two independent variables, x and t. However, the presence of the extra variable w provides a wider selection of solutions, which can be sought by prescribing the structure of the function ψ. It will be shown below how to find two classes of solutions to the nonlinear heat equation with a source (34.4.3.1) on the basis of equation (34.4.3.5).

Case 1. First, we look for particular solutions to equation (34.4.3.5), independent of x, in the product form

$$\psi = \alpha(t)h(w). \tag{34.4.3.6}$$

Formula (34.4.3.6) defines the solution structure; the functions $\alpha(t)$ and $h(w)$ are yet unknown and will be determined in the course of the subsequent analysis. Substituting (34.4.3.6) into (34.4.3.5) yields

$$\dot\alpha(t) = \alpha^3(t)h(w)\big(f(w)h(w)\big)'' + \alpha(t)h(w)\big(g(w)/h(w)\big)',$$

where the dot denotes a derivative with respect to t. The equation has a nontrivial solution if the relations

$$\begin{aligned} h(w)\big(f(w)h(w)\big)'' &= A,\\ h(w)\big(g(w)/h(w)\big)' &= B \end{aligned} \tag{34.4.3.7}$$

hold; A and B are arbitrary constants. Equations (34.4.3.7) involve three unknown functions; given one of them, one can find the other two.

The function (34.4.3.6) generates a solution to the original equation (34.4.3.1). By virtue of the second equation in (34.4.3.2), this solution can be represented in implicit form:

$$\int \frac{dw}{h(w)} = \alpha(t)x + \beta(t), \tag{34.4.3.8}$$

where the function $\alpha = \alpha(t)$ satisfies the Bernoulli equation

$$\dot\alpha = A\alpha^3 + B\alpha,$$

which is easy to integrate. The function $\beta(t)$ is determined from the ordinary differential equation resulting from the substitution of solution (34.4.3.8) into the original equation (34.4.3.1).

Assuming that the function $h = h(w)$ is prescribed (it can be defined arbitrarily), we integrate equations (34.4.3.7) to obtain the forms of the functions that determine the equation under study (34.4.3.1):

$$f(w) = \frac{A}{h(w)}\int Q(w)\,dw + \frac{C_1 w + C_2}{h(w)}, \quad g(w) = h(w)\big[BQ(w) + C_3\big], \quad Q(w) = \int \frac{dw}{h(w)},$$

where C_1, C_2, and C_3 are arbitrary constants.

Case 2. Now we look for particular solutions to equation (34.4.3.5), independent of t, in the product form

$$\psi = \theta(x)p(w). \tag{34.4.3.9}$$

Substituting (34.4.3.9) into (34.4.3.5) and rearranging, we obtain

$$\theta''_{xx} fp + \theta\theta'_x p(2fp'_w + 3f'_w p) + \theta^3 p^2 (fp)''_{ww} + \theta(pg'_w - p'_w g) = 0. \tag{34.4.3.10}$$

Such functional differential equations are discussed in detail in Chapter 29. Solutions to equation (34.4.3.10)—there are several solutions—can be obtained by the splitting method using the results of Section 29.5 [see the functional equation (29.5.2.3) and its solutions (29.5.2.4) and (29.5.2.5)].

Without performing a complete analysis of equation (34.4.3.10), we write out one exact solutions:

$$\theta(x) = x, \quad f(w) = \frac{aw + b}{p(w)}, \quad g(w) = -3(aw + b) - 2p(w)\int \frac{(aw + b)p'_w(w)\,dw}{p^2(w)}, \tag{34.4.3.11}$$

where $p = p(w)$ is an arbitrary function; a and b are arbitrary constants. Substituting (34.4.3.9) into the second differential constraint in (34.4.3.2) and taking into account that $\theta(x) = x$ [see (34.4.3.11)], we find that $w_x = xp(w)$. Integrating the last relation yields

$$\int \frac{dw}{p(w)} = \frac{1}{2}x^2 + \xi(t). \tag{34.4.3.12}$$

Differentiating (34.4.3.12) with respect to t and taking into account the form of the first differential constraint in (34.4.3.2), we find that $\varphi = \xi_t' p(w)$. Substituting the expressions of φ and ψ into (34.4.3.4) and bearing in mind relations (34.4.3.11) and (34.4.3.12), we obtain a linear differential equation for $\xi(t)$. The solution of this equation results in the exponential dependence

$$\xi(t) = Ce^{-2at}, \tag{34.4.3.13}$$

where C is an arbitrary constant.

Formulas (34.4.3.12) and (34.4.3.13) define a solution to the nonlinear heat equation (34.4.3.1) in implicit form; the determining functions $f(w)$ and $g(w)$ are given by (34.4.3.11), where $p(w)$ is an arbitrary function.

Example 34.20. Consider the nonlinear system of four equations

$$\frac{\partial^2 F}{\partial t\partial x} + F\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\left(\frac{\partial F}{\partial x}\right)^2 = \nu\frac{\partial^3 F}{\partial x^3} + 2(uv + w^2) - \alpha - \beta, \tag{34.4.3.14}$$

$$\frac{\partial u}{\partial t} + F\frac{\partial u}{\partial x} - u\frac{\partial F}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + \gamma, \tag{34.4.3.15}$$

$$\frac{\partial v}{\partial t} + F\frac{\partial v}{\partial x} - v\frac{\partial F}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2} + \gamma, \tag{34.4.3.16}$$

$$\frac{\partial w}{\partial t} + F\frac{\partial w}{\partial x} - w\frac{\partial F}{\partial x} = \nu\frac{\partial^2 w}{\partial x^2} + \frac{\alpha - \beta}{2}, \tag{34.4.3.17}$$

which describes a broad class of exact solutions to the unsteady three-dimensional Navier–Stokes equations (Aristov and Polyanin, 2009). Equations (34.4.3.14)–(34.4.3.17) involve three arbitrary functions of time, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$, and four unknown functions, $F = F(t, x)$, $u = u(t, x)$, $v = v(t, x)$, and $w = w(t, x)$.

Equations (34.4.3.14)–(34.4.3.17) will be analyzed in conjunction with three first-order differential constraints

$$u = m\frac{\partial F}{\partial x} + A, \quad v = n\frac{\partial F}{\partial x} + B, \quad w = k\frac{\partial F}{\partial x} + C, \tag{34.4.3.18}$$

where m, n, k, A, B, and C are unknown functions of t. By demanding that the four equations (34.4.3.14)–(34.4.3.17) must coincide after inserting (34.4.3.18), one arrives at a nonlinear system for the unknowns, consisting of one algebraic and six ordinary differential equations:

$$mn + k^2 = \frac{1}{4}, \tag{34.4.3.19}$$

$$\frac{A - m_t'}{m} = \frac{B - n_t'}{n} = \frac{C - k_t'}{k} = 2(An + Bm + 2Ck), \tag{34.4.3.20}$$

$$\frac{\gamma - A_t'}{m} = \frac{\gamma - B_t'}{n} = \frac{\alpha - \beta - 2C_t'}{2k} = -\alpha - \beta + 2AB + 2C^2. \tag{34.4.3.21}$$

This system has seven equations for nine unknowns: six functions m, n, k, A, B, and C from (34.4.3.18) and three functions α, β, and γ from (34.4.3.14)–(34.4.3.17); in this case the last three functions are also treated as unknown. It can be shown that the last equation in (34.4.3.20) is a consequence of the other three equations in (34.4.3.19) and (34.4.3.20). Therefore, three unknown functions in system (34.4.3.19)–(34.4.3.21) can, in general, be taken arbitrarily.

In view of (34.4.3.18)–(34.4.3.21), system (34.4.3.14)–(34.4.3.17) is reduced to a single equation

$$\frac{\partial^2 F}{\partial t\partial x} + F\frac{\partial^2 F}{\partial x^2} - \left(\frac{\partial F}{\partial x}\right)^2 = \nu\frac{\partial^3 F}{\partial x^3} + q\frac{\partial F}{\partial x} + p, \tag{34.4.3.22}$$

where the functions $p = p(t)$ and $q = q(t)$ are determined by

$$p = \frac{\gamma - A_t'}{m}, \quad q = \frac{A - m_t'}{m}. \tag{34.4.3.23}$$

For $m = n$, the general solution to system (34.4.3.19)–(34.4.3.21) can be represented in the form

$$\begin{aligned}
&m = n = \frac{1}{2}\sin\varphi, \quad k = \frac{1}{2}\cos\varphi,\\
&A = B = \frac{1}{2}(q\sin\varphi + \varphi_t'\cos\varphi), \quad C = \frac{1}{2}(q\cos\varphi - \varphi_t'\sin\varphi),\\
&\alpha = \frac{1}{4}q^2 + \frac{1}{4}(\varphi_t')^2 - \frac{1}{2}p(1-\cos\varphi) + C_t',\\
&\beta = \frac{1}{4}q^2 + \frac{1}{4}(\varphi_t')^2 - \frac{1}{2}p(1+\cos\varphi) - C_t',\\
&\gamma = \frac{1}{2}p\sin\varphi + A_t',
\end{aligned} \tag{34.4.3.24}$$

where $p = p(t)$, $q = q(t)$, and $\varphi = \varphi(t)$ are arbitrary functions. For convenience, the free functions p and q in (34.4.3.24) are chosen so that system (34.4.3.14)–(34.4.3.17) in conjunction with the differential constraints (34.4.3.18) and (34.4.3.24) is reduced to a single equation (34.4.3.22) with the same functions $p = p(t)$ and $q = q(t)$.

Thus, we have proved the following important statement: any solution to equation (34.4.3.22) for any $p = p(t)$ and $q = q(t)$ generates a solution to the system of equations (34.4.3.14)–(34.4.3.17).

The case $m \neq n$ and a large number of exact solutions to equation (34.4.3.22) can be found in Aristov and Polyanin (2009, 2010), Polyanin and Aristov (2009), and Aristov, Knyazev, and Polyanin (2009) (see also equation 13.5.1.4).

Example 34.21. Consider the nonlinear hydrodynamic-type system of two equations

$$\begin{aligned}
\frac{\partial^2 F}{\partial t\partial x} + F\frac{\partial^2 F}{\partial x^2} - \Big(\frac{\partial F}{\partial x}\Big)^2 &= \nu\frac{\partial^3 F}{\partial x^3} + q\frac{\partial F}{\partial x} + p,\\
\frac{\partial G}{\partial t} + F\frac{\partial G}{\partial x} - G\frac{\partial F}{\partial x} &= \nu\frac{\partial^2 G}{\partial x^2},
\end{aligned} \tag{34.4.3.25}$$

the first of which is independent of G and coincides with (34.4.3.22) and the second is linear in G. The functions $p = p(t)$ and $q = q(t)$ in the first equation in (34.4.3.25) can be chosen arbitrarily.

It can be shown that exact solutions to system (34.4.3.25) generate some exact solutions to system (34.4.3.14)–(34.4.3.17) and, hence, exact solutions to the unsteady three-dimensional Navier–Stokes equations (Aristov and Polyanin 2009, 2010).

Let us supplement system (34.4.3.25) with the second-order differential constraint

$$G = a(t) + b(t)\frac{\partial F}{\partial x} + c(t)\frac{\partial^2 F}{\partial x^2}. \tag{34.4.3.26}$$

In the second equation of system (34.4.3.25), we eliminate G using (34.4.3.26) and compare the resulting equation with the first equation in (34.4.3.25) (as well as with the equations obtained by differentiating the first equation in (34.4.3.25) with respect to x). This results in the compatibility conditions of system (34.4.3.25) and the differential constraint (34.4.3.26):

$$\begin{aligned}
a_t' + bp &= 0,\\
b_t' + bq - a &= 0,\\
c_t' + qc &= 0.
\end{aligned} \tag{34.4.3.27}$$

The last equation in (34.4.3.27) is easy to integrate: $c = C_1\exp(-\int q\,dt)$, where C_1 is an arbitrary constant. From the first two equations in (34.4.3.27), we obtain a linear second-order equation for b:

$$b_{tt}'' + qb_t' + (p + q_t')b = 0. \tag{34.4.3.28}$$

On solving this equation, one finds $a = a(t)$ from the second equation in (34.4.3.27) without integration.

The result obtained can be rephrased as follows. Suppose an exact solution $F = F(t, x)$ to the first equation in (34.4.3.25) is known. Then the corresponding exact solution to the second equation in (34.4.3.25) can be obtained by formula (34.4.3.26), where $a = a(t)$, $b = b(t)$, and $c = c(t)$ are determined by solving the system of ordinary differential equations (34.4.3.27).

It is noteworthy that formula (34.4.3.26) can be used for the analysis of nonlinear stability of solutions to system (34.4.3.25) (Polyanin, 2009 *a*).

Remark 1. The books by Sidorov, Shapeev, and Yanenko (1984) and Meleshko (2005) illustrate the construction of exact solutions to the system of gas dynamics equations with the aid of differential constraints.

TABLE 34.1
Second-order differential constraints corresponding to some classes of exact solutions representable in explicit form

No.	Type of solution	Structure of solution	Differential constraints
1	Additive separable solution	$w = \varphi(x) + \psi(y)$	$w_{xy} = 0$
2	Multiplicative separable solution	$w = \varphi(x)\psi(y)$	$ww_{xy} - w_x w_y = 0$
3	Generalized separable solution	$w = \varphi(x)y^2 + \psi(x)y + \chi(x)$	$w_{yy} - f(x) = 0$
4	Generalized separable solution	$w = \varphi(x)\psi(y) + \chi(x)$	$w_{yy} - f(y)w_y = 0$ $w_{xy} - g(x)w_y = 0$
5	Functional separable solution	$w = f(z),\ z = \varphi(x)y + \psi(x)$	$w_{yy} - g(w)w_y^2 = 0$
6	Functional separable solution	$w = f(z),\ z = \varphi(x) + \psi(y)$	$ww_{xy} - g(w)w_x w_y = 0$

Remark 2. The nonclassical method of symmetry reductions for partial differential equations is reduced to the analysis of equations using two differential constraints, one of which is a first-order constraint and the order of the other is determined by the order of the equation under study (see Subsection 34.5.3).

⊙ *References for Section* 34.4: V. A. Galaktionov (1994), O. V. Kaptsov (1995, 1998), P. J. Olver and E. M. Vorob'ev (1996), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), O. V. Kaptsov and I. V. Verevkin (2003), A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov (2005), B. Kruglikov (2008), S. N. Aristov and A. D. Polyanin (2009, 2010), A. D. Polyanin and S. N. Aristov (2009), A. D. Polyanin (2009 *a*), S. N. Aristov, D. V. Knyazev, and A. D. Polyanin (2009).

34.5. Connection Between the Method of Differential Constraints and Other Methods

The method of differential constraints is one of the most general methods for the construction of exact solutions to nonlinear partial differential equations. Many other methods can be treated as its particular case.*

34.5.1. Generalized/Functional Separation of Variables vs. Differential Constraints

Table 34.1 lists examples of second-order differential constraints that are essentially equivalent to the most common forms of separable solutions. For functional separable solutions (rows 5 and 6), the function g can be expressed through f.

Table 34.2 lists examples of third-order differential constraints which may be regarded as equivalent to direct specifications of the most common forms of functional separable solutions.

Searching for a generalized separable solution of the form $w(x, y) = \varphi_1(x)\psi_1(y) + \cdots + \varphi_n(x)\psi_n(y)$, with $2n$ unknown functions, is equivalent to prescribing a differential constraint of order $2n$; in general, the number of unknown functions $\varphi_i(x)$, $\psi_i(y)$ corresponds to the order of the differential equation representing the differential constraint.

For the types of solutions listed in Tables 34.1 and 34.2, it is preferable to use the methods of generalized and functional separation of variables, since these methods require less steps where it is necessary to solve intermediate differential equations. Furthermore, the method of differential constraints is ill-suited for the construction of exact solutions of third- and higher-order equations since they lead to cumbersome computations and rather complex equations (often, the original equations are simpler).

Remark. The direct specification of a solution structure, on which the methods of generalized and functional separation of variables are based, may be treated as the use of a zeroth-order differential constraint.

* The basic difficulty in applying the method of differential constraints is due to the great generality of its statements and the necessity of selecting differential constraints suitable for specific classes of equations. This is why for the construction of exact solutions of nonlinear equations, it is often preferable to use simpler (but less general) methods.

TABLE 34.2
Third-order differential constraints corresponding to some classes of exact solutions representable in explicit form

Type of solution	Structure of solution	Differential constraint
Generalized separable	$w = \varphi(x)y^2 + \psi(x)y + \chi(x)$	$w_{yyy} = 0$
Generalized separable	$w = \varphi(x)\psi(y) + \chi(x)$	$w_y w_{xyy} - w_{xy} w_{yy} = 0$
Functional separable	$w = f\big(\varphi(x)y + \psi(x)\big)$	$w_y(w_x w_{yyy} - w_y w_{xyy}) = 2w_{yy}(w_x w_{yy} - w_y w_{xy})$
Functional separable	$w = f\big(\varphi(x) + \psi(y)\big)$	$w_x w_y w_{xyy} - w_y w_{xxy} = w_{xy}(w_x^2 w_{yy} - w_y^2 w_{xx})$

34.5.2. Direct Method of Symmetry Reductions and Differential Constraints

Consider a symmetry reduction based on a prescribed form of the desired solution,

$$w(x,t) = F\big(x,t,u(z)\big), \qquad z = z(x,t), \tag{34.5.2.1}$$

where $F(x,t,u)$ and $z(x,t)$ should be selected so as to obtain ultimately a single ordinary differential equation for $u(z)$; see Section 31.1.

Let us show that employing the solution structure (34.5.2.1) is equivalent to searching for a solution with the help of a first-order quasilinear differential constraint

$$\xi(x,t)\frac{\partial w}{\partial t} + \eta(x,t)\frac{\partial w}{\partial x} = \zeta(x,t,w). \tag{34.5.2.2}$$

Indeed, first integrals of the characteristic system of ordinary differential equations

$$\frac{dt}{\xi(x,t)} = \frac{dx}{\eta(x,t)} = \frac{dw}{\zeta(x,t,w)}$$

have the form

$$z(x,t) = C_1, \qquad \varphi(x,t,w) = C_2, \tag{34.5.2.3}$$

where C_1 and C_2 are arbitrary constants. Therefore, the general solution of equation (34.5.2.2) can be written as follows:

$$\varphi(x,t,w) = u\big(z(x,t)\big), \tag{34.5.2.4}$$

where $u(z)$ is an arbitrary function. On solving (34.5.2.4) for w, we obtain a representation of the solution in the form (34.5.2.1).

34.5.3. Nonclassical Method of Symmetry Reductions and Differential Constraints

The nonclassical method of symmetry reductions can be restated in terms of the method of differential constraints. This can be demonstrated by the following example with a general second-order equation

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{34.5.3.1}$$

Let us supplement equation (34.5.3.1) with two differential constraints

$$\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} = \zeta, \tag{34.5.3.2}$$

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial w} + \zeta_1\frac{\partial F}{\partial w_x} + \zeta_2\frac{\partial F}{\partial w_y} + \zeta_{11}\frac{\partial F}{\partial w_{xx}} + \zeta_{12}\frac{\partial F}{\partial w_{xy}} + \zeta_{22}\frac{\partial F}{\partial w_{yy}} = 0, \tag{34.5.3.3}$$

where $\xi = \xi(x,y,w)$, $\eta = \eta(x,y,w)$, and $\zeta = \zeta(x,y,w)$ are unknown functions, and the coordinates of the first and the second prolongations ζ_i and ζ_{ij} are defined by formulas (32.1.3.2) and (32.1.3.7). The differential constraint (34.5.3.3) coincides with the invariance condition for equation (34.5.3.1); see (32.2.1.3)–(32.2.1.4).

The method for the construction of exact solutions to equation (34.5.3.1) based on using the first-order partial differential equation (34.5.3.2) and the invariance condition (34.5.3.3) corresponds to the nonclassical method of symmetry reductions (see Section 33.1).

Remark. When the classical schemes of group analysis are employed, one first considers two equations, (34.5.3.1) and (34.5.3.3). From these, one eliminates one of the highest-order derivatives, say w_{yy}, while the remaining derivatives (w_x, w_y, w_{xx}, and w_{xy}) are assumed "independent." The resulting expression splits into powers of independent derivatives (see Subsection 32.2.1). As a result, one arrives at an overdetermined system of equations, from which the functions ξ, η, and ζ are found. Then, these functions are inserted into the quasilinear first-order equation (34.5.3.2), whose solution allows us to determine the general form of a solution (this solution contains some arbitrary functions). Next, using (34.5.3.1), one can refine the structure of the solution obtained in the preceding step.

The classical scheme may result in the loss of some solutions, since at the first step of splitting it is assumed that the first derivatives w_x and w_y are independent, whereas these derivatives are in fact linearly dependent due to equation (34.5.3.2).

⊙ *References for Section* 34.5: P. J. Olver (1994), A. D. Polyanin and V. F. Zaitsev (2004), S. V. Meleshko (2005), A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov (2005).

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Chapter 35
Painlevé Test for Nonlinear Equations of Mathematical Physics

35.1. Movable Singularities of Solutions of Ordinary Differential Equations

35.1.1. Examples of Solutions Having Movable Singularities

The connection between the structure of differential equations and singularities of their solutions was established more than a hundred years ago. The singularities of solutions of linear ordinary differential equations are completely determined by singularities of the coefficients of the equations. Since the position of such singularities does not depend on integration constants, they are called fixed singularities. In the case of nonlinear equations, their solutions may also possess movable singularities, whose position depends on the initial conditions (integration constants).

Below, we give simplest examples of first-order ordinary differential equations and their solutions with movable singularities.

Equation	*Solution*	*Type of singularity of the solution*
$u'_z = -u^2$	$u = 1/(z - z_0)$	movable pole
$u'_z = 1/u$	$u = 2\sqrt{z - z_0}$	algebraic branch point
$u'_z = e^{-u}$	$u = \ln(z - z_0)$	logarithmic branch point
$u'_z = -u \ln^2 u$	$u = \exp[1/(z - z_0)]$	essentially singular point

Algebraic branch points, logarithmic branch points, and essentially singular points are called "critical singular points."

35.1.2. Classification Results for Nonlinear First- and Second-Order Equations

1°. In 1884, L. L. Fuchs established the following fact: the first-order nonlinear differential equation

$$u'_z = R(z, u),$$

where the function R is rational in the second argument and analytic with respect to the first, admits solutions without movable critical points (other than movable poles) only if it coincides with the general Riccati equation $u'_z = A_0(z) + A_1(z)u + A_2(z)u^2$.

2°. The second-order ordinary differential equations (in the complex plane) of the form

$$u''_{zz} = R(z, u, u'_z),$$

where $R = R(z, u, w)$ is a rational function of u and w and is analytic in z, were classified by P. Painlevé (1900) and B. Gambier (1910). These authors showed that all equations of this form whose solutions have no movable critical points (other than fixed singular points and movable poles) can be divided into 50 classes. The equations of 44 out of these classes can be integrated by quadrature or their order can be reduced. The remaining 6 classes, in canonical form, are irreducible and are called Painlevé equations (their solutions are called Painlevé transcendents).

35.1.3. Painlevé Equations

1°. The first Painlevé equation has the form

$$u''_{zz} = 6u^2 + z.$$

The equation has a movable pole z_0; in its neighborhood, the solutions can be represented by the series

$$u = \frac{1}{(z-z_0)^2} + \sum_{n=2}^{\infty} a_n (z-z_0)^n,$$

$$a_2 = -\tfrac{1}{10} z_0, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = C, \quad a_5 = 0, \quad a_6 = \tfrac{1}{300} z_0^2,$$

where z_0 and C are arbitrary constants; the coefficients a_n ($n \geq 7$) are uniquely determined by z_0 and C.

2°. The second Painlevé equation is expressed as

$$u''_{zz} = 2u^3 + zu + a.$$

In a neighborhood of the movable pole z_0, its solutions admit the following expansions:

$$u = \frac{m}{z-z_0} + \sum_{n=1}^{\infty} b_n (z-z_0)^n,$$

$$b_1 = -\tfrac{1}{6} m z_0, \quad b_2 = -\tfrac{1}{4}(m+\alpha), \quad b_3 = C, \quad b_4 = \tfrac{1}{72} z_0 (m+3\alpha),$$

$$b_5 = \tfrac{1}{3024}\left[(27 + 81\alpha^2 - 2z_0^3)m + 108\alpha - 216 C z_0\right],$$

where $m = \pm 1$; z_0 and C are arbitrary constants; and the coefficients b_n ($n \geq 6$) are uniquely determined by z_0 and C. The classical second Painlevé transcendent corresponds to $m = 1$.

3°. The other four Painlevé equations have a quadratic nonlinearity in the first derivative and are not written out here. It should be observed that the solution of the fourth Painlevé equation has a movable pole, while the solutions of the third, the fifth, and the sixth Painlevé equations have fixed logarithmic branch points. More detailed information about the Painlevé equations can be found in the literature cited at the end of this subsection.

4°. An ordinary differential equation (of any order) is called a Painlevé type equation if its solutions do not have movable singularities. Such equations are said to possess the *Painlevé property*.

35.1.4. Painlevé Test for Ordinary Differential Equations

Consider the nth-order ordinary differential equation

$$u_z^{(n)} = F\big(z, u, u'_z, \ldots, u_z^{(n-1)}\big), \qquad u_z^{(n)} \equiv \frac{d^n u}{dz^n}. \tag{35.1.4.1}$$

The Painlevé test involves searching for a solution to equation (35.1.4.1) in the form of a series with a movable pole type singularity:

$$u(z) = \frac{1}{(z-z_0)^p} \sum_{m=0}^{\infty} A_m (z-z_0)^m, \tag{35.1.4.2}$$

where z_0 is any number and p is a positive integer. Solution (35.1.4.2) must be general, and therefore the coefficients A_m must contain $n-1$ arbitrary constants (in this case, by virtue of the arbitrariness of z_0, the solution will depend, in accordance with the equation order, on n arbitrary constants). If there is more than one solution (35.1.4.2), all of them must satisfy the above requirements.

Example 35.1. Let us illustrate the characteristic features of using the Painlevé test by an example of a nonlinear second-order ordinary differential equation

$$u''_{zz} = u^2 + au + bz + cz^2. \tag{35.1.4.3}$$

In order to determine the leading term of the series (35.1.4.2), characterized by the exponent $-p$ and coefficient A_0, we substitute the single term

$$u = \frac{A_0}{\xi^p}, \quad \xi = z - z_0, \tag{35.1.4.4}$$

into equation (35.1.4.3) and multiply the resulting expression by ξ^{p+2} (the product $\xi^{p+2}u''_{zz}$ gives a quantity of degree 0) to obtain

$$p(p+1)A_0 = A_0^2\xi^{2-p} + A_0a\xi^2 + b(\xi+z_0)\xi^{p+2} + c(\xi+z_0)^2\xi^{p+2}. \tag{35.1.4.5}$$

As $\xi \to 0$ ($p > 0$), the only term on the right-hand side that makes a nonzero contribution is the first term $A_0^2\xi^{2-p}$. For this term to be an identical constant, one must set $p = 2$. In this case, it follows from (35.1.4.5) that $A_0 = 6$. Hence, the leading term of the series is given by (35.1.4.4) with

$$p = 2, \quad A_0 = 6.$$

With the above in mind, we look for a solution to equation (35.1.4.3) in the series form

$$u = 6\xi^{-2} + A_1\xi^{-1} + A_2 + A_3\xi + A_4\xi^2 + A_5\xi^3 + A_6\xi^4 + \cdots. \tag{35.1.4.6}$$

Substituting (35.1.4.6) into (35.1.4.3), collecting the terms of like powers of ξ, and equating the coefficients to zero, we arrive at a system of algebraic equations for A_m:

$$\begin{aligned}
\xi^{-3}&: \quad 10A_1 = 0,\\
\xi^{-2}&: \quad 12A_2 + A_1^2 + 6a = 0,\\
\xi^{-1}&: \quad 12A_3 + 2A_1A_2 + A_1a = 0,\\
1&: \quad 10A_4 + 2A_1A_3 + A_2^2 + A_2a + bz_0 + cz_0^2 = 0,\\
\xi&: \quad 6A_5 + 2A_1A_4 + 2A_2A_3 + A_3a + 2cz_0 + b = 0,\\
\xi^2&: \quad 0\times A_6 + 2A_1A_5 + 2A_2A_4 + A_3^2 + A_4a + c = 0.
\end{aligned} \tag{35.1.4.7}$$

The left column specifies the powers of ξ and the right column gives the corresponding coefficients. The first five equations in (35.1.4.7) give

$$A_1 = 0, \quad A_2 = -\tfrac{1}{2}a, \quad A_3 = 0, \quad A_4 = \tfrac{1}{10}\left(\tfrac{1}{4}a^2 - bz_0 - cz_0^2\right), \quad A_5 = -\tfrac{1}{6}b - \tfrac{1}{3}cz_0. \tag{35.1.4.8}$$

Substituting these values into the last equation in (35.1.4.7), we get

$$0\times A_6 + c = 0. \tag{35.1.4.9}$$

Depending on the value of c, there are two alternative possibilities.

1°. If $c = 0$, equation (35.1.4.9) is satisfied identically for any A_6. In this case, equation (35.1.4.3) passes the Painlevé test and its solution can be represented as the series expansion (35.1.4.6) with two arbitrary constants, z_0 and A_6; the first five coefficients are given by (35.1.4.8) and the coefficients $A_7, A_8, \ldots$ can be determined successively from recurrence relations (not written out here).

2°. If $c \neq 0$, equation (35.1.4.9) is not satisfied for any values of A_6. In this case, equation (35.1.4.3) does not have solutions of the form (35.1.4.2) and, hence, fails the Painlevé test.

Remark 1. In the equation, the terms determined by the constants p and A_0 of the leading term in the series (35.1.4.4) are called the *leading terms*. In equation (35.1.4.3), the leading terms are u''_{zz} and u^2.

35.1.5. Remarks on the Painlevé Test. Fuchs Indices. Examples

In general, just as in Example 35.1, the recurrence relations for determining the expansion coefficients have the form

$$k_mA_m = \Phi_m(A_0, A_1, \ldots, A_{m-1}), \qquad m = 1, 2, \ldots. \tag{35.1.5.1}$$

In order to obtain an arbitrary constant A_m, we take into account the fact that the equations

$$k_m = 0, \quad \Phi_m(A_0, A_1, \ldots, A_{m-1}) = 0 \tag{35.1.5.2}$$

must hold simultaneously for the solution.

In Example 35.1, the first seven terms of the series and a fairly large amount of calculations were required to carry out the Painlevé test. It was unknown in advance how many terms would be required for the analysis. For practical purposes, it would be useful to know, without voluminous calculations, how many terms of the expansion (35.1.4.2) should be taken as well as how many coefficients A_m and which of them could be arbitrary. To answer these questions, one should be able to calculate the factor k_m in formulas (35.1.5.1).

From the analysis in Example 35.1 it follows that the multipliers k_m of the coefficients A_m are only determined by the leading terms of equation (35.1.4.3); see the first terms in the second column in (35.1.4.7). A similar situation is the case for the general nth-order equation (35.1.4.1). Bearing in mind that the left-hand

sides of the relations in (35.1.5.1) are linear in A_m and the right-hand sides are independent of A_m, the multipliers k_m can be found by substituting the two-term expression

$$u = A_0\xi^{-p} + A_m\xi^{m-p}, \quad \xi = z - z_0 \tag{35.1.5.3}$$

into the leading terms of equation (35.1.4.1). Collecting the terms with like powers of A_m gives

$$A_m k_m \xi^q + O(A_m^2) = 0, \qquad q \geq m - n - p, \tag{35.1.5.4}$$

where k_m is the desired multiplier. It can be shown that k_m is a polynomial of degree n (n is the order of the differential equation) in the integer index m:

$$k_m = b_m m^n + b_{n-1}m^{n-1} + \cdots + b_1 m + b_0. \tag{35.1.5.5}$$

The equation $k_m = 0$ always has a root $m = 1$ (which corresponds to the arbitrariness in the choice of z_0) and the other roots determine the so-called *Fuchs indices* (also known as *resonances*)—the numbers $m_1, \ldots, m_{n-1}$ of the coefficients A_m in the expansion (35.1.4.2) that can be arbitrary.

In practice, the Painlevé test is convenient to carry out in three sequential stages.

First stage. First, one determines the leading term of the expansion (35.1.4.2), characterized by the exponent $-p$ and coefficient A_0. To this end, the monomial (35.1.4.4) is substituted into equation (35.1.4.1) and then both sides of the resulting equation are multiplied by ξ^{p+n}; the left-hand side will be constant. A value of p is chosen so that the limiting value of the right-hand side as $\xi \to 0$ is also constant (other than zero). The equality of the left- and right-hand sides as $\xi \to 0$ enables one to find A_0.

If all obtained values of p are positive integers (which is the first necessary condition of the Painlevé test), one can continue with the analysis of equation (35.1.4.1). If at least one of the values of p is noninteger or complex, then the equation under study fails the Painlevé test.

Second stage. The Fuchs indices are determined at this step. To this end, the binomial (35.1.5.3) is substituted into the leading terms and the multiplier k_m appearing on the left-hand side of the relations in (35.1.5.1) is determined. The second necessary condition of the Painlevé test is as follows: the equation $k_m = 0$ must have $n - 1$ distinct nonnegative integer roots, which means that the representation

$$k_m = (m + 1)(m - m_1)\ldots(m - m_{n-1}), \qquad 0 \leq m_1 < m_2 < \cdots < m_{n-1}, \tag{35.1.5.6}$$

must hold; m_j are integer numbers. If this condition is satisfied, one can proceed to the third stage.

If the equation $k_m = 0$ has noninteger, negative integer (except for $m = -1$) or complex roots, then the equation under study fails the Painlevé test and the analysis is stopped.

Third stage. One checks whether conditions (35.1.5.2) hold simultaneously and calculates the expansion coefficients A_m up to $m = m_{n-1}$ inclusive. If a condition (35.1.5.2) is not satisfied for some $m_j \leq m_{n-1}$, then the equation fails the Painlevé test.

Example 35.2. Let us find the Fuchs indices for equation (35.1.4.3) from Example 35.1.

Substituting the binomial (35.1.5.3) with $p = 2$ and $A_0 = 6$ into the leading terms of equation (35.1.4.3) (the corresponding "truncated" equation is $u''_{zz} = u^2$), we obtain the desired multiplier

$$k_m = m^2 - 5m - 6 = (m + 1)(m - 6).$$

From the equation $k_m = 0$ we find the Fuchs index $m_1 = 6$. Hence, the second necessary condition of the Painlevé test is satisfied and we should consider seven terms in the expansion (35.1.4.6) (the coefficient of the last term is A_6). The subsequent analysis is performed in the same manner as in Example 35.1.

Example 35.3. Consider the equation

$$u''_{zz} + 2uu'_z + au + bz + cz^2 = 0. \tag{35.1.5.7}$$

First stage. Substituting the monomial (35.1.4.4) into (35.1.5.7), we find the constants the leading term in the expansion

$$p = 1, \quad A_0 = 1. \tag{35.1.5.8}$$

Since p is a positive integer, equation (35.1.5.7) satisfies the first necessary condition of the Painlevé test.

Second stage. Substituting the binomial (35.1.5.3) into the leading terms of equation (35.1.5.7), taking into account the values (35.1.5.8) (the terms containing derivatives), and rearranging, we obtain

$$A_m(m + 1)(m - 2)\xi^{m-3} + O(A_m^2) = 0.$$

From the quadratic equation

$$k_m = (m + 1)(m - 2) = 0$$

we find the Fuchs index (other than –1):

$$m_1 = 2.$$

Since this is a positive integer, we conclude that equation (35.1.5.7) satisfies the second necessary condition of the Painlevé test.

Third stage. Substituting the expansion

$$u = \xi^{-1} + A_1 + A_2\xi + \cdots,$$

where three terms should only be considered (up to the term proportional to A_2, which follows from the second stage), into the equation, we obtain

$$\begin{aligned} \xi^{-2}&: \quad -2A_1 = 0, \\ \xi^{-1}&: \quad 0 \times A_2 + a = 0. \end{aligned} \tag{35.1.5.9}$$

It follows from the first equation that $A_1 = 0$. The second equation gives $a = 0$. The value of A_2 can be arbitrary, and so equation (35.1.5.7) passes the Painlevé test for arbitrary b and c.

Example 35.4. Consider the equation

$$uu''_{zz} + (u'_z)^2 = 0. \tag{35.1.5.10}$$

Substituting the monomial (35.1.4.4) into it, we find two values of the exponent:

$$p = 0, \quad p = -\frac{1}{2}.$$

Hence, equation (35.1.5.10) fails the Painlevé test.

The general solution of equation (35.1.5.10) is expressed as $u = A\sqrt{z - z_0}$ and has an algebraic branch point.

35.1.6. The Painlevé Test for Systems of Ordinary Differential Equations

The Painlevé test can be used for analyzing systems of ordinary differential equations. For a system consisting of N equations, a solution is sought by expanding all unknowns into series with movable poles:

$$u_k = (z - z_0)^{-p_k}\left[A_{k0} + A_{k1}(z - z_0) + A_{k2}(z - z_0)^2 + \cdots\right], \quad k = 1, \ldots, N, \tag{35.1.6.1}$$

where the p_k are positive integers. The generality of the solution must be ensured by the required number of arbitrary expansion coefficients (consistent with the system order) and the free parameter z_0. If the highest orders of the derivatives in the individual equations are $n_1, \ldots, n_N$, then the number of arbitrary constants a_{km} in solution (35.1.6.1) must be equal to $n_1 + \cdots + n_N - 1$.

Remark. In 1888, S. V. Kovalevskaya succeeded in integrating the equations of motion of a rigid body having a fixed point and subject to gravity, in a case previously unknown. She examined solutions of a system of six first-order nonlinear ordinary differential equations. Solutions were sought in the form of series expansions in powers of each unknown quantity with movable poles; see (35.1.6.1) with $N = 6$. The generality of the solution was ensured by a suitable (corresponding to the order of the system) number of arbitrary coefficients in the expansions and the free parameter z_0.

It should be mentioned that the studies of S. V. Kovalevskaya preceded the works of Painlevé on the classification of second-order ordinary differential equations, where similar expansions were used. For this reason, the more precise term *Kovalevskaya–Painlevé test* is sometimes used along with the term *Painlevé test* in the literature.

⊙ *References for Section* 35.1: V. V. Golubev (1950), G. M. Murphy (1960), M. J. Ablowitz, A. Ramani, H. Segur (1980), A. R. Its and V. Yu. Novokshenov (1986), M. Tabor (1989), V. I. Gromak and N. A. Lukashevich (1990), A. R. Chowdhury (2000), V. I. Gromak (2002), A. D. Polyanin and V. F. Zaitsev (2003), N. A. Kudryashov (2004), V. I. Gromak and A. S. Zinchenko (2004).

35.2. Solutions of Partial Differential Equations with a Movable Pole. Method Description

Basic idea: by analogy with ordinary differential equations, solutions of partial differential equations are sought in the form of series expansions containing a movable pole singularity. The position of the pole is defined by an arbitrary function.

For a clearer exposition, we will be considering equations of mathematical physics in two independent variables, x and t, and one dependent variable, w, explicitly independent of x and t.

35.2.1. Simple Scheme for Studying Nonlinear Partial Differential Equations

A solution sought in a small neighborhood of a manifold $x - x_0(t) = 0$ is the form of the following series expansion (Jimbo, Kruskal, and Miwa, 1982):

$$w(x,t) = \frac{1}{\xi^p}\sum_{m=0}^{\infty} w_m(t)\xi^m, \qquad \xi = x - x_0(t). \tag{35.2.1.1}$$

Here, the exponent p is a positive integer, so that the movable singularity is of the pole type. The function $x_0(t)$ is assumed arbitrary, and the w_m are assumed to depend on derivatives of $x_0(t)$.

The representation (35.2.1.1) is substituted into the given equation. The exponent p and the leading term $u_0(t)$ are first determined from the balance of powers in the expansion. Terms with like powers of ξ are further collected. In the resulting polynomial, the coefficients of the different powers of ξ are all equated to zero to obtain a system of ordinary differential equations for the functions $w_m(t)$.

The solution obtained is general if the expansion (35.2.1.1) involves arbitrary functions, with the number of them equal to the order of the equation in question.

35.2.2. General Scheme for Analysis of Nonlinear Partial Differential Equations

A solution of a partial differential equation is sought in a neighborhood of a singular manifold $\xi(x,t) = 0$ in the form of a generalized series expansion symmetric in the independent variables (Weiss, Tabor, and Carnevalle, 1983):

$$w(x,t) = \frac{1}{\xi^p}\sum_{m=0}^{\infty} w_m(x,t)\xi^m, \qquad \xi = \xi(x,t), \tag{35.2.2.1}$$

where $\xi_t\xi_x \neq 0$. Here and henceforth, the subscripts x and t denote partial derivatives; the function $\xi(x,t)$ is assumed arbitrary and the w_m are assumed to be dependent on derivatives of $\xi(x,t)$.

Expansion (35.2.1.1) is a special case of expansion (35.2.2.1), when the equation of the singular manifold has been resolved for x.

The requirement that there are no movable singularities implies that p is a positive integer. The solution will be general if the number of arbitrary functions appearing in the coefficients $w_m(x,t)$ of the expansion variable $\xi(x,t)$ coincides with the order of the equation.

Substituting (35.2.2.1) into the equation, collecting terms with like powers of ξ, and equating the coefficients to zero, one arrives at the following recurrence relations for the expansion coefficients:

$$k_m w_m = \Phi_m(w_0, w_1, \ldots, w_{m-1}, \xi_t, \xi_x, \ldots). \tag{35.2.2.2}$$

Here, k_m are polynomials of degree n with integer argument m of the form

$$k_m = (m+1)(m-m_1)(m-m_2)\ldots(m-m_{n-1}), \tag{35.2.2.3}$$

where n is the order of the equation concerned.

If the roots of the polynomial, $m_1, m_2, \ldots, m_{n-1}$, called *Fuchs indices* (*resonances*), are all nonnegative integers and the consistency conditions

$$\Phi_{m=m_j} = 0 \qquad (j = 1, 2, \ldots, n-1) \tag{35.2.2.4}$$

hold, the equation is said to pass the *Painlevé test*. Equations that pass the Painlevé test are often classified as integrable, which is supported by the fact that such equations are reducible to linear equations in many known cases.

35.2.3. Basic Steps of the Painlevé Test for Nonlinear Equations

For nonlinear equations of mathematical physics, the Painlevé test is convenient to carry out in several steps. At the first and second steps, one determines the leading term in the expansion (35.2.1.1) and the Fuchs indices; this allows to verify the necessary conditions for the Painlevé test without making full computations. For the sake of clarity, the basic steps in performing the Painlevé test for nonlinear equations, using the expansion (35.2.1.1), are shown in Figure 35.1.

Remark. An equation fails to pass the Painlevé test if any of the following conditions holds: $p < 0$, p is noninteger, p is complex, $m_j < 0$, m_j is noninteger, or m_j is complex (at least for one j, where $j = 1, \ldots, n-1$).

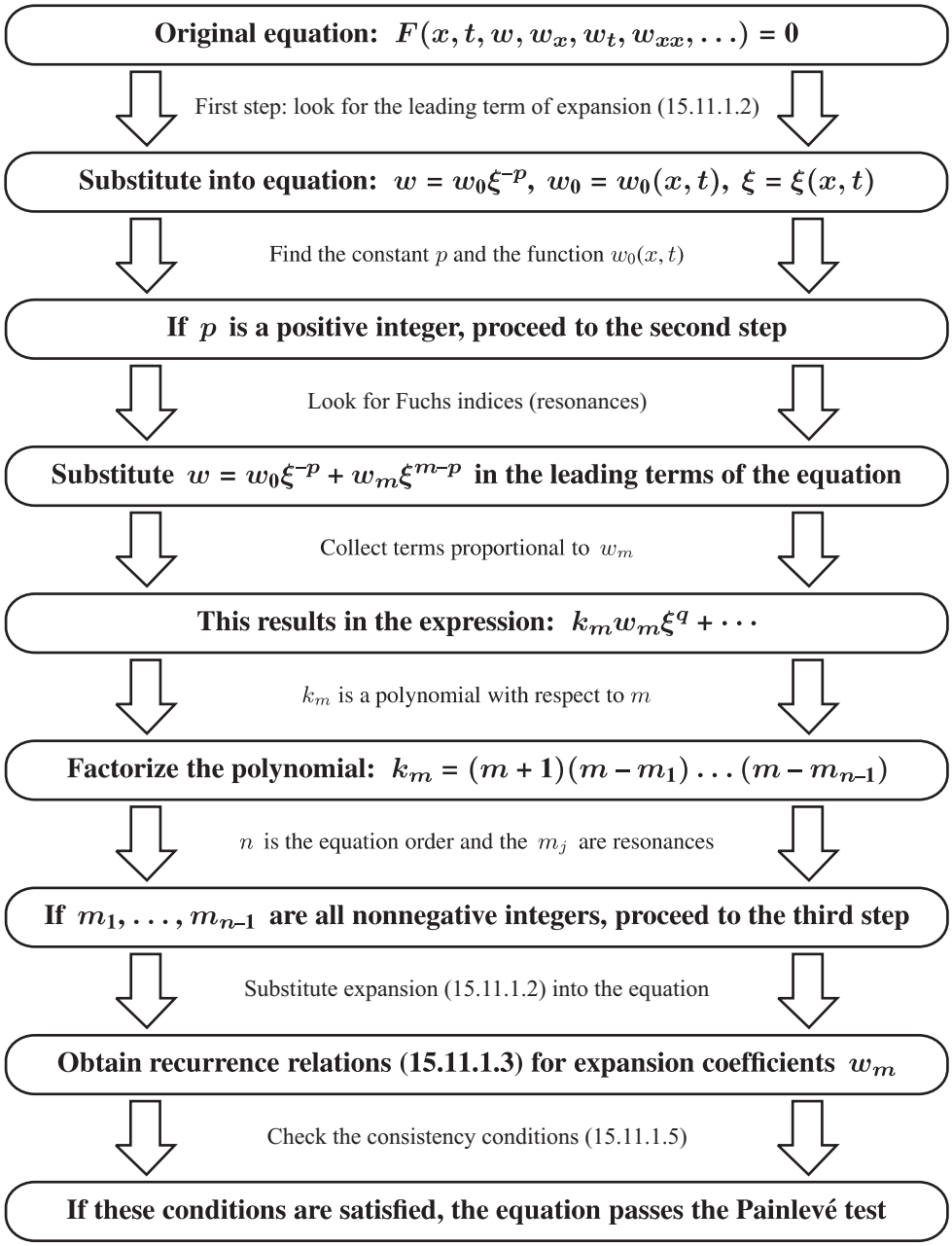

Figure 35.1. Basic steps of the Painlevé test for nonlinear equations of mathematical physics. It is assumed that $\xi = x - x_0(t)$ for the simple scheme and $\xi = \xi(x,t)$ for the general scheme.

35.2.4. Some Remarks. Truncated Expansions

Numerous researchers have established that many known integrable nonlinear equations of mathematical physics pass the Painlevé test. New equations possessing this property have also been found.

As a simple check whether a specific equation passes the Painlevé test, one may use the simple scheme based on the expansion (35.2.1.1). The associated important technical simplifications as compared with the expansion (35.2.2.1) are due to the fact that $(w_m)_x = 0$ and $\xi_x = 1$.

The general expansion (35.2.2.1), involving more cumbersome but yet more informative computations, can prove useful after the Painlevé property has been established at the simple check. It may help reveal many important properties of equations and their solutions.

In some cases, a truncated expansion,

$$w = \frac{w_0}{\xi^p} + \frac{w_1}{\xi^{p-1}} + \cdots + w_p, \tag{35.2.4.1}$$

can be useful for constructing exact solutions and finding a Bäcklund transformation linearizing the original equation. This expansion corresponds to zero values of the expansion coefficients w_m with $m > p$ in (35.2.2.1); see Examples 35.6 and 35.7 in Subsection 35.3.1.

The Painlevé test for a nonlinear partial differential equation can be performed for special classes of its exact solutions, usually traveling-wave solutions and self-similar solutions, which are determined by ordinary differential equations. If the ordinary differential equation obtained fails the Painlevé test, then the original partial differential equation also fails the test. If the ordinary differential equation passes the Painlevé test, then the original partial differential equation normally also passes the test.

Remark. It is widely believed that a nonlinear equation is integrable only if it passes the Painlevé test. This view is mistaken, which is illustrated by the counterexample below.

Example 35.5. Consider the nonlinear second-order equation

$$w_t = w^2 w_{xx}. \tag{35.2.4.2}$$

This equation is integrable and reducible to the linear heat equation (see equation 5.1.9.13).

However, equation (35.2.4.2) fails the Painlevé test. Searching for a solution in the series expansion form (35.2.1.1) leads to a negative fractional value of the exponent, $p = -1/2$. Note also that some particular solutions of (35.2.4.2) feature moving algebraic branch-point singularities. For example, the simplest traveling-wave solutions of the form $w = w(z)$ with $z = x - \lambda t$ are expressed as

$$w = \pm\sqrt{2\lambda(z - z_0)},$$

where λ and z_0 are arbitrary constants.

⊙ *References for Section* 35.2: M. J. Ablowitz, A. Ramany, and H. Segur (1980), M. Jimbo, M. D. Kruskal, T. Miwa (1982), J. Weiss, M. Tabor, G. Carnevalle (1983), J. Weiss (1983, 1984, 1985, 1986), W.-H. Steeb, N. Euler (1988), R. Conte (1989, 1999), R. Conte, M. Musette (1989, 1993), M. Musette (1998), A. D. Polyanin and V. F. Zaitsev (2004), N. A. Kudryashov (2004).

35.3. Performing the Painlevé Test and Truncated Expansions for Studying Some Nonlinear Equations

35.3.1. Equations Passing the Painlevé Test

This Subsection treats some common nonlinear equations of mathematical physics. For their analysis, the simple scheme of the Painlevé test will be used first; this scheme is based on the expansion (35.2.1.1) [see also the scheme in Fig. 35.1 with $\xi = x - x_0(t)$]. The truncated expansion (35.2.4.1) will then be used for constructing Bäcklund transformations.

Example 35.6. Consider the Burgers equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \nu\frac{\partial^2 w}{\partial x^2}. \tag{35.3.1.1}$$

First step. Substitute the leading term of expansion (35.2.1.1) into equation (35.3.1.1) and multiply the resulting relation by ξ^{p+2} (the product $\xi^{p+2}w_{xx}$ is equal to unity in order of magnitude). This results in

$$w_0'\xi^2 + pw_0x_0'\xi - pw_0^2\xi^{1-p} = \nu p(p+1)w_0,$$

where $\xi = x - x_0$, $x_0 = x_0(t)$, $w_0 = w_0(t)$, and the prime denotes a derivative with respect to t. We find from the balance of the leading terms, which corresponds to dropping two leftmost terms, that

$$p = 1, \qquad w_0 = -2\nu \qquad (m = 0). \tag{35.3.1.2}$$

Since p is a positive integer, the first necessary condition of the Painlevé test is satisfied.

Second step. In order to find the Fuchs indices (resonances), we substitute the binomial

$$w = -2\nu\xi^{-1} + w_m\xi^{m-1}$$

into the leading terms ww_x and νw_{xx} of equation (35.3.1.1). Collecting coefficients of like powers of w_m, we get

$$\nu(m+1)(m-2)w_m\xi^{m-3} + \cdots = 0.$$

Equating $(m+1)(m-2)$ to zero yields the Fuchs index

$$m_1 = 2.$$

Since it is a positive integer, the second necessary condition of the Painlevé test is satisfied.

Third step. We substitute the expansion (35.2.1.1) (according to the second step, we have to consider terms up to number $m = 2$ inclusive),

$$w = -2\nu\xi^{-1} + w_1 + w_2\xi + \cdots,$$

into the Burgers equation (35.3.1.1), collect terms of like powers of $\xi = x - x_0(t)$, and then equate the coefficients of the different powers to zero to obtain a system of equations for the w_m:

$$\begin{aligned} &\xi^{-2}\colon \quad 2\nu(w_1 - x_0') = 0,\\ &\xi^{-1}\colon \quad 0 \times w_2 = 0. \end{aligned} \tag{35.3.1.3}$$

From the second relation in (35.3.1.3) it follows that the function $w_2 = w_2(t)$ can be chosen arbitrarily. Therefore the Burgers equation (35.3.1.1) passes the Painlevé test and its solution has two arbitrary functions, $x_0 = x_0(t)$ and $w_2 = w_2(t)$, as required.

It follows from the first relation in (35.3.1.3) that $w_1 = x_0'(t)$. The solution to equation (35.3.1.1) can be written as

$$w(x,t) = -\frac{2\nu}{x - x_0(t)} + x_0'(t) + w_2(t)[x - x_0(t)]^2 + \cdots,$$

where $x_0(t)$ and $w_2(t)$ are arbitrary functions.

Cole–Hopf transformation. For further analysis of the Burgers equation (35.3.1.1), we use a truncated expansion of the general form (35.2.4.1) with $p = 1$:

$$w = \frac{w_0}{\xi} + w_1, \tag{35.3.1.4}$$

where $w_0 = w_0(x,t)$, $w_1 = w_1(x,t)$, and $\xi = \xi(x,t)$. Substitute (35.3.1.4) in (35.3.1.1) and collect the terms of equal powers in ξ to obtain

$$\begin{aligned} \xi^{-3}w_0\xi_x\big(w_0 + 2\nu\xi_x\big) + \xi^{-2}\big[w_0\xi_t - w_0(w_0)_x + w_0w_1\xi_x - 2\nu(w_0)_x\xi_x - \nu w_0\xi_{xx}\big]\\ + \xi^{-1}\big[-(w_0)_t - w_0(w_1)_x - w_1(w_0)_x + \nu(w_0)_{xx}\big] - (w_1)_t - w_1(w_1)_x + \nu(w_1)_{xx} = 0, \end{aligned}$$

where the subscripts x and t denote partial derivatives. Equating the coefficients of like powers of ξ to zero, we get the system of equations

$$\begin{aligned} w_0 + 2\nu\xi_x &= 0,\\ w_0\xi_t - w_0(w_0)_x + w_0w_1\xi_x - 2\nu(w_0)_x\xi_x - \nu w_0\xi_{xx} &= 0,\\ (w_0)_t + w_0(w_1)_x + w_1(w_0)_x - \nu(w_0)_{xx} &= 0,\\ (w_1)_t + w_1(w_1)_x - \nu(w_1)_{xx} &= 0, \end{aligned} \tag{35.3.1.5}$$

where it has been taken into account that $\xi_x \not\equiv 0$.

If follows from the first equation of (35.3.1.5) that

$$w_0 = -2\nu\xi_x. \tag{35.3.1.6}$$

Substituting this into the second and third equations of (35.3.1.5), after some rearrangements we obtain

$$\begin{aligned} &\xi_t + w_1\xi_x - \nu\xi_{xx} = 0,\\ &(\xi_t + w_1\xi_x - \nu\xi_{xx})_x = 0. \end{aligned} \tag{35.3.1.7}$$

It is obvious that if the first equation of (35.3.1.7) is valid, then the second is satisfied identically. The last equation in (35.3.1.5) is the Burgers equation.

Hence, formula (35.3.1.4) in view of (35.3.1.6) can be rewritten as

$$w = -2\nu\frac{\partial}{\partial x}\ln\xi + w_1, \tag{35.3.1.8}$$

where the functions w and w_1 satisfy the Burgers equation and the function ξ is described by the first equation of (35.3.1.7). Given a solution w_1 of the Burgers equation, formula (35.3.1.8) allows obtaining other solutions of it by solving the first equation in (35.3.1.7), which is linear in ξ.

Taking into account that $w_1 = 0$ is a particular solution of the Burgers equation, let us substitute it into (35.3.1.7) and (35.3.1.8). This results in the Cole–Hopf transformation

$$w = -2\nu\frac{\xi_x}{\xi}.$$

This transformation allows constructing solutions of the nonlinear Burgers equation (35.3.1.1) via solutions of the linear heat equation

$$\xi_t = \nu\xi_{xx}.$$

Example 35.7. Consider the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0. \tag{35.3.1.9}$$

First step. Let us substitute the leading term of the expansion (35.2.1.1) into equation (35.3.1.9) and then multiply the resulting relation by ξ^{p+3} (the product $\xi^{p+3}w_{xxx}$ gives a zeroth order quantity) to obtain

$$w_0'\xi^3 + pw_0x_0'\xi^2 - pw_0^2\xi^{2-p} - p(p+1)(p+2)w_0 = 0,$$

where $\xi = x - x_0$, $x_0 = x_0(t)$, and $w_0 = w_0(t)$. From the balance of the highest-order terms (only the last two terms are taken into account) it follows that

$$p = 2, \quad w_0 = -12 \qquad (m = 0).$$

Since p is a positive integer, the first necessary condition of the Painlevé test is satisfied.

Second step. To find the Fuchs indices (resonances), we substitute the binomial

$$w = -12\xi^{-2} + w_m\xi^{m-2}$$

into the leading terms of equation (35.3.1.9), where the second and the third term are taken into account. Isolating the term proportional to w_m, we have

$$(m+1)(m-4)(m-6)w_m\xi^{m-5} + \cdots = 0.$$

Equating $(m+1)(m-4)(m-6)$ to zero gives the Fuchs indices

$$m_1 = 4, \quad m_2 = 6.$$

Since they are both positive integers, the second necessary condition of the Painlevé test is satisfied.

Third step. We substitute the expansion (35.2.1.1), while considering, according to the second step, the terms up to number $m = 6$ inclusive,

$$w = -12\xi^{-2} + w_1\xi^{-1} + w_2 + w_3\xi + w_4\xi^2 + w_5\xi^3 + w_6\xi^4 + \cdots, \qquad \xi = x - x_0(t) \tag{35.3.1.10}$$

into equation (35.3.1.9). Then we collect the terms of equal powers in ξ and equate the coefficients of the different powers of ξ to zero to arrive at a system of equations for the w_m:

$$\begin{aligned}
&\xi^{-4}: \quad 2w_1 = 0,\\
&\xi^{-3}: \quad 24w_2 - 24x_0' - w_1^2 = 0,\\
&\xi^{-2}: \quad 12w_3 + w_1x_0' - w_1w_2 = 0,\\
&\xi^{-1}: \quad 0\times w_4 + w_1' = 0,\\
&1: \quad -6w_5 + w_2' - w_3x_0' + w_1w_4 + w_2w_3 = 0,\\
&\xi: \quad 0\times w_6 + w_3' - 2w_4x_0' + w_3^2 + 2w_1w_5 + 2w_2w_4 = 0.
\end{aligned} \tag{35.3.1.11}$$

Simple computations show that the equations with resonances corresponding to the powers ξ^{-1} and ξ are satisfied identically. Hence, the Korteweg–de Vries equation (35.3.1.9) passes the Painlevé test.

The solution of equation (35.3.1.11) results in the following expansion coefficients in (35.3.1.10):

$$w_1 = 0, \quad w_2 = x_0'(t), \quad w_3 = 0, \quad w_4 = w_4(t), \quad w_5 = \tfrac{1}{6}x_0''(t), \quad w_6 = w_6(t),$$

where $x_0(t)$, $w_4(t)$, and $w_6(t)$ are arbitrary functions.

Truncated series expansion and the Bäcklund transformation. For further analysis, let us use a truncated expansion of the general form (35.2.4.1) with $p = 2$:

$$w = \frac{w_0}{\xi^2} + \frac{w_1}{\xi} + w_2. \tag{35.3.1.12}$$

Substituting (35.3.1.12) into (35.3.1.9) and equating the functional coefficients of the different powers of ξ to zero, in the same way as in Example 35.6, we arrive at the Bäcklund transformation

$$\begin{aligned} &w = 12(\ln\xi)_{xx} + w_2,\\ &\xi_t\xi_x + w_2\xi_x^2 + 4\xi_x\xi_{xxx} - 3\xi_{xx}^2 = 0,\\ &\xi_{xt} + w_2\xi_{xx} + \xi_{xxx} = 0,\\ &(w_2)_t + w_2(w_2)_x + (w_2)_{xxx} = 0. \end{aligned} \tag{35.3.1.13}$$

It relates the solutions w and w_2 of the Korteweg–de Vries equations. Eliminating w_2 from the second and third equations in (35.3.1.13), one can derive an equation for ξ, which can further be reduced, via suitable transformations, to a system of linear equations.

Example 35.8. Consider the Kadomtsev–Petviashvili equation

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3}\right) + a\frac{\partial^2 w}{\partial y^2} = 0,$$

which can be regarded as an integrable generalization of the Korteweg–de Vries equation of a higher dimension and a higher order.

1°. In many-dimensional cases, one utilizes an analogue of the expansion (35.2.1.1):

$$w(x,y,t) = \frac{1}{\varepsilon^\alpha}\sum_{n=0}^{\infty} w_n(y,t)\varepsilon^n, \quad \varepsilon = x - x_0(y,t). \tag{35.3.1.14}$$

Equating the leading singular terms for the Kadomtsev–Petviashvili equation, we obtain the same result as that for the Korteweg–de Vries equation,

$$\alpha = 2, \quad w_0 = -12 \qquad (n = 0).$$

Substituting the expansion (35.3.1.14) into the original equation, we obtain

$$w_{tx} + ww_{xx} + w_x^2 + w_{xxxx} + aw_{yy} = \sum_{n=0}^{\infty}\varepsilon^{n-6}E_n(y,t) = 0,$$

$$E_n(y,t) = (n+1)(n-4)(n-5)(n-6)w_n + \cdots.$$

It is apparent that there are three resonances: $n = 4,\ 5,\ 6$. In order to verify the conditions of the Painlevé test, let us write out recurrence relations for the first seven terms of the expansion,

$$\begin{aligned}
&E_0 = 10w_0(w_0+12) = 0 && (n=0),\\
&E_1 = 12w_1(w_0+2) = 0 && (n=1),\\
&E_2 = 3[2(\varepsilon_t + a\varepsilon_y^2 + w_2)w_0 + w_1^2] = 0 && (n=2),\\
&E_3 = a(w_1)_{yy} - 2(w_0)_t - 4a(w_0)_y\varepsilon_y - 2[aw_0\varepsilon_{yy} - (\varepsilon_t + a\varepsilon_y^2 + w_2)w_1 - w_3w_0] = 0 && (n=3),\\
&E_4 = a(w_0)_{yy} - (w_1)_t - 2a(w_1)_y\varepsilon_y - aw_1\varepsilon_{yy} = 0 && (n=4),\\
&E_5 = a(w_1)_{yy} = 0 && (n=5),\\
&E_6 = a(w_2)_{yy} + \big\{(w_3)_t + 2a(w_3)_y\varepsilon_y + aw_3\varepsilon_{yy}\big)\\
&\qquad\qquad + 2[(\varepsilon_t + a\varepsilon_y^2 + w_2w_4 + \tfrac{1}{2}w_3^2 + w_5w_1 + (w_0+12)w_6]\big\} = 0 && (n=6).
\end{aligned}$$

The last three relations (corresponding to resonances), in view of the preceding relations, hold identically and do not contain w_4, w_5, w_6. There are four arbitrary functions (ε, w_4, w_5, w_6) in the solution of the forth-order equation under consideration, which indicates that the Painlevé property holds.

2°. The utilization of the general expansion, with the series truncated so that $w_n = 0$ for $n > 2$, leads us to the Bäcklund transformation (for simplicity, we set $a = 1$)

$$\begin{aligned} &w = 12(\ln\varepsilon)_{xx} + w_2,\\ &\varepsilon_t\varepsilon_x + 4\varepsilon_x\varepsilon_{xxx} - 3\varepsilon_{xx}^2 + \varepsilon_y^2 + w_2\varepsilon_x^2 = 0,\\ &\varepsilon_{xt} + \varepsilon_{xxxx} + \varepsilon_{yy} - w_2\varepsilon_{xx} = 0,\\ &(w_2)_{tx} + w_2(w_2)_{xx} + (w_2)_x^2 + (w_2)_{xxxx} + (w_2)_{yy} = 0. \end{aligned}$$

Eliminating w_2 from the second and the third equations, we obtain an equation for the function ε, which allows us to pass to a solution of a system of linear equations.

35.3.2. Checking Whether Nonlinear Systems of Equations of Mathematical Physics Pass the Painlevé Test

Systems of partial differential equations are also useful to be verified whether they pass the Painlevé test. This is illustrated below by a specific example.

Example 35.9. Consider the model system of equations

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = -\frac{1}{2}\frac{\partial c^2}{\partial x} + \nu\frac{\partial^2 w}{\partial x^2},$$
$$\frac{\partial c}{\partial t} + \frac{\partial (wc)}{\partial x} = \chi\frac{\partial^2 c}{\partial x^2}$$

that describes convective mass transfer of an active substance in a viscous fluid in the case where the flow is affected by the substance through the pressure quadratically dependent on its concentration. Such equations are used for describing one-dimensional flows of electrically conducting fluids in a magnetic field with high magnetic pressure.

1°. *Checking whether the system passes the Painlevé test.* By analogy with the expansion (35.2.1.1), let us represent the desired quantities in the form

$$w(x,t) = \frac{1}{\varepsilon^\alpha}\sum_{n=0}^{\infty} w_n(t)\varepsilon^n, \quad c(x,t) = \frac{1}{\varepsilon^\beta}\sum_{n=0}^{\infty} c_n(t)\varepsilon^n, \quad \varepsilon \equiv x - x_0(t).$$

Equating the leading singular terms of the equations, we find that

$$\alpha = \beta = 1, \quad w_0 = -\chi, \quad c_0^2 = \chi(2\nu - \chi).$$

Let us write the recurrence relations for the series terms in matrix form

$$\begin{pmatrix} -(n-2)[\chi + \nu(n-1)] & (n-2)c_0 \\ (n-2)c_0 & -(n-2)n\chi \end{pmatrix}\begin{pmatrix} w_n \\ c_n \end{pmatrix} = \begin{pmatrix} f_{n-1} \\ g_{n-1} \end{pmatrix}.$$

The quantities f_{n-1}, g_{n-1} depend on the functions $w_0, \ldots, w_{n-1}, c_0, \ldots, c_{n-1}, x_0$. The condition of unique solvability of the matrix equation for the specified higher-order coefficients is violated if the characteristic determinant is equal to zero (the case of degenerate matrix), and then these coefficients may turn out to be arbitrary. Thus, the resonances are determined from the condition

$$\nu\chi(n+1)(n-2)^2(n-2+\chi/\nu) = 0.$$

All these resonances are positive integers (except for the special resonance $n = -1$) only if the Prandtl number is equal to unity, $\mathrm{Pr} \equiv \nu/\chi = 1$. One resonance, $n = 1$, is simple, and the other, $n = 2$, is multiple, so that the overall number of resonances is equal to four.

Writing out the first three recurrence relations

$$\begin{aligned} &c_0^2 + w_0(w_0 + 2\nu) = 0, && w_0 + \nu = 0 && (n = 0),\\ &c_0c_1 + w_0(\varepsilon_t + w_1) = 0, && w_0c_1 + c_0(\varepsilon_t + w_1) = 0 && (n = 1),\\ &(w_0)_t = 0, && (c_0)_t = 0 && (n = 2), \end{aligned}$$

we see that the compatibility condition holds for the resonance $n = 1$, since the two relations for $n = 1$ coincide by virtue of the expressions for $n = 0$ ($w_0 = \pm c_0$). The multiple resonance $n = 2$ also satisfies the compatibility condition, since both coefficients w_0, c_0 are constant. Therefore, the Painlevé property takes place for the equations of a fluid with an active substance (for $\nu/\chi = 1$).

2°. *Truncated expansions and Bäcklund transformations.* Using the general expansion with the series truncated so that $w_2 = w_3 = \cdots = 0$ and $c_2 = c_3 = \cdots = 0$, we obtain a Bäcklund transformation for the equations of a fluid with an active substance

$$w = \frac{w_0}{\varepsilon} + w_1, \quad c = \frac{c_0}{\varepsilon} + c_1,$$
$$w_0 = -\nu\varepsilon_x, \quad c_0 = \pm\nu\varepsilon_x, \quad \varepsilon_t + (w_1 \mp c_1)\varepsilon_x = \nu\varepsilon_{xx},$$
$$(w_1)_t + w_1(w_1)_x = -c_1(c_1)_x + \nu(w_1)_{xx}, \quad (c_1)_t + (w_1c_1)_x = \nu(c_1)_{xx}.$$

Comparing this with the Bäcklund transformation for the Burgers equation, we see that if, in the above transformation, we pass to the new variables equal to the sum and the difference of the original variables, we

obtain identical equations. Indeed, passing to such variables in the original equations with unit Prandtl number, we obtain a pair of identical Burgers equations,

$$s_t + ss_x = \nu s_{xx}, \quad s = w + c,$$
$$r_t + rr_x = \nu r_{xx}, \quad r = w - c,$$

each of which reduces to the linear heat equation (see Example 35.6).

Numerous investigations show that many known integrable nonlinear equations of mathematical physics possess the Painlevé property. Some new equations with this property have also been found. During the verification of the conditions of the Painlevé test for more complex equations and systems, resonances with higher n may arise. In such situations, analytical solution becomes more and more difficult. However, the Painlevé test is highly adapted for algorithmization and allows for the utilization of symbolic computation methods.

35.3.3. Construction of Solutions of Nonlinear Equations That Fail the Painlevé Test, Using Truncated Expansions

In some cases truncated expansions of the form (35.2.4.1) may be effective in finding exact solutions to nonlinear equations of mathematical physics that fail the Painlevé test. In such cases, the expansion parameter p must be a positive integer; it is determined in the same way as at the first step of performing the Painlevé test. We illustrate this by a specific example below.

Example 35.10. Consider the nonlinear diffusion equation with a cubic source

$$w_t = w_{xx} - 2w^3. \tag{35.3.3.1}$$

First step. Let us substitute the leading term of the expansion (35.2.1.1) into equation (35.3.3.1) and then multiply the resulting relation by ξ^{p+2} to obtain

$$w_0'\xi^2 + pw_0x_0'\xi = p(p+1)w_0 - 2w_0^3\xi^{2-2p},$$

where $\xi = x - x_0$, $x_0 = x_0(t)$, and $w_0 = w_0(t)$. From the balance of the highest-order terms (both terms on the right-hand side are taken into account) it follows that

$$p = 1, \quad w_0 = \pm 1 \qquad (m = 0). \tag{35.3.3.2}$$

Since p is a positive integer, the first necessary condition of the Painlevé test is satisfied.

Second step. The equation is invariant under the substitution of $-w$ for w. Hence, it suffices to consider only the positive value of w_0 in (35.3.3.2). Therefore, in order to find resonances, we substitute the binomial

$$w = \xi^{-1} + w_m\xi^{m-1}$$

in the leading terms w_{xx} and bw^3 of equation (35.3.3.1). Collecting the terms proportional to w_m, we get

$$(m+1)(m-4)w_m\xi^{m-3} + \cdots = 0.$$

Equating $(m+1)(m-4)$ to zero gives the Fuchs index $m_1 = 4$. Since it is a positive integer, the second necessary condition of the Painlevé test is satisfied.

Third step. We substitute the expansion (35.2.1.1) into equation (35.3.3.1); according to the second step, the terms up to number $m = 4$ inclusive must be taken into account. It can be shown that the consistency condition (35.2.2.4) is not satisfied, and therefore the equation in question fails the Painlevé test.

Using a truncated expansion for finding exact solutions. For further analysis, we use a truncated expansion of the general form (35.2.4.1) with $p = 1$, which is from the first step. This results in formula (35.3.1.4). Substituting (35.3.1.4) into the diffusion equation (35.3.3.1) and collecting the terms of equal powers in ξ, we obtain

$$\xi^{-3}\left(2w_0\xi_x^2 - 2w_0^3\right) + \xi^{-2}\left[w_0\xi_t - 2(w_0)_x\xi_x - w_0\xi_{xx} - 6w_0^2w_1\right]$$
$$+ \xi^{-1}\left[-(w_0)_t + (w_0)_{xx} - 6w_0w_1^2\right] - (w_1)_t + (w_1)_{xx} - 2w_1^3 = 0.$$

Equating the coefficients of like powers of ξ to zero, we arrive at the system of equations

$$\begin{aligned} w_0(\xi_x^2 - w_0^2) &= 0,\\ w_0\xi_t - 2(w_0)_x\xi_x - w_0\xi_{xx} - 6w_0^2w_1 &= 0,\\ -(w_0)_t + (w_0)_{xx} - 6w_0w_1^2 &= 0,\\ (w_1)_t - (w_1)_{xx} + 2w_1^3 &= 0. \end{aligned} \tag{35.3.3.3}$$

From the first equation in (35.3.3.3) we have

$$w_0 = \xi_x. \tag{35.3.3.4}$$

The other solution differs in sign only and gives rise to the same result, and therefore is not considered. Substituting (35.3.3.4) into the second and third equations of (35.3.3.3) and canceling by nonzero factors, we obtain

$$\begin{aligned} \xi_t - 3\xi_{xx} - 6\,w_1\xi_x &= 0, \\ -\xi_{xt} + \xi_{xxx} - 6w_1^2\xi_x &= 0. \end{aligned} \tag{35.3.3.5}$$

The latter equation in (35.3.3.3), which coincides with the original equation (35.3.3.1), is satisfied if

$$w_1 = 0. \tag{35.3.3.6}$$

On inserting (35.3.3.4) and (35.3.3.6) in (35.3.1.4), we get the following representation for solutions:

$$w = \frac{\xi_x}{\xi}, \tag{35.3.3.7}$$

where the function ξ is determined by an overdetermined linear system of equations resulting from the substitution of (35.3.3.6) into (35.3.3.5):

$$\begin{aligned} \xi_t - 3\xi_{xx} &= 0, \\ -\xi_{xt} + \xi_{xxx} &= 0. \end{aligned} \tag{35.3.3.8}$$

Differentiate the first equation with respect to x and then eliminate the mixed derivative w_{xt} using the second equation to obtain $\xi_{xxx} = 0$. It follows that

$$\xi = \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t). \tag{35.3.3.9}$$

In order to determine the functions $\varphi_k(t)$, let us substitute (35.3.3.9) into equations (35.3.3.8) to obtain

$$\begin{aligned} \varphi_2'x^2 + \varphi_1'x + \varphi_0' - 6\varphi_2 &= 0, \\ -\varphi_2'x - \varphi_1' &= 0. \end{aligned}$$

Equating the functional coefficients of the different powers of x to zero and integrating the resulting equations, we get

$$\varphi_2 = C_2, \quad \varphi_1 = C_1, \quad \varphi_0 = 6C_2t + C_0, \tag{35.3.3.10}$$

where C_0, C_1, and C_2 are arbitrary constants. Substituting (35.3.3.9) into (35.3.3.7) and taking into account (35.3.3.10), we find an exact solution of equation (35.3.3.1) in the form

$$w = \frac{2C_2x + C_1}{C_2x^2 + C_1x + 6C_2t + C_0}.$$

⊙ *References for Section* 35.3: M. Jimbo, M. D. Kruskal, T. Miwa (1982), J. Weiss, M. Tabor, G. Carnevalle (1983), J. Weiss (1983), R. Conte (1989), M. Musette (1998), V. A. Gorodtsov (1998, 2000), D. M. Klimov, V. G. Baydulov, and V. A. Gorodtsov (2001), N. A. Kudryashov and M. B. Sukharev (2001), N. A. Kudryashov (1990 *a*, 1990 *b*, 1993, 2004), A. D. Polyanin and V. F. Zaitsev (2004).

Chapter 36
Methods of the Inverse Scattering Problem (Soliton Theory)

Preliminary remarks. The methods of the inverse scattering problem rely on "implicit" linearization of equations. Main idea: Instead of the original nonlinear equation in the unknown w, one considers an auxiliary overdetermined linear system of equation for a (vector) function $\boldsymbol{\varphi}$, with the coefficients of this system generally dependent on w and the derivatives of w with respect to the independent variables. The linear system for $\boldsymbol{\varphi}$ is chosen so that the compatibility condition for its equations gives rise to the original nonlinear equation for w.

36.1. Method Based on Using Lax Pairs

36.1.1. Method Description. Consistency Condition. Lax Pairs

We will be studying a nonlinear evolution equation of the form

$$\frac{\partial w}{\partial t} = \mathrm{F}(w), \tag{36.1.1.1}$$

where the right-hand side $\mathrm{F}(w)$ depends on w and its derivatives with respect to x.

Consider two auxiliary linear differential equations, one corresponding to an eigenvalue problem and involving derivatives with respect to the space variable x only,

$$\mathrm{L}\varphi = \lambda\varphi, \tag{36.1.1.2}$$

and the other describing the evolution of the eigenfunction in time,

$$\frac{\partial \varphi}{\partial t} = -\mathrm{M}\varphi. \tag{36.1.1.3}$$

The coefficients of the linear differential operators L and M in equations (36.1.1.2) and (36.1.1.3) depend on w and its derivatives with respect to x.

Since system (36.1.1.2)–(36.1.1.3) is overdetermined (there are two equations for φ), the operators L and M cannot be arbitrary—they must satisfy a compatibility condition. In order to find this condition, let us first differentiate (36.1.1.2) with respect to t. Assuming that the eigenvalues λ are independent of time t, we have

$$\mathrm{L}_t\varphi + \mathrm{L}\varphi_t = \lambda\varphi_t.$$

Replacing φ_t here by the right-hand side (36.1.1.3), we get

$$\mathrm{L}_t\varphi - \mathrm{LM}\varphi = -\lambda\mathrm{M}\varphi.$$

Taking into account the relations $\lambda\mathrm{M}\varphi = \mathrm{M}(\lambda\varphi)$ and $\lambda\varphi = \mathrm{L}\varphi$, we arrive at the compatibility condition

$$\mathrm{L}_t\varphi = \mathrm{LM}\varphi - \mathrm{ML}\varphi,$$

which can be rewritten in the form of an operator equation:

$$\frac{\partial \mathrm{L}}{\partial \mathrm{t}} = \mathrm{LM} - \mathrm{ML}. \tag{36.1.1.4}$$

The linear operators L and M [or the linear equations (36.1.1.2) and (36.1.1.3)] are said to form a *Lax pair* for the nonlinear equation (36.1.1.1) if the compatibility condition (36.1.1.4) coincides with equation (36.1.1.1). The right-hand side of equation (36.1.1.4) represents the *commutator of the operators* L and M, which is denoted by $[\mathrm{L}, \mathrm{M}] = \mathrm{LM} - \mathrm{ML}$ for short.

Thus, if a suitable Lax pair is found, the analysis of the nonlinear equation (36.1.1.1) can be reduced to that of two simpler, linear equations, (36.1.1.2) and (36.1.1.3).

Remark. The operator M in equations (36.1.1.3) and (36.1.1.4) is defined to an additive function of time; it can be changed according to the rule

$$\mathrm{M} \Longrightarrow \mathrm{M} + p(t),$$

where $p(t)$ is an arbitrary function. This function is found in solving a Cauchy problem for equation (36.1.1.1); see Subsection 36.4.2.

36.1.2. Examples of Lax Pairs for Nonlinear Equations of Mathematical Physics

Example 36.1. Let us show that a Lax pair for the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t}+\frac{\partial^3 w}{\partial x^3}-6w\frac{\partial w}{\partial x}=0 \tag{36.1.2.1}$$

is formed by the operators

$$\mathrm{L}=w-\frac{\partial^2}{\partial x^2},\quad \mathrm{M}=4\frac{\partial^3}{\partial x^3}-6w\frac{\partial}{\partial x}-3\frac{\partial w}{\partial x}+p(t), \tag{36.1.2.2}$$

which generate the linear equations

$$\begin{aligned}\varphi_{xx}+(\lambda-w)\varphi&=0,\\ \varphi_t+4\varphi_{xxx}-6w\varphi_x-3w_x\varphi+p(t)\varphi&=0.\end{aligned} \tag{36.1.2.3}$$

Here, $p(t)$ is an arbitrary function.

It is not difficult to verify that the following formulas hold:

$$\begin{aligned}\mathrm{LM}(\varphi)=&-4\varphi_{xxxxx}+10w\varphi_{xxx}+[15w_x-p(t)]\varphi_{xx}+(12w_{xx}-6w^2)\varphi_x\\&+[3w_{xxx}-3ww_x+wp(t)]\varphi,\\ \mathrm{ML}(\varphi)=&-4\varphi_{xxxxx}+10w\varphi_{xxx}+[15w_x-p(t)]\varphi_{xx}+(12w_{xx}-6w^2)\varphi_x\\&+[4w_{xxx}-9ww_x+wp(t)]\varphi,\end{aligned}$$

where $\varphi(x,t)$ is an arbitrary function. It follows that

$$\mathrm{LM}(\varphi)-\mathrm{ML}(\varphi)=(-w_{xxx}+6ww_x)\varphi. \tag{36.1.2.4}$$

From (36.1.2.2) and (36.1.2.4) we obtain

$$\mathrm{L}_t=w_t,\quad \mathrm{LM}-\mathrm{ML}=-w_{xxx}+6ww_x.$$

On inserting these expressions into (36.1.1.4), we arrive at the Korteweg–de Vries equation (36.1.2.1).

Remark. A procedure for solving the Cauchy problem for equation (36.1.2.1) is outlined in Section 36.4.

The linear equations (36.1.1.2) and (36.1.1.3) for the auxiliary function φ, which form a Lax pair, can have a vector form; in this case, the linear operators L and M are represented by matrices. In other words, the individual equations (36.1.1.2) and (36.1.1.3) may be replaced by appropriate systems of linear equations.

Example 36.2. The sinh-*Gordon equation*

$$w_{xt}=4a\sinh w$$

can be represented as a vector Lax pair

$$\begin{aligned}\mathrm{L}\boldsymbol{\varphi}&=\lambda\boldsymbol{\varphi},\\ \boldsymbol{\varphi}_t&=-\mathrm{M}\boldsymbol{\varphi},\end{aligned}$$

where

$$\boldsymbol{\varphi}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad \mathrm{L}=\begin{pmatrix}0 & \partial_x+\frac{1}{2}w_x\\ \partial_x-\frac{1}{2}w_x & 0\end{pmatrix},\quad \mathrm{M}=\frac{a}{\lambda}\begin{pmatrix}0 & e^{w}\\ e^{-w} & 0\end{pmatrix}.$$

Example 36.3. The nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+\alpha|w|^2w=0$$

can be represented as the vector Lax pair

$$\begin{aligned}\mathrm{L}\boldsymbol{\varphi}&=\lambda\boldsymbol{\varphi},\\ \boldsymbol{\varphi}_t&=-\mathrm{M}\boldsymbol{\varphi},\end{aligned}$$

where

$$\boldsymbol{\varphi}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad \mathrm{L}=\begin{pmatrix}i(1+p)\frac{\partial}{\partial x} & \bar{w}\\ w & i(1-p)\frac{\partial}{\partial x}\end{pmatrix},\quad \mathrm{M}=i\begin{pmatrix}p\frac{\partial^2}{\partial x^2}-\frac{|w|^2}{1+p} & -i\bar{w}_x\\ iw_x & p\frac{\partial^2}{\partial x^2}+\frac{|w|^2}{1-p}\end{pmatrix},$$

with $\alpha=2/(1-p^2)$; the bar or a symbol denote the complex conjugate.

The determination of a Lax pair for a given nonlinear equation is a very complex problem that is basically solvable for isolated equations only. Therefore, the "implicit" linearization of equations is usually realized using a simpler method, described in Subsection 36.2.

⊙ *References for Section* 36.1: C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura (1967), P. D. Lax (1968), V. E. Zakharov and L. D. Faddeev (1971), V. E. Zakharov and A. B. Shabat (1972, 1974), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), L. D. Faddeev and L. A. Takhtajan (1987), K. Chadan, D. Colton, L. Paivarinta, and W. Rundell (1997), M. J. Ablowitz and P. A. Clarkson (1991), R. Pike and P. Sabatier (2002).

36.2. Method Based on a Compatibility Condition for Systems of Linear Equations

36.2.1. General Scheme. Compatibility Condition. Systems of Two Equations

Consider two systems of linear equations

$$\boldsymbol{\varphi}_x = \mathrm{A}\boldsymbol{\varphi}, \tag{36.2.1.1}$$

$$\boldsymbol{\varphi}_t = \mathrm{B}\boldsymbol{\varphi}, \tag{36.2.1.2}$$

where $\boldsymbol{\varphi}$ is an n-dimensional vector and A and B are $n\times n$ matrices. The right-hand sides of systems (36.2.1.1) and (36.2.1.2) cannot be arbitrary—they must satisfy a compatibility condition. To find this condition, let us differentiate systems (36.2.1.1) and (36.2.1.2) with respect to t and x, respectively, and eliminate the mixed derivative $\boldsymbol{\varphi}_{xt}$ from the resulting equations. Then replacing the derivatives $\boldsymbol{\varphi}_x$ and $\boldsymbol{\varphi}_t$ by the right-hand sides of (36.2.1.1) and (36.2.1.2), we obtain

$$\mathrm{A}_t - \mathrm{B}_x + [\mathrm{A},\mathrm{B}] = 0, \tag{36.2.1.3}$$

where $[\mathrm{A},\mathrm{B}] = \mathrm{AB} - \mathrm{BA}$. It turns out that, given a matrix A, there is a simple deductive procedure for finding B as a result of which the compatibility condition (36.2.1.3) becomes a nonlinear evolution equation.

Let us dwell on the special case where the vector function $\boldsymbol{\varphi}$ has two components, $\boldsymbol{\varphi} = \binom{\varphi_1}{\varphi_2}$. We choose a linear system of equations (36.2.1.1) in the form

$$\begin{aligned}(\varphi_1)_x &= -i\lambda\varphi_1 + f\varphi_2,\\ (\varphi_2)_x &= g\varphi_1 + i\lambda\varphi_2,\end{aligned} \tag{36.2.1.4}$$

where λ is the spectral parameter, f and g are some (generally complex valued) functions of two variables x and t, and $i^2 = -1$. As system (36.2.1.2) we take the most general linear system involving the derivatives with respect to t:

$$\begin{aligned}(\varphi_1)_t &= A\varphi_1 + B\varphi_2,\\ (\varphi_2)_t &= C\varphi_1 + D\varphi_2,\end{aligned} \tag{36.2.1.5}$$

where A, B, C, and D are some functions (dependent on the variables x, t and the parameter λ) to be determined in the subsequent analysis. Differentiating equations (36.2.1.4) with respect to t and equations (36.2.1.5) with respect to x and assuming that $(\varphi_{1,2})_{xt} = (\varphi_{1,2})_{tx}$, we obtain compatibility conditions in the form

$$\begin{aligned}A_x &= Cf - Bg,\\ B_x + 2i\lambda B &= f_t - (A-D)f,\\ C_x - 2i\lambda C &= g_t + (A-D)g,\\ -D_x &= Cf - Bg.\end{aligned} \tag{36.2.1.6}$$

For simplicity, we set

$$D = -A.$$

In this case, the first and last equations in (36.2.1.6) coincide, so that (36.2.1.6) turns into a system of three determining equations:

$$\begin{aligned}A_x &= Cf - Bg,\\ B_x + 2i\lambda B &= f_t - 2Af,\\ C_x - 2i\lambda C &= g_t + 2Ag.\end{aligned} \tag{36.2.1.7}$$

The functions A, B, and C must be expressed in terms of f and g. The general solution of system (36.2.1.7) for arbitrary functions f and g cannot be found. So let us look for particular solutions in the form finite expansions in positive and negative powers of the parameter λ.

36.2.2. Solution of the Determining Equations in the Form of Polynomials in λ

The simplest polynomial representations of the unknown functions that give rise to nontrivial results are quadratic in the spectral parameter λ:

$$\begin{aligned} A &= A_2\lambda^2 + A_1\lambda + A_0, \\ B &= B_2\lambda^2 + B_1\lambda + B_0, \\ C &= C_2\lambda^2 + C_1\lambda + C_0. \end{aligned} \tag{36.2.2.1}$$

Let us substitute (36.2.2.1) into (36.2.1.7) and collect the terms with equal powers in λ to obtain

$$\lambda^2(A_{2x} - C_2 f + B_2 g) + \lambda(A_{1x} - C_1 f + B_1 g) + A_{0x} - C_0 f + B_0 g = 0, \tag{36.2.2.2}$$

$$\begin{aligned} &2i\lambda^3 B_2 + \lambda^2(B_{2x} + 2iB_1 + 2A_2 f) \\ &\quad + \lambda(B_{1x} + 2iB_0 + 2A_1 f) + B_{0x} + 2A_0 f - f_t = 0, \end{aligned} \tag{36.2.2.3}$$

$$\begin{aligned} &-2i\lambda^3 C_2 + \lambda^2(C_{2x} - 2iC_1 - 2A_2 g) \\ &\quad + \lambda(C_{1x} - 2iC_0 - 2A_1 g) + C_{0x} - 2A_0 g - g_t = 0. \end{aligned} \tag{36.2.2.4}$$

Let us equate the coefficients of like powers of λ to zero starting from the highest power. Setting the coefficients of λ^3 to zero gives

$$B_2 = C_2 = 0. \tag{36.2.2.5}$$

Equating the coefficients of λ^2 to zero and taking into account (36.2.2.5), we find that

$$A_2 = a = \text{const}, \quad B_1 = iaf, \quad C_1 = iag. \tag{36.2.2.6}$$

Setting the coefficient of λ in (36.2.2.2) to zero and then replacing B_1 and C_1 in accordance with (36.2.2.6), we have $A_{1x} = 0$, whence $A_1 = a_1 = \text{const}$. For simplicity, we dwell on the special case $a_1 = 0$ (arbitrary a_1 gives rise to more general results), so that

$$A_1 = 0. \tag{36.2.2.7}$$

By equating to zero the coefficients of λ in the equations obtained from (36.2.2.3) and (36.2.2.4) and taking into account (36.2.2.6) and (36.2.2.7), we get

$$B_0 = -\tfrac{1}{2}af_x, \quad C_0 = \tfrac{1}{2}ag_x. \tag{36.2.2.8}$$

Setting the coefficient of λ^0 in (36.2.2.2) to zero and then integrating, we find that $A_0 = \frac{1}{2}afg + a_0$, where a_0 is an arbitrary constant. As before, we set $a_0 = 0$ for simplicity, which results in

$$A_0 = \tfrac{1}{2}afg. \tag{36.2.2.9}$$

Then equations (36.2.2.3) and (36.2.2.4) in view of (36.2.2.8) and (36.2.2.9) become

$$\begin{aligned} f_t &= -\tfrac{1}{2}af_{xx} + af^2 g, \\ g_t &= \tfrac{1}{2}ag_{xx} - afg^2. \end{aligned} \tag{36.2.2.10}$$

Substituting (36.2.2.1) and (36.2.2.5)–(36.2.2.10) into (36.2.1.5) yields

$$\begin{aligned} (\varphi_1)_t &= a(\lambda^2 + \tfrac{1}{2}fg)\varphi_1 + a(i\lambda f - \tfrac{1}{2}f_x)\varphi_2, \\ (\varphi_2)_t &= a(i\lambda g + \tfrac{1}{2}g_x)\varphi_1 - a(\lambda^2 + \tfrac{1}{2}fg)\varphi_2. \end{aligned} \tag{36.2.2.11}$$

Thus, two linear systems (36.2.1.4) and (36.2.2.11) are compatible if the functions f and g satisfy the cubically nonlinear system (36.2.2.10). For

$$g = -k\bar{f}, \quad a = 2i, \tag{36.2.2.12}$$

where k is a real constant and the bar over a symbol stands for its complex conjugate, both equations (36.2.2.11) turn into one and the same nonlinear Schrödinger equation

$$if_t = f_{xx} + 2k|f^2|f \qquad (f\bar{f} = |f|^2). \tag{36.2.2.13}$$

Likewise, one can use other polynomials in λ and determine the associated linear systems generating nonlinear evolution equations.

Example 36.4. Searching for solutions of the determining system (36.2.1.7) in the form third-order polynomials in λ results in

$$\begin{aligned} A &= a_3\lambda^3 + a_2\lambda^2 + \tfrac{1}{2}(a_3fg + a_1)\lambda + \tfrac{1}{2}a_2fg - \tfrac{1}{4}ia_3(fg_x - gf_x) + a_0, \\ B &= ia_3f\lambda^2 + (ia_2f - \tfrac{1}{2}a_3f_x)\lambda + ia_1f + \tfrac{1}{2}ia_3f^2g - \tfrac{1}{2}a_2f_x - \tfrac{1}{4}a_3f_{xx}, \\ C &= ia_3g\lambda^2 + (ia_2g + \tfrac{1}{2}a_3g_x)\lambda + ia_1g + \tfrac{1}{2}ia_3fg^2 + \tfrac{1}{2}a_2g_x - \tfrac{1}{4}ia_3g_{xx}, \end{aligned} \tag{36.2.2.14}$$

where a_0, a_1, a_2, and a_3 are arbitrary constants. The evolution equations for f and g corresponding to (36.2.2.14) are

$$\begin{aligned} &f_t + \tfrac{1}{4}ia_3(f_{xxx} - 6fgf_x) + \tfrac{1}{2}a_2(f_{xx} - 2f^2g) - ia_1f_x - 2a_0f = 0, \\ &g_t + \tfrac{1}{4}ia_3(g_{xxx} - 6fgg_x) - \tfrac{1}{2}a_2(g_{xx} - 2fg^2) - ia_1g_x + 2a_0g = 0. \end{aligned} \tag{36.2.2.15}$$

Consider two important special cases leading to interesting nonlinear equations of mathematical physics.

1°. If

$$a_0 = a_1 = a_2 = 0, \quad a_3 = -4i, \quad g = 1$$

the second equation of (36.2.2.15) is satisfied identically, and the first equation of (36.2.2.15) translates into the Korteweg–de Vries equation

$$f_t + f_{xxx} - 6ff_x = 0.$$

2°. If

$$a_0 = a_1 = a_2 = 0, \quad a_3 = -4i, \quad g = \pm f,$$

both equations (36.2.2.15) translate into one and the same modified Korteweg–de Vries equation

$$f_t + f_{xxx} \mp 6f^2f_x = 0.$$

Example 36.5. Now we will look for a solution of the determining system (36.2.1.7) in the form of the simple one-term expansion in negative powers of λ:

$$A = a(x,t)\lambda^{-1}, \quad B = b(x,t)\lambda^{-1}, \quad C = c(x,t)\lambda^{-1}. \tag{36.2.2.16}$$

This results in the relations

$$a_x = \tfrac{1}{2}i(fg)_t, \quad b = -\tfrac{1}{2}if_t, \quad c = \tfrac{1}{2}ig_t. \tag{36.2.2.17}$$

The evolution equations for functions f and g corresponding to (36.2.2.17) are written as

$$\begin{aligned} f_{xt} &= -4iaf, \\ g_{xt} &= -4iag. \end{aligned} \tag{36.2.2.18}$$

If we set

$$a = \tfrac{1}{4}i\cos w, \quad b = c = \tfrac{1}{4}i\sin w, \quad f = -g = -\tfrac{1}{2}w_x, \tag{36.2.2.19}$$

then the three relations (36.2.2.17) are reduced to one and the same equation, the sine-Gordon equation

$$w_{xt} = \sin w, \tag{36.2.2.20}$$

and the two equations (36.2.2.18) coincide and give a differential consequence of equation (36.2.2.20):

$$w_{xxt} = w_x\cos w.$$

Thus, the linear system of equations (36.2.1.4) and (36.2.1.5), whose coefficients are defined by formulas (36.2.2.16) and (36.2.2.19) with $D = -A$, is compatible if the function w satisfies the sine-Gordon equation (36.2.2.20).

Remark. Sometimes the determining equations (36.2.1.1)–(36.2.1.2) are also called a Lax pair, by analogy with (36.1.1.2)–(36.1.1.3).

⊙ *References for Section* 36.2: V. E. Zakharov and A. B. Shabat (1972), M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur (1974), M. J. Ablowitz and H. Segur (1981), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), M. J. Ablowitz, P. A. Clarkson (1991).

36.3. Method Based on Linear Integral Equations

36.3.1. Description of the Method

Below we outline the approach proposed by Zakharov and Shabat (1974) based on using linear integral equations of the form*

$$K(x,y) = F(x,y) + \int_x^\infty K(x,z)N(x;z,y)\,dz, \quad y \ge x, \tag{36.3.1.1}$$

where F and N are given functions and K is the unknown function. In each specific case, the function N is explicitly expressed through F. Apart from the explicitly specified arguments x, y, and z, the functions F, N, and K can also depend on some additional parameters: $t, \lambda, \ldots$. The derivatives with respect to these parameters can appear in the differential equations for F and K, but equation (36.3.1.1) is treated at fixed values of these parameters.

Define an operator A_x such that

$$\mathrm{A}_x f(y) = \begin{cases} \int_x^\infty f(z)N(x;z,y)\,dz & \text{if } y \ge x, \\ 0 & \text{if } y < x \end{cases}$$

and assume that for each chosen N, it is possible to prove that the operator $\mathrm{I} - \mathrm{A}_x$ is invertible and its inverse, $(\mathrm{I} - \mathrm{A}_x)^{-1}$, is continuous, where I is the identity operator.

The following three steps represent an algorithm for finding a nonlinear equation that can then be solved by the inverse scattering method.

1°. One specifies the exact form of the integral equation (36.3.1.1). To this end, one specifies the relation between N and F (N is expressed in terms of F).

2°. One chooses two suitable linear (ordinary or partial) differential equations for F:

$$\mathrm{L}_m F = 0, \quad m = 1,\ 2. \tag{36.3.1.2}$$

3°. The function K is related to F by equation (36.3.1.1), which can be rewritten in the form

$$(\mathrm{I} - \mathrm{A}_x)K = F. \tag{36.3.1.3}$$

Applying the operators L_m, appearing in (36.3.1.2), to equation (36.3.1.3), one obtains

$$\mathrm{L}_m(\mathrm{I} - \mathrm{A}_x)K = 0, \quad m = 1,\ 2.$$

This equation can be rewritten as

$$(\mathrm{I} - \mathrm{A}_x)(\mathrm{L}_m K) = \mathrm{R}_m, \quad m = 1,\ 2,$$

where R_m contains all nonzero terms of the commutator $[\mathrm{L}_m, (\mathrm{I} - \mathrm{A}_x)]$. Equations (36.3.1.1) and (36.3.1.2) must be chosen such that R_m is representable in the form

$$\mathrm{R}_m = (\mathrm{I} - \mathrm{A}_x)\mathrm{M}_m(K), \quad m = 1,\ 2,$$

where $\mathrm{M}_m(K)$ is a nonlinear functional of K. The operator $\mathrm{I} - \mathrm{A}_x$ is invertible, and hence the function K must satisfy the nonlinear differential equations

$$\mathrm{L}_m K - \mathrm{M}_m(K) = 0, \quad m = 1, 2. \tag{36.3.1.4}$$

Consequently, each solution of the linear integral equation (36.3.1.1) also solves the nonlinear differential equations (36.3.1.4). As a rule, special cases or corollaries of one of the equations in (36.3.1.4) are of most interest.

Remark. The first two stages of the algorithm are the most important and most difficult. The linear differential equations (36.3.1.2) usually correspond to an eigenvalue problem (for $m = 1$) or the time evolution of eigenfunctions (for $m = 2$).

The fundamental features of the above algorithms are illustrated below by specific examples.

* Such equations are called Gelfand–Levitan–Marchenko type integral equations.

36.3.2. Specific Examples

Example 36.6. Let us consider the linear integral equation

$$K(x,y) = F(x,y) + \int_x^\infty K(x,z)F(z,y)\,dz, \tag{36.3.2.1}$$

which follows from (36.3.1.1) if $N(x;z,y) = F(z,y)$. It is implied that the functions F and K also depend on a parameter t (for brevity, this is not specified implicitly).

Note some identities that will be used in the sequel,

$$\partial_x^n \int_x^\infty K(x,z)F(z,y)\,dz = \int_x^\infty F(z,y)\partial_x^n K(x,z)\,dz + A_n, \tag{36.3.2.2}$$

$$\int_x^\infty K(x,z)\partial_x^n F(z,y)\,dz = (-1)^n \int_x^\infty F(z,y)\partial_z^n K(x,z)\,dz + B_n, \tag{36.3.2.3}$$

where the A_n and B_n are defined by the formulas

$$\begin{aligned} &A_1 = -K(x,x)F(x,y), \quad A_n = (A_{n-1})_x - F(x,y)[\partial_x^{n-1}K(x,z)]_{z=x},\\ &B_1 = -K(x,x)F(x,y), \quad B_2 = -K(x,x)\partial_x F(x,y) + F(x,y)[\partial_z K(x,z)]_{z=x}, \quad \ldots. \end{aligned}$$

Corollaries of the above identities:

$$\begin{aligned} &A_1 - B_1 = 0,\\ &A_2 - B_2 = -2F(x,y)\partial_x K(x,x),\\ &A_3 - B_3 = -3\partial_x F(x,y)\partial_x K(x,x) - 3F(x,y)\big[(\partial_x^2 + \partial_x\partial_z)K(x,z)\big]_{z=x}. \end{aligned} \tag{36.3.2.4}$$

Let us introduce an operator L_1 and require that F satisfy the linear wave equation

$$L_1 F \equiv (\partial_x^2 - \partial_y^2)F(x,y) = 0. \tag{36.3.2.5}$$

Applying the operator L_1 to (36.3.2.1) and taking into account (36.3.2.2), (36.3.2.5), we obtain

$$\begin{aligned} (\partial_x^2 - \partial_y^2)K(x,y) &= (\partial_x^2 - \partial_y^2)\int_x^\infty K(x,z)F(z,y)\,dz\\ &= \int_x^\infty K_{xx}(x,z)F(z,y)\,dz - \int_x^\infty K(x,z)F_{yy}(z,y)\,dz + A_2. \end{aligned}$$

Using relations (36.3.2.3) and (36.3.2.4), we get

$$(\partial_x^2 - \partial_y^2)K(x,y) = \int_x^\infty F(z,y)(\partial_x^2 - \partial_z^2)K(x,z)\,dz - 2F(x,y)\frac{d}{dx}K(x,x). \tag{36.3.2.6}$$

Taking advantage of the operator form of representation of the equation concerned, $F = (\mathrm{I} - \mathrm{A}_x)K$, we rewrite (36.3.2.6) in the form

$$(\mathrm{I} - \mathrm{A}_x)\Big[(\partial_x^2 - \partial_y^2)K(x,y) + 2K_x(x,x)K(x,y)\Big] = 0.$$

Further, taking into account that the operator $\mathrm{I} - \mathrm{A}_x$ is invertible, we get

$$(\partial_x^2 + \partial_y^2)K(x,y) + w(x)K(x,y) = 0, \tag{36.3.2.7}$$

where the function $w(x)$ is defined by

$$w(x) = -2\frac{d}{dx}K(x,x). \tag{36.3.2.8}$$

Require that F satisfy also the other linear equation

$$L_2 F = (\partial_t + (\partial_x + \partial_y)^3)F = 0 \tag{36.3.2.9}$$

and apply the operator L_2 to (36.3.2.1) to obtain

$$\big(\partial_t + (\partial_x + \partial_y)^3\big)K(x,y) = \big(\partial_t + (\partial_x + \partial_y)^3\big)\int_x^\infty K(x,z)F(z,y)\,dz.$$

A procedure similar to the above calculations for the operator L_1 yields

$$K_t + (\partial_x + \partial_y)^3 K - 3w(\partial_x + \partial_y)K = 0. \tag{36.3.2.10}$$

For the characteristic $y = x$, equation (36.3.2.10) reduces, after differentiating with respect to x and taking into account relation (36.3.2.8), to the Korteweg–de Vries equation

$$w_t - 6ww_x + w_{xxx} = 0.$$

Thus, any function F satisfying the linear equations (36.3.2.5), (36.3.2.9) and rapidly decaying as $x \to +\infty$ generates a solution of the Korteweg–de Vries equation. To this end, one should solve the linear integral equation (36.3.2.1) for function K and express w through K by formula (36.3.2.8).

Example 36.7. Now consider the more complex linear integral equation

$$K(x,y) = F(x,y) + \frac{\sigma}{4}\int_x^\infty \int_x^\infty K(x,z)F(z,u)F(u,y)\,dz\,du, \tag{36.3.2.11}$$

where $\sigma = \pm 1$. Here and in what follows, the coefficients are chosen with a view to simplifying the calculations. Equation (36.3.2.11) is a special case of equation (36.3.1.1) with

$$N(x;z,y) = \frac{\sigma}{4}\int_x^\infty F(z,u)F(u,y)\,du.$$

Let the operator L_1 and the associated linear differential equation have the form

$$L_1 F = (\partial_x - \partial_y)F = 0. \tag{36.3.2.12}$$

The general solution of this equation can be written as

$$F(x,y) = F\Big(\frac{x+y}{2}\Big).$$

Shifting the lower limit of integration to zero, we rewrite equation (36.3.2.11) in the form

$$K(x,y) = F\Big(\frac{x+y}{2}\Big) + \frac{\sigma}{4}\int_0^\infty \int_0^\infty K(x,x+\zeta)F\Big(\frac{2x+\zeta+\eta}{2}\Big)F\Big(\frac{x+\eta+y}{2}\Big)\,d\zeta\,d\eta, \tag{36.3.2.13}$$

or, equivalently,

$$[(\mathrm{I} - \sigma \mathrm{A}_x)K](x,y) = F\Big(\frac{x+y}{2}\Big),$$

where the operator A_x is defined by

$$\mathrm{A}_x f(y) = \frac{1}{4}\int_0^\infty \int_0^\infty f(\zeta)F\Big(\frac{2x+\zeta+\eta}{2}\Big)F\Big(\frac{x+\eta+y}{2}\Big)\,d\zeta\,d\eta.$$

Introducing the function

$$K_2(x,z) = \int_0^\infty K(x,x+\zeta)F\Big(\frac{x+\zeta+z}{2}\Big)\,d\zeta \tag{36.3.2.14}$$

we can rewrite equation (36.3.2.11) as

$$K(x,y) = F\Big(\frac{x+y}{2}\Big) + \frac{\sigma}{4}\int_0^\infty K_2(x,x+\eta)F\Big(\frac{x+\eta+y}{2}\Big)\,d\eta. \tag{36.3.2.15}$$

Applying the operator L_1 of (36.3.2.12) to equation (36.3.2.15), and the operator $\partial_x + \partial_z$ to (36.3.2.14), and taking into account the invertibility of $\mathrm{I} - \sigma \mathrm{A}_x$, we find, after appropriate calculations, that

$$(\partial_x + \partial_y)K_2(x,y) = -2K(x,x)K(x,y), \tag{36.3.2.16}$$

$$(\partial_x - \partial_y)K(x,y) = -\frac{\sigma}{2}K(x,x)K_2(x,y). \tag{36.3.2.17}$$

Applying the operator $\partial_x + \partial_y$ to (36.3.2.13), we get

$$F'\Big(\frac{x+y}{2}\Big) = (\mathrm{I} - \sigma \mathrm{A}_x)\Big[(\partial_x + \partial_y)K(x,y) + \frac{\sigma}{2}K(x,y)K_2(x,x)\Big]. \tag{36.3.2.18}$$

Let us require that the function F satisfy the second linear equation

$$L_2 F = (\partial_t + (\partial_x + \partial_y)^3)F = 0. \tag{36.3.2.19}$$

Applying the operator L_2 to equation (36.3.2.13) and taking into account the above auxiliary relations (36.3.2.16)–(36.3.2.18), we ultimately find that

$$[\partial_t + (\partial_x + \partial_y)^3]K(x,y) = 3\sigma K(x,x)K(x,y)\partial_x K(x,x) + 3\sigma K^2(x,x)(\partial_x + \partial_y)K(x,y) \tag{36.3.2.20}$$

for $y \geq x$. Now, by setting $w(x,t) = K(x,x;t)$, we rewrite equation (36.3.2.20), for $y = x$, in terms of the dependent variable w to obtain the modified Korteweg–de Vries equation

$$w_t + w_{xxx} = 6\sigma w^2 w_x. \tag{36.3.2.21}$$

Thus, each solution that satisfies two linear differential equations (36.3.2.12) and (36.3.2.19) simultaneously, with a sufficiently fast decay rate as $x \to \infty$, determines a solution of nonlinear equation (36.3.2.21), which is found from the linear integral equation (36.3.2.11).

Example 36.8. Now consider the Kadomtsev–Petviashvili equation:

$$(w_t + w_{xxx} - 6ww_x)_x + 3aw_{yy} = 0. \tag{36.3.2.22}$$

It can be shown that any sufficiently rapidly decaying function $F = F(x,z;y,t)$ as $x \to +\infty$ that satisfies simultaneously the two linear differential equations

$$\sqrt{a}\,F_y + F_{xx} - F_{zz} = 0,$$
$$F_t + 4F_{xxx} + 4F_{zzz} = 0$$

generates a solution to the Kadomtsev–Petviashvili equation (36.3.2.22) in the form

$$w = -2\frac{d}{dx}K(x,x;y,t),$$

where $K = K(x,z;y,t)$ is a solution to the Gelfand–Levitan–Marchenko linear integral equation

$$K(x,z;y,t) + F(x,z;y,t) + \int_x^\infty K(x,s;y,t)F(s,z;y,t)\,ds = 0.$$

The variables y and t appear in it as parameters.

⊙ *References for Subsection* 36.3: V. E. Zakharov and A. B. Shabat (1974), M. J. Ablowitz and H. Segur (1981), M. J. Ablowitz and P. A. Clarkson (1991).

36.4. Solution of the Cauchy Problem by the Inverse Scattering Problem Method

36.4.1. Preliminary Remarks. Direct and Inverse Scattering Problems

The solution of the Cauchy problem for nonlinear equations admitting a Lax pair (see Section 36.1) or another "implicit" linearization (see Section 36.2) falls into several successive steps. Two of them involve the solution of the direct and the inverse scattering problem for auxiliary linear equations. Summarized below are relevant results for the linear stationary Schrödinger equation

$$\varphi''_{xx} + [\lambda - f(x)]\varphi = 0 \qquad (-\infty < x < \infty). \tag{36.4.1.1}$$

It is assumed that the function $f(x)$, called the *potential*, vanishes as $x \to \pm\infty$. It is also assumed that the condition $\int_{-\infty}^{\infty}(1+|x|)\,|f(x)|\,dx < \infty$ holds.

Direct scattering problem. Consider the linear eigenvalue problem for the ordinary differential equation (36.4.1.1).

Eigenvalues can be of two types:

$$\begin{aligned} &\lambda_n = -\kappa_n^2, \quad n = 1, 2, \ldots, N \quad \text{(discrete spectrum)},\\ &\lambda = k^2, \qquad -\infty < k < \infty \qquad \text{(continuous spectrum)}. \end{aligned} \tag{36.4.1.2}$$

It is known that if $\min f(x) < \lambda < 0$, equation (36.4.1.1) has a discrete spectrum of eigenvalues, and if $f(x) < 0$ and $\lambda > 0$, it has a continuous spectrum.

Let $\lambda_n = -\kappa_n^2$ be discrete eigenvalues numbered so that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0,$$

and let $\varphi_n = \varphi_n(x)$ be the associated eigenfunctions, which vanish as $x \to \pm\infty$ and are square summable. Eigenfunctions are defined up to a constant factor. If an eigenfunction is fixed by its asymptotic behavior for negative x,

$$\varphi_n \to \exp(\kappa_n x) \quad \text{as} \quad x \to -\infty,$$

then the leading asymptotic term in the expansion of φ_n for large positive x is expressed as

$$\varphi_n \to c_n \exp(-\kappa_n x) \quad \text{as} \quad x \to \infty. \tag{36.4.1.3}$$

The eigenfunction φ_n has $n-1$ zero and moreover

$$c_n = (-1)^{n-1}|c_n|.$$

For continuous spectrum, $\lambda = k^2$, the behavior of the wave function φ at infinity is specified by a linear combination of the exponentials $\exp(\pm ikx)$, since $f \to 0$ as $|x| \to \infty$. The conditions at infinity are

$$\begin{aligned} &\varphi \to e^{-ikx} && \text{as} \quad x \to -\infty, \\ &\varphi \to a(k)e^{-ikx} + b(k)e^{ikx} && \text{as} \quad x \to \infty, \end{aligned} \tag{36.4.1.4}$$

and equation (36.4.1.1) uniquely determines the functions $a(k)$ and $b(k)$; note that they are related by the simple constraint $|a|^2 - |b|^2 = 1$. The first term in the second condition of (36.4.1.4) corresponds to a transmitted wave, and the second term to a reflected wave. Therefore, the quotient

$$r(k) = \frac{b(k)}{a(k)} \tag{36.4.1.5}$$

is called the *coefficient of reflection* (also *reflection factor* or *reflectance*).

The set of quantities

$$S = \{\kappa_n,\ c_n,\ r(k);\ n = 1, \ldots, N\} \tag{36.4.1.6}$$

appearing in relations (36.4.1.3)–(36.4.1.5) is called the *scattering data*. It is the determination of these data for a given potential $f(x)$ that is the purpose of the direct scattering problem. The scattering data completely determine the eigenvalue spectrum of the stationary Schrödinger equation (36.4.1.1).

Note the useful formulas

$$\begin{aligned} |a(k)| &= \left(1 - |r(k)|^2\right)^{-1}, \\ \arg a(k) &= \frac{1}{i}\sum_{n=1}^{N} \frac{k - i\kappa_n}{k + i\kappa_n} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\ln|a(s)|}{s - k}\, ds, \end{aligned} \tag{36.4.1.7}$$

which can be used to restore the function $a(k)$ for a given reflection factor $r(k)$. The integral in the second formula of (36.4.1.7) is understood as a Cauchy principal value integral, and $i\kappa_n$ are zeros of the function $a(k)$ analytic in the upper half-plane [the κ_n appear in the asymptotic formulas (36.4.1.3)].

Inverse scattering problem. The mapping of the potentials of equation (36.4.1.1) into the scattering data (36.4.1.6) is unique and invertible. The procedure of restoring $f(x)$ for given S is the subject matter of the inverse scattering problem. Summarized below are the most important results of studying this problem.

To determine the potential, one has to solve first the Gelfand–Levitan–Marchenko integral equation

$$K(x, y) + \Phi(x, y) + \int_x^{\infty} K(x, z)\Phi(z, y)\, dz = 0, \tag{36.4.1.8}$$

where the function $\Phi(x, y) = \overline{\Phi}(x + y)$ is determined via the scattering data (36.4.1.6) as follows:

$$\Phi(x, y) = \sum_{n=1}^{N} \frac{c_n}{ia'(i\kappa_n)} e^{-\kappa_n(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)e^{ik(x+y)}\, dk, \quad a'(k) = \frac{da}{dk}. \tag{36.4.1.9}$$

The potential is expressed in terms of the solution to the linear integral equation (36.4.1.8) as

$$f(x) = -2\frac{d}{dx}K(x, x). \tag{36.4.1.10}$$

Note that the direct and inverse scattering problems are usually taught in courses on quantum mechanics. For more details about these problems, see, for example, the paper by Gelfand and Levitan (1951) and the book by Novikov, Manakov, Pitaevskii, and Zakharov (1984).

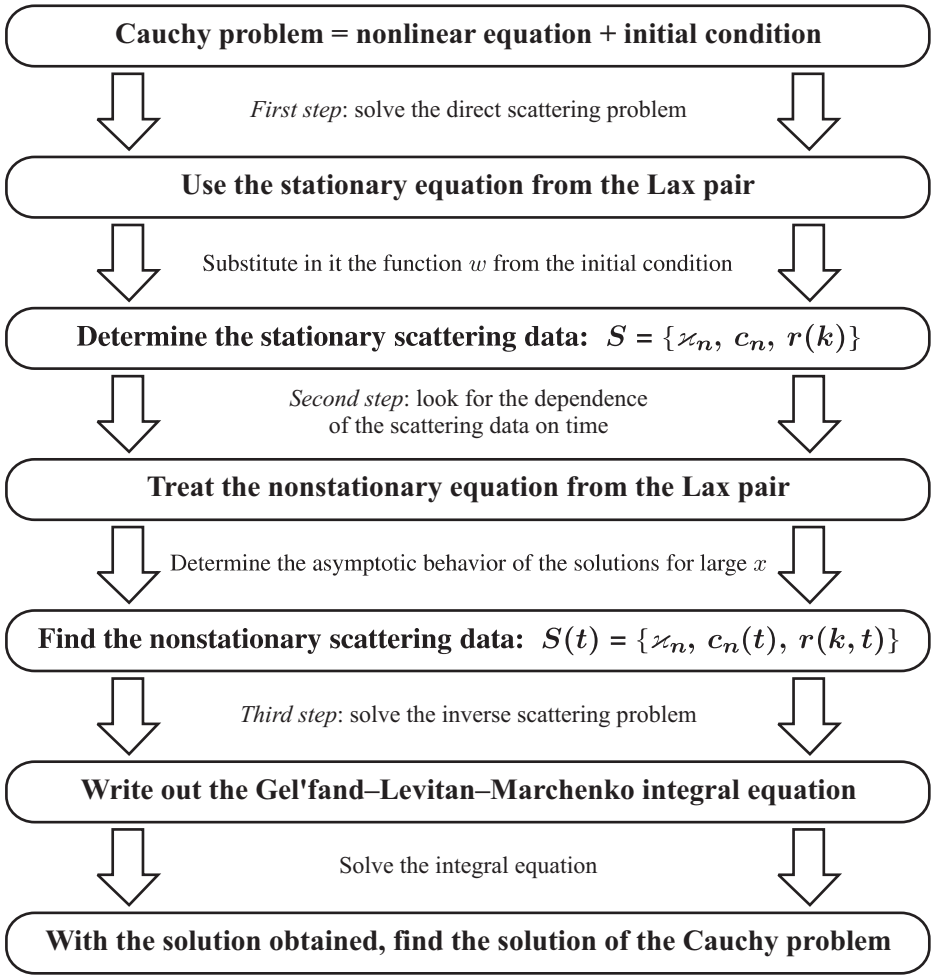

Figure 36.1. Main stages of solving the Cauchy problem for integrable nonlinear equations using the method of the inverse scattering problem.

36.4.2. Solution of the Cauchy Problem for Nonlinear Equations by the Inverse Scattering Problem Method

Figure 36.1 depicts the general scheme of solving the Cauchy problem for nonlinear equations using the method of the inverse scattering problem. It is assumed that either a Lax pair (36.1.1.2)–(36.1.1.3) for the given equation has been obtained at an earlier stage or a representation of the equation as the compatibility condition for the overdetermined system of linear equations (36.2.1.1)–(36.2.1.2) is known.

Let us employ this scheme for the solution of the Cauchy problem for the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0 \tag{36.4.2.1}$$

with the initial condition

$$w = f(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty). \tag{36.4.2.2}$$

The function $f(x) < 0$ is assumed to satisfy the same conditions as the potential in equation (36.4.1.1).

The solution of the Cauchy problem (36.4.2.1), (36.4.2.2) falls into several successive steps (see Figure 36.1). The Korteweg–de Vries equation is represented in the form of a Lax pair (36.1.2.3) and the results for the solution of the direct and inverse scattering problems are used.

First step. One considers the linear eigenvalue problem—the direct scattering problem—for the auxiliary ordinary differential equation (36.4.1.1) that results from the substitution of the initial profile (36.4.2.2) into

the first equation of the Lax pair (36.1.2.3) for the Korteweg–de Vries equation (36.4.2.1). On solving this problem, one determines the scattering data (36.4.1.6).

Second step. For $t > 0$, the function $w = w(x, t)$ appears in the first equation (36.1.2.3) instead of the initial profile $f(x)$. In the associated nonstationary problem, the eigenvalues (36.4.1.2) are preserved, since they are independent of t (see Subsection 36.1.1), but the eigenfunctions $\varphi = \varphi(x, t)$ are changed.

For continuous eigenvalues $\lambda > 0$, the asymptotic behavior of the eigenfunctions, according to (36.4.1.4), is as follows:

$$\begin{aligned} &\varphi \to e^{-ikx} && \text{as} \quad x \to -\infty, \\ &\varphi \to a(k,t)e^{-ikx} + b(k,t)e^{ikx} && \text{as} \quad x \to \infty. \end{aligned} \tag{36.4.2.3}$$

Substitute the first asymptotic relation of (36.4.2.3) into the second equation of the Lax pair (36.1.2.3) and take into account that $w \to 0$ as $x \to -\infty$ to obtain the function $p(t)$:

$$p(t) = -4ik^3 = \text{const}. \tag{36.4.2.4}$$

In order to determine the time dependences, $a = a(k, t)$ and $b = b(k, t)$, let us consider the second equation of the Lax pair (36.1.2.3), where $p(t)$ is given by (36.4.2.4). Taking into account that $w \to 0$ as $x \to \infty$, we get the linearized equation

$$\varphi_t + 4\varphi_{xxx} - 4ik^3\varphi = 0. \tag{36.4.2.5}$$

Substituting the second asymptotic relation of (36.4.2.3) into (36.4.2.5) and equating the coefficients of the exponentials e^{-ikx} and e^{ikx}, we arrive at the linear ordinary differential equations

$$a'_t = 0, \qquad b'_t - 8ik^3 b = 0.$$

Integrating them gives the coefficient of reflection

$$r(k,t) = \frac{b(t,k)}{a(t,k)} = r(k,0)e^{8ik^3 t}. \tag{36.4.2.6}$$

The asymptotic behavior of the eigenfunctions for discrete eigenvalues, with $k = i\kappa_n$, is determined in a similar way using the linearized equation (36.4.2.5).

Consequently, we obtain the time dependences of the scattering data:

$$S(t) = \{\kappa_n,\ c_n(t) = c_n(0)e^{8\kappa_n^3 t},\ r(k,t) = r(k,0)e^{8ik^3 t};\ n = 1, \ldots, N\}. \tag{36.4.2.7}$$

Third step. Using the scattering data (36.4.2.7), we introduce the following function by analogy with (36.4.1.9):

$$\Phi(x,y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k,0)e^{i[8k^3 t + k(x+y)]}\,dk + \sum_{n=1}^{N} \frac{c_n(0)}{ia'(i\kappa_n)} e^{8\kappa_n^3 t - \kappa_n(x+y)}. \tag{36.4.2.8}$$

In this case, Φ depends on time parametrically. This function plays the same role for the first equation of the Lax pair (36.1.2.3) for $t > 0$ as the function (36.4.1.9) for equation (36.4.1.1); recall that equation (36.4.1.1) coincides with the first equation of the Lax pair (36.1.2.3), where w is substituted for by its value in the initial condition (36.4.2.2).

In order to recover the function $w(x, t)$ from the scattering data (36.4.2.7), one has first to solve the linear integral equation

$$K(x,y;t) + \Phi(x,y;t) + \int_x^{\infty} K(x,z;t)\Phi(z,y;t)\,dz = 0, \tag{36.4.2.9}$$

obtained from (36.4.1.8) by simple renaming. This equation contains the function Φ of (36.4.2.8). Then, one should use the formula

$$w(x,t) = -2\frac{d}{dx}K(x,x;t), \tag{36.4.2.10}$$

obtained by renaming from (36.4.1.10), to determine the solution of the Cauchy problem for the Korteweg–de Vries equation (36.4.2.1)–(36.4.2.2).

36.4.3. N-Soliton Solution of the Korteweg–de Vries Equation

Let us find an exact solution of the Korteweg–de Vries equation (36.4.2.1) for the case of nonreflecting potentials, with zero coefficient of reflection (36.4.1.5). We proceed from the linear integral equation (36.4.2.9).

Setting $r(k,0)=0$ in (36.4.2.8), we have

$$\Phi(x,y;t)=\sum_{n=1}^{N}\gamma_n e^{-\kappa_n(x+y)+8\kappa_n^3 t},\qquad \gamma_n=\frac{c_n(0)}{ia'(i\kappa_n)}>0.$$

Substituting this function into (36.4.2.9), after simple rearrangements we obtain the equation

$$K(x,y;t)+\sum_{n=1}^{N}\Gamma_n(t)e^{-\kappa_n(x+y)}+\sum_{n=1}^{N}\Gamma_n(t)e^{-\kappa_n y}\int_x^\infty e^{-\kappa_n z}K(x,z;t)\,dz=0, \tag{36.4.3.1}$$

where

$$\Gamma_n(t)=\gamma_n e^{8\kappa_n^3 t}. \tag{36.4.3.2}$$

The solution of the integral equation (36.4.3.1) is sought in the form of a finite sum:

$$K(x,y;t)=\sum_{n=1}^{N}e^{-\kappa_n y}K_n(x,t). \tag{36.4.3.3}$$

Inserting (36.4.3.3) in (36.4.3.1) and integrating, we have

$$\sum_{n=1}^{N}e^{-\kappa_n y}K_n(x,t)+\sum_{n=1}^{N}\Gamma_n(t)e^{-\kappa_n(x+y)}+\sum_{n=1}^{N}\Gamma_n(t)e^{-\kappa_n y}\sum_{m=1}^{N}K_m(x,t)\frac{e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m}=0.$$

Rewriting this equation in the form $\sum\limits_{n=1}^{N}\psi_n(x,t)e^{-\kappa_n y}=0$ and then setting $\psi_n(x,t)=0$, we arrive at a nonhomogeneous system of linear algebraic equations for $K_n(x,t)$:

$$K_n(x,t)+\Gamma_n(t)\sum_{m=1}^{N}\frac{1}{\kappa_n+\kappa_m}e^{-(\kappa_n+\kappa_m)x}K_m(x,t)=-\Gamma_n(t)e^{-\kappa_n x},\quad n=1,\ldots,N. \tag{36.4.3.4}$$

Using Cramer's rule, we rewrite the solution to system (36.4.3.4) as the ratio of determinants

$$K_n(x,t)=\frac{\det\mathbb{A}^{(n)}(x,t)}{\det\mathbb{A}(x,t)}, \tag{36.4.3.5}$$

where $\mathbb{A}$ is the matrix with entries

$$\begin{aligned}A_{n,m}(x,t)&=\delta_{nm}+\frac{\Gamma_n(t)}{\kappa_n+\kappa_m}e^{-(\kappa_n+\kappa_m)x}\\&=\delta_{nm}+\frac{\gamma_n}{\kappa_n+\kappa_m}e^{-(\kappa_n+\kappa_m)x+8\kappa_n^3 t},\\ \delta_{nm}&=\begin{cases}1 & \text{if } n=m,\\ 0 & \text{if } n\neq m,\end{cases}\end{aligned} \tag{36.4.3.6}$$

and $\mathbb{A}^{(n)}(x,t)$ is the matrix obtained from $\mathbb{A}$ by substituting the right-hand sides of equations (36.4.3.4) for the nth column. Substituting (36.4.3.5) in (36.4.3.3) yields

$$K(x,y;t)=\sum_{n=1}^{N}e^{-\kappa_n y}\frac{\det\mathbb{A}^{(n)}(x,t)}{\det\mathbb{A}(x,t)}.$$

Further, by setting $y=x$, we get

$$K(x,x;t)=\frac{1}{\det\mathbb{A}(x,t)}\sum_{n=1}^{N}e^{-\kappa_n x}\det\mathbb{A}^{(n)}(x,t)=\frac{\partial}{\partial x}\ln\det\mathbb{A}(x,t).$$

In view of (36.4.2.10), we now find a solution to the Korteweg–de Vries equation (36.4.2.1) for the case of nonreflecting potentials in the form

$$w(x,t)=-2\frac{\partial^2}{\partial x^2}\ln\det\mathbb{A}(x,t). \tag{36.4.3.7}$$

This solution contains $2N$ free parameters κ_n, γ_n $(n=1,\ldots,N)$ and is called an *N-soliton solution*.

Example. In the special case $N = 1$, the matrix $\mathbb{A}(x,t)$ is characterized by a single element, defined by $n = m = 1$ in (36.4.3.6):

$$A_{1,1}(x,t) = 1 + \frac{\gamma}{2\kappa} e^{-2\kappa x + 8\kappa^3 t}.$$

Substituting this expression into (36.4.3.7), we get the *one-soliton solution* of the Korteweg–de Vries equation

$$w(x,t) = -\frac{2\kappa^2}{\cosh^2[\kappa(x - 4\kappa^2 t) + \xi_0]}, \qquad \xi_0 = \frac{1}{2\kappa} \ln \frac{\gamma}{2\kappa}. \tag{36.4.3.8}$$

This solution represents a solitary wave traveling to the right with a constant $v = 4\kappa^2$ and rapidly decaying at infinity.

For the N-soliton solution (36.4.3.7), the following asymptotic relations hold:

$$w(x,t) \approx -2 \sum_{n=1}^{N} \frac{\kappa_n^2}{\cosh^2\left[\kappa_n(x - v_n t) + \xi_n^{\pm}\right]} \quad \text{as} \quad t \to \pm\infty, \tag{36.4.3.9}$$

where $v_n = 4\kappa_n^2$ is the speed of the nth component. From the comparison of formulas (36.4.3.8) and (36.4.3.9) it follows that for large times, the N-soliton solution gets broken into N independent one-soliton solutions, and solitons with a higher amplitude travel with a higher speed.

⊙ *References for Section* 36.4: C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura (1967), P. D. Lax (1968), V. E. Zakharov and L. D. Faddeev (1971), V. E. Zakharov and A. B. Shabat (1972, 1974), M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur (1973, 1974), M. J. Ablowitz and H. Segur (1981), B. M. Levitan (1984), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), V. A. Marchenko (1986), L. D. Faddeev and L. A. Takhtajan (1987), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), M. J. Ablowitz, P. A. Clarkson (1991), E. D. Belokolos, A. I. Bobenko, et al. (1994), N. A. Kudryashov (2004).

Chapter 37
Conservation Laws

37.1. Basic Definitions and Examples

37.1.1. General Form of Conservation Laws. Integrals of Motion

1°. Consider a partial differential equation with two independent variables

$$F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial t}, \frac{\partial^2 w}{\partial x^2}, \ldots\right) = 0. \tag{37.1.1.1}$$

A conservation law for this equation has the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{37.1.1.2}$$

where

$$T = T\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \ldots\right), \quad X = X\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \ldots\right). \tag{37.1.1.3}$$

The left-hand side of the conservation law (37.1.1.2) must vanish for all (sufficiently smooth) solutions of equation (37.1.1.1). In simplest cases, the substitution of relations (37.1.1.3) into the conservation law (37.1.1.2) followed by differentiation and elementary transformations leads to a relation that coincides with (37.1.1.1) up to a functional factor. The quantities T and X in (37.1.1.2) are called a *density* and a *flow*, respectively.

The density and the flow in the conservation law (37.1.1.2) are not uniquely determined; they can be changed according to the rules $T \Longrightarrow T + \varphi(x)$, $X \Longrightarrow X + \psi(t)$, where $\varphi(x)$ and $\psi(t)$ are arbitrary functions, or, in general, by the rules

$$T \Longrightarrow aT + \frac{\partial \Phi}{\partial x}, \quad X \Longrightarrow aX - \frac{\partial \Phi}{\partial t},$$

where $\Phi = \Phi(x, t)$ is an arbitrary function of two variables and $a \neq 0$ is any number.

For nonstationary equations with n spatial variables $x_1, \ldots, x_n$, conservation laws have the form

$$\frac{\partial T}{\partial t} + \sum_{k=1}^{n} \frac{\partial X_k}{\partial x_k} = 0.$$

Partial differential equations can have several (sometimes infinitely many) conservation laws or none at all.

2°. If the total variation of the quantity X on the interval $a \le x \le b$ is equal to zero, i.e., $X(a) = X(b)$, then the following "integral of motion" takes place:

$$\int_a^b T\,dx = \text{const} \qquad (\text{for all } t). \tag{37.1.1.4}$$

For many specific equations, relations of the form (37.1.1.4) have a clear physical meaning and are used for approximate analytical solution of the corresponding problems, as well as for the verification of results obtained by numerical methods.

37.1.2. Conservation Laws for Some Nonlinear Equations of Mathematical Physics

Example 37.1. The Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0$$

admits infinitely many conservation laws of the form (37.1.1.2). The simplest of them are determined by the expressions

$$\begin{aligned}
&T_1 = w, && X_1 = w_{xx} - 3w^2;\\
&T_2 = w^2, && X_2 = 2ww_{xx} - w_x^2 - 4w^3;\\
&T_3 = w_x^2 + 2w^3, && X_3 = 2w_xw_{xxx} - w_{xx}^2 + 6w^2w_{xx} - 12ww_x^2 - 9w^4;\\
&T_4 = 3tw^2 + xw, && X_4 = t\big(6ww_{xx} - 3w_x^2 - 12w^3\big) - w_x + xw_{xx} - 3xw^2,
\end{aligned}$$

which must be substituted into (37.1.1.2); the subscripts denote partial derivatives with respect to x.

Example 37.2. The sine-Gordon equation

$$\frac{\partial^2 w}{\partial x\partial t} - \sin w = 0$$

also has infinitely many conservation laws. The first three are described by the formulas

$$\begin{aligned}
&T_1 = w_x^2, && X_1 = 2\cos w;\\
&T_2 = w_x^4 - 4w_{xx}^2, && X_2 = 4w_x^2\cos w;\\
&T_3 = 3w_x^6 - 12w_x^2w_{xx}^2 + 16w_x^3w_{xxx} + 24w_{xxx}^2, && X_3 = (2w_x^4 - 24w_{xx}^2)\cos w.
\end{aligned}$$

Example 37.3. The nonhomogeneous Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = F(x,y),$$

where $f(x,y)$ is an arbitrary function of two variables, admits the conservation law

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y} + \int F(x,y)\,dy\right) = 0.$$

Example 37.4. The Schrödinger equation with a cubic nonlinearity

$$i\frac{\partial w}{\partial t} = \Delta w + |w|^2 w, \qquad \Delta w = \sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2},$$

where $i^2 = 1$ and w is a complex function of real arguments, has two conservation laws ($n \ge 2$):

$$\begin{aligned}
\big(|w|^2\big)_t + i\nabla\cdot\big(\bar w\nabla w - w\nabla\bar w\big)_x &= 0,\\
\big(|\nabla w|^2 - \tfrac12|w|^4\big)_t + i\nabla\cdot\big[(\Delta w + |w|^2 w)\nabla\bar w - (\Delta\bar w + |w|^2\bar w)\nabla w\big]_x &= 0.
\end{aligned}$$

Here $\nabla = \big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$ and the bar over a symbol denotes the complex conjugate.

In the one-dimensional case, which corresponds to $n = 1$, the equation in question has infinitely many conservation laws.

Example 37.5. The nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$$

has generally two conservation laws:

$$\begin{aligned}
&w_t + [-f(w)w_x]_x = 0 \qquad \text{(trivial conservation law)},\\
&[(C_1x + C_2)w]_t + [C_1F(w) - (C_1x + C_2)f(w)w_x]_x = 0, \qquad F(w) = \int f(w)\,dw,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Example 37.6. The nonlinear heat equation with a quadratic source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + aw^2 \qquad (a \ne 0)$$

does not have conservation laws.

⊙ *References for Section* 37.1: G. B. Whitham (1965), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), M. D. Kruskal, R. M. Miura, C. S. Gardner, and N. J. Zabusky (1970), A. C. Scott, F. Y. Chu, and D. W. McLaughlin (1973), J. L. Lamb (1974), R. K. Dodd and R. K. Bullough (1977), P. J. Olver (1986), N. H. Ibragimov (1994), S. E. Harris (1996), A. M. Vinogradov and I. S. Krasilshchik (1997), B. J. Cantwell (2002), A. N. Kara and F. M. Mahomed (2002).

37.2. Equations Admitting Variational Form. Noetherian Symmetries

37.2.1. Lagrangian. Euler–Lagrange Equation. Noetherian Symmetries

Here, we consider second-order equations in two independent variables,

$$F\Big(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial x^2}\Big) = 0, \tag{37.2.1.1}$$

that admit the variational formulation of minimizing a functional of the form

$$Z[w] = \int_S L(x, y, w, w_x, w_y)\, dx\, dy. \tag{37.2.1.2}$$

The function $L = L(x, y, w, w_x, w_y)$ is called a *Lagrangian*.

It is well known that a minimum of the functional (37.2.1.2) corresponds to the *Euler–Lagrange equation*

$$\frac{\partial L}{\partial w} - D_x\left(\frac{\partial L}{\partial w_x}\right) - D_y\left(\frac{\partial L}{\partial w_y}\right) = 0, \tag{37.2.1.3}$$

where D_x and D_y are the total differential operators in x and y. Therefore, the original equation (37.2.1.1) must be a consequence of equation (37.2.1.3).

A symmetry that preserves the differential form

$$\Omega = L(x, y, w, w_x, w_y)\, dx\, dy$$

is called a *Noetherian symmetry* of the Lagrangian L. In order to obtain Noetherian symmetries, one should find point transformations

$$\bar{x} = f_1(x, y, w, \varepsilon), \quad \bar{y} = f_2(x, y, w, \varepsilon), \quad \bar{w} = g(x, y, w, \varepsilon) \tag{37.2.1.4}$$

such that preserve the differential form, $\bar{\Omega} = \Omega$, i.e.,

$$\bar{L}\, d\bar{x}\, d\bar{y} = L\, dx\, dy. \tag{37.2.1.5}$$

Calculating the differentials $d\bar{x}$, $d\bar{y}$ and taking into account (37.2.1.4), we obtain

$$d\bar{x} = D_x f_1\, dx, \quad d\bar{y} = D_y f_2\, dy,$$

and, therefore, relation (37.2.1.5) can be rewritten as

$$(L - \bar{L} D_x f_1 D_y f_2)\, dx\, dy = 0,$$

which is equivalent to

$$L - \bar{L} D_x f_1 D_y f_2 = 0. \tag{37.2.1.6}$$

Let us associate the point transformation (37.2.1.4) with the prolongation operator

$$X = \xi \partial_x + \eta \partial_y + \zeta \partial_w + \zeta_1 \partial_{w_x} + \zeta_2 \partial_{w_y}, \tag{37.2.1.7}$$

where the coordinates of the first prolongation, ζ_1 and ζ_2, are defined by formulas (32.1.3.2) from Subsection 32.1.3. Then, by the usual procedure, from (37.2.1.6) one obtains the *invariance condition* in the form

$$X(L) + L(D_x \xi + D_y \eta) = 0. \tag{37.2.1.8}$$

Noetherian symmetries are determined by (37.2.1.8).

Each Noetherian symmetry operator X generates a conservation law,

$$D_x\left(L\xi + (\zeta - \xi w_x - \eta w_y)\frac{\partial L}{\partial w_x}\right) + D_y\left(L\eta + (\zeta - \xi w_x - \eta w_y)\frac{\partial L}{\partial w_y}\right) = 0. \tag{37.2.1.9}$$

This formula will be used later on for constructing conservation laws for some nonlinear PDEs.

37.2.2. Examples of Constructing Conservation Laws Using Noetherian Symmetries

Example 37.7. Consider the stationary heat equation with nonlinear source

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-f(w)=0. \tag{37.2.2.1}$$

This equation admits the variational formulation of minimizing the functional (37.2.1.2) with the Lagrangian

$$L=w_x^2+w_y^2+2F(w),\qquad F(w)=\int f(w)\,dw. \tag{37.2.2.2}$$

This can be verified by substituting (37.2.2.2) into the Euler–Lagrange equation (37.2.1.3). In (37.2.2.2), it is assumed that $F(w)\ge 0$.

Substituting (37.2.2.2) into the invariance condition (37.2.1.8) and taking into account (37.2.1.7) and formulas (32.1.3.2) from Subsection 32.1.3, after some rearrangements we obtain a polynomial in the derivatives w_x and w_y:

$$\begin{aligned}&-\xi_w w_x^3-\eta_w w_x^2 w_y-\xi_w w_x w_y^2-\eta_w w_y^3+(2\zeta_w-\xi_x+\eta_y)w_x^2-2(\eta_x+\xi_y)w_x w_y\\&\quad+(2\zeta_w-\eta_y+\xi_x)w_y^2+2(\zeta_x+F\xi_w)w_x+2(\zeta_y+F\eta_w)w_y+2f\zeta+2F(\xi_x+\eta_y)=0.\end{aligned} \tag{37.2.2.3}$$

Equating the functional coefficients of w_x^3, $w_x^2 w_y$, w_x, and w_y to zero, we find that $\xi_w=0$, $\eta_w=0$, $\zeta_x=0$, and $\zeta_y=0$. Consequently,

$$\xi=\xi(x,y),\qquad \eta=\eta(x,y),\qquad \zeta=\zeta(w). \tag{37.2.2.4}$$

In (37.2.2.3), equating the coefficients of the remaining powers of the derivatives to zero, we obtain

$$\begin{aligned} w_x^2:&\quad 2\zeta_w-\xi_x+\eta_y=0,\\ w_y^2:&\quad 2\zeta_w-\eta_y+\xi_x=0,\\ w_x w_y:&\quad \eta_x+\xi_y=0,\\ 1:&\quad f\zeta+F(\xi_x+\eta_y)=0.\end{aligned} \tag{37.2.2.5}$$

For arbitrary $f=f(w)$, it follows from the last equation in (37.2.2.5) that

$$\zeta=0,\qquad \xi_x+\eta_y=0. \tag{37.2.2.6}$$

From the first equation in (37.2.2.5) and the second equation in (37.2.2.6), with $\zeta=0$, it follows that $\xi_x=0$, $\eta_y=0$ or $\xi=\xi(y)$, $\eta=\eta(x)$. Substituting these expressions into the third equation of (37.2.2.5), we get

$$\xi=C_1 y+C_2,\qquad \eta=-C_1 x+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants. Therefore, for arbitrary $f(w)$, Noetherian symmetries of the Lagrangian (37.2.2.2) are defined by the three operators

$$\begin{aligned}&\mathrm{X}_1=\partial_x && (C_2=1,\ C_1=C_3=0);\\ &\mathrm{X}_2=\partial_y && (C_3=1,\ C_1=C_2=0);\\ &\mathrm{X}_3=y\partial_x-x\partial_y && (C_1=1,\ C_2=C_3=0).\end{aligned} \tag{37.2.2.7}$$

In accordance with (37.2.1.9), these operators determine three conservation laws:

$$\begin{aligned}&D_x(-w_x^2+w_y^2+2F)+D_y\big(-2w_x w_y\big)=0 && (\xi=1,\ \eta=\zeta=0);\\ &D_x\big(-2w_x w_y\big)+D_y\big(w_x^2-w_y^2+2F\big)=0 && (\eta=1,\ \xi=\zeta=0);\\ &D_x\big(-yw_x^2+yw_y^2+2xw_x w_y+2yF\big)\\ &\qquad+D_y\big(-xw_x^2+xw_y^2-2yw_x w_y-2xF\big)=0 && (\xi=y,\ \eta=-x,\ \zeta=0),\end{aligned} \tag{37.2.2.8}$$

with the function $F=F(w)$ defined by (37.2.2.2).

Remark 1. The operators (37.2.2.7) could be found by a symmetry analysis of the original differential equation (37.2.2.1), as in Example 32.1.

Remark 2. In the variational formulation of equation (37.2.2.2), it is assumed that $F(w) \geq 0$. However, the conservation laws (37.2.2.8) are valid for any $F(w)$. This situation is also typical of other equations; to obtain conservation laws, it usually suffices that the equation in question is representable as the Euler–Lagrange equation (37.2.1.3) (the variational formulation may be unnecessary).

Example 37.8. The *equation of minimal surfaces*

$$(1 + w_y^2)w_{xx} - 2w_x w_y w_{xy} + (1 + w_x^2)w_{yy} = 0 \tag{37.2.2.9}$$

corresponds to the functional (37.2.1.2) with Lagrangian

$$L = \sqrt{1 + w_x^2 + w_y^2}.$$

The admissible point operators

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + y\partial_y + w\partial_w, \quad X_4 = y\partial_x - x\partial_y, \quad X_5 = \partial_w \tag{37.2.2.10}$$

are found from the invariance condition (37.2.1.8), as was the case in Example 37.7 (the procedure is also described in Section 32.2). With operators (37.2.2.10) and formula (37.2.1.9), we have the five conservation laws for equation (37.2.2.9):

$$X_1: \quad D_x\left(L - w_x\frac{\partial L}{\partial w_x}\right) + D_y\left(-w_x\frac{\partial L}{\partial w_y}\right) = 0,$$

$$X_2: \quad D_x\left(-w_y\frac{\partial L}{\partial w_x}\right) + D_y\left(L - w_y\frac{\partial L}{\partial w_y}\right) = 0,$$

$$X_3: \quad D_x\left(Lx + (w - xw_x - yw_y)\frac{\partial L}{\partial w_x}\right) + D_y\left(Ly + (w - xw_x - yw_y)\frac{\partial L}{\partial w_y}\right) = 0,$$

$$X_4: \quad D_x\left(Ly + (yw_x - xw_y)\frac{\partial L}{\partial w_x}\right) + D_x\left(-Ly + (yw_x - xw_y)\frac{\partial L}{\partial w_y}\right) = 0,$$

$$X_5: \quad D_x\left(\frac{w_x}{\sqrt{1 + w_x^2 + w_y^2}}\right) + D_y\left(\frac{w_y}{\sqrt{1 + w_x^2 + w_y^2}}\right) = 0.$$

⊙ *References for Section* 37.2: A. M. Vinogradov (1984), P. J. Olver (1986), J. A. Cavalcante and K. Tenenblat (1988), N. H. Ibragimov (1994), A. M. Vinogradov and I. S. Krasilshchik (1997).

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Chapter 38
Nonlinear Systems of Partial Differential Equations

38.1. Overdetermined Systems of Two Equations

38.1.1. Overdetermined Systems of First-Order Equations in One Unknown

Consider an overdetermined system of two quasilinear first-order equations

$$\begin{aligned} z_x &= F(x, y, z), \\ z_y &= G(x, y, z), \end{aligned} \tag{38.1.1.1}$$

in one unknown function $z = z(x, y)$.

In the general case, the system is unsolvable. To derive a necessary consistency condition for system (38.1.1.1), let us differentiate the first equation with respect to y and the second with respect to x. Eliminate the first derivatives from the resulting relations using the initial equations of the system to obtain

$$\begin{aligned} z_{xy} &= F_y + F_z z_y = F_y + GF_z, \\ z_{yx} &= G_x + G_z z_x = G_x + FG_z. \end{aligned}$$

Equating the expressions of the second derivatives z_{xy} and z_{yx} to each other, we obtain a necessary condition for consistency of system (38.1.1.1):

$$F_y + GF_z = G_x + FG_z. \tag{38.1.1.2}$$

Consider two possible situations.

First case. Suppose the substitution of the right-hand sides of equations (38.1.1.1) into the necessary condition (38.1.1.2) results in an equation with x, y, and z. Treating it as an algebraic (transcendental) equation for z, we find $z = z(x, y)$. The direct substitution of the expression $z = z(x, y)$ into both equations (38.1.1.1) gives an answer to the question whether it is a solution of the system in question or not.

Example 38.1. Consider the quadratically nonlinear system

$$\begin{aligned} z_x &= yz, \\ z_y &= z^2 + axz. \end{aligned} \tag{38.1.1.3}$$

In this case, the consistency condition (38.1.1.2) can be written, after simple rearrangements, in the form

$$z(a - 1 + yz) = 0.$$

This equation is not satisfied identically and gives rise to two possible expressions for z:

$$z = 0 \quad \text{or} \quad z = \frac{1 - a}{y}.$$

Substituting them into (38.1.1.3), we see that the former is a solution of the system and the latter is not.

Second case. Suppose the substitution of the right-hand sides of equations (38.1.1.1) into the necessary condition (38.1.1.2) turns it into an identity. In this case, the system has a family of solutions that depends at least on one arbitrary constant. If the partial derivatives F_y, F_z, G_x, G_z are continuous and condition (38.1.1.2) is satisfied identically, then system (38.1.1.1) has a unique solution $z = z(x, y)$ that takes a given value, $z = z_0$, at $x = x_0$ and $y = y_0$.

A simple way to find solutions in this case is as follows. The first equation of system (38.1.1.1) is treated as an ordinary differential equation in x with parameter y. One finds its solution, where the role of the arbitrary constant is played by an arbitrary function $\varphi(y)$. Substituting this solution into the second equation of system (38.1.1.1), one determines the function $\varphi(y)$.

TABLE 38.1
Some overdetermined systems of equations

No.	First equation of system	Second equation of system	Consistency conditions
1	$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + g(w)$	$\frac{\partial w}{\partial t} = \varphi(w)$	(34.3.1.8)
2	$\frac{\partial^2 w}{\partial x \partial t} = f(w)$	$\frac{\partial w}{\partial x} = \varphi(t)g(w)$	(34.3.2.12)
3	$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right] + f_2(w)$	$\frac{\partial^2 w}{\partial x^2} = g_1(w)\left(\frac{\partial w}{\partial x}\right)^2 + g_2(w)$	(34.4.1.8)
4	$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = a\frac{\partial^3 w}{\partial y^3}$	$\frac{\partial w}{\partial x} = \varphi(y)$	(34.2.1.8)

Example 38.2. Consider the nonlinear system

$$\begin{aligned} z_x &= ae^{y-z}, \\ z_y &= be^{y-z} + 1. \end{aligned} \tag{38.1.1.4}$$

In this case, the consistency condition (38.1.1.2) is satisfied identically.

To solve the first equation, let us make the change of variable $w = e^z$ to obtain the linear equation $w_x = ae^y$. Its general solution has the form $w = ae^y x + \varphi(y)$, where $\varphi(y)$ is an arbitrary function. Going back to the original variable, we find the general solution of the first equation of system (38.1.1.4):

$$z = \ln[ae^y x + \varphi(y)]. \tag{38.1.1.5}$$

Substituting this solution into the second equation of system (38.1.1.4), we obtain a linear first-order equation for $\varphi = \varphi(y)$:

$$\varphi'_y = \varphi + be^y.$$

Its general solution is $\varphi = (by + C)e^y$, where C is an arbitrary constant. Substituting this solution into (38.1.1.5), we obtain the following solution of system (38.1.1.4):

$$z = \ln[ae^y x + (by + C)e^y] = y + \ln(ax + by + C).$$

38.1.2. Other Overdetermined Systems of Equations in One Unknown

Chapter 34 describes a consistency analysis for overdetermined systems of partial differential equations in the context of the differential constraints method; one of the equations there is called a differential constraint. Detailed consideration is given to systems consisting of one second-order equation and one first-order equation (see Section 34.3), and also to some systems consisting of two second-order equations (see Section 34.4). Examples of solving such systems are also given.

Table 38.1 presents some overdetermined systems of equations with consistency conditions; the derivation of the consistency conditions for these systems can be found in Chapter 34.

38.2. Pfaffian Equations and Their Solutions. Connection with Overdetermined Systems

38.2.1. Pfaffian Equations

A *Pfaffian equation* is an equation of the form

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0. \tag{38.2.1.1}$$

Equation (38.2.1.1) always has a solution $x = x_0$, $y = y_0$, $z = z_0$, where x_0, y_0, and z_0 are arbitrary constants. Such simple solutions are not considered below.

We will distinguish between the following two cases:

1. Find a two-dimensional solution to the Pfaffian equation, when the three variables x, y, z are connected by a single relation (a certain condition must hold for such a solution to exist).

2. Find a one-dimensional solution to the Pfaffian equation, when the three variables x, y, z are connected by two relations.

38.2.2. Condition for Integrability of the Pfaffian Equation by a Single Relation

Let a solution of the Pfaffian equation be representable in the form $z = z(x, y)$, where z is the unknown function and x, y are independent variables. From equation (38.2.1.1) we find the expression for the differential:

$$dz = -\frac{P}{R}\,dx - \frac{Q}{R}\,dy. \tag{38.2.2.1}$$

On the other hand, since $z = z(x, y)$, we have

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{38.2.2.2}$$

Equating the right-hand sides of (38.2.2.1) and (38.2.2.2) to each other and taking into account the independence of the differentials dx and dy, we obtain an overdetermined system of equations of the form (38.1.1.1):

$$z_x = -P/R, \quad z_y = -Q/R. \tag{38.2.2.3}$$

A consistency condition for the system can be found by setting $F = -P/R$ and $G = -Q/R$ in (38.1.1.2). After some rearrangements we obtain the equation

$$R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = 0. \tag{38.2.2.4}$$

If condition (38.2.2.4) is satisfied identically, the Pfaffian equation (38.2.1.1) is integrable by one relation of the form

$$U(x, y, z) = C, \tag{38.2.2.5}$$

where C is an arbitrary constant. In this case, the Pfaffian equation is said to be *completely integrable*. The left-hand side of a completely integrable Pfaffian equation (38.2.1.1) can be represented in the form

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz \equiv \mu(x, y, z)\,dU(x, y, z),$$

where $\mu(x, y, z)$ is an *integrating factor*.

A solution of a completely integrable Pfaffian equation can be found by solving the overdetermined system (38.2.2.3) employing the method presented in Section 38.1. Alternatively, the following equivalent technique can be used: it is first assumed that x = const in equation (38.2.1.1), which corresponds to $dx = 0$. Then the resulting ordinary differential equation is solved for $z = z(y)$, where x is treated as a parameter, and the constant of integration is regarded as an arbitrary function of x: $C = \varphi(x)$. Finally, by substituting the resulting solution into the original equation (38.2.1.1), one finds the function $\varphi(x)$.

Example 38.3. Consider the Pfaffian equation

$$y(xz + a)\,dx + x(y + b)\,dy + x^2 y\,dz = 0. \tag{38.2.2.6}$$

The integrability condition (38.2.2.4) is satisfied identically. Let us set $dx = 0$ in equation (38.2.2.6) to obtain the ordinary differential equation

$$(y + b)\,dy + xy\,dz = 0. \tag{38.2.2.7}$$

Treating x as a parameter, we find the general solution of the separable equation (38.2.2.7):

$$z = -\frac{1}{x}(y + b\ln|y|) + \varphi(x), \tag{38.2.2.8}$$

where $\varphi(x)$ is the constant of integration, dependent on x. On substituting (38.2.2.8) into the original equation (38.2.2.6), we arrive at a linear ordinary differential equation for $\varphi(x)$:

$$x^2\varphi'_x + x\varphi + a = 0.$$

Its general solution is expressed as

$$\varphi(x) = -\frac{a}{x}\ln|x| + \frac{C}{x},$$

where C is an arbitrary constant. Substituting this $\varphi(x)$ into (38.2.2.8) yields the solution of the Pfaffian equation (38.2.2.6):

$$z = -\frac{1}{x}(y + a\ln|x| + b\ln|y| - C).$$

This solution can be equivalently represented in the form of an integral (38.2.2.5) as

$$xz + y + a\ln|x| + b\ln|y| = C.$$

Geometric interpretation. Let us introduce the vector $\mathbf{F} = \{P, Q, R\}$. Then the condition of complete integrability (38.2.2.4) of the Pfaffian equation (38.2.1.1) can be written in the dot product form: $\mathbf{F} \cdot \operatorname{curl}\mathbf{F} = 0$. Solution (38.2.2.5) represents a one-parameter family of surfaces orthogonal to $\mathbf{F}$.

38.2.3. Pfaffian Equations Not Satisfying the Integrability Condition

Consider now Pfaffian equations that do not satisfy the condition of integrability by one relation. In this case, relation (38.2.2.4) is not satisfied identically and there are two different methods for the investigation of such equations.

First method. Relation (38.2.2.4) is treated as an algebraic (transcendental) equation for one of the variables. For example, solving it for z, we get a relation $z = z(x, y)$.* The direct substitution of this solution into (38.2.1.1), with x and y regarded as independent variables, answers the question whether it is a solution of the Pfaffian equation. The thus obtained solutions (if any) do not contain a free parameter.

Second method. One-dimensional solutions are sought in the form of two relations. One relation is prescribed, for example, in the form

$$z = f(x), \tag{38.2.3.1}$$

where $f(x)$ is an arbitrary function. Using it, one eliminates z from the Pfaffian equation (38.2.1.1) to obtain an ordinary differential equation for $y = y(x)$:

$$Q(x, y, f(x))y'_x + P(x, y, f(x)) + R(x, y, f(x))f'_x(x) = 0.$$

Suppose the general solution of this equation has the form

$$\Phi(x, y, C) = 0, \tag{38.2.3.2}$$

where C is an arbitrary constant. Then formulas (38.2.3.1), (38.2.3.2) define a one-dimensional solution of the Pfaffian equation in the form of two relations involving one arbitrary function and one free parameter.

Example 38.4. Consider the Pfaffian equation

$$y\,dx + dy + x\,dz = 0. \tag{38.2.3.3}$$

Substituting $P = y$, $Q = 1$, and $R = x$ into the left-hand side of condition (38.2.2.4), we find that $x + 1 \not\equiv 0$. Therefore, equation (38.2.3.3) is not integrable by one relation.

Let us look for one-dimensional solutions by choosing one relation in the form (38.2.3.1). Consequently, we arrive at the ordinary differential equation

$$y'_x + y + xf'_x(x) = 0.$$

Its general solution has the form

$$y = Ce^{-x} - e^{-x}\int e^x x f'_x(x)\,dx, \tag{38.2.3.4}$$

where C is an arbitrary constant. Formulas (38.2.3.1) and (38.2.3.4) represent a one-dimensional solution of the Pfaffian equation (38.2.3.3) involving an arbitrary function $f(x)$ and an arbitrary constant C.

Remark. For equations that are not completely integrable, the first relation can be chosen in a more general form than (38.2.3.1):

$$z = f(x, y),$$

where $f(x, y)$ is an arbitrary function of two arguments. Using it to eliminate z from the Pfaffian equation, we obtain an ordinary differential equation for $y = y(x)$. However, it is impossible to find the general solution of this equation in closed form even for very simple equations, including equation (38.2.3.3).

38.3. Systems of First-Order Equations Describing Convective Mass Transfer with Volume Reaction

38.3.1. Traveling-Wave Solutions and Some Other Invariant Solutions

This subsection presents exact solutions to some classes of nonlinear systems of first-order equations of the form

$$\frac{\partial u}{\partial x} = F_1(u, w), \quad \frac{\partial w}{\partial t} = F_2(u, w). \tag{38.3.1.1}$$

Remark. Equations of convective mass transfer in two-component systems with volume chemical reaction without diffusion,

$$\frac{\partial u}{\partial \tau} + a_1\frac{\partial u}{\partial \xi} = F_1(u, w), \quad \frac{\partial w}{\partial \tau} + a_2\frac{\partial w}{\partial \xi} = F_2(u, w),$$

* There may be several such relations, and even infinitely many.

TABLE 38.2
Systems equations of the form (38.3.1.1) admitting invariant solutions that are found using combinations of translation and scaling transformations (C is an arbitrary constant)

Systems equations	Invariant transformations	Form of invariant solutions
$\frac{\partial u}{\partial x}=e^{au}f(au-bw)$, $\frac{\partial w}{\partial t}=e^{bw}g(au-bw)$	$t=C^{ab}\bar{t},\ x=C^{ab}\bar{x}$, $u=\bar{u}-b\ln C,\ w=\bar{w}-a\ln C$	$u=y(\xi)-\frac{1}{a}\ln t,\ w=z(\xi)-\frac{1}{b}\ln t,\ \xi=\frac{x}{t}$
$\frac{\partial u}{\partial x}=u^kf(u^nw^m)$, $\frac{\partial w}{\partial t}=w^sg(u^nw^m)$	$t=C^{n(s-1)}\bar{t},\ x=C^{m(1-k)}\bar{x}$, $u=C^m\bar{u},\ w=C^{-n}\bar{w}$	$u=t^{\frac{m}{n(s-1)}}y(\xi),\ w=t^{\frac{1}{1-s}}z(\xi),\ \xi=xt^{\frac{m(k-1)}{n(s-1)}}$
$\frac{\partial u}{\partial x}=u^kf(u^nw^m)$, $\frac{\partial w}{\partial t}=wg(u^nw^m)$	$t=\bar{t}+\ln C,\ x=C^{m(1-k)}\bar{x}$, $u=C^m\bar{u},\ w=C^{-n}\bar{w}$	$u=e^{mt}y(\xi),\ w=e^{-nt}z(\xi),\ \xi=e^{m(k-1)t}x$

are reduced to such systems of equations. This is achieved by using the characteristic variables

$$x=\frac{\xi-a_2\tau}{a_1-a_2},\qquad t=\frac{\xi-a_1\tau}{a_2-a_1}\qquad (a_1\neq a_2).$$

It is apparent that system (38.3.1.1) admits exact traveling-wave solutions

$$u=u(z),\quad w=w(z),\quad z=kx-\lambda t,$$

where k and λ are arbitrary constants, and the functions $u(z)$ and $w(z)$ are determined by the autonomous system of ordinary differential equations

$$ku_z'-F_1(u,w)=0,\qquad \lambda w_z'+F_2(u,w)=0.$$

Further on, $f(\ldots)$ and $g(\ldots)$ are arbitrary functions of their arguments.

Table 38.2 lists some systems of the form (38.3.1.1) admitting other invariant solutions that are found using combinations of translation and scaling transformations in the independent and dependent variables; these transformations preserve the form of the equations (i.e., a given system of equations is converted into an identical one). The procedure for constructing such solutions for a single equation is detailed in Chapters 27 and 28.

38.3.2. Systems Reducible to an Ordinary Differential Equation

Consider the system of equations of the special form

$$\frac{\partial u}{\partial x}=f(w)u,\qquad \frac{\partial w}{\partial t}=g(w)u^k. \tag{38.3.2.1}$$

In two special cases, $k=0$ and $f(w)=\text{const}$, one of the two equations can be solved independently from the other, and consequently system (38.3.2.1) is easy to integrate. Further on, it is assumed that $k\neq 0$ and $f(w)\neq\text{const}$.

Let us first simplify system (38.3.2.1) via the change of variables

$$U=u^k,\quad W=\int\frac{dw}{g(w)}. \tag{38.3.2.2}$$

We obtain

$$\frac{\partial U}{\partial x}=\Phi(W)U,\qquad \frac{\partial W}{\partial t}=U, \tag{38.3.2.3}$$

where the function $\Phi(W)$ is defined parametrically by the formulas

$$\Phi=kf(w),\qquad W=\int\frac{dw}{g(w)}, \tag{38.3.2.4}$$

with w treated as a parameter. Substituting U in the first equation of system (38.3.2.3) by the left-hand side of the second equation of that system, we arrive at a second-order equation for W:

$$\frac{\partial^2W}{\partial x\partial t}=\Phi(W)\frac{\partial W}{\partial t}. \tag{38.3.2.5}$$

Integrating with respect to t gives

$$\frac{\partial W}{\partial x} = \int \Phi(W)\,dW + \theta(x), \tag{38.3.2.6}$$

where $\theta(x)$ is an arbitrary function.

Going back in (38.3.2.6) to the original variable w by formulas (38.3.2.2), we get

$$\frac{\partial w}{\partial x} = kg(w)\int \frac{f(w)}{g(w)}\,dw + \theta(x)g(w). \tag{38.3.2.7}$$

Equation (38.3.2.7) may be treated as a first-order ordinary differential equation in x. On finding its general solution, one should replace the constant of integration C in it by an arbitrary function of time $\psi(t)$, since w is dependent on x and t.

Example 38.5. Consider the quadratically nonlinear system

$$\frac{\partial u}{\partial x} = awu, \qquad \frac{\partial w}{\partial t} = bwu, \tag{38.3.2.8}$$

which is a special case of system (38.3.2.1) with $k = 1$, $f(w) = aw$, and $g(w) = bw$. Substituting these functions into (38.3.2.7), we obtain the Bernoulli equation

$$\frac{\partial w}{\partial x} = aw^2 + b\theta(x)w.$$

Its general solution is given by

$$w = \frac{\xi}{\psi(t) - a\int \xi\,dx}, \qquad \xi = \exp\left(b\int \theta(x)\,dx\right), \tag{38.3.2.9}$$

where $\psi(t)$ is an arbitrary function; it has been taken into account that w is dependent on x and t. Since formula (38.3.2.9) involves an arbitrary function $\theta(x)$, it is convenient to introduce the notation $\varphi(x) = -\int \xi\,dx$. Consequently, we have the general solution of system (38.3.2.8) in the form

$$u = -\frac{\psi_t'(t)}{a\varphi(x) + b\psi(t)}, \qquad w = -\frac{\varphi_x'(x)}{a\varphi(x) + b\psi(t)},$$

where $\varphi(x)$ and $\psi(t)$ are arbitrary functions; here, ψ has been renamed $b\psi$ as compared with (38.3.2.9).

Example 38.6. To the special case $\theta(x) = \text{const}$ in (38.3.2.7) there correspond special solutions of system (38.3.2.1) of the form

$$w = w(z), \quad u = [\psi'(t)]^{1/k}v(z), \quad z = x + \psi(t),$$

involving an arbitrary function $\psi(t)$; the prime denotes a derivative. The functions $w(z)$ and $v(z)$ are determined by the autonomous system of ordinary differential equations

$$v_z' = f(w)v, \qquad w_z' = g(w)v^k.$$

Its general solution can be written out in implicit form:

$$\int \frac{dw}{g(w)[kF(w) + C_1]} = z + C_2, \qquad v = [kF(w) + C_1]^{1/k}, \qquad F(w) = \int \frac{f(w)}{g(w)}\,dw.$$

▷ *Exact solutions to a number of other first-order nonlinear systems of the form (38.3.1.1) can be found in Section 16.1.*

38.3.3. Some Nonlinear Problems of Suspension Transport in Porous Media*

1°. Deep bed filtration of particle suspensions in porous media occurs during sea/produced water injection in oil reservoirs, drilling fluid invasion into reservoir productive zones, sand filtration in gravel packs, fines migration in oil fields, bacteria, virus or contaminant transport in groundwater, industrial filtering, etc. The basic features of the process are the particle capture by the porous medium and the consequent permeability reduction. The particles are captured due to size exclusion, surface sorption, electrical forces, sedimentation, diffusion, etc.

* Subsections 38.3.3 and 38.3.4 were written by P. G. Bedrikovetsky.

For a single component system, the governing equations consist of the mass balance for retained and suspended particles and the equation of capture kinetics:

$$\begin{aligned} &\frac{\partial}{\partial t}(u+w)+\frac{\partial u}{\partial x}=0,\\ &\frac{\partial w}{\partial t}=f(w)u, \end{aligned} \tag{38.3.3.1}$$

where u is the suspended concentration and w is a retained (deposited) concentration, and $f(w)$ is the filtration coefficient.

Let us transform the first equation of system (38.3.3.1) in the following way. Replace w_t in the first equation by the right-hand side of the second equation and then pass from x, t to the new characteristic variables $\xi = -x$, $\eta = x - t$. Consequently, we obtain a system of equations of the form (38.3.2.1):

$$\frac{\partial u}{\partial \xi}=f(w)u, \qquad \frac{\partial w}{\partial \eta}=-f(w)u.$$

Its solution is reduced to integrating an ordinary differential equation.

2°. The problem of particle suspension injection into a particle-free porous reservoir is described by system (38.3.3.1) with the following initial and boundary conditions:

$$u=w=0 \quad \text{at} \quad t=0, \qquad u=1 \quad \text{at} \quad x=0. \tag{38.3.3.2}$$

The problem (38.3.3.1)–(38.3.3.2) has an exact analytical solution. The introduction of the potential

$$\Phi(w)=\int_0^w \frac{dz}{f(z)} \tag{38.3.3.3}$$

allows transforming the second equation (38.3.3.1) to the form

$$u=\frac{\partial \Phi}{\partial t}. \tag{38.3.3.4}$$

The substitution of (38.3.3.4) into the first equation (38.3.3.1) and its integration with respect to t from 0 to t taking into account the initial conditions (38.3.3.2) results in the following first-order quasilinear partial differential equation:

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}=-f(w)w. \tag{38.3.3.5}$$

Fixing $u = 1$ in the second equation of (38.3.3.1) for the inlet $x = 0$ provides a boundary condition for equation (38.3.3.5):

$$\Phi(w)=t \quad \text{at} \quad x=0. \tag{38.3.3.6}$$

The solution of problem (38.3.3.5)–(38.3.3.6) is obtained by the method of characteristics (see Subsection 24.3.5) and can be represented in the implicit form

$$\int_w^{\Phi^{-1}(t-x)} \frac{dz}{zf(z)}=x. \tag{38.3.3.7}$$

Here, $\Phi^{-1}(w)$ is the inverse of the integral (38.3.3.3).

Now let us obtain the expression for the suspended concentration $u(x, t)$. Taking the time derivative of both sides of (38.3.3.7), we obtain

$$\left(\frac{1}{wf(w)}\frac{\partial w}{\partial t}\right)_{x,t}-\left(\frac{1}{wf(w)}\frac{\partial w}{\partial t}\right)_{0,t-x}=0.$$

Replacing here the partial derivative by the left-hand side of the second equation in (38.3.3.1), we have

$$\frac{u(x,t)}{w(x,t)}=\frac{u(0,t-x)}{w(0,t-x)}, \tag{38.3.3.8}$$

so the ratio u/w does not change along the characteristics. Taking into account the boundary conditions (38.3.3.2) and (38.3.3.6) in (38.3.3.8), we obtain the expression for the suspended concentration

$$u(x,t)=\frac{w(x,t)}{\Phi^{-1}(t-x)}. \tag{38.3.3.9}$$

Finally, formulas (38.3.3.7), (38.3.3.9), and (38.3.3.3) form an exact solution for problem (38.3.3.1)–(38.3.3.2) for $x < t$. For $x > t$, we have $u = w = 0$.

Example 38.7. Consider system (38.3.3.1) with the linear filtration coefficient

$$f(w) = 1 - w. \tag{38.3.3.10}$$

In this case the potential (38.3.3.3) is

$$\Phi(w) = -\ln(1 - w).$$

From (38.3.3.6) it is possible to calculate the value on the boundary

$$w(0, t) = \Phi^{-1}(t) = 1 - e^{-t}$$

and the left-hand side of (38.3.3.7) can be determined explicitly. Finally, in view of (38.3.3.9), we find the solution to problem (38.3.3.1)–(38.3.3.2) with condition (38.3.3.10) in the form

$$u = \frac{e^{t-x}}{e^x + e^{t-x} - 1}, \qquad w = \frac{e^{t-x} - 1}{e^x + e^{t-x} - 1}.$$

38.3.4. Some Generalizations

1°. Consider a more general system than (38.3.3.1),

$$\frac{\partial}{\partial t} g(u, w) + a \frac{\partial u}{\partial x} = f_1(w)u,$$
$$\frac{\partial w}{\partial t} = f_2(w)u.$$

On eliminating u from the first equation using the second equation and on integrating with respect to t, we obtain a nonlinear first-order partial differential equation for $w(x, t)$:

$$g\left(\frac{1}{f_2(w)}\frac{\partial w}{\partial t}, w\right) + \frac{a}{f_2(w)}\frac{\partial w}{\partial x} = \int \frac{f_1(w)}{f_2(w)}\, dw + \theta(x), \tag{38.3.4.1}$$

where $\theta(x)$ is an arbitrary function. If θ = const, a complete integral of equation (38.3.4.1) is sought in the form $w = w(C_1x + C_2t + C_3)$. In this case, one can obtain the general integral (containing an arbitrary function) of equation (38.3.4.1) in parametric form by the method described in Section 24.1, after solving the associated ordinary differential equation.

2°. System (38.3.3.1) admits the following generalization:

$$\frac{\partial}{\partial t}\Big[u + G(w_1, \ldots, w_n)\Big] + \frac{\partial u}{\partial x} = 0, \tag{38.3.4.2}$$

$$\frac{\partial w_k}{\partial t} = F_k(w_1, \ldots, w_n)u, \qquad k = 1, \ldots, n. \tag{38.3.4.3}$$

From equations (38.3.4.3) we have

$$\frac{1}{F_1(w_1, \ldots, w_n)}\frac{\partial w_1}{\partial t} = \cdots = \frac{1}{F_n(w_1, \ldots, w_n)}\frac{\partial w_n}{\partial t} = u. \tag{38.3.4.4}$$

We look for solutions of system (38.3.4.2)–(38.3.4.3) of the special form

$$w_1 = w_1(w_n), \quad \ldots, \quad w_{n-1} = w_{n-1}(w_n). \tag{38.3.4.5}$$

The functions $w_1, \ldots, w_{n-1}$ are thus assumed to be expressible in terms of w_n. From (38.3.4.4), in view of (38.3.4.5), we find the following system of $n - 1$ ordinary differential equations:

$$\frac{dw_k}{dw_n} = \frac{F_k(w_1, \ldots, w_n)}{F_n(w_1, \ldots, w_n)}, \qquad k = 1, \ldots, n - 1. \tag{38.3.4.6}$$

Further, assuming that a solution of system (38.3.4.6) has been obtained and the functions (38.3.4.5) are known, we substitute them into (38.3.4.2)–(38.3.4.3) to arrive at the system of two equations

$$\begin{aligned} &\frac{\partial}{\partial t}\big[u + g(w_n)\big] + \frac{\partial u}{\partial x} = 0, \\ &\frac{\partial w_n}{\partial t} = f_n(w_n)u, \end{aligned} \tag{38.3.4.7}$$

where

$$g(w_n) = G\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big), \qquad f_n(w_n) = F_n\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big).$$

Introducing the new dependent variable

$$w = g(w_n) \equiv G\big(w_1(w_n), \ldots, w_{n-1}(w_n), w_n\big)$$

in (38.3.4.7), we obtain system (38.3.3.1) in which the function $f = f(w)$ is defined parametrically,

$$f = g'(w_n)f_n(w_n), \qquad w = g(w_n),$$

with w_n being the parameter.

Remark 1. Solutions of the above special form arise, for example, in problems with initial and boundary conditions,

$$w_1 = \cdots = w_n = u = 0 \quad \text{at} \quad t = 0,$$
$$u = 1 \quad \text{at} \quad x = 0.$$

(The initial conditions correspond to the absence of particles in the reservoir at the initial time and the boundary condition corresponds to a given suspended concentration in the injected fluid.) In this case, the system of ordinary differential equations (38.3.4.6) is solved under the following initial conditions:

$$w_1 = \cdots = w_{n-1} = 0 \quad \text{at} \quad w_n = 0.$$

Remark 2. In problems on flows of n-component fluids through porous media, the mass balance for retained and suspended particles is governed by equation (38.3.4.2) in which the function G is the sum of individual components:

$$G(w_1, \ldots, w_n) = \sum_{k=1}^{n} w_k;$$

equations (38.3.4.3) define the capture kinetics.

38.4. First-Order Hyperbolic Systems of Quasilinear Equations. Systems of Conservation Laws of Gas Dynamic Type*

38.4.1. Systems of Two Quasilinear Equations. Systems in the Form of Conservation Laws

1°. Consider the system of two quasilinear equations of the form

$$\begin{aligned} f_1(u,w)\frac{\partial u}{\partial t} + g_1(u,w)\frac{\partial w}{\partial t} + h_1(u,w)\frac{\partial u}{\partial x} + k_1(u,w)\frac{\partial w}{\partial x} = 0, \\ f_2(u,w)\frac{\partial u}{\partial t} + g_2(u,w)\frac{\partial w}{\partial t} + h_2(u,w)\frac{\partial u}{\partial x} + k_2(u,w)\frac{\partial w}{\partial x} = 0, \end{aligned} \tag{38.4.1.1}$$

where x and t are independent variables, $u = u(x,t)$ and $w = w(x,t)$ are two unknown functions, and $f_k(u,w)$ and $g_k(u,w)$ are prescribed functions ($k = 1, 2, 3, 4$).

For any $f_k(u,w)$ and $g_k(u,w)$, system (38.4.1.1) admits the following simplest solutions:

$$u = C_1, \quad w = C_2, \tag{38.4.1.2}$$

where C_1 and C_2 are arbitrary constants.

2°. The main mathematical models in continuum mechanics and theoretical physics have the form of systems of conservation laws. Usually mass, momentum, and energy for phases and/or components are conserved.

A quasilinear system of two conservation laws in two independent variables has the form

$$\begin{aligned} \frac{\partial F_1(u,w)}{\partial t} + \frac{\partial G_1(u,w)}{\partial x} = 0, \\ \frac{\partial F_2(u,w)}{\partial t} + \frac{\partial G_2(u,w)}{\partial x} = 0. \end{aligned} \tag{38.4.1.3}$$

It is a special case of system (38.4.1.1). It admits simple solutions of the form (38.4.1.2).

38.4.2. Self-Similar Continuous Solutions. Hyperbolic Systems

System (38.4.1.1) is invariant under the transformation $(x,t) \to (ax, at)$, where a is any number ($a \neq 0$). Therefore, it admits a self-similar solution of the form

$$u = u(\xi), \quad w = w(\xi), \quad \xi = \frac{x}{t}. \tag{38.4.2.1}$$

* Section 38.4 was written by A. P. Pires and P. G. Bedrikovetsky.

The substitution of (38.4.2.1) into (38.4.1.1) results in the following system of ordinary differential equations:

$$\begin{aligned}(h_1 - \xi f_1)u'_\xi + (k_1 - \xi g_1)w'_\xi = 0,\\ (h_2 - \xi f_2)u'_\xi + (k_2 - \xi g_2)w'_\xi = 0,\end{aligned} \tag{38.4.2.2}$$

where the arguments of the functions f_m, g_m, h_m, and k_m ($m = 1, 2$) are omitted for brevity.

System (38.4.2.2) may be treated as an algebraic system for the derivatives u'_ξ and w'_ξ; for it to have a nonzero solution, its determinant must be equal to zero:

$$(f_1 g_2 - f_2 g_1)\xi^2 - (f_1 k_2 + g_2 h_1 - f_2 k_1 - g_1 h_2)\xi + h_1 k_2 - h_2 k_1 = 0. \tag{38.4.2.3}$$

System (38.4.1.1) is called *strictly hyperbolic* (or, for short, *hyperbolic*) if the discriminant of the quadratic equation (38.4.2.3), with respect to ξ, is positive:

$$(f_1 k_2 + g_2 h_1 - f_2 k_1 - g_1 h_2)^2 - 4(f_1 g_2 - f_2 g_1)(h_1 k_2 - h_2 k_1) > 0. \tag{38.4.2.4}$$

In this case, equation (38.4.2.3) has two different roots:

$$\xi_{1,2} = \xi_{1,2}(u, w). \tag{38.4.2.5}$$

Substituting either root (38.4.2.5) into (38.4.2.2) and taking u to be a parameter along the integral curve, we obtain

$$(k_1 - \xi_{1,2} g_1)\frac{dw}{du} + h_1 - \xi_{1,2} f_1 = 0. \tag{38.4.2.6}$$

The ordinary differential equation (38.4.2.6) provides two sets of continuous solutions for system (38.4.1.1). The procedure for constructing these solutions consists of two steps: (i) the dependence $w = w(u)$, different for each set, is found from (38.4.2.6); and (ii) this dependence is then substituted into formula (38.4.2.5), in the left-hand side of which $\xi_{1,2}$ is replaced by x/t. This results in a solution defined in implicit form:

$$\xi_{1,2}(u, w) = x/t, \qquad w = w(u).$$

Example 38.8. The system of equations describing one-dimensional longitudinal oscillations of an elastic bar consists of the equations of balance of mass and momentum:

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial w}{\partial x} = 0,\\ \frac{\partial w}{\partial t} - \frac{\partial \sigma(u)}{\partial x} = 0.\end{aligned} \tag{38.4.2.7}$$

Here, u is the deformation gradient (strain), v is the strain rate, and $\sigma(u)$ is the stress.

Self-similar solutions are sought in the form (38.4.2.1). This results in the system of ordinary differential equations

$$\begin{aligned}\xi u'_\xi + w'_\xi = 0,\\ \sigma'_u(u)u'_\xi + \xi w'_\xi = 0.\end{aligned} \tag{38.4.2.8}$$

Equating the determinant of this system to zero, we obtain the quadratic equation $\xi^2 = \sigma'_u(u)$, whose roots are

$$\xi_{1,2} = \pm\sqrt{\sigma'_u(u)}. \tag{38.4.2.9}$$

The system is assumed to be hyperbolic, which implies that $\sigma'_u(u) > 0$. Substituting (38.4.2.9) into (38.4.2.8) yields a separable first-order ordinary differential equation:

$$w'_u \pm \sqrt{\sigma'_u(u)} = 0.$$

Integrating gives its general solution

$$w = C \mp \int \sqrt{\sigma'_u(u)}\, du, \tag{38.4.2.10}$$

where C is an arbitrary constant (two different constants, corresponding to the plus and minus sign). Formula (38.4.2.10) together with

$$\pm\sqrt{\sigma'_u(u)} = x/t \tag{38.4.2.11}$$

defines two one-parameter self-similar solutions of system (38.4.2.7) of the form (38.4.2.1) in implicit form.

38.4.3. Simple Riemann Waves

In the case of self-similar solutions of the form (38.4.2.1), one of the unknowns can be expressed in terms of the other by eliminating the independent variable ξ to obtain, for example,

$$w = w(u). \tag{38.4.3.1}$$

Assuming initially that the unknowns are functionally related via (38.4.3.1), one can obtain a wider class of exact solutions to system (38.4.1.1) than the class of self-similar solutions. Indeed, substituting (38.4.3.1) into (38.4.1.1) gives two equations for one function $u = u(x, t)$:

$$\begin{aligned}(f_1 + g_1 w'_u)\frac{\partial u}{\partial t} + (h_1 + k_1 w'_u)\frac{\partial u}{\partial x} &= 0,\\ (f_2 + g_2 w'_u)\frac{\partial u}{\partial t} + (h_2 + k_2 w'_u)\frac{\partial u}{\partial x} &= 0.\end{aligned} \tag{38.4.3.2}$$

The dependence (38.4.3.1) is chosen so that the two equations of (38.4.3.2) coincide. This condition results in the following first-order ordinary differential equation for $w = w(u)$ quadratically nonlinear in the derivative:

$$(g_1k_2 - g_2k_1)(w'_u)^2 + (f_1k_2 + g_1h_2 - f_2k_1 - g_2h_1)w'_u + f_1h_2 - f_2h_1 = 0. \tag{38.4.3.3}$$

Treating (38.4.3.3) as a quadratic equation in w'_u, we assume its discriminant to be positive:

$$(f_1k_2 + g_1h_2 - f_2k_1 - g_2h_1)^2 - 4(f_1h_2 - f_2h_1)(g_1k_2 - g_2k_1) > 0.$$

This condition is equivalent to condition (38.4.2.4) and implies that the system in question, (38.4.1.1), is strictly hyperbolic. In this case, equation (38.4.3.3) has two different real roots and is reducible to first-order ordinary differential equations of the standard form

$$w'_u = \Lambda_m(u, w), \quad m = 1,\ 2. \tag{38.4.3.4}$$

Having found a solution $w = w(u)$ of this equation for any m, we substitute it into the first (or the second) equation (38.4.3.2). As a result, we obtain a quasilinear first-order partial differential equation for $u(x, t)$:

$$(f_1 + g_1\Lambda_m)\frac{\partial u}{\partial t} + (h_1 + k_1\Lambda_m)\frac{\partial u}{\partial x} = 0, \qquad w = w(u). \tag{38.4.3.5}$$

The general solution of this equation can be obtained by the method of characteristics (see Subsection 24.3.5). This solution depends on one arbitrary function and is called a *simple Riemann wave*. To each of the two equations (38.4.3.4) there corresponds an equation (38.4.3.5) and a Riemann wave.

Suppose equations (38.4.3.4) have the integrals

$$\mathcal{R}_m(u, w) = C_m, \quad m = 1,\ 2,$$

where C_1 and C_2 are arbitrary constants. The functions $\mathcal{R}_m(u, w)$ are called *Riemann invariants*. Another definition of Riemann functions is given in Subsection 38.4.6.

Example 38.9. We look for exact solutions of system (38.4.2.7) of the special form (38.4.3.1), implying that one of the unknown functions can be expressed via the other. Substituting (38.4.3.1) into (38.4.2.7) gives

$$\frac{\partial u}{\partial t} - w'_u\frac{\partial u}{\partial x} = 0, \quad w'_u\frac{\partial u}{\partial t} - \sigma'_u(u)\frac{\partial u}{\partial x} = 0. \tag{38.4.3.6}$$

Requiring that the two equations coincide, we get the condition

$$(w'_u)^2 = \sigma'_u(u).$$

It follows that there are two possible relations between the unknowns:

$$w \mp \int \sqrt{\sigma'_u(u)}\,du = C_m, \quad m = 1,\ 2. \tag{38.4.3.7}$$

The left-hand sides of these relations represent Riemann functions. On eliminating w from (38.4.3.6), using (38.4.3.7), we arrive at the quasilinear first-order partial differential equation

$$\frac{\partial u}{\partial t} \mp \sqrt{\sigma'_u(u)}\,\frac{\partial u}{\partial x} = 0. \tag{38.4.3.8}$$

A detailed analysis of this equation is given in Section 23.3; the general solution is found by the method of characteristics and can be represented in implicit form:

$$x \pm t\sqrt{\sigma'_u(u)} = \Phi(u), \tag{38.4.3.9}$$

where $\Phi(u)$ is an arbitrary function.

Formulas (38.4.3.7) and (38.4.3.9) define two exact solutions of system (38.4.2.7), each containing one arbitrary function and one arbitrary constant. In the special case $\Phi(u) \equiv 0$, this solution turns into the self-similar solution considered above. In the case of a linear system, with $\sigma(u) = a^2 u$, formulas (38.4.3.9) give two traveling-wave solutions

$$u = \Psi_{1,2}(x \pm at),$$

which represent waves propagating in opposite directions and having an arbitrary undistorted profile.

The evolution of a Riemann wave with the initial profile

$$u = \varphi(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty) \tag{38.4.3.10}$$

for the lower sign in equation (38.4.3.8) is described parametrically by the formulas

$$x = \xi + \mathcal{F}(\xi)t, \quad u = \varphi(\xi), \tag{38.4.3.11}$$

where $\mathcal{F}(\xi) = \sqrt{\sigma'_u(u)}\,\big|_{u=\varphi(\xi)}$. The second unknown, w, is expressed in terms of u by formula (38.4.3.7).

Example 38.10. A one-dimensional ideal adiabatic (isentropic) gas flow is governed by the system of two equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} &= 0, \\ \frac{\partial(\rho v)}{\partial t} + \frac{\partial[\rho v^2 + p(\rho)]}{\partial x} &= 0. \end{aligned} \tag{38.4.3.12}$$

Here, $\rho = \rho(x,t)$ is the density, $v = v(x,t)$ is the velocity, and p is the pressure. The first equation (38.4.3.12) represents the law of conservation of mass in fluid mechanics and is referred to as a continuity equation. The second equation (38.4.3.12) represents the law of conservation of momentum. The equation of state is given in the form $p = p(\rho)$. For an ideal polytropic gas, $p = A\rho^\gamma$, where the constant γ is the adiabatic exponent.

Exact solutions of system (38.4.3.12) of the form $v = v(\rho)$ are given by the formulas

$$\begin{aligned} v &= \pm \int \sqrt{p'_\rho(\rho)}\,\frac{d\rho}{\rho} + C_m, \\ x &\mp t\left[\int \sqrt{p'_\rho(\rho)}\,\frac{d\rho}{\rho} + \sqrt{p'_\rho(\rho)} + C_m\right] = \Phi(\rho), \end{aligned}$$

where $\Phi(\rho)$ is an arbitrary function and the C_m are arbitrary constants (different solutions have the different constants, $m = 1, 2$). The Riemann functions are obtained from the first relation by expressing C_m via ρ and v.

For an ideal polytropic gas, which corresponds to $p = A\rho^\gamma$, the exact solution of system (38.4.3.12) with $\gamma \neq 1$ becomes

$$\begin{aligned} v &= \pm\frac{2}{\gamma-1}\sqrt{A\gamma}\,\rho^{\frac{\gamma-1}{2}} + C_m, \\ x &\mp t\left(\frac{\gamma+1}{\gamma-1}\sqrt{A\gamma}\,\rho^{\frac{\gamma-1}{2}} + C_m\right) = \Phi(\rho). \end{aligned}$$

Remark. System (38.4.3.12) with $\rho = h$ and $p(\rho) = \frac{1}{2}gh^2$, where v is the horizontal velocity averaged over the height h of the water level and g is the acceleration due to gravity, governs the *dynamics of shallow water*.

38.4.4. Linearization of Systems of Gas Dynamic Type by the Hodograph Transformation

The hodograph transformation is used in gas dynamics and the theory of jets for the linearization of equations and finding solutions of certain boundary value problems.

For a quasilinear system of two equations (38.4.1.1), the hodograph transformation has the form

$$x = x(u,w), \quad t = t(u,w); \tag{38.4.4.1}$$

this means that u, w are now treated as the independent variables and x, t as the dependent variables.

Differentiating each relation in (38.4.4.1) with respect to x and t (as composite functions) and eliminating the partial derivatives u_t, w_t, u_x, w_x from the resulting relations, we obtain

$$\frac{\partial u}{\partial t} = -J\frac{\partial x}{\partial w}, \quad \frac{\partial w}{\partial t} = J\frac{\partial x}{\partial u}, \quad \frac{\partial u}{\partial x} = J\frac{\partial t}{\partial w}, \quad \frac{\partial w}{\partial x} = -J\frac{\partial t}{\partial u}, \tag{38.4.4.2}$$

where $J = \dfrac{\partial u}{\partial x}\dfrac{\partial w}{\partial t} - \dfrac{\partial u}{\partial t}\dfrac{\partial w}{\partial x}$ is the Jacobian of the functions $u = u(x,t)$ and $w = w(x,t)$. Replacing the derivatives in (38.4.1.1) with the help of relations (38.4.4.2) and dividing by J, we arrive at the linear system

$$\begin{aligned} g_1(u,w)\frac{\partial x}{\partial u} - k_1(u,w)\frac{\partial t}{\partial u} - f_1(u,w)\frac{\partial x}{\partial w} + h_1(u,w)\frac{\partial t}{\partial w} = 0,\\ g_2(u,w)\frac{\partial x}{\partial u} - k_2(u,w)\frac{\partial t}{\partial u} - f_2(u,w)\frac{\partial x}{\partial w} + h_2(u,w)\frac{\partial t}{\partial w} = 0. \end{aligned} \tag{38.4.4.3}$$

Remark. The hodograph transformation (38.4.4.1) is unusable if $J \equiv 0$. In this degenerate case, the quantities u and w are functionally related, and hence they cannot be taken as independent variables. In this case, relation (38.4.3.1) holds; it determines simple Riemann waves. This is why in using the hodograph transformation (38.4.4.1), one loses solutions corresponding to simple Riemann waves.

38.4.5. Cauchy and Riemann Problems. Qualitative Features of Solutions

Cauchy problem ($t \ge 0$, $-\infty < x < \infty$). Find functions $u = u(x,t)$, $w = w(x,t)$ that solve system (38.4.1.1) for $t > 0$ and satisfy the initial conditions

$$u(x,0) = \varphi_1(x), \qquad w(x,0) = \varphi_2(x), \tag{38.4.5.1}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are prescribed functions. The Cauchy problem is also often referred to as an initial value problem.

Riemann problem ($t \ge 0$, $-\infty < x < \infty$). Find functions $u = u(x,t)$, $w = w(x,t)$ that solve system (38.4.1.1) for $t > 0$ and satisfy piecewise-smooth initial conditions of the special form

$$u(x,0) = \begin{cases} u_{\mathrm{L}} & \text{if } x < 0,\\ u_{\mathrm{R}} & \text{if } x > 0, \end{cases} \qquad w(x,0) = \begin{cases} w_{\mathrm{L}} & \text{if } x < 0,\\ w_{\mathrm{R}} & \text{if } x > 0. \end{cases} \tag{38.4.5.2}$$

Here, u_{L}, u_{R}, w_{L}, and w_{R} are prescribed constant quantities.

Since equations (38.4.1.1) and the boundary conditions (38.4.5.2) do not change under the transformation $(x,t) \to (ax, at)$, where a is any positive number, the solution of the Riemann problem (38.4.1.1), (38.4.5.2) is self-similar. The unknown functions are sought in the form (38.4.2.1) and satisfy the ordinary differential equations (38.4.2.2). When passing to the self-similar variable $\xi = x/t$, the initial conditions (38.4.5.2) are transformed into boundary conditions of the form

$$u \to u_{\mathrm{L}}, \quad w \to w_{\mathrm{L}} \quad \text{as} \quad \xi \to -\infty; \qquad u \to u_{\mathrm{R}}, \quad w \to w_{\mathrm{R}} \quad \text{as} \quad \xi \to \infty.$$

Solutions of the Riemann problem (38.4.1.1), (38.4.5.2) can be either continuous (such solutions are called *rarefaction waves*) or discontinuous (in this case, they describe *shock waves*). In both cases, the solution for each of the unknowns (e.g., for u) consists of two types of segments: those where the solution is constant, $u = \text{const}$, and those where it smoothly changes according to the law $u = u(x/t)$. The endpoints of these segments lie on straight lines $x = a_n t$ in the plane (x,t). At these points, individual segments of solutions either join together (for continuous solutions) or agree with one another based on conservation laws (for discontinuous solutions); for details see Subsections 38.4.8 and 38.4.10.

Let us explain why shock waves, corresponding to discontinuous solutions, arise by an example. Consider the system of equations for one-dimensional longitudinal oscillations of an elastic bar (38.4.2.7). It was shown above that this system admits simple Riemann waves, which are described by the quasilinear first-order partial differential equation (38.4.3.8). This equation coincides, up to notation, with the model equation of gas dynamics (23.3.1.1), which was studied in detail in Section 23.3. If the function $\mathcal{F}(\xi)$ in solution (38.4.3.11) is nonmonotonic (this function is determined by the initial profile of the wave), then the characteristic lines defined by the first formula in (38.4.3.11) will intersect in the (x,t) plane for various ξ. This results in nonuniqueness in determining the function $u = u(x,t)$, which contradicts physical laws. Therefore, in order to avoid nonuniqueness of physical quantities, discontinuous solutions have to be considered.

38.4.6. Reduction of Systems to the Canonical Form. Riemann Invariants

1°. Eliminating the derivative w_t and then the derivative u_t from (38.4.1.1), we arrive after simple rearrangements to the *canonical form* of a gas dynamics system:

$$\begin{aligned} \frac{\partial u}{\partial t} + p_1(u,w)\frac{\partial u}{\partial x} + q_1(u,w)\frac{\partial w}{\partial x} = 0,\\ \frac{\partial w}{\partial t} + p_2(u,w)\frac{\partial u}{\partial x} + q_2(u,w)\frac{\partial w}{\partial x} = 0, \end{aligned} \tag{38.4.6.1}$$

where

$$p_1 = \frac{h_1 g_2 - g_1 h_2}{f_1 g_2 - g_1 f_2}, \quad q_1 = \frac{k_1 g_2 - g_1 k_2}{f_1 g_2 - g_1 f_2}, \quad p_2 = \frac{f_1 h_2 - h_1 f_2}{f_1 g_2 - g_1 f_2}, \quad q_2 = \frac{f_1 k_2 - k_1 f_2}{f_1 g_2 - g_1 f_2}.$$

The functions f_m, g_m, h_m, k_m, p_m, and q_m ($m = 1, 2$) depend on u and w; it is assumed that $f_1 g_2 - g_1 f_2 \not\equiv 0$.

2°. For further simplifications of system (38.4.6.1), we proceed as follows. Multiply the first equation by $b_1 = b_1(u, w)$ and the second by $b_2 = b_2(u, w)$ and add up to obtain

$$b_1 \frac{\partial u}{\partial t} + b_2 \frac{\partial w}{\partial t} + (b_1 p_1 + b_2 p_2) \frac{\partial u}{\partial x} + (b_1 q_1 + b_2 q_2) \frac{\partial w}{\partial x} = 0. \tag{38.4.6.2}$$

Impose the following constraints of the functions b_1 and b_2:

$$\begin{aligned} b_1 p_1 + b_2 p_2 &= \lambda b_1, \\ b_1 q_1 + b_2 q_2 &= \lambda b_2. \end{aligned} \tag{38.4.6.3}$$

These constraints represent a linear homogeneous algebraic system of equations for b_m. For this system to have nontrivial solutions, its determinant must be zero. This results in a quadratic equation for the eigenvalues $\lambda = \lambda(u, w)$:

$$\lambda^2 - (p_1 + q_2)\lambda + p_1 q_2 - p_2 q_1 = 0. \tag{38.4.6.4}$$

For hyperbolic systems (38.4.6.1), equation (38.4.6.4) has two different real roots:

$$\lambda_{1,2} = \tfrac{1}{2}(p_1 + q_2) \pm \tfrac{1}{2}\sqrt{(p_1 - q_2)^2 + 4 p_2 q_1}. \tag{38.4.6.5}$$

For each eigenvalue λ_m of (38.4.6.5), from (38.4.6.3) we find the associated eigenfunction

$$b_1^{(m)} = \varphi_m(\lambda_m - q_2), \qquad b_2^{(m)} = \varphi_m q_1, \tag{38.4.6.6}$$

where $\varphi_m = \varphi_m(u, w)$ is an arbitrary function (it will be defined later); $m = 1, 2$.

From (38.4.6.2), in view of (38.4.6.3), we obtain two equations for the two roots (38.4.6.5):

$$b_1^{(m)} \left(\frac{\partial u}{\partial t} + \lambda_m \frac{\partial u}{\partial x} \right) + b_2^{(m)} \left(\frac{\partial w}{\partial t} + \lambda_m \frac{\partial w}{\partial x} \right) = 0, \qquad m = 1, 2. \tag{38.4.6.7}$$

The function φ_m in (38.4.6.6) can be determined from the conditions

$$b_1^{(m)} = \frac{\partial \mathcal{R}_m}{\partial u}, \qquad b_2^{(m)} = \frac{\partial \mathcal{R}_m}{\partial w}. \tag{38.4.6.8}$$

On differentiating the first relation (38.4.6.8) with respect to w and the second with respect to u, we equate the mixed derivatives $(\mathcal{R}_m)_{uw}$ and $(\mathcal{R}_m)_{wu}$. In view of (38.4.6.6), we obtain the following linear first-order partial differential equation for φ_m:

$$\frac{\partial}{\partial w}[\varphi_m(\lambda_m - q_2)] = \frac{\partial}{\partial u}(\varphi_m q_1). \tag{38.4.6.9}$$

This equation can be solved by the method of characteristics; see Subsection 24.3.5. Assuming that a solution of equation (38.4.6.9) has been obtained (any nontrivial solution can be taken) and taking into account formulas (38.4.6.6), we find the functions $\mathcal{R}_m = \mathcal{R}_m(u, w)$ from system (38.4.6.8). Replacing $b_1^{(m)}$ and $b_2^{(m)}$ in (38.4.6.7) by the right-hand sides of (38.4.6.8), we get two equations:

$$\begin{aligned} \frac{\partial \mathcal{R}_1}{\partial t} + \widetilde{\lambda}_1(\mathcal{R}_1, \mathcal{R}_2) \frac{\partial \mathcal{R}_1}{\partial x} &= 0, \\ \frac{\partial \mathcal{R}_2}{\partial t} + \widetilde{\lambda}_2(\mathcal{R}_1, \mathcal{R}_2) \frac{\partial \mathcal{R}_2}{\partial x} &= 0, \end{aligned} \tag{38.4.6.10}$$

where $\widetilde{\lambda}_m(\mathcal{R}_1, \mathcal{R}_2) = \lambda_m(u, w)$, $m = 1, 2$. The functions $\mathcal{R}_1$ and $\mathcal{R}_2$ appearing in system (38.4.6.10) are called *Riemann invariants*.

System (38.4.6.10) admits two exact solutions:

$$\begin{aligned} &\mathcal{R}_1 = C_1, \quad x - \widetilde{\lambda}_2(C_1, \mathcal{R}_2)t = \Phi_1(\mathcal{R}_2), \\ &\mathcal{R}_2 = C_2, \quad x - \widetilde{\lambda}_1(\mathcal{R}_1, C_2)t = \Phi_2(\mathcal{R}_1), \end{aligned}$$

where C_m are arbitrary constants and $\Phi_m(\mathcal{R}_{3-m})$ are arbitrary functions ($m = 1, 2$).

If the function $\widetilde{\lambda}_1$ in (38.4.6.10) is independent of $\mathcal{R}_2$, then the solution of system (38.4.6.10) is reduced to successive integration of two quasilinear first-order partial differential equations.

Remark 1. Sometimes it is more convenient to use the formulas

$$b_1^{(m)} = \varphi_m p_2, \quad b_2^{(m)} = \varphi_m(\lambda_m - p_1) \tag{38.4.6.6a}$$

rather than (38.4.6.6). In this case, equation (38.4.6.9) is replaced by

$$\frac{\partial}{\partial w}(\varphi_m p_2) = \frac{\partial}{\partial u}[\varphi_m(\lambda_m - p_1)]. \tag{38.4.6.9a}$$

Remark 2. The Riemann functions are not uniquely defined. Transformations of the dependent variables

$$\mathcal{R}_1 = \Psi_1(z_1), \quad \mathcal{R}_2 = \Psi_2(z_2),$$

where $\Psi_1(z_1)$ and $\Psi_2(z_2)$ are arbitrary functions, preserve the general form of system (38.4.6.10); what changes are the functions $\widetilde{\lambda}_m$ only. This is due to the functional arbitrariness in determining the functions φ_m from equations (38.4.6.9) or (38.4.6.9a). This arbitrariness can be used for the selection of most simple Riemann functions.

Example 38.11. Consider the system of equations of longitudinal oscillations of an elastic bar (38.4.2.7), which is a special case of system (38.4.6.1) with

$$p_1 = 0, \quad q_1 = -1, \quad p_2 = -\sigma'(u), \quad q_2 = 0. \tag{38.4.6.11}$$

To determine the eigenvalues and the corresponding eigenfunctions, we use formulas (38.4.6.5), (38.4.6.6), and (38.4.6.11) and equation (38.4.6.9). As a result, we obtain

$$\lambda_{1,2} = \pm\sqrt{\sigma'(u)}\,; \quad b_1^{(m)} = \sqrt{\sigma'(u)}, \quad b_2^{(m)} = \mp 1, \quad \varphi_m = \pm 1 \qquad (m = 1,\ 2).$$

The Riemann functions are found from equations (38.4.6.8):

$$\mathcal{R}_1 = -w + \int \sqrt{\sigma'(u)}\,du, \quad \mathcal{R}_2 = w + \int \sqrt{\sigma'(u)}\,du.$$

Note that the Riemann functions coincide, up to notation, with the integrals (38.4.3.7) obtained earlier from other considerations.

Example 38.12. Consider the system of equations (38.4.3.12) describing one-dimensional flow of an ideal adiabatic gas. First, reduce the system to the canonical form

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} + \frac{p'(\rho)}{\rho}\frac{\partial \rho}{\partial x} + v\frac{\partial v}{\partial x} = 0.$$

This is a special case of system (38.4.6.1) with

$$u = \rho, \quad w = v, \quad p_1 = v, \quad q_1 = \rho, \quad p_2 = p'(\rho)/\rho, \quad q_2 = v. \tag{38.4.6.12}$$

To determine the eigenvalues and the corresponding eigenfunctions, we use formulas (38.4.6.5), (38.4.6.6), and (38.4.6.12) and equation (38.4.6.9). As a result, we obtain

$$\lambda_{1,2} = v \pm \sqrt{p'(\rho)}\,; \quad b_1^{(m)} = \frac{\sqrt{p'(\rho)}}{\rho}, \quad b_2^{(m)} = \pm 1, \quad \varphi_m = \pm\frac{1}{\rho} \qquad (m = 1,\ 2).$$

The Riemann functions are found from equations (38.4.6.8):

$$\mathcal{R}_1 = v + \int \frac{\sqrt{p'(\rho)}}{\rho}\,d\rho, \quad \mathcal{R}_2 = -v + \int \frac{\sqrt{p'(\rho)}}{\rho}\,d\rho.$$

3°. Let us show that the Riemann invariants are constant along the rarefaction waves for hyperbolic systems. The substitution of the self-similar solution forms $\mathcal{R}_m = \mathcal{R}_m(\xi)$, where $\xi = x/t$, into system (38.4.6.10) results in the following system of two ordinary differential equations:

$$\big[\xi - \lambda_m(\mathcal{R}_1, \mathcal{R}_2)\big]\frac{d\mathcal{R}_m}{d\xi} = 0 \qquad (m = 1,\ 2). \tag{38.4.6.13}$$

The equality $\xi = \lambda_1(\mathcal{R}_1, \mathcal{R}_2)$ takes place along the first rarefaction wave. Hence, the first factor in the second equation of (38.4.6.13) is nonzero. Therefore, the second factor in the second equation of (38.4.6.13) is zero. It follows that $\mathcal{R}_2 = \text{const}$ along the first rarefaction wave. Along the second rarefaction wave, $\mathcal{R}_1$ is constant.

38.4.7. Hyperbolic $n \times n$ Systems of Conservation Laws. Exact Solutions

Here we consider systems of conservation laws of the form

$$\frac{\partial \mathbf{F}(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial x} = 0, \tag{38.4.7.1}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ is a vector function of two scalar variables, and $\mathbf{F} = \mathbf{F}(\mathbf{u})$ and $\mathbf{G} = \mathbf{G}(\mathbf{u})$ are vector functions,

$$\begin{aligned}
\mathbf{u} &= (u_1, \ldots, u_n)^{\mathrm{T}}, \quad && u_i = u_i(x,t);\\
\mathbf{F} &= (F_1, \ldots, F_n)^{\mathrm{T}}, \quad && F_i = F_i(\mathbf{u});\\
\mathbf{G} &= (G_1, \ldots, G_n)^{\mathrm{T}}, \quad && G_i = G_i(\mathbf{u}).
\end{aligned}$$

Here and henceforth, $(u_1, \ldots, u_n)^{\mathrm{T}}$ stands for a column vector with components $u_1, \ldots, u_n$.

Note three important types of exact solutions to system (38.4.7.1):

1. For any $\mathbf{F}$ and $\mathbf{G}$, system (38.4.7.1) admits the following simplest solutions:

$$\mathbf{u} = \mathbf{C}, \tag{38.4.7.2}$$

where $\mathbf{C}$ is an arbitrary constant vector.

2. System (38.4.7.1) admits self-similar solutions of the form

$$\mathbf{u} = \mathbf{u}(\xi), \quad \xi = x/t. \tag{38.4.7.3}$$

The procedure for finding them is analogous to that used in Subsection 38.4.2 (see also Subsection 38.4.10).

3. System (38.4.7.1) also admits more general exact solutions of the form

$$u_1 = u_1(u_n), \quad \ldots, \quad u_{n-1} = u_{n-1}(u_n), \tag{38.4.7.4}$$

where all the components are functionally related and can be expressed in terms of one of them. After substituting (38.4.7.4) into the original system (38.4.7.1), we require that all the resulting equations coincide. As a result, we obtain a system of $(n-1)$ ordinary differential equations for determining the dependences (38.4.7.4) and one first-order partial differential equation for $u_n = u_n(x,t)$. The mentioned procedure is described in detail in Subsection 38.4.3 for the case of a two-equation system.

Let us show that some systems of conservation laws can be represented as systems of ordinary differential equations along curves $x = x(t)$ called characteristic curves.

Differentiating both sides of system (38.4.7.1) yields

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0, \tag{38.4.7.5}$$

where $\mathbf{A} = \widetilde{\mathbf{F}}^{-1}(\mathbf{u})\widetilde{\mathbf{G}}(\mathbf{u})$, $\widetilde{\mathbf{F}}(\mathbf{u})$ is the matrix with entries $\frac{\partial F_i}{\partial u_j}$, $\widetilde{\mathbf{G}}(\mathbf{u})$ is the matrix with entries $\frac{\partial G_i}{\partial u_j}$, and $\widetilde{\mathbf{F}}^{-1}$ is the inverse of the matrix $\widetilde{\mathbf{F}}$.

Let us multiply each scalar equation in (38.4.7.5) by $b_i = b_i(\mathbf{u})$ and take the sum. On rearranging terms under the summation sign, we obtain

$$\sum_{i=1}^{n} b_i \frac{\partial u_i}{\partial t} + \sum_{i,j=1}^{n} b_j a_{ji} \frac{\partial u_i}{\partial x} = 0, \tag{38.4.7.6}$$

where $a_{ij} = a_{ij}(\mathbf{u})$ are the entries of the matrix $\mathbf{A}$.

If $\mathbf{b} = (b_1, \ldots, b_n)$ is a left eigenvector of the matrix $\mathbf{A}(\mathbf{u})$ that corresponds to an eigenvalue $\lambda = \lambda(\mathbf{u})$, so that

$$\sum_{j=1}^{n} b_j a_{ji} = \lambda b_i,$$

then equation (38.4.7.6) can be rewritten in the form

$$\sum_{i=1}^{n} b_i \left(\frac{\partial u_i}{\partial t} + \lambda \frac{\partial u_i}{\partial x} \right) = 0. \tag{38.4.7.7}$$

Thus, system (38.4.7.5) is transformed to a linear combination of total derivatives of the unknowns u_i with respect to t along the direction $(\lambda, 1)$ on the plane (x,t), i.e., the total time derivatives are taken along the trajectories having the velocity λ:

$$\sum_{i=1}^{n} b_i \frac{du_i}{dt} = 0, \quad \frac{dx}{dt} = \lambda, \tag{38.4.7.8}$$

where

$$b_i = b_i(\mathbf{u}), \quad \lambda = \lambda(\mathbf{u}), \quad x = x(t), \quad \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{dx}{dt}\frac{\partial u_i}{\partial x}.$$

Equations (38.4.7.8) are called *differential relations on characteristics*. The second equation in (38.4.7.8) explains why an eigenvalue λ is called a *characteristic velocity*.

System (38.4.7.5) is called *hyperbolic* if the matrix $\mathbf{A}$ has n real eigenvalues and n linearly independent left eigenvectors. If all eigenvalues are distinct for any $\mathbf{u}$, then system (38.4.7.5) is said to be *strictly hyperbolic*.

Remark 1. If the hyperbolic system (38.4.7.5) is linear and the coefficients of the matrix $\mathbf{A}$ are constant, then the eigenvalues λ_k are constant and the characteristic lines in the (x, t) plane become straight lines:

$$x = \lambda_k t + \text{const}.$$

Since all eigenvalues λ_k are different, the general solution of system (38.4.7.5) can be represented as the sum of particular solutions as follows:

$$\mathbf{u} = \phi_1(x - \lambda_1 t)\mathbf{r}^1 + \cdots + \phi_n(x - \lambda_n t)\mathbf{r}^n,$$

where the $\phi_k(\xi_k)$ are arbitrary functions, $\xi_k = x - \lambda_k t$, and $\mathbf{r}^k$ is the right eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_k, $k = 1, \ldots, n$. The particular solutions $\mathbf{u}_k = \phi_k(x - \lambda_k t)\mathbf{r}^k$ are called traveling-wave solutions. Each of these solutions represents a wave that travels in the $\mathbf{r}^k$-direction with velocity λ_k.

Remark 2. The characteristic form (38.4.7.7) of the hyperbolic system (38.4.7.5) forms the basis for the numerical characteristics method which allows the solution of system (38.4.7.5) in its domain of continuity. Suppose that we already have a solution $\mathbf{u}(x, t)$ for all values of x and a fixed time t. To construct a solution at a point $(x,\ t + \Delta t)$, we find the points $(x - \lambda_k \Delta t,\ t)$ from which the characteristics arrive at the point $(x,\ t + \Delta t)$. Since the $\mathbf{u}(x - \lambda_k \Delta t,\ t)$ are known, relations (38.4.7.8) can be regarded as a system of n linear equations in the n unknowns $\mathbf{u}(x, t + \Delta t)$. Thus, a solution for the time $t + \Delta t$ can be found.

Remark 3. The origin of hyperbolic systems of conservation laws as mathematical models for physical phenomena is discussed extensively in the literature. The classical treatises by Courant and Hilbert (1989), Landau and Lifshitz (1987), and Whitham (1974) and also a comprehensive monograph by Dafermos (2000) should be mentioned. Conservation law systems for various gas flow regimes in Eulerian and Lagrangian coordinates are treated in the monographs by Courant and Friedrichs (1985), Landau and Lifshitz (1987), Logan (1994), and Zel'dovich and Raizer (1968). Gas flows with chemical reactions (combustion and phase transitions) are discussed in the books by Zel'dovich and Raizer (1966, 1967), Zel'dovich, Barenblatt, Librovich, and Makhviladze (1985). Hyperbolic systems for chromatography are dealt with in Rhee, Aris, and Amundson (1970, 1986, 1989). The monographs by Hanyga (1985) and Kulikovskii and Sveshnikova (1995) give a comprehensive presentation of the theory of elastic media. Both Barenblatt, Entov, and Ryzhik (1991) and Bedrikovetsky (1993) discuss hyperbolic systems for two-phase multi-component flows in porous media describing oil recovery processes. Traffic flow and shallow water mechanics are treated in Logan (1994) and Whitham (1974).

Methods of analytical integration for self-similar Riemann problems are presented in the monographs by Smoller (1983) and Dafermos (2000); non-self-similar problem integration methods for wave interactions are given in Glimm (1989), LeVeque (2002), and Bedrikovetsky (1993).

38.4.8. Shock Waves. Rankine–Hugoniot Jump Conditions

Let us consider a discontinuity along a trajectory $x_{\mathrm{f}}(t)$ and obtain the mass balance condition along a discontinuity (shock wave). The region $x > x_{\mathrm{f}}(t)$ is conventionally assumed to lie ahead of the shock, and the region $x < x_{\mathrm{f}}(t)$ is assumed to lie behind the shock. The shock speed D is determined by the relation

$$D = \frac{dx_{\mathrm{f}}}{dt}. \tag{38.4.8.1}$$

To refer to the value of a quantity, A, behind the shock, the minus superscript will be used, A^-, since this value corresponds to negative x in the initial value formulation. Likewise, the value of A ahead of the shock will be denoted A^+.

For an arbitrary quasilinear system in the form of conservation laws (38.4.7.1), the balance equations for a shock can be represented as

$$[F_i(\mathbf{u})]D = [G_i(\mathbf{u})], \quad i = 1, \ldots, n, \tag{38.4.8.2}$$

where $[A] = A^+ - A^-$ stands for the jump of a quantity A at the shock. Formulas (38.4.8.2) are called the *Rankine–Hugoniot jump conditions*; they are derived from integral consequences of the equations in question in much the same ways as for a single equation (see Subsection 23.3.4).

Example 38.13. The Rankine–Hugoniot conditions for an isentropic gas flow (38.4.3.12) follow from (38.4.8.2). We have

$$\begin{aligned} [\rho]D &= [\rho v], \\ [\rho v]D &= [\rho v^2 + p(\rho)]. \end{aligned} \tag{38.4.8.3}$$

Here, the density and the velocity ahead of the shock are denoted ρ^+ and v^+, while those behind the shock are ρ^- and v^-.

Eliminating the shock speed D from (38.4.8.3), we obtain the relation

$$v^+ - v^- = \pm\sqrt{\frac{(\rho^+ - \rho^-)\big[p(\rho^+) - p(\rho^-)\big]}{\rho^-\rho^+}}. \tag{38.4.8.4}$$

Each of the signs before the square root in (38.4.8.4) corresponds to a branch of the locus of points that can be connected with a given point (v^-, ρ^-) by a shock (see Fig. 38.1*a*). For an ideal polytropic gas, one should set $p = A\rho^\gamma$ in formulas (38.4.8.3) and (38.4.8.4).

Let us determine the set of states (v^+, ρ^+) reachable by a shock from a given point (v^-, ρ^-). Express the point (v^+, ρ^+) ahead of the shock via the solution of the transcendental system (38.4.8.3) to obtain

$$v^+ = v^+(v^-, \rho^-, D), \qquad \rho^+ = \rho^+(v^-, \rho^-, D).$$

The graphs of the solution determined by (38.4.8.4) are shown in Figs. 38.1*a* and 38.1*b*. The solid lines correspond to the minus sign before the radical and represent stable (evolutionary) shocks, while the dashed lines correspond to the plus sign and represent unstable (nonevolutionary) shocks; see below.

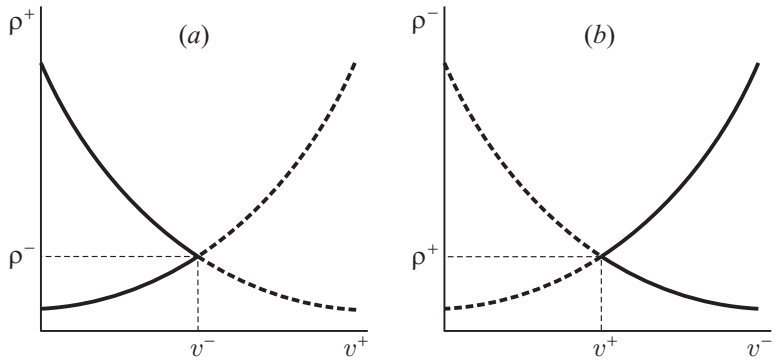

Figure 38.1. Loci of points that can be connected by a shock wave: (*a*) with a given state (v^-, ρ^-) and (*b*) with a given state (v^+, ρ^+). The solid lines correspond, respectively, to the minus and plus signs in formula (38.4.8.4) (for $p = A\rho^\gamma$) before the radical for cases (*a*) and (*b*).

38.4.9. Shock Waves. Evolutionary Conditions. Lax Condition

In general, discontinuities of solutions are surfaces where conditions are imposed that relate the quantities on both sides of the surfaces. For hyperbolic systems in the conservation-law form (38.4.7.1), these relations have the form (38.4.8.2) and involve the discontinuity velocity D.

The evolutionary conditions are necessary conditions for unique solvability of the problem of the discontinuity interaction with small perturbations depending on the x-coordinate normal to the discontinuity surface. For hyperbolic systems, a one-dimensional small perturbation can be represented as a superposition of n waves, each being a traveling wave propagating at a characteristic velocity $\lambda_i^{\mp}$. This allows us to classify all these waves into incoming and outgoing ones, depending on the sign of the difference $\lambda_i^{\mp} - D$. Incoming waves are fully determined by the initial conditions, while outgoing ones must be determined from the linearized boundary conditions at the shock.

We consider below the stability of a shock with respect to a small perturbation. This kind of stability is determined by incoming waves. For this reason, we focus below on incoming waves.

Let m^+ and m^- be the numbers of incoming waves from the right and left of the shock, respectively. It can be shown that if the relation

$$m^+ + m^- - 1 = n \tag{38.4.9.1}$$

holds, the problem of the discontinuity interaction with small perturbations is uniquely solvable. Relation (38.4.9.1) is called the *Lax condition*. If (38.4.9.1) holds, the corresponding discontinuity is called *evolutionary*; otherwise, it is called *nonevolutionary*. For evolutionary discontinuities, small incoming perturbations generate small outgoing perturbations and small changes in the discontinuity velocity.

If

$$m^+ + m^- - 1 > n,$$

then either such discontinuities do not exist or the perturbed quantities cannot be uniquely determined (the given conditions are underdetermined).

If

$$m^+ + m^- - 1 < n,$$

then the problem of the discontinuity interaction with small perturbations has no solution in the linear approximation. Previous studies of various physical problems have shown that the interaction of nonevolutionary discontinuities with small perturbations results in their disintegration into two or more evolutionary discontinuities.

The evolutionary condition (38.4.9.1) can be rewritten in the form of inequalities relating the shock speed D and the velocities $\lambda_i^{\mp}$ of small disturbances. Let us enumerate the characteristic velocities on both sides of the discontinuity so that

$$\lambda_1(\mathbf{u}) \le \lambda_2(\mathbf{u}) \le \cdots \le \lambda_n(\mathbf{u}).$$

A shock is called a k-shock if both kth characteristics are incoming:

$$\begin{aligned} &\text{if } i > k, \text{ then } D < \lambda_i^{\mp};\\ &\text{if } i < k, \text{ then } D > \lambda_i^{\mp};\\ &\text{if } i = k, \text{ then } \lambda_i^+ < D < \lambda_i^-. \end{aligned} \tag{38.4.9.2}$$

Below is another, equivalent statement of the Lax condition: $n + 1$ inequalities out of the $2n$ inequalities

$$\lambda_k^+ \le D \le \lambda_k^- \qquad (k = 1, \ldots, n) \tag{38.4.9.3}$$

must hold.

For two-equation systems, the shock evolutionarity requires that three out of four inequalities

$$\lambda_1^+ \le D \le \lambda_1^-, \quad \lambda_2^+ \le D \le \lambda_2^-, \tag{38.4.9.4}$$

must hold.

Example 38.14. The Lax condition for the adiabatic gas flow equations (38.4.3.12) are obtained by substituting the eigenvalue expressions $\lambda_{1,2} = v \pm \sqrt{p'(\rho)}$ (see Example 38.12) into inequalities (38.4.9.4). We have

$$v^+ \pm \sqrt{p'(\rho^+)} < D < v^- \pm \sqrt{p'(\rho^-)}. \tag{38.4.9.5}$$

The shock evolutionarity requires that three of the four inequalities in (38.4.9.5) hold. Substituting the equation of state for a polytropic ideal gas, $p = Ar^\gamma$, into (38.4.9.5), we obtain the following evolutionarity criterion:

$$v^+ \pm \sqrt{A\gamma(\rho^+)^{\gamma-1}} < D < v^- \pm \sqrt{A\gamma(\rho^-)^{\gamma-1}}. \tag{38.4.9.6}$$

For the adiabatic gas flow system (38.4.3.12), the solution vector is $\mathbf{u} = (\rho, v)^{\mathrm{T}}$. Figure 38.1*a* shows the locus of points $\mathbf{u}^+$ that can be connected by a shock to the point $\mathbf{u}^-$; it is divided into the evolutionary part (solid line) and nonevolutionary part (dashed line). It can be shown that a shock issuing from the point (v^-, ρ^-) and passing through any point (v^+, ρ^+) of the solid part of the locus of first-family shocks obeys the Lax conditions (38.4.9.6). The shock speed D of the first family decreases from $\lambda_1(\mathbf{u}^-)$, for points $\mathbf{u}^+$ tending to point $\mathbf{u}^-$, to v^- as $\rho^+ \to 0$ and $v^+ \to -\infty$. Along the locus of the second-family shocks, the speed decreases from $\lambda_2(\mathbf{u}^-)$, for points $\mathbf{u}^+$ tending to $\mathbf{u}^-$, to $-\infty$ as $\rho^+ \to \infty$ and $v^+ \to -\infty$.

Figure 38.1*b* depicts the locus of points $\mathbf{u}^-$ that can be connected by a shock to the point $\mathbf{u}^+$. The evolutionary part of the locus is shown by a solid line; the dashed line shows the nonevolutionary part. The shock speed D of the first family increases from $\lambda_1(\mathbf{u}^+)$, for points $\mathbf{u}^-$ tending to $\mathbf{u}^+$, to ∞ as $\rho^- \to \infty$ and $v^- \to \infty$. Along the locus of the second family shocks, the speed increases from $\lambda_2(\mathbf{u}^+)$, for points $\mathbf{u}^-$ tending to $\mathbf{u}^+$, to v^+ as $\rho^- \to 0$ and $v^- \to \infty$.

38.4.10. Solutions for the Riemann Problem. Solutions Describing Shock Waves

1°. Self-similar solutions (38.4.7.3) together with the simplest solutions (38.4.7.2) enable the solution of the Riemann problem. This problem is formulated as follows: find a function $\mathbf{u} = \mathbf{u}(x,t)$ that solves system (38.4.7.1) for $t > 0$ and satisfies the following initial condition of a special form:

$$\mathbf{u} = \begin{cases} \mathbf{u}_{\mathrm{L}} & \text{if } x < 0 \\ \mathbf{u}_{\mathrm{R}} & \text{if } x > 0 \end{cases} \quad \text{at} \quad t = 0. \tag{38.4.10.1}$$

Here, $\mathbf{u}_{\mathrm{L}}$ and $\mathbf{u}_{\mathrm{R}}$ are two prescribed constant vectors.

With the self-similar variable $\xi = x/t$, the initial condition (38.4.10.1) is transformed into the boundary conditions

$$\mathbf{u} \to \mathbf{u}_{\mathrm{L}} \quad \text{as} \quad \xi \to -\infty, \qquad \mathbf{u} \to \mathbf{u}_{\mathrm{R}} \quad \text{as} \quad \xi \to \infty. \tag{38.4.10.2}$$

2°. Without loss of generality, we will be considering the case $\mathbf{F}(\mathbf{u}) = \mathbf{u}$. The substitution of the self-similar form $\mathbf{u}(x,t) = \mathbf{u}(\xi)$ into (38.4.7.1) with $\mathbf{F}(\mathbf{u}) = \mathbf{u}$ yields

$$\big(\widetilde{\mathbf{G}} - \xi\mathbf{I}\big)\mathbf{u}'_\xi = 0, \tag{38.4.10.3}$$

where $\widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}(\mathbf{u})$ is the matrix with entries $G_{ij} = \frac{\partial G_i}{\partial u_j}$ and $\mathbf{I}$ is the identity matrix.

Hence, the velocity vector for the continuous solution $\mathbf{u}(\xi)$ is a right eigenvector of the matrix $\widetilde{\mathbf{G}}$ for any point $\mathbf{u}$, and the corresponding eigenvalue equals the self-similar coordinate:

$$\xi = \lambda_k, \quad \mathbf{u}'_\xi = \alpha\mathbf{r}^k. \tag{38.4.10.4}$$

Here, $\lambda_k = \lambda_k(\mathbf{u})$ is a root of the algebraic equation $\det|\widetilde{\mathbf{G}} - \lambda\mathbf{I}| = 0$, $\mathbf{r}^k = \mathbf{r}^k(\mathbf{u})$ is a solution of the corresponding degenerate linear system of equations $\big(\widetilde{\mathbf{G}} - \lambda\mathbf{I}\big)\mathbf{r} = 0$, and $\alpha = \alpha(\mathbf{u})$ is a positive function that will be defined below.

Differentiating both sides of the first equation (38.4.10.4) with respect to ξ yields

$$\alpha = \frac{1}{\langle \nabla\lambda_k, \mathbf{r}^k\rangle}, \quad \langle \nabla\lambda_k, \mathbf{r}^k\rangle = \frac{\partial\lambda_k}{\partial u_1}r_1^k + \cdots + \frac{\partial\lambda_k}{\partial u_n}r_n^k.$$

Any $n \times n$ hyperbolic system allows for n continuous solutions [of system (38.4.10.3)] corresponding to n characteristic velocities $\lambda = \lambda_k$. The continuous solutions are determined by n systems of ordinary differential equations. Each system is represented by a phase portrait in the n-dimensional $\mathbf{u}$-space. A solution/trajectory that corresponds to a characteristic velocity λ_k is called a kth rarefaction wave.

3°. A trajectory of solution (38.4.7.3) in the space $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$ is called a *solution path*. The path is parametrized by the self-similar coordinate ξ. The path connects the point $\mathbf{u} = \mathbf{u}_{\mathrm{L}}$ with the point $\mathbf{u} = \mathbf{u}_{\mathrm{R}}$. The self-similar coordinate ξ monotonically increases along the path varying from $-\infty$ at $\mathbf{u} = \mathbf{u}_{\mathrm{L}}$ to $+\infty$ at $\mathbf{u} = \mathbf{u}_{\mathrm{R}}$. The path consists of continuous segments representing solutions of the ordinary differential equations (38.4.10.3) (rarefaction waves), line segments that connect two points $\mathbf{u}^-$ and $\mathbf{u}^+$ satisfying the Rankine–Hugoniot conditions (38.4.8.2) and evolutionary conditions (38.4.9.3), and rest points $\mathbf{u}(\xi) = \text{const}$.

Example 38.15. Consider a solution consisting of two shocks and one rarefaction. The structural formula* for the solution path is $\mathbf{u}_{\mathrm{L}} \to 1 \text{ — } 2 \to \mathbf{u}_{\mathrm{R}}$; specifically,

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_{\mathrm{L}} & \text{if } -\infty < x/t < D_1, \\ \mathbf{u}_1 & \text{if } D_1 < x/t < \lambda_2(\mathbf{u}_1), \\ \mathbf{u}^{(2)}(\xi) & \text{if } \lambda_2(\mathbf{u}_1) < x/t < \lambda_2(\mathbf{u}_2), \\ \mathbf{u}_2 & \text{if } \lambda_2(\mathbf{u}_2) < x/t < D_2, \\ \mathbf{u}_{\mathrm{R}} & \text{if } D_2 < x/t < \infty. \end{cases}$$

The shock speed D_1 (resp., D_2) can be found from the Hugoniot condition by setting $\mathbf{u}^- = \mathbf{u}_{\mathrm{L}}$ and $\mathbf{u}^+ = \mathbf{u}_1$ (resp., $\mathbf{u}^- = \mathbf{u}_2$ and $\mathbf{u}^+ = \mathbf{u}_{\mathrm{R}}$). Points 1 and 2 are located on the same rarefaction curve. The vector $\mathbf{u}^{(2)}(\xi)$ is a second-family rarefaction wave that is described by the system of ordinary differential equations (38.4.10.3) with $\xi = \lambda_2(\mathbf{u})$.

Figure 38.2*a* depicts a sequence of rarefactions and shocks in the plane (x,t). Figure 38.2*b* shows the profile of the solution component u_i along the x-axis. The self-similar curves $\mathbf{u} = \mathbf{u}(\xi)$ coincide with the

* In structural formulas like $\mathbf{u}_{\mathrm{L}} \to 1 \text{ — } 2 \to \mathbf{u}_{\mathrm{R}}$, the symbol "→" stands for a shock wave and "—" stands for a rarefaction.

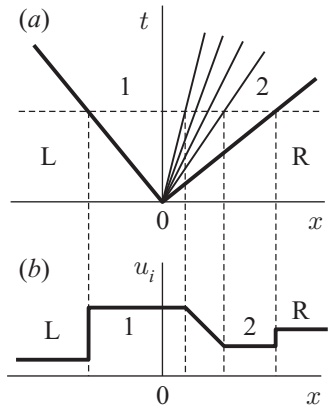

Figure 38.2. Solution for the Riemann problem: (a) centered waves in the (x, t) plane; (b) the u_i profile.

profiles $\mathbf{u}(x, t = 1)$. For $t > 1$, the graphs of $\mathbf{u}(x, t)$ are obtained from the self-similar curves by extending them along the axis x by a factor of t.

Example 38.16. Let us discuss an adiabatic gas flow in a tube in front of an impermeable piston moving with a velocity v_{L} (see Fig. 38.3). The initial state is defined by prescribing initial values of the velocity and density:

$$v = 0, \quad \rho = \rho_{\mathrm{R}} \quad \text{at} \quad \xi = \infty \qquad (\xi = x/t). \tag{38.4.10.5}$$

The piston is impermeable; therefore, the gas velocity in front of the shock is equal to the piston velocity (Fig. 38.4a):

$$v = v_{\mathrm{L}} \quad \text{at} \quad \xi = v_{\mathrm{L}}. \tag{38.4.10.6}$$

The gas density in front of the piston is unknown in this problem.

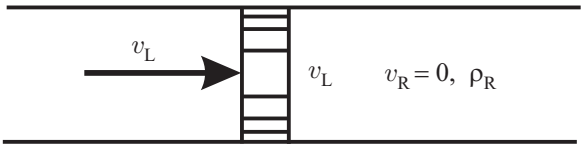

Figure 38.3. Constant velocity piston motion in a tube.

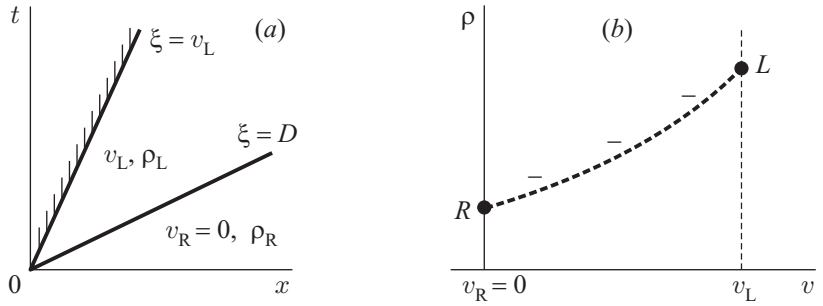

Figure 38.4. Solution of the piston problem: (a) shock wave in the (x, t) plane; (b) shock locus in the phase plane.

Figure 38.4b shows the locus of points that can be connected by a shock to the point $(v_{\mathrm{R}} = 0, \rho_{\mathrm{R}})$. This locus is a first-family shock. The intersection of the locus with the line $v = v_{\mathrm{L}}$ defines the value ρ_{L}. Hence, ρ_{L} can be found from the equation

$$v_{\mathrm{L}}^2 = \frac{[p(\rho_{\mathrm{L}}) - p(\rho_{\mathrm{R}})](\rho_{\mathrm{L}} - \rho_{\mathrm{R}})}{\rho_{\mathrm{L}}\rho_{\mathrm{R}}}, \tag{38.4.10.7}$$

which has been obtained by taking the square of equation (38.4.8.4). There exists a root $\rho_{\rm L}$ of (38.4.10.7) such that $\rho_{\rm L} > \rho_{\rm R}$. Hence, the gas is compressed ahead of the piston (Fig. 38.4*a*). The expression for the shock speed can be found from the first Hugoniot condition (38.4.8.3):

$$D = \frac{\rho_{\rm L} v_{\rm L}}{\rho_{\rm L} - \rho_{\rm R}} > v_{\rm L}. \tag{38.4.10.8}$$

The shock speed exceeds the piston velocity of (38.4.10.8) for $\rho_{\rm L} > \rho_{\rm R}$. Both characteristics of the first family as well as the characteristic ahead of the shock from the second family arrive at the shock, so that the Lax condition is satisfied. It can be proved that there are no other configurations that satisfy the initial-boundary conditions (38.4.10.5), (38.4.10.6).

⊙ *References for Section* 38.4: S. E. Buckley and M. C. Leverett (1942), O. A. Oleinik (1957), I. M. Gelfand (1959), C. Wachmann (1964), P. Lax (1973), T. P. Liu (1974), G. B. Whitham (1974), A. Jeffrey (1976), A. G. Kulikovskii (1979), F. John (1982), C. M. Dafermos (1983, 2000), B. L. Rozhdestvenskii and N. N. Yanenko (1983), J. Smoller (1983), R. Courant and D. Hilbert (1985), R. Courant and R. Friedrichs (1985), H. Rhee, R. Aris, and N. R. Amundson (1986, 1989), L. D. Landau and E. M. Lifshitz (1987), L. W. Lake (1989), P. G. Bedrikovetsky and M. L. Chumak (1992*a,b*), P. G. Bedrikovetsky (1993), D. J. Logan (1994), E. Godlewski and P.-A. Raviart (1996), D. Serre (1996), A. Bressan (2000), C. M. Dafermos (2000), A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov (2001), R. J. LeVeque (2002), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

38.5. Systems of Second-Order Equations of Reaction-Diffusion Type

38.5.1. Traveling-Wave Solutions and Some Other Invariant Solutions

In this subsection, we consider systems of nonlinear second-order equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= a_1 \frac{\partial^2 u}{\partial x^2} + F_1(u, w), \\ \frac{\partial w}{\partial t} &= a_2 \frac{\partial^2 w}{\partial x^2} + F_2(u, w), \end{aligned} \tag{38.5.1.1}$$

which often arise in the theory of mass exchange of reactive media, combustion theory, mathematical biology, and biophysics.

It is apparent that systems (38.5.1.1) admit exact traveling-wave solutions

$$u = u(z), \quad w = w(z), \quad z = kx - \lambda t,$$

where k and λ are arbitrary constants, and the functions $u(z)$ and $w(z)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} a_1 k^2 u''_{zz} + \lambda u'_z + F_1(u, w) &= 0, \\ a_2 k^2 w''_{zz} + \lambda w'_z + F_2(u, w) &= 0. \end{aligned}$$

Table 38.3 lists some systems of the form (38.5.1.1) admitting other invariant solutions that are found using combinations of translation and scaling transformations in the independent and dependent variables; these transformations preserve the form of the equations (i.e., a given system of equations is converted into an identical one). The procedure for constructing such solutions for a single equation is detailed in Chapter 27 and Section 28.4.

38.5.2. Generalized Separable Solutions

In some cases, nonlinear systems admit generalized separable solutions of the form

$$\begin{aligned} u &= \varphi_1(t)\theta(x, t) + \psi_1(t), \\ w &= \varphi_2(t)\theta(x, t) + \psi_2(t), \end{aligned} \tag{38.5.2.1}$$

where the functions $\varphi_1(t)$, $\varphi_2(t)$, $\psi_1(t)$, and $\psi_2(t)$ are selected so that both equations of system (38.5.1.1) are reduced to one and the same equation for $\theta(x, t)$.

TABLE 38.3
Systems of equations of the form (38.5.1.1) admitting invariant solutions that are found using combinations of translation and scaling transformations (C is an arbitrary constant)

Systems of equations	Invariant transformations	Form of invariant solutions
$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+e^{\lambda u}f(\lambda u-\sigma w),$ $\frac{\partial w}{\partial t}=b\frac{\partial^2 w}{\partial x^2}+e^{\sigma w}g(\lambda u-\sigma w)$	$t=C^{2\lambda\sigma}\bar{t},\ x=C^{\lambda\sigma}\bar{x},$ $u=\bar{u}-2\sigma\ln C,$ $w=\bar{w}-2\lambda\ln C$	$u=y(\xi)-\frac{1}{\lambda}\ln t,\ w=z(\xi)-\frac{1}{\sigma}\ln t,\ \xi=\frac{x}{\sqrt{t}}$
$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+u^{1+kn}f\left(u^n w^m\right),$ $\frac{\partial w}{\partial t}=b\frac{\partial^2 w}{\partial x^2}+w^{1-km}g\left(u^n w^m\right)$	$t=C^{-kmn}\bar{t},$ $x=C^{-\frac{1}{2}kmn}\bar{x},$ $u=C^m\bar{u},\ w=C^{-n}\bar{w}$	$u=t^{-\frac{1}{kn}}y(\xi),\ w=t^{\frac{1}{km}}z(\xi),\ \xi=\frac{x}{\sqrt{t}}$
$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+uf\left(u^n w^m\right),$ $\frac{\partial w}{\partial t}=b\frac{\partial^2 w}{\partial x^2}+wg\left(u^n w^m\right)$	$t=\bar{t}+\ln C,$ $x=\bar{x}+\lambda\ln C,$ $u=C^m\bar{u},\ w=C^{-n}\bar{w}$	$u=e^{mt}y(\xi),\ w=e^{-nt}z(\xi),\ \xi=x-\lambda t$

Example 38.17. Consider the system

$$\begin{aligned}\frac{\partial u}{\partial t}&=a\frac{\partial^2 u}{\partial x^2}+uf(bu-cw)+g_1(bu-cw),\\ \frac{\partial w}{\partial t}&=a\frac{\partial^2 w}{\partial x^2}+wf(bu-cw)+g_2(bu-cw),\end{aligned}\tag{38.5.2.2}$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions.

Exact solutions are sought in the form (38.5.2.1). Let us require that the arguments of the functions appearing in the system must depend on time t only, so that

$$\frac{\partial}{\partial x}(bu-cw)=[b\varphi_1(t)-c\varphi_2(t)]\frac{\partial}{\partial x}\theta(x,t)=0.$$

It follows that

$$\varphi_1(t)=c\varphi(t),\qquad \varphi_2(t)=b\varphi(t).\tag{38.5.2.3}$$

Substituting (38.5.2.1), in view of (38.5.2.3), into (38.5.2.2) after elementary rearrangements, we obtain

$$\begin{aligned}\frac{\partial\theta}{\partial t}&=a\frac{\partial^2\theta}{\partial x^2}+\left[f(b\psi_1-c\psi_2)-\frac{\varphi'}{\varphi}\right]\theta+\frac{1}{c\varphi}\left[\psi_1 f(b\psi_1-c\psi_2)+g_1(b\psi_1-c\psi_2)-\psi_1'\right],\\ \frac{\partial\theta}{\partial t}&=a\frac{\partial^2\theta}{\partial x^2}+\left[f(b\psi_1-c\psi_2)-\frac{\varphi'}{\varphi}\right]\theta+\frac{1}{b\varphi}\left[\psi_2 f(b\psi_1-c\psi_2)+g_2(b\psi_1-c\psi_2)-\psi_2'\right].\end{aligned}\tag{38.5.2.4}$$

For the equations of (38.5.2.4) to coincide, let us equate the expressions in square brackets to zero. As a result, we obtain an autonomous system of ordinary differential equations for $\varphi=\varphi(t)$, $\psi_1=\psi_1(t)$, and $\psi_2=\psi_2(t)$:

$$\begin{aligned}\varphi'&=\varphi f(b\psi_1-c\psi_2),\\ \psi_1'&=\psi_1 f(b\psi_1-c\psi_2)+g_1(b\psi_1-c\psi_2),\\ \psi_2'&=\psi_2 f(b\psi_1-c\psi_2)+g_2(b\psi_1-c\psi_2).\end{aligned}\tag{38.5.2.5}$$

From (38.5.2.4)–(38.5.2.5) it follows that $\theta=\theta(x,t)$ satisfies the linear heat equation

$$\frac{\partial\theta}{\partial t}=a\frac{\partial^2\theta}{\partial x^2}.\tag{38.5.2.6}$$

From the first equation in (38.5.2.5) we can express the function φ via the other two:

$$\varphi=C_1\exp\left(\int f(b\psi_1-c\psi_2)\,dt\right).\tag{38.5.2.7}$$

(The constant of integration C_1 can be set equal to 1, since φ and θ appear in the solution in the form of a product.) As a result, we obtain an exact solution of system (38.5.2.1) in the form

$$\begin{aligned}u&=\psi_1(t)+c\exp\left[\int f(b\psi_1-c\psi_2)\,dt\right]\theta(x,t),\\ w&=\psi_2(t)+b\exp\left[\int f(b\psi_1-c\psi_2)\,dt\right]\theta(x,t),\end{aligned}\tag{38.5.2.8}$$

where the functions $\psi_1 = \psi_1(t)$ and $\psi_2 = \psi_2(t)$ are described by the last two equations of system (38.5.2.5), and the function $\theta(x, t)$ is a solution of the linear heat equation (38.5.2.6).

Remark. Now we show how the functions $\varphi(t)$ and $\psi(t)$, appearing in (38.5.2.8), can be found. Multiply the second equation (38.5.2.5) by b and the third by $-c$ and add up to obtain a separable first-order ordinary differential equation for the linear combination of the unknowns:

$$z'_t = zf(z) + bg_1(z) - cg_2(z), \qquad z = b\psi_1 - c\psi_2. \tag{38.5.2.9}$$

Its general solution is expressed in implicit form as

$$\int \frac{dz}{zf(z) + bg_1(z) - cg_2(z)} = t + C_2. \tag{38.5.2.10}$$

For given functions $f(z)$, $g(z)$, and $h(z)$, from (38.5.2.10) we find $z = z(t)$. Substituting it into the second equation (38.5.2.5), we get a linear equation for ψ_1. Solving this equation and taking into account the relation for z, ψ_1, and ψ_2 in (38.5.2.9), we obtain

$$\psi_1 = C_3F(t) + F(t)\int \frac{g_1(z)}{F(t)}\,dt, \quad \psi_2 = \frac{b}{c}\psi_1 - \frac{1}{c}z, \quad F(t) = \exp\left[\int f(z)\,dt\right], \quad z = z(t).$$

Example 38.18. Consider now the system

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right).$$

An exact solution is sought in the form (38.5.2.1) with $\varphi_2(t) = 1$ and $\psi_1(t) = \psi_2(t) = 0$. After simple rearrangements we get

$$u = \varphi(t)\exp\left[\int g(\varphi(t))\,dt\right]\theta(x, t), \quad w = \exp\left[\int g(\varphi(t))\,dt\right]\theta(x, t),$$

where the function $\varphi = \varphi(t)$ is described by the separable first-order nonlinear ordinary differential equation

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \tag{38.5.2.11}$$

and the function $\theta = \theta(x, t)$ satisfies the linear heat equation (38.5.2.6).

The general solution of equation (38.5.2.11) is written out in implicit form as

$$\int \frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

▷ *Exact solutions to some other second-order nonlinear systems of the form (38.5.1.1) can be found in Section 17.1.*

⊙ *References for Section* 38.5: A. D. Polyanin and V. F. Zaitsev (2004), A. D. Polyanin and A. V. Manzhirov (2007).

Part III

Symbolic and Numerical Solutions of Nonlinear PDEs with Maple, Mathematica, and MATLAB

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Chapter 39
Nonlinear Partial Differential Equations with Maple

39.1. Introduction

Although there exist various analytic solution methods for special nonlinear PDEs, there is no central theory for nonlinear PDEs in the general case. Moreover, the functions and data in nonlinear PDE problems are frequently defined at discrete points. Therefore, we have to study numerical approximation methods for nonlinear PDEs.

Many of the most interesting phenomena (e.g., singularity, stability, chaos, and solitons) are related to nonlinear PDEs [e.g., see Ablowitz and Clarkson (1991), Logan (1994), Taylor (1996), Debnath (2005), Galaktionov and Svirshchevskii (2006), and Kudryashov (2010d)]. However, the understanding of these phenomena required such technology that was either not available or limited in the pre-modern era. In modern mathematics, there exist computers, supercomputers, computer algebra systems (such as Maple and Mathematica), and an interactive programming environment for scientific computing, MATLAB, that can perform various mathematical operations of which humans are incapable and where symbolic and numerical computations play a central role in scientific progress [see Akritas (1989), Calmet and van Hulzen (1983), Davenport, Siret, and Tournier (1993), and Wester (1999)].

In this chapter, following the most important ideas and methods, we propose and develop new computer algebra ideas and methods to obtain analytical, symbolic, numerical, and graphical solutions for studying nonlinear partial differential equations. We compute analytical and numerical solutions in terms of predefined functions (which are an implementation of known methods for solving PDEs) and develop new procedures for constructing new solutions using Maple. We show the very helpful role that computer algebra systems play in the analytical derivation of numerical methods, calculation of numerical solutions, and comparison of numerical and asymptotic solutions.

⊙ *References for Section* 39.1: J. Calmet and J. A. van Hulzen (1983), A. G. Akritas (1989), J. H. Davenport, Y. Siret, and E. Tournier (1993), M. J. Wester (1999), L. Debnath (2005).

39.2. Brief Introduction to Maple

Maple is a general-purpose computer algebra system (CAS) in which symbolic computation can readily be combined with exact and approximate (floating-point) numerical computation as well as with arbitrary-precision numerical computation. Maple provides powerful scientific graphics capabilities [for details, see Kreyszig (2000), Corless (1995), Heck (2003), Richards (2002), Abel and Braselton (2005), Meade et al. (2009), Shingareva and Lizárraga-Celaya (2009), etc.].

The first concept of Maple and its initial versions were developed by the Symbolic Computation Group at the University of Waterloo in the early 1980s. The Maplesoft company was created in 1988. Maple was mainly developed in research labs at the University of Waterloo and University of Western Ontario [see Char et al. (1990), Geddes, Czapor, and Labahn (1992)], with important contributions from worldwide research groups at other universities.

The most important features of Maple are as follows: fast symbolic and numerical computation and interactive visualization; easiness to use; easiness to incorporate new user-defined capabilities; understandable, open-source software development path; availability for almost all operating systems; powerful programming language, intuitive syntax, and easy debugging; extensive library of mathematical functions and specialized packages; free resources and the collaborative character of development (e.g., see Maple website `www.maplesoft.com` (`MWS`), Maple Application Center `MWS/applications`, Teacher Resource Center `MWS/TeacherResource`, Student Help Center `MWS/studentcenter`, and Maple Community `MWS/community`).

Maple consists of three parts: the interface, the kernel (basic computational engine), and the library. The interface and the kernel (written in the C programming language) form a smaller part of the system (they are loaded when a Maple session is started). The interface handles the input of mathematical expressions, output display, plotting of functions, and support of other user communication with the system. The interface medium is the Maple worksheet. The kernel interprets the user input, carries out the basic algebraic operations, and deals with storage management. The library consists of two parts: the main library and a collection of packages. The main library (written in the Maple language) includes many functions in which most of the common mathematical knowledge of Maple resides.

Basic concepts. The *prompt symbol* $(>)$ serves for typing a Maple function; the semicolon $(;)$ or colon $(:)$ symbols* with pressing `Enter` at the end of the function serve for evaluating the Maple function, displaying the result, and inserting a new prompt.

Maple contains a complete *online help system* and a command line help system, which can be used, e.g., by typing `?NameOfFunction` or `help(NameOfFunction);` or `?help`; reference information can also be accessed by using the `Help` menu, by highlighting a function and then pressing Ctrl+F1 or F1 or F2 (for Ver. ≥ 9), and by pressing Ctrl+F2.

Maple worksheets are files that keep track of the working process and organize it as a collection of expandable groups (see `?worksheet`, `?shortcut`). It is best to begin a new worksheet (or a new problem) with the statement `restart` for cleaning Maple's memory. All examples and problems in the book are assumed to begin with `restart`.

Previous results (during a session) can be referred to with symbols `%` (the last result), `%%` (the penultimate result), `%%...%` (k times) (the kth preceding result).

Comments can be included with the sharp sign `#` and all characters following it up to the right end of a line. Also the text can be inserted with `Insert` $\rightarrow$ `Text`.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed the function incorrectly. Do correct the function or interrupt the computation (the stop button in the Tool Bar menu).

*In earlier versions of Maple and in *Classic Worksheet Maple*, we have to end a function with a colon or semicolon. In these chapters, we follow this tradition in every example and problem.

Maple source code can be viewed for most of the functions, general and specialized (package functions); e.g., `interface(verboseproc=2); print(factor);`

Palettes can be used for building or editing mathematical expressions without needing to remember the Maple syntax.

The Maplet User Interface (for Ver. ≥ 8) consists of *Maplet applications* that are collections of windows, dialogs, and actions (see `?Maplets`).

A number of specialized functions are available in various specialized *packages (subpackages)* (see `?index[package]`, `with`).

Numerical approximations: numerical approximation of `expr` to 10 significant digits, `evalf(expr)`; global change of precision `Digits:=n` (see `?environment`); local change of the precision, `evalf(expr,n)`; numerical approximation of `expr` using a binary hardware floating-point system, `evalhf(expr)`; performing numerical approximations using the hardware or software floating-point systems, `UseHardwareFloats:=value` (for details, see `?UseHardwareFloats`, `?environment`).

39.2.1. Maple Language

Maple language is a high-level programming language, which is well-structured and comprehensible. It supports a large collection of data structures, or Maple objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). The Maple procedures in the library are available in readable form. The library can be supplemented with locally developed user programs and packages.

Arithmetic operators: `+ - * / ^`.

Logic operators: `and`, `or`, `xor`, `implies`, `not`.

Relation operators: `<`, `<=`, `>`, `>=`, `=`, `<>`.

A variable name is a combination of letters, digits, or the underline symbol (_), starting from a letter; e.g., `a12_new`.

Abbreviations for the longer Maple functions or any expressions: `alias`, for example, `alias(H=Heaviside); diff(H(t),t);` to remove this abbreviation, type `alias(H=H);`

Maple is case sensitive; i.e., there is a difference between lowercase and uppercase letters; e.g., `evalf(Pi)` and `evalf(pi)` are different commands.

Various reserved keywords, symbols, names, and functions; these words cannot be used as variable names, e.g., operator keywords, additional language keywords, global names that start with (_) (see `?reserved`, `?ininames`, `?inifncs`, `?names`).

The assignment/unassignment operators: a variable can be "free" (with no assigned value) or can be assigned any value (symbolic or numeric) by the assignment operators `a:=b` or `assign(a=b)`. To unassign (clear) an assigned variable (see `?:=` and `?'`), type, e.g., `x:='x'`, `evaln(x)`, or `unassign('x')`.

The difference between the *operators* `:=` and `=` is as follows: the operator `A:=B` is used to assign `B` to the variable `A`, and the operator `A=B` is used to indicate equality (not assignment) between the left- and the right-hand sides (see `?rhs`), for example, `Equation:=A=B; Equation; rhs(Equation); lhs(Equation);`

The range operator (`..`), an expression of type range `expr1..expr2`, for example, `a[i]$ i=1..9; plot(sin(x),x=-Pi..Pi);`

Statements are input instructions from the keyboard that are executed by Maple (e.g., `break`, `by`, `do`, `end`, `for`, `function`, `if`, `proc`, `restart`, `return`, `save`, `while`).

The statement separators semicolon (`;`) and colon (`:`). The result of a statement followed with a semicolon (`;`) will be displayed, and it will not be displayed if it is followed by a colon (`:`), e.g., `plot(sin(x),x=0..Pi); plot(sin(x),x=0..Pi):`

An expression is a valid statement and is formed as a combination of constants, variables, operators and functions. Every expression is represented as a tree structure in which each node (and leaf) has a particular data type. For the analysis of any node and branch, the functions `type`, `whattype`, `nops`, `op` can be used. *A boolean expression* is formed with *logical operators* and relation operators.

An equation is represented using the binary operator `(=)` and has two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

Inequalities are represented using the relation operators and have two operands, the left-hand side, `lhs`, and the right-hand side, `rhs`.

A string is a sequence of characters having no value other than itself; it cannot be assigned to, and will always evaluate to itself. For instance, `x:="string";` and `sqrt(x);` is an invalid function. Names and strings can be used with the `convert` and `printf` functions.

Maple is sensitive to types of brackets and quotes.

Types of brackets: parentheses for grouping expressions, `(x+9)*3`, for delimiting the arguments of functions, `sin(x)`; square brackets for constructing lists, `[a,b,c]`, vectors, matrices, arrays; curly brackets for constructing sets, `{a,b,c}`.

Types of quotes: forward-quotes to delay the evaluation of expressions, `'x+9+1'`, to clear variables, `x:='x';` back-quotes to form a symbol or a name, `` `a name:=7`; k:=5; print(`the value of k is`,k); `` double quotes to create strings, and a single double quote `"`, to delimit strings.

Types of numbers: integer, rational, real, complex, root, e.g., `-55`, `5/6`, `3.4`, `-2.3e4`, `Float(23,-45)`, `3-4*I, Complex(2/3,3);` `RootOf(_Z^3-2,index=1);`

Predefined constants: symbols for commonly used mathematical constants, `gamma`, `Pi`, `I`, `true`, `false`, `infinity`, `FAIL`, `exp(1)` (for details, see `?ininames`, `?constants`).

Functions or function expressions have the form `f(x)` or `expr(args)` and represent a function call, or an application of a function (or procedure) to arguments (`args`). *Active functions* (beginning with a lowercase letter) are used for computing, e.g., `diff`, `int`, `limit`. *Inert functions* (beginning with a capital letter) are used for showing steps in the problem-solving process; e.g., `Diff`, `Int`, `Limit`.

Library functions (or predefined functions) and user-defined functions.

Predefined functions: most of the well-known functions are predefined by Maple and known to some Maple functions (e.g., `diff`, `evalc`, `evalf`, `expand`, `series`, `simplify`, etc.). Numerous special functions are defined (for details, see `?FunctionAdvisor`).

User-defined functions: the functional operator (`->`) (see `?->`); e.g., the function $f(x) = \sin x$ is defined as `f:=x->sin(x);`

Alternative definitions of functions: `unapply` converts an expression to a function, and a procedure is defined with `proc`.

Evaluation of function $f(x)$ at $x = a$, $\{x = a, y = b\}$, e.g., `f(a); subs(x=a, f(x));` `eval(f(x),x=a);`

In the Maple language, there are two forms of modularity: *procedures* and *modules*. *A procedure* (see `?procedure`) is a block of statements which one needs to use repeatedly. A procedure can be used to define a function (if the function is too complicated to be written with the use of the arrow operator) or to create a matrix, graph, or logical value. *A module* (see `?module`) is a generalization of the procedure concept. While a procedure groups a sequence of statements into a single statement (block of statements), a module groups related functions and data.

In the Maple language, there are essentially *two control structures*: the selection structure `if` and the repetition structure `for`.

Maple objects, sequences, lists, sets, tables, arrays, vectors, and matrices are used for representing more complicated data. *Sequences* `a1,a2,a3`, *lists* `[a1,a2,a3]`, and *sets* $\{a1,a2,a3\}$ are groups of expressions. Maple preserves the order and repetitions in sequences and lists and does not preserve them in sets. The order in sets can change during a Maple session. A *table* is a group of expressions represented in tabular form. Each entry has an index (an integer or any arbitrary expression) and a value (see `?table`). An *array* is a table with integer range of indices (see `?Array`). In Maple, arrays can be of any dimension (depending on computer memory). A *vector* is a one-dimensional array with positive integer range of indices (see `?vector`, `?Vector`). A *matrix* is a two-dimensional array with positive integer range of indices (see `?matrix`, `?Matrix`).

⊙ *References for Section* 39.2: B. W. Char, K. O. Geddes, G. H. Gonnet, M. B. Monagan, and S. M. Watt (1990), K. O. Geddes, S. R. Czapor, and G. Labahn (1992), R. M. Corless (1995), E. Kreyszig (2000), D. Richards (2002), A. Heck (2003), M. L. Abel and J. P. Braselton (2005), D. B. Meade, S. J. M. May, C-K. Cheung, and G. E. Keough (2009), I. K. Shingareva and C. Lizárraga-Celaya (2009).

39.3. Analytical Solutions and Their Visualizations

39.3.1. Constructing Analytical Solutions in Terms of Predefined Functions

The computer algebra system Maple has various predefined functions based on symbolic algorithms for constructing analytical solutions of linear and nonlinear PDEs [see a more detailed description in Cheb-Terrab and von Bulow (1995)]. Although the predefined functions are an implementation of known methods for solving PDEs, it allows us to solve nonlinear equations and obtain solutions automatically (in terms of predefined functions) as well as develop new methods and procedures for constructing new solutions.

Consider the most important functions for finding all possible analytical solutions of a given PDE problem.

```
pdsolve(PDE);          pdsolve(PDE,build);        pdetest(sol,PDEs);
pdsolve(PDE,funcs,HINT=val,INTEGRATE,build,singsol=val);
infolevel[procname]:=val;  pdsolve(PDEsys);           with(PDEtools);
declare(funcs);        dchange(rules,PDE);        diff_table(funcs);
separability(PDE,func);    casesplit(PDEs);      Infinitesimals(PDE);
SimilaritySolutions(PDE,ops);      F:=TWSolutions(functions_allowed);
TWSolutions(PDE,parameters=val,singsol=false,functions=F);
TWSolutions(PDE,output=ODE);              TWSolutions(PDE,extended);
```

Comment. `HINT=val` gives some hints; with `build` one can construct an explicit expression for the indeterminate function `func`; with `'+'` or `'*'` one can construct a solution by separation of variables (in the form of sum or product, respectively); with `'TWS'` or `'TWS(MathFuncName)'` one can construct a traveling-wave solution as a power series in $\tanh(\xi)$ or several mathematical functions (including special functions), where ξ represents a linear combination of the independent variables, etc.

`pdsolve`, finding analytical solutions for a given partial differential equation `PDE` and for systems of `PDEs`,

`PDEtools`, a collection of functions for finding analytical solutions for `PDEs`, e.g.,

`declare`, declaring functions and derivatives on the screen for a simple, compact display.

`casesplit`, splitting into cases and successively decouple a system of differential equations.

`separability`, determining under what conditions it is possible to obtain a complete solution by separation of variables.

`SimilaritySolutions`, determining group-invariant (point-symmetry) solutions for given PDEs and systems.

`TWSolutions`, constructing traveling-wave solutions for autonomous PDEs and systems of them, etc.

Notation. In this chapter, we use the following conventions introduced in Maple: `_Cn` (`n=1,2,...`), for arbitrary constants; `_Fn`, for arbitrary functions; `_c[n]`, for arbitrary constants while separating the variables; `_s`, for a parameter in the characteristic system; `&where`, for a solution structure, `_ε`, for a Lie group parameter. Also we introduce the following notation for the Maple solutions: `Eqn`, for equations; `PDEn/ODEn`, for PDEs/ODEs; `trn`, for transformations; `sysn`, for systems; `IC`, `BC`, `IBC`, for initial or/and boundary conditions; `Ln`, for lists of expressions; `Gn`, for graphs of solutions.

Let us assume that we have obtained exact solutions and we wish to verify whether these solutions are exact solutions of given nonlinear PDEs.

Example 39.1. For example, for the modified Liouville equation 7.2.1.1

$$u_{tt} = a^2 u_{xx} + be^{\beta u},$$

we verify that the traveling-wave solution and the functional separable solution,

$$u(x,t)=\frac{1}{\beta}\ln\left(\frac{2(B^2-a^2A^2)}{b\beta(Ax+Bt+C)^2}\right), \quad u(x,t)=\frac{1}{\beta}\ln\left(\frac{8a^2C}{b\beta}\right)-\frac{2}{\beta}\ln|(x+A)^2-a^2(t+B)^2+C|,$$

are exact solutions of the modified Liouville equation as follows:

```
with(PDEtools); declare(u(x,t)); U:=diff_table(u(x,t));
PDE1:=U[t,t]=a^2*U[x,x]+b*exp(beta*U[]);
Sol11:=u(x,t)=1/beta*ln(2*(B^2-a^2*A^2)/(b*beta*(A*x+B*t+C)^2));
Sol12:=S->u(x,t)=1/beta*ln(8*a^2*C/(b*beta))
  -2/beta*ln(S*(x+A)^2-S*a^2*(t+B)^2+S*C);
Test11:=pdetest(Sol11,PDE1);
Test12:=pdetest(Sol12(1),PDE1); Test13:=pdetest(Sol12(-1),PDE1);
```

where a, b, β, A, B, C are arbitrary real constants.

Example 39.2. For the nonlinear Schrödinger equation 5.7.1.1

$$iu_t + u_{xx} + k|u|^2 u = 0,$$

we verify that the solutions

$$u(x,t) = C_1 \exp\left[i(C_2 x + (kC_1^2 - C_2^2)t + C_3)\right], \quad u(x,t) = A\sqrt{\frac{2}{k}}\frac{\exp\left[iBx + i(A^2 - B^2)t + iC_1\right]}{\cosh(Ax - 2ABt + C_2)}$$

are exact solutions of this nonlinear equation as follows:

```
with(PDEtools);  declare(u(x,t)); U:=diff_table(u(x,t));
interface(showassumed=0); assume(k>0);
PDE1:=I*U[t]+U[x,x]+k*abs(U[])^2*U[]=0;
Sol1:=u(x,t)=C1*exp(I*(C2*x+(k*C1^2-C2^2)*t+C3));
Test1:=pdetest(Sol1,PDE1); Test11:=simplify(evalc(Test1));
Sol2:=u(x,t)=A*sqrt(2/k)*(exp(I*B*x+I*(A^2-B^2)*t+I*C1))/(cosh(A*x-2*A*B*t+C2));
Test2:=pdetest(Sol2,PDE1); Test21:=simplify(evalc(Test2));
```

Here A, B, C_1, C_2, and C_3 are arbitrary real constants, and the second solution is valid for $k > 0$.

Example 39.3. The general solution of the first-order nonlinear PDE

$$xu_y - u_x = g(x)u^2$$

can be found and tested as follows:

```
with(PDEtools); declare(u(x,y));
PDE1:=x*diff(u(x,y),y)-diff(u(x,y),x)=g(x)*u(x,y)^2;
Sol1:=pdsolve(PDE1); pdetest(Sol1,PDE1);
```

where the Maple result reads:

$$Sol1 := u = \frac{1}{\int g(x)\,dx + _F1\left(\frac{1}{2}x^2 + y\right)}$$

Example 39.4. The first-order nonlinear PDE

$$uu_y = u_x$$

can be solved by the method of characteristics as follows:

```
with(PDEtools);  declare(u(x,y));
PDE1:=u(x,y)*diff(u(x,y),y)=diff(u(x,y),x);
sysCh:=charstrip(PDE1,u(x,y)); funcs:=indets(sysCh,Function);
Sol1:=dsolve(sysCh,funcs,explicit);
```

where the Maple result reads:

$$sysCh := \{u_s = 0, x_s = -1, y_s = u(_s)\}, \quad funcs := \{u(_s), x(_s), y(_s)\}$$

$$Sol1 := \{u(_s) = _C2, x(_s) = -_s + _C3, y(_s) = _C2\,_s + _C1\}$$

Comment. For the given equation (`PDE1`), we first obtain the characteristic system (depending on the parameter `_s`) via `charstrip` and then solve this system via `dsolve`.

Example 39.5. For the *Burgers equation*,

$$u_t + uu_x = \nu u_{xx},$$

which describes wave processes in various media (e.g., diffusive waves in fluid dynamics, sound waves in viscous fluid, waves in liquid-filled viscoelastic tubes, etc.), we can construct various forms of analytical solutions of the Burgers equation; e.g.,

```
with(PDEtools); declare(u(x,t)); infolevel[pdsolve]:=5;
PDE1:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=nu*diff(u(x,t),x$2);
casesplit(PDE1); separability(PDE1,u(x,t));
separability(PDE1,u(x,t),`*`); Sol1:=pdsolve(PDE1,HINT='TWS');
Sol2:=pdsolve(PDE1,HINT='TWS(tan)');
for i from 1 to 2 do Test||i:=pdetest(Sol||i,PDE1) od;
```

with Maple results

$$Sol1 := u = -\frac{_C3}{_C2} - 2\nu\,_C2\tanh(_C1 + _C2\,x + _C3\,t)$$

$$Sol2 := u = -\frac{_C3}{_C2} + 2\nu\,_C2\tan(_C1 + _C2\,x + _C3\,t)$$

Comment. First, we can split the nonlinear problem into all related regular cases (the general case and all possible singular cases). In our situation, we have the general case. Then we check the separability conditions (additive, by default, and multiplicative): if there is a separable solution, the result is 0. For our problem, we have the expression $-\nu u_t(u_t + uu_x)$ that must vanish for the PDE to become separable. Finally, we construct exact solutions: traveling-wave solutions with `HINT='TWS'`, `HINT='TWS(tan)'`. If we try to construct separable solutions with the functional `HINT`, i.e., `HINT=f(x)*g(t)`, then Maple cannot find the solution of this form (the degenerate solution $u = (x+C)/t$), and, according to the general strategy [see Cheb-Terrab and von Bulow (1995)], applies another method and gives the result that coincides to one of the known solutions, i.e., the traveling-wave solution (`Sol1`).

Example 39.6. By applying Maple predefined functions to the nonlinear wave equation of the form

$$u_{tt} = a(e^{\lambda u}u_x)_x,$$

we can find additive separable and self-similar solutions and verify that these solutions are exact solutions:

```
with(PDEtools); declare(u(x,t));
PDE1:=diff(u(x,t),t$2)=a*(diff(exp(lambda*u(x,t))*diff(u(x,t),x),x));
Sol1:=pdsolve(PDE1); Sol2:=pdsolve(PDE1,build);
Sol3:=pdsolve(PDE1,HINT=`+`,build); Sol4:=[SimilaritySolutions(PDE1)];
for i from 1 to 4 do pdetest(Sol||i,PDE1); od;
```

with Maple results

$$Sol1 := (u = _F1(x) + _F2(t))\ \&\mathrm{where}\left[\left\{_F1_{x,x} = e^{-\lambda\,_F1(x)}_c_1 - \lambda\,_F1_x{}^2, _F2_{t,t} = a e^{\lambda\,_F2(t)}_c_1\right\}\right]$$

$$Sol2 := u = -2\frac{\ln(2)}{\lambda} + \frac{\ln\left(\lambda\left(_c_1 x^2 - 2\,_C1\,x + 2\,_C2\right)\right)}{\lambda} + \ln\left(\frac{_C3}{a\,_c_1\left(\cos\left(1/2\sqrt{_C3\,\lambda}\,(t+_C4)\right)\right)^2}\right)\lambda^{-1}$$

$$Sol4 := \left[u = _C1\,t + _C2, u = \frac{\ln(-_C1\,\lambda\,(x+_C2))}{\lambda}, u = \left(-2\ln(t) + \ln\left(\frac{-\lambda\,(_C1\,x - _C2)\,a + x^2}{a}\right)\right)\lambda^{-1},\right.$$
$$\left. u = \left(2\ln(x) - 2\ln(2) + \ln\left(a^{-1}_C1^{-2}\left(\cos\left(1/2\,\frac{t+_C2}{_C1}\right)\right)^{-2}\right)\right)\lambda^{-1}\right]$$

Comment. We start from the function `pdsolve(PDE1)` (without options) and obtain the structure of the exact solution (additive separable solution) `Sol1`, for which we build an explicit expression in `Sol2`. Then we find the additive separable solution `Sol3` via the option `HINT`. The solutions `Sol2` and `Sol3` are equivalent. And then we construct similarity solutions in `Sol4`.

Example 39.7. *Kolmogorov–Petrovskii–Piskunov equation (KPP) and KPP-type equation.* By applying Maple predefined functions to the KPP-type equations

$$u_t = u_{xx} + au + bu^2,$$
$$u_t = u_{xx} + au + bu^2 + cu^3$$

(where a, b, and c are arbitrary parameters), we can find traveling-wave solutions and verify that these solutions are exact solutions of the given PDEs as follows:

```
with(PDEtools); declare(u(x,t));
K1:=a*u(x,t); K2:=K1+b*u(x,t)^2; K3:=K2+c*u(x,t)^3;
Eq1:=diff(u(x,t),t)=diff(u(x,t),x$2)+K2;
Eq2:=diff(u(x,t),t)=diff(u(x,t),x$2)+K3;
for i from 1 to 2 do Sol||i:=pdsolve(Eq||i,build);
pdetest(Sol||i,Eq||i); od;
```

with Maple results

$$Sol1 := u = -\frac{3}{4}\frac{a}{b} - \frac{1}{2}a\tanh\left(-_C1 - \frac{1}{12}\sqrt{-6a}x + \frac{5}{12}at\right)b^{-1} + \frac{1}{4}a\left(\tanh\left(-_C1 - \frac{1}{12}\sqrt{-6a}x + \frac{5}{12}at\right)\right)^2 b^{-1}$$

$$Sol2 := u = -\frac{1}{2}\frac{b}{c} + \frac{1}{2}\sqrt{-4ac+b^2}\tanh\left(-_C1 - \frac{1}{4}\frac{\sqrt{2}\sqrt{c(4ac-b^2)}x}{c} + \frac{1}{4}\frac{\sqrt{-4ac+b^2}bt}{c}\right)c^{-1}$$

Comment. For more general forms of the equations considered above, i.e., $u_t = u_{xx} + au + bu^m$ (where m is an arbitrary parameter, $m \neq 1, 2, 3$) and $u_t = u_{xx} + au + bu^m + cu^{2m-1}$ (where m is an arbitrary parameter, $m \neq 1, 2$), we cannot find exact solutions with the aid of the predefined function `pdsolve` and various options (for Maple Ver. $\leq$ 14). However, such solutions are known (see equations 5.1.3.2 and 5.1.3.3).

By applying the function `TWSolutions` (with various options) of the package `PDEtools`, one can construct various types of traveling-wave solutions in Maple; e.g., for the nonlinear equation

$$u_t = u_{xx} + au + bu^2 + cu^3,$$

we have

```
infolevel[TWSolutions]:=2;  Sol20:=pdsolve(Eq2,HINT='TWS');
Fs:=TWSolutions(functions_allowed);
Sol21:=TWSolutions(Eq2,singsol=false,functions=Fs): N:=nops([Sol21]);
for i from 1 to N do op(i,[Sol21]) od;
Sol22:=TWSolutions(Eq2,singsol=false,parameters=[a,b,c],
       functions=[coth],remove_redundant=true);
Sol23:=TWSolutions(Eq2,function=identity,output=ODE);
```

with Maple results

$$Sol22 := \left\{a = -2b_C4 + 8_C2^2, b = b, c = -2\frac{_C2^2}{_C4^2}, u = _C4 - _C4\coth\left(-_C1 - _C2x - \left(-b_C4 + 6_C2^2\right)t\right)\right\}, \ \ldots$$

$$Sol23 := \left[\left\{_C2\,u_\tau - _C1^2 u_{\tau,\tau} - au(\tau) - b(u(\tau))^2 - c(u(\tau))^3\right\}, \{\tau = _C3 + _C1x + _C2t, u(\tau) = u\}\right]$$

Comment. By adding the function `infolevel` with values (0–5), we can obtain increasingly detailed information on each of the steps of the solving process. `Sol20` is equivalent to `pdsolve(Eq2,build)`.

Then, with the aid of the predefined function `TWSolutions`, we can find various types of traveling-wave solutions (of nonlinear PDEs and systems) according to a list of functions `Fs` (instead of the `tanh` function, by default). The algorithm is based on the tanh-function method and its various extensions (see Section 39.3.4). Including the options `parameters` (splitting into cases with respect to the parameters), `singsol` (avoiding singular solutions), `functions` (expanding in series of different functions), `remove_redundant=true` (removing all redundant solutions, i.e., all particular cases of more general solutions being constructed), we can obtain numerous traveling-wave solutions (`Sol21`). For example, by specifying the `coth` function and other options, we can obtain various traveling-wave solutions. (We present just one solution in `Sol22`.)

Sometimes it is useful to obtain a reduction (without computing a solution) that converts a nonlinear PDE into an ODE form and the corresponding ODE.

By default, the predefined function `TWSolutions` generates solutions of the form $f_i(\tau)=\sum_{k=0}^{n_i} A_{ik}\tau^k$, where the n_i are finite, the A_{ik} are constants with respect to the x_j, and $\tau=\tanh(\sum_{i=0}^{j} C_i x_i)$. In `Sol23`, we obtain an ODE by using a simpler form, i.e., by choosing the identity as the function (the option `function=identity`), where τ takes the form $\tau=\sum_{i=0}^{j} C_i x_i$. In our case, we have $u(x,t)=u(\tau)$, where $\tau=C_1x+C_2t+C_3$, and the ODE acquires the form $-C_1^2u_{\tau\tau}+C_2u_\tau-au-bu^2-cu^3=0$ (`Sol23`). It should be noted that there is an equal sign missing in the ODE of the displayed solution `Sol23`; this can lead to a misunderstanding on the part of the reader.

Exact solutions of various types of nonlinear PDEs. Finally, by applying Maple predefined functions, we find exact solutions (general, traveling-wave, separable, similarity, and symmetry) and characteristic systems of various types of nonlinear PDEs (a and b are arbitrary parameters). The results are presented in Table 39.1.

TABLE 39.1

Maple predefined functions and exact solutions of various types of nonlinear PDEs

No.	Nonlinear PDE	Exact solution	Maple function
1	$u_x^2u_y=1$	$\{u(s)=3s+C_5, x(s)=2C_3C_4s+C_1, y(s)={C_3}^2s+C_2, p_1(s)=C_3, p_2(s)=C_4\}$	`pdsolve(Eq1,HINT=strip)`
2	$u_x^2+u_y^2=1$	$u=-\sqrt{y^2+x^2}+_C1,\ u=\sqrt{y^2+x^2}+_C1$	`SimilaritySolutions(Eq2,S[5])`
3	$u_t=(uu_x)_x$	$(u=_F1(x)\,_F2(t))$ `&where` $\left[\left\{_F1_{x,x}=_c_1-\frac{{_F1_x}^2}{_F1(x)}, _F2_t=(_F2(t))^2\,_c_1\right\}\right]$	`pdsolve(Eq3)`
4	$u_t=u_{xx}+au+bu^3$	$u=-\frac{1}{2}\frac{a}{\sqrt{-ba}}+\frac{1}{2}\frac{\sqrt{-ba}\coth\left(-\frac{1}{2}_C1-\frac{1}{4}\sqrt{2}\sqrt{a}x+\frac{3}{4}at\right)}{b}$	`pdsolve(Eq4,HINT='TWS(coth)')`
5	$u_{tt}=(e^uu_x)_x$	$u=-2\ln(2)+\ln\left(_c_1x^2-2\,_C1x+2\,_C2\right)+\ln\left(\frac{1}{_c_1\,_C3\left(\cos\left(\frac{1}{2}\frac{t+_C4}{_C3}\right)\right)^2}\right)$	`pdsolve(Eq5,HINT=f(x)+g(t),ops)`
6	$u_{xx}+u_{yy}=au^2$	$u=\frac{6\,\mathrm{WeierstrassP}(2\,_C3+_C1x+_C2y,0,_C4)\left(_C1^2+_C2^2\right)}{a}$	`pdsolve(Eq6,build)`
7	$u_t+uu_x=au_{xxx}$	$u=\frac{\left(-_C3-8a\,_C2^3+12a\,_C2^3\left(\tanh\left(_C1+_C2x\,_\varepsilon+_C3t\,_\varepsilon^3\right)\right)^2\right)_\varepsilon^2}{_C2}$	`SymmetrySolutions(Sol70,S[4])`
8	$uu_t=au_{xxx}$	$(u=f(x)g(t))$ `&where` $\left[\left\{f_{x,x,x}=_c_1(f(x))^2, g_t=a\,_c_1\right\}\right]$	`pdsolve(Eq8,HINT=f(x)*g(t))`
9	$u_{tt}+(uu_x)_x+u_{xxxx}=0$	$u=\frac{-_C3^2+8\,_C2^4}{_C2^2}-12\,_C2^2(\tanh(_C1+_C2x+_C3t))^2$	`TWSolutions(Eq9,singsol=false)`

The nonlinear partial differential equations and some additional solutions listed in the table are specified in Maple as follows:

```
with(PDEtools); declare(u(x,t));
Eq1:=diff(u(x,y),x)^2*diff(u(x,y),y)=1;
Eq2:=diff(u(x,y),x)^2+diff(u(x,y),y)^2=1; S:=Infinitesimals(Eq2);
Eq3:=diff(u(x,t),t)=diff(diff(u(x,t),x)*u(x,t),x);
Eq4:=diff(u(x,t),t)=diff(u(x,t),x$2)+a*u(x,t)+b*u(x,t)^3;
Eq5:=diff(u(x,t),t$2)=diff(exp(u(x,t))*diff(u(x,t),x),x); ops:=INTEGRATE,build;
Eq6:=diff(u(x,y),x$2)+diff(u(x,y),y$2)=a*u(x,y)^2;
Eq7:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=a*diff(u(x,t),x$3);
Sol70:=pdsolve(Eq7); S:=Infinitesimals(Eq7);
Eq8:=u(x,t)*diff(u(x,t),t)=a*diff(u(x,t),x$3);
Eq9:=diff(u(x,t),t,t)+diff(u(x,t)*diff(u(x,t),x),x)+diff(u(x,t),x$4)=0;
Sol11:=subs(seq(_C||i=C[i],i=1..5),_s=s,seq(_p[i]=p[i],i=1..2),Sol1);
```

Comment. In the above table, the original solution `Sol1` of the first equation is too long, and we present it in a more concise form after some substitutions (e.g., $_C1 = C_1$, $_s = s$, $_p_1 = p_1$, etc.; see `Sol11`). Here $p_1 = u_x$, $p_2 = u_y$. For the fifth equation, an additive separable solution is presented. (The automatic integration of ODEs, found by separation of variables, is performed with the option `INTEGRATE`, and an explicit expression for the unknown function is constructed with the option `build`.) The sixth equation is obtained in terms of the Weierstrass elliptic function, `WeierstrassP`, which is a doubly periodic function with periods $2\omega_1$ and $2\omega_2$, defined in Maple as $P(z,g_2,g_3) = z^{-2} + \sum\limits_{m,n}\Big((z-\Omega_{m,n})^{-2} - \Omega_{m,n}^{-2}\Big)$, where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2$. The sum ranges over all integers m and n except for the term for which $m = n = 0$. The function $P(z,g_2,g_3)$ satisfies the differential equation $(P')^2 = 4P^3 - g_2P - g_3$, where g_2 and g_3, called invariants, are related to $\Omega_{m,n}$ by the equations $g_2 = 60\sum\limits_{m,n}\Omega_{m,n}^{-4}$, $g_3 = 140\sum\limits_{m,n}\Omega_{m,n}^{-6}$. The infinitesimals, introduced for solving the second and the seventh equations, refer to the list with the components of the infinitesimal generator of a symmetry group. The solution of equation 7 contains an arbitrary constant $_\varepsilon$ (Lie group parameter). Equations 3, 5, and 8 are solved by separation of variables; in this case, according to Maple notation, the arbitrary constant is denoted by c_1 (so as to distinguish it from the usual arbitrary constant C_1).

39.3.2. Constructing Solutions via Transformations

Now we apply some special transformations (e.g., the Bäcklund transformations and the Cole–Hopf transformation) to nonlinear equations of mathematical physics and obtain exact solutions of these equations.

Example 39.8. *Bäcklund transformations. Sine–Gordon equation.* Let us verify that the sine–Gordon equations

$$u_{xt} = \sin u, \qquad v_{xt} = \sin v$$

are related by the Bäcklund transformations

$$\tfrac{1}{2}(u-v)_x = a\sin\big(\tfrac{1}{2}(u+v)\big),$$
$$\tfrac{1}{2}(u+v)_t = \tfrac{1}{a}\sin\big(\tfrac{1}{2}(u-v)\big),$$

where $a \neq 0$ is an arbitrary constant. Since both u and v satisfy the same sine–Gordon equation, the pair of Bäcklund transformations is referred to as an *auto-Bäcklund transformation* for the sine–Gordon equation.

```
interface(showassumed=0); assume(a<>0); with(PDEtools):
declare(u(x,t),v(x,t)); alias(u=u(x,t),v=v(x,t)); SGEq:=diff(u,x,t)=sin(u);
BT:=[diff(u-v,x)/2=a*sin(1/2*(u+v)),diff(u+v,t)/2=sin(1/2*(u-v))/a];
Eq1:=subs(BT,diff(BT[1],t)); Eq2:=subs(BT,diff(BT[2],x));
SGEqu:=combine(Eq2+Eq1,trig); SGEqv:=combine(Eq2-Eq1,trig);
```

By applying the Bäcklund transformations, we can find a solution of the sine–Gordon equation as follows:

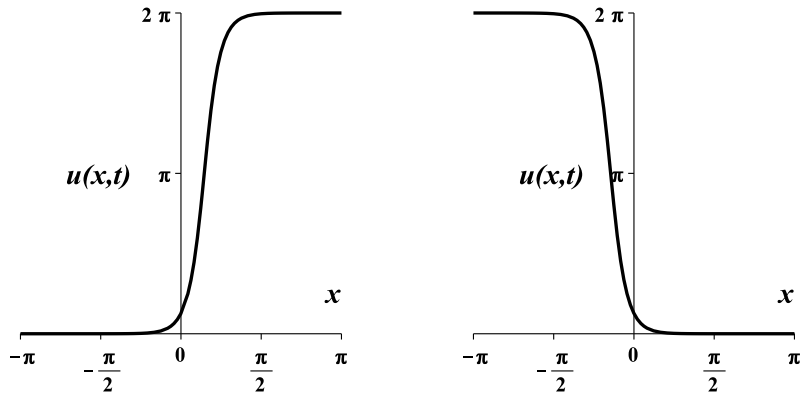

Figure 39.1. The kink and antikink solutions of the sine–Gordon equation.

```
with(plots): setoptions(plot,scaling=constrained);
Eq3:=subs(v=0,factor(BT*2)); I1:=int(1/sin(U/2),U);
Eq4:=int(2*a,x)=convert(simplify(convert(I1,tan)),tan)+A(t);
Eq5:=int(2/a,t)=convert(simplify(convert(I1,tan)),tan)+B(x);
t1:=tan(U/4); Eq6:=expand(isolate(Eq4,t1));
Eq7:=expand(isolate(Eq5,t1)); Eq8:=lhs(Eq6)=rhs(Eq6)*rhs(Eq7);
C:=select(has,rhs(Eq8),[A,B]); Eq9:=lhs(Eq8)=beta*rhs(Eq8/C);
Eq10:=u=solve(Eq9,U); KinkSol:=unapply(rhs(Eq10),x,t,a,beta);
beta1:=0.1; a1:=10; xR:=-Pi..Pi; tR:=0..1;
plot3d(evalf(KinkSol(x,t,a1,beta1)),x=xR,t=tR,orientation=[71,73]);
plot(evalf(KinkSol(x,0,a1,beta1)),x=xR,tickmarks=[spacing(Pi/2),spacing(Pi/2)]);
```

and verify that the new solution of the sine–Gordon equation describes a *soliton or kink* solution and has the following Maple form:

$$Eq10 := u = 4\arctan\left(\beta e^{ax} e^{\frac{t}{a}}\right).$$

Let us visualize the kink solution in 3D and 2D spaces.

By considering the standard form of the sine–Gordon equation

$$u_{XX} - \beta u_{TT} = \sin u, \quad \text{where} \quad \beta = \frac{1}{c^2}$$

with the characteristic coordinate transformations

$$x = \tfrac{1}{2}(X + cT), \quad t = \tfrac{1}{2}(X - cT),$$

we can obtain a *soliton or kink* and an *antisoliton or antikink* solution. Let us visualize the kink and antikink solutions obtained (see Fig. 39.1).

```
c1:=1; Eq11:=simplify(subs({x=(X+c*T)/2,t=(X-c*T)/2},rhs(Eq10)));
tr1:=[a=epsilon*sqrt((1-U)/(1+U)),c=U*m/(2*(a-1/a))];
Eq12:=simplify(subs(tr1[1],simplify(subs(tr1,Eq11))));
KinkSols:=unapply(Eq12,m,U,epsilon,T,X,beta); m1:=a->1/2*(a+1/a);
U1:=(c,a)->c*((a^2-1)/(a^2+1)); m1(a1); U1(c1,a1);
Kink:=plot(evalf(KinkSols(m1(a1),U1(c1,a1),1,0,X,beta1)),X=xR,
      tickmarks=[spacing(Pi/2),spacing(Pi/2)]):
AntiKink:=plot(evalf(KinkSols(m1(a1),U1(c1,a1),-1,0,X,beta1)),
          X=xR,color=blue,tickmarks=[spacing(Pi/2),spacing(Pi/2)]):
display({Kink,AntiKink});
```

Example 39.9. *Cole–Hopf transformation for Burgers equation.* Let $u(x,t)$, $\psi(x,t)$, and $\phi(x,t)$ be real functions. By applying the Cole–Hopf transformation, $u = \psi_x = -2\nu\dfrac{\phi_x}{\phi}$, $\psi = -2\nu\ln\phi$, we can transform any solution $\phi = \xi(x,t)$ of the linear heat equation

$$\phi_t = \nu\phi_{xx}$$

into the solution $u = -2\nu\dfrac{\phi_x}{\phi}$ of the Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

as follows:

```
with(PDEtools): declare(u(x,t),psi(x,t),phi(x,t));
alias(u=u(x,t),psi=psi(x,t),phi=phi(x,t));
BEq:=diff(u,t)+u*diff(u,x)=nu*diff(u,x$2);
ConservLaw:=diff(u,t)+Diff((u^2/2-nu*diff(u,x)),x)=0;
Eq1:=u=Diff(psi,x); Eq2:=-op(1,op(2,lhs(ConservLaw)))=Diff(psi,t);
Eq3:=algsubs(Eq1,Eq2);
tr1:={psi=-2*nu*log(phi)}; tr2:=diff(psi,x)=value(subs(tr1,Eq1));
Eq4:=[lhs(tr2)=rhs(rhs(tr2)),diff(psi,x$2)=diff(rhs(rhs(tr2)),x),diff(op(tr1),t)];
Eq5:=expand(subs(Eq4,value(Eq3))); DiffusEq:=Eq5*(-1/2)*phi/nu;
```

The opposite is not generally true (see Example 26.21),

```
with(PDEtools): declare(u(x,t),phi(x,t)); alias(u=u(x,t),phi=phi(x,t));
BEq:=diff(u,t)+u*diff(u,x)-nu*diff(u,x$2)=0; tr1:={u=-2*nu*diff(phi,x)/phi};
Eq1:=collect(expand(subs(tr1,BEq)),diff); Eq2:=expand(Eq1*phi^2/2);
Eq11:=expand(-diff(1/phi*(diff(phi,t)-nu*diff(phi,x$2)),x)=0);
Eq21:=expand(Eq11*phi^2*nu); Eq2-Eq21;
Eq3:=int(-Diff(1/phi*(diff(phi,t)-nu*diff(phi,x$2)),x),x)=f(t); Eq4:=Eq3*phi;
```

since a solution of the Burgers equation $u_t + uu_x = \nu u_{xx}$ generates a solution of the more general equation $\phi_t - \nu\phi_{xx} = -\phi f(t)$ (`Eq4`).

39.3.3. Constructing Traveling-Wave Solutions

Traveling-wave solutions often occur in various problems of mathematical physics. These solutions are invariant under translation. A traveling-wave solution of the wave equation was first obtained by d'Alembert in 1747 [see Lindsay (1973)]. In practice, this method is simple and useful in finding solutions of both linear and nonlinear PDEs [see Polyanin (2002)]. As before, we illustrate the method of finding traveling-wave solutions by solving problems.

Example 39.10. *Burgers equation.* Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}.$$

We derive *traveling-wave solutions* of the Burgers equation, i.e., solutions of the form $u(x,t) = U(z)$, $z = x - \lambda t$. Investigating the shock-wave structure of the solutions, we assume that there exist constant values $u_1 > 0$ and $u_2 > 0$ ($u_1 > u_2$) such that $\lim\limits_{z\to-\infty} U(z) = u_1$ and $\lim\limits_{z\to+\infty} U(z) = u_2$. We determine the shape of the waveform $U(z)$ and plot the graphs of traveling-wave solutions $U(z)$ varying the diffusion parameter ν as follows:

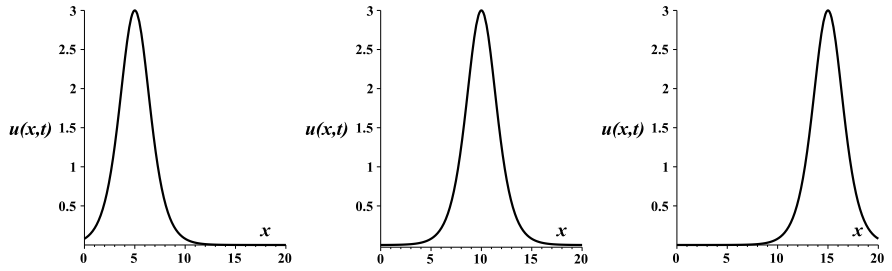

Figure 39.2. The traveling-wave solution of the Korteweg-de Vries equation.

```
interface(showassumed=0); assume(nu>0,U(z)>0,u1>0,u2>0,u1>u2);
A:=2*nu; tr1:={lambda=1/2*(u1+u2),C=-1/2*u1*u2};
tr2:=U(z)-u1=u1-U(z); tr3:=x-lambda*t=z;
Eq1:=u->diff(u,t)+u*diff(u,x)-nu*diff(u,x$2);
Eq2:=Eq1(U(x-lambda*t)); Eq3:=convert(algsubs(tr3,Eq2),diff);
Eq4:=int(Eq3,z)=C; dU:=diff(U(z),z); Sol1:=solve(Eq4,dU) assuming z<>0;
Sol2:=(dU=factor(subs(tr1,Sol1)))*A; Sol3:=Sol2/rhs(Sol2)/A*dz;
Sol3L:=lhs(Sol3); Sol3R:=rhs(Sol3);
Sol4:=combine(int(Sol3L,z),symbolic)=int(coeff(Sol3R,dz),z);
Sol5:=subs(tr2,Sol4); Sol6:=solve(1=solve(Sol5,dz),U(z)) assuming z<>0;
SolU:=unapply(Sol6,z,u1,u2,nu); SolU(0,5,1,0.3);
plot(SolU(z,5,1,0.3),z=-5..5,0..6,thickness=3);
plot(SolU(z,5,1,0.001),z=-1..1,0..6,thickness=3);
```

and we obtain the Maple form of the traveling-wave solution of the Burgers equation:

$$SolU := (z, u1, u2, \nu) \mapsto \left(e^{1/2\frac{z(u1-u2)}{\nu}} u2 + u1\right)\left(1 + e^{1/2\frac{z(u1-u2)}{\nu}}\right)^{-1}$$

Example 39.11. *Korteweg-de Vries equation.* Consider the Korteweg-de Vries (KdV) equation

$$u_t + auu_x + bu_{xxx} = 0,$$

where a and b are constants. The KdV equation admits a special form of the exact solution, a soliton, which arises in many physical processes (e.g., water waves, internal gravity waves in a stratified fluid, ion-acoustic waves in plasma, etc.) We seek a special type of traveling-wave solutions, i.e., *one-soliton solution*, $u(x,t) = U(z)$, $z = x - ct$, of the KdV equation, where c is a constant and $U(z) \to 0$ as $|z| \to \infty$:

```
with(PDEtools): with(plots): tr1:=x-c*t=z; tr2:={a=1,b=1,c=1};
Eq1:=u->diff(u,t)+a*u*diff(u,x)+b*diff(u,x$3)=0;
Eq2:=expand(Eq1(U(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2);
Eq4:=map(convert,Eq3,diff); Eq5:=map(int,lhs(Eq4),z)-C1=0;
Eq6:=expand(Eq5*2*diff(U(z),z)); Eq7:=map(int,Eq6,z);
Eq8:=lhs(Eq7)=C2; Eq9:=subs({C1=0,C2=0},Eq8);
Sol1:=[dsolve(Eq9,U(z))]; Sol11:=subs(_C1=0,simplify(Sol1[2]));
Sol12:=convert(Sol11,sech); Sol2:=eval(subs(z=x-c*t,Sol11),tr2);
convert(Sol2,sech);
animate(plot,[rhs(Sol2),x=-20..20,color=blue],t=0..20,numpoints=100,
        frames=50,thickness=2);
pdetest(u(x,t)=rhs(Sol2),subs(tr2,Eq1(u(x,t))));
```

Comment. By substituting this solution form into the KdV equation, we arrive at the third-order nonlinear ODE $-cU_z + aUU_z + bU_{zzz} = 0$ (`Eq4`). By integrating this equation with respect to z twice (and introducing the

integration constants C_1 and C_2), we obtain the first-order nonlinear ODE $-cU^2+\frac{1}{3}aU^3+bU_z^2-2C_1U=C_2$ (Eq8). Considering the special case where $U(z)$ and its derivatives tend to zero at infinity and $C_1=C_2=0$, we can determine that the exact solution has the Maple form:

$$Sol12 := U(z) = 3c\left(\operatorname{sech}\left(1/2\,\frac{\sqrt{cb}z}{b}\right)\right)^2 a^{-1}$$

By setting, e.g., $a=b=c=1$, we confirm that the solution obtained is indeed an exact solution of the KdV equation and travels at a constant velocity without changing the shape (see Fig. 39.2).

Example 39.12. *Sine–Gordon equation.* Consider the sine–Gordon equation

$$u_{xx}-\alpha u_{tt}=\sin u, \quad \text{where} \quad \alpha=\frac{1}{c^2}.$$

We construct traveling-wave solutions of the sine–Gordon equation in the form of one-soliton solutions as follows:

```
with(PDEtools): with(plots): declare(u(x,t),phi(xi));
alias(u=u(x,t),phi=phi(xi));  interface(showassumed=0); assume(Phi>Phi0);
tr1:=x-lambda*t=xi; tr2:=lambda^2/c^2=U^2;
SGEq:=u->diff(u,x$2)-1/c^2*diff(u,t$2)=sin(u);
Eq1:=expand(SGEq(phi(lhs(tr1)))); Eq2:=expand(subs(tr1,Eq1));
Eq3:=convert(algsubs(tr2,Eq2),diff);
Eq4:=expand(normal(Eq3*diff(phi(xi),xi)/(1-U^2)));
Eq5:=Diff(1/2*(Diff(phi,xi)^2)+cos(phi)/(1-U^2),xi)=0;
Eq41:=lhs(Eq4)-rhs(Eq4)=0; factor(Eq41-value(Eq5));
Eq6:=int(value(op(1,Eq5)),xi)=B;
Eq7:=normal(convert(isolate(Eq6,diff(phi,xi)),radical)); Eq7R:=rhs(Eq7);
Eq80:=subs(phi=psi,1/sqrt(collect(-numer(op(1,op(2,Eq7R))),B)));
Eq8:=Int(Eq80,psi=Phi0..Phi);
Eq90:=op(1,rhs(Eq7))/sqrt(-denom(op(1,op(2,rhs(Eq7)))));
Eq9:=int(Eq90,eta=xi0..xi); Eq10:=Eq8=Eq9; Eq11:=subs({B*(1-U^2)=1},Eq10);
Eq12:=simplify(subs(1-cos(psi)=2*sin(psi/2)^2,Eq11),symbolic);
Eq13:=normal(convert(value(Eq12),tan)); Eq13L:=lhs(Eq13);
Eq14:=expand(op(3,op(3,Eq13L))*op(1,Eq13L)*op(2,Eq13L)=rhs(Eq13));
Sol:=expand(isolate(Eq14,Phi));
tr3:={lambda=0.1,c=10.}; tr4:={tan(Phi0/4)=1,xi0=0}; tr5:=x-U*t;
Kink:=unapply(subs(xi=tr5,expand(subs(tr4,rhs(Sol)))),x,t,U);
AntiKink:=unapply(subs(xi=-tr5,expand(subs(tr4,rhs(Sol)))),x,t,U);
U1:=subs(tr3,lhs(tr2)); xR:=-2*Pi..2*Pi; K:=plot(Kink(x,0,U1),x=xR):
AK:=plot(AntiKink(x,0,U1),x=xR,color=blue): display({K,AK});
```

These one-soliton solutions have the following Maple form:

$$Kink := (x,t,U)\mapsto 4\arctan\left(e^{\frac{x-Ut}{\sqrt{1-U^2}}}\right), \quad AntiKink := (x,t,U)\mapsto 4\arctan\left(e^{\frac{-x+Ut}{\sqrt{1-U^2}}}\right),$$

where $U=\lambda/c$. These solutions are called the soliton (or kink) and antisoliton (or antikink) solutions, respectively (depending on the sign). The solitons propagate in the positive or negative x-direction, respectively, at the velocity U. We visualize the soliton and antisoliton solutions.

Example 39.13. *Fisher equation. Phase portraits.* Consider the Fisher equation

$$u_t-u_{xx}=u(1-u).$$

It is known that this equation admits a traveling-wave solution of the form $u(x,t)=U(z)$, $z=x-ct$ ($0\le u(x,t)\le 1$), where c is the wave speed and the waveform $U(z)$ satisfies the boundary conditions $\lim\limits_{z\to-\infty}U(z)=1$

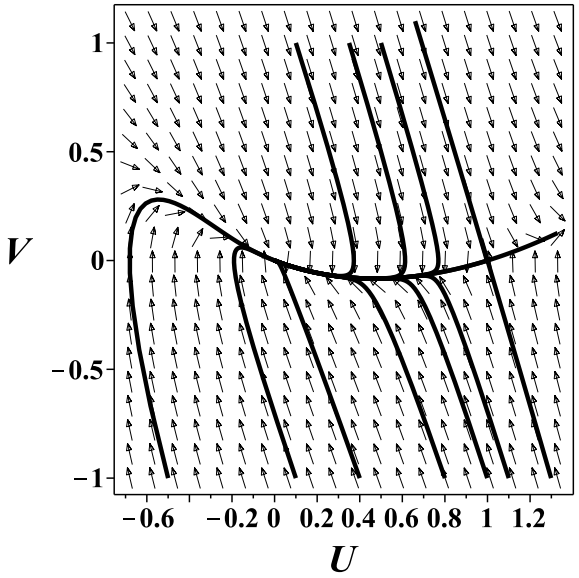

Figure 39.3. The phase portrait on the plane (U,V) for the Fisher equation.

and $\lim_{z\to\infty} U(z) = 0$. Let us show that the Fisher equation can be reduced to the nonlinear ODE $-U''(z) - cU'(z) - U(z) + U(z)^2 = 0$ (Eq4) or to the system of two first-order ODEs $U' = V$, $V' = -cV - U + U^2$ (Eq5, Eq6), or, equivalently, $\dfrac{dV}{dU} = \dfrac{-cV - U + U^2}{V}$ (Eq7) as follows:

```
with(PDEtools): with(plots): with(DEtools): with(LinearAlgebra):
Ops:=arrows=medium,dirgrid=[20,20],stepsize=0.1,thickness=2,
     linecolour=blue,color=green;
tr1:=x-c*t=z; tr2:={V(z)=V,U(z)=U};
Eq1:=u->diff(u,t)-diff(u,x$2)-u*(1-u);
Eq2:=expand(Eq1(U(lhs(tr1)))); Eq3:=algsubs(tr1,Eq2)=0;
Eq4:=map(convert,Eq3,diff); Eq5:=diff(U(z),z)=V(z);
Eq6:=isolate(subs(Eq5,Eq4),diff(V(z),z));
Eq7:=subs(tr2,Diff(V,U)=rhs(Eq6)/rhs(Eq5));
```

This is a system of first-order autonomous ODEs, $\frac{du}{d\xi} = P(u,v)$, $\frac{dv}{d\xi} = Q(u,v)$, where P and Q are nonlinear functions of the two variables u and v and the independent variable is ξ, which can be eliminated by dividing one equation by the other $\frac{du}{dv} = \frac{P(u,v)}{Q(u,v)}$. The solution of this equation can be represented as a *phase diagram* on the plane (u,v) with some *phase trajectories* along which the system will evolve as ξ increases. Since the integration for finding $u(v)$ may not be possible, the phase plane analysis can be useful to determine *stationary* or *singular points* u_0, v_0 in some regions of the *Poincaré phase plane* and to analyze the nature of solutions. In practice, since the phase plane portraits may be complicated, it is convenient to obtain them with Maple [see Lynch (2009)].

Let us study the dynamical system obtained. We claim that the singular points (W_0, V_0) of this dynamical system are $(0,0)$ and $(1,0)$. Following the phase plane analysis, we study the nature of the given nonlinear system and show that the matrix associated with this system at the singular points $(0,0)$ and $(1,0)$ has the eigenvalues $-\frac{1}{2}(c \mp \sqrt{c^2-4})$ and $\frac{1}{2}(-c \pm \sqrt{c^2+4})$, respectively. According to the theory of dynamical systems, we show that the point $(0,0)$ is a stable node for $c \geq 2$ and the point $(1,0)$ is a saddle point. Finally, we construct the phase portrait (see Fig. 39.3) on the plane (U,V), showing that there exists a unique separatrix joining the stable node $(0,0)$ with the saddle point $(1,0)$ (for $c \geq 2$).

```
Eq81:=denom(rhs(Eq7))=0; Eq82:=[solve(subs(Eq81,numer(rhs(Eq7))),U)];
```

```
SingPoints:=[[Eq82[1],rhs(Eq81)],[Eq82[2],rhs(Eq81)]];
P:=subs(tr2,rhs(Eq5)); Q:=subs(tr2,rhs(Eq6));
A:=<diff(P,U),diff(Q,U)|diff(P,V),diff(Q,V)>;
A1:=subs({U=0,V=0},A); A2:=subs({U=1,V=0},A);
Eigenvalues(A1); Eigenvalues(A2); c>=2;
Eqs:=subs(c=3,[Eq5,Eq6]); vars:=[U(z),V(z)];
IC:=[[U(0)=-0.5,V(0)=-1.],[U(0)=0.1,V(0)=1.],[U(0)=0.96968212,V(0)=0.1],
     [U(0)=0.35,V(0)=1.],[U(0)=0.5,V(0)=1.],[U(0)=0.8,V(0)=-1.],
     [U(0)=0.4,V(0)=-1.],[U(0)=0.1,V(0)=-1.],[U(0)=0.999,V(0)=-1.],
     [U(0)=1.1,V(0)=-1.],[U(0)=1.29889,V(0)=-1.],[U(0)=0.65999,V(0)=1.1]];
phaseportrait(Eqs,vars,z=0..60,IC,Ops,view=[-0.7..1,-1..1]);
```

39.3.4. Ansatz Methods (Tanh-Coth Method, Sine-Cosine Method, and Exp-Function Method) for Constructing Traveling-Wave Solutions

In this section, we consider the most important ansatz methods (the tanh-function method, sine-cosine method, and exp-function method) for constructing traveling-wave solutions of nonlinear PDEs. Recently developed, these methods became the most powerful and effective algebraic methods for finding exact solutions of nonlinear PDEs partly due to the modern computer algebra systems Maple and Mathematica, which allow performing a lot of cumbersome analytical calculations.

Tanh-coth method. The tanh-function expansion method was developed by Malfliet (1992) and Hereman (1996). The main idea of the tanh method is based on the assumption that traveling-wave solutions can be expressed in terms of the tanh function; i.e., a new variable, e.g., $Y = \tanh(z)$, can be introduced and all derivatives of Y are expressed in terms of tanh function. Later on, the tanh method was modified, extended, and generalized for finding more and more exact solutions. In recent years, various forms of the tanh method have been developed, e.g., the tanh-coth method has been proposed by Wazwaz (2006); see also Parkes (2010) and Zhang (2010). Many researchers developed symbolic programs and packages to deal with the tedious algebraic computations that arise in the solution process (e.g., see Parkes and Duffy (1996)).

Example 39.14. Consider the *Klein–Gordon equation* of the form

$$u_{tt} - u_{xx} + u - u^3 = 0. \tag{39.3.4.1}$$

We seek traveling-wave solutions of the Klein–Gordon equation, i.e., solutions of the form $u(x,t) = U(z)$, $z = \mu(x+ct)$. First, we convert the PDE into the following ODE: $\mu^2 c^2 U_{zz} - \mu^2 U_{zz} + U - U^3 = 0$ (`Eq2`). If all terms of this ODE contain derivatives in z, then the ODE should be simplified (by integrating and considering the integration constants to be zero).

By introducing a new variable $Y = \tanh(z)$ or $Y = \coth(z)$, we propose the following ansatz (`Ansatz1`):

$$U(z) = S(Y) = \sum_{i=0}^{M} a_i Y^i,$$

where M ($M > 0$, $M \in \mathbb{Z}$) is to be determined.

By substituting this series expansion into the ODE, we obtain an equation in powers of Y:

$$(-\mu^2 S_{YY} + \mu^2 c^2 S_{YY})Y^4 + (2\mu^2 c^2 S_Y - 2\mu^2 S_Y)Y^3 + (-2\mu^2 c^2 S_{YY} + 2\mu^2 S_{YY})Y^2 + (2\mu^2 S_Y - 2\mu^2 c^2 S_Y)Y + S(Y) + \mu^2 c^2 S_{YY} - \mu^2 S_{YY} - S(Y)^3 = 0.$$

To determine the parameter M, we balance the linear terms of highest order in the resulting equation with the highest-order nonlinear terms according to the following formulas for the highest exponents of the function $U(z)$ and its derivatives: $U(z) \to M$, $U^n(z) \to nM$, $U'(z) \to M+1$, $U''(z) \to M+2$, $U^{(k)}(z) \to M+k$. In our case, we have $3M = M+2$, $M = 1$ and $S(Y) = a_0 + a_1 Y$.

Then, by matching the coefficients of like powers of Y in the simplified equation (`Eq41`)

$$(2\mu^2 c^2 a_1 - a_1^3 - 2\mu^2 a_1)Y^3 - 3a_0 a_1^2 Y^2 + (-3a_0^2 a_1 + 2\mu^2 a_1 + a_1 - 2\mu^2 c^2 a_1)Y - a_0^3 + a_0 = 0,$$

where these coefficients have to vanish, we obtain the system of algebraic equations (`sys1`) for the unknowns a_i $(i = 0, \ldots, M)$, μ, and c:

$$-3a_0 a_1^2 = 0,\ -a_0^3 + a_0 = 0,\ 2\mu^2 c^2 a_1 - a_1^3 - 2\mu^2 a_1 = 0,\ -3a_0^2 a_1 + 2\mu^2 a_1 + a_1 - 2\mu^2 c^2 a_1 = 0.$$

Finally, by determining these unknowns, μ, a_0, and a_1 (`Sols`), and by using the series expansion (`Ansatz1`), we obtain the exact solutions $u(x,t) = \pm\tanh\left(\sqrt{1/(-2+2c^2)}(x+ct)\right)$ (`SolsTF`) and verify that these solutions are the exact solutions.

```
with(PDEtools): with(plots): declare(u(x,t),U(z));
alias(u=u(x,t),U=U(z)); interface(showassumed=0); assume(n>1);
f:=tanh; tr1:=mu*(x+c*t)=z; tr2:=f(z)=Y; sys1:={}; SolsT:={}:
PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0;
Eq1:=expand(PDE1(U(lhs(tr1)))); Eq2:=convert(expand(subs(tr1,Eq1)),diff);
tr3:=U(z)=S(lhs(tr2)); Ansatz1:=S(Y)=Sum(a[i]*Y^i,i=0..M);
Eq3:=convert(algsubs(tr2,algsubs(tr3,Eq2)),diff);
Eq31:=collect(Eq3,Y); tr5:=isolate(3*M=M+2,M);
tr6:=value(subs(tr5,Ansatz1)); Eq4:=algsubs(tr6,Eq3); Eq41:=collect(Eq4,Y);
for i from 0 to 3 do sys1:=sys1 union {coeff(lhs(Eq4),Y,i)=0}; od: sys1;
vars:=indets(sys1) minus {c}; Sols:=[allvalues([solve(sys1,vars)])];
NSols:=nops(Sols);
for i from 1 to NSols do Op||i:=op(i,Sols): Nops:=nops(Op||i):
 for j from 1 to Nops do
   SolsT:=SolsT union {u=subs(op(j,Op||i),subs(z=lhs(tr1),
          subs(Y=lhs(tr2),rhs(tr6))))};
od: od: SolsT; SolsTF:=select(has,SolsT,f);
for i from 1 to nops(SolsTF) do simplify(PDE1(rhs(SolsTF[1]))); od;
```

Sine-cosine method. The main idea of the sine-cosine method is based on the assumption that traveling-wave solutions can be expressed in terms of the sine or cosine functions; i.e., a new variable, e.g., $Y = \cos(z)$ or $Y = \sin(z)$, can be introduced. As in the tanh method considered above, the main advantage of the sine-cosine method is in simplifying the solution process and reducing the amount of computational work (compared to modern analytical methods); i.e., we have to solve a system of algebraic equations rather than a nonlinear differential equation. Moreover, we can apply computer algebra systems, Maple and Mathematica, for this purpose.

Example 39.15. For the Klein–Gordon equation (39.3.4.1) considered in the previous example, let us find traveling-wave solutions, i.e., solutions of the form $u(x,t) = U(z)$, $z = x + ct$. As before, we convert the PDE into the following ODE: $c^2 U_{zz} - U_{zz} + U - U^3 = 0$ (`Eq2`). If all terms of this ODE contain derivatives in z, then the ODE should be simplified (by integrating and by assuming the integration constants to be zero).

According to the sine-cosine method, the traveling-wave solutions can be expressed in the form $u(x,t) = \lambda\cos^\beta(\mu z)$ for $|z| \le \frac{1}{2}\pi/\mu$ or in the form $u(x,t) = \lambda\sin^\beta(\mu z)$ for $|z| \le \pi/\mu$, where the parameters λ, μ (wave number), and β are to be determined. By introducing a new variable $Y = \cos(z)$ or $Y = \sin(z)$, we propose the following ansatz: $U(z) = \lambda\cos^\beta(\mu z)$ (`AnsatzC`), $U(z) = \lambda\sin^\beta(\mu z)$ (`AnsatzS`).

By substituting this ansatz into the ODE, we obtain the trigonometric equation (Eq3). Then, by simplifying this equation and by introducing the new variable $Y = \cos(\mu z)$, we obtain (Eq33)

$$-\lambda^3 Y^{3\beta} + \lambda\beta^2\mu^2 c^2 Y^{\beta-2} - \lambda\beta^2\mu^2 c^2 Y^{\beta} - \lambda\beta\mu^2 c^2 Y^{\beta-2} - \lambda\beta^2\mu^2 Y^{\beta-2} + \lambda\beta^2\mu^2 Y^{\beta} + \lambda\beta\mu^2 Y^{\beta-2} + \lambda Y^{\beta} = 0.$$

To determine the parameter β, we balance the exponents of each pair of Y. Then, to determine λ and μ, we match the coefficients of like powers of Y, where these coefficients have to vanish. This gives a system of algebraic equations for the unknown parameters β, λ, and μ: $Y^{3\beta} = Y^{\beta-2}$, (Eqbeta), $-\lambda\beta^2\mu^2c^2 + \lambda\beta^2\mu^2 + \lambda = 0$ (Eqmu), $-\lambda^3 + \lambda\beta^2\mu^2c^2 - \lambda\beta\mu^2c^2 - \lambda\beta^2\mu^2 + \lambda\beta\mu^2 = 0$ (Eqlambda).

Finally, by determining these parameters, $\beta = -1$ (trbeta), $\mu = 1/\sqrt{c^2-1}$ (trmu), and $\lambda = \sqrt{2}$ (trlambda), and by using the proposed ansatz (AnsatzC, AnsatzS), we obtain the exact solutions and verify that these solutions are exact solutions.

Let $d = \sqrt{c^2-1}$. If $c^2 - 1 > 0$, then we have (SolC, SolS):

$$u(x,t) = \sqrt{2}\sec\left(d^{-1}(x+ct)\right) \quad \text{for} \quad |d^{-1}z| \le \frac{1}{2}\pi$$

and

$$u(x,t) = \sqrt{2}\csc\left(d^{-1}(x+ct)\right) \quad \text{for} \quad 0 < |d^{-1}z| \le \pi.$$

If $c^2 - 1 < 0$, then we have (SolCC, SolSC)

$$u(x,t) = \sqrt{2}\,\mathrm{sech}\left(d^{-1}(x+ct)\right) \quad \text{and} \quad u(x,t) = i\sqrt{2}\,\mathrm{csch}\left(d^{-1}(x+ct)\right).$$

```
with(PDEtools): with(plots): declare(u(x,t),U(z)); alias(u=u(x,t),U=U(z));
f:=cos: tr1:=x+c*t=z; tr2:=f(z)=Y;
PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0;
Eq1:=expand(PDE1(U(lhs(tr1)))); Eq2:=convert(expand(subs(tr1,Eq1)),diff);
AnsatzC:=U(z)=lambda*cos(mu*z)^beta; AnsatzS:=U(z)=lambda*sin(mu*z)^beta;
Eq3:=algsubs(AnsatzC,Eq2); Eq31:=expand(simplify(Eq3,trig),cos(mu*z));
Eq32:=simplify(Eq31,power); Eq33:=combine(subs(cos(mu*z)=Y,lhs(Eq32)));
term11:=select(has,Eq33,Y^(3*beta)); term12:=select(has,term11,Y);
term21:=select(has,Eq33,Y^(beta-2)); term22:=select(has,op(1,term21),Y);
trbeta:=isolate(term12=term22,beta);
trmu:=sort(subs(trbeta,[solve(coeff(Eq33,Y^beta),mu)]))[1];
Eqlambda:=coeff(Eq33,Y^(3*beta))+coeff(Eq33,Y^(beta-2))=0;
Eqlambda1:=subs(mu=trmu,trbeta,Eqlambda);
Solslambda:=expand([solve(Eqlambda1,lambda)]);
trlambda:=sort(convert(Solslambda,set) minus {0})[1];
SolC:=simplify(subs(z=lhs(tr1),trbeta,lambda=trlambda,mu=trmu,u=rhs(AnsatzC)));
SolS:=simplify(subs(z=lhs(tr1),trbeta,lambda=trlambda,mu=trmu,u=rhs(AnsatzS)));
SolC:=convert(SolC,sec);  SolS:=convert(SolS,csc);
trC:=1/sqrt(c^2-1)=1/(I*sqrt(1-c^2));
SolCC:=convert(simplify(subs(trC,SolC)),sech);
SolSC:=convert(simplify(subs(trC,SolS)),csch);
simplify([PDE1(rhs(SolC)),PDE1(rhs(SolS)),PDE1(rhs(SolCC)),PDE1(rhs(SolSC))],trig);
Sol1G:=simplify(subs(c=2,n=3,SolC)); Sol2G:=simplify(subs(c=1/2,n=3,SolC));
animate(rhs(Sol1G),x=0..Pi,t=0..5,view=[default,-200..200]);
animate(rhs(Sol2G),x=-3..3,t=0..1,frames=50);
```

Exp-function method. The Exp-function method was proposed by He and Wu (2006) to obtain exact solutions of nonlinear evolution equations. The method has been applied to various nonlinear PDEs of mathematical physics, e.g., the Burgers and KdV equations as well as the Kuramoto–Sivashinsky, Boussinesq, and other equations; see He and Wu (2006), He and Abdon (2007), Chun (2008), Erbaş and Yusufoğlu (2009), Ganji et al. (2009), Soliman (2009), Salas (2008b), Kudryashov (2009b, 2010a), and Kudryashov and Loginova (2009).

Example 39.16. Let us find exact solutions of the Klein–Gordon equation (39.3.4.1) (considered in the preceding two examples) and present the solution procedure of the Exp-function method with the aid of the Maple computer algebra system.

We seek traveling-wave solutions, i.e., solutions of the form $u(x,t)=U(z)$, $z=\mu(x+ct)$, where μ and c are constants to be determined. As before, we convert the PDE into the following ODE: $\mu^2c^2U_{zz}-\mu^2U_{zz}+U-U^3=0$ (`Eq2`).

According to the Exp-function method, we propose an ansatz; i.e., we assume that the traveling-wave solutions can be expressed in the form [see He and Abdou (2007)] (`tr3`)

$$U(z)=\frac{\sum_{k=-r}^{s}a_ke^{kz}}{\sum_{j=-p}^{q}b_je^{jz}}=\frac{a_re^{rz}+\cdots+a_{-s}e^{-sz}}{b_pe^{pz}+\cdots+b_{-q}e^{-qz}}, \tag{39.3.4.2}$$

where r, s, p, and q are unknown positive integers to be determined and a_k and b_j are unknown constants.

By considering the resulting ODE, we can determine the values of r, p and s, q. First, by balancing the highest order linear term with the highest order nonlinear term, we obtain $-3r-3p=-r-5p$; i.e., $r=p$. In a similar way, by balancing the lowest order linear term with the lowest order nonlinear term, we have $3s+3q=s+5q$; i.e., $s=q$. These symbolic computations can be performed with Maple.

Consider the special case in which $r=p=1$ and $s=q=1$. Then the ansatz becomes (`tr4`)

$$U(z)=\frac{a_{-1}e^{-z}+a_0+a_1e^{z}}{b_{-1}e^{-z}+b_0+b_1e^{z}}.$$

By substituting this expression into the ODE and by matching the coefficients of like powers of e^{kz}, we generate the system of algebraic equations (`sys1`) for the unknowns c, μ, a_{-1}, a_0, a_1, b_{-1}, b_0, b_1.

To determine all unknown constants, we solve this system of algebraic equations and obtain 128 solutions (`Sols`). It should be noted that among these solutions there are many equivalent forms of solutions (represented in different forms and for different values of integration constants).* Without full analysis of all solutions, let us compare some solutions with the exact solutions obtained by the tanh-function method. Thus, we choose the following solutions (`Sol1`, `Sol2`):

$$c=c_0,\quad \mu=-\sqrt{\frac{1}{2(c_0^2-1)}},\quad a_{-1}=b_{-1},\quad a_0=0,\quad a_1=-b_1,\quad b_{-1}=b_{-1},\quad b_0=0,\quad b_1=b_1;$$

$$c=c_0,\quad \mu=-\sqrt{\frac{1}{2(c_0^2-1)}},\quad a_{-1}=-b_{-1},\quad a_0=0,\quad a_1=b_1,\quad b_{-1}=b_{-1},\quad b_0=0,\quad b_1=b_1.$$

Let $d=\sqrt{1/(c^2-1)}$. The corresponding traveling-wave solutions acquire the form:

$$u(x,t)=\pm\frac{b_{-1}+b_{-1}\tanh(\frac{\sqrt{2}}{2}d(x+ct))-b_1+b_1\tanh(\frac{\sqrt{2}}{2}d(x+ct))}{b_{-1}+b_{-1}\tanh(\frac{\sqrt{2}}{2}d(x+ct))+b_1-b_1\tanh(\frac{\sqrt{2}}{2}d(x+ct))}.$$

By setting $b_1=-1$ and $b_{-1}=-1$, we rewrite the above solutions in the form

$$u(x,t)=\pm\tanh(\tfrac{\sqrt{2}}{2}d(x+ct)),$$

which coincide with the exact solutions obtained above by the tanh-function method. It is possible to find other type of exact solutions by analyzing all sets of parameters or considering other special cases, e.g., $r=p=2$ and $s=q=2$.

```
with(PDEtools): with(plots): declare(u(x,t),U(z)); alias(u=u(x,t),U=U(z));
tr1:=mu*(x+c*t)=z; PDE1:=u->diff(u,t$2)-diff(u,x$2)+u-u^3=0;
Eq1:=expand(PDE1(U(lhs(tr1)))); Eq2:=convert(expand(subs(tr1,Eq1)),diff);
tr3:=U(z)=Sum(a[k]*exp(k*z),k=-r..s)/Sum(b[j]*exp(j*z),j=-p..q);
trpc:=isolate(-3*r-3*p=-r-5*p,p); trqd:=isolate(3*s+3*q=s+5*q,q);
```

*The obtained ODE (`Eq2`) can readily be integrated, and its general solution can be expressed in terms of elliptic integrals.

```
params:={p=1,r=1,q=1,s=1}; tr4:=value(subs(params,tr3));
Eq3:=factor(value(algsubs(subs(params,tr3),Eq2)));
Eq31:=simplify(op(2,lhs(Eq3)));
for i from 1 to 3 do
  E||i:=coeff(Eq31,exp(i*z));
  E||(i+3):=coeff(Eq31,exp(-i*z)); od;
E7:=remove(has,Eq31,exp); E8:=c=c0; sys1:={seq(E||i,i=1..8)};
vars:={mu,c,seq(a[i],i=-1..1),seq(b[j],j=-1..1)};
Sols:=[allvalues([solve(sys1,vars)])]; NSols:=nops(Sols);
Sol1:=op(11,op(7,Sols)); Sol2:=op(15,op(7,Sols));
SolF1:=U=subs(c0=c,subs(Sol1,subs(z=lhs(tr1),rhs(tr4))));
SolF2:=U=subs(c0=c,subs(Sol2,subs(z=lhs(tr1),rhs(tr4))));
SolF12:=simplify(convert(convert(SolF1,trig),tanh),tanh);
SolF22:=simplify(convert(convert(SolF2,trig),tanh),tanh);
SolF13:=collect(subs(b[1]=-1,b[-1]=-1,SolF12),tanh);
SolF23:=collect(subs(b[1]=-1,b[-1]=-1,SolF22),tanh);
factor(PDE1(rhs(SolF13))); factor(PDE1(rhs(SolF23)));
```

39.3.5. Constructing Self-Similar Solutions

Self-similar solutions often occur in numerous problems of mathematical physics. The similarity method is based on symmetry properties of a physical system and the algebraic symmetry of a nonlinear PDE (see Section 27.3). Self-similar solutions are obtained by solving an associated ODE. These solutions are invariant under a *scaling or similarity transformation*. For example, the self-similar solution of the problem of an expanding blast wave was first obtained (on the basis of dimensional analysis) by Taylor (1950) and Sedov (1993). The similarity method is important and useful when finding solutions of both linear and nonlinear PDEs. As before, we illustrate the method of finding self-similar solutions by solving problems.

Example 39.17. *Nonlinear diffusion equation.* Consider the nonlinear diffusion equation

$$u_t = au_{xx} + bu^n.$$

We consider the *scaling transformation*, $t = CT$, $x = C^k X$, $W = C^m W$, where $C \neq 0$ is an arbitrary constant and m and k are some unknown constants. We assume that this equation is invariant under the scaling transformation for suitable values of m and k. We verify that this scaling transformation converts the nonlinear diffusion equation into the equation $C^{m-1}W_T = aC^{m-2k}W_{XX} + bC^{mn}W^n$ (`Eq1`). We show that $k = \frac{1}{2}$, $m = \frac{1}{1-n}$ (`tr2`), and the *self-similar variables* acquire the form $u(x,t) = t^{\alpha}U(\xi)$, $\xi = xt^{\beta}$ (`tr3`), where $\alpha = m$ and $\beta = -k$.

```
with(PDEtools): declare(u(x,t),W(X,T),U(xi)); alias(u=u(x,t),W=W(X,T),U=U(xi));
interface(showassumed=0); assume(k>0,m>0,n>0,C>0,t>0);
DiffusEq:=(t,x,u)->diff(u,t)=a*diff(u,x$2)+b*u^n;
tr1:={t=T*C,x=X*C^k,u=C^m*W};
Eq1:=combine(dchange(tr1,DiffusEq(t,x,u),[T,X,W]));
Ex21:=select(has,lhs(Eq1),C); Ex22:=select(has,rhs(Eq1),k);
Ex23:=select(has,expand(rhs(Eq1)),n);
Ex31:=op(2,Ex21); Ex32:=op(2,select(has,Ex22,k));
Ex33:=op(2,select(has,Ex23,C)); Eqs:={Ex31=Ex32,Ex32=Ex33};
tr2:=convert((solve(Eqs,{k,m}) assuming n<>1),list);
alpha:=rhs(tr2[2]); beta:=-rhs(tr2[1]);
tr3:={xi=x*(t^beta),u=U(xi)*t^(alpha)}; tr31:=x*(t^beta)=xi;
Eq21:=DiffusEq(t,x,U(lhs(tr31))*t^alpha);
```

```
ODE1:=convert(algsubs(tr31,Eq21),diff); c1:=select(has,op(2,lhs(ODE1)),[t,n]);
ODE11:=expand(simplify(ODE1/c1*t));
ODEFin:=map(factor,lhs(collect(factor(ODE11-rhs(ODE11)),xi)))=0;
```

Therefore, the nonlinear diffusion equation is reduced to the ordinary differential equation that has the following Maple form:

$$ODEFin := -\frac{U}{-1+n} - \frac{1}{2}U_{\xi}\xi - bU^n - aU_{\xi,\xi} = 0$$

Example 39.18. *Sine–Gordon equation.* Consider the sine–Gordon equation

$$u_{xt} = \sin u$$

and the scaling transformation $t = C^n T$, $x = C^m X$, $W = C^k W$ (where $C \neq 0$ is an arbitrary constant). We assume that this equation is invariant under the scaling transformation for suitable values of n, m, and k. We verify that this scaling transformation converts the sine–Gordon equation into the equation $C^{-k-n+m} W_{XT} = \sin(C^m W)$ (`Eq1`). We show that $m = -n$, $k = k$ (`tr2`), where we set $k = 0$. The self-similar variables acquire the form $u(x,t) = t^{\alpha} U(\xi)$, $\xi = x/t^{\beta}$ (`tr3`), where $\alpha = -k/(2n)$ and $\beta = n/m$.

```
with(PDEtools): declare(u(x,t),W(X,T),U(xi)); alias(u=u(x,t),W=W(X,T),U=U(xi));
interface(showassumed=0); assume(C>0,t>0);
SGEq:=(t,x,u)->diff(u,x,t)=sin(u);
tr1:={t=T*C^n,x=X*C^m,u=C^k*W};
Eq1:=combine(dchange(tr1,SGEq(t,x,u),[T,X,W]));
Ex21:=select(has,lhs(Eq1),C); Ex22:=select(has,op(rhs(Eq1)),C);
Ex31:=op(2,Ex21); Ex32:=op(2,Ex22); tr2:=solve(Ex31=Ex32,{n,k});
alpha:=subs(k=0,-k/(2*n)); beta:=subs(tr2,n/m);
tr3:={xi=x/(t^beta),u=U(xi)*t^(alpha)}; tr31:=x/(t^beta)=xi;
Eq21:=SGEq(t,x,U(lhs(tr31))*t^alpha);
ODE1:=convert(algsubs(tr31,Eq21),diff);
```

Finally, the sine–Gordon equation is reduced to the ordinary differential equation that has the following Maple form:

$$ODE1 := U_{\xi,\xi}\xi + U_{\xi} = \sin(U)$$

Example 39.19. *Similarity and invariant solutions.* Consider the nonlinear equation

$$u_{yy} + u_y u_{xx} = 0.$$

The similarity and invariant transformations, as well as similarity and invariant solutions, can be obtained with the aid of the predefined functions of the package `PDEtools` as follows:

```
with(PDEtools): declare(u(x,y)); alias(u=u(x,y));
PDE1:=diff(u,y$2)+diff(u,y)*diff(u,x$2)=0;
DepVar:=u; S:=Infinitesimals(PDE1,DepVar); S6:=S[6];
IS:=InfinitesimalGenerator(S6,u,expanded,prolongation=1);
Invariants(S6,u); SimTr:=SimilarityTransformation(S6,u,V(X,T));
SimSols:=SimilaritySolutions(PDE1,u); NewVars:=S6;
InvTr0:=InvariantTransformation(S6,u,NewVars);
InvTr:=subs(_p1=T,_t1=X,_u1=V,[InvTr0]);
InvSols:=[InvariantSolutions(PDE1,dependency={x,y},numberofsolutions=all)];
```

with Maple results

$$SimTr := \{T = \ln(y), X = x, V(X) = yu(x,y)\}, \left\{x = X, y = \mathrm{e}^T, u(x,y) = \frac{V(X)}{\mathrm{e}^T}\right\}$$

$$SimSols := u(x,y)=0, u(x,y)=_C1\,y+_C2, u(x,y)=\frac{x^2+_C1\,x+_C2}{y}, u(x,y)=\tanh\left(\frac{y+_C2}{_C1}\right)x^2\,_C1^{-1}$$

$$InvTr := \left[\{T=y, X=x, V(X)=yu(x,y)\}, \left\{x=X, y=T, u(x,y)=\frac{V(X)}{T}\right\}\right]$$

$$InvSols := \left[u(x,y)=\frac{x^2+_C1\,x+_C2}{y}, u(x,y)=\tanh\left(\frac{y+_C2}{_C1}\right)x^2\,_C1^{-1}\right]$$

Comment. In `SimSols`, we calculate four exact solutions: two trivial solutions ($u=0$ and a solution depending on one independent variable x) and two nontrivial solutions. In `InvSols`, by including additional options (`numberofsolutions`, `dependency`), we obtain just the two desired nontrivial solutions. In `SimTr` and `InvTr`, we determine the transformations that reduce the given PDE to the corresponding ODE.

39.3.6. Constructing Solutions along Characteristics

Consider the methods for finding general solutions of first-order linear, quasilinear, and nonlinear equations, the *method of characteristics*, and the *generalized method of characteristics*. These methods allows reducing a PDE to a system of ODEs along which the given PDE with some initial data (the Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the *characteristic curves* and transformed into a general solution of the original PDE.

By applying the generalized method of characteristics for solving first-order nonlinear PDEs of the form $F(x,y,u,u_x,u_y)=0$ or $F(x,y,u,p,q)=0$ (where $u_x=p$ and $u_y=q$), we show how to obtain a *complete solution* or a *complete integral* of nonlinear equations, i.e., a two-parameter family of surfaces of the form $f(x,y,u,a,b)=0$ (where a and b are parameters). The general solution of a nonlinear PDE can be determined from its complete integral. But in the nonlinear case the solution surface, or integral surface, through which the complete integral has to pass is a cone, known as the *Monge cone*. The characteristic curves depend on the orientation of the tangent planes to the Monge cone at each point. Thus, there exists a Monge cone of characteristics, and the *characteristic equations* or the *Charpit equations* for nonlinear equations have the form

$$\frac{dx}{dt}=F_p,\ \frac{dy}{dt}=F_q,\ \frac{du}{dt}=pF_p+qF_q,\ \frac{dp}{dt}=-(F_x+pF_u),\ \frac{dq}{dt}=-(F_y+qF_u).$$

See also Chapters 23 and 24.

Example 39.20. *Method of characteristics. Quasilinear PDEs. Classical Cauchy problems.*
1°. Consider the initial value problem

$$u_t+t^2uu_x=7, \quad u(x,0)=x.$$

By applying the method of characteristics, we show that the solution of this Cauchy problem is

$$u(x,t)=7t-\frac{3}{4}\frac{7t^4-4x}{t^3+3}$$

and plot the characteristic curves (see Fig. 39.4).

```
with(plots); tR:=0..4; xR:=-40..40; ODE:=diff(U(t),t)=7;
Sol_Ch:=dsolve({ODE,U(0)=X[0]}); Eq_Ch:=diff(x(t),t)=t^2*U(t);
Eq_Ch:=subs(Sol_Ch,Eq_Ch); Cur_Ch:=dsolve({Eq_Ch, x(0)=X[0]});
display([seq(plot([subs(X[0]=x,eval(x(t),Cur_Ch)),t,t=tR],
         color=blue,thickness=2),x=xR)],view=[xR,tR]);
u:=unapply(subs(X[0]=solve(subs(x(t)=x,Cur_Ch),X[0]),eval(U(t),Sol_Ch)),x,t);
```

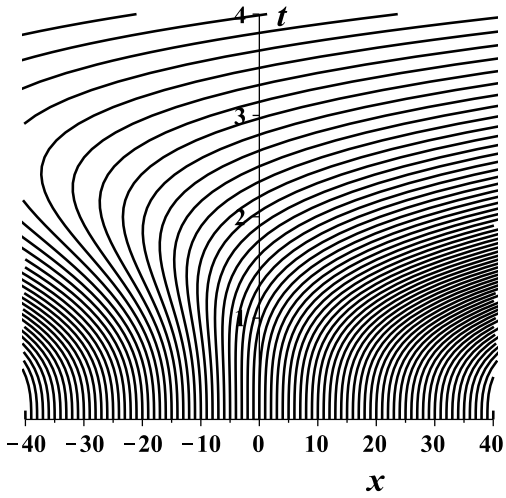

Figure 39.4. Characteristic curves for $u_t + t^2 u u_x = 7$.

2°. Consider the initial value problem

$$u_t - u^2 u_x = 3u, \quad u(x,0) = f(x).$$

We apply the method of characteristics as follows:

```
interface(showassumed=0);
assume(t>0); assume(X[0],constant);
with(plots); tR:=0..1; xR:=-10..10; Ops1:=color=blue,thickness=2;
Ops2:=view=[xR,tR]; tr1:=x=x(t); tr2:=x(t)=x; f1:=x->1; f2:=x->x;
tr3:=f(X[0])=f1(X[0]); tr4:=f(X[0])=f2(X[0]);
ODE1:=diff(x(t),t)=-U(t)^2; ODE2:=diff(U(t),t)=3*U(t);
Sol2:=dsolve(ODE2,U(t)); Sol21:=subs(_C1=C,rhs(Sol2));
IniCond:=u(X[0],0)=f(X[0]); Const2:=evala(subs(t=0,Sol21))=rhs(IniCond);
Sol22:=U(t)=subs(Const2,Sol21); ODE11:=combine(subs(Sol22,ODE1));
Sol1:=dsolve(ODE11,x(t)); X0:=expand(subs(t=0,Sol1));
Const1:=isolate(X0,_C1); Chars:=subs(Const1,x(0)=X[0],Sol1);
GenSol:=rhs(Sol22); Char1:=subs(tr2,tr3,Chars);
Char2:=subs(tr2,tr4,Chars); X01:=isolate(Char1,X[0]);
X02:=[solve(Char2,X[0]) assuming X[0]<>0];
u1:=unapply(subs(f(X[0])=f1(X01),GenSol),x,t);     u1(x,t);
u2:=unapply(subs(f(X[0])=f2(X02[1]),GenSol),x,t);  u2(x,t);
u3:=unapply(subs(f(X[0])=f2(X02[2]),GenSol),x,t);  u3(x,t);
for i from 1 to 2 do
 display([seq(plot([subs(X[0]=x,eval(x(t),Char||i)),t,t=tR],Ops1),x=xR)],Ops2); od;
for i from 1 to 3 do display([seq(plot([u||i(x,t),t,t=tR],Ops1),x=xR)],Ops2); od;
```

We show that the *implicit form of the solution* (or parametric representation of the solution) of this Cauchy problem has the Maple form

$$Sol22 := U(t) = f(X_0)\mathrm{e}^{3t} \quad Chars := x(t) = -1/6\,(f(X_0))^2\,\mathrm{e}^{6t} + X_0 + 1/6\,(f(X_0))^2$$

The characteristic curves for $f(x) = 1$ and $f(x) = x$ and the solutions $u_i(x,t)$ $(i = 1,2,3)$ for $f(x) = 1$ and $f(x) = x$ are presented in Figs. 39.5 and 39.6, respectively.

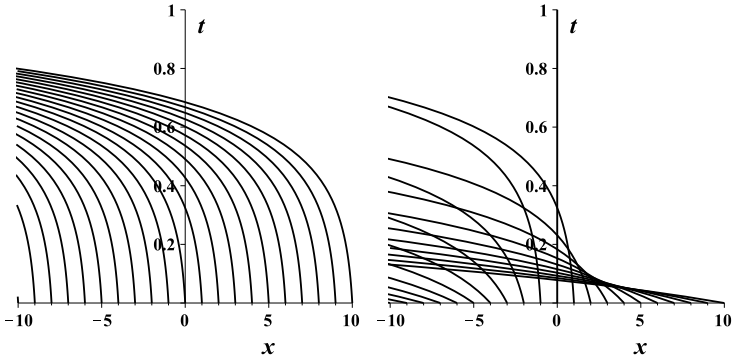

Figure 39.5. Characteristic curves for $f(x) = 1$ and $f(x) = x$.

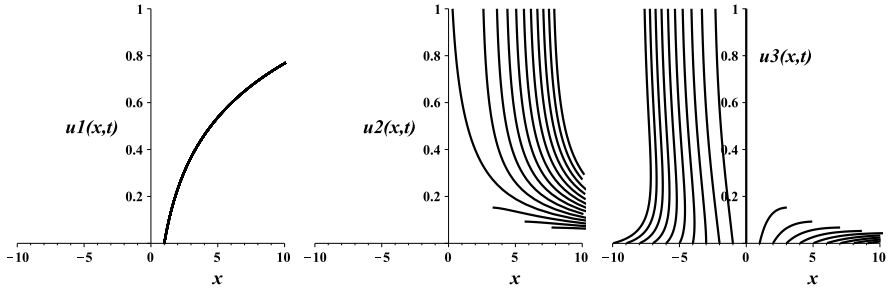

Figure 39.6. Solution $u_1(x,t)$ for $f(x) = 1$ and solutions $u_2(x,t)$, $u_3(x,t)$ for $f(x) = x$.

Example 39.21. *Method of characteristics. Quasilinear PDEs. Generalized Cauchy problem.* Consider the first-order quasilinear PDE

$$uu_x + u_y = 1$$

and assume that an integral surface of this equation passes through the initial curve parametrically represented by the equations

$$x = x_0(r) = r, \quad y = y_0(r) = r, \quad u = u_0(r) = 0 \quad (r \text{ is a parameter}).$$

By applying the method of characteristics, we show that the integral surface of this equation has the form

$$(u-1)^2 + 2(y-x) = 1.$$

```
with(PDEtools): declare(u(x,y));
PDE:=u->u(x,y)*diff(u(x,y),x)+diff(u(x,y),y)=1; expand(PDE(u));
IniCurve:=[X=X0(r),Y=Y0(r),U=U0(r)];
CharEqs:=DX/U=(DY/1=DU/1);
CharEqsParamForm:=[diff(X(t),t)=U(t),diff(Y(t),t)=1,diff(U(t),t)=1];
IniConds:=[x(r,0)=rhs(IniCurve[1]),y(r,0)=rhs(IniCurve[2]),
          u(r,0)=rhs(IniCurve[3])];
Eq3:=U(r,t)=int(rhs(CharEqsParamForm[3]),t)+rhs(IniCurve[3]);
Eq2:=Y(r,t)=int(rhs(CharEqsParamForm[2]),t)+rhs(IniCurve[2]);
Eq1:=X(r,t)=int(subs(U(t)=rhs(Eq3),rhs(CharEqsParamForm[1])),t)+rhs(IniCurve[1]);
IniData:=[r,r,0];
IniCondsPart:=[x(r,0)=IniData[1],y(r,0)=IniData[2],u(r,0)=IniData[3]];
tr1:=[X0(r)=rhs(IniCondsPart[1]),Y0(r)=rhs(IniCondsPart[2]),
      U0(r)=rhs(IniCondsPart[3])];
sys1:={Eq1,Eq2,Eq3}; sys2:=subs(tr1,sys1);
tr2:=[X(r,t)=x,Y(r,t)=y,U(r,t)=u];
```

```
sys3:=subs(tr2,sys2);
SolFin:=eliminate(sys3,{r,t});
with(Student[Precalculus]):
CompleteSquare(op(SolFin[2]))=0;
```

Example 39.22. *Generalized method of characteristics. Nonlinear PDEs. Complete integrals.* Consider the first-order nonlinear PDE

$$u_x^2 + yu_y = u.$$

By applying the generalized method of characteristics, we show that the complete integral of this nonlinear equation has the form

$$u(x,t) = \tfrac{1}{4}(x+b)^2 + ay.$$

```
with(Student[Precalculus]): F:=p^2+q*y-u;
CharEqs:=[diff(x(t),t)=diff(F,p),diff(y(t),t)=diff(F,q),
          diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
          diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
          diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:=q(t)=q; tr2:=du=p*dx+q*dy; tr3:=u-a*y=v;
Eq1:=map(int,CharEqs[5],t); Eq2:=lhs(Eq1)=rhs(Eq1)+a;
Eq3:=[solve(subs(subs(tr1,Eq2),F),p)]; Eq4:=p=Eq3[1];
Eq5:=(dx/rhs(CharEqs[1])=dy/rhs(CharEqs[2]))=du/rhs(CharEqs[3]);
Eq6:=subs(tr2,Eq5); Eq7:=subs(p=rhs(Eq4),q=rhs(Eq2),tr2);
Eq8:=expand(isolate(Eq7,dx));
Eq81:=rhs(Eq8)=d(lhs(tr3))/denom(rhs(Eq8)); Eq82:=subs(tr3,Eq81);
Eq83:=int(1/denom(rhs(Eq82)),v)=int(lhs(Eq8)/dx,x)+b;
Eq84:=subs(v=lhs(tr3),lhs(Eq83))=rhs(Eq83);
Sol:=solve(Eq84,u); CompleteInt:=u(x,y)=CompleteSquare(Sol);
```

Example 39.23. *Generalized method of characteristics. Classical Cauchy problem.* Consider the first-order nonlinear PDE

$$(u_x)^2 u_y = 1$$

with the initial data $u(x,0) = x$. By applying the generalized method of characteristics, we show that the solution of this Cauchy problem has the form $u(x,t) = x + y$.

```
F:=p^2*q-1; IniData:={u=x,y=0};
CharEqs:=[diff(x(t),t)=diff(F,p),diff(y(t),t)=diff(F,q),
          diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
          diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
          diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:={q(t)=q,p(t)=p}; tr2:=du=p*dx+q*dy;
Eq1:=dsolve(CharEqs[4],p(t)); Eq2:=subs(tr1,_C1=a,Eq1);
Eq3:=dsolve(CharEqs[5],q(t)); Eq4:=subs(tr1,_C1=c,Eq3);
Eq5:=subs(Eq2,Eq4,F); Eq51:=isolate(Eq5,c);
Eq6:=subs(Eq51,tr2); Eq7:=subs(p=rhs(Eq2),q=rhs(Eq4),Eq51,tr2);
Sol:=int(lhs(Eq7)/du,u)=int(op(1,rhs(Eq7))/dx,x)+int(op(2,rhs(Eq7))/dy,y)+b;
Sol1:=subs(IniData,Sol); Consts:={a=coeff(lhs(Sol1),x),b=coeff(lhs(Sol1),x,0)};
SolCauchy:=subs(Consts,Sol);
```

Example 39.24. *Generalized method of characteristics. Generalized Cauchy problem.* Consider the first-order nonlinear PDE

$$(u_x)^2 + u_y + u = 0.$$

Let us find a solution of this equation that passes through the initial curve represented parametrically by the equations $x(r,0) = r$, $y(r,0) = r$, $u(r,0) = 2r - 1$, where r is a parameter. Applying the generalized method of characteristics, we state the Cauchy problem for this nonlinear equation to determine the family of curves in

the parametric form $u = u(r,t)$, $x = x(r,t)$, $y = y(r,t)$ and the values $p = p(r,t)$ and $q = q(r,t)$, where r and t are parameters. When seeking the solution surface $u = u(x,y)$ with given initial data at $t = 0$, we first determine the initial conditions for $p(r,t)$ and $q(r,t)$ at $t = 0$ (`sols`):

$$p(r,0) = \tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{-3-8r}, \quad q(r,0) = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{-3-8r},$$

which must satisfy the original equation and the *strip condition* (`StripCond`)

$$\frac{du(r,0)}{dr} = p(r,0)\frac{dx(r,0)}{dr} + q(r,0)\frac{dy(r,0)}{dr}.$$

Take the first pair of values of $p(r,0)$ and $q(r,0)$. Then we can determine the values $p = p(r,t)$ and $q = q(r,t)$ (`Solp`, `Solq`)

$$p(r,t) = -\tfrac{1}{2}(-1+S)e^{-t}, \quad q(r,t) = \tfrac{1}{2}(3+S)e^{-t},$$

where $S = \sqrt{-3-8r}$, and the solution of the Cauchy problem as the family of curves in the parametric form $u = u(r,t)$, $x = x(r,t)$, $y = y(r,t)$ (`SolFin`):

$$u(r,t) = -\left(\tfrac{3}{2} + \tfrac{1}{2}S\right)e^{-t} + \left(\tfrac{1}{2}(1+S) + 2r\right)e^{-2t}, \quad x(r,t) = (1-S)(1-e^{-t}) + r, \quad y(r,t) = t + r.$$

Finally, we verify that this solution is an exact solution of the given nonlinear PDE (`test1`).

```
interface(showassumed=0); assume(t>0); F:=p^2+q+u; F1:=p(r,t)^2+q(r,t)+u(r,t);
IniData:=[x(r,0)=r,y(r,0)=r,u(r,0)=2*r-1]; IniData1:=[x(0)=r,y(0)=r,u(0)=2*r-1];
CharEqs:=[diff(x(t),t)=diff(F,p), diff(y(t),t)=diff(F,q),
          diff(u(t),t)=p*diff(F,p)+q*diff(F,q),
          diff(p(t),t)=-(diff(F,x)+p*diff(F,u)),
          diff(q(t),t)=-(diff(F,y)+q*diff(F,u))];
tr1:={p(t)=p(r,t),q(t)=q(r,t)}; tr2:={p=p(r,t),q=q(r,t)};
Eq1:=subs(IniData,subs(t=0,F1))=0;
StripCond:=diff(u(r,0),r)=p(r,0)*diff(x(r,0),r)+q(r,0)*diff(y(r,0),r);
IniData2:=diff(IniData,r); Eq2:=subs(IniData2,StripCond);
sys1:={Eq1,Eq2}; vars:=indets(sys1) minus {r};
sols:=[allvalues(solve(sys1,vars))];
for k from 4 to 5 do
 Eq||k||1:=subs(tr1,lhs(CharEqs[k]))=subs(tr2,rhs(CharEqs[k])); od;
Solp:=dsolve({Eq41} union {op(1,sols[1])},p(r,t));
Solq:=dsolve({Eq51} union {op(2,sols[1])},q(r,t));
tr4:={p=rhs(Solp),q=rhs(Solq)}; tr5:={x(t)=x,y(t)=y,u(t)=u};
Eq6:=combine(subs(tr4,[CharEqs[i] $ i=1..3]));
Eq7:=dsolve({op(Eq6),op(IniData1)},{x(t),y(t),u(t)});
Eq71:= Eq7 union {Solp} union {Solq};
SolFin:=subs({u(t)=u(r,t),x(t)=x(r,t),y(t)=y(r,t)},Eq71);
test1:=simplify(subs(SolFin,F1));
```

39.3.7. Constructing Separable Solutions

Separation of variables is one of the most important methods for solving linear PDEs, in which the structure of a PDE allows us to seek *multiplicatively separable* or *additively separable* exact solutions; e.g., $u(x,t) = \phi(x) \circ \psi(t)$ (where the multiplication or addition is denoted by $\circ$). Although for some specific nonlinear PDEs one can directly apply this idea of separation of variables (just as for linear PDEs), it does not apply to wide classes of nonlinear PDEs. We consider the generalizations of the *classical method of separation of variables* developed in recent years by Galaktionov (1990, 1995), Polyanin and Zhurov (1998), Polyanin and Zaitsev (2004), Galaktionov and Svirshchevskii (2006), and Polyanin and Manzhirov (2007).

Generalized separable solutions can be written in the form

$$u(x,t)=\phi_1(x)\psi_1(t)+\cdots+\phi_n(x)\psi_n(t).$$

Unlike linear equations, in nonlinear equations the functions $\phi_i(x)$ with different subscripts i are usually related to one another [and to the functions $\psi_j(y)$]. In general, the functions $\varphi_i(x)$ and $\psi_j(y)$ are not known in advance and are to be identified. See Chapter 29 for details.

Then, for nonlinear PDEs it is of great interest to find *functional separable solutions*, i.e., exact solutions of the form

$$u(x,t)=F(z) \quad \text{or} \quad F(u(x,t))=z,$$

where $z=\phi_1(x)\psi_1(t)+\cdots+\phi_n(x)\psi_n(t)$. These solutions can be found if the functions $F(z)$ (or $F(u)$), $\phi_i(x)$, and $\psi_i(t)$ $(i=1,\ldots,n)$ exist. See Chapter 30 for details.

In recent years, numerous approaches have been proposed for studying generalized and functional separation of variables, e.g., by specifying some systems of coordinate functions $\{\phi_i(x)\}$ or $\{\psi_i(t)\}$ and by finding special functional separable solutions, by differentiating and splitting, by theory of linear subspaces invariant under nonlinear differential operators, by extending the Lie theory, etc. We note that the classical additive and multiplicative separable solutions and the generalized separable solutions can be treated as special cases of the above functional separable solution if $u(x,t)=F(z)=z$ (or $F(u(x,t))=u(x,t)=z$).

We consider various problems in which one can construct generalized and functional separable solutions by using various approaches, e.g., by differentiating, splitting, etc.

Example 39.25. *Generalized separable solutions by differentiation: nth-order nonlinear PDE.* Consider the nth-order nonlinear PDE

$$u_t u_{xt}-u_x u_{tt}=f(x)u_t^{(n)},$$

where $f(x)$ is an arbitrary function. By applying the method of generalized separation of variables and by seeking exact solutions in the form $u(x,t)=\phi_1(x)\psi_1(t)+\phi_2(x)$, we arrive at the following equations (`Eq71`, `Eq72`):

$$-\frac{-\psi_1{}_t^2+\psi_{1tt}\psi_1}{\psi_{1tt}}=C_1, \quad \frac{\phi_{2x}f_x(x)-\phi_{2xx}f(x)}{\phi_{1x}f_x(x)-\phi_{1xx}f(x)}=C_1.$$

```
with(PDEtools): declare(u(x,t),W(x,t),phi1(x),phi2(x),psi1(t));
interface(showassumed=0): assume(n,'integer',n>0): tr1:=phi1(x)*psi1(t)+phi2(x);
PDE1:=u->diff(u(x,t),t)*diff(u(x,t),x,t)
        -diff(u(x,t),x)*diff(u(x,t),t$2)=f(x)*diff(u(x,t),t$n);
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=collect(expand(Eq3/phi1(x)),diff(phi1(x),x));
Eq5:=expand(Eq4/f(x)); Eq6:=diff(Eq5,x);
Eq61:=isolate(isolate(Eq6,phi1(x)),phi2(x)); Eq62:=factor(Eq61);
TermX:=select(has,rhs(Eq62),f(x)); Eq7:=evala(Eq62/TermX);
Eq71:=rhs(Eq7)=_C1; Eq72:=lhs(Eq7)=_C1;
```

We seek exact solutions of this equation. Consider the following two cases:

(1) *Nondegenerate case*: the separation constant is nonzero.

We find that $\psi_1(t)=-C_1+C_4e^{\lambda t}$ (`Sol2`), $\phi_2(x)=\phi_1(x)C_1+C_2\int f(x)\,dx+C_3$ (`Sol11`), and the exact solution acquires the form $u(x,t)=\phi_1(x)e^{\lambda t}-\lambda^{n-2}\int f(x)\,dx+C$ (`SolFin1`). Here $\phi_1(x)$ is an arbitrary function, and C and λ are arbitrary constants.

```
Sol1:=dsolve(Eq72,phi2(x)); Sol11:=simplify(map(value,Sol1));
Sol2:=simplify(subs({_C2=lambda,_C3=_C4},dsolve(Eq71,psi1(t))));
Eq8:=algsubs(Sol11,algsubs(Sol2,Eq4)); C34:={_C3=C,_C4=1};
Eq81:=subs(C34,expand(Eq8/exp(lambda*t)));
C2:=simplify(isolate(Eq81,_C2)); Sol21:=subs(C2,C34,Sol2);
Sol12:=subs(C2,C34,Sol11);
SolFin1:=u(x,t)=algsubs(Sol12,algsubs(Sol21,tr1));
pdetest(SolFin1,PDE1(u));
```

(2) *Degenerate case*: the separation constant is zero.
We find that $\phi_1(x) = C_1 + C_2 \int f(x)\,dx$ (Sol3) and $\phi_2(x) = C_3 + C_4 \int f(x)\,dx$ (Sol4). The ODE $C_2(\psi_{1t})^2 - \psi_{1tt}(C_2\psi_1 + C_4) = \psi_1^{(n)}$ (Eq11) for the function $\psi_1 = \psi_1(t)$ describes the exact solution of the form $u(x,t) = \phi_1(x)\psi_1(t) + \phi_2(x)$.

```
Eq9:=denom(lhs(Eq7))=0; Eq10:=numer(lhs(Eq7))=0;
Sol3:=dsolve(Eq9,phi1(x)); Sol4:=subs({_C1=_C3,_C2=_C4},
dsolve(Eq10,phi2(x)));
Eq11:=simplify(algsubs(Sol3,algsubs(Sol4,Eq4))/f(x));
```

Example 39.26. *Generalized separable solutions by splitting.* Consider the third-order nonlinear PDE

$$u_{xt} + (u_{xx})^2 - uu_{xx} = \nu u_{xxx}.$$

By applying the method of generalized separation of variables, by seeking exact solutions in the form $u(x,t) = \phi_1(x)\psi_1(t) + \psi_2(t)$, and by applying the splitting procedure, we arrive at the functional equation $\Phi_1(x)\Psi_1(t) + \cdots + \Phi_4(x)\Psi_4(t) = 0$, where $\Phi_1 = \phi_{1x}$, $\Psi_1 = \psi_{1t}$, $\Phi_2 = -\phi_{1xx}$, $\Psi_2 = \psi_1\psi_2$, $\Phi_3 = -\phi_1\phi_{1xx} + (\phi_{1x})^2$, $\Psi_3 = \psi_1^2$, $\Phi_4 = -\phi_{1xxx}$, and $\Psi_4 = \nu\psi_1$. Knowing the solutions of this functional equation, e.g., the solution $\Psi_1 = A_1\Psi_3 + A_2\Psi_4$, $\Psi_2 = A_3\Psi_3 + A4\Psi_4$, $\Phi_3 = -A_1\Phi_1 - A_3\Phi_2$, $\Phi_4 = -A_2\Phi_1 - A_4\Phi_2$, we arrive at the following system of ODEs (sys1):

$$\psi_{1t} = A_1\psi_1^2 + A_2\nu\psi_1, \quad \psi_1\psi_2 = A_3\psi_1^2 + A_4\nu\psi_1,$$
$$-\phi_1\phi_{1xx} + (\phi_{1x})^2 = -A_1\phi_{1x} + A_3\phi_{1xx}, \quad -\phi_{1xxx} = -A_2\phi_{1x} + A_4\phi_{1xx}.$$

```
with(PDEtools): declare(u(x,t),W(x,t),psi1(t),psi2(t),phi1(x));
PDE1:=u->diff(u(x,t),x,t)+(diff(u(x,t),x))^2
        -u(x,t)*diff(u(x,t),x$2)=nu*diff(u(x,t),x$3);
tr1:=psi1(t)*phi1(x)+psi2(t); Eq2:=expand(PDE1(W));
Eq3:=sort(collect(expand(subs(W(x,t)=tr1,Eq2)),psi1(t)));
Eq31:=Eq3-rhs(Eq3); FunDiffEq1:=add(Phi[i]*Psi[i],i=1..4)=0;
SolFunDiffEq1:=[Psi[1]=A1*Psi[3]+A2*Psi[4],Psi[2]=A3*Psi[3]+A4*Psi[4],
                Phi[3]=-A1*Phi[1]-A3*Phi[2],Phi[4]=-A2*Phi[1]-A4*Phi[2]];
L1:=[selectremove(has,op(2,lhs(Eq31)),[psi1])];
L2:=[selectremove(has,op(1,lhs(Eq31)),[psi1,psi2])];
L3:=[selectremove(has,op(3,lhs(Eq31)),[psi1])];
L4:=[selectremove(has,op(4,lhs(Eq31)),[nu,psi1])];
tr2:=[Phi[1]=L1[2],Psi[1]=L1[1],Phi[2]=L2[2],Psi[2]=L2[1],
      Phi[3]=L3[2],Psi[3]=L3[1],Phi[4]=L4[2],Psi[4]=L4[1]];
sys1:=subs(tr2,SolFunDiffEq1);
```

The last two equations in the system are consistent if and only if the function $\phi_1(x)$ satisfies the equation $\phi_{1x} = B_1\phi_1 + B_2$. (We show this after the solution procedure.) By integrating these equations, we find the solutions $\phi_1 = B_2x + B_3$ (if $B_1 = 0$, Solphi12), $\phi_1 = -B_2/B_1 + B_3e^{B_1x}$ (if $B_1 \neq 0$, Solphi11), $\psi_1 = -1/(A_1t - B_4)$ (if $A_2 = 0$, Solpsi12), $\psi_1 = -\nu A_2/(A_1 - \nu A_2B_4e^{-\nu A_2t})$ (if $A_2 \neq 0$, Solpsi11), and $\psi_2 = A_3\psi_1 + \nu A_4$ (Solpsi2).

By determining the relations between the arbitrary constants A_1, A_2, A_3, A_4, B_1, and B_2, we arrive at the equations $-A_4B_1 + A_2 - B_1^2 = 0$ (Eq621), $B_1(B_2 - A_3B_1 + A_1) = 0$ (Eq63), and $B_2(B_2 - A_3B_1 + A_1) = 0$ (Eq64).

All these formulas allow us to determine the exact solutions of the form $u(x,t)=\phi_1(x)\psi_1(t)+\psi_2(t)$. If we consider, for example, the case in which $A_2=0$, $B_1=0$, and $B_2=-A_1$, then we obtain the final form of the exact solution $u(x,t)=\dfrac{x+C_1}{t+C_2}+C_3$ (SolFin13), where $C_1=\dfrac{B_3+A_3}{B_2}$, $C_2=\dfrac{B_4}{B_2}$, $C_3=A_4\nu$.

```
tr3:=_C1=B3; tr4:=_C1=B4; Eqsys1:=diff(phi1(x),x)=B1*phi1(x)+B2;
Eq41:=algsubs(Eqsys1,sys1[3]); Eq42:=algsubs(Eqsys1,sys1[4]);
Eq51:=isolate(Eq41,phi1(x)); Eq52:=isolate(Eq42,phi1(x));
Eq61:=denom(rhs(Eq52))=0; Eq611:=select(has,lhs(Eq61),A4)=0;
Eq62:=numer(rhs(Eq52))=0; Eq621:=select(has,lhs(Eq62),A4)=0;
Eq63:=denom(rhs(Eq51))=0; Eq64:=numer(rhs(Eq51))=0;
vars:=indets(Eq611) union indets(Eq63);
SolConst:=[solve({Eq611,Eq63,Eq64},vars)]; SolConst1:=convert(SolConst[1],list);
SolConst2:=convert(SolConst[2],list); Sol1:=solve({Eq611,Eq63,Eq64},vars);
Solphi11:=subs(tr3,dsolve(Eqsys1,phi1(x)));
Solphi12:=subs(tr3,dsolve(subs(B1=0,Eqsys1),phi1(x)));
Solpsi11:=subs(tr4,dsolve(sys1[1],psi1(t)));
Solpsi12:=subs(tr4,dsolve(subs(A2=0,sys1[1]),psi1(t)));
Solpsi2:=simplify(isolate(sys1[2],psi2(t)));
SolFin11:=collect(subs(Solpsi2,tr1),psi1(t));
SolFin12:=subs(SolConst2[2],SolConst2[5],subs(SolConst1[1],
          subs(Solphi12,Solpsi12,SolFin11)));
SolFin13:=(expand(numer(op(1,SolFin12))/B2))/
          (expand(denom(op(1,SolFin12))/B2))+op(2,SolFin12);
```

Let us show that if the last two equations in the system obtained above, i.e., $-\phi_1\phi_{1xx}+(\phi_{1x})^2=-A_1\phi_{1x}+A_3\phi_{1xx}$ and $-\phi_{1xxx}=-A_2\phi_{1x}+A_4\phi_{1xx}$, are consistent, then the function $\phi_1(x)$ satisfies the equation $\phi_{1x}=B_1\phi_1+B_2$; and if the function $\phi_1(x)$ satisfies the equation $\phi_{1x}=B_1\phi_1+B_2$, then these two equations are consistent.

```
E1:=lhs(sys1[3])-rhs(sys1[3]); E2:=lhs(sys1[4])-rhs(sys1[4]);
E21:=expand(isolate(E2,select(has,E2,A4))/(-A4));
E11:=factor(isolate(E1,select(has,E1,A3)));
E12:=E11-op(1,rhs(E11)); E13:=factor(E12*(-1));
E14:=isolate(E13,diff(phi1(x),x$2)); E3:=expand(rhs(E21)-rhs(E14));
E4:=collect(E3,diff(phi1(x),x)); E5:=subs(diff(phi1(x),x$3)=0,E4);
E6:=simplify(expand(E5)); E7:=select(has,E6,A2)=0;
E8:=expand(isolate(E7,diff(phi1(x),x))); sys21:=[E1=0,E2=0];
trphi1:=diff(phi1(x),x)=B1*phi1(x)+B2; sys22:=algsubs(trphi1,sys21);
Solsys21:=op(solve(sys22,phi1(x))); expand(subs(E8,trphi1));
```

Example 39.27. *Generalized separable solutions. Simplified approach.* Consider the third-order nonlinear partial differential equation

$$u_t u_{xt}-u_x u_{tt}=\nu u_{ttt}.$$

By applying the method of generalized separation of variables and the simplified approach, i.e., by specifying one of the systems of coordinate functions $\{\phi_i(x)\}$ and by seeking exact solutions in a simplified form, e.g., $u(x,t)=x\psi_1(t)+\psi_2(t)$, we arrive at the following ODEs (Eq51, Eq52):

$$(\psi_{1t})^2-\psi_1\psi_{1tt}-\nu\psi_{1ttt}=0,\quad \psi_{1t}\psi_{2t}-\psi_1\psi_{2tt}-\nu\psi_{2ttt}=0.$$

Then, by applying Maple predefined functions (odeadvisor, symgen, symtest, and equinv contained in the package DEtools), we can solve the first ODE performing a symmetry reduction, i.e., reduction of the order of the equation. This means that we can seek the most general ODE invariant under some transformations. In our case, these transformations are $\{\xi=1,\eta=0\}$, $\{\xi=-t,\eta=\psi_1\}$ (Symms). Then, by solving the most general ODE that is invariant under these transformations, i.e., the equation $\psi_{1t}=C_1(\psi_1)^2$ (ODE51), we arrive at the solution of the first ODE $\psi_1=-1/(C_1t-C_2)$ (Sol1), where C_1 and C_2 are arbitrary constants. Then,

by determining the constant $C_1 = -1/(6\nu)$ (`SolConsts`) and by taking $B_1 = 6\nu C_2$, we arrive at the solution $\psi_1 = 6\nu/(t+B_1)$ (`Solpsi1`) of the first ODE. By solving the second ODE, we obtain the solution $\psi_2 = C_1 + C_3/(t+B_1) + C_2/(t+B_1)^2$ (`Solpsi2`).

Finally, the exact solution of the given nonlinear PDE in the desired simplified form is (`SolFin1`)

$$u(x,t) = \frac{6\nu x}{t+B_1} + \frac{C_3}{t+B_1} + \frac{C_2}{(t+B_1)^2} + C_1,$$

where B_1, C_1, C_2, C_3 are arbitrary constants.

```
with(PDEtools): with(DEtools): declare(u(x,t),W(x,t),psi1(t),psi2(t));
tr1:=x*psi1(t)+psi2(t);
PDE1:=u->diff(u(x,t),t)*diff(u(x,t),x,t)
        -diff(u(x,t),x)*diff(u(x,t),t$2)-nu*diff(u(x,t),t$3)=0;
Eq2:=expand(PDE1(W)); Eq3:=expand(subs(W(x,t)=tr1,Eq2));
Eq4:=collect(Eq3,x); ODEs:=[selectremove(has,lhs(Eq4),x)];
Eq51:=expand(ODEs[1]/x)=0; Eq52:=ODEs[2]=0; odeadvisor(Eq51);
infolevel[symgen]:=2: Symms:=symgen(Eq51,psi1(t));
map(symtest,[Symms],Eq51); ODE51:=equinv([Symms],psi1(t));
Sol1:=dsolve(ODE51,psi1(t)); Consts:=algsubs(Sol1,Eq51);
SolConsts:=_C1=solve(select(has,lhs(Consts),nu)=0,_C1);
Solpsi1:=subs(6*nu*_C2=B1,simplify(subs(SolConsts,Sol1)));
Eq521:=algsubs(simplify(Solpsi1),Eq52); Solpsi2:=dsolve(Eq521,psi2(t));
SolFin1:=u(x,t)=subs(Solpsi1,Solpsi2,tr1); pdetest(SolFin1,PDE1(u));
```

Example 39.28. *Functional separation of variables. Simplified approach.* Consider the nonlinear diffusion equation

$$u_t = u_{xx} + F(u),$$

where $F(u)$ is an arbitrary nonlinear function. By applying the method of *functional separation of variables*, $u(x,t) = W(z)$, and the simplified approach, i.e., by specifying one of the systems of coordinate functions $\{\phi_i(x)\}$ and by seeking exact solutions in a simplified form, e.g., $z = x\psi_1(t) + \psi_2(t)$, we arrive at the following ODE (`Eq51`):

$$-\frac{\psi_{1t} z}{\psi_1} + \frac{\psi_{1t}\psi_2}{\psi_1} - \psi_{2t} + \frac{\psi_1^2 W_{zz}}{W_z} + \frac{F(W(z))}{W_z} = 0.$$

Then, by applying the splitting procedure, i.e., by rewriting this equation in the form of a functional equation, we obtain $\Phi_1(x)\Psi_1(t) + \cdots + \Phi_4(x)\Psi_4(t) = 0$, where $\{\Phi_1 = 1,\ \Psi_1 = \psi_{1t}\psi_2/\psi_1 - \psi_{2t},\ \Phi_2 = z,\ \Psi_2 = -\psi_{1t}/\psi_1,\ \Phi_3 = W_{zz}/W_z,\ \Psi_3 = \psi_1^2,\ \Phi_4 = F(W(z))/W_z,\ \Psi_4 = 1\}$ (`tr3`). Knowing the solutions of this functional equation, e.g., we arrive at the following system of ODEs (`sys1`): $\{\psi_{1t}\psi_2/\psi_1 - \psi_{2t} = A_1\psi_1^2 + A_2,\ -\psi_{1t}/\psi_1 = A_3\psi_1^2 + A_4,\ W_{zz}/W_z = -A_1 - A_3 z,\ F(W(z))/W_z = -A_2 - A_4 z\}$, where the A_i $(i = 1,\ldots,4)$ are arbitrary constants.

```
with(PDEtools):
declare(psi1(t),psi2(t),W(z)); interface(showassumed=0); assume(z>0):
for i from 1 to 4 do assume(A||i,constant); od:
tr1:=psi1(t)*x+psi2(t)=z; tr2:=expand(isolate(tr1,x));
PDE1:=u->diff(u,t)=diff(u,x$2)+F(u);
Eq2:=expand(PDE1(W(lhs(tr1)))); Eq3:=convert(algsubs(tr1,Eq2),diff);
Eq4:=expand(Eq3/diff(W(z),z)); Eq5:=expand(subs(tr2,Eq4));
Eq51:=-Eq5+rhs(Eq5); FunDiffEq1:=add(Phi[i]*Psi[i],i=1..4)=0;
SolFunDiffEq1:=[Psi[1]=A1*Psi[3]+A2*Psi[4], Psi[2]=A3*Psi[3]+A4*Psi[4],
                Phi[3]=-A1*Phi[1]-A3*Phi[2],Phi[4]=-A2*Phi[1]-A4*Phi[2]];
L11:=[selectremove(has,op(2,lhs(Eq51)),[psi1,psi2])];
L12:=[selectremove(has,op(3,lhs(Eq51)),[-1,psi2])];
L2:=[selectremove(has,op(1,lhs(Eq51)),[-1,psi1])];
L3:=[selectremove(has,op(4,lhs(Eq51)),[psi1])];
```

```
L4:=[selectremove(has,op(5,lhs(Eq51)),[1])];
tr3:=[Phi[1]=L11[2]*L12[2],Psi[1]=L11[1]+L12[1],Phi[2]=L2[2],Psi[2]=L2[1],
      Phi[3]=L3[2],Psi[3]=L3[1],Phi[4]=L4[2],Psi[4]=L4[1]];
sys1:=subs(tr3,SolFunDiffEq1);
```

By integrating these equations, we can find the unknown functions $\psi_1(t)$, $\psi_2(t)$, $W(z)$, and $F(W(z))$ with four integrating constants B_i $(i=1,\ldots,4)$ and the exact solution of the form $u(x,t)=W(x\psi_1(t)+\psi_2(t))$. If we consider, say, the special case in which $A_1=-1$, $A_2\neq 0$, $A_3=0$, $A_4\neq 0$, $B_1\neq 0$, $B_2\neq 0$, $B_3=1$, and $B_4=0$, then we obtain the following explicit form of the exact solution (`Sol11`, `Sol12`):

$$u(x,t)=W(z)=e^z=e^{x\psi_1(t)+\psi_2(t)},\quad F(W(z))=-W(z)\bigl(A_2+A_4\ln(W(z))\bigr),$$
$$\psi_1(t)=\pm\frac{\sqrt{A_4^2B_1e^{2A_4t}}}{A_4B_1e^{2A_4t}},\quad \psi_2(t)=-\frac{A_2B_1e^{2A_4t}+1\pm B_2\sqrt{A_4^2B_1e^{2A_4t}}}{e^{2A_4t}A_4B_1}.$$

```
tr4:=_C1=B1; tr5:=_C1=B2;
S2:=subs(tr4,[dsolve(sys1[2],psi1(t))]); S2[1];
Expr:=subs(tr5,map(expand,dsolve(sys1[1],psi2(t))));
S1[1]:=simplify(subs(S2[1],Expr)); S1[2]:=simplify(subs(S2[2],Expr));
Eq3sys1:=int(lhs(sys1[3]),z)=int(rhs(sys1[3]),z);
Eq3sys11:=isolate(Eq3sys1,diff(W(z),z));
S3:=int(lhs(Eq3sys11),z)=B3*Int(rhs(Eq3sys11),z)+B4;
Eq4sys1:=expand(subs(S3,(subs(F(W(z))=F1,sys1[4]))));
S4:=F(W(z))=rhs(simplify(isolate(Eq4sys1,F1)));
Sol1:=[S1[1],S2[1],S3,S4]; Sol2:=[S1[2],S2[2],S3,S4];
Consts:=[A1=-1,A3=0,B3=1,B4=0];
Sol11:=factor(value(subs(Consts,Sol1)));
Sol12:=factor(value(subs(Consts,Sol2)));
tr6:=expand(ln(rhs(Sol11[3])))=combine(ln(lhs(Sol11[3])));
F1:=F(W(z))=factor(expand(subs(tr6,rhs(Sol11[4]))));
```

Example 39.29. *Functional separation of variables by differentiation. Klein–Gordon equation.* Consider the nonlinear Klein–Gordon equation

$$u_{tt}-u_{xx}=F(u),$$

where $F(u)$ is a nonlinear function. By seeking exact solutions in the form $u(x,t)=W(z)$, $z=\phi(x)+\psi(t)$, we arrive at the following functional-differential equation with three variables (`Eq8`):

$$\psi_{tt}-\phi_{xx}+(\psi_t^2-\phi_x^2)G(z)=H(z), \tag{39.3.7.1}$$

where $G(z)=W_{zz}/W_z$, $H(z)=F(W(z))/W_z$. By differentiating this equation with respect to t and x and by simplifying the resulting equation, we obtain the following functional-differential equation with two variables (`Eq14`): $(\psi_t^2-\phi_x^2)(G_{zz}-2G(z)G_z)=H_{zz}-2H(z)G_z$.

```
with(PDEtools):
declare(phi(x),psi(t));
tr1:=phi(x)+psi(t)=z;
PDE1:=u->diff(u,t$2)-diff(u,x$2)=F(u);
Eq2:=expand(PDE1(W(lhs(tr1)))); Eq3:=subs(tr1,Eq2);
Eq4:=convert(expand(Eq3/diff(W(z),z)),diff);
Eq5:=collect(Eq4,diff); Eq6:=map(normal,lhs(Eq5))=rhs(Eq5);
trH:= H(z)=rhs(Eq4); trG:=normal(select(has,lhs(Eq4),W));
trG1:=[selectremove(has,trG,W)]; trG2:=G(z)=trG1[1];
Eq7:=eval(Eq6,rhs(trH)=H(z));
Eq8:=collect(simplify(Eq7,{rhs(trG2)=G(z)}),G(z));
Eq9:=subs(z=z(x,t),Eq8);
Eq10:=diff(Eq9,t,x);
```

```
trD:={z(x,t)=z,diff(z(x,t),x)=diff(lhs(tr1),x),
      diff(z(x,t),t)=diff(lhs(tr1),t),diff(z(x,t),t,x)=diff(lhs(tr1),t,x)};
Eq11:=subs(trD,Eq10);
Eq12:=convert(expand(Eq11/(diff(psi(t),t)*diff(phi(x),x))),diff);
Eq13:=expand(Eq12/2/diff(G(z),z))-Eq8;
Eq14:=factor(Eq13*2*diff(G(z),z));
```

This equation holds in two cases.
The *first case*: $\{G_{zz}-2G(z)G_z=0$ and $H_{zz}-2H(z)G_z=0\}$ (sys1).
By integrating this system of ODEs, we can obtain the following solutions for $G(z)$: $G(z)=C_1$ (SolG1), $G(z)=-1/(z+C_1)$ (SolG2), $G(z)=-B_1\tanh(B_1z+B_2)$ (SolG31), $G(z)=-B_1/\coth(B_1z+B_2)$ (SolG4), and $G(z)=E_1\tan(E_1z+E_2)$ (SolG51), where C_1, B_1, B_2, E_1, and E_2 are arbitrary constants.

```
sys1:=[-select(has,lhs(Eq14),G)=0,rhs(Eq14)=0];
Eq1sys1:=(sys1[1]-diff(G(z),z$2))*(-1);
SolG1:=G(z)=C1;
Eq1sys11:=int(lhs(Eq1sys1),z)=int(rhs(Eq1sys1),z)+C1;
Eq1sys12:=expand(Eq1sys11/G(z)^2);
Eq1sys13:=int(lhs(Eq1sys12),z)=map(int,rhs(Eq1sys12),z)+C2;
SolG:=isolate(eval(Eq1sys13,C1=0),G(z));
Q1:=numer(rhs(SolG))*(-1); Q2:=denom(rhs(SolG))*(-1);
SolG2:=G(z)=subs(-C2=C1,Q1/Q2);
SolG3:=subs(_C1=C2,dsolve(Eq1sys11,G(z)));
SolG31:=algsubs(C2*B1=B2,eval(SolG3,sqrt(C1)=B1));
SolG4:=convert(SolG31,coth);
SolG5:=subs(_C1=C1,_C2=C2,dsolve(sys1[1],G(z)));
SolG51:=subs(E1*C2=E2,map(combine,subs(1/C1=E1,SolG5)));
for i from 1 to 5 do algsubs(SolG||i,sys1[1]); od;
```

Without full analysis, let us take the second solution (SolG2), $G(z)=-1/(z+C_1)$, and solve this system of ODEs for one special case, $C_1=0$, $B_1=1$, and $B_2=0$. Then we can determine $G(z)=-1/z$ (SolG21), $H(z)=A_2z^2+A_1/z$ (SolH1), $W(z)=\ln|z|$ (SolW1), and $F(W)=A_2e^W+A_1e^{-2W}$ (SolF1). To determine $\phi(x)$ and $\psi(t)$, we substitute these results into the functional-differential equation (39.3.7.1), and then we arrive at the equation (EqPhiPsi):

$$-\psi_t^2+\phi_x^2+\phi\psi_{tt}+\psi\psi_{tt}-\phi\phi_{xx}-\psi\phi_{xx}=A_2\phi^3+3A_2\psi\phi^2+3A_2\phi\psi^2+A_2\psi^3+A_1. \qquad (39.3.7.2)$$

Then, by differentiating it with respect to t and x and by separating the variables, we arrive at the following system of ODEs (ODE1, (ODE2):

$$\psi_{ttt}-6A_2\psi\psi_t=C\psi_t, \quad \phi_{xxx}+6A_2\phi\phi_x=C\phi_x,$$

where C is the separation constant.

```
with(Student[Precalculus]): Consts1:={C1=0,B1=1,B2=0};
Eq2sys1:=simplify(algsubs(SolG2,sys1[2]));
CompleteSquare(lhs(Eq2sys1));
SolH:=subs(_C1=A2,_C2=A1,dsolve(Eq2sys1,H(z)));
SolW:=subs(_C1=B2,_C2=B1,dsolve(eval(trG2,SolG2),W(z)));
SolW1:=eval(SolW,Consts1);
SolF:=expand(isolate(eval(trH,{F(W(z))=F1,SolH,SolW1}),F1));
EqFW:=op(op(2,eliminate({SolF,W=rhs(SolW1)},z)))=0;
SolF1:=F(W)=combine(expand(eval(solve(EqFW,F1),Consts1)));
SolG21:=eval(SolG2,Consts1); SolH1:=eval(SolH,Consts1);
SolGH:={SolG21,SolH1}; pp:=phi(x)+psi(t);
EqPhiPsi:=simplify(subs(z=lhs(tr1),eval(Eq8,SolGH)))*pp;
```

```
EqPhiPsi1:=diff(EqPhiPsi,t,x);
EqPhiPsi2:=expand(EqPhiPsi1/(diff(phi(x),x)*diff(psi(t),t)));
EqPhiPsi3:=lhs(EqPhiPsi2)-EqPhiPsi2;
ODE1:=expand((select(has,rhs(EqPhiPsi3),psi)=C)*diff(psi(t),t));
ODE2:=expand((-select(has,rhs(EqPhiPsi3),phi)=C)*diff(phi(x),x));
```

These ODEs can be integrated twice (ODE14, ODE24):

$$\psi_t^2 = 2A_1\psi^3 + C\psi^2 + 2C_1\psi + 2C_2, \quad \phi_x^2 = -2A_1\phi^3 + C\phi^2 + 2C_3\phi + 2C_4,$$

where the C_i ($i = 1,\ldots,4$) are arbitrary constants. By substituting these equations into the above Eq. (39.3.7.2) for determining $\phi(x)$ and $\psi(t)$, we can remove some constants; i.e., we obtain the relations $C_3 = -C_1$, and $C_4 = C_2 + A_2/2$ (trC3, trC4), and the equations become (ODE15, ODE25)

$$\psi_t^2 = 2A_1\psi^3 + C\psi^2 + 2C_1\psi + 2C_2, \quad \phi_x^2 = -2A_1\phi^3 + C\phi^2 - 2C_1\phi + 2C_2 + A_2.$$

The solutions of these equations can be found and analyzed. For the other cases of solutions for $G(z)$, the analysis can be performed in a similar way.

```
ODE11:=map(int,lhs(ODE1),t)=int(rhs(ODE1),t)+C1;
ODE12:=expand(ODE11*diff(psi(t),t));
ODE13:=expand(map(int,lhs(ODE12),t)=map(int,rhs(ODE12),t)+C2)*2;
ODE21:=map(int,lhs(ODE2),x)=int(rhs(ODE2),x)+C3;
ODE22:=expand(ODE21*diff(phi(x),x));
ODE23:=expand(map(int,lhs(ODE22),x)=map(int,rhs(ODE22),x)+C4)*2;
ODE14:=isolate(ODE13,diff(psi(t),t)^2);
ODE24:=isolate(ODE23,diff(phi(x),x)^2);
trPsiDD:=expand(diff(ODE14,t)/diff(psi(t),t))/2;
trPhiDD:=expand(diff(ODE24,x)/diff(phi(x),x))/2;
EqConsts:=expand(eval(EqPhiPsi,{ODE14,ODE24,trPsiDD,trPhiDD}));
EqConsts1:=EqConsts-rhs(EqConsts);
trC3:=isolate(coeff(subs(phi(x)=0,lhs(EqConsts1)),psi(t),1)=0,C3);
trC4:=isolate(subs([psi(t)=0,phi(x)=0],lhs(EqConsts1))=0,C4);
ODE15:=ODE14;
ODE25:=eval(ODE24,{trC3,trC4});
```

The *second case*. First, let us represent the functional-differential equation (39.3.7.1) in the form $f(x) + g(t) = q(z)$ (FunDiffEq1), whose solution is $\{f(x) = A\phi(x) + B,\ g(t) = A\psi(t) - B + C,\ q(z) = Az + C\}$ (SolFunDiffEq1). Therefore, we arrive at the equation (Eq141)

$$\psi_t^2 - \phi_x^2 = (H_{zz} - 2H(z)G_z)/(G_{zz} - 2G(z)G_z), \quad \text{i.e.,} \quad f(x) + g(t) = q(z),$$

where $g(t) = \psi_t^2$ (trg), $f(x) = -\phi_x^2$ (trf), and $q(z) = (H_{zz} - 2H(z)G_z)/(G_{zz} - 2G(z)G_z)$ (trq). For the second case, the functional-differential equation (39.3.7.1) holds if (sys2)

$$\psi_t^2 = A\psi + B, \quad \phi_x^2 = -A\phi + B - C, \quad \frac{H_{zz} - 2H(z)G_z}{G_{zz} - 2G(z)G_z} = Az + C.$$

By integrating the first two ODEs in the above system, we obtain the following solutions for $\phi(x)$ and $\psi(t)$ (SolPhiPsi20, SolPhiPsi21):

$$\phi(x) = \pm\sqrt{B - C}\,x + D_2, \quad \psi(t) = \pm\sqrt{B}\,t + D_1 \quad \text{if} \quad A = 0,$$

$$\phi(x) = \left(-\tfrac{1}{4}D_2^2 - \tfrac{1}{4}x^2 + \tfrac{1}{2}xD_2\right)A + \frac{B - C}{A}, \quad \psi(t) = \left(-\tfrac{1}{2}tD_1 + \tfrac{1}{4}t^2 + \tfrac{1}{4}D_1^2\right)A - \frac{B}{A} \quad \text{if} \quad A \neq 0,$$

where D_1 and D_2 are arbitrary constants. We can observe that the function $F(W)$ is arbitrary in both cases.

Finally, we obtain the traveling-wave solution $W = W(z) = W(E_1x + E_2t)$ ($E_1 = E_2 = -\sqrt{B}$, S1) of the nonlinear Klein–Gordon equation for $A = 0$ and the solution of the form $W = W(z) = W(x^2 - t^2)$ (S2) for $A \neq 0$.

```
Eq14; termG:=-select(has,lhs(Eq14),G);
Eq141:=simplify(Eq14/(termG));
trg:=g(t)=select(has,lhs(Eq141),psi(t));
trf:=f(x)=select(has,lhs(Eq141),phi(x));
trq:=q(z)=rhs(Eq141);
FunDiffEq1:=f(x)+g(t)=q(z);
SolFunDiffEq1:=[g(t)=A*psi(t)+B,f(x)=A*phi(x)-B+C,q(z)=A*z+C];
sys2:=subs(trf,trg,trq,SolFunDiffEq1);
sys20:=subs(A=0,sys2);
fs:={phi(x),psi(t)};
SolPhiPsi20:=subs(_C1=D1,_C2=D2,dsolve({sys20[1],sys20[2]},fs));
SolPhiPsi2:=subs(_C1=D1,_C2=D2,[dsolve({sys2[1],sys2[2]},fs)]);
for i from 1 to nops(SolPhiPsi2) do SolPhiPsi2[i]; od;
SolPhiPsi21:=map(collect,SolPhiPsi2[4],A);
rhs(SolPhiPsi20[1][1]+SolPhiPsi20[2][1]); trG2; trH;
trF:=F(W(z))=sin(W);
trFD:=(D(F))(W(z))=diff(rhs(trF),W);
trFD2:=((D@@2)(F))(W(z))=diff(rhs(trF),W$2);
EqW:=simplify(eval(eval(sys20[3],{trH,trG2}),{trF,trFD,trFD2}));
EqW11:=factor(convert(EqW,tan));
EqW12:=subs(tan(W/2)=tan(W(z)/2),EqW11);
dsolve(EqW12,W(z));
Consts2:={A=-4,C=0,D1=0,D2=0};
S1:=subs(Consts2,rhs(SolPhiPsi20[1][1]+SolPhiPsi20[2][1]));
S2:=subs(Consts2,expand(rhs(op(SolPhiPsi2[4][1])+op(SolPhiPsi2[4][2]))));
```

⊙ *References for Section* 39.3: G. I. Taylor (1950), R. B. Lindsay (1973), V. A. Galaktionov (1990, 1995), W. Malfliet (1992), L. I. Sedov (1993), E. S. Cheb-Terrab and K. von Bulow (1995), R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth (1996), E. J. Parkes and B. R. Duffy (1996), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002), J. H. He and X. H. Wu (2006), A. D. Polyanin and A. V. Manzhirov (2006), J. H. He and M. A. Abdou (2007), S. Lynch (2009), A. M. Wazwaz (2006).

39.4. Analytical Solutions of Nonlinear Systems

In this section, we consider the most important analytical methods for solving nonlinear systems of partial differential equations. In particular, we construct traveling-wave solutions of nonlinear systems of first-order and second-order equations and generalized separable solutions of second-order nonlinear systems [see Polyanin and Manzhirov (2007) and Schiesser and Griffiths (2009) as well as Subsections 38.3.1, 38.4.2, 38.5.1, and 38.5.2].

39.4.1. Constructing Traveling-Wave Solutions

Nonlinear systems of first-order PDEs. Consider nonlinear systems of the form

$$u_x = F(u,v),$$
$$v_t = G(u,v).$$

Such systems can describe various physical, chemical, and biological processes. It is known that this system can admit *traveling-wave solutions*; i.e., we seek a solution in the form $u = W_1(\xi)$, $v = W_2(\xi)$, $\xi = kx - \lambda t$, where k and λ are arbitrary constants.

Example 39.30. *Nonlinear system of first-order PDEs.* As a particular case, consider the system

$$u_x = u\ln(v), \quad v_t = v\ln(u).$$

We verify that this system admits exact traveling-wave solutions and the functions $W_1(\xi)$ and $W_2(\xi)$ are described by the autonomous system of ordinary differential equations (`Sys4`)

$$W_{1\xi}k - W_1\ln(W_2)=0, \quad -W_{2\xi}\lambda - W_2\ln(W_1)=0.$$

We construct the traveling-wave solution (`trW1`, `trW2`)

$$W_1(\xi) = e^{\lambda\sin(\xi/\sqrt{k\lambda})}, \quad W_2(\xi) = e^{\sqrt{k\lambda}\cos(\xi/\sqrt{k\lambda})}$$

and verify that it is an exact solution of the given nonlinear system.

```
with(PDEtools):
interface(showassumed=0): assume(k>0,lambda>0,x>0,t>0,xi>0);
declare(W1(xi),W2(xi),w1(x,t),w2(x,t),u(x,t),v(x,t),F(u,v),G(u,v));
U,V:=diff_table(u(x,t)),diff_table(v(x,t));
F:=(u,v)->u*ln(v); G:=(u,v)->v*ln(u);
Sys1:=(w1,w2)->[diff(w1,x)-F(w1,w2)=0,diff(w2,t)-G(w1,w2)=0];
tr1:=k*x-lambda*t=xi;
Sys2:=expand(Sys1(W1(lhs(tr1)),W2(lhs(tr1))));
Sys3:=algsubs(tr1,Sys2);
Sys4:=map(convert,Sys3,diff);
Sol1:=combine(dsolve(Sys4,{W1(xi),W2(xi)}));
Sol11:=simplify(subs(_C1=0,_C2=1,Sol1));
trW1:=expand(subs(Sol11[1],Sol11[2]));
trW2:=Sol11[1];
expand(subs(trW1,trW2,Sys4));
```

Nonlinear systems of second-order PDEs. Consider nonlinear systems of the form

$$u_t = a_1 u_{xx} + F(u,v),$$
$$v_t = a_2 v_{xx} + G(u,v),$$

which describe reaction-diffusion phenomena. These systems can admit traveling-wave solutions; i.e., we seek a solution in the form $u = W_1(\xi)$, $v = W_2(\xi)$, $\xi = kx - \lambda t$, where k and λ are arbitrary constants.

Example 39.31. *Nonlinear system of second-order PDEs.* As a particular case, consider the system

$$u_t = a_1 u_{xx} + u\ln v, \quad v_t = a_2 v_{xx} + v\ln u.$$

We show that this system admits exact traveling-wave solutions and the functions $W_1(\xi)$ and $W_2(\xi)$ are described by the autonomous system of ordinary differential equations (`Sys4`)

$$-\lambda W_{1\xi} - a_1 k^2 W_{1\xi\xi} - W_1\ln(W_2) = 0, \quad -\lambda W_{2\xi} - a_2 k^2 W_{2\xi\xi} - W_2\ln(W_1) = 0.$$

```
with(PDEtools):
interface(showassumed=0):
assume(k>0,lambda>0,x>0,t>0,xi>0,a1>0,a2>0);
declare(W1(xi),W2(xi),w1(x,t),w2(x,t),u(x,t),v(x,t),F(u,v),G(u,v));
U,V:=diff_table(u(x,t)),diff_table(v(x,t));
F:=(u,v)->u*ln(v);
G:=(u,v)->v*ln(u);
Sys1:=(w1,w2)->[diff(w1,t)-a1*diff(w1,x$2)-F(w1,w2)=0,
                diff(w2,t)-a2*diff(w2,x$2)-G(w1,w2)=0];
tr1:=k*x-lambda*t=xi;
Sys2:=expand(Sys1(W1(lhs(tr1)),W2(lhs(tr1))));
Sys3:=algsubs(tr1,Sys2);
Sys4:=map(convert,Sys3,diff);
```

39.4.2. Constructing Generalized Separable Solutions

Now let us generalize the method of generalized separation of variables considered in Section 39.3.7. to systems of second-order nonlinear partial differential equations.

Example 39.32. *Nonlinear systems of second-order PDEs.* Consider the systems of nonlinear second-order PDEs of the following form:

$$u_t = au_{xx} + uF(bu-cv) + G_1(bu-cv),$$
$$v_t = av_{xx} + vF(bu-cv) + G_2(bu-cv),$$

where $F(z)$, $G_1(z)$ and $G_2(z)$ are arbitrary functions.

By applying the method of generalized separation of variables and by seeking exact solutions in the form

$$u(x,t) = \phi_1(t)\theta(x,t) + \psi_1(t), \quad v(x,t) = \phi_2(t)\theta(x,t) + \psi_2(t), \tag{39.4.2.1}$$

we find exact solutions of this nonlinear system. We assume that the functions $\phi_i(t)$ and $\psi_i(t)$ $(i = 1,2)$ are selected so that the two equations of the original nonlinear system can be reduced to a single equation for the function $\theta(x,t)$. Also we assume that the arguments $bu-cw$ of the functions F, G_1, and G_2 in the original system depend only on t; i.e., $(bu-cw)_x = (b\phi_1 - c\phi_2)\theta_x = 0$. By introducing the new function $\phi(t)$, we can obtain the following conditions on $\phi_i(t)$ $(i = 1,2)$: $\phi_1 = c\phi$ and $\phi_2 = b\phi$.

By substituting the relations (39.4.2.1) describing the solution form and the above-derived conditions on $\phi_i(t)$ $(i = 1,2)$ into the original system, we show that the original system can be reduced to the form (`Sys6`)

$$\left(\frac{\phi_t}{\phi} - F(z)\right)\theta - a\theta_{xx} + \theta_t + \frac{1}{c\phi}\Big(\psi_{1t} - G_1(z) - F(z)\psi_1\Big) = 0,$$
$$\left(\frac{\phi_t}{\phi} - F(z)\right)\theta - a\theta_{xx} + \theta_t + \frac{1}{b\phi}\Big(\psi_{2t} - G_2(z) - F(z)\psi_2\Big) = 0, \tag{39.4.2.2}$$

where $z = b\psi_1 - c\psi_2$. By solving the resulting system of equations, we obtain the following system of ordinary differential equations for the functions $\phi(t)$ and $\psi_i(t)$ $(i = 1,2)$: $\{\phi_t/\phi = F(z)$, $\psi_{1t} = G_1(z) + F(z)\psi_1$, $\psi_{2t} = G_2(z) + F(z)\psi_2\}$ (`Eq1`, `Eq2`, `Eq3`) and the linear diffusion equation $-a\theta_{xx} + \theta_t = 0$ (`Eq4`) for the function $\theta(x,t)$.

Finally, we construct the exact solution of the original nonlinear system: $u = c\phi\theta + \psi_1$ (`Solu`), $v = b\phi\theta + \psi_2$ (`Solv`), where $\phi = \exp\Big(\int F(b\psi_1 - c\psi_2)\,dt\Big)$ (`Solphi`), the functions $\psi_i(t)$ $(i = 1,2)$ are described by the ODEs obtained above, and the function $\theta(x,t)$ is a solution of the linear diffusion equation.

```
with(PDEtools):
declare(u(x,t),v(x,t),W1(x,t),W2(x,t),phi1(t),phi2(t),
        psi1(t),psi2(t),phi(t),theta(x,t));
tr1:=phi1(t)*theta(x,t)+psi1(t); tr2:=phi2(t)*theta(x,t)+psi2(t);
Sys1:=(u,v)->[diff(u(x,t),t)-a*diff(u(x,t),x$2)
             -u(x,t)*F(b*u(x,t)-c*v(x,t))-G1(b*u(x,t)-c*v(x,t)),
              diff(v(x,t),t)-a*diff(v(x,t),x$2)
             -v(x,t)*F(b*u(x,t)-c*v(x,t))-G2(b*u(x,t)-c*v(x,t))];
Sys2:=expand(Sys1(W1,W2));
Sys3:=expand(subs(W1(x,t)=tr1,W2(x,t)=tr2,Sys2));
Cond1:=diff(b*u(x,t)-c*v(x,t),x);
Cond11:=collect(expand(subs(u(x,t)=tr1,v(x,t)=tr2,Cond1)),diff)=0;
Sol1:=isolate(Cond11,phi2(t));
tr31:={phi1(t)=c*phi(t)}; tr32:={subs(tr31,Sol1)};
Sys4:=collect(expand(subs(tr31,tr32,Sys3)),diff);
Sys5:=expand([Sys4[1]/c/phi(t),Sys4[2]/b/phi(t)]);
Sys6:=collect(expand(factor(Sys5)),[theta,c,b]);
Eq1:=remove(has,expand(select(has,Sys6[1],[F])/theta(x,t)),theta);
Eq2:=remove(has,Sys6[1],[theta]);
Eq3:=remove(has,Sys6[2],[theta]);
Eq4:=remove(has,Sys6[1],[b,c]);
```

```
Solphi:=subs(_C1=1,dsolve(Eq1,phi(t)));
Solu:=u(x,t)=subs(Solphi,subs(tr31,tr1));
Solv:=v(x,t)=subs(Solphi,subs(tr32,tr2));
```

⊙ *References for Section* 39.4: A. D. Polyanin and A. V. Manzhirov (2006), W. E. Schiesser and G. W. Griffiths (2009).

39.5. Constructing Exact Solutions Using Symbolic Computation. What Can Go Wrong?

In this section, we discuss some critical and important aspects of constructing exact solutions of nonlinear PDEs with the aid of computer algebra systems, e.g., Maple. We present some selected examples where wrong results are obtained and give some advice on how to avoid this probable situation.

First, let us formulate three definitions related to exact solutions.

In this Section, we consider exact solutions of nonlinear PDEs that can be represented in terms of elementary functions. (Infinite function series and products are not included.)

We say that an exact solution of a nonlinear PDE is *not new* if it is possible to reduce it to a known exact solution.

We say that an exact solution of a nonlinear PDE is *redundant* if there exist more general solutions such that this redundant solution can be considered as a particular case of these more general solutions.

39.5.1. Constructing New Exact Solutions

Consider some difficulties that can arise before, during, and after solving a nonlinear PDE. Before starting to generate exact solutions of a nonlinear PDE, you have to choose an appropriate method. There exist several methods (often called ansatz methods) and their generalizations for generating exact solutions of nonlinear PDEs with the aid of computer algebra systems, e.g., the tanh-function method, the tanh-coth method, the sine-cosine method, the exp-function method, the F-expansion method, and the (G'/G)-expansion method. These methods allow one to find exact solutions of nonlinear nonintegrable PDEs.

Nowadays, it is possible to perform cumbersome analytical computation for finding exact solutions of nonlinear PDEs by applying various ansatz methods and by using various computer algebra systems (e.g., Maple and Mathematica). It should be noted that within the last few years there have been numerous scientific papers published, describing the construction of new solutions of various nonlinear PDEs and application of these methods. Various generalizations of these methods and their implementations in computer algebra systems for finding new exact solutions of nonlinear PDEs have been developed by many researchers. However, some exact solutions are often either not new or redundant.

Comparing exact solution generation methods. There are close connections between the truncated expansion method introduced by WTC [see Weiss, Tabor, and Carnevale (1983)] and ansatz methods; i.e., many ansatz methods can be considered as a consequence of the truncated expansion method [see Kudryashov (2009)]. Thus, the exact solutions obtained by applying these methods are often not new; i.e., there exist coincidences between different forms of solutions. Since any expression containing exponential, trigonometric, or

hyperbolic functions can be rewritten in different but equivalent forms, let us show how to derive such equivalent forms with Maple and compare exact solutions obtained by applying various ansatz methods.

Example 39.33. For the *KdV–Burgers equation*,

$$u_t + uu_x + au_{xxx} = bu_{xx},$$

we can construct various equivalent forms of exact solutions.

1. By applying the truncated expansion method, we can obtain the exact solution of the form [for details, see Kudryashov (1988)] (`Sol1`)

$$u(x,t) = C_2 - \frac{12}{5}b(\ln v)_x + 12a(\ln v)_{xx}, \tag{39.5.1.1}$$

where C_1 and C_2 are arbitrary constants, $v = 1 + C_1 e^z$, $z = kx - \lambda t$, $k_{1,2} = \pm\frac{b}{5a}$, $\lambda_{1,2} = \pm\frac{bC_2}{5a} - \frac{6b^3}{125a^2}$ (`tr21`, `tr22`). This solution satisfies the nonlinear ODE of the form $ak^3V_{zzz} - bk^2V_{zz} + kVV_z - \lambda V_z = 0$ (`ODE1`, `Eq2`), where $u(x,t) = V(z)$ and $z = kx - \lambda t$.

The exact solution (39.5.1.1) can be rewritten in the form (`Eq4`)

$$U(z) = C_2 + \tfrac{12}{5}k(-b+5ak)\frac{C_1e^z}{1+C_1e^z} - 12ak^2\left(\frac{C_1e^z}{1+C_1e^z}\right)^2 \tag{39.5.1.2}$$

```
with(PDEtools); declare(u(x,t),V(z));
U,V1:=diff_table(u(x,t)),diff_table(V(z));
interface(showassumed=0); assume(z>0);
PDE1:=U[t]+U[]*U[x]+a*U[x,x,x]=b*U[x,x];
tr1:=z=k*x-lambda*t;
tr21:={k=b/(5*a),lambda=b*C2/(5*a)-6*b^3/(125*a^2)};
tr22:={k=-b/(5*a),lambda=-b*C2/(5*a)-6*b^3/(125*a^2)};
v:=subs(tr1,1+C1*exp(z)); v1:=1+C1*exp(z);
Sol1:=u(x,t)=C2-12/5*b*diff(ln(v),x)+12*a*diff(ln(v),x$2);
ODE1:=a*k^3*V1[z,z,z]-b*k^2*V1[z,z]+k*V1[]*V1[z]-lambda*V1[z]=0;
Eq1:=expand(subs(rhs(tr1)=z,Sol1));
Eq2:=algsubs(V(z)=rhs(Eq1),ODE1);
Eq3:=simplify(subs(tr21,subs(tr1,Eq2)));
term23:=factor(select(has,rhs(Eq1),[b,12]));
Eq4:=u(x,t)=op(1,rhs(Eq1))+term23+op(4,rhs(Eq1));
```

2. Let us derive the exact solution corresponding to the F-expansion method [see Kudryashov (2005)] from the exact solution obtained by the truncated expansion method.

If we denote by $H(z)$ the expression

$$H(z) = \frac{C_1e^z}{1+C_1e^z}, \tag{39.5.1.3}$$

then the exact solution acquires the form (`Eq5`)

$$U(z) = C_2 + \tfrac{12}{5}k(-b+5ak)H(z) - 12ak^2H(z)^2;$$

i.e., this means that this expression corresponds to the ansatz of the F-expansion method: $U(z) = A_0 + \sum_{i=0}^{M} A_iH(z)^i$. In our case, for the KdV–Burgers equation we have $U(z) = A_0 + A_1H(z) + A_2H(z)^2$ (`Eq6`), where $H(z)$ is the solution of the Riccati equation $H_z = A + BH(z) + CH^2$ ($A = 0, B = 1, C = -1$). Thus, comparing the results obtained by the F-expansion method and the truncated expansion method, we can conclude that we have the same exact solutions.

```
tr3:=H(z)=C1*exp(z)/v1; tr31:=isolate(tr3,v1); tr32:=lhs(tr3)^2=rhs(tr3)^2;
tr4:=diff(tr3,z); tr5:=subs({rhs(tr3)=lhs(tr3),rhs(tr32)=lhs(tr32)},tr4);
Eq5:=subs(tr31,Eq4);
tr6:=[A0=C2,A1=select(has,rhs(Eq5),b)/H(z),A2=select(has,rhs(Eq5),k^2)/H(z)^2];
Eq6:=u(x,t)=add(A||i*H(z)^i,i=0..2); subs(tr6,Eq6);
```

3. Let us derive the exact solution corresponding to the tanh-function method of Malfliet and Hereman (1996) from the exact solution obtained by the truncated expansion method.

By applying the transformation $H(z)=\frac{1}{2}\left(\tanh(\frac{1}{2}(z-z_0))+1\right)$ and by substituting k_1 and λ_1 (see `tr7`, `tr21`) into (39.5.1.2), we can rewrite the exact solution (39.5.1.2) of the KdV–Burgers equation in the form

$$U(z)=C_2-\frac{6b^2}{25a}\tanh(\tfrac{1}{2}(z-z_0))-\frac{3b^2}{25a}+\frac{3b^2}{25a}\tanh(\tfrac{1}{2}(z-z_0))^2.$$

A similar solution can be obtained by substituting k_2, λ_2 into (39.5.1.2). Here, as before, $z=kx-\lambda t$, $k_{1,2}=\pm\frac{b}{5a}$, and $\lambda_{1,2}=\pm\frac{bC_2}{5a}-\frac{6b^3}{125a^2}$ (`tr21`, `tr22`). Thus, this form of the solution of the KdV–Burgers equation corresponds to the ansatz of the tanh-function method $U(z)=\sum_{i=0}^{M}A_i\tanh(z)^i$, and in our case we have the following expression: $U(z)=A_0+A_1\tanh(\frac{1}{2}(z-z_0))+A_2\tanh(\frac{1}{2}(z-z_0))^2$ (`Eq10`).

```
tr7:=H(z)=1/2*(tanh((z-z0)/2)+1);
Eq71:=convert(rhs(tr7),exp); Eq72:=combine(subs(C1=exp(-z0),rhs(tr3)));
simplify(Eq71-Eq72); Eq8:=factor(subs(tr7,Eq5)); Eq9:=expand(subs(tr21,Eq8),tanh);
tr8:=[A0=remove(has,rhs(Eq9),z),A1=remove(has,select(has,select(has,rhs(Eq9),
      [-3/25]),z),z),A2=remove(has,select(has,rhs(Eq9),[6/25]),z)];
Eq10:=u(x,t)=add(A||i*tanh(mu*(z-z0))^i,i=0..2); subs(mu=1/2,tr8,Eq10);
```

4. Let us derive the exact solution corresponding to the exp-function method He and Wu (2006) from the exact solution obtained by the truncated expansion method.

By multiplying the numerator and denominator in (39.5.1.2) by e^{-z}, we can rewrite the exact solution (39.5.1.2) of the KdV–Burgers equation in the form (`Eq13`)

$$U(z)=\frac{5C_2e^{-z}+(10C_1C_2-12C_1bk+60C_1ak^2)+(5C_1^2C_2-12C_1^2bk)e^{z}}{5e^{-z}+10C_1+5C_1^2e^{z}}.$$

This solution corresponds to the ansatz (39.3.4.2) of the exp-function method, and in our case we have

$$U(z)=\frac{A_0e^{z}+A_1+A_2e^{-z}}{B_0e^{z}+B_1+B_2e^{-z}}.$$

```
Eq11:=normal(rhs(Eq4));
Eq11:=combine(expand(numer(rhs(Eq4)))/expand(denom(rhs(Eq4))));
Eq12N:=combine(expand(numer(Eq11*exp(-z)))); Eq12D:=denom(Eq11);
Eq12D1:=combine(expand(Eq12D*exp(-z))); Eq13:=u(x,t)=Eq12N/Eq12D1;
```

Solving nonlinear ODEs. Constructing traveling-wave reductions of PDEs, we have to solve nonlinear ODEs. To find new exact solutions of PDEs, one has to apply integration methods for ODEs and determine (if possible) the general solutions of ODEs.

Example 39.34. For the *Benjamin–Bona–Mahony equation*,

$$u_t-u_{xxt}+u_x+\left(\tfrac{1}{2}u^2\right)_x=0,$$

we can construct traveling-wave solutions. By substituting $u(x,t)=U(z)$, $z=kx-ct$ into the Benjamin–Bona–Mahony equation, we obtain the nonlinear ODE $-cU_z+ck^2U_{zzz}+kU_z+kUU_z=0$ (`Eq3`). But then, instead of finding traveling-wave solutions of this ODE [see Z. Z. Ganji, D. D. Ganji, and Bararnia (2009)], one can determine the general solution of this equation. By integrating this ODE over z, we have $-cU+ck^2U_{zz}+kU+\frac{1}{2}kU^2+C_1=0$ (`Eq4`). Then, by multiplying the resulting equations by $2U_z$ and by integrating over z, we obtain the ODE $-cU^2+ck^2U_z^2+kU^2+\frac{1}{3}kU^3+2C_1U+C2=0$ (`Eq5`, `Eq6`). Finally, we can find the solution of this equation by quadrature (`Sol1`).

```
with(PDEtools); declare(U(z),F(z)); tr1:=k*x-c*t=z;
Eq1:=u->diff(u,t)-diff(u,x,x,t)+diff(u,x)+diff(u^2/2,x);
Eq2:=Eq1(U(k*x-c*t)); Eq3:=convert(subs(tr1,Eq2),diff);
Eq4:=int(Eq3,z)+C1=0; Eq5:=expand(Eq4*2*diff(U(z),z));
Eq6:=int(lhs(Eq5),z)+C2=0; Sol1:=dsolve(Eq6,U(z));
```

Simplifying exact solutions. To prove that the exact solutions obtained are new, one should simplify them.

Example 39.35. For the *fifth-order KdV equation*,

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0,$$

we can construct numerous traveling-wave solutions by applying the exp-function method. For example, consider two exact solutions of the form [for details, see Chun (2008)]

$$U_1(z) = \frac{-k^4e^z + 2k^2a_0 - k^4b_{-1}e^{-z}}{2k^2e^z - 4a_0 + 2k^2b_{-1}e^{-z}}, \quad U_2(z) = \frac{a_1e^z + a_1b_0 + a_1b_{-1}e^{-z}}{e^z + b_0 + b_{-1}e^{-z}}. \tag{39.5.1.4}$$

These solutions can be simplified to the trivial solutions: $U_1(z) = -\frac{1}{2}k^2$ (`Sol12`) if $z = k(x - ct)$, $c = 7k^4/4$, and $U_2(z) = a_1$ (`Sol22`) if $z = k(x + ct)$.

```
with(PDEtools);
declare(u(x,t)); U:=diff_table(u(x,t)):
PDE1:=U[t]+30*U[]^2*U[x]+20*U[x]*U[x,x]+10*U[]*U[x,x,x]+U[x,x,x,x,x]=0;
Sol1:=(-k^4*exp(z)+2*k^2*a[0]-k^4*b[-1]*exp(-z))/(2*k^2*exp(z)
      -4*a[0]+2*k^2*b[-1]*exp(-z));
tr1:=z=k*(x-7*k^4*t/4);
Sol11:=subs(tr1,Sol1); Sol12:=factor(Sol11);
Sol2:=(a[1]*exp(z)+a[1]*b[0]+a[1]*b[-1]*exp(-z))/(exp(z)+b[0]+b[-1]*exp(-z));
tr2:=z=k*(x+c*t);
Sol21:=subs(tr2,Sol2); Sol22:=factor(Sol21);
pdetest(u(x,t)=Sol1,PDE1);
pdetest(u(x,t)=Sol2,PDE1);
```

39.5.2. Removing Redundant Exact Solutions

Adding integration constants. Constructing traveling-wave reductions of nonlinear PDEs, we obtain nonlinear ODEs that can be integrated. To remove redundant solutions (i.e., to obtain more general solutions) of nonlinear PDEs, it is necessary to add arbitrary constants after the integration of ODEs.

Example 39.36. For the *Boussinesq equation*,

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0,$$

we can construct traveling-wave solutions. By substituting $u(x,t) = U(z)$, $z = x - ct$ into the Boussinesq equation, we obtain the nonlinear ODE $c^2U_{zz} - U_{zz} - 2U_z^2 - 2UU_{zz} + U_{zzzz} = 0$ (`Eq3`). After integrating this ODE twice over z, we have $c^2U - U - U^2 + U_{zz} + C_2 = C_1z$ (`Eq5`). However, some researchers [see Bekir (2008)] often omit integration constants (in our case C_1 and C_2) and find less general (redundant) solutions.

```
with(PDEtools); declare(U(z)); tr1:=x-c*t=z;
Eq1:=u->diff(u,t$2)-diff(u,x$2)-diff(u^2,x$2)+diff(u,x$4)=0;
Eq2:=Eq1(U(x-c*t)); Eq3:=convert(subs(tr1,Eq2),diff);
Eq4:=int(lhs(Eq3),z)=C1; Eq5:=int(lhs(Eq4),z)+C2=int(rhs(Eq4),z);
Sol1:=dsolve(subs(C1=0,Eq5),U(z)); Sol2:=dsolve(subs(C1=0,C2=0,Eq5),U(z));
```

Generalizing exact solutions. To remove redundant solutions (i.e., to obtain more general solutions) of nonlinear PDEs, one should generalize these solutions (if possible) by introducing arbitrary constants.

Example 39.37. For the *the KdV–Burgers equation*,

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0,$$

one can construct traveling-wave solutions by applying the modified extended tanh-function method [for details, see Soliman (2009)] and obtain, for example, the two following exact solutions (`Sol1`, `Sol2`):

$$U(z) = K\big(9 - 6\tanh(Z) - 3\tanh^2(Z)\big), \quad U(z) = K\big(9 - 6\coth(Z) - 3\coth^2(Z)\big),$$

where $u(x,t) = U(z)$, $z = x - ct$, $Z = \nu z/(10\mu)$, $c = 6\nu^2/(25\mu)$, and $K = \nu^2/(25\varepsilon\mu)$. However, by introducing the arbitrary constant z_0, we can generalize these solutions as follows (`SolG`):

$$U(z) = 3K(3 - 2\tanh(Z - z_0) - \tanh^2(Z - z_0)),$$

where we have the first exact solution for $z_0 = i\pi/2$ and the second exact solution for $z_0 = 0$.

```
with(PDEtools); with(RealDomain): declare(u(x,t)); U:=diff_table(u(x,t)):
PDE1:=U[t]+epsilon*U[]*U[x]-nu*U[x,x]+mu*U[x,x,x]=0;
Z:=nu*z/(10*mu); K:=nu^2/(25*epsilon*mu); tr1:=z=x-6*nu^2*t/(25*mu);
Sol1:=K*(9-6*tanh(Z)-3*tanh(Z)^2); Sol2:=K*(9-6*coth(Z)-3*coth(Z)^2);
SolG:=3*K*(3-2*tanh(Z-z0)-tanh(Z-z0)^2);
Sol11:=subs(tr1,Sol1); Sol21:=subs(tr1,Sol2);
pdetest(u(x,t)=Sol11,PDE1); pdetest(u(x,t)=Sol21,PDE1); pdetest(u(x,t)=SolG,PDE1);
T11:=expand(Sol1,tanh); T12:=expand(subs(z0=I*Pi/2,SolG),tanh); T11-T12;
T21:=:-expand(Sol2,tanh); T22:=:-expand(subs(z0=0,SolG),tanh); T21-T22;
```

Introducing and removing additional arbitrary constants. To remove redundant solutions (i.e., obtain more general solutions) of nonlinear PDEs, sometimes one has to introduce additional arbitrary constants such that $m = n$. (The number m of arbitrary constants in the general solution of an n-order ODE is n.) If the number of arbitrary constants is greater than n, i.e., $m > n$, we can decrease it by transformations.

Example 39.38. For the *Sharma–Tasso–Olver equation*,

$$u_t + \alpha(u^3)_x + \tfrac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0,$$

we can construct a traveling-wave solution in the form (`Sol1`)

$$U(z) = \frac{\sqrt{c}(C_2 e^Z - e^{-Z})}{\sqrt{\alpha}(C_1 + C_2 e^Z + e^{-Z})},$$

where $u(x,t) = U(z)$, $z = \mu(x - ct)$, $K = \sqrt{c}z/(\mu\sqrt{\alpha})$, and $-cU + \alpha U^3 + 3\alpha\mu U U_z + \alpha\mu^2 U_{zz} = C$ (`Eq5`) is the associated ODE. To compare this solution with the solution known in the literature, we set $C = 0$ in this ODE. According to the results obtained by Erbaş and Yusufoğlu (2009), their solution acquires the form (`Sol2`)

$$U(z) = \frac{\sqrt{c}(C_2 e^Z - C_3 e^{-Z})}{\sqrt{\alpha}(C_1 + C_2 e^Z + C_3 e^{-Z})}.$$

For this case, the number of arbitrary constants is 3, and the order of ODE (`Eq5`) is 2; i.e., $m > n$. One can remove one of the constants by the transformations $C_2/C_1 = A$, $C_3/C_1 = B$ (`tr1`, `SolFin`).

```
with(PDEtools); declare(u(x,t),U(z),F(z)); tr1:=mu*(x-c*t)=z;
Eq1:=u->diff(u,t)+alpha*diff(u^3,x)+3/2*alpha*diff(u^2,x$2)+alpha*diff(u,x$3);
Eq2:=Eq1(U(mu*(x-c*t))); Eq3:=convert(subs(tr1,Eq2),diff);
Eq4:=simplify(Eq3/mu); Eq5:=int(Eq4,z)=C; Sol1:=dsolve(subs(C=0,Eq5),U(z));
K:=sqrt(c)/sqrt(alpha); Z:=K*z/mu;
Sol2:=K*(C2*exp(Z)-C3*exp(-Z))/(C1+C2*exp(Z)+C3*exp(-Z)); Eq1(Sol2);
Sol2N:=combine(expand(numer(Sol2)/C1)); Sol2D:=combine(expand(denom(Sol2)/C1));
tr1:={C2=C1*A,C3=C1*B}; SolFin:=U(z)=subs(tr1,Sol2N)/subs(tr1,Sol2D);
Test1:=algsubs(SolFin,subs(C=0,Eq5));
```

Verifying exact solutions. To remove redundant and wrong solutions of nonlinear partial differential equations, it is necessary to verify that the exact solution obtained satisfies the given nonlinear PDE. In Maple, it can be done with the function `pdetest` or by directly substituting the exact solution obtained into the given equation.

Example 39.39. For the *Burgers equation*,

$$u_t + \alpha u u_x = \nu u_{xx},$$

we can construct a traveling-wave solution; e.g., consider three solutions of the Burgers equation obtained by Soliman (2009)] (`Sol1`, `Sol2`, `Sol3`):

$$U(z) = \frac{c}{\alpha} + \frac{2\nu}{\alpha}\tanh(z), \quad U(z) = \frac{c}{\alpha} + \frac{c^2}{2\nu\alpha}\coth(z), \quad U(z) = \frac{c}{\alpha} + \frac{2\nu}{\alpha}\tanh(z) + \frac{c^2}{8\nu\alpha}\coth(z),$$

where $u(x,t) = U(z)$, $z = x - ct$. Unfortunately, these solutions are wrong (do not satisfy the Burgers equation). The correct solutions have the form

$$u(x,t) = -\frac{C_3}{C_2\alpha} - \frac{2\nu C_2}{\alpha}\tanh(C_1 + C_2 x + C_3 t), \quad u(x,t) = \frac{c}{\alpha} - \frac{2\nu}{\alpha}\tanh(x_0 + x - ct).$$

```
with(PDEtools); declare(u(x,t));
PDE1:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=nu*diff(u(x,t),x$2);
xi:=x-c*t;
Sol1:=u(x,t)=c/alpha+2*nu/alpha*tanh(xi);
Sol2:=u(x,t)=c/alpha+c^2/(2*nu*alpha)*coth(xi);
Sol3:=u(x,t)=c/alpha+2*nu/alpha*tanh(xi)+c^2/(8*nu*alpha)*coth(xi);
Sol4:=pdsolve(PDE1,HINT='TWS');
Sol5:=subs(_C2=alpha,_C3=-c,_C1=x0,Sol4);
for i from 1 to 5 do Test||i:=pdetest(Sol||i,PDE1) od;
```

Removing redundant solutions via predefined functions. To remove all redundant solutions and ignore the singular cases of nonlinear PDEs via Maple's predefined functions, it is necessary to add the options `remove_redundant=true` and `singsol=false` into the function `TWSolutions`. This means that all singular cases and particular cases of more general solutions being constructed are removed. By default, the predefined function `TWSolutions` removes constant solutions that are redundant.

Example 39.40. For the *KdV–Burgers equation*,

$$u_t + \varepsilon u u_x + \mu u_{xxx} = \nu u_{xx},$$

we can generate traveling-wave solutions via predefined function `TWSolutions` with and without additional options `remove_redundant=true` and `singsol=false`. As a result, we have 3 exact solutions (without additional options) and only one exact solution (with additional options).

```
with(PDEtools); declare(u(x,t));
U:=diff_table(u(x,t));
PDE1:=U[t]+epsilon*U[]*U[x]-nu*U[x,x]+mu*U[x,x,x]=0;
Sol1:=TWSolutions(PDE1);
Sol2:=TWSolutions(PDE1,singsol=false,remove_redundant=true);
```

⊙ *References for Section* 39.5: J. Weiss, M. Tabor, and G. Carnevale (1983). N. A. Kudryashov (1988, 2005, 2009), A. Bekir (2008), C. Chun (2008), B. Erbaş and E. Yusufoğlu (2009), Z. Z. Ganji, D. D. Ganji, and H. Bararnia (2009), A. A. Soliman (2009).

39.6. Some Errors That People Commonly Do When Constructing Exact Solutions with the Use of Symbolic Computations*

39.6.1. General Description of the Problem

In this section, we discuss some critical and important aspects of constructing exact solutions of nonlinear PDEs with the aid of computer algebra systems such as Maple and Mathematica.

We say that an exact solution of a nonlinear PDE *is not new* if it is possible to reduce it to an earlier-known exact solution.

In what follows, we restrict ourselves to exact solutions of traveling-wave type; these are the simplest and widely used solutions of the form $w = w(z)$, where $z = kx - \lambda t$, whose construction can be reduced to studying ordinary differential equations. It is for these solutions that major advances have recently been achieved in the framework of computer algebra systems with a wide use of ansatz methods (the Exp-function method, the tanh-coth method, the sine-cosine method, etc.), where the solution is sought in the form of an a priori given function depending on a number of free parameters (indeterminate coefficients), which are then found by the substitution of the would-be solution into the resulting ODE.

The use of computer algebra systems usually results in quite a few (dozens of) solutions. One often has the erroneous impression that all of these solutions are distinct. Having briefly compared them with earlier-known solutions, researchers sometimes leap to the conclusion that they have obtained new solution(s) and send these solutions to scientific journals. In turn, some referees do not put enough effort into assessing these papers. As a result, recently there have been many "new" solutions that are actually not new at all but nevertheless get published in scientific journals.

This is what actually happens: computer algebra systems often simultaneously give a number of equivalent (coinciding) solutions written in different forms. To reveal the coincidence, the researcher needs to carry out additional analysis (which often proves rather laborious).

Here is an English translation of a small excerpt from the book (2010 *d*) by Kudryashov, who has drawn the present authors' attention to this problem.

Once upon a time, in the hot summer of 2008, I was looking through the journal "Chaos, Solitons and Fractals" and spotted an article by Soliman (2008), who reported finding new exact solutions of the Korteweg–de Vries–Burgers equation. These solutions were seemingly new indeed, but they appeared strange to me, because I was well acquainted with all known solutions of this equation. This article could change my understanding of methods for finding exact solutions of nonlinear differential equations, and so I decided to check the solutions given there. I used Maple to substitute Soliman's solutions into the Korteweg–de Vries–Burgers equation, and they proved to be correct indeed. However, there was an essential drawback to these solutions: they were cumbersome. By studying them more attentively, I discovered that both the numerator and the denominator contained the same function factor, which could be canceled! I had to do this operation, reminiscent of

*Section 39.6 has been written in collaboration with A. I. Zhurov.

algebra school lessons, in the author's stead. After the cancellation, the solutions acquired a form well known to me. Then it became clear to me how many "new solutions" had been obtained. The numerator and denominator of solutions obtained by the Exp-function method usually contained an additional clumsy factor, which could be canceled yielding a known expression as a result. But researchers often failed to do this, because the computer algebra systems Maple and Mathematica cannot find this cumbersome additional factor on their own. The author who discovered "new solutions" was not alone in his delusion. There were even a number of embarrassing cases where the cancellation produced a constant solution.

In what follows, we present some typical examples where such "new solutions" were obtained and published.

39.6.2. Examples in Which "New Solutions" Are Obtained

Following Kudryashov (2011 *b*), let us analyze some traveling-wave solutions of the system

$$\begin{aligned} u_{ty} &= (u^2 - u_x)_{xy} + 2v_{xxx}, \\ v_t &= v_{xx} + 2uv_x. \end{aligned} \tag{39.6.2.1}$$

given in a paper by Wazwaz and Mehanna (2010). The solutions are sought in the form $u = u(\eta)$, $v = v(\eta)$, where $\eta = \mu(x+y-ct)$ (μ and c are free parameters). As a result, the authors obtained the autonomous ordinary differential equations of the special form

$$\mu^2 u''_{\eta\eta} - 2u^3 - 3cu^2 - c^2 u = 0 \tag{39.6.2.2}$$

for $u = u(\eta)$, which can readily be integrated. In particular, Eq. (39.6.2.2) admits the simple particular solutions

$$u = \frac{c}{1 + Ae^{\pm(c/\mu)\eta}}, \tag{39.6.2.3}$$

where A is an arbitrary constant.

To find an exact solution of equation (39.6.2.2), Wazwaz and Mehanna used several methods, including the Exp-function method with the subsequent application of computer algebra systems. Let us see what solutions they obtained, and let us simplify these solutions. (We consider only some of the solutions and preserve the authors' numbering.) The first solution has the form

$$\begin{aligned} u_1 &= \frac{a_0 - cb_{-1}e^{-\eta}}{-\frac{a_0(a_0+b_0c)}{c^2 b_{-1}}e^{\eta} + b_0 + b^{-1}e^{-\eta}} \\ &= -\frac{c(a_0 - cb_{-1}e^{-\eta})}{(1 + \frac{a_0+b_0c}{cb_{-1}}e^{\eta})(a_0 - cb_{-1}e^{-\eta})} = -\frac{c}{1 + \frac{a_0+b_0c}{cb_{-1}}e^{\eta}} = -\frac{c}{1+Ae^{\eta}}, \end{aligned} \tag{39.6.2.4}$$

where $A = \frac{a_0+b_0c}{cb_{-1}}$ is one arbitrary constant (rather than the three arbitrary constants a_0, b_0, and b_{-1} occurring in the original formula). We see that the solution (39.6.2.4) is a special case of (39.6.2.3) for $c = \pm\mu$.

The solution u_2 can be simplified and is a special case of the solution (39.6.2.3) for $\mu = \mp\frac{1}{2}c$. In this case, we have

$$u_2(\eta) = -\frac{b_1 c e^{\eta}}{b_1 e^{\eta} + b_{-1} e^{-\eta}} = -\frac{c}{1 + A e^{-2\eta}}, \tag{39.6.2.5}$$

where $A = b_{-1}/b_1$ is an arbitrary constant.

The solution u_5 is a special case of (39.6.2.3) for $\mu = \mp\frac{1}{3}c$. This follows from the relations

$$u_5(\eta) = -\frac{b_2 c e^{2\eta}}{b_2 e^{2\eta} + b_{-1} e^{-\eta}} = -\frac{c}{1 + A e^{-3\eta}}, \tag{39.6.2.6}$$

where $A = b_{-1}/b_2$ is an arbitrary constant.

We observe that u_6 is a special case of the solution (39.6.2.3) for $\mu = \mp\frac{1}{2}c$ if we take into account the following transformations:

$$\begin{aligned} u_6(\eta) &= \frac{-b_2 c e^{2\eta} - \frac{c b_2 b_{-1}}{b_0} e^{\eta}}{b_2 e^{2\eta} + \frac{b_2 b_{-1}}{b_0} e^{\eta} + b_0 + b_{-1} e^{-\eta}} \\ &= -\frac{\frac{c b_2}{b_0} e^{2\eta}(b_0 + b_{-1} e^{-\eta})}{(b_0 + b_{-1} e^{-\eta})(1 + \frac{b_2}{b_0} e^{2\eta})} = -\frac{\frac{c b_2}{b_0} e^{2\eta}}{1 + \frac{b_2}{b_0} e^{2\eta}} \\ &= -\frac{c}{1 + \frac{b_0}{b_2} e^{-2\eta}} = -\frac{c}{1 + A e^{-2\eta}}, \end{aligned} \tag{39.6.2.7}$$

where $A = b_0/b_2$ is an arbitrary constant.

We conclude that u_7 is a special case of the solution (39.6.2.3) for $\mu = \mp c$ by using the following transformations:

$$\begin{aligned} u_7(\eta) &= \frac{-b_2 c e^{2\eta} + a_1 e^{\eta} + a_0}{b_2 e^{2\eta} + b_1 e^{\eta} - \frac{a_1 b_1 c + a_0 b_2 c + a_1{}^2}{b_2 c^2} - \frac{a_0(a_1 + b_1 c)}{b_2 c^2} e^{-\eta}} \\ &= \frac{-b_2 c^2 (b_2 c e^{\eta} - a_1 - a_0 e^{-\eta}) e^{\eta}}{(b_1 c + a_1 + b_2 c e^{\eta})(b_2 c e^{\eta} - a_1 - a_0 e^{-\eta})} \\ &= -\frac{c}{1 + \frac{b_1 c + a_1}{b_2 c} e^{-\eta}} = -\frac{c}{1 + A e^{-\eta}}, \end{aligned} \tag{39.6.2.8}$$

where $A = \frac{b_1 c + a_1}{b_2 c}$ is an arbitrary constant.

We see that the solution u_8 is a special case of (39.6.2.3) for $\mu = \pm\frac{1}{3}c$ if we reduce u_8 to the form

$$u_8(\eta) = -\frac{c b_{-1} e^{-\eta}}{b_2 e^{2\eta} + b_{-1} e^{-\eta}} = -\frac{c}{1 + \frac{b_2}{b_{-1}} e^{3\eta}} = -\frac{c}{1 + A e^{3\eta}}, \tag{39.6.2.9}$$

where $A = b_2/b_{-1}$ is an arbitrary constant.

The solution u_9 can be simplified as well. As result, we see that u_9 is a special case of the solution (39.6.2.3) for $\mu = \pm\frac{1}{2}c$ by using the transformations

$$u_9(\eta) = \frac{-b_0c - cb_{-1}e^{-\eta}}{b_2e^{2\eta} + \frac{b_2b_{-1}}{b_0}e^{\eta} + b_0 + b_{-1}e^{-\eta}}$$
$$= -\frac{c(b_0 + b_{-1}e^{-\eta})}{\left(1 + \frac{b_2}{b_0}e^{2\eta}\right)(b_0 + b_{-1}e^{-\eta})} = -\frac{c}{1 + \frac{b_2}{b_0}e^{2\eta}} = -\frac{c}{1 + C_1e^{2\eta}}, \qquad (39.6.2.10)$$

where $A = b_2/b_0$ is an arbitrary constant.

The solution u_{10} can be simplified by the transformations

$$u_{10}(\eta) = \frac{a_1e^{\eta} - \frac{c(cb_2b_{-1} + b_0a_1)}{a_1} - cb_{-1}e^{-\eta}}{b_2e^{2\eta} - \frac{c^3b_2^2b_{-1} + a_1b_0b_2c^2 + a_1^3}{ca_1^2}e^{\eta} + b_0 + b_{-1}e^{-\eta}}$$
$$= \frac{c\left[a_1^3e^{\eta} - ca_1(cb_2b_{-1} + b_0a_1) - ca_1^2b_{-1}e^{-\eta}\right]}{\left(\frac{b_2c}{a_1}e^{\eta} - 1\right)\left[a_1^3e^{\eta} - ca_1(cb_2b_{-1} + b_0a_1) - ca_1^2b_{-1}e^{-\eta}\right]}$$
$$= -\frac{c}{1 - \frac{b_2c}{a_1}e^{\eta}} = -\frac{c}{1 + Ae^{\eta}}, \qquad (39.6.2.11)$$

where $A = -\frac{b_2c}{a_1}$ is an arbitrary constant. We see that the solution u_{10} is a special case of the solution (39.6.2.3) for $\mu = \pm c$.

There are many other similar examples of "new solutions" obtained in Wazwaz and Mehanna (2010) (see Kudryashov (2011 *b*) for details). All of them are of very simple structure and have the form of the fraction

$$u = \frac{\varphi(\eta)}{\psi(\eta)} = \frac{\varphi(\eta)f_n(\eta)}{\psi(\eta)f_n(\eta)}, \quad n = 1, 2, \ldots,$$

whose numerator and denominator are simultaneously multiplied by the same function depending on an additional set of irrelevant parameters. As a result, they claim to obtain "new many-parameter exact solutions."

Remark 39.1. There are plenty of other examples of this sort. (See the literature cited at the end of this section, where a classification and examples of other typical errors of some authors can be found; e.g., see the very useful and informative papers by Kudryashov (2009 *b*) and Kudryashov and Loguinova (2009).)

Remark 39.2.. It would be very useful if the symbolic computation systems Maple and Mathematica could output the following warning when producing several solutions of one equation: "The solutions obtained can be equivalent but written in different form." (This would save inexperienced researchers the embarrassment of publishing unnecessarily.)

Even better, the developers of these systems could put some effort in the development of programs capable of revealing equivalent solutions and retaining only one of them, simplest in the form.

Remark 39.3. To compare two solutions $u_1 = f(x)$ and $u_2 = g(x)$ of one and the same equation, it is expedient to construct the dependence $u_1 = \Phi(u_2)$ (in closed analytic form or in the form of a curve). This simple procedure permits revealing equivalent solutions in many cases.

⊙ *References for Section* 39.6: A. A. Soliman (2008), N. A. Kudryashov (2009 *a, b, c*, 2010 *a, b, c, d*, 2011 *a, b*), N. A. Kudryashov and N. B. Loguinova (2009), N. A. Kudryashov and M. B. Soukharev (2009, 2010), N. A. Kudryashov, P. N. Ryabov, and D. I. Sinelshchikov (2010, 2011), R. O. Popovich and O. O. Vaneeva (2010), N. A. Kudryashov and D. I. Sinelshchikov (2010), A. M. Wazwaz and M. S. Mehanna (2010).

39.7. Numerical Solutions and Their Visualizations

In this section, first, we consider the construction of numerical and graphical solutions of various initial-boundary value problems using predefined Maple functions and default methods, e.g., for second-order PDEs (the nonlinear diffusion and wave equations). In addition, describing the classical numerical methods, we show how to find numerical and graphical solutions specifying a particular numerical method and numerical boundary conditions.

Then we consider finite-difference approximations and solve various important initial-boundary value problems, e.g., for the nonlinear diffusion equation, the nonlinear wave equation, and the nonlinear Poisson equation.

39.7.1. Constructing Numerical Solutions in Terms of Predefined Functions

First, consider the predefined Maple functions with the aid of which we can obtain approximate numerical solutions when solving various linear and nonlinear PDE problems. With the aid of the predefined function `pdsolve` (the option `numeric`), we can numerically solve initial-boundary value problems.

The predefined function `pdsolve` allows us to solve a single PDE (higher order) and PDE systems by the default method or specify a particular method for solving a single PDE.

It is possible to impose Dirichlet, Neumann, Robin, or periodic boundary conditions.

Below are typical Maple commands useful for constructing numerical solutions in terms of predefined functions:

```
infolevel[all]:=5;          Sol:=pdsolve(PDEs,IBCs,numeric,funcs,ops);
Num_vals:=Sol:-value();               Sol:-plot3d(func,t=t0..t1,ops);
Num_vals(num1,num2);         Sol:-animate(func,t=t0..t1,x=x0..x1,ops);
pdsolve(PDE,IBCs, numeric,numericalbcs=val,method=M1,startup=M2,ops);
```

Description of some functions:

`pdsolve,numeric`, finding numerical solutions of a partial differential equation `PDE` or a system of `PDEs`.

`plot3d`, `animate`, visualizing the numerical solution `Sol` obtained by `pdsolve,numeric`.

`value`, displaying numerical values of a numerical solution `Sol`.

Comment. The solution obtained is represented as a module (similar to a procedure, with the operator `:-`), which can be used for obtaining visualizations (`plot`, `plot3d`, `animate`, `animate3d`) and numerical values (`value`); for more detail, see `?pdsolve[numeric]`.

Numerical and graphical solutions by default methods. Numerical solutions can be obtained in Maple automatically (without specifying a numerical method) by default θ-methods [see Larsson and Thomée (2008), Morton and Mayers (1995)]. The θ-method is a generalization of the known finite difference approximations (explicit and implicit) by introducing a parameter θ ($0 \le \theta \le 1$) and by taking a weighted average of the two formulas, where the special case $\theta = \frac{1}{2}$ corresponds to the Crank–Nicholson method [see

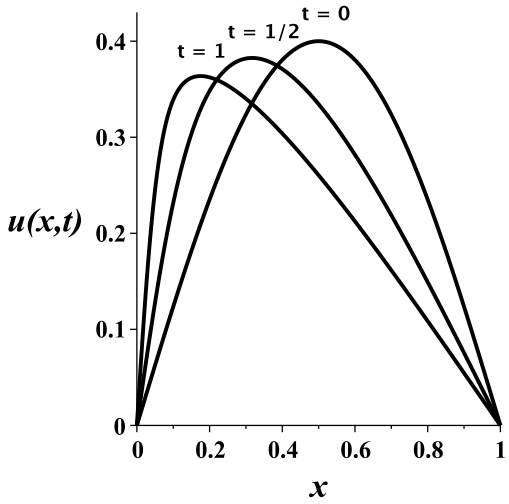

Figure 39.7. Numerical solution of the Burgers equation at $t = 0, \frac{1}{2}, 1$.

Crank and Nicolson (1947)] and $\theta = 0$ and $\theta = 1$ are just the explicit and implicit methods, respectively.

It should be noted that in Maple (Ver. ≤ 14) one can numerically solve only evolution equations via the predefined functions.

Example 39.41. *Numerical and graphical solutions. Burgers equation.* We find numerical and graphical solutions of the following initial-boundary value problem for the Burgers equation:

$$u_t = \nu u_{xx} + u u_x; \quad u(x,0) = A\sin(\pi x), \quad u(0,t) = u(1,t) = 0,$$

with initial amplitude $A = 0.4$ and kinematic viscosity $\nu = 0.009$. The numerical solution of this initial-boundary value problem at times $t = 0, \frac{1}{2}, 1$ is shown in Fig. 39.7

```
infolevel[all]:=5;
with(plots): with(PDEtools): declare(u(x,t));
nu:=0.009; A:=0.4: S:=1/100; tR:=0..40; xR:=0..1; NF:=30; NP:=100;
N:=3; L:=1; L1:=[red,blue,green]; L2:=[0,1/2,1];  Ops:=spacestep=S,timestep=S;
PDE1:=diff(u(x,t),t)-nu*diff(u(x,t),x$2)-u(x,t)*diff(u(x,t),x)=0;
f:=x->A*sin(Pi*x/L); IC:={u(x,0)=f(x)}; BC:={u(0,t)=0,u(L,t)=0};
Sol1:=pdsolve(PDE1,IC union BC,numeric,u(x,t),Ops);
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Num_vals1:=Sol1:-value(); Num_vals1(1/2,Pi);
Sol1:-plot3d(u(x,t),t=tR,shading=zhue,axes=boxed);
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Example 39.42. *Numerical and graphical solutions. Nonlinear wave equation.* We find numerical and graphical solutions of the nonlinear wave equation subject to the following initial and boundary conditions:

$$u_{tt} = a^2 (e^{\lambda u} u_x)_x; \quad u(x,0) = 0, \quad u_t(x,0) = \sin(2\pi x), \quad u(0,t) = u(1,t) = 0,$$

in the domain $\{0 < x < 1,\ 0 < t < \infty\}$, where $a = 0.1$. The numerical solution of this initial-boundary value problem at times $t = 0.1, 0.5, 1.$ is shown in Fig. 39.8

```
with(plots): with(PDEtools): declare(u(x,t)); a:=0.1: lambda:=-0.4:
S:=1/100; tR:=0..10; xR:=0..1; NF:=30; NP:=100; Ops:=spacestep=S,timestep=S;
```

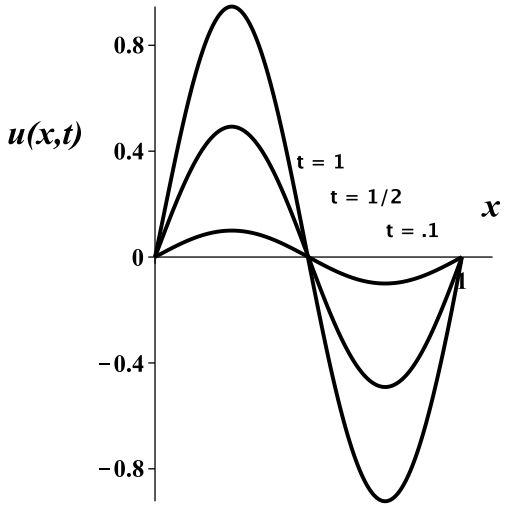

Figure 39.8. Numerical solution of the nonlinear wave equation at $t = 0.1, 0.5, 1$.

```
PDE1:=diff(u(x,t),t$2)-a^2*diff(exp(lambda*u(x,t))*diff(u(x,t),x),x)=0;
Ics:={D[2](u)(x,0)=sin(2*Pi*x),u(x,0)=0}; Bcs:={u(0,t)=0,u(1,t)=0};
Sol1:=pdsolve(PDE1,Ics union Bcs,numeric,u(x,t),Ops);
N:=3; L1:=[red,blue,green]; L2:=[0.1,1/2,1];
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)}); Num_vals1:=Sol1:-value(); Num_vals1(1/2,Pi);
Sol1:-plot3d(u(x,t),t=tR,shading=zhue,axes=boxed);
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Numerical methods embedded in Maple. As was mentioned above, in Maple one can find numerical solutions specifying one of the eleven *classical methods* (e.g., Euler, Crank–Nicholson, Lax–Friedrichs, etc.), specifying numerical boundary conditions and finite difference schemes (for two-stage methods). But there is some restriction with respect to these classical methods: a single PDE must be parabolic or hyperbolic and of first-order in time. PDEs that are greater than first-order in time can be solved by converting to an equivalent first-order system. More detailed information about numerical methods embedded in Maple is presented in Table 39.2.

Example 39.43. *Numerical and graphical solutions: specifying a single-stage numerical method.* Consider the following initial-boundary value problem for the *inviscid Burgers equation*

$$u_t = uu_x; \quad u(x,0) = e^{-10(2x-1)^2}, \quad u(0,t) = 0.$$

We would like to obtain the numerical and graphical solutions of this initial-boundary value problem by applying the single-stage explicit `ForwardTime1SpaceBackward` method. Since the boundary condition is given on the left (according to the method), we consider the domain $\{0 \le x \le 1,\ t > 0\}$. We can find the numerical and graphical solutions of this initial-boundary value problem, e.g., at times $t = 0, 0.15, 0.3$, as follows:

```
with(PDEtools): with(plots): declare(u(x,t));
NF:=30; NP:=100; xR:=0..1; tR:=0..1;
S:=1/100; Ops:=timestep=S,spacestep=S; N:=3;
L1:=[red,blue,magenta]; L2:=[0,0.15,0.3];
PDE1:=diff(u(x,t),t)+diff(u(x,t),x)*u(x,t)=0;
```

TABLE 39.2
Numerical methods embedded in Maple with brief description and some references

Numerical method	Brief description	References
`FTime1Space[b]` `FTime1Space[f]`	Explicit 1-stage method for first-order time/space PDEs. Accuracy: $O(h,k)$. Stability: $k < ah$ (a depends upon the problem). `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`CTime1Space[b]` `CTime1Space[f]`	Implicit 1-stage method for PDEs: $F(u,u_x,u_t,u_{xt})=0$. Accuracy: $O(h,k^2)$. Stability: unconditionally stable for many problems. `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`BTime1Space[b]` `BTime1Space[f]`	Implicit 1-stage method for PDEs: $F(u,u_x,u_t,u_{xt})=0$. Accuracy: $O(h,k)$. Stability: unconditionally stable for many problems. `[b/f]`: to describe right/left TW, (to specify BC at the left/right boundary). Numerical BCs are not required.	Lapidus and Pinder (1999) LeVeque (2007) Morton and Mayers (1995)
`Euler` or `FTimeCSpace`	Explicit 1-stage method for PDEs: 1st-order in t, nth-order in space (no mixed derivative). Accuracy: $O(h^2,k)$. Stability: some restriction on h and k. Numerical BCs are required depending upon the order of PDE in space.	Lapidus and Pinder (1999) Strikwerda (2004) Morton and Mayers (1995)
`CrankNicholson` `CTimeCSpace`	Implicit 1-stage method for PDEs: 1st-order in t, nth-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of PDE in space.	Lapidus and Pinder (1999) Thomas (1995) Morton and Mayers (1995)
`BackwardEuler` or `BTimeCSpace`	Implicit 1-stage method for PDEs: 1st-order in t, nth-order in space (no mixed derivative). Accuracy: $O(h^2,k)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of PDE in space.	Lapidus and Pinder (1999) Strikwerda (2004) Morton and Mayers (1995)
`Box[b]` `Box[f]`	Implicit 1-stage method for PDEs: 1st-order in t, nth-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Stability: unconditionally stable for many problems. Numerical BCs are required depending upon the order of PDE in space.	Strikwerda (2004) LeVeque (2007) Larsson and Thomée (2008)
`LaxFriedrichs`	Explicit 1-stage method for PDEs: 1st-order in time, odd-order in space (no mixed derivative). Accuracy: $O(h^2,k,M)$. Stability: restriction of the form $k < ah^P$. Numerical BCs are required depending upon the order of PDE in space.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)
`LaxWendroff`	Explicit 1-stage method for linear PDEs that are first-order in time and space. Accuracy: $O(h^2,k,h,k^2)$. Stability: restriction of the form $k < ah$. Numerical BCS are required so that one BC is specified for each boundary.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)
`Leapfrog`	Explicit 2-stage method for PDEs: 1st-order in time, nth-order in space (no mixed derivative). Accuracy: $O(h^2,k^2)$. Startup method. Stability: restriction. Numerical BCs are required depending upon the order of PDE in space.	Morton and Mayers (1995) Strikwerda (2004) Thomas (1995)
`DuFortFrankel`	Explicit 2-stage method for linear/nonlinear PDEs: 1st-order in t, even-order in space (no mixed derivative). Accuracy: $O(h^2,k^2,Q)$. Startup method is required. No numerical BCs are required. Stability: restriction of the form $k < ah^P$.	Larsson and Thomée (2008) LeVeque (2007) Strikwerda (2004)

```
f:=x->exp(-10*(2*x-1)^2); IBC:={u(x,0)=f(x),u(0,t)=0};
M1:=ForwardTime1Space[backward];
Sol1:=pdsolve(PDE1,IBC,numeric,time=t,range=xR,method=M1,Ops);
Num_vals1:=Sol1:-value(); Num_vals1(0,0.5);
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP*2): od:
display({seq(G||i,i=1..N)});
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Comment. If $f'(x_{cr}) = (-80x+40)e^{-10(2x-1)^2} < 0$ for $x_{cr} \in (\frac{1}{2},\infty)$, then the solutions of this initial value problem break down at time t_{cr} by generating a vertical line in the wave profile, which corresponds to a discontinuity in the solution. For $t > t_{cr}$, the solution must be continued as a *weak solution*, one example of which is a *shock wave*.

The abbreviated Maple's names of the embedded numerical methods are presented in the first column of the table. The following abbreviations in the table are adopted: `F`, for-

ward; `B`, backward; `C`, centered; `b`, backward; `f`, forward; `TW`, traveling-wave; `BC(s)`, boundary condition(s); h, space step; k, time step; P, the differential order of the PDE in the spatial variable; *ceil*, smallest integer greater than or equal to a number; $M = h^{2N}/k$, $Q = k^2/h^{2N}$, and $N = ceil(\frac{1}{2}P)$. For example, for the *forward time backward space method*, the abbreviated Maple's name is `FTime1Space[b]` and the corresponding complete Maple's name is `ForwardTime1Space[backward]`.*

Example 39.44. *Numerical and graphical solutions: specifying a single-stage numerical method and numerical boundary condition (NBC).* Consider the following initial-boundary value problem for the first-order nonlinear PDE

$$u_t = \nu u^3 u_x; \quad u(x,0) = \cos(\tfrac{1}{2}\pi x), \quad u(-1,t) = u(1,t),$$

defined in the domain $\{-1 \le x \le 1,\ 0 \le t \le 5\}$. We would like to obtain the numerical and graphical solutions of this problem by applying the well-known, popular `CrankNicholson` method [see Crank and Nicolson (1947)]. Since this PDE is of odd order in space, it requires a numerical boundary condition, which we can choose as Box discretization of the PDE at the right boundary, `[Box,n]`. Then we can find the numerical and graphical solutions of this initial-boundary value problem, e.g., at times $t = 0, \frac{1}{2}$ as follows:

```
with(PDEtools): with(plots): declare(u(x,t)); NF:=30; NP:=100;
xR:=-1..1; tR:=0..5; S:=1/100; Ops:=timestep=S,spacestep=S;
N:=2; L1:=[red,blue]; L2:=[0,1/2]; nu:=0.1; NBCs:=[Box,n];
PDE1:=diff(u(x,t),t)=nu*diff(u(x,t),x)*u(x,t)^3;
IBC:={u(x,0)=cos(Pi*x/2),u(-1,t)=u(1,t)}; M1:=CrankNicholson;
Sol1:=pdsolve(PDE1,IBC,numeric,numericalbcs=NBCs,method=M1,Ops);
for i from 1 to N do G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP): od:
display({seq(G||i,i=1..N)});
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

Example 39.45. *Numerical and graphical solutions: specifying a two-stage numerical method, startup method, and NBC.* Consider the initial-boundary value problem

$$u_t + u u_x = 0; \quad u(x,0) = f(x), \quad u_x(-10,t) = 0$$

defined in the domain $\{-10 \le x \le 10,\ t > 0\}$, where $f(x) = \begin{cases} 0, & x < -1, \\ x+1, & -1 \le x \le 0, \\ 1, & x > 0. \end{cases}$

We would like to obtain the numerical and graphical solutions of this problem by applying the two-stage explicit `DuFortFrankel` method. Therefore, we have to indicate how to compute the additional stage required for two-stage methods, i.e., the `startup` option. Since the inviscid Burgers equation is of odd order in space, we have to indicate a numerical boundary condition (NBC), which we can choose as an NBC that forces the value of the solution on the right boundary to be the same as the value of the solution at the first interior point; i.e., `u[1, n]-u[1,n-1]`. Then we can find the numerical and graphical solutions of this initial-boundary value problem at times $t = 0, 0.4, 0.5$ as follows:

```
with(PDEtools): with(plots): declare(u(x,t));
NF:=30; NP:=100; xR:=-10..10; tR:=0..5; S:=1/50;
Ops:=timestep=S,spacestep=S; N:=3;
L1:=[red,blue,magenta]; L2:=[0,0.4,0.5];
PDE1:=diff(u(x,t),t)+u(x,t)*diff(u(x,t),x)=0;
f:=x->piecewise(x<-1,0,x>=-1 and x<=0,x+1,1);
IBC:={u(x,0)=f(x),(D[1](u))(-10,t)=0}; M1:=DuFortFrankel;
```

*Maple's names `CrankNicholson`, `LaxFriedrichs`, `LaxWendroff`, and `DuFortFrankel` correspond to numerical methods known in scientific literature: `Crank-Nicolson` (this name is slightly different from Maple's name), `Lax-Friedrichs`, `Lax-Wendroff`, and `DuFort-Frankel`.

```
Sol1:=pdsolve(PDE1,IBC,type=numeric,time=t,range=xR,
      numericalbcs=u[1,n]-u[1,n-1],method=M1,startup=Euler,Ops);
for i from 1 to N do
 G||i:=Sol1:-plot(t=L2[i],color=L1[i],numpoints=NP,thickness=2):
od:
display({seq(G||i,i=1..N)});
Sol1:-animate(u(x,t),x=xR,t=tR,frames=NF,numpoints=NP,thickness=3);
```

39.7.2. Constructing Finite Difference Approximations

Now we show a helpful role of computer algebra systems for generating and applying various finite difference approximations for constructing numerical solutions of linear and nonlinear PDEs.

To approximate linear and nonlinear PDEs by finite differences, we have to generate a *mesh* (or grid) in a domain $\mathcal{D}$, e.g., $\mathcal{D} = \{a < x < b,\, c < t < d\}$. The mesh can be of various types, e.g., rectangular, along the characteristics, polar, etc. We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable in a given PDE is $u(x,t)$.

We write h and k for the line spacings and define the *mesh points* as follows: $X_i = a + ih$, $T_j = c + jk$ $(i = 0,\ldots,NX,\ j = 0,\ldots,NT)$, and $h = (b-a)/NX$, $k = (d-c)/NT$. We calculate approximations to the solution at these mesh points; these approximate points will be denoted by $U_{i,j} \approx u(X_i, T_j)$. We approximate the derivatives in a given equation by finite differences (of various types) and then solve the resulting difference equations.

Example 39.46. *Nonlinear parabolic equations. Burgers equation. Explicit method.* Consider the following initial-boundary value problem for the Burgers equation:

$$u_t = \nu u_{xx} + uu_x; \quad u(x,0) = \sin(\pi x), \quad u(0,t) = 0, \quad u(1,t) = 0$$

in the domain $\{0 \le x \le 1,\ 0 < t < T\}$, where $T = 0.4$ and $\nu = 0.009$.

Let us generate the rectangular mesh: $X = a + ih$, $T = jk$ $(i = 0,\ldots,NX,\ j = 0,\ldots,NT$, $h = (b-a)/NX$, $k = T/NT)$. We denote the approximate solution of $u(x,t)$ at the mesh point (i,j) as $U_{i,j}$.

In *forward difference method*, the second derivatives u_{xx} are replaced by a central difference approximation (CDA) and the first-order derivatives u_t and u_x by a forward difference approximation (FWDA). The FD scheme for the nonlinear Burgers equation has the form

$$U_{i,j+1} = U_{i,j} + r(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) + (k/h)U_{i,j}(U_{i+1,j} + U_{i,j}),$$

where $r = \nu k/h^2$. In this explicit FD scheme, the unknown value $U_{i,j+1}$ (on $j+1$ step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the jth step). We construct an approximate numerical solution of the initial-boundary value problem by applying the forward difference method as follows:

```
with(plots): nu:=0.009: NX:=50: NT:=25: a:=0: b:=1: L:=1.:
T:=0.4; h:=evalf((b-a)/NX); k:=evalf(T/NT); r:=nu*k/h^2;
f:=x->evalf(sin(Pi*x/L)); for i from 0 to NX do X[i]:=a+i*h od:
IC:={seq(U(i,0)=f(X[i]),i=0..NX)};
BC:={seq(U(a,j)=0,j=0..NT),seq(U(NX,j)=0,j=0..NT)}:
IBC:=IC union BC:
FD:=(i,j)->U(i,j)+r*(U(i+1,j)-2*U(i,j)+U(i-1,j))+k/h*U(i,j)*(U(i+1,j)-U(i,j));
for j from 0 to NT do for i from 1 to NX-1 do U(i,j+1):=subs(IBC,FD(i,j)); od: od:
G:=j->plot([seq([X[i],subs(IBC,U(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,thickness=3,labels=["X","U"]);
```

Example 39.47. *Inviscid Burgers equation. Explicit method.* Now consider a particular case of the Burgers equation, the inviscid Burgers equation; i.e., $u_t + uu_x = 0$. This equation is of hyperbolic type. The initial-boundary value problem for this equation has the form

$$u_t + uu_x = 0; \quad u(x,0) = \arctan(4x) + 2, \quad u(a,t) = 0$$

in the domain $\{a \le x \le b,\ 0 < t < T\}$, where $a = -4$, $b = 6$, and $T = 0.4$. Note that this equation can be rewritten in the *conservation form*, i.e., as the physical conservation law of the form $u_t + F_x(u) = 0$, where $F(u) = \frac{1}{2}u^2$. For the discretization of the problems, we use this form of the Burgers equation.

Let us generate the rectangular mesh $X = a + ih$, $T = jk$ ($i = 0,\ldots,NX$, $j = 0,\ldots,NT$, $h = (b-a)/NX$, $k = T/NT$). We denote the approximate solution of $u(x,t)$ at the mesh point (i,j) by $U_{i,j}$.

In the *Lax method*, the derivative u_t is replaced by the forward difference approximation (FWDA) and the derivative $F_x(u)$ by the central difference approximation (CDA) (together with the corresponding substitution). The FD scheme for the nonlinear diffusion equation has the form

$$U_{i,j+1} = \frac{1}{2}\left(U(i+1,j) + U(i-1,j)\right) - \frac{1}{4}(k/h)\left(U^2(i+1,j) - U^2(i-1,j)\right).$$

In this explicit FD scheme, the unknown value $U_{i,j+1}$ (on $j+1$ step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the jth step). This FD scheme is stable for $k/h < 1/|u|$; i.e., the stability condition depends on the solution. We construct an approximate numerical solution of the initial-boundary value problem by applying the Lax method as follows:

```
with(plots):
NX:=50: NT:=9: a:=-4; b:=6; T:=0.4;
h:=evalf((b-a)/NX); k:=evalf(T/NT); f:=x->arctan(4*x)+2;
for i from 0 to NX do X[i]:=a+i*h od:
IC:={seq(U[i,0]=f(X[i]),i=0..NX)}; BC:={seq(U[a,j]=0,j=0..NT)}:
IBC:=IC union BC:
FD:=(i,j)->1/2*(U[i+1,j]+U[i-1,j])-k/(2*h)*(U[i+1,j]^2/2-U[i-1,j]^2/2);
for j from 0 to NT do for i from 1 to NX-1 do U[i,j+1]:=subs(IBC,FD(i,j)); od: od:
L1:=[1,3,5,7,9]; NL1:=nops(L1);
for i from 1 to NL1 do G||(L1[i]):=[seq([X[j],U[j,L1[i]]],j=1..NX+1)]; od:
plot([seq(G||(L1[i]),i=1..NL1)]);
```

Example 39.48. *Nonlinear wave equation. Explicit method.* Consider the initial-boundary value problem for the nonlinear wave equation describing the motion of a fixed string:

$$u_{tt} = \frac{1}{c^2}u_{xx} + e^u; \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = 0, \quad u(\tfrac{1}{2},t) = 0$$

in the domain $\{0 \le x \le L,\ 0 \le t \le T\}$, where $f(x) = 0$, $g(x) = \sin(4\pi x)$, $T = 0.5$, and $c = 1/(4\pi)$.

In *explicit central difference method*, each second derivative is replaced by the central difference approximation (CDA). The FD scheme for the nonlinear wave equation has the form

$$U_{i,j+1} = 2(1-r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1} + \exp(U(i,j))(ck)^2,$$

where $r = (ck/h)^2$. In this FD scheme, we have one unknown value $U_{i,j+1}$ that depends explicitly on the four known values $U_{i,j}$, $U_{i+1,j}$, $U_{i-1,j}$, $U_{i,j-1}$ at the previous time steps j and $j-1$. To start the process, we have to know the values of U at the time steps $j = 0$ and $j = 1$. Thus, we can define the initial conditions at these times steps: $U_{i,0} = f(X_i)$ and $U(X_i,0)_t \approx (U_{i,1} - U_{i,0})/k = g(X_i)$, $U_{i,1} = f(X_i) + kg(X_i)$.

We construct an approximate numerical solution of the initial-boundary value problem by applying an explicit finite difference method and plot the numerical solution as follows:

```
with(plots): L:=0.5: T:=2.: NX:=15: NT:=20: c:=0.25; h:=L/NX;
k:=T/NT; r:=(c*k/h)^2; f:=x->0: g:=x->evalf(sin(4*Pi*x)):
IC:={seq(U(i,0)=f(i*h),i=1..NX-1),seq(U(i,1)=f(i*h)+k*g(i*h),i=1..NX-1)}:
BC:={seq(U(0,j)=0,j=0..NT),seq(U(NX,j)=0,j=0..NT)}:
IBC:=IC union BC:
```

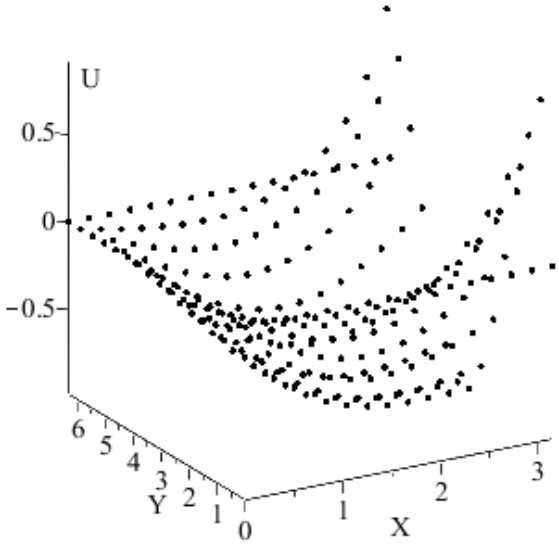

Figure 39.9. Numerical solution of the nonlinear Poisson equation.

```
FD:=(i,j)->evalf(2*(1-r)*U(i,j)+r*(U(i+1,j)+U(i-1,j))-U(i,j-1)+exp(U(i,j))*(c*k)^2);
for j from 1 to NT-1 do for i from 1 to NX-1 do U(i,j+1):=subs(IBC,FD(i,j)); od:od:
G:=j->plot([seq([i*h,subs(IBC,U(i,j))],i=0..NX)],color=blue):
display([seq(G(j),j=0..NT)],insequence=true,thickness=3,labels=["X","U"]);
```

Example 39.49. *Nonlinear Poisson equation. Boundary value problem.* Consider the boundary value problem for the two-dimensional nonlinear Poisson equation

$$u_{xx}+u_{yy}=\sin u; \quad u(x,c)=0, \quad u(x,d)=0, \quad u(a,y)=0, \quad u(b,y)=\cos y,$$

describing a potential field $u(x,y)$ in a bounded rectangular object occupying the domain $\{a \le x \le b,\ c \le y \le d\}$, where $a=0$, $b=\pi$, $c=0$, and $d=2\pi$.

Let us generate the rectangular mesh $x=a+ih$, $y=c+jk$ ($i=0,\ldots,NX$, $j=0,\ldots,NY$, $h=(b-a)/NX$, $k=(d-c)/NY$). We denote the approximate solution $u(x,y)$ at the mesh point (i,j) by $U_{i,j}$. The second derivatives in the Poisson equation can be replaced by the central difference approximation (CDA). The FD scheme has the form

$$2(1+r)U_{i,j}-U_{i+1,j}-U_{i-1,j}-rU_{i,j+1}-rU_{i,j-1}=h^2\sin(U_{i,j}),$$

where $r=(h/k)^2$.

We construct the approximate numerical solution of the boundary value problem by applying the above explicit finite-difference scheme and plot (see Fig. 39.9) the numerical solution inside the domain as follows:

```
with(plots):
a:=0; b:=Pi; c:=0; d:=2*Pi; NX:=15; NY:=15;
h:=(b-a)/NX; k:=(d-c)/NY; r:=(h/k)^2;
XY:=seq(x[i]=a+i*h,i=0..NX),seq(y[j]=c+j*k,j=0..NY);
FD:=(i,j)->evalf(2*(1+r)*U[i,j]-U[i+1,j]-U[i-1,j]-r*U[i,j+1]
                -r*U[i,j-1]-h^2*sin(U[i,j]))=0;
F1:=i->0; F2:=i->0; F3:=j->0; F4:=j->cos(j*k);
BC:=[seq(U[i,0]=F1(i),i=0..NX),seq(U[i,NY]=F2(i),i=0..NX),
     seq(U[0,j]=F3(j),j=0..NY),seq(U[NX,j]=F4(j),j=0..NY)];
```

```
Eqs:={seq(seq(FD(i,j),i=1..NX-1),j=1..NY-1)}: Eqs1:=subs(BC,Eqs);
vars:={seq(seq(U[i,j],i=1..NX-1),j=1..NY-1)};
Sol:=fsolve(Eqs1,vars);
Points:=[seq(seq([x[i],y[j],U[i,j]],i=0..NX),j=0..NY)];
Points1:=evalf(subs(XY,BC,op(Sol),Points));
pointplot3d(Points1,symbol=solidsphere,shading=z,orientation=[136,-60],axes=frame);
```

⊙ *References for Section* 39.7: J. Crank and P. Nicolson (1947), K. W. Morton and D. F. Mayers (1995), J. W. Thomas(1995), L. Lapidus and G. F. Pinder (1999), L. Strikwerda (2004), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008).

39.8. Analytical-Numerical Solutions

In this section, we show the very helpful role of computer algebra systems for deriving numerical methods and testing the results obtained. By way of example, we consider an analytical-numerical approach for constructing an approximate numerical solution of the problem describing the nonlinear standing wave motion on the free surface of a fluid in *Eulerian coordinates* [see Birkhoff (1950)]. We compare this numerical solution with the asymptotic solution of this problem obtained in Eulerian and *Lagrangian coordinates* [for details, see Sekerzh-Zenkovich (1947), Shingareva, Lizárraga-Celaya, and Ochoa Ruiz (2006)].

39.8.1. Analytical Derivation of Numerical Methods

By way of example, we construct a *spectral collocation method* for solving an initial-boundary value problem for two-dimensional nonlinear standing waves of finite amplitude on the free surface of a fluid. The higher-order asymptotic solution of this nonlinear problem in Lagrangian coordinates with weak capillarity was obtained by using computer algebra systems [see Shingareva and Lizárraga-Celaya (2004), Shingareva and Lizárraga-Celaya (2007)].

There are two ways for describing the fluid motion: the Eulerian approach (in which the coordinates are fixed in the reference frame of the observer) and the Lagrangian approach (in which the coordinates are fixed in the reference frame of the moving fluid). In this section, we follow the Eulerian approach for constructing a numerical solution and compare the results with the asymptotic solution obtained in Lagrangian and Eulerian coordinates.

We first describe the statement of the problem in Eulerian coordinates and the analytical construction of the numerical method, and then we describe its implementation in Maple for deriving formulas of the method, obtaining numerical results, and comparing the numerical solution with the asymptotic solution.

Statement of the problem. Consider the two-dimensional inviscid potential flow of a fluid bounded below by a horizontal bottom and above by a free surface. We assume the fluid motion to be periodic in time t and in the horizontal x-direction. For simplicity, we consider a special case of infinite-depth pure gravity surface water waves (i.e., the fluid depth is infinite and the surface tension constant is zero). The horizontal size of the domain L is given.

We define Cartesian coordinates according to the requirements: the fluid motion is symmetric with respect to the vertical plane $x = 0$; the free surface is $y = \eta(x,t)$; the line

$y=0$ corresponds to the mean water level; at $t=0$, the fluid is at rest; and the infinite depth is defined as $y=-h=-\infty$.

Investigating this 2D nonlinear standing wave problem and assuming that the flow is irrotational, we have to solve the following dimensionless *Euler equations* for a fluid with initial and boundary conditions in the domain $\mathcal{D}=\{0\le x\le\pi,-\infty\le y\le C\eta(x,t)\}$ for the wave profile $\eta(x,t)$, the velocity potential $\phi(x,y,t)$, and the angular frequency ω of the nonlinear standing wave:

$$\begin{aligned}
&\phi_{xx}+\phi_{yy}=0 \quad \text{in } \{0\le x\le\pi,-\infty\le y\le C\eta(x,t)\};\\
&\eta+\omega\phi_t+\tfrac{1}{2}C[(\phi_x)^2+(\phi_y)^2]=0,\phi_y-\omega\eta_t-C\phi_x\eta_x=0 \quad \text{at } y=C\eta(x,t);\\
&\phi_x=0 \quad \text{at } x=0,\pi; \quad \phi_y=0 \quad \text{at } y=-h;\\
&\int_0^{\pi}\eta(x,t)\,dx=0,\ \nabla\phi(x,y,t+2\pi)=\nabla\phi(x,y,t),\ \eta(0,0)-\eta(\pi,0)=2. \qquad (39.8.1.1)
\end{aligned}$$

Description of the method. Analytical derivation of formulas of the method. Let us study periodic solutions (in x and t) of the standing wave problem.

First step. To solve the *exact nonlinear problem* numerically, let us represent the unknown functions $\eta(x,t)$ and $\phi(x,y,t)$ in the form of series:

$$\eta(x,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nk}\cos(nx)\cos[(m-1)t], \qquad (39.8.1.2)$$

$$\phi(x,y,t)=A_0t+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin(mt)\cos[(n-1)x]\cosh[(n-1)(y+h)], \qquad (39.8.1.3)$$

where A_0, $A_{11},\ldots$, $B_{11},\ldots$, are the unknown coefficients. We note that these forms of the functions $\eta(x,t)$ and $\phi(x,y,t)$ allow us to convert the first, fourth, fifth, sixth, and seventh equations in the governing system (39.8.1.1) to the identities. Additionally, it can be shown that, according to the symmetry and periodicity of the problem, some unknown coefficients are zero; i.e., $A_{nm}=0$ and $B_{nm}=0$ if $n+m$ is even. Therefore, we have simplified our problem: instead of solving eight equations we have to find (for a given value of the amplitude C) the unknown nonzero coefficients A_0, A_{nm}, B_{nm} and the nonlinear frequency ω satisfying the following three equations:

$$\begin{aligned}
&\eta+\omega\phi_t+\tfrac{1}{2}C[(\phi_x)^2+(\phi_y)^2]=0, \quad \phi_y-\omega\eta_t-C\phi_x\eta_x=0 \quad \text{at } y=C\eta(x,t);\\
&\eta(0,0)-\eta(\pi,0)=2. \qquad (39.8.1.4)
\end{aligned}$$

We solve this problem approximately by replacing the infinite series by finite sums of the form

$$\eta(x,t)=\sum_{n=1}^{N-1}\sum_{m=1}^{N}B_{nk}\cos(nx)\cos[(m-1)t], \qquad (39.8.1.5)$$

$$\phi(x,y,t)=A_0t+\sum_{n=1}^{N}\sum_{m=1}^{N-1}A_{nm}\sin(mt)\cos[(n-1)x]\cosh[(n-1)(y+h)]. \qquad (39.8.1.6)$$

If N and $n+m$ are even, then $A_{nm}=0$, $B_{nm}=0$ and we can determine that the number of nonzero coefficients of the A_{nm} and of the B_{nm} is $N(N-1)$. Therefore, for a given

value of the amplitude C, the total number of unknown coefficients (A_{nm}, B_{nm}, A_0, ω) is $N(N-1)+2$. These unknown coefficients have to satisfy the governing system (39.8.1.4). We analytically derive the governing system (39.8.1.4) as follows:

```
N:=4: S:=0:
tr1:={y=C*eta};
tr2:={x=0,t=0};
tr3:={x=0,t=Pi};
for n from 1 to N-1 do
  for m from 1 to N do
    if type(n+m,odd) then S:=S+B||n||m*cos(n*x)*cos((m-1)*t) fi:
od: od:
eta:=S; S:=0:
for n from 1 to N do for m from 1 to N-1 do
  if type(n+m,odd) then S:=S+A||n||m*cos((n-1)*x)*cosh((n-1)*(y+h))*sin(m*t) fi:
od: od:
phi:=A0*t+S;
Eq1:=eval(eta+omega*diff(phi,t)+1/2*C*((diff(phi,x))^2+(diff(phi,y))^2),tr1);
Eq2:=eval(diff(phi,y)-omega*diff(eta,t)-C*diff(phi,x)*diff(eta,x),tr1);
Eq3:=2-eval(eta,tr2)+eval(eta,tr3);
Eq4:=eval(Eq1,tr3);
```

Second step. Now let us find these unknown coefficients by the collocation method, i.e., by satisfying the governing system (39.8.1.4). Therefore, we define the mesh (X_j, T_i) ($j = 1,\ldots,N-1$, $i = 1,\ldots,N/2$) in $\mathcal{D}$, in which we choose the mesh points as follows:

$$X_j = \begin{cases} \pi(j-1)/(N-2) & \text{if } N \neq 4, \\ \pi(j-\frac{1}{2})/(N-1) & \text{if } N = 4, \end{cases} \qquad T_i = \pi(i-\tfrac{1}{2})/N,$$

where N is even (e.g., $N = 4,6,8,\ldots$). By substituting the truncated series (39.8.1.5), and (39.8.1.6) into the governing system (39.8.1.4) obtained at the first step, at the mesh points we can obtain $N(N-1)+1$ nonlinear transcendental equations; we can obtain the last equation by substituting the initial values $x = 0$ and $t = 0$ into the first equation of the governing system (39.8.1.4). Therefore, we analytically construct the system of $N(N-1)+2$ nonlinear transcendental equations for $N(N-1)+2$ unknown coefficients A_{nm}, B_{nm}, A_0, and ω as follows:

```
tr4:=(j,i)->[x=X||j,t=T||i];
L1:=NULL: L2:=NULL:
for i from 1 to N/2 do
  T||i:=Pi/N*(i-1/2):
  for j to N-1 do
    if N=4 then X||j:=Pi/3*(j-1/2) else X||j:=Pi/(N-2)*(j-1) fi;
    Eq1||i||j:=eval(subs(tr4(j,i),Eq1));
    Eq2||i||j:=eval(subs(tr4(j,i),Eq2));
    print("Eq1"||i||j,Eq1||i||j); print("Eq2"||i||j,Eq2||i||j);
od; od;
for n to N do for m to N-1 do if type(n+m,odd) then L1:=L1,A||n||m fi: od: od:
for n to N-1 do for m to N do if type(n+m,odd) then L2:=L2,B||n||m fi: od: od:
LA:=[L1]; LB:=[L2]; k:=1:
for i from 1 to N/2 do for j from 1 to N-1 do
  F||k:=simplify(Eq1||i||j); F||(k+N/2*(N-1)):=simplify(Eq2||i||j); k:=k+1:
```

```
od; od;
F||(N*(N-1)+1):=simplify(Eq3);
F||(N*(N-1)+2):=simplify(Eq4); LF:=NULL:
for i from 1 to N*(N-1)+2 do LF:=LF,F||i; od:
LFforNum:={LF}:
```

Thus, we have analytically constructed the system of $N(N-1)+2$ nonlinear transcendental equations for $N(N-1)+2$ unknown coefficients A_{nm}, B_{nm}, A_0, and ω. In our case, $N=4$ and the number of equations is 14. The two steps of the solution process described above can be performed for various even values of N, e.g., $N=6,8$, which correspond, respectively, to 32 and 58 equations. Then we solve this system numerically in Maple.

39.8.2. Constructing Numerical Solutions

The system of nonlinear equations can be solved numerically; e.g., we solve it by Newton's iteration method in Maple. We apply the spectral collocation method constructed above for calculating standing waves for various values of the amplitude C. In particular, we are interested in determining the nonlinear frequency and wave profiles of standing waves. The process of constructing the numerical solution is as follows: Newton's iterations are performed starting from a small value of C corresponding to the solution of the given linearized problem. It is known that the convergence of Newton's method depends on an appropriate initial guess. As follows from Bryant and Stiassnie (1994), there exist infinitely many solutions of this problem. But we restrict ourselves to the construction of an approximate numerical *one-mode solution*, in which the only main harmonic dominates, $\cos x\cos t$ and $\cos x\sin t\cosh(y+h)$.

Therefore, to specify an appropriate initial guess, i.e., to enter to the neighborhood of a point in the space $\mathbb{R}^{N(N-1)+2}$, that corresponds to the neighborhood of the point of the linear approximation, it is important to specify the coefficients A_{21} (of the main harmonic $\phi(x,y,t)$), B_{12} (of the main harmonic $\eta(x,t)$), and ω for all values of N ($N=4,6,8,\ldots$) according to the following formulas: $A_{21}=-\omega/\sinh h$, $B_{12}=1$, $\omega=\sqrt{\tanh h}$. These formulas can be obtained as a particular one-mode solution, $\eta(x,t)=\cos x\cos t$, $\phi(x,y,t)=-(\omega/\sinh h)\sin t\cos x\cosh(y+h)$, $\omega^2=\tanh h$, of the corresponding linearized problem $\eta+\omega\phi_t=0$, $\phi_y-\omega\eta_t=0$ at $y=0$. The other coefficients, A_{nm} and B_{nm}, i.e., the coordinates of a point in the space $\mathbb{R}^{N(N-1)+2}$, can be specified as small values (or zeros) compared with A_{21} and B_{12}. We define the solution procedure `SolOmega0` for the first calculation by specifying an appropriate initial guess as follows:

```
SolOmega0:=proc(H,c,LEqs::set,A::list,B::list)
local param,param1,LF1,IVA,IVB,IVA1,IVB1,IVA0,IVals,Sol0,OmegaNum0,tr1;
  param:=[C=c,h=H]; param1:=omega0=sqrt(tanh(H));
  LF1:=evalf(eval(LEqs,param));
  IVA:={op(A)} minus {A21}; IVB:={op(B)} minus {B12};
  IVA:=[op(IVA)]; IVB:=[op(IVB)];
  IVA1:={seq(IVA[i]=0.,i=1..nops(IVA))};
  IVB1:={seq(IVB[i]=0.,i=1..nops(IVB))}; IVA0:={A0=0.};
  tr1:={A21=-omega0/sinh(H),B12=1.,omega=omega0};
  IVals:=`union`(eval(eval(tr1,param1),param),IVA1,IVB1,IVA0);
  Sol0:=fsolve(LF1,IVals);
  OmegaNum0:=eval(rhs(op(select(has,Sol0,omega)))/omega0,param1);
```

```
  RETURN(OmegaNum0,Sol0);
end proc;
```

Also we define the solution procedure `SolOmega` for the subsequent calculations using solutions obtained at the previous step as an appropriate initial guess as follows:

```
SolOmega:=proc(H,c,LEqs::set,IVals::set)
local param,param1,LF1,Sol,OmegaNum;
   param:=[C=c,h=H];
   param1:=omega0=sqrt(tanh(H));
   LF1:=evalf(eval(LEqs,param));
   Sol:=fsolve(LF1,IVals);
   OmegaNum:=eval(rhs(op(select(has,Sol,omega)))/omega0,param1);
   RETURN(OmegaNum,Sol);
end proc;
```

The general procedure of constructing the subsequent numerical solutions consists of the following steps:

1°. Finding a solution by applying Newton's method (`SolOmega0`) for a small value of C after a few Newton's iterations with a given accuracy (it can be changed varying the environment variable `Digits`).

2°. Considering the solution obtained at the previous step as an initial guess, gradually changing the wave amplitude, and applying Newton's method (`SolOmega`).

3°. Repeating step 2° until the wave amplitude will have the maximum desired value.

```
with(plots):
Cn:=10; C0:=0.07; Cf:=0.39; Ch:=(Cf-C0)/Cn;
LAmpl:=[seq(C0+i*Ch,i=0..Cn)];
LNum:=NULL:
for i to nops(LAmpl) do
  if i=1 then
    Sol||i:=SolOmega0(3.,LAmpl[i],LFforNum,LA,LB);
    LNum:=LNum,Sol||i[1];
  else
    Sol||i:=SolOmega(3.,LAmpl[i],LFforNum,Sol||(i-1)[2]);
    LNum:=LNum,Sol||i[1];
  fi;
od:
LOmegaNum:=[LNum];
PNum:=[seq([LOmegaNum[i],LAmpl[i]],i=1..nops(LOmegaNum))];
GNum:=plot(PNum,color=green,thickness=3): display(GNum);
```

39.8.3. Comparison of Asymptotic and Numerical Solutions

The asymptotic solution of the problem of capillary-gravity waves in an infinite depth fluid in Lagrangian variables was obtained by Shingareva and Lizárraga-Celaya (2007), and we write out the frequency-amplitude dependence up to the third order:

$$\frac{\omega}{\omega_0}=1+\varepsilon^2\left[\frac{81\delta^3+36\delta^2+27\delta-8}{64(3\delta+1)(1-3\delta)}\right]+O(\varepsilon^3),$$

where ω_0 is the frequency of linear standing waves with the dispersion relation $\omega_0^2=g\kappa(1+T_z)$. Here the dimensionless surface tension is $T_z=\kappa^2 ST/(\rho g)$. We note that we follow the

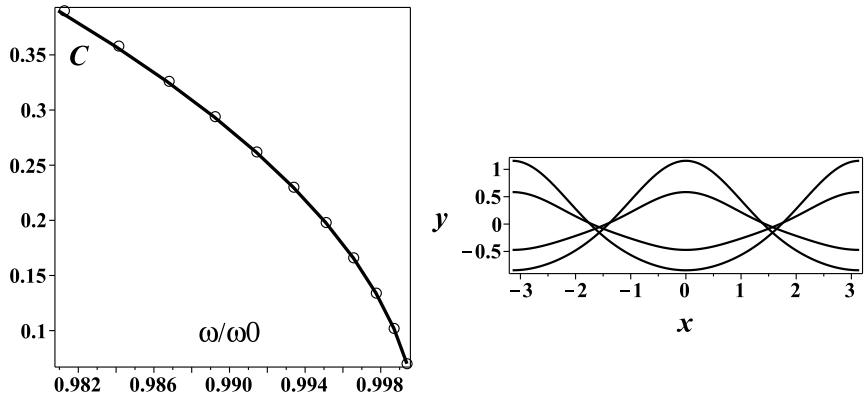

Figure 39.10. Frequency-amplitude dependence obtained numerically and asymptotically; surface profiles at times $t = 0, \pi, \frac{5\pi}{3}, \frac{10\pi}{3}$.

notation of Concus [see Concus (1962)], $\delta = \dfrac{T_z}{1+T_z}$, for more compact presentation of the result. Let us consider the particular case in which the surface tension is neglected [see Andreev (1992)]. Therefore, $\delta = 0$, and $\omega/\omega_0 = 1 - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3)$. This result is equal to the previous results obtained by Shingareva (1995) in Lagrangian variables and equal to the analytic solution obtained by Rayleigh (1915), Tadjbakhsh and Keller (1960), Penney and Price (1952), Aoki (1980), and Okamura (1997) in Eulerian variables. The coincidence of the results was explained by Shingareva and Lizárraga-Celaya (2007).

We compare the numerical solution obtained in Eulerian variables with the asymptotic solutions obtained in Lagrangian and Eulerian variables (up to the third order) and visualize the results as follows:

```
trH:={h=3.};
LAsym:=NULL:
LPP:=NULL:
Omega2:=(81*deltaˆ3+36*deltaˆ2+27*delta-8)/(64*(3*delta+1)*(1-3*delta));
omega0:=evalf(subs(trH,sqrt(tanh(h))));
for i from 1 to nops(LAmpl) do
  LAsym:=LAsym,eval(1+Omega2*epsilonˆ2/omega0,{delta=0,epsilon=LAmpl[i]}); od:
LOmegaAsym:=[LAsym];
PAsym:=[seq([LOmegaAsym[i],LAmpl[i]],i=1..nops(LOmegaAsym))];
GAsym:=plot(PAsym,color=blue,thickness=3):
for i from 1 to nops(LAmpl) do LPP:=LPP,eval(1-1/8*Aˆ2,{A=LAmpl[i]}); od:
LOmegaPP:=[LPP];
PPP:=[seq([LPP[i],LAmpl[i]],i=1..nops(LOmegaPP))];
GPP:=plot(PPP,color=magenta,thickness=3):
display({GNum,GAsym,GPP});
display({GNum,GAsym});
display({GNum,GPP});
```

In particular, the frequency-amplitude dependences obtained numerically in Eulerian variables and asymptotically in Lagrangian variables (up to the third order) are shown in Fig 39.10.

Comment. To compare the numerical and analytical results up to the third approximation, it is necessary to choose $N = 4$ in the numerical procedure. Also, we choose the value $h = 3$ which corresponds to the problem of standing waves in an infinite depth fluid.

Finally, we construct the surface profiles, $y = \eta(x,t)$ (see Fig. 39.10), and the velocity potentials, $\phi(x,y,t)$ at $y = C\eta(x,t)$, using the approximate numerical solutions obtained up to the third approximation. Therefore, we can observe the standing wave motions as follows:

```
Sol11; C11:=LAmpl[11]; Prof11:=eval(eta,Sol11[2]);
VelPoten:=eval(eval(phi,Sol11[2]),trH);
VelPot11:=eval(VelPoten,{y=C11*Prof11});
animate(Prof11,x=-Pi..Pi,t=0..4*Pi,color=blue,frames=30,
        numpoints=100,thickness=3,scaling=constrained);
animate(VelPot11,x=-Pi..Pi,t=0..4*Pi,color=blue,frames=30,
        numpoints=100,thickness=3,scaling=constrained);
```

⊙ *References for Section* 39.8: Lord Rayleigh (1915), Y. I. Seker (1947), G. Birkhoff (1950), W. G. Penney and A. T. Price (1952), P. Concus (1962), H. Aoki (1980), I. Tadjbakhsh and J. B. Keller (1980), P. J. Bryant and M. Stiassnie (1994), I. K. Shingareva (1995), M. Okamura (1997), I. K. Shingareva and C. Lizárraga-Celaya (2004, 2007), I. K. Shingareva, C. Lizárraga-Celaya, and A. D. Ochoa Ruiz (2006).

Chapter 40
Nonlinear Partial Differential Equations with Mathematica

40.1. Introduction

Partial differential equations had already been studied as early as in the 18th century; e.g., see Poincaré's seminal work (Poincaré, 1890). However, there were quite a few concepts in partial differential equations which were not revealed until the advent much later of important theoretical tools on the one hand and modern technology on the other hand. For example, even though stability issues were already considered by Poincaré and Lyapunov, they dealt almost exclusively with ordinary differential equations, and it was not until the late 20th century that the stability theory for PDEs really blossomed. The same can be said of singularities, solitons, chaos, etc., just to name a few. The new technologies include not only hardware (including high-performance computers) but also sophisticated software capable of doing really fine mathematics, with symbolic computations going hand in hand with numerical ones, both complementing each other.

In the preceding chapter, we have already seen one interactive programming environment for scientific computing, Maple. Here we dwell upon another tremendous system, Wolfram Research's Mathematica, and try to reveal at least some of its breathtaking capabilities.

⊙ *References for Section* 40.1: H. Poincaré (1890), J. Calmet and J. A. van Hulzen (1983), M. V. Grosheva and G. B. Efimov (1988), J. H. Davenport, Y. Siret, and E. Tournier (1993), M. J. Wester (1999).

40.2. Brief Introduction to Mathematica

Mathematica is a general-purpose computer algebra system in which symbolic computation can readily be combined with exact, approximate (floating-point), and arbitrary-precision numerical computation. Mathematica provides powerful scientific graphics capabilities [for details, see Bahder (1995), Getz and Helmstedt (2004), Gray (1994), Gray and Glynn (1991), E. Green and Johnson (1994), Ross (1995), Shingareva and Lizárraga-Celaya (2009), Vvedensky (1993), Zimmerman and Olness (1995), etc.].

The first concept of Mathematica and its first versions were developed by Stephen Wolfram in 1979–1988. The Wolfram Research company, which continues to develop Mathematica, was founded in 1987 [Wolfram (2002, 2003)].

The most important features of Mathematica are: fast symbolic, numerical, acoustic, parallel computation; static and dynamic computation, and interactive visualization; it is possible to incorporate new user-defined capabilities; available for almost all operating systems; powerful and logical programming language; extensive library of mathematical

functions and specialized packages; interactive mathematical typesetting system; free resources (e.g., see the Mathematica Learning Center `www.wolfram.com/support/learn`, Wolfram Demonstrations Project `demonstrations.wolfram.com`, and Wolfram Information Center `library.wolfram.com`).

Mathematica consists of two basic parts: the *kernel*, computational engine, and the *interface*, *front end*. These two parts are separate, but communicate with each other via the *MathLink* protocol. The kernel interprets the user input and performs all computations. The kernel assigns the labels `In[number]` to the input expression and `Out[number]` to the output. These labels can be used for keeping the computation order. In this chapter, we do not include these labels in the examples. The result of the kernel's work can be viewed with the function `InputForm`. The interface between the user and the kernel is called the *front end* and is used to display the input and the output generated by the kernel. The medium of the front end is the Mathematica *notebook*.

There are significant changes to numerous Mathematica functions incorporated to the new versions of the system. The description of important differences for Ver. < 6 and Ver. ≥ 6 is reported in the literature [e.g., see Shingareva and Lizárraga-Celaya (2009)].*

Basic concepts. If we type a Mathematica command and press the `RightEnter` key or `Shift+ Enter` (or `Enter` to continue the command on the next line), Mathematica evaluates the command, displays the result, and inserts a horizontal line (for the next input).

Mathematica contains many sources of online help; e.g., see Wolfram Documentation Center, Wolfram Demonstrations Project (for Ver. ≥ 6), Mathematica Virtual Book (for Ver. ≥ 7), and the `Help` menu. It is possible to mark a function and press F1; to type `?func`, `??func`, `Options[func]`; to use the symbols (`?`) and (`*`); e.g., `?Inv*`, `?*Plot`, or `?*our*`.

Mathematica notebooks are electronic documents that may contain Mathematica output, text, and graphics (see `?Notebook`). It is possible to work with many notebooks simultaneously. A Mathematica notebook consists of a list of cells. Cells are indicated along the right edge of the notebook by brackets. Cells can contain subcells, and so on. The kernel evaluates a notebook cell by cell. There are *different types of cells:* input cells (for evaluation) and text cells (for comments); Title, Subtitle, Section, Subsection, etc., can be found in the menu `Format` → `Style`.

Previous results (during a session) can be referred to with symbols `%` (the last result), `%%` (the next-to-last result), and so on.

Comments can be included within the characters `(*comments*)`.

Incorrect response: if some functions take an "infinite" computation time, you may have entered or executed the command incorrectly. To terminate a computation, you can use `Evaluation` → `Quit Kernel` → `Local`.

Palettes can be used for building or editing mathematical expressions, texts, and graphics and allows one to access the most common mathematical symbols by mouse clicks.

In Mathematica, there exist many specialized functions and modules that are not loaded initially. They must be loaded separately from files in the Mathematica directory. These files are of the form `filename.m`. The full name of a package consists of a `context`

*A complete list of all changes can be found in the Documentation Center and on the Wolfram website `www.wolfram.com`.

and a `short` name, and it is written as `context`short`. To load a package corresponding to a context, type `<<context`.` To get a list of the functions in a package, type `Names["context`*"]`.

Numerical approximations: `N[expr]`, `expr//N` (numerical approximation of `expr` to 6 significant digits), `N[expr,n]`, `NumberForm[expr,n]` (numerical approximation of the expression to *n* significant digits), `ScientificForm[expr,n]`, scientific notation of numerical approximation of `expr` to *n* significant digits.

40.2.1. Mathematica Language

Mathematica language is a very powerful programming language based on systems of transformation rules and on functional, procedural, and object-oriented programming techniques [see Maeder (1996)]. This distinguishes it from traditional programming languages. It supports a large collection of data structures, or Mathematica objects (functions, sequences, sets, lists, arrays, tables, matrices, vectors, etc.), and operations on these objects (type-testing, selection, composition, etc.). The library can be enlarged with custom programs and packages.

Symbol refers to a symbol with a specified name, e.g., expressions, functions, objects, optional values, results, argument names. The *name of symbol* is a combination of letters, digits, or certain special characters, not beginning with a digit, e.g., `a12new`. Once defined, a symbol retains its value until it is changed or removed.

Expression is a symbol that represents an ordinary Mathematica expression `expr` in readable form. The head of `expr` can be obtained with `Head[expr]`. The structure and various forms of `expr` can be analyzed with `TreeForm`, `FullForm[expr]`, `InputForm[expr]`. *A boolean expression* is formed with *logical operators* and relation operators.

Basic arithmetic operators and the corresponding functions: `+ - * / ^`, `Plus`, `Minus`, `Subtract`, `Times`, `Divide`, `Power`.

Logic and relation operators and their equivalent functions: `&&`, `||`, `!`, `=>`, `==`, `!= <`, `>`, `<= >=`, `And`, `Or`, `Xor`, `Not`, `Implies`, `Equal`, `Unequal`, `Less`, `Greater`, `LessEqual`, `GreaterEqual`.

Mathematica is case sensitive, and there is a difference between lowercase and uppercase letters; e.g., `Sin[Pi]` and `sin[Pi]` are different. All Mathematica functions begin with a capital letter. Some functions (e.g., `PlotPoints`) use more than one capital. To avoid conflicts, it is best to begin with a lower-case letter for all user-defined symbols.

The result of each calculation is displayed, but it can be suppressed by using a semicolon (;), e.g., `Plot[Sin[x],x,0,2*Pi]; a=9; b=3; c=a*b;`.

Patterns: Mathematica language is based on pattern matching. A pattern is an expression that contains an underscore character (_). The pattern can stand for any expression. Patterns can be constructed from templates; e.g., `x_`, `x_/;cond`, `pattern?test`, `x_:IniValue`, `x^n_`, `x_^n_`, `f[x_]`, `f_[x_]`.

Basic transformation rules: `->`, `:>`, `=`, `:=`, `^:=`, `^=`.

The rule `lhs->rhs` transforms `lhs` to `rhs`. Mathematica regards the left-hand side as a pattern. The rule `lhs:>rhs` transforms `lhs` to `rhs`, evaluating `rhs` only after the rule has been used. The assignment `lhs=rhs` (or `Set`) specifies that the rule `lhs->rhs` should be used whenever it applies. The assignment `lhs:=rhs` (or `SetDelayed`) specifies

that `lhs:>rhs` should be used whenever it applies, i.e., `lhs:=rhs` does not evaluate `rhs` immediately but leaves it unevaluated until the rule is actually called. The rule `lhs^:=rhs` assigns `rhs` to be the delayed value of `lhs`, and associates the assignment with symbols that occur at level one in `lhs`. The rule `lhs^=rhs` assigns `rhs` to be the value of `lhs`, and associates the assignment with symbols that occur at level one in `lhs`. Transformation rules are useful for making substitutions without making the definitions permanent and are applied to an expression using the operator `/.` (`ReplaceAll`) or `//.` (`ReplaceRepeated`).

The difference between the operators (=) and (==) is as follows: the operator `lhs= rhs` is used to assign `rhs` to `lhs`, and the equality operator `lhs==rhs` indicates equality (not assignment) between `lhs` and `rhs`.

Unassignment of definitions: `Clear[symb]`, `ClearAll[symb]`, `Clear["Global`*"]`, `Clearall["Global`*"]`, `Remove[symb]`, `symb=.`, `Remove["`*"]`; (to clear all global symbols defined in a Mathematica session), `?symb`, `?`*` (to recall a symbol's definition).

`ClearAll["Global‘*"]; Remove["Global‘*"];` is a useful initialization to start working on a problem.

An equation is represented using the binary operator == and has two operands, the left-hand side `lhs` and the right-hand side `rhs`.

Inequalities are represented using relational operators and have two operands, the left-hand side `lhs` and the right-hand side `rhs`.

A string is a sequence of characters having no value other than itself and can be used as labels for graphs, tables, and other displays. The strings are enclosed within double-quotes, e.g., `"abc"`.

Data types: every expression is represented as a tree structure in which each node (and leaf) has a particular data type. A variety of functions can be used for the analysis of any node and branch, e.g., `Length`, `Part`, a group of functions ending in the letter `Q` (`DigitQ`, `IntegerQ`, etc.).

Types of brackets: parentheses for grouping, `(x+9)*3`; square brackets for function arguments, `Sin[x]`; curly brackets for lists, `{a,b,c}`.

Types of quotes: back-quotes for context mark, format string character, number mark, precision mark, accuracy mark; double-quotes for strings.

Types of numbers: integer, rational, real, complex, root; e.g., `-5, 5/6, -2.3^-4, ScientificForm[-2.3^-4], 3-4*I, Root[#^2+#+1&,2]`.

Mathematical constants: symbols for definitions of selected mathematical constants, e.g., `Catalan`, `Degree`, `E`, `EulerGamma`, `I`, `Pi`, `Infinity`, `GoldenRatio`; for example, `{60Degree//N, N[E,30]}`.

Two classes of functions: *pure functions* and functions defined in terms of a variable (*predefined* and *user-defined* functions).

Pure functions are defined without a reference to any specific variable. The arguments are labeled `#1`, `#2`, ..., and an ampersand `&` is used at the end of definition. Most of the mathematical functions are predefined. Mathematica includes all common special functions of mathematical physics.

The names of mathematical functions are complete English words or the traditional abbreviations (for a few very common functions), e.g., `Conjugate`, `Mod`. Mathematical functions named after a person have the form `PersonSymbol`; for example, the Legendre polynomials $P_n(x)$ are named `LegendreP[n,x]`.

User-defined functions are defined using the pattern `x_`; e.g., the function $f(x) =$`expr` of one variable is defined as `f[x_]:=expr;`

Evaluation of a function or an expression without assigning a value can be performed using the replacement operator `/.`, e.g., `f[a]`, `expr/.x->a`.

Function application: `expr//func` is equivalent to `fun[expr]`.

A module is a local object that consists of several functions which one needs to use repeatedly (see `?Module`). A module can be used to define a function (if the function is too complicated to write by using the notation `f[x_]:=expr`), to create a matrix, a graph, a logical value, etc. *Block* is similar to `Module`; the main difference between them is that `Block` treats the values assigned to symbols as local but the names as global, whereas `Module` treats the names of local variables as local. *With* is similar to `Module`, the principal difference between them is that `With` uses local constants that are evaluated only once, but `Module` uses local variables whose values may change many times.

In Mathematica language, there are the following *two control structures*: the selection structures `If`, `Which`, `Switch` and the repetition structures `Do`, `While`, `For`.

Mathematica objects: *lists* are the fundamental objects in Mathematica. The other objects (e.g., sets, matrices, tables, vectors, arrays, tensors, objects containing data of mixed type) are represented as lists. A list is an ordered set of objects separated by commas and enclosed in curly braces, {`elements`}, or defined with the function `List[elements]`. *Nested lists* are lists that contain other lists. There are many functions which manipulate lists, and here we review some of the most basic ones. *Sets* are represented as lists. *Vectors* are represented as lists; vectors are simple lists. Vectors can be expressed as single columns with `ColumnForm[list,horiz,vert]`. *Tables, matrices, and tensors* are represented as nested lists. There is no difference between the way they are stored: they can be generated using the functions `MatrixForm[list]`, `TableForm[list]`, or using the nested list functions. Matrices and tables can also be conveniently generated using the *Palettes* or `Insert` menu. *A matrix* is a list of vectors. *A tensor* is a list of matrices with the same dimension.

In Mathematica (for Ver. ≥ 6), the new kind of output, the *dynamic output*, has been introduced allowing to create dynamic interfaces of different types. Numerous new functions for creating various dynamic interfaces have been developed.

⊙ *References for Section* 40.2: T. Gray and J. Glynn (1991), D. D. Vvedensky (1993), T. B. Bahder (1994), J. W. Gray (1994), E. Green, B. Evans, and J. Johnson (1994), C. C. Ross (1995), R. L. Zimmerman and F. Olness (1995), S. Wolfram (2002, 2003), C. Getz and J. Helmstedt (2004), I. K. Shingareva and C. Lizárraga-Celaya (2009).

40.3. Analytical Solutions and Their Visualizations

Although finding analytical solutions of most nonlinear PDEs requires new proper methods, in many cases it is possible to apply various analytical methods (developed for some classes of nonlinear PDEs), for example, a change of dependent and independent variables, various transformation methods, methods for constructing traveling-wave and self-similar solutions, classical and new methods of separation of variables, generalized method of characteristics, qualitative analysis methods, etc.

40.3.1. Constructing Solutions Using Predefined Functions

In computer algebra system Mathematica, analytical solutions of a given nonlinear PDE or systems of nonlinear PDEs can be found with the aid of the predefined function `DSolve`:

```
DSolve[PDE,u,{x1,..,xn}]  DSolve[PDE,u[x1,...,xn],{x1,...,xn}]
DSolve[PDE, u[x1,...,xn], {x1,...,xn}, GeneratedParameters->C]
```

Description:

`DSolve`, finding analytical solutions of a PDE for the function `u`, with independent variables `x1,...xn` ("pure function" solution).

`DSolve`, finding analytical solutions of a PDE for the function `u[x1,...,xn]`, with independent variables `x1`, ..., `xn`.

`DSolve`, `GeneratedParameters`, finding analytical solutions of a PDE for the function `u[x1,...,xn]`, with independent variables `x1`, ..., `xn` and specifying the arbitrary constants.

Notation. In this chapter, we use the following conventions introduced in Mathematica: `C[n]` (`n` $= 1,2,\ldots$), for arbitrary constants or arbitrary functions.

In general, arbitrary parameters can be specified, e.g., $F_1, F_2, \ldots$, by applying the option `GeneratedParameters->(Subscript[F,#]&)` of the predefined function `DSolve`.

Also we introduce the following notation for the Mathematica solutions: `eqn`, for equations; `pden/oden`, for PDEs/ODEs; `trn`, for transformations; `sysn`, for systems; `ic`, `bc`, `ibc`, for initial or/and boundary conditions; `listn`, for lists of expressions; `gn`, for graphs of solutions (where `n` $= 1,2,\ldots$).

Let us assume that we have obtained exact solutions and we are interested to verify whether these solutions are exact solutions of given nonlinear PDEs.

Example 40.1. For example, for the modified Liouville equation 7.2.1.1

$$u_{tt} = a^2 u_{xx} + be^{\beta u}$$

we verify that the traveling-wave solution and the functional separable solution

$$u(x,t) = \frac{1}{\beta}\ln\left(\frac{2(B^2 - a^2A^2)}{b\beta(Ax+Bt+C)^2}\right), \quad u(x,t) = \frac{1}{\beta}\ln\left(\frac{8a^2C}{b\beta}\right) - \frac{2}{\beta}\ln|(x+A)^2 - a^2(t+B)^2 + C|$$

are exact solutions of the modified Liouville equation as follows:

```
pde1=D[u[x,t],{t,2}]-a^2*D[u[x,t],{x,2}]-b*Exp[beta*u[x,t]]==0
sol11=u->Function[{x,t},1/beta*Log[2*(B^2-a^2*A^2)/(b*beta*(A*x+B*t+C)^2)]]
sol12[S_]:=u->Function[{x,t},1/beta*Log[8*a^2*C/(b*beta)]
                        -2/beta*Log[S*(x+A)^2-S*a^2*(t+B)^2+S*C]];
{test11=pde1/.sol11//FullSimplify, test12=pde1/.sol12[1]//FullSimplify,
 test13=pde1/.sol12[-1]//FullSimplify}
```

where a, b, β, A, B, and C are arbitrary real constants.

Example 40.2. For the nonlinear Schrödinger equation 5.7.1.1

$$iu_t + u_{xx} + k|u|^2 u = 0,$$

we verify that the solutions

$$u(x,t) = C_1 \exp\left[i(C_2 x + (kC_1^2 - C_2^2)t + C_3)\right], \quad u(x,t) = A\sqrt{\frac{2}{k}}\,\frac{\exp\left[iBx + i(A^2 - B^2)t + iC_1\right]}{\cosh(Ax - 2ABt + C_2)}$$

are exact solutions of this nonlinear equation as follows:

```
pde1=I*D[u[x,t],t]+D[u[x,t],{x,2}]+k*Abs[u[x,t]]^2*u[x,t]==0
sol1=u->Function[{x,t},C1*Exp[I*(C2*x+(k*C1^2-C2^2)*t+C3)]]
test1=pde1/.sol1
test11=Assuming[{C1>0}, test1//ComplexExpand//FullSimplify]
sol2=u->Function[{x,t},
        A*Sqrt[2/k]*(Exp[I*B*x+I*(A^2-B^2)*t+I*C1])/(Cosh[A*x-2*A*B*t+C2])]
test2=pde1/.sol2
test21=Assuming[{k>0,{A,B,C,C1,C2,x,t}\[Element] Reals},FullSimplify[test2]]
```

Here A, B, C_1, C_2, and C_3 are arbitrary real constants, and the second solution is valid for $k > 0$.

Example 40.3. The general solution of the first-order nonlinear PDE

$$xu_y - u_x = g(x)u^2$$

can be found and tested as follows:

```
pde1=x*D[u[x,y],y]-D[u[x,y],x]==g[x]*u[x,y]^2
sol1=DSolve[pde1,u,{x,y}]//FullSimplify//First
Print["sol1=",sol1]
DSolve[pde1,u,{x,y}]//FullSimplify
HoldForm[sol1]==sol1
test1=pde1/.sol1//FullSimplify
sol2=u[x,y]/.sol1
Print["sol2=",sol2]
sol3=(u[x,y]/.sol1)/.{C[1][var_]:>F[var],K[1]->s}//FullSimplify
```

where the Mathematica result reads:

$$\text{sol1} = \left\{u \to \text{Function}\left[\{x,y\}, \frac{1}{-\int_1^x -g[K[1]]\,dK[1] - C[1]\left[\frac{1}{2}\left(x^2+2y\right)\right]}\right]\right\},$$

$$\text{sol2} = \frac{1}{-\int_1^x -g[K[1]]\,dK[1] - C[1]\left[\frac{1}{2}\left(x^2+2y\right)\right]},$$

$$\text{sol3} = -\frac{1}{\text{F}\left[\frac{x^2}{2}+y\right] + \int_1^x -g[s]\,ds}.$$

According to the Mathematica notation, `sol1` is the "pure function" solution for $u(x,y)$ (where `C[1]` is an arbitrary function and `K[1]` is the integration variable), `sol2` represents the solution $u(x,y)$, and `sol3` represents the solution $u(x,y)$ in a more convenient form, with arbitrary function F and integration variable s.

Example 40.4. For the *Burgers equation*

$$u_t + uu_x = \nu u_{xx},$$

we construct an exact solution (or a special solution in the Mathematica terminology), namely, the traveling-wave solution (`sol1`)

```
Off[DSolve::"nlpde"];
pde1=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==\[Nu]*D[u[x,t],{x,2}]
sol1=DSolve[pde1,u,{x,t}]//FullSimplify//First
test1=pde1/.sol1//FullSimplify
```

with the Mathematica result

$$\text{sol1} = \left\{ u \to \text{Function}\left[\{x,t\}, -\frac{C[2] + 2\nu C[1]^2 \text{Tanh}[xC[1] + tC[2] + C[3]]}{C[1]} \right] \right\}$$

When solving this nonlinear PDE, Mathematica generates a warning message, which can be ignored or suppressed with the `Off` function.

Example 40.5. *Kolmogorov–Petrovskii–Piskunov equation (KPP) and KPP-type equation.* By applying the Mathematica predefined functions to the KPP-type equations

$$u_t = u_{xx} + au + bu^2,$$
$$u_t = u_{xx} + au + bu^2 + cu^3$$

(where a, b, and c are arbitrary parameters), we can find traveling-wave solutions and verify that these solutions are exact solutions of the given PDEs as follows:

```
Off[DSolve::"nlpde"]; Off[Reduce::"useq"];
l1={a->1,b->1,c->1}; l2={a->1,b->1,c->-1};
{K1=a*u[x,t], K2=K1+b*u[x,t]^2, K3=K2+c*u[x,t]^3}
{eq1=(D[u[x,t],t]==D[u[x,t],{x,2}]+K2)/.l1,
 eq2=(D[u[x,t],t]==D[u[x,t],{x,2}]+K3)/.l2}
{sol1=DSolve[eq1,u,{x,t}]//FullSimplify,
 sol2=DSolve[eq2,u,{x,t}]//FullSimplify}
{HoldForm[sol1]==sol1, HoldForm[sol2]==sol2}
{n1=Length[sol1], n2=Length[sol2]}
test1=Table[eq1/.sol1[[i]],{i,1,n1}]//FullSimplify;
test2=Table[eq2/.sol2[[i]],{i,1,n2}]//FullSimplify;
Print[HoldForm[sol1]==test1]; Print[HoldForm[sol2]==test2];
```

with the Mathematica results

$$\text{sol1[[1]]} = \left\{ u \to \text{Function}\left[\{x,t\}, -\frac{1}{4}\left(1 + \text{Tanh}\left[\frac{1}{12}\left(5t - \sqrt{6}x - 12C[3]\right)\right]\right)^2 \right] \right\}$$
$$\text{sol2[[5]]} = \left\{ u \to \text{Function}\left[\{x,t\}, \frac{1}{2}\left(1 + \sqrt{5}\,\text{Tanh}\left[\frac{1}{4}\left(\sqrt{5}t - \sqrt{10}x - 4C[3]\right)\right]\right) \right] \right\}$$

When solving these nonlinear PDEs, Mathematica generates many warning messages, which can be ignored or suppressed with the `Off` function. Since Mathematica cannot find explicit forms of traveling-wave solutions of these equations for arbitrary parameters a, b, and c, we set some values, e.g., $a = 1$, $b = 1$, and $c = 1$ for the first equation and $a = 1$, $b = 1$, and $c = -1$ for the second equation. Mathematica generates 8 and 12 traveling-wave solutions, respectively, for the first and the second nonlinear equation. As a result, we present the simplest exact solutions for each equation, i.e., `sol1[[1]]` and `sol2[[5]]`.

For more general forms of the equations considered above, i.e., $u_t = u_{xx} + au + bu^m$ (m is an arbitrary parameter, $m \neq 1, 2, 3$) and $u_t = u_{xx} + au + bu^m + cu^{2m-1}$ (m is an arbitrary parameter, $m \neq 1, 2$), we cannot find exact solutions with the aid of the predefined function `DSolve` (for Mathematica Ver. ≤ 7). However, such solutions are known (see equations 5.1.3.2 and 5.1.3.3).

40.3.2. Constructing Solutions via Transformations

Now we apply some special transformations (e.g., the Bäcklund transformations and the Cole–Hopf transformation) to nonlinear equations of mathematical physics and obtain exact solutions of these equations.

Example 40.6. *Bäcklund transformations. Sine–Gordon equation.* Let us verify that the sine–Gordon equations

$$u_{xt} = \sin u, \quad v_{xt} = \sin v$$

are related by the Bäcklund transformations,

$$\tfrac{1}{2}(u-v)_x = a\sin\Big(\tfrac{1}{2}(u+v)\Big), \quad \tfrac{1}{2}(u+v)_t = \tfrac{1}{a}\sin\Big(\tfrac{1}{2}(u-v)\Big),$$

where $a \neq 0$ is an arbitrary constant. Since both u and v satisfy the same sine–Gordon equation, the pair of Bäcklund transformations is referred to as an *auto-Bäcklund transformation* for the sine–Gordon equation.

```
SGEq[u_]:=D[u,{x,t}]==Sin[u];
BT={D[u[x,t]-v[x,t],x]/2->a*Sin[1/2*(u[x,t]+v[x,t])],
    D[u[x,t]+v[x,t],t]/2->Sin[1/2*(u[x,t]-v[x,t])]/a}
{eq1=D[BT[[1]],t]/.BT, eq2=D[BT[[2]],x]/.BT}
SGEqu=(eq2[[1]]+eq1[[1]]==eq2[[2]]+eq1[[2]])//FullSimplify
SGEqv=(eq2[[1]]-eq1[[1]]==eq2[[2]]-eq1[[2]])//FullSimplify
```

By applying the Bäcklund transformations, we can find a solution of the sine–Gordon equation as follows:

```
Off[Solve::"ifun"]; SetOptions[Plot,ImageSize->500,PlotStyle->
                    {Hue[0.9],Thickness[0.01]},PlotPoints->100,PlotRange->All];
SetOptions[Plot3D,BoxRatios->{1,1,1},PlotRange->All,ViewPoint->{-1,2,2}];
SGEq[u_]:=D[u,{x,t}]==Sin[u];
BT:={D[u[x,t]-v[x,t],x]/2->a*Sin[1/2*(u[x,t]+v[x,t])],
     D[u[x,t]+v[x,t],t]/2->Sin[1/2*(u[x,t]-v[x,t])]/a};
{eq1=D[BT[[1]],t]/.BT, eq2=D[BT[[2]],x]/.BT}
SGEqu=(eq2[[1]]+eq1[[1]]==eq2[[2]]+eq1[[2]])//FullSimplify
SGEqv=(eq2[[1]]-eq1[[1]]==eq2[[2]]-eq1[[2]])//FullSimplify
eq3=BT/.{v[x,t]->0, D[v[x,t],x]->0, D[v[x,t],t]->0}
eq4=Assuming[Cos[U/4]>0,Simplify[Integrate[2*a,x]==Integrate[1/Sin[U/2],U]+A[t]]]
eq5=Assuming[Cos[U/4]>0,Simplify[Integrate[2/a,t]-Integrate[1/Sin[U/2],U]==B[x]]]
{eq6=Solve[eq4,Tan[U/4]], eq7=Solve[eq5,Tan[U/4]]}
eq8=eq6[[1,1,1]]==eq6[[1,1,2]]*eq7[[1,1,2]]
const=Exp[Select[eq8[[2,2]],MemberQ[#1,A[t]]||MemberQ[#1,B[x]]&]]
eq9=eq8[[1]]==alpha*(Thread[eq8/const,Equal])[[2]]
eq10=Solve[eq9,U]
KinkSol[X_,T_,A_,Alpha_]:=eq10[[1,1,2]]/.{x->X,t->T,a->A,alpha->Alpha};
{alpha1=0.1, a1=10}
Plot3D[N[KinkSol[x,t,a1,alpha1]],{x,-Pi,Pi},{t,0,1}]
Plot[N[KinkSol[x,0,a1,alpha1]],{x,-Pi,Pi}]
```

and verify that the new solution of the sine–Gordon equation describes a *soliton or kink* solution and has the following Mathematica form:

$$\text{eq10} = \Big\{\Big\{U \to 4\,\text{ArcTan}\Big[\text{alpha}e^{\frac{t}{a}+ax}\Big]\Big\}\Big\}$$

We visualize the kink solution in 3D and 2D spaces (see Fig. 40.1).

Considering the standard form of the sine–Gordon equation

$$u_{XX} - \beta u_{TT} = \sin u, \quad \text{where} \quad \beta = \frac{1}{c^2}$$

with the characteristic coordinate transformations

$$x = \tfrac{1}{2}(X + cT), \quad t = \tfrac{1}{2}(X - cT),$$

we can obtain a *soliton or kink* solution and an *antisoliton or antikink* solution. We visualize the kink and antikink solutions thus obtained.

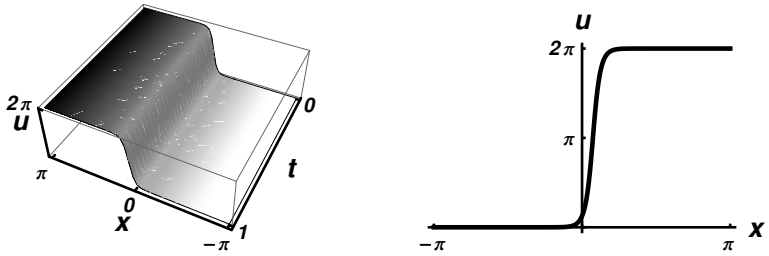

Figure 40.1. The kink solution of the sine–Gordon equation in 3D and 2D spaces.

```
c1=1; eq11=eq10[[1,1,2]]/.{x->(X+c*T)/2,t->(X-c*T)/2}//FullSimplify
tr1={a->\[Epsilon]*Sqrt[(1-U)/(1+U)],c->U*m/(2*(a-1/a))}
eq12=Assuming[U>1,Simplify[(eq11/.tr1)/.tr1[[1]]]]
KinkSols[M1_,U1_,Epsilon1_,T1_,X1_,Alpha1_]:=eq12/.
      {m->M1,U->U1,\[Epsilon]->Epsilon1,T->T1,X->X1,alpha->Alpha1};
m1[a_]:=1/2*(a+1/a); V1[c_,a_]:=c*((a^2-1)/(a^2+1));
{m1[a1],V1[c1,a1]}
Kink=Plot[N[KinkSols[m1[a1],V1[c1,a1],1,0,X,alpha1]],{X,-Pi,Pi}];
AntiKink=Plot[N[KinkSols[m1[a1],V1[c1,a1],-1,0,X,alpha1]],
             {X,-Pi,Pi},PlotStyle->{Hue[0.7],Thickness[0.01]}];
Show[{Kink,AntiKink}]
```

Example 40.7. *Cole–Hopf transformation for the Burgers equation.* Let $u(x,t)$, $\psi(x,t)$, and $\phi(x,t)$ be real functions. By applying the Cole–Hopf transformation, $u = \psi_x = -2\nu\dfrac{\phi_x}{\phi}$, $\psi = -2\nu\ln\phi$, we can transform any solution $\phi = \xi(x,t)$ of the linear heat equation

$$\phi_t = \nu\phi_{xx}$$

into the solution $u = -2\nu\dfrac{\phi_x}{\phi}$ of the Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

as follows:

```
BEq=D[u[x,t],t]+u[x,t]*D[u[x,t],x]==nu*D[u[x,t],{x,2}]
ConservLaw=D[u[x,t],t]+Hold[D[(u[x,t])^2/2-nu*D[u[x,t],x],x]]==0
eq1=u[x,t]->Hold[D[psi[x,t],x]]
eq2=ConservLaw[[1,1]]==Hold[D[psi[x,t],t]]//Simplify
{eq3=eq2/.eq1, L3=Level[eq3,{3}]}
eq31=-L3[[1]]==eq3[[2]]
tr1=psi[x,t]->-2*nu*Log[phi[x, t]]
tr2=D[psi[x,t],x]==ReleaseHold[(eq1/.tr1)]
eq4={tr2[[1]]->(tr2[[2]])[[2]],
     D[psi[x,t],{x,2}]->D[(tr2[[2]])[[2]],x], D[tr1,t]}
eq5=(ReleaseHold[eq31]/.eq4)//Expand
DiffusEq=Thread[eq5*(-1/2)*phi[x,t]/nu,Equal]
```

The opposite is not generally true (see Example 26.21),

```
BEq=D[u[x,t],t]+u[x,t]*D[u[x,t],x]-nu*D[u[x,t],{x,2}]==0
{tr1=u[x,t]->-2*nu*D[phi[x,t],x]/phi[x,t],tr11=D[tr1,t],
 tr12=Table[D[tr1,{x,i}],{i,1,2}]//Flatten}
```

```
eq1=(BEq/.tr1/.tr11/.tr12)//Expand
eq2=Thread[eq1*phi[x,t]^2/2,Equal]//Expand
eq11=-D[1/phi[x,t]*(D[phi[x,t],t]-nu*D[phi[x,t],{x,2}]),x]==0
eq21=Thread[eq11*phi[x,t]^2*nu,Equal]
eq22=eq2[[1]]-eq21[[1]]//Expand
eq3=Integrate[-D[1/phi[x,t]*(D[phi[x,t],t]-nu*D[phi[x,t],{x,2}]),x],x]==f[t]
eq4=Thread[eq3*phi[x,t],Equal]//Expand
```

since a solution of the Burgers equation $u_t + uu_x = \nu u_{xx}$ generates a solution of the more general equation $\phi_t - \nu\phi_{xx} = -\phi f(t)$ (eq4).

40.3.3. Constructing Traveling-Wave Solutions

Traveling-wave solutions often occur in various problems of mathematical physics. These solutions are translation invariant. A traveling-wave solution of the wave equation was first obtained by d'Alembert in 1747 [see Lindsay (1973)]. In practice, this method is simple and useful in finding solutions of both linear and nonlinear PDEs [see Section 27.2 as well as Logan (1994), Polyanin and Manzhirov (2007), and Griffiths and Schiesser (2011)]. As before, we illustrate the method of finding traveling-wave solutions by solving problems.

Example 40.8. *Burgers equation.* Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}.$$

We derive *traveling-wave solutions* of the Burgers equation, i.e., solutions of the form $u(x,t) = U(z)$, $z = x - \lambda t$. When studying the shock-wave structure of the solutions, we assume that there exist constant values $u_1 > 0$ and $u_2 > 0$ ($u_1 > u_2$) such that $\lim_{z\to-\infty} U(z) = u_1$ and $\lim_{z\to+\infty} U(z) = u_2$.

```
SetOptions[Plot,ImageSize->300,PlotPoints->100,
           PlotStyle->{Hue[0.9],Thickness[0.01]},PlotRange->{All,{0,6}}];
{a=2*nu,tr1={lambda->1/2*(u1+u2),C->-1/2*u1*u2},
        tr2=U[z]-u1->u1-U[z],tr3=x-lambda*t->z}
eq1[u_]:=D[u,t]+u*D[u,x]-nu*D[u,{x,2}];
{eq2=eq1[U[x-lambda*t]], eq3=eq2/.tr3}
{eq4=Integrate[eq3,z]==C, dU=D[U[z],z]}
sol1=Assuming[z!=0,Solve[eq4,dU]]
sol2=Thread[(Factor[sol1/.tr1][[1,1]])*a,Rule]
sol3=Thread[sol2/sol2[[2]]/a*dz,Rule]
sol4=Integrate[sol3[[1]],z]==Integrate[Coefficient[sol3[[2]],dz],z]
sol5=sol4/.tr2
sol6=Assuming[z!=0,Solve[1==Solve[sol5,dz][[1,1,2]],U[z]]]//FullSimplify
solU[Z_,U1_,U2_,NU_]:=sol6[[1,1,2]]/.{z->Z,u1->U1,u2->U2,nu->NU};
solU[0,5,1,0.3]
Show[GraphicsRow[{Plot[solU[Z,5,1,0.999],{Z,-Pi,Pi}],
     Plot[solU[Z,5,1,0.3],{Z,-Pi,Pi}],Plot[solU[Z,5,1,0.09],{Z,-Pi,Pi}]}]]
```

We determine the shape of waveform $U(z)$, and the Mathematica result reads:

$$\text{sol6} = \left\{\left\{U[z] \to \frac{\text{u1} - \text{u2}}{1 + e^{\frac{(\text{u1}-\text{u2})z}{2\text{nu}}}} + \text{u2}\right\}\right\}$$

We plot the traveling-wave solutions $U(z)$ varying the diffusion parameter ν (see Fig. 40.2 for $\nu = 0.999$, 0.3, 0.09).

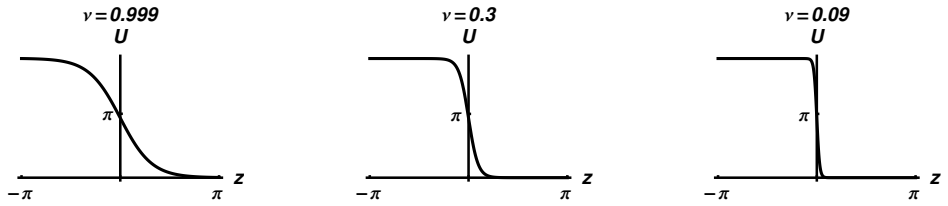

Figure 40.2. Traveling-wave solutions of the Burgers equation.

Example 40.9. *Korteweg–de Vries equation.* Consider the Korteweg–de Vries (KdV) equation

$$u_t + auu_x + bu_{xxx} = 0,$$

where a and b are constants. The KdV equation admits a special form of the exact solution, a soliton, which arises in many physical processes (e.g., water waves, internal gravity waves in a stratified fluid, ion-acoustic waves in plasma, etc.) We seek a special type of traveling-wave solutions, i.e., a *one-soliton solution* $u(x,t) = U(z)$, $z = x - ct$, of the KdV equation, where c is a constant and $U(z) \to 0$ as $|z| \to \infty$:

```
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.9],Thickness[0.01]}];
{tr1=x-c*t->z, tr2={a->1,b->1,c->1}}
eq1[u_]:=D[u,t]+a*u*D[u,x]+b*D[u,{x,3}]==0;
{eq2=eq1[U[tr1[[1]]]]//Expand, eq3=eq2/.tr1, eq4=eq3//TraditionalForm}
eq5=Integrate[eq3[[1]],z]-C1==0
eq6=Thread[eq5*2*D[U[z],z],Equal]//Expand
eq7=Thread[Integrate[eq6,z],Equal]
{eq8=eq7[[1]]==C2, eq9=eq8/.{C1->0,C2->0}}
{sol1=DSolve[eq9,U[z],z], sol11=Simplify[sol1[[2]]]/.{C[1]->0}}
sol2=(sol11/.{z->x-c*t})/.tr2
sol3[X_,T_]:=sol2[[1,2]]/.{x->X,t->T};
Animate[Plot[sol3[x,t],{x,-20,20},PlotRange->{All,{0,Pi}}],{t,0,20}]
(eq1[u]/.tr2)/.{u->sol2[[1,2]]}
```

By substituting this solution form into the KdV equation, we arrive at the third-order nonlinear ODE $-cU_z + aUU_z + bU_{zzz} = 0$ (`eq3`, `eq4`). By integrating this equation with respect to z twice (and by introducing the integration constants C_1 and C_2), we obtain the first-order nonlinear ODE $-cU^2 + \frac{1}{3}aU^3 + bU_z^2 - 2C_1U = C_2$ (`eq8`). Considering a special case where $U(z)$ and its derivatives tend to zero at infinity and $C_1 = C_2 = 0$, we can determine that the exact solution has the Mathematica form

$$\text{sol11} = \left\{ U[z] \to \frac{3c\,\text{Sech}\left[\frac{\sqrt{c}z}{2\sqrt{b}}\right]^2}{a} \right\}$$

By setting, say, $a = b = c = 1$, we confirm that the solution obtained is indeed an exact solution of the KdV equation and it travels at a constant velocity without changing the shape.

Example 40.10. *Sine–Gordon equation.* Consider the sine–Gordon equation

$$u_{xx} - \alpha u_{tt} = \sin u, \quad \text{where} \quad \alpha = \frac{1}{c^2}.$$

We construct traveling-wave solutions of the sine–Gordon equation in the form of one-soliton solution as follows:

```
trS1[eq_,var_]:=Select[eq,MemberQ[#,var,Infinity]&];
trS3[eq_,var_]:=Select[eq,FreeQ[#,var]&]; Off[Solve::"ifun"];
```

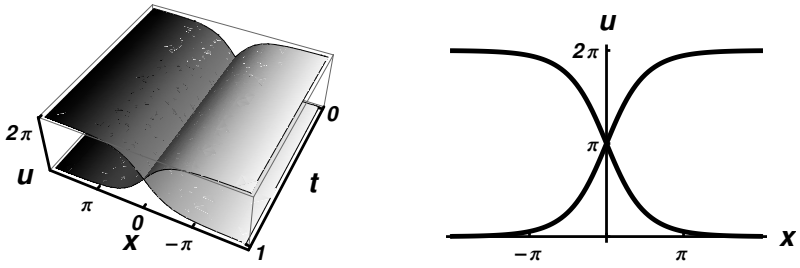

Figure 40.3. The kink and antikink solutions of the SG equation in 3D and 2D spaces.

```
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.9],Thickness[0.01]},PlotRange->
 {All,{0,2*Pi}}]; sGEq[u_]:=D[u,{x,2}]-1/c^2*D[u,{t,2}]==Sin[u];
{tr1=x-lambda*t->xi, tr2=lambda^2/c^2->uN^2, eq1=sGEq[phi[tr1[[1]]]]//Expand,
 eq2=eq1/.tr1//Expand, eq3=eq2/.tr2, eq4=Thread[eq3*D[phi[xi],xi]/(1-uN^2),
 Equal]//Simplify, eq5=Hold[D[1/2*(D[phi[xi],xi]^2)+Cos[phi[xi]]/(1-uN^2),xi]==0],
 eq41=eq4[[1]]-eq4[[2]]==0, eq41==ReleaseHold[Eq5], eq51=D[Level[eq5,{3}][[1]],xi]}
{eq6=Integrate[eq51,xi]==b//FullSimplify, eq7=Flatten[Solve[eq6,D[phi[xi],xi]]],
 eq7R=eq7[[2,2]], term7=trS3[eq7R,Sqrt[2]], term70=Factor[term7^2],
 term71=Numerator[term70], term72=Denominator[term70],
 integrd=1/Sqrt[Collect[-term71,b]]/.{phi[xi]->psi},
 integrd1=integrd/.{b*(1-uN^2)->1}}
{integrd2=Assuming[Sin[psi/2]>0,PowerExpand[integrd1/.{1-Cos[psi]->2*Sin[psi/2]^2}]],
 eq8=Assuming[phi0<phi && (phi<=0||phi0>0) && (phi<=-2*Pi||phi0>=-2*Pi) &&
 (phi<=2*Pi||phi0>=2*Pi),Integrate[integrd2,{psi,phi0,phi}]]}
{eq90=trS1[eq7R,Sqrt[2]]/Sqrt[-term72], eq9=Integrate[eq90,{eta,xi0,xi}],
 eq10=eq8==eq9, sol=First[Solve[eq10,phi]], tr3={lambda->0.1,c->10.},
 tr4={Cot[phi0/4]->1,xi0->0,C[1]->0}, tr5=x-uN*t}
kink[xN_,tN_,uNU_]:=Expand[(sol[[1,2]]/.tr4)/.{xi->tr5}]/.{x->xN,t->tN,uN->uNU};
antiKink[xN_,tN_,uNU_]:=Expand[(sol[[1,2]]/.tr4)/.{xi->-tr5}]/.{x->xN,t->tN,uN->uNU};
uN1=tr2[[1]]/.tr3
kS=Plot[kink[x,0,uN1],{x,-2*Pi,2*Pi}];
aKS=Plot[antiKink[x,0,uN1],{x,-2*Pi,2*Pi},PlotStyle->{Hue[0.7],Thickness[0.01]}];
Show[{kS,aKS}]
```

These one-soliton solutions have the following Mathematica form:

$$\text{Kink[x,y,u]} = 4\text{ArcCot}\left[e^{\sqrt{\frac{1}{1-u^2}}\left(-1+u^2\right)(x-uy)}\right]$$

$$\text{AntiKink[x,y,u]} = 4\text{ArcCot}\left[e^{\sqrt{\frac{1}{1-u^2}}\left(-1+u^2\right)(-x+uy)}\right]$$

These solutions are called the soliton (or kink) and antisoliton (or antikink) solutions, respectively (depending on the sign). The solitons propagate in the positive or negative x-direction, respectively, at the velocity U. We visualize the soliton and antisoliton solutions in 3D and 2D spaces (see Fig. 40.3).

Example 40.11. *Fisher equation. Phase portraits.* Consider the Fisher equation

$$u_t - u_{xx} = u(1-u).$$

It is known that this equation admits a traveling-wave solution of the form $u(x,t) = U(z)$, $z = x - ct$ $(0 \le u(x,t) \le 1)$, where c is the wave speed and the waveform $U(z)$ satisfies the boundary conditions $\lim\limits_{z\to-\infty} U(z) = 1$, $\lim\limits_{z\to\infty} U(z) = 0$.

We show that the Fisher equation can be reduced to the nonlinear ODE $-U''(z)-cU'(z)-U(z)+U(z)^2=0$ (eq3, eq4) or to the system of two first-order ODEs $U'=V$, $V'=-cV-U+U^2$ (tr3[[1]], Eq6), or, equivalently, $\frac{dV}{dU}=\frac{-cV-U+U^2}{V}$ (eq7) as follows:

```
{tr1=x-c*t->z,tr2={V[z]->V,U[z]->U},
 tr3={D[U[z],z]->V[z],D[U[z],{z,2}]->D[V[z],z]}}
eq1[u_]:=D[u,t]-D[u,{x,2}]-u*(1-u);
eq2=eq1[U[tr1[[1]]]]//Expand
{eq3=(eq2/.tr1)==0, eq4=eq3//TraditionalForm}
{eq5=eq3/.tr3, eq6=Solve[eq5,D[V[z],z]]}
eq7=Hold[D[V,U]]==eq6[[1,1,2]]/V[z]
```

This is a system of first-order autonomous ODEs: $\frac{du}{d\xi}=P(u,v)$, $\frac{dv}{d\xi}=Q(u,v)$, where P and Q are nonlinear functions of the two variables u and v, and the independent variable is ξ, which can be eliminated by dividing one equation by the other, $\frac{du}{dv}=\frac{P(u,v)}{Q(u,v)}$. The solution of this equation can be represented as a *phase diagram* on the (u,v)-plane with some *phase trajectories* along which the system will evolve as ξ increases. Since the integration for finding $u(v)$ may not be possible, the phase plane analysis can be useful to determine *stationary* or *singular points* u_0, v_0 in some regions of the *Poincaré phase plane* and analyze the nature of solutions. In practice, since the phase plane portraits may be complicated, it is convenient to obtain them with Mathematica.

Let us study the dynamical system obtained. We show that the singular points (W_0,V_0) of this dynamical system are $(0,0)$ and $(1,0)$. Following the phase plane analysis, we study the nature of the given nonlinear system and show that the matrix associated with this system at singular points, $(0,0)$ and $(1,0)$, has the eigenvalues $-\frac{1}{2}(c\mp\sqrt{c^2-4})$ and $\frac{1}{2}(-c\pm\sqrt{c^2+4})$, respectively. According to the theory of dynamical systems, we show that the point $(0,0)$ is a stable node for $c\geq 2$ and the point $(1,0)$ is a saddle point. Finally, we construct the phase portrait on the plane (U,V), showing that there exists a unique separatrix joining the stable node $(0,0)$ with the saddle point $(1,0)$ (for $c\geq 2$).

```
{eq81=eq7[[2,1,1]]->0, eq82=Solve[eq7[[2,2]]==0/.eq81,U[z]]}
SingPoints={{eq82[[1,1,2]],eq81[[2]]},{eq82[[2,1,2]],eq81[[2]]}}
{P1=tr3[[1,2]]/.tr2, Q1=eq6[[1,1,2]]/.tr2}
MA={{D[P1,U],D[P1,V]},{D[Q1,U],D[Q1,V]}}
{A1=MA/.{U->0,V->0}, A2=MA/.{U->1,V->0}}
{Eigenvalues[A1], Eigenvalues[A2], c>=2}
{eqs=({tr3[[1]],eq6[[1,1]]}/.c->3)/.Rule->Equal, vars={U,V}}
IC={{-0.5,-1.},{0.1,1.},{0.96968212,0.1},{0.35,1.},{0.5,1.},{0.8,-1.},{0.4,-1.},
    {0.1,-1.},{0.999,-1.},{1.1,-1.},{1.29889,-1.},{0.65999,1.1}};
n=Length[IC]; zF=60;
Do[{sys[i]={eqs[[1]],eqs[[2]],U[0]==IC[[i,1]],V[0]==IC[[i,2]]};
    sols=NDSolve[sys[i],vars,{z,0,zF}]; cu=U/.sols[[1]]; cv=V/.sols[[1]];
    c[i]=ParametricPlot[Evaluate[{cu[z],cv[z]}],{z,0,zF},
                        PlotStyle->{Hue[0.1*i+0.2],Thickness[.01]}];}, {i,1,n}]
fu=eqs[[1,2]]/.tr2; fv=eqs[[2,2]]/.tr2;
fd=VectorPlot[{fu,fv},{U,-0.7,1.},{V,-1.,1.},VectorColorFunction->Hue];
Show[fd,Table[c[i],{i,1,n}]]
```

40.3.4. Ansatz Methods (Tanh-Coth Method, Sine-Cosine Method, and Exp-Function Method) for Constructing Traveling-Wave Solutions

In this section, we consider the most important ansatz methods (the tanh function method, the sine-cosine method, and the exp-function method) for constructing traveling-wave solutions of nonlinear PDEs. Recently developed, these methods became the most powerful

and effective algebraic methods for finding exact solutions of nonlinear PDEs partly due to the modern computer algebra systems Maple and Mathematica, which allow performing a lot of cumbersome analytical calculations.

Tanh-coth method. The tanh function expansion method was proposed by Malfliet (1992). The main idea of the tanh method is based on the assumption that traveling-wave solutions can be expressed in terms of the tanh function; i.e., a new variable, e.g., $Y = \tanh(z)$, can be introduced and all derivatives of Y are expressed in terms of the tanh function. Then the tanh method has been modified, extended, and generalized for finding more exact solutions. In recent years, various forms of the tanh method have been developed, e.g., the tanh-coth method has been proposed by Wazwaz (2006); see also Parkes (2010) and Zhang (2010). Many researchers developed symbolic programs and packages to deal with the tedious algebraic computations that arise in the solution process, e.g., Parkes and Duffy (1996).

Example 40.12. Consider the *Klein–Gordon equation* of the form

$$u_{tt} - u_{xx} + u - u^3 = 0. \tag{40.3.4.1}$$

We seek traveling-wave solutions of the Klein–Gordon equation, i.e., solutions of the form $u(x,t) = U(z)$, $z = \mu(x+ct)$. First, we convert the PDE into the following ODE: $\mu^2c^2U_{zz} - \mu^2U_{zz} + U - U^3 = 0$ (`eq2`). If all terms of this ODE contain derivatives in z, then the ODE should be simplified (by integrating and by assuming the integration constants to be zero).

Introducing a new variable $Y = \tanh(z)$ or $Y = \coth(z)$, we propose the following ansatz (`ansatz1`):

$$U(z) = S(Y) = \sum_{i=0}^{M} a_i Y^i,$$

where M ($M > 0$, $M \in \mathbb{Z}$) is to be determined.

By substituting this series expansion into the ODE, we obtain an equation in powers of Y:

$$(-\mu^2 S_{YY} + \mu^2c^2S_{YY})Y^4 + (2\mu^2c^2S_Y - 2\mu^2S_Y)Y^3 + (-2\mu^2c^2S_{YY} + 2\mu^2S_{YY})Y^2$$
$$+ (2\mu^2S_Y - 2\mu^2c^2S_Y)Y + S(Y) + \mu^2c^2S_{YY} - \mu^2S_{YY} - S(Y)^3 = 0.$$

To determine the parameter M, we balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms according to the following formulas for the highest exponents of the function $U(z)$ and its derivatives: $U(z) \to M$, $U^n(z) \to nM$, $U'(z) \to M+1$, $U''(z) \to M+2$, $U^{(k)}(z) \to M+k$. In our case, we have $3M = M+2$, $M = 1$ and $S(Y) = a_0 + a_1Y$.

Then, by matching the coefficients of like powers of Y in the simplified equation (`eq4`)

$$(2\mu^2c^2a_1 - a_1^3 - 2\mu^2a_1)Y^3 - 3a_0a_1^2Y^2 + (-3a_0^2a_1 + 2\mu^2a_1 + a_1 - 2\mu^2c^2a_1)Y - a_0^3 + a_0 = 0,$$

where these coefficients have to vanish, we obtain the system of algebraic equations (`sys1`) for the unknowns a_i ($i = 0,\ldots,M$), μ, and c:

$$-3a_0a_1^2 = 0,\ -a_0^3 + a_0 = 0,\ 2\mu^2c^2a_1 - a_1^3 - 2\mu^2a_1 = 0,\ -3a_0^2a_1 + 2\mu^2a_1 + a_1 - 2\mu^2c^2a_1 = 0.$$

Finally, determining these unknowns, μ, a_0, and a_1 (`sols`), and using the series expansion (`ansatz1`), we obtain the exact solutions $u(x,t) = \pm\tanh\Big(\sqrt{1/(-2+2c^2)}(x+ct)\Big)$ (`solsT`) and verify that these solutions are exact solutions.

```
Off[Solve::"svars"]; pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0;
{tr1=mu*(x+c*t)->z,tr2=Tanh[z]->yN, sys1={}, solsT={}}
{eq1=pde1[uN[tr1[[1]]]]//Expand, eq2=eq1/.tr1//Expand,"eq2="eq2}
{tr3=uN[z]->s[tr2[[1]]], tr31=D[uN[z],{z,2}]->D[s[tr2[[1]]],{z,2}],
 ansatz1=s[yN]->Sum[a[i]*yN^i,{i,0,mN}], "ansatz1="ansatz1, eq3=(eq2/.tr3)/.tr31,
```

```
 tr32=Table[(Sech[z]^2)^i->(1-Tanh[z]^2)^i,{i,1,2}]}
{eq31=(eq3/.tr32)/.tr2, eq32=Collect[eq31,yN], tr5=Solve[3*mN==mN+2,mN]//First,
 tr6=ansatz1/.tr5, tr61=Table[D[ansatz1,{yN,i}]/.tr5,{i,1,2}],
 eq4=(eq32/.tr6)/.tr61,"eq4="eq4}
Do[sys1=Union[sys1,{Coefficient[eq4[[1]],yN,i]==0}],{i,0,3}];
{sys1, vars=Complement[Variables[Table[sys1[[i]]//First,{i,1,Length[sys1]}]],{c}],
 sols=Solve[sys1,vars], nSols=Length[sols]}
Do[solsT=Union[solsT,{u[x,t]->((tr6[[2]]/.yN->tr2[[1]])/.z->tr1[[1]])/.sols[[i]]}],
 {i,nSols-3,nSols}]; solsT
Table[pde1[solsT[[i]][[2]]]//FullSimplify,{i,1,Length[solsT]}]
```

Sine-cosine method. The main idea of the sine-cosine method is based on the assumption that traveling-wave solutions can be expressed via the sine or cosine functions, and hence a new variable, e.g., $Y=\cos(z)$ or $Y=\sin(z)$, can be introduced. As in the tanh method considered above, the main advantage of the sine-cosine method consists in simplifying the solution process and reducing the amount of computational work (compared with modern analytical methods); i.e., we have to solve a system of algebraic equations instead of a nonlinear differential equation. Moreover, we can apply computer algebra systems, Maple and Mathematica, for this purpose.

Example 40.13. For the Klein–Gordon equation (40.3.4.1) considered in the preceding example, let us find traveling-wave solutions, i.e., solutions of the form $u(x,t)=U(z)$, $z=x+ct$. As before, we convert the PDE into the following ODE: $c^2U_{zz}-U_{zz}+U-U^3=0$ (`eq2`). If all terms of this ODE contain derivatives in z, then the ODE should be simplified (by integrating and by assuming the integration constants to be zero).

According to the sine-cosine method, traveling-wave solutions can be expressed in the form $u(x,t)=\lambda\cos^\beta(\mu z)$ for $|z|\le\frac{1}{2}\pi/\mu$) or in the form $u(x,t)=\lambda\sin^\beta(\mu z)$ for $|z|\le\pi/\mu$, where the parameters λ, μ (wave number), and β are to be determined. By introducing a new variable $Y=\cos(z)$ or $Y=\sin(z)$, we propose the following ansatz: $U(z)=\lambda\cos^\beta(\mu z)$ (`ansatzC`) or $U(z)=\lambda\sin^\beta(\mu z)$ (`ansatzS`).

By substituting this ansatz into the ODE, we obtain the trigonometric equation (`eq3`). Then, by simplifying this equation and by introducing the new variable $Y=\cos(\mu z)$, we obtain (`eq32`)

$$-\lambda^3Y^{3\beta}+\lambda\beta^2\mu^2c^2Y^{\beta-2}-\lambda\beta^2\mu^2c^2Y^\beta-\lambda\beta\mu^2c^2Y^{\beta-2}$$
$$-\lambda\beta^2\mu^2Y^{\beta-2}+\lambda\beta^2\mu^2Y^\beta+\lambda\beta\mu^2Y^{\beta-2}+\lambda Y^\beta=0.$$

To determine the parameter β, we balance the exponents of each pair of Y. Then, to determine λ and μ, we match the coefficients of like powers of Y, where these coefficients have to vanish. This gives a system of algebraic equations for the unknown parameters β, λ, and μ: $Y^{3\beta}=Y^{\beta-2}$, (`term12==term22`), $-\lambda\beta^2\mu^2c^2+\lambda\beta^2\mu^2+\lambda=0$ (`eqmu1`), $-\lambda^3+\lambda\beta^2\mu^2c^2-\lambda\beta\mu^2c^2-\lambda\beta^2\mu^2+\lambda\beta\mu^2=0$ (`eqlambda`).

Finally, determining these parameters, $\beta=-1$ (`trbeta`), $\mu=1/\sqrt{c^2-1}$ (`trmu`), and $\lambda=\sqrt{2}$ (`trlambda`), and using the proposed ansatz (`ansatzC`, `ansatzS`), we obtain exact solutions and verify that these solutions are indeed exact solutions.

Let $d=\sqrt{c^2-1}$. If $c^2-1>0$, then we have (`solC`, `solS`)

$$u(x,t)=\sqrt{2}\sec\left(d^{-1}(x+ct)\right)\quad\text{for}\quad|d^{-1}z|\le\frac{1}{2}\pi$$

and

$$u(x,t)=\sqrt{2}\csc\left(d^{-1}(x+ct)\right)\quad\text{for}\quad0<|d^{-1}z|\le\pi.$$

If $c^2-1<0$, then we have (`solCC`, `solSC`)

$$u(x,t)=\sqrt{2}\,\mathrm{sech}\left(d^{-1}(x+ct)\right)\quad\text{and}\quad u(x,t)=i\sqrt{2}\,\mathrm{csch}\left(d^{-1}(x+ct)\right).$$

```
pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0; {tr1=x+c*t->z, tr2=Cos[z]->Y}
{eq1=pde1[U[tr1[[1]]]]//Expand, eq2=eq1/.tr1,"eq2="eq2}
{ansatzC=U[z]->lambda*Cos[mu*z]^beta, ansatzS=U[z]->lambda*Sin[mu*z]^beta}
```

```
ansatzC1=Table[D[U[z],{z,i}]->D[lambda*Cos[mu*z]^beta,{z,i}],{i,1,2}]
{eq3=(eq2/.ansatzC)/.ansatzC1, eq31=eq3/.{Sin[x_]^2:>1-Cos[x]^2}//Factor}
{eq32=eq31[[1]]/.{Cos[mu*z]->Y}//Expand, Y3b=Y^(3*beta), Yb2=Y^(beta-2)}
{term11=Select[eq32,MemberQ[#1,Y3b]&], term12=Select[term11,MemberQ[#1,Y]&]}
{term21=Select[eq32,MemberQ[#1,Yb2]&],
 term22=Select[Factor[term21],MemberQ[#1,Y]&]}
trbeta=Solve[term12[[2]]==term22[[2]], beta]
{eqmu=Reduce[Select[eq32,MemberQ[#1,Y^(beta)]&]==0,mu], eqmu1=eqmu[[3]]}
trmu=(eqmu1[[2,2]]//ToRules)/.trbeta
eqlambda=Select[eq32,MemberQ[#1,Y3b]&]+Select[eq32,MemberQ[#1,Yb2]&]==0
eqlambda1=(eqlambda/.trbeta)/.trmu//First
{solslambda=Solve[eqlambda1,lambda]//Simplify, trlambda=solslambda[[3]]}
solC=(u[x,t]->ansatzC[[2]]/.trmu/.trlambda/.trbeta/.z->tr1[[1]])
solS=(u[x, t]->ansatzS[[2]]/.trmu/.trlambda/.trbeta/.z->tr1[[1]])
{trC=1/Sqrt[c^2-1]->1/(I*Sqrt[1-c^2]), solCC=solC/.trC, solSC=solS/.trC}
Map[Simplify[pde1[#1[[1,1,2]]]]&,{solC,solS,solCC,solSC}]
{sol1G=solC/.{c->2,n->3}, sol2G=solC/.{c->1/2,n->3}}
F1[X_,T_]:=sol1G[[1,1,2]]/.x->X/.t->T; F1[x,t]
F2[X_,T_]:=sol2G[[1,1,2]]/.x->X/.t->T; F2[x,t]
Animate[Plot[F1[x,t],{x,0,Pi},PlotRange->{{0,Pi},{-200,200}},
        PlotStyle->Hue[0.7]],{t,0,5}]
Animate[Plot[F2[x,t],{x,-3,3},PlotRange->{{-3,3},{0,2}},
        PlotStyle->Hue[0.7]],{t,0,5}]
```

Exp-function method. The Exp-function method was proposed by He and Wu (2006) to obtain exact solutions of nonlinear evolution equations. Then the method was applied to various nonlinear PDEs of mathematical physics, e.g., the Burgers and KdV equations as well as the Kuramoto–Sivashinsky (KS) and Boussinesq equations.

Example 40.14. Let us find exact solutions of the Klein–Gordon equation (40.3.4.1) (considered in the two preceding examples) and present the solution procedure of the Exp-function method with the aid of computer algebra system Mathematica.

We seek traveling-wave solutions, i.e., solutions of the form $u(x,t) = U(z)$, $z = \mu(x + ct)$, where μ and c are constants to be determined. As before, we convert the PDE into the following ODE: $\mu^2 c^2 U_{zz} - \mu^2 U_{zz} + U - U^3 = 0$ (eq2).

According to the Exp-function method, we propose an ansatz; i.e., we assume that the traveling-wave solutions can be expressed in the form [see He and Abdou (2007)] (tr3)

$$U(z) = \frac{\sum_{k=-r}^{s} a_k e^{kz}}{\sum_{j=-p}^{q} b_j e^{jz}} = \frac{a_r e^{rz} + \cdots + a_{-s} e^{-sz}}{b_p e^{pz} + \cdots + b_{-q} e^{-qz}}. \tag{40.3.4.2}$$

where r, s, p, and q are unknown positive integers to be determined and a_k and b_j are unknown constants.

Considering the resulting ODE, we can determine the values of r, p and s, q. First, by balancing the highest-order linear term with the highest-order nonlinear term, we obtain $-3r - 3p = -r - 5p$; i.e., $r = p$. In a similar way, by balancing the least-order linear term with the least-order nonlinear term, we have $3s + 3q = s + 5q$; i.e., $s = q$. These symbolic computations can be performed with Mathematica.

Consider the particular case in which $r = p = 1$ and $s = q = 1$. Then the ansatz acquires the form (tr4)

$$U(z) = \frac{a_{-1} e^{-z} + a_0 + a_1 e^{z}}{b_{-1} e^{-z} + b_0 + b_1 e^{z}}.$$

By substituting this expression into the ODE and by matching the coefficients of like powers of e^{kz}, we generate the system of algebraic equations (sys1) for the unknowns c, μ, a_{-1}, a_0, a_1, b_{-1}, b_0, and b_1.

To determine all unknown constants, we solve this system of algebraic equations and obtain 75 solutions (sols). It should be noted that among these solutions there are many equivalent forms of solutions (represented in different forms and for different values of integration constants), since the obtained ODE (eq2) can readily

be integrated and its solution can be expressed via elliptic integrals. Without full analysis of all solutions, let us compare some solutions with the exact solutions obtained by the tanh function method. Thus, we choose the following solutions (`sol1`, `sol2`):

$$c = c_0, \quad \mu = -\sqrt{\frac{1}{2(c_0^2-1)}}, \quad -b_{-1} = a_{-1}, \quad a_0 = 0, \quad b_1 = a_1, \quad b_{-1} = b_{-1}, \quad b_0 = 0, \quad b_1 = b_1;$$

$$c = c_0, \quad \mu = -\sqrt{\frac{1}{2(c_0^2-1)}}, \quad b_{-1} = a_{-1}, \quad a_0 = 0, \quad -b_1 = a_1, \quad b_{-1} = b_{-1}, \quad b_0 = 0, \quad b_1 = b_1,$$

and the corresponding Mathematica results read

$$\text{solF12} = u[x,t] \to -\frac{a[-1]+a[1]+a[-1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]-a[1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]}{a[-1]-a[1]+a[-1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]+a[1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]}$$

$$\text{solF22} = u[x,t] \to \frac{a[-1]+a[1]+a[-1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]-a[1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]}{a[-1]-a[1]+a[-1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]+a[1]\text{Tanh}\left[\frac{ct+x}{\sqrt{-2+2c^2}}\right]}$$

To compare with the Maple results, we set `a[1]->b[1]` and `a[-1]->-b[-1]`. The corresponding traveling-wave solutions take the form (`solF14`, `solF24`)

$$u(x,t) = \pm\frac{b_{-1}+b_{-1}\tanh(\frac{\sqrt{2}}{2}d(x+ct))-b_1+b_1\tanh(\frac{\sqrt{2}}{2}d(x+ct))}{b_{-1}+b_{-1}\tanh(\frac{\sqrt{2}}{2}d(x+ct))+b_1-b_1\tanh(\frac{\sqrt{2}}{2}d(x+ct))},$$

where $d = \sqrt{1/(c^2-1)}$. By setting $b_1 = -1$ and $b_{-1} = -1$, we rewrite the above solutions in the form (`solF15`, `solF25`):

$$u(x,t) = \pm\tanh(\tfrac{\sqrt{2}}{2}d(x+ct)),$$

which coincide with the exact solutions obtained above by the tanh function method. Finally, we verify that the solutions obtained are exact solutions of the nonlinear Klein–Gordon equation.

One can find other types of exact solutions by analyzing all sets of parameters or by considering other particular cases, e.g., $r = p = 2$ and $s = q = 2$.

```
pde1[u_]:=D[u,{t,2}]-D[u,{x,2}]+u-u^3==0; tr1=mu*(x+c*t)->z
{eq1=pde1[U[tr1[[1]]]]//Expand, eq2=eq1/.tr1}
tr3=U[z]->Sum[a[k]*Exp[k*z],{k,-r,s}]/Sum[b[j]*Exp[j*z],{j,-p,q}]
{trpc=Solve[-3*r-3*p==-r-5*p,p], trqd=Solve[3*s+3*q==s+5*q,q]}
{params={p->1,r->1,q->1,s->1}, tr4=tr3/.params}
tr41=Table[D[tr4[[1]],{z,i}]->D[tr4[[2]],{z,i}],{i,1,2}]//Expand
{eq3=((eq2/.tr4)/.tr41)//Factor, eq31=eq3[[1,3]]}
Do[Eq[i]=Coefficient[eq31,Exp[i*z]]==0;Print[Eq[i]],{i,1,6}];
s1=0; nL:=Length[eq31];
Do[If[D[eq31[[i]],z]==0,s1=s1+eq31[[i]],s1=s1+0],{i,1,nL}];
{Eq[7]=s1==0, Eq[8]=c==c0}; sys1=Table[Eq[i],{i,1,8}]
vars={mu,c,Table[a[i],{i,-1,1}],Table[b[j],{j,-1,1}]}//Flatten
sols=Reduce[sys1//FullSimplify,vars]
sol12[S_]:=Map[ToRules,{sols[[S,2,1]],sols[[S,3]],sols[[S,5]],
          sols[[S,6]],sols[[S,7]],sols[[S,8]]}]//Flatten; nsols=Length[sols]
{sol1=sol12[45], sol2=sol12[46]}
solF1=u[x,t]->((tr4[[2]]/.{z->tr1[[1]]})/.sol1)/.{c0->c}
solF2=u[x,t]->((tr4[[2]]/.{z->tr1[[1]]})/.sol2)/.{c0->c}
ruleSCH={Sinh[x_]:>Tanh[x]/(Sqrt[1-Tanh[x]^2]),Cosh[x_]:>1/(Sqrt[1-Tanh[x]^2])};
solF12=(solF1//ExpToTrig)/.ruleSCH//Factor
solF22=(solF2//ExpToTrig)/.ruleSCH//Factor
```

```
{solF13=solF12/.{a[1]->-1,a[-1]->-1}, solF23=solF22/.{a[1]->-1,a[-1]->-1}}
solF14=solF12/.{a[1]->b[1],a[-1]->-b[-1]}
solF24=solF22/.{a[1]->-b[1], a[-1]->b[-1]}
{solF15=solF14/.{b[1]->-1,b[-1]->-1}, solF25=solF24/.{b[1]->-1,b[-1]->-1}}
{pde1[solF15[[2]]],pde1[solF25[[2]]]}//FullSimplify
```

40.3.5. Constructing Self-Similar Solutions

Self-similar solutions often occur in numerous problems of mathematical physics. The method of similarity is based on some symmetry properties of a physical system and the algebraic symmetry of a nonlinear PDE. Self-similar solutions are obtained by solving an associated ODE. These solutions are invariant under a *scaling or similarity transformation*. For example, the self-similar solution of the problem of expanding blast wave was first obtained (based on dimensional analysis) by Taylor (1950) and Sedov (1993). The similarity method is important and useful when finding solutions of both linear and nonlinear PDEs. As before, we illustrate the method of finding self-similar solutions by solving problems.

Example 40.15. *Nonlinear diffusion equation.* Consider the nonlinear diffusion equation

$$u_t = au_{xx} + bu^n.$$

We consider the *scaling transformation* $t = CT, x = C^k X, W = C^m W$, where $C \neq 0$ is an arbitrary constant and m and k are some unknown constants. We assume that this equation is invariant under the scaling transformation for suitable values of m and k. We verify that this scaling transformation converts the nonlinear diffusion equation into the equation $C^{m-1}W_T = aC^{m-2k}W_{XX} + bC^{mn}W^n$ (`eq1`). We show that $k = \frac{1}{2}$, $m = \frac{1}{1-n}$ (`tr2`), and the *self-similar variables* acquire the form $u(x,t) = t^{\alpha}U(\xi)$, $\xi = xt^{\beta}$ (`tr3`), where $\alpha = m$ and $\beta = -k$.

```
DiffusEq[x_,t_]:=D[u[x,t],t]==a*D[u[x,t],{x,2}]+b*u[x,t]^n;
{tr11={x->X*C^k,t->T*C}, tr12=u->W*C^m,
 tr13={(C^m*W)[X,T]->C^m*W[X,T], D[(C^m*W)[X,T],{X,2}]->C^m*D[W[X,T],{X,2}],
       D[(C^m*W)[X,T],T]->C^m*D[W[X,T],T]}}
eq1T[v_]:=((Simplify[DiffusEq[x,t]/.
            u->Function[{x,t},u[x/C^k,t/C]]])/.tr11//ExpandAll)/.{u->v};
eq1=eq1T[W*C^m]/.tr13//PowerExpand
{ex21=Select[eq1[[2]],MemberQ[#1,C]&],ex22=Select[eq1[[1]],MemberQ[#1,a]&],
 ex23=Select[eq1[[1]],MemberQ[#1,b]&]}
{ex31=ex21[[2]], ex32=ex22[[2,2]], ex33=ex23[[2,2]]}
tr2=Assuming[n!=1,Solve[{ex31==ex32,ex32==ex33},{k,m}]]
{alpha=tr2[[1,1,2]], beta=-tr2[[1,2,2]]}
{tr3={xi->x*t^(beta),u->U[xi]*t^(alpha)}, tr31=x*t^(beta)->xi}
DEq[x_,t_,u_]:=D[u,t]==a*D[u,{x,2}]+b*u^n;
eq21=DEq[x,t,U[tr31[[1]]]*t^(alpha)]//PowerExpand
{ODE1=eq21/.tr31, c1=Select[ODE1[[2,1]],MemberQ[#1,t]&]}
{ODE11=Thread[ODE1/c1,Equal]/.tr31//FullSimplify, ODEFin=ODE11/.tr31}
```

Therefore the nonlinear diffusion equation is reduced to the ordinary differential equation that has the following Mathematica form:

$$\text{ODEFin} = \frac{U[\text{xi}]}{-1+n} + bU[\text{xi}]^n + \frac{1}{2}\text{xi}U'[\text{xi}] + aU''[\text{xi}] == 0$$

Example 40.16. *Sine–Gordon equation.* Consider the sine–Gordon equation

$$u_{xt} = \sin u$$

and the scaling transformation $t = C^n T, x = C^m X, W = C^k W$ (where $C \neq 0$ is an arbitrary constant). We assume that this equation is invariant under the scaling transformation for suitable values of n, m, and k. We verify that this scaling transformation converts the sine–Gordon equation into the equation $C^{-k-n+m} W_{XT} = \sin(C^m W)$ (`eq1`). We show that $m = -n$ and $k = k$ (`tr2`), where we set $k = 0$. The self-similar variables acquire the form $u(x,t) = t^{\alpha} U(\xi), \xi = x/t^{\beta}$ (`tr3`), where $\alpha = -k/(2n)$ and $\beta = n/m$.

```
SGEq[x_,t_]:=D[u[x,t],x,t]==Sin[u[x,t]];
{tr11={t->T*C^n,x->X*C^m}, tr12=u->C^k*W,
 tr13={(C^k*W)[X,T]->C^k*W[X,T],D[(C^k*W)[X,T],X,T]->C^k*D[W[X,T],X,T]}}
eq1T[v_]:=((Simplify[SGEq[x,t]/.
            u->Function[{x,t},u[x/C^m,t/C^n]]])/.tr11//ExpandAll)/.{u->v};
eq1=eq1T[W*C^k]/.tr13//PowerExpand
{ex21=Select[eq1[[2]],MemberQ[#1,C]&], ex22=Select[eq1[[1,1]],MemberQ[#1,C]&]}
{ex31=ex21[[2]]==0, ex32=ex22[[2]]==0}
tr2=Solve[{ex31,ex32},{n,k}]
{alpha=(-k/(2*n)/.tr2)[[1]], beta=(n/m/.tr2)[[1]]}
{tr3={xi->x/t^(beta),u->U[xi]*t^(alpha)}, tr31=x/t^(beta)->xi}
SGEq1[x_,t_,u_]:=D[u,x,t]==Sin[u];
ODE1=SGEq1[x,t,U[tr31[[1]]]*t^alpha]/.tr31
```

Finally, the sine–Gordon equation is reduced to the ordinary differential equation that has the following Mathematica form:

$$\text{ODE1} = U'[\text{xi}] + \text{xi}\,U''[tx] == \text{Sin}[U[\text{xi}]]$$

40.3.6. Constructing Solutions Along Characteristics

Now we consider methods for finding general solutions of first-order linear, quasilinear, and nonlinear equations, the method of characteristics, and the generalized method of characteristics. These methods allow reducing a PDE to a system of ODEs along which the given PDE with some initial data (Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the characteristic curves and transformed into a general solution of the original PDE.

By applying the generalized method of characteristics for solving first-order nonlinear PDEs of the form $F(x,y,u,u_x,u_y) = 0$ or $F(x,y,u,p,q) = 0$ (where $u_x = p$ and $u_y = q$), we show how to obtain a *complete solution* or a *complete integral* of nonlinear equations, i.e., a two-parameter family of surfaces of the form $f(x,y,u,a,b) = 0$ (where a and b are parameters).

The general solution of a nonlinear PDE can be determined from its complete integral. But in the nonlinear case, the solution surface or integral surface through which the complete integral has to pass is a cone, known as the *Monge cone*. The characteristic curves depend on the orientation of tangent planes to the Monge cone at each point. Thus, there exists a Monge cone of characteristics, and the *characteristic equations*, or the *Charpit equations*, for the nonlinear equations have the form

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{du}{dt} = pF_p + qF_q, \quad \frac{dp}{dt} = -(F_x + pF_u), \quad \frac{dq}{dt} = -(F_y + qF_u).$$

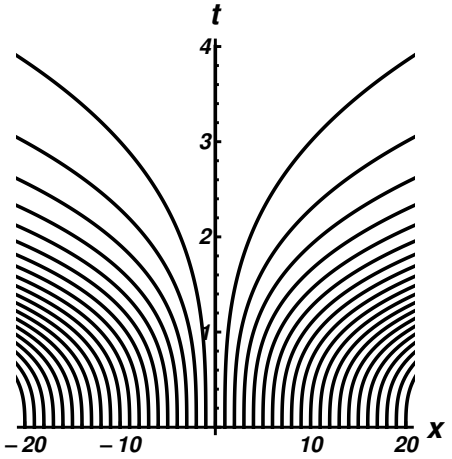

Figure 40.4. Characteristic curves for $u_t + t^2 u u_x = 0$.

Example 40.17. *Method of characteristics. Quasilinear PDEs. Classical Cauchy problems.*
1°. Consider the initial value problem

$$u_t + t^2 u u_x = 0, \quad u(x,0) = x.$$

By applying the method of characteristics, we show that the solution of this Cauchy problem is

$$u(x,t) = \frac{3x}{3+t^3}$$

and plot the characteristic curves (see Fig. 40.4).

```
SetOptions[Plot,ImageSize->500,PlotStyle->{Hue[0.7],Thickness[0.001]}];
ODE=D[U[t],t]==0
SolCh=DSolve[{ODE,U[0]==X[0]},U[t],t]
EqCh=D[x[t],t]==t^2*U[t]/.SolCh[[1]]
CurCh=DSolve[{EqCh,x[0]==X[0]},x[t],t]//Simplify
g=Table[ParametricPlot[{(CurCh[[1,1,2]]/.X[0]->x),t},{t,0,4}],{x,-20,20}];
Show[g,PlotRange->{{-20,20},{0,4}},AspectRatio->1]
uu=SolCh[[1]]/.Solve[CurCh[[1,1,2]]==x,X[0]]
u[X_,T_]:=uu[[1,1,2]]/.{x->X,t->T}; u[X,T]
```

2°. Consider the initial value problem

$$u_t - u^2 u_x = 3u, \quad u(x,0) = f(x).$$

We apply the method of characteristics as follows:

```
SetOptions[ParametricPlot,ImageSize->300,AspectRatio->1,
           PlotStyle->{Hue[0.7],Thickness[0.005]},PlotRange->{{-10,10},{0,1}}];
f1[x_]:=1; f2[x_]:=x; {tr1=x->x[t], tr2=x[t]->x}
{tr3=f[X[0]]->f1[X[0]], tr4=f[X[0]]->f2[X[0]]}
{ODE1=D[x[t],t]==-U[t]^2, ODE2=D[U[t],t]==3*U[t]}
{Sol2=DSolve[ODE2,U[t],t], Sol21=Sol2/.C[1]->C}
IniCond=u[X[0],0]==f[X[0]]
Const2=(Sol21[[1,1,2]]/.t->0)->IniCond[[2]]
{Sol22=Sol21[[1,1]]/.Const2, ODE11=ODE1/.Sol22}
Sol1=DSolve[ODE11,x[t],t]
```

```
X0=Sol1[[1]]/.t->0/.Rule->Equal//Expand
{Const1=Solve[X0,C[1]], Chars=Sol1/.Const1/.x[0]->X[0]}
GenSol=Sol22[[2]]
Char[1]=Chars/.tr2/.tr3/.Rule->Equal
Char[2]=Chars/.tr2/.tr4/.Rule->Equal
X01=Solve[Flatten[Char[1]],X[0]]//Flatten
X02=Solve[Flatten[Char[2]],X[0]]//Flatten
u[1][X_,T_]:=GenSol/.f[X[0]]->f1[X01[[1,2]]]/.{x->X,t->T};
u[2][X_,T_]:=GenSol/.f[X[0]]->f2[X02[[1,2]]]/.{x->X,t->T};
u[3][X_,T_]:=GenSol/.f[X[0]]->f2[X02[[2,2]]]/.{x->X,t->T};
{u[1][X,T], u[2][X,T], u[3][X,T]}
Do[g[i]=Table[ParametricPlot[{Char[i][[1,1,1,2]]/.X[0]->x,t},
              {t,0,1}],{x,-10,10}];,{i,1,2}];
GraphicsRow[{Show[g[1]],Show[g[2]]}]
Do[gu[i]=Table[ParametricPlot[{u[i][X,T],T},
              {T,0.001,1}],{X,{-10,-8,-6,-5,5,6,8,10}}];,{i,1,3}];
GraphicsRow[{Show[gu[1]],Show[gu[2]],Show[gu[3]]}]
```

We show that the *implicit form of the solution* (or parametric representation of the solution) of this Cauchy problem has the Mathematica form

$$\text{Sol22} = U[t] \to e^{3t} f[X[0]]$$

$$\text{Chars} = \left\{\left\{\left\{x[t] \to -\frac{1}{6}e^{6t} f[X[0]]^2 + \frac{1}{6}\left(f[X[0]]^2 + 6X[0]\right)\right\}\right\}\right\}$$

Example 40.18. *Method of characteristics. Quasilinear PDEs. Generalized Cauchy problem.* Consider the first-order quasilinear PDE

$$uu_x + u_y = 1$$

and assume that the integral surface of this equation passes through the initial curve parametrically represented by the equations

$$x = x_0(r) = r, \quad y = y_0(r) = r, \quad u = u_0(r) = 0 \quad (r \text{ is a parameter}).$$

By applying the method of characteristics, we show that the integral surface of this equation has the form

$$(u-1)^2 + 2(y-x) = 1.$$

```
PDE[u_]:=u[x,y]*D[u[x,y],x]+D[u[x,y],y]==1; PDE[u]//Expand
IniCurve={X->X0[r],Y->Y0[r],U->U0[r]}
CharEqs=DX/U==(DY/1==DU/1)
CharEqsParamForm={D[X[t],t]==U[t],D[Y[t],t]==1,D[U[t],t]==1}
IniConds={x[r,0]->IniCurve[[1,2]],y[r,0]->IniCurve[[2,2]],
          u[r,0]->IniCurve[[3,2]]}
eq3=U[r,t]==Integrate[CharEqsParamForm[[3,2]],t]+IniCurve[[3,2]]
eq2=Y[r,t]==Integrate[CharEqsParamForm[[2,2]],t]+IniCurve[[2,2]]
eq1=X[r,t]==Integrate[CharEqsParamForm[[1,2]]/.U[t]->eq3[[2]],t]+IniCurve[[1,2]]
IniData={r,r,0}
IniCondsPart={x[r,0]->IniData[[1]],y[r,0]->IniData[[2]],u[r,0]->IniData[[3]]}
tr1={X0[r]->IniCondsPart[[1,2]],Y0[r]->IniCondsPart[[2,2]],U0[r]->IniCondsPart[[3,2]]}
{sys1={eq1,eq2,eq3}, sys2=sys1/.tr1}
{tr2={X[r,t]->x,Y[r,t]->y,U[r,t]->u}, sys3=sys2/.tr2}
SolFin=Eliminate[sys3,{r,t}]//FullSimplify
CS[a_.x_^2+b_.x_+c_.]:=a*((x+b/(2*a))^2-(b^2-4*a*c)/(4*a^2));
CompleteSquare[X_]:=If[TrueQ[X==Expand[X]],X,CS[Expand[X]]];
SolFin1=Map[Factor,CompleteSquare[SolFin[[1]]-SolFin[[2]]]]
```

Example 40.19. *Generalized method of characteristics. Nonlinear PDEs. Complete integrals.* Consider the first-order nonlinear PDE

$$u_x^2 + yu_y = u.$$

By applying the generalized method of characteristics, we show that the complete integral of this nonlinear equation has the form

$$u(x,y) = \tfrac{1}{4}(x+b)^2 + ay.$$

```
CS[a_.x_^2+b_.x_+c_.]:=a*((x+b/(2*a))^2-(b^2-4*a*c)/(4*a^2));
CompleteSquare[X_]:=If[TrueQ[X==Expand[X]],X,CS[Expand[X]]];
F=p^2+q*y-u
CharEqs={D[x[t],t]==D[F,p], D[y[t],t]==D[F,q], D[u[t],t]==p*D[F,p]+q*D[F,q],
         D[p[t],t]==-(D[F,x]+p*D[F,u]), D[q[t],t]==-(D[F,y]+q*D[F,u])}
{tr1=q[t]->q, tr2=du->p*dx+q*dy, tr3=u-a*y->v}
eq1=Thread[Integrate[CharEqs[[5]],t],Equal]
eq2=eq1[[1]]->eq1[[2]]+a
eq3=Solve[F==0/.{eq2/.tr1},p]
eq4=p==eq3[[1,1,2]]
eq5=(dx/CharEqs[[1,2]]==dy/CharEqs[[2,2]])==du/CharEqs[[3,2]]
eq6=eq5/.tr2
eq7=du==(tr2/.{p->eq4[[2]],q->eq2[[2]]})[[2]]
eq8=Solve[eq7,dx]//Expand
eq81=eq8[[1,1,2]]==d[tr3[[1]]]*eq8[[1,1,2,1,3]]
eq82=eq81/.tr3
eq83=Integrate[eq82[[2,1]],v]==Integrate[eq8[[1,1,1]]/dx,x]+b
eq84=eq83[[1]]==eq83[[2]]/.v->tr3[[1]]
Sol=Solve[eq84,u]
CompleteInt=u[x,y]==CompleteSquare[Sol[[1,1,2]]]
```

Example 40.20. *Generalized method of characteristics. Classical Cauchy problem.* Consider the first-order nonlinear PDE

$$(u_x)^2 u_y = 1$$

with the initial data $u(x,0) = x$. By applying the generalized method of characteristics, we show that the solution of this Cauchy problem has the form $u(x,t) = x + y$.

```
{F=p^2*q-1, IniData={u->x,y->0}}
CharEqs={D[x[t],t]==D[F,p], D[y[t],t]==D[F,q],
         D[u[t],t]==p*D[F,p]+q*D[F,q], D[p[t],t]==-(D[F,x]+p*D[F,u]),
         D[q[t],t]==-(D[F,y]+q*D[F,u])}
{tr1={q[t]->q,p[t]->p}, tr2=du->p*dx+q*dy}
{eq1=DSolve[CharEqs[[4]],p[t],t], eq2=eq1/.tr1/.C[1]->a}
{eq3=DSolve[CharEqs[[5]],q[t],t], eq4=eq3/.tr1/.C[1]->c}
{eq5=Flatten[F==0/.eq2/.eq4], eq51=Solve[eq5,c]}
eq6=tr2/.eq51
eq7=tr2/.p->eq2[[1,1,2]]/.q->eq4[[1,1,2]]/.eq51/.Rule->Equal
Sol=Integrate[eq7[[1,1]]/du,u]==
    Integrate[eq7[[1,2,1]]/dx,x]+Integrate[eq7[[1,2,2]]/dy,y]+b
Sol1=Sol/.IniData
Consts={a->Coefficient[Sol1[[1]],x],b->Coefficient[Sol1[[1]],x,0]}
SolCauchy=Sol/.Consts
```

Example 40.21. *Generalized method of characteristics. Generalized Cauchy problem.* Consider the first-order nonlinear PDE

$$(u_x)^2 + u_y + u = 0.$$

Let us find the solution of this equation that passes through the initial curve represented parametrically by the equations $x(r,0) = r$, $y(r,0) = r$, $u(r,0) = 2r - 1$, where r is a parameter. By applying the generalized method

of characteristics, we state the Cauchy problem for this nonlinear equation to determine the family of curves in the parametric form $u = u(r,t)$, $x = x(r,t)$, $y = y(r,t)$ and the values $p = p(r,t)$ and $q = q(r,t)$, where r and t are parameters. When seeking the solution surface $u = u(x,y)$ with given initial data at $t = 0$, we first determine the initial conditions for $p(r,t)$ and $q(r,t)$ at $t = 0$ (`sols`):

$$p(r,0) = \tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{-3-8r}, \quad q(r,0) = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{-3-8r},$$

which must satisfy the original equation and the *strip condition* (`StripCond`)

$$\frac{du(r,0)}{dr} = p(r,0)\frac{dx(r,0)}{dr} + q(r,0)\frac{dy(r,0)}{dr}.$$

We take the first pair of values of $p(r,0)$ and $q(r,0)$ (`sols1`). Then we can determine the values of $p = p(r,t)$ and $q = q(r,t)$ (`Solp`, `Solq`)

$$p(r,t) = -\tfrac{1}{2}(-1+S)e^{-t}, \quad q(r,t) = \tfrac{1}{2}(3+S)e^{-t},$$

where $S = \sqrt{-3-8r}$, and the solution of the Cauchy problem as the family of curves in the parametric form $u = u(r,t)$, $x = x(r,t)$, $y = y(r,t)$ (`SolFin`),

$$u(r,t) = -\left(\tfrac{3}{2} + \tfrac{1}{2}S\right)e^{-t} + \left(\tfrac{1}{2}(1+S) + 2r\right)e^{-2t}, \quad x(r,t) = (1-S)(1-e^{-t}) + r, \quad y(r,t) = t + r.$$

Finally, we verify that this solution is an exact solution of the given nonlinear PDE (`test1`).

```
{F=p^2+q+u, F1=p[r,t]^2+q[r,t]+u[r,t]}
IniData={x[r,0]->r,y[r,0]->r,u[r,0]->2*r-1}
IniData1={x[0]==r,y[0]==r,u[0]==2*r-1}
CharEqs={D[x[t],t]==D[F,p],D[y[t],t]==D[F,q], D[u[t],t]==p*D[F,p]+q*D[F,q],
         D[p[t],t]==-(D[F,x]+p*D[F,u]), D[q[t],t]==-(D[F,y]+q*D[F,u])}
{tr1={p->p[r,t],q->q[r,t]}, tr12={p'[t]->D[p[r,t],t],q'[t]->D[q[r,t],t]},
 tr2={p->p[r,t],q->q[r,t]}}
{eq1=(F1/.t->0/.IniData)==0,
 StripCond=D[u[r,0],r]==p[r,0]*D[x[r,0],r]+q[r,0]*D[y[r,0],r],
 IniData2=D[IniData,r], eq2=StripCond/.IniData2}
{sys1={eq1,eq2}, vars=Complement[Variables[{eq1[[1]],eq2[[1]]}],{r}],
 sols=Solve[sys1,vars], sols1=sols//First//Sort}
Table[eq[k]=(CharEqs[[k]][[1]]/.tr12)==(CharEqs[[k]][[2]]/.tr2),{k,4,5}]
sols2={vars[[1]]==sols1[[1,2]], vars[[2]]==sols1[[2,2]]}
Solp=DSolve[{eq[4],sols2[[1]]},p[r,t],{r,t}]
Solq=DSolve[{eq[5],sols2[[2]]},q[r,t],{r,t}]
{tr4={p->Solp[[1,1,2]], q->Solq[[1,1,2]]}, tr5={x[t]->x,y[t]->y,u[t]->u},
 tr6={u[t]->u[r,t], x[t]->x[r,t], y[t]->y[r,t]}}
eq6=Table[CharEqs[[i]],{i,1,3}]/.tr4//Together
{eq7=DSolve[{eq6,IniData1},{x[t],y[t],u[t]},{t}], eq71={eq7,Solp,Solq}}
{SolFin=eq71/.tr6//Flatten, test1=F1/.SolFin//Simplify}
```

40.3.7. Constructing Separable Solutions

Separation of variables is one of the most important methods for solving linear PDEs, in which the structure of a PDE allows us to seek *multiplicative separable* or *additive separable* exact solutions; e.g., $u(x,t) = \phi(x) \circ \psi(t)$ (where the multiplication or addition is denoted by $\circ$). Although for some specific nonlinear PDEs it is possible to apply this idea of separation of variables directly (as for linear PDEs), it does not apply to wide classes of nonlinear PDEs. We consider the generalizations of the *classical method of separation of variables* developed in the recent years by Galaktionov (1990, 1995), Polyanin and Zhurov (1998), and Polyanin and Manzhirov (2006).

The *generalized separable solutions* can be written in the form

$$u(x,t) = \phi_1(x)\psi_1(t) + \cdots + \phi_n(x)\psi_n(t).$$

Unlike linear equations, in the nonlinear equations the functions $\phi_i(x)$ with different subscripts i are usually related to one another [and to functions $\psi_j(y)$]. In general, the functions $\varphi_i(x)$ and $\psi_j(y)$ are not known in advance and are to be identified.

Then, for nonlinear PDEs, it is of great interest to find *functional separable solutions*, i.e., exact solutions of the form

$$u(x,t) = F(z) \quad \text{or} \quad F(u(x,t)) = z,$$

where $z = \phi_1(x)\psi_1(t) + \cdots + \phi_n(x)\psi_n(t)$. These solutions can be found if the functions $F(z)$ (or $F(u)$), $\phi_i(x)$, and $\psi_i(t)$ $(i = 1,\ldots,n)$ exist.

In the recent years, many approaches have been proposed for studying the generalized and functional separation of variables, e.g., by specifying some of the systems of coordinate functions $\{\phi_i(x)\}$ or $\{\psi_i(t)\}$ and finding special functional separable solutions, by differentiating and splitting, by the theory of linear subspaces invariant under nonlinear differential operators, by extending the Lie theory, etc. Note that the classical additive and multiplicative separable solutions and the generalized separable solutions can be treated as special cases of the above functional separable solution if $u(x,t) = F(z) = z$ (or $F(u(x,t)) = u(x,t) = z$).

We consider various problems in which one can construct generalized and functional separable solutions by using various approaches, e.g., by differentiating, splitting, etc.

Example 40.22. *Generalized separable solutions by differentiation: n th-order nonlinear PDE.* Consider the n th-order nonlinear PDE

$$u_t u_{xt} - u_x u_{tt} = f(x)u_t^{(n)},$$

where $f(x)$ is an arbitrary function. By applying the method of generalized separation of variables and by seeking exact solutions in the form $u(x,t) = \phi_1(x)\psi_1(t) + \phi_2(x)$, we arrive at the following equations (`eq71`, `eq72`):

$$-\frac{-\psi_1{}_t^2 + \psi_{1tt}\psi_1}{\psi_{1tt}} = C_1, \quad -\frac{\phi_{2x}f_x(x) - \phi_{2xx}f(x)}{-\phi_{1x}f_x(x) + \phi_{1xx}f(x)} = C_1.$$

```
tr1=phi1[x]*psi1[t]+phi2[x]
Dphi1={phi1[x],phi1'[x],phi1''[x]}
Dpsi1={psi1[t],psi1'[t],psi1''[t]}
Dphi2={phi2[x],phi2'[x],phi2''[x]}
PDE1[u_]:=D[u,t]*D[u,x,t]-D[u,x]*D[u,{t,2}]==f[x]*D[u,{t,n}];
tr11=D[phi1[x]*psi1[t],{t,n}]->phi1[x]*Derivative[n][psi1][t]
{eq2=PDE1[W[x,t]]//Expand, eq3=PDE1[tr1]/.tr11//Expand}
eq4=Collect[Expand[Thread[eq3/phi1[x],Equal]],phi1'[x]]
eq5=Thread[eq4/f[x],Equal]//Expand
eq6=Thread[D[eq5,x],Equal]
eq61=Collect[Collect[eq6,Dphi1],Dpsi1]
eq612=Thread[eq61/eq61[[1,3,2]],Equal]//Simplify
eq613=Collect[eq612,Dphi2]
eq62=Thread[f[x]^2*(eq613[[1,1;;2]]==-eq613[[1,3]]),Equal]//Factor
TermX=eq62[[2,1]]
eq7=Thread[eq62/TermX,Equal]
{eq71=-eq7[[2]]==C[1], eq72=-eq7[[1]]==C[1]}
eq721=Thread[eq72/eq72[[1,1]],Equal]//Expand
```

We seek the exact solutions of this equation. Consider the following two cases:

(1) *Nondegenerate case*: the separation constant is not zero.

We find that $\psi_1(t) = -C_1 + C_4 e^{\lambda t}$ (Sol2), $\phi_2(x) = \phi_1(x)C_1 + C_2 \int f(x)\,dx + C_3$ (Sol1), and the exact solution acquires the form $u(x,t) = \phi_1(x)e^{\lambda t} - \lambda^{n-2} \int f(x)\,dx + C$ (SolFin1). Here $\phi_1(x)$ is an arbitrary function, and C and λ are arbitrary constants.

```
Sol11=DSolve[eq721,phi2[x],x]/.K[1]->x
Sol12=Sol11/.Sol11[[1,1,2,2,2]]->x
Sol1=phi2[x]->C[3]+Map[Integrate[#,x]&,Sol12[[1,1,2,2,1]]]
Sol211=Collect[(DSolve[eq71,psi1[t],t]//ExpToTrig)/.
       {C[2]*C[3]->C2}//TrigExpand,{Cosh[t*C[2]],Sinh[t*C[2]]}]
Sol212=Sol211/.{Sol211[[1,1,2,2,2]]->C[3]}//TrigToExp
Sol2=(Sol212/.C[2]->lambda/.C[3]->C[4])[[1,1]]
C34={C[3]->C,C[4]->1}
eq8=eq4/.Sol1/.D[Sol1,x]/.Sol2/.D[Sol2,t]/.D[Sol2,{t,2}]/.
    Map[D[#,{t,n}]&,Sol2]/.{n->10}//Expand
eq81=Assuming[{lambda,t}\[Element] Reals,
     ExpandAll[Thread[eq8/Exp[lambda*t],Equal]/.C34/.{10->n}]]
C2=(Solve[eq81,C[2]])[[1,1]]
{Sol21=Sol2/.C2/.C34, Sol12=Sol1/.C2/.C34}
SolFin1=Assuming[{lambda,t}\[Element] Reals,
                  u[x,t]->tr1/.Sol21/.Sol12//FullSimplify]
PDEtest=((PDE1[SolFin1[[2]]]/.{n->10})/.{10->n}//FullSimplify)/.{10->n}
```

(2) *Degenerate case*: the separation constant is zero.

We find that $\phi_1(x) = C_1 + C_2 \int f(x)\,dx$ (Sol3) and $\phi_2(x) = C_3 + C_4 \int f(x)\,dx$ (Sol4). The ODE $C_2(\psi_{1t})^2 - \psi_{1tt}(C_2\psi_1 + C_4) = \psi_1^{(n)}$ (eq11) for the function $\psi_1 = \psi_1(t)$ describes the exact solution of the form $u(x,t) = \phi_1(x)\psi_1(t) + \phi_2(x)$.

```
{eq9=1/eq7[[1,1]]==0, eq10=eq7[[1,2]]==0}
Sol31=DSolve[eq9,phi1[x],x]/.K[1]->x
Sol3=(Sol31/.Sol31[[1,1,2,2,2]]->x/.{C[2]->C[1],C[1]->C[2]})[[1,1]]
Sol41=DSolve[eq10,phi2[x],x]/.K[1]->x
Sol4=(Sol41/.Sol41[[1,1,2,2,2]]->x/.{C[2]->C[3],C[1]->C[4]})[[1,1]]
eq11=Thread[(eq4/.Sol4/.Sol3/.D[Sol3,x]/.D[Sol4,x])/f[x],Equal]//Expand
```

Example 40.23. *Generalized separable solutions by splitting.* Consider the third-order nonlinear PDE

$$u_{xt} + (u_{xx})^2 - uu_{xx} = \nu u_{xxx}.$$

By applying the method of generalized separation of variables, by seeking exact solutions in the form $u(x,t) = \phi_1(x)\psi_1(t) + \psi_2(t)$, and by applying the splitting procedure, we arrive at the functional equation

$$\Phi_1(x)\Psi_1(t) + \cdots + \Phi_4(x)\Psi_4(t) = 0,$$

where $\Phi_1 = \phi_{1x}$, $\Psi_1 = \psi_{1t}$, $\Phi_2 = -\phi_{1xx}$, $\Psi_2 = \psi_1\psi_2$, $\Phi_3 = -\phi_1\phi_{1xx} + (\phi_{1x})^2$, $\Psi_3 = \psi_1^2$, $\Phi_4 = -\phi_{1xxx}$, and $\Psi_4 = \nu\psi_1$. Knowing the solutions of this functional equation, e.g., the solution $\Psi_1 = A_1\Psi_3 + A_2\Psi_4$, $\Psi_2 = A_3\Psi_3 + A4\Psi_4$, $\Phi_3 = -A_1\Phi_1 - A_3\Phi_2$, $\Phi_4 = -A_2\Phi_1 - A_4\Phi_2$, we arrive at the following system of ODEs (sys1):

$$\psi_{1t} = A_1\psi_1^2 + A_2\nu\psi_1, \quad \psi_1\psi_2 = A_3\psi_1^2 + A_4\nu\psi_1,$$
$$-\phi_1\phi_{1xx} + (\phi_{1x})^2 = -A_1\phi_{1x} + A_3\phi_{1xx}, \quad -\phi_{1xxx} = -A_2\phi_{1x} + A_4\phi_{1xx}.$$

```
PDE1[u_]:=D[u,x,t]+(D[u,x])^2-u*D[u,{x,2}]==nu*D[u,{x,3}];
tr1=psi1[t]*phi1[x]+psi2[t]
eq2=PDE1[W[x,t]]//Expand
eq3=Collect[PDE1[tr1],psi1[t]]
eq31=eq3[[1]]-eq3[[2]]==0
FunDiffeq1=Sum[Phi[i]*Psi[i],{i,1,4}]==0
SolFunDiffeq1={Psi[1]->A1*Psi[3]+A2*Psi[4], Psi[2]->A3*Psi[3]+A4*Psi[4],
               Phi[3]->-A1*Phi[1]-A3*Phi[2],Phi[4]->-A2*Phi[1]-A4*Phi[2]}
L11=Select[eq31[[1]],MemberQ[#1,psi1'[t]]&]
L21=Select[eq31[[1]],MemberQ[#1,phi1''[x]]&]
L31=Select[eq31[[1]],MemberQ[#1,psi1[t]^2]&]
L41=Select[eq31[[1]],MemberQ[#1,nu]&]
L1={L11[[2]],L11[[1]]}
L2={L21[[2]]*L21[[3]],L21[[1]]*L21[[4]]}
L3={L31[[1]],L31[[2]]}
L4={L41[[2]]*L41[[3]],L41[[1]]*L41[[4]]}
tr2={Phi[1]->L1[[2]], Psi[1]->L1[[1]], Phi[2]->L2[[2]], Psi[2]->L2[[1]],
     Phi[3]->L3[[2]], Psi[3]->L3[[1]], Phi[4]->L4[[2]], Psi[4]->L4[[1]]}
sys1=SolFunDiffeq1/.tr2/.Rule->Equal
```

The last two equations in the system are consistent if and only if the function $\phi_1(x)$ satisfies the equation $\phi_{1x} = B_1\phi_1 + B_2$. (We shall show this after the solution procedure.) By integrating these equations, we find the solutions: $\phi_1 = B_2 x + B_3$ if $B_1 = 0$ (Solphi12), $\phi_1 = -\dfrac{B_2}{B_1} + B_3 e^{B_1 x}$ if $B_1 \neq 0$ (Solphi11), $\psi_1 = \dfrac{1}{-B_4 - A_1 t}$ if $A_2 = 0$ (Solpsi12), $\psi_1 = -\dfrac{A_2 e^{A_2\nu(B_4+t)}\nu}{A_1 e^{A_2\nu(B_4+t)} - 1}$ if $A_2 \neq 0$ (Solpsi11), and $\psi_2 = A_3\psi_1 + \nu A_4$ (Solpsi2).

Determining the relations between the arbitrary constants A_1, A_2, A_3, A_4, B_1, and B_2, we arrive at the equations: $-A_4B_1 + A_2 - B_1^2 = 0$ (eq621), $B_1(B_2 - A_3B_1 + A_1) = 0$ (eq63), and $B_2(B_2 - A_3B_1 + A_1) = 0$ (eq64).

All these formulas allow us to determine the exact solutions of the form $u(x,t) = \phi_1(x)\psi_1(t) + \psi_2(t)$. If we consider, for example, the case in which $A_2 = 0$, $B_1 = 0$, and $B_2 = -A_1$, then we obtain the final form of the exact solution $u(x,t) = \dfrac{x + C_1}{t + C_2} + C_3$ (SolFin13), where $C_1 = \dfrac{B_3 + A_3}{B_2}$, $C_2 = \dfrac{B_4}{B_2}$, $C_3 = A_4\nu$.

```
{tr3=C[1]->B3, tr4=C[1]->B4}
eqsys1=Table[D[phi1[x],{x,i}]->D[B1*phi1[x]+B2,{x,i-1}],{i,1,3}]
{eq41=sys1[[3]]//.eqsys1, eq42=sys1[[4]]//.eqsys1}
eq512=Collect[Thread[(eq41[[1]]-eq41[[2]])/phi1[x],Equal],phi1[x]]//Apart
eq51=phi1[x]->-eq512[[2]]/eq512[[1]]*phi1[x]
eq522=Collect[Thread[(eq42[[1]]-eq42[[2]])/phi1[x],Equal],phi1[x]]//Apart
eq52=phi1[x]->-eq522[[2]]/eq522[[1]]*phi1[x]
{eq61=Denominator[eq52[[2]]]==0, eq611=eq61[[1,2]]==0}
{eq62=Numerator[eq52[[2]]]==0//Factor, eq621=eq62[[1,2]]==0}
eq63=Denominator[eq51[[2]]]==0
eq64=Numerator[eq51[[2]]]==0//Factor
vars=Variables[{eq611[[1]],eq63[[1]]}]
SolConst1=Solve[{eq611,eq63,eq64,A3==a3,A4==a4,B1==b1,B2==b2},vars]
SolConst2=Solve[{eq611,eq63,eq64,A1==a1,A2==0,A3==a3,A4==a4,B2==0},vars]
Sol1={SolConst1[[1]],SolConst2[[1]]}
{tr5={a3->A3,a4->A4,b1->B1,b2->B2},tr6={A2->0,B1->0,A1->-B2}}
eqsys11=eqsys1[[1]]/.Rule->Equal
Solphi11=DSolve[eqsys11,phi1[x],x]/.tr3
Solphi12=DSolve[eqsys11/.tr6,phi1[x],x]/.tr3
Solpsi11=DSolve[sys1[[1]],psi1[t],t]/.tr4//FullSimplify
Solpsi12=DSolve[sys1[[1]]/.{A2->0},psi1[t],t]/.tr4
Solpsi2=Solve[sys1[[2]],psi2[t]]
```

```
SolFin11=tr1/.Solpsi2//Apart
SolFin12=(SolFin11/.Solpsi12/.Solphi12/.tr6/.tr5)[[1,1,1]]
SolFin13=Expand[Numerator[SolFin12[[2]]]/B2]/Expand[
                Denominator[SolFin12[[2]]]/B2]+SolFin12[[1]]
```

Let us show that if the last two equations in the system obtained above, i.e., $-\phi_1\phi_{1xx}+(\phi_{1x})^2=-A_1\phi_{1x}+A_3\phi_{1xx}$ and $-\phi_{1xxx}=-A_2\phi_{1x}+A_4\phi_{1xx}$, are consistent, then the function $\phi_1(x)$ satisfies the equation $\phi_{1x}=B_1\phi_1+B_2$; and if the function $\phi_1(x)$ satisfies the equation $\phi_{1x}=B_1\phi_1+B_2$, then these two equations are consistent.

```
{E1=sys1[[3,1]]-sys1[[3,2]]==0, E2=sys1[[4,1]]-sys1[[4,2]]==0}
{E21=Solve[E2,phi1''[x]]//Expand, E11=Solve[E1,phi1''[x]]//Factor}
E3=E21[[1,1,2]]-E11[[1,1,2]]//Expand
{E4=Collect[E3,phi1'[x]], E5=E4/.phi1'''[x]->0}
{E6=Expand[E5]//Simplify, E7=Select[E6,MemberQ[#,A2*A3]&]==0}
E8=Solve[E7,phi1'[x]]//Expand
{sys21={E1,E2}, sys22=sys21//.eqsys1}
Solsys21=Solve[sys22,phi1[x]]
{eqsys1[[1]]/.E8, eqsys1[[1]]}
```

Example 40.24. *Generalized separable solutions. Simplified approach.* Consider the third-order nonlinear PDE

$$u_tu_{xt}-u_xu_{tt}=\nu u_{ttt}.$$

By applying the method of generalized separation of variables and the simplified approach, i.e., by specifying one of the systems of coordinate functions $\{\phi_i(x)\}$ and by seeking exact solutions in a simplified form, e.g., $u(x,t)=x\psi_1(t)+\psi_2(t)$, we arrive at the following ODEs (`eq51`, `eq52`):

$$(\psi_{1t})^2-\psi_1\psi_{1tt}-\nu\psi_{1ttt}=0,$$
$$\psi_{1t}\psi_{2t}-\psi_1\psi_{2tt}-\nu\psi_{2ttt}=0.$$

Then we can solve the first nonlinear ODE performing a symmetry reduction, i.e., a reduction of the order of the equation. This means that solving the most general ODE $\psi_{1t}=C_1(\psi_1)^2$ (`ODE51`) that is invariant under the transformations $\{\xi=1,\eta=0\}$, $\{\xi=-t,\eta=\psi_1\}$, we arrive at the solution of the first ODE $\psi_1=1/(-C_1t-C_2)$ (`Sol1`), where C_1 and C_2 are arbitrary constants.

Then, by determining the constant $C_1=-1/(6\nu)$ (`SolConsts`) and by choosing $B_1=-6\nu C_2$, we arrive at the solution $\psi_1=6\nu/(t+B_1)$ (`Solpsi1`) of the first ODE. By solving the second ODE, we obtain the solution $\psi_2=C_5-C_4/(t+B_1)^2-C_3/(t+B_1)$ (`Solpsi2`).

Finally, the exact solution of the given nonlinear PDE in the desired simplified form is (`SolFin1`)

$$u(x,t)=\frac{6\nu x}{t+B_1}-\frac{C_1}{t+B_1}-\frac{C_2}{(t+B_1)^2}+C_3,$$

where B_1, C_1, C_2, and C_3 are arbitrary constants.

```
PDE1[u_]:=D[u,t]*D[u,x,t]-D[u,x]*D[u,{t,2}]-nu*D[u,{t,3}]==0;
tr1=x*psi1[t]+psi2[t]
{eq2=PDE1[W[x,t]]//Expand, eq3=PDE1[tr1]//Expand}
eq4=Collect[eq3,x]
{ODE1=Select[eq4[[1]],MemberQ[#,x]&], ODE2=eq4[[1]]-ODE1}
ODEs={ODE1,ODE2}
{eq51=(ODE1/x//Expand)==0, eq52=ODE2==0}
ODE51=psi1'[t]==C[1]*psi1[t]^2
Sol1=DSolve[ODE51,psi1[t],t]
Consts=eq51/.Table[D[Sol1[[1,1]],{t,i}],{i,0,3}]//Simplify
eqConsts=Select[Consts[[1]],MemberQ[#,1]&]
SolConsts=Solve[eqConsts==0,C[1]]
```

```
Solpsi1=((Simplify[Sol1/.SolConsts[[2]]])/.{6*nu*C[2]->B1})[[1,1]]
eq521=eq52/.Solpsi1/.D[Solpsi1,t]
Solpsi2=DSolve[eq521,psi2[t],t]
SolFin1=u[x,t]->tr1/.Solpsi1/.Solpsi2
PDE1[SolFin1[[1,2]]]//FullSimplify
```

Example 40.25. *Functional separation of variables. Simplified approach.* Consider the nonlinear diffusion equation

$$u_t = u_{xx} + F(u),$$

where $F(u)$ is an arbitrary nonlinear function. By applying the method of *functional separation of variables*, $u(x,t) = W(z)$, and a simplified approach, i.e., by specifying one of the systems of coordinate functions $\{\phi_i(x)\}$ and by seeking exact solutions in a simplified form, e.g., $z = x\psi_1(t) + \psi_2(t)$, we arrive at the following ODE (`eq51`):

$$-\frac{\psi_{1t}z}{\psi_1} + \frac{\psi_{1t}\psi_2}{\psi_1} - \psi_{2t} + \frac{\psi_1^2 W_{zz}}{W_z} + \frac{F(W(z))}{W_z} = 0.$$

Then, by applying the splitting procedure, i.e., by rewriting this equation in the form of a functional equation, we obtain

$$\Phi_1(x)\Psi_1(t) + \cdots + \Phi_4(x)\Psi_4(t) = 0,$$

where $\{\Phi_1 = 1, \Psi_1 = \psi_{1t}\psi_2/\psi_1 - \psi_{2t}, \Phi_2 = z, \Psi_2 = -\psi_{1t}/\psi_1, \Phi_3 = W_{zz}/W_z, \Psi_3 = \psi_1^2, \Phi_4 = F(W(z))/W_z, \Psi_4 = 1\}$ (`tr3`). Knowing the solutions of this functional equation, e.g., we arrive at the following system of ODEs (`sys1`):

$$\begin{aligned}
\frac{\psi_{1t}\psi_2}{\psi_1} - \psi_{2t} &= A_1\psi_1^2 + A_2,\\
-\frac{\psi_{1t}}{\psi_1} &= A_3\psi_1^2 + A_4,\\
\frac{W_{zz}}{W_z} &= -A_1 - A_3 z,\\
\frac{F(W(z))}{W_z} &= -A_2 - A_4 z,
\end{aligned}$$

where the A_i $(i = 1,\ldots,4)$ are arbitrary constants.

```
PDE1[u_]:=D[u,t]==D[u,{x,2}]+F[u];
tr1=psi1[t]*x+psi2[t]->z
tr2=Solve[tr1/.Rule->Equal,x]//Expand
{eq2=PDE1[W[tr1[[1]]]]//Expand, eq3=eq2/.tr1}
eq4=Thread[eq3/W'[z],Equal]//Expand
{eq5=eq4/.tr2//Expand, eq51=-eq5[[1,1]]+eq5[[1,2]]==0}
FunDiffeq1=Sum[Phi[i]*Psi[i],{i,1,4}]==0
SolFunDiffeq1={Psi[1]->A1*Psi[3]+A2*Psi[4], Psi[2]->A3*Psi[3]+A4*Psi[4],
               Phi[3]->-A1*Phi[1]-A3*Phi[2],Phi[4]->-A2*Phi[1]-A4*Phi[2]}
L11={Select[eq51[[1]],MemberQ[#,psi1'[t]] && MemberQ[#,psi2[t]]&],1}
L12={Select[eq51[[1]],MemberQ[#,psi2'[t]]&],1}
L2={Select[eq51[[1]],MemberQ[#,psi1'[t]] && MemberQ[#,z]&]/z,z}
L31=Select[eq51[[1]],MemberQ[#,psi1[t]^2]&]
L3={L31[[1]],L31/psi1[t]^2}
L4={1,Select[eq51[[1]],MemberQ[#,F[W[z]]]&]}
tr3={Phi[1]->L11[[2]]*L12[[2]], Psi[1]->L11[[1]]+L12[[1]],
     Phi[2]->L2[[2]], Psi[2]->L2[[1]], Phi[3]->L3[[2]],
     Psi[3]->L3[[1]], Phi[4]->L4[[2]], Psi[4]->L4[[1]]}
sys1=SolFunDiffeq1/.tr3/.Rule->Equal
```

By integrating these equations, we can find the unknown functions $\psi_1(t)$, $\psi_2(t)$, $W(z)$, and $F(W(z))$ with four integrating constants B_i $(i = 1,\ldots,4)$ and the exact solution of the form $u(x,t) = W(x\psi_1(t) + \psi_2(t))$ as follows:

```
{tr4=C[1]->B1, tr5=C[1]->B2}
{S2=DSolve[sys1[[2]],psi1[t],t]/.tr4, S2[[1]]}
S11=DSolve[sys1[[1]],psi2[t],t]/.K[1]->t
{S12=S11/.S11[[1,1,2,2,1,2]]->t, S13=S12/.tr5}
S14=psi2[t]->B2*psi1[t]+Hold[Integrate[
    (-(A2*psi1[t])-A1*psi1[t]^3)/psi1[t]^2,t]]*psi1[t]
{S1[1]=S14/.S2[[1]], S1[2]=S14/.S2[[2]]}
eq3sys1=Integrate[sys1[[3,1]],z]==Integrate[sys1[[3,2]],z]
eq3sys11=Solve[eq3sys1,W'[z]]
S3=Integrate[eq3sys11[[1,1,1]],z]->B4+B3*Hold[Integrate[eq3sys11[[1,1,2]],z]]
eq4sys1=(sys1[[4]]/.F[W[z]]->F1)/.D[S3,z]//ReleaseHold//Simplify
S4=F[W[z]]->Solve[eq4sys1,F1][[1,1,2]]
Consts={A1->-1, A3->0, B3->1, B4->0}
s11=S1[1]/.Consts//ReleaseHold//Simplify
s21=S2[[1]]/.Consts
s31=W[z]->B4+B3*Integrate[eq3sys11[[1,1,2]]/.Consts,z]/.Consts
s41=S4/.Consts
Sol1={s11,s21,s31,s41}
s12=S1[2]/.Consts//ReleaseHold//Simplify
s22=S2[[2]]/.Consts
s32=W[z]->B4+B3*Integrate[eq3sys11[[1,1,2]]/.Consts,z]/.Consts
s42=S4/.Consts
Sol2={s12,s22,s32,s42}
tr6=Log[Sol1[[3,2]]]->Log[Sol1[[3,1]]]//PowerExpand
F11=F[W[z]]==(Sol1[[4,2]]/.tr6)//Together
```

If we consider, for example, the special case in which $A_1=-1$, $A_2\neq 0$, $A_3=0$, $A_4\neq 0$, $B_1\neq 0$, $B_2\neq 0$, $B_3=1$, and $B_4=0$, then we obtain the explicit form of the exact solution (`Sol1`, `Sol2`, `F11`)

$$u(x,t)=W(z)=e^z=e^{x\psi_1(t)+\psi_2(t)},\quad F(W(z))=-W(z)\bigl(A_2+A_4\ln(W(z))\bigr),$$
$$\psi_1(t)=\pm\frac{\sqrt{A_4^2B_1e^{2A_4t}}}{A_4B_1e^{2A_4t}},\quad \psi_2(t)=-\frac{A_2B_1e^{2A_4t}+1\pm B_2\sqrt{A_4^2B_1e^{2A_4t}}}{e^{2A_4t}A_4B_1}.$$

Example 40.26. *Functional separation of variables by differentiation. Klein–Gordon equation.* Consider the nonlinear Klein–Gordon equation

$$u_{tt}-u_{xx}=F(u),$$

where $F(u)$ is a nonlinear function. By seeking exact solutions in the form $u(x,t)=W(z)$, $z=\phi(x)+\psi(t)$, we arrive at the following functional-differential equation with three variables (`eq8`):

$$\psi_{tt}-\phi_{xx}+(\psi_t^2-\phi_x^2)G(z)=H(z), \tag{40.3.7.1}$$

where $G(z)=W_{zz}/W_z$, $H(z)=F(W(z))/W_z$. By differentiating this equation with respect to t and x and by simplifying the resulting equation, we obtain the functional-differential equation with two variables (`eq14`) $(\psi_t^2-\phi_x^2)(G_{zz}-2G(z)G_z)=H_{zz}-2H(z)G_z$.

```
PDE1[u_]:=D[u,{t,2}]-D[u,{x,2}]==F[u];
tr1=phi[x]+psi[t]->z
{eq2=PDE1[W[tr1[[1]]]]//Expand, eq3=eq2/.tr1}
{eq4=Thread[eq3/W'[z],Equal]//Expand, eq5=Collect[eq4,W'[z]]}
eq6=Map[Simplify[#]&,eq5[[1]]]==eq5[[2]]
{trH=H[z]->eq6[[2]], trG=Select[eq6[[1]],MemberQ[#,W''[z]]&]}
LtrG1=Level[trG,1]
{trG1={LtrG1[[2]]*LtrG1[[3]],LtrG1[[1]]}, trG2=G[z]->trG1[[1]]}
{eq7=eq6/.trH[[2]]->H[z], eq8=eq7/.trG2[[2]]->G[z]}
{eq9=eq8/.z->z[x,t], eq10=D[eq9,t,x]}
```

```
eq11=eq10/.{z[x,t]->z,D[z[x,t],x]->D[tr1[[1]],x],
            D[z[x,t],t]->D[tr1[[1]],t],
            D[z[x,t],t,x]->D[tr1[[1]],t,x]}
eq12=Thread[eq11/(psi'[t]*phi'[x]),Equal]//Expand
eq131=Thread[eq12/2/G'[z],Equal]//Expand
eq13=eq131[[1]]-eq8[[1]]==eq131[[2]]-eq8[[2]]
eq14=Thread[eq13*2*G'[z],Equal]//Factor
```

This equation holds in two cases.

The *first case*: $\{G_{zz}-2G(z)G_z=0$ and $H_{zz}-2H(z)G_z=0\}$ (sys1).
By integrating this system of ODEs, we can obtain the following solutions for $G(z)$: $G(z)=C_1$, $G(z)=-1/(z+C_1)$ (SolG2), $G(z)=B_1\tanh(B_2-B_1z)$ (SolG31), $G(z)=B_1/\coth(B_2-B_1z)$ (SolG4), and $G(z)=E_1\tan(E_1z+E_2)$ (SolG51), where C_1, B_1, B_2, E_1, and E_2 are arbitrary constants.

```
sys1={eq14[[1,3]]==0,eq14[[2]]==0}
{eq1sys1=Thread[sys1[[1]]+G''[z],Equal], SolG1=C1}
eq1sys11=Integrate[eq1sys1[[1]],z]==Integrate[eq1sys1[[2]],z]+C1
eq1sys12=Thread[eq1sys11/G[z]^2,Equal]//Expand
eq1sys13=Integrate[eq1sys12[[1]],z]==Map[Integrate[#,z]&,eq1sys12[[2]]]+C2
SolG=Solve[eq1sys13/.C1->0,G[z]]
Q1=Numerator[SolG[[1,1,2]]]*(-1)
Q2=Denominator[SolG[[1,1,2]]]*(-1)
SolG2=G[z]->Q1/Q2/.-C2->C1
SolG3=DSolve[eq1sys11,G[z],z]/.C[1]->C2
SolG31=SolG3/.Sqrt[C1]->B1/.C2*B1->B2
SolG4=SolG31/.{Tanh[x_]:>1/HoldForm[Coth[x]]}
SolG5=DSolve[sys1[[1]],G[z],z]/.{C[1]->C1,C[2]->C2}
SolG51=SolG5/.Sqrt[C1]->E1/.E1*C2->E2
```

Without full analysis, let us take the second solution (SolG2) $G(z)=-1/(z+C_1)$ and solve this system of ODEs for one special case in which $C_1=0$, $B_1=1$, and $B_2=0$. Then we can determine $G(z)=-1/z$ (SolG21), $H(z)=A_1z^2+A_2/z$ (SolH1), $W(z)=\ln|z|$ (SolW1), and $F(W)=A_1e^W+A_2e^{-2W}$ (SolF1). To determine $\phi(x)$ and $\psi(t)$, we substitute these results into the functional-differential equation (40.3.7.1) and arrive at the equation (EqPhiPsi):

$$(\phi+\psi)[A_1(\phi+\psi)^2+\phi_{xx}-\psi_{tt}]+A_2+(\psi_t)^2=(\phi_x)^2. \tag{40.3.7.2}$$

Then, by differentiating it with respect to t and x and by separating the variables, we arrive at the following system of ODEs (ODE1, ODE2):

$$\psi_{ttt}-6A_2\psi\psi_t=C\psi_t, \quad \phi_{xxx}+6A_2\phi\phi_x=C\phi_x,$$

where C is the separation constant.

```
Consts1={C1->0,B1->1,B2->0}
eq2sys1=sys1[[2]]/.SolG2/.D[SolG2,z]//Simplify
SolH=DSolve[eq2sys1,H[z],z]/.{C[2]->A2,C[1]->A1}
SolW=DSolve[SolG2/.trG2/.Rule->Equal,W[z],z]/.{C[2]->B2,C[1]->B1}
SolW1=SolW/.Consts1
SolF=Solve[(trH/.{F[W[z]]->F1}/.SolH/.SolW1/.Rule->Equal)[[1,1]],F1]//Expand
zSol=Solve[W==SolW1[[1,1,2]],z]
EqFW=(SolF/.W'[z]->1/z/.zSol)[[1,1]]
SolF1=F[W]->(EqFW/.Consts1)[[1,2]]
{SolG21=SolG2/.Consts1, SolH1=SolH/.Consts1}
EqPhiPsi=Thread[(eq8/.SolG21/.SolH1/.
         z->tr1[[1]])[[1]]*(phi[x]+psi[t]),Equal]//FullSimplify
```

```
EqPhiPsi1=D[EqPhiPsi,t,x]
EqPhiPsi2=Thread[EqPhiPsi1/(phi'[x]*psi'[t]),Equal]//Expand
EqPhiPsi3=EqPhiPsi2[[1]]-EqPhiPsi2[[2]]
ODE0=Thread[(Select[EqPhiPsi3,MemberQ[#,psi[t]] ||
             MemberQ[#,psi'''[t]]&]==C)*psi'[t],Equal]//Expand
ODE1=ODE0[[1]]*(-1)==ODE0[[2]]
ODE2=Thread[(Select[EqPhiPsi3,MemberQ[#,phi[x]] ||
             MemberQ[#,phi'''[x]]&]==C)*phi'[x],Equal]//Expand
```

These ODEs can be integrated twice (`ODE14`, `ODE24`):

$$\psi_t^2 = 2A_1\psi^3 + C\psi^2 + 2C_1\psi + 2C_2, \quad \phi_x^2 = -2A_1\phi^3 + C\phi^2 + 2C_3\phi + 2C_4,$$

where the C_i $(i = 1,\ldots,4)$ are arbitrary constants. By substituting these equations into the above Eq. (40.3.7.2) for $\phi(x)$ and $\psi(t)$, we can remove some constants; i.e., we obtain the relations $C_3 = -C_1$, $C_4 = C_2 + A_2/2$ (`trC3`, `trC4`), and the equations become (`ODE15`, `ODE25`)

$$\psi_t^2 = 2A_1\psi^3 + C\psi^2 + 2C_1\psi + 2C_2, \quad \phi_x^2 = -2A_1\phi^3 + C\phi^2 - 2C_1\phi + 2C_2 + A_2.$$

The solutions of these equations can be found and analyzed. For the other cases of solutions for $G(z)$, the analysis can be performed in a similar way.

```
ODE11=Map[Integrate[#,t]&,ODE1[[1]]]==Integrate[ODE1[[2]],t]+C1
ODE12=Thread[ODE11*psi'[t],Equal]//Expand
ODE13=Thread[(Map[Integrate[#,t]&,ODE12[[1]]]==
              Map[Integrate[#,t]&,ODE12[[2]]]+C2)*2,Equal]//Expand
ODE21=Map[Integrate[#,x]&,ODE2[[1]]]==Integrate[ODE2[[2]],x]+C3
ODE22=Thread[ODE21*phi'[x],Equal]//Expand
ODE23=Thread[(Map[Integrate[#,x]&,ODE22[[1]]]==
              Map[Integrate[#,x]&,ODE22[[2]]]+C4)*2,Equal]//Expand
ODE14=Solve[ODE13/.psi'[t]^2->Psi2,Psi2]/.Psi2->psi'[t]^2
ODE24=Solve[ODE23/.phi'[x]^2->Phi2,Phi2]/.Phi2->phi'[x]^2
trPsiDD=Thread[D[ODE14[[1,1]],t]/psi'[t]/2,Rule]//Expand
trPhiDD=Thread[D[ODE24[[1,1]],x]/phi'[x]/2,Rule]//Expand
EqConsts=EqPhiPsi/.ODE14/.ODE24/.trPsiDD/.trPhiDD//Expand
EqConsts1=EqConsts[[1,1,2]]-EqConsts[[1,1,1]]
trC3=Solve[Coefficient[EqConsts1/.phi[x]->0,psi[t],1]==0,C3]
trC4=Solve[(EqConsts1/.psi[t]->0/.phi[x]->0)==0,C4]
ODE15=(ODE14/.Consts1)[[1,1]]
ODE25=(ODE24/.trC3/.trC4)[[1,1,1,1]]
```

The *second case*. First, let us represent the functional-differential equation (40.3.7.1) in the form $f(x) + g(t) = q(z)$ (`FunDiffEq1`), whose solution is $\{f(x) = A\phi(x) + B,\ g(t) = A\psi(t) - B + C,\ q(z) = Az + C\}$ (`SolFunDiffEq1`). Therefore, we arrive at the equation (`eq141`)

$$-\psi_t^2 + \phi_x^2 = \frac{H_{zz} - 2H(z)G_z}{-G_{zz} + 2G(z)G_z}, \quad \text{i.e.,} \quad f(x) + g(t) = q(z),$$

where $g(t) = -\psi_t^2$ (`trg`), $f(x) = \phi_x^2$ (`trf`), $q(z) = (H_{zz} - 2H(z)G_z)/(-G_{zz} + 2G(z)G_z)$ (`trq`). For the second case, the functional-differential equation (40.3.7.1) holds if (`sys2`)

$$-\psi_t^2 = A\psi - B + C, \quad \phi_x^2 = A\phi + B, \quad \frac{H_{zz} - 2H(z)G_z}{-G_{zz} + 2G(z)G_z} = Az + C.$$

By integrating the first two ODEs in the above system, we can obtain the following solutions for $\phi(x)$ and $\psi(t)$ (`SolPhiPsi20`, `SolPhiPsi21`):

$$\phi(x) = \pm\sqrt{B}x + D_2, \quad \psi(t) = \pm\sqrt{B - C}\,t + D_1 \quad \text{if} \quad A = 0,$$

$$\phi(x) = \frac{A}{4}(D_2^2 \pm 2D_2x + x^2) - \frac{B}{A}, \quad \psi(t) = \frac{A}{4}(-D_1^2 \pm 2D_1t - t^2) + \frac{B - C}{A} \quad \text{if} \quad A \neq 0,$$

where D_1 and D_2 are arbitrary constants. We can observe that the function $F(W)$ is arbitrary in both cases.

Finally, we can obtain the traveling-wave solution $W = W(z) = W(E_1 x + E_2 t)$ (where $E_1 = E_2 = -\sqrt{B}$, S1) of the nonlinear Klein–Gordon equation for $A = 0$ and a solution of the form $W = W(z) = W(x^2 - t^2)$ (S2) for $A \neq 0$.

```
{eq14, termG=Level[eq14[[1]],1][[3]]}
eq141=Thread[eq14/termG,Equal]//Simplify
trg=g[t]->-Select[eq141[[2]],MemberQ[#,psi'[t]]&]
trf=f[x]->eq141[[1]]
trq=q[z]->Select[eq141[[2]],MemberQ[#,1/termG]&]
FunDiffEq1=f[x]+g[t]==q[z]
SolFunDiffEq1={f[x]->A*phi[x]+B,g[t]->A*psi[t]-B+C,q[z]->A*z+C}
sys2=SolFunDiffEq1/.trf/.trg/.trq/.Rule->Equal
sys20=sys2/.A->0/.Rule->Equal
SolPhiPsi20={DSolve[sys20[[1]],phi[x],x]/.{C[1]->D2},
             DSolve[sys20[[2]],psi[t],t]}/.{C[1]->D1}
SolPhiPsi2={DSolve[sys2[[1]],phi[x],x]/.{C[1]->D2},
            DSolve[sys2[[2]],psi[t],t]}/.{C[1]->D1}
SolPhiPsi21=Collect[SolPhiPsi2,A]
S10=SolPhiPsi20[[1,1,1,2]]+SolPhiPsi20[[2,1,1,2]]
{trG2, trH, trF=F[W[z]]->Sin[W]}
trFD=F'[W[z]]->D[trF[[2]],W]
trFD2=F''[W[z]]->D[trF[[2]],{W,2}]
EqW=sys20[[3]]/.trH/.trG2/.D[trH,{z,2}]/.D[trG2,z]/.
                D[trG2,{z,2}]/.trF/.trFD/.trFD2//Simplify
EqW11=EqW/.{Sin[x_]:>2*Tan[x/2]/(1+Tan[x/2]^2),
            Cos[x_]:>(1-Tan[x/2]^2)/(1+Tan[x/2]^2)}//Factor
EqW12=EqW11/.Tan[W/2]->Tan[W[z]/2]
DSolve[EqW12,W[z],z]
{Consts2={A->4,C->0,D1->0,D2->0}, S10/.Consts2}
SolPhiPsi2[[1,1,1,2]]+SolPhiPsi2[[2,1,1,2]]/.Consts2//Expand
```

⊙ *References for Section* 40.3: G. I. Taylor (1950), R. B. Lindsay (1973), V. A. Galaktionov (1990, 1995), W. Malfliet (1992), L. I. Sedov (1993), E. J. Parkes and B. R. Duffy (1996), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002), A. D. Polyanin and A. V. Manzhirov (2006), A. M. Wazwaz (2006), J. H. He and X. H. Wu (2006), J. H. He and M. A. Abdou (2007).

40.4. Analytical Solutions of Nonlinear Systems

In this section, we consider the most important analytical methods for solving nonlinear systems of partial differential equations. In particular, we construct traveling-wave solutions of nonlinear systems of first- and second-order equations and generalized separable solutions of second-order nonlinear systems [see Polyanin and Manzhirov (2006), Schiesser and Griffiths (2009)].

40.4.1. Constructing Traveling-Wave Solutions

Nonlinear systems of first-order PDEs. Consider nonlinear systems of the form

$$u_x = F(u,v),$$
$$v_t = G(u,v).$$

Such systems can describe various physical, chemical, and biological processes. It is known that this system can admit *traveling-wave solutions*; i.e., we seek a solution in the form $u = W_1(\xi)$, $v = W_2(\xi)$ with $\xi = kx - \lambda t$, where k and λ are arbitrary constants.

Example 40.27. *A nonlinear system of first-order PDEs.* As a special case, consider the system

$$u_x = u\ln(v), \quad v_t = v\ln(u).$$

We verify that this system admits exact traveling-wave solutions and the functions $W_1(\xi)$ and $W_2(\xi)$ are described by the autonomous system of ordinary differential equations (`sys3`)

$$W_{1\xi}k - W_1\ln(W_2) = 0, \quad -W_{2\xi}\lambda - W_2\ln(W_1) = 0.$$

We construct a traveling-wave solution and verify that it is an exact solution of the given nonlinear system as follows:

```
Off[Solve::"ifun"]; tr1=k*x-\[Lambda]*t->\[Xi]
F[u_,v_]:=u*Log[v]; G[u_,v_]:=v*Log[u];
sys1[w1_,w2_]:={D[w1,x]-F[w1,w2]==0, D[w2,t]-G[w1,w2]==0};
sys2=sys1[W1[tr1[[1]]],W2[tr1[[1]]]]//Expand
sys3=sys2/.tr1
Sol1=DSolve[sys3,{W1[\[Xi]],W2[\[Xi]]},\[Xi]];
Sol11=Assuming[{{k,\[Lambda],\[Xi]}>0},Sol1/.{C[1]->1,C[2]->0}//
              TrigToExp//PowerExpand]
n=Length[Sol11]
Sols=Table[Assuming[{{k,\[Lambda],\[Xi]}>0},Sol11[[i]]//FullSimplify],{i,1,n}]
Sol12=Sols[[4]]
Sol13=Assuming[{{k,\[Lambda],\[Xi]}>0},ExpandAll[Sol12,Trig->True]//PowerExpand]
trW1=Sol13[[2]]/.Sin[2*x_]:>2*Sin[x]*Cos[x]
trW2=(Sol13[[1]]/.\[Lambda]*Cos[x_]^2-\[Lambda]*Sin[x_]^2:>\[Lambda]*Cos[2*x])
sys4=(sys3/.trW1/.D[trW1,\[Xi]]/.trW2/.D[trW2,\[Xi]])//FullSimplify
Assuming[{{k,\[Lambda],\[Xi]}>0},
   (sys4//TrigToExp//PowerExpand//FullSimplify)//PowerExpand]
```

The result, obtained with Mathematica Ver. 7, reads

$$\text{trW1} = \text{W1}[\xi] \to e^{\frac{\sqrt{2}\sqrt{\lambda}\,\text{Sin}\left[\frac{\xi}{\sqrt{k}\sqrt{\lambda}}\right]}{\sqrt{k}}}$$

$$\text{trW2} = \text{W2}[\xi] \to e^{\frac{\sqrt{\lambda+\lambda\,\text{Cos}\left[\frac{2\xi}{\sqrt{k}\sqrt{\lambda}}\right]}}{\sqrt{\lambda}}}$$

Nonlinear systems of second-order PDEs. Consider nonlinear systems of the form

$$u_t = a_1 u_{xx} + F(u,v),$$
$$v_t = a_2 v_{xx} + G(u,v),$$

which describe reaction-diffusion phenomena. These systems can admit traveling-wave solutions; i.e., we seek a solution in the form $u = W_1(\xi)$, $v = W_2(\xi)$, $\xi = kx - \lambda t$, where k and λ are arbitrary constants.

Example 40.28. *A nonlinear system of second-order PDEs.* As a special case, consider the system

$$u_t = a_1 u_{xx} + u\ln v, \quad v_t = a_2 v_{xx} + v\ln u.$$

We show that this system admits exact traveling-wave solutions and the functions $W_1(\xi)$ and $W_2(\xi)$ are described by the autonomous system of ordinary differential equations (`sys3`)

$$-\lambda W_{1\xi} - a_1 k^2 W_{1\xi\xi} - W_1\ln(W_2) = 0, \quad -\lambda W_{2\xi} - a_2 k^2 W_{2\xi\xi} - W_2\ln(W_1) = 0.$$

```
F[u_,v_]:=u*Log[v]; G[u_,v_]:=v*Log[u];
sys1[w1_,w2_]:={D[w1,t]-a1*D[w1,{x,2}]-F[w1,w2]==0,
                D[w2,t]-a2*D[w2,{x,2}]-G[w1,w2]==0};
tr1=k*x-lambda*t->xi
sys2=sys1[W1[tr1[[1]]],W2[tr1[[1]]]]//Expand
sys3=sys2/.tr1
```

40.4.2. Constructing Generalized Separable Solutions

Now let us generalize the method of generalized separation of variables considered in Section 40.3.7. to systems of second-order nonlinear partial differential equations.

Example 40.29. *Nonlinear systems of second-order PDEs.* Consider a system of nonlinear second-order PDEs of the following form:

$$u_t = au_{xx} + uF(bu-cv) + G_1(bu-cv),$$
$$v_t = av_{xx} + vF(bu-cv) + G_2(bu-cv),$$

where $F(z)$, $G_1(z)$, and $G_2(z)$ are arbitrary functions.

By applying the method of generalized separation of variables and by seeking exact solutions in the form

$$u(x,t) = \phi_1(t)\theta(x,t) + \psi_1(t), \quad v(x,t) = \phi_2(t)\theta(x,t) + \psi_2(t), \tag{40.4.2.1}$$

we find exact solutions of this nonlinear system. We assume that the functions $\phi_i(t)$ and $\psi_i(t)$ $(i = 1,2)$ are selected so that the two equations in the original nonlinear system can be reduced to a single equation for the function $\theta(x,t)$. We also assume that the arguments $bu-cw$ of the functions F, G_1, and G_2 in the original system depend only on t; i.e., $(bu-cw)_x = (b\phi_1 - c\phi_2)\theta_x = 0$. By introducing the new function $\phi(t)$, we can obtain the following conditions on $\phi_i(t)$ $(i = 1,2)$: $\phi_1 = c\phi$ and $\phi_2 = b\phi$.

By substituting relations (40.4.2.1) describing the solution form and the above-derived conditions on $\phi_i(t)$ $(i = 1,2)$ into the original system, we show that the original system can be reduced to the form (`sys6`)

$$\left(\frac{\phi_t}{\phi} - F(z)\right)\theta - a\theta_{xx} + \theta_t + \frac{1}{c\phi}\Big(\psi_{1t} - G_1(z) - F(z)\psi_1\Big) = 0,$$
$$\left(\frac{\phi_t}{\phi} - F(z)\right)\theta - a\theta_{xx} + \theta_t + \frac{1}{b\phi}\Big(\psi_{2t} - G_2(z) - F(z)\psi_2\Big) = 0, \tag{40.4.2.2}$$

where $z = b\psi_1 - c\psi_2$.

By solving the resulting system of equations, we obtain the following system of ordinary differential equations for the functions $\phi(t)$ and $\psi_i(t)$ $(i = 1,2)$: $\{\phi_t/\phi = F(z),\ \psi_{1t} = G_1(z) + F(z)\psi_1,\ \psi_{2t} = G_2(z) + F(z)\psi_2\}$ (`eq1`, `eq2`, `eq3`) and the linear diffusion equation $-a\theta_{xx} + \theta_t = 0$ (`eq4`) for the function $\theta(x,t)$.

Finally, we construct the exact solution of the original nonlinear system: $u = c\phi\theta + \psi_1$ (`Solu`), $v = b\phi\theta + \psi_2$ (`Solv`), where $\phi = \exp\left(\int F(b\psi_1 - c\psi_2\,dt\right)$ (`Solphi`), the functions $\psi_i(t)$ $(i = 1,2)$ are described by the ODEs obtained above, and the function $\theta(x,t)$ is a solution of the linear diffusion equation.

```
sys1[u_,v_]:={D[u,t]-a*D[u,{x,2}]-u*F[b*u-c*v]-G1[b*u-c*v],
              D[v,t]-a*D[v,{x,2}]-v*F[b*u-c*v]-G2[b*u-c*v]};
{tr1=phi1[t]*theta[x,t]+psi1[t], tr2=phi2[t]*theta[x,t]+psi2[t]}
{sys2=sys1[W1,W2]//Expand, sys3=sys1[tr1,tr2]//Expand}
cond1=D[b*u[x,t]-c*v[x,t],x]
cond11=(cond1/.{D[u[x,t],x]->D[tr1,x],D[v[x,t],x]->D[tr2,x]}//Expand)==0
Sol1=Solve[cond11,phi2[t]]
{tr31=phi1[t]->c*phi[t], tr32=Sol1/.tr31}
sys4=(sys3/.tr31/.tr32/.D[tr31,t]/.D[tr32,t]//Simplify)[[1,1]]
sys5={sys4[[1]]/c/phi[t],sys4[[2]]/b/phi[t]}//Expand
sys6=Collect[Expand[Factor[sys5]],{theta[x,t],c,b}]
```

```
{sys61L=Level[sys6[[1]],1], sys62L=Level[sys6[[2]],1]}
{eq1=sys61L[[1]]/theta[x,t]==0, eq2=sys61L[[2]]}
{eq3=sys62L[[2]],   eq4=sys61L[[3]]+sys61L[[4]]}
Solphi11=DSolve[eq1,phi[t],t]/.C[1]->1
Solphi12=Solphi11/.K[1]->t
Solphi=Solphi12/.Solphi12[[1,1,2,2,2]]->t
Solu=u[x,t]->tr1/.tr31/.Solphi
Solv=v[x,t]->tr2/.tr32/.Solphi
```

⊙ *References for Section* 40.4: A. D. Polyanin and A. V. Manzhirov (2006), W. E. Schiesser and G. W. Griffiths (2009).

40.5. Numerical Solutions and Their Visualizations

In this section, we first consider the construction of numerical and graphical solutions of various initial-boundary value problems using predefined Mathematica functions, e.g., for second-order PDEs (the Burgers equation and the nonlinear wave equation). Then we consider finite-difference approximations and solve various important initial-boundary value problems, e.g., for the Burgers equation, the inviscid Burgers equation, the nonlinear wave equation, and the nonlinear Poisson equation.

40.5.1. Constructing Numerical Solutions in Terms of Predefined Functions

In Mathematica, with the aid of the predefined function `NDSolve`, it is possible to obtain approximate numerical solutions of various linear and nonlinear PDE problems (initial-boundary value problems). This can be done by the Mathematica system applying the *method of lines*. We note that in Mathematica (Ver. ≤ 7) it is only possible to solve evolution equations numerically by using `NDSolve`. In addition, it is possible to explicitly specify the method of lines for solving PDEs and the proper suboptions for the method of lines.

```
NDSolve[{PDEs,IC,BC}, DepVars, IndVars, Ops]
NDSolve[{PDE,IC,BC},  u, {x,x1,x2},{t,t1,t2},...]
NDSolve[{PDEs,IC,BC}, {u1,...,un}, {x,x1,x2},{t,t1,t2},...]
NDSolve[{PDE,IC,BC},u,{x,x1,x2},{t,t1,t2},Method->{m,subOps}]
Options[NDSolve‘MethodOfLines]
```

Description:

`NDSolve`, finding numerical solutions to PDEs problems (initial boundary value problems), where `DepVars` and `IndVars` are the dependent and independent variables, respectively.

`NDSolve,Method`, finding numerical solutions to PDEs problems by the method of lines with some specific suboptions `subOps`.

The option `Method` and the most important suboptions:

```
Method->{"MethodOfLines","SpatialDiscretization"->{
 "TensorProductGrid","MinPoints"->val,"MaxPoints"->val,
 "MaxStepSize"->val,"PrecisionGoal"->val,"DifferenceOrder"->val}}
```

Now we schematically write out the numerical solution `Sol` with the aid of `NDSolve` and then use it to obtain various visualizations (e.g., `Plot`, `Plot3D`, `Animate`) and numerical values (`NumVals`).

```
Sol=NDSolve[{PDE,IC,BC},u,{x,x1,x2},{t,t1,t2},ops]
Plot[Evaluate[u[x,tk]/.Sol],{x,x1,x2},ops]
NumVals=Evaluate[u[xk,tk]/.Sol]
Plot3D[Evaluate[u[x,t]/.Sol],{x,x1,x2},{t,t1,t2},ops]
Animate[Plot[Evaluate[u[x,t]/.Sol],{x,x1,x2},ops],{t,t1,t2},ops]
```

Example 40.30. *Numerical and graphical solutions. Burgers equation.* We find numerical and graphical solutions of the following initial-boundary value problem for the Burgers equation:

$$u_t = \nu u_{xx} + u u_x; \qquad u(x,0) = A\sin(\pi x), \quad u(0,t) = u(1,t) = 0,$$

with initial amplitude $A = 0.4$ and kinematic viscosity $\nu = 0.009$. The numerical solution of this initial-boundary value problem for $x \in [0,1]$ and $t \in [0,4]$ (the 3D plot and the contour plot) is shown in Fig. 40.5.

```
f[x_]:=A*Sin[Pi*x/L];
{nu=0.009, A=0.4, S=0.01, NP=100, nN=3, L=1, L1={Red,Blue,Green}, L2={0,1/2,1}}
PDE1=D[u[x,t],t]-nu*D[u[x,t],{x,2}]-u[x,t]*D[u[x,t],x]==0
{IC={u[x,0]==f[x]}, BC={u[0,t]==0,u[L,t]==0}}
Sol1=NDSolve[{PDE1,IC,BC},u,{x,0,1},{t,0,4},MaxStepSize->S];
Do[G[i]=Plot[Evaluate[u[x,L2[[i]]]/.Sol1],{x,0,1},PlotRange->All,
        PlotStyle->{L1[[i]],Thickness[0.01]},PlotPoints->NP*2],{i,1,nN}];
Show[Table[G[i],{i,1,nN}]]
NumVals1=Evaluate[u[1/2,Pi]/.Sol1]; Print[NumVals1];
G3D=Plot3D[Evaluate[u[x,t]/.Sol1],{x,0,1},{t,0,4},
    ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->{-1,2,1}];
GCP=ContourPlot[Evaluate[u[x,t]/.Sol1],{x,0,1},{t,0,4},
    ColorFunction->Hue,ImageSize->300];
GraphicsRow[{G3D,GCP}]
Animate[Plot[Evaluate[u[x,t]/.Sol1],{x,0,1},PlotRange->{0,0.4},
        PlotStyle->{Blue,Thickness[0.01]},PlotPoints->NP],
        {t,0,4},AnimationRate->0.5]
```

Example 40.31. *Numerical and graphical solutions. Nonlinear wave equation.* We find numerical and graphical solutions of the nonlinear wave equation subject to the following initial and boundary conditions:

$$u_{tt} = a^2 (e^{\lambda u} u_x)_x; \quad u(x,0) = 0, \quad u_t(x,0) = \sin(2\pi x), \quad u(0,t) = u(1,t) = 0,$$

in the domain $\{0 < x < 1,\ 0 < t < \infty\}$, where $a = 0.1$. Numerical solution of this initial-boundary value problem for $x \in [0,1]$ and $t \in [0,10]$ (the 3D and contour plots) is shown in Fig. 40.6.

```
<<DifferentialEquations`InterpolatingFunctionAnatomy`
{a=0.1, lambda=-0.4, S=0.01, NP=100}
PDE1=D[u[x,t],{t,2}]-a^2*D[Exp[lambda*u[x,t]]*D[u[x,t],x],x]==0
IC={(D[u[x,t],t]/.{t->0})==Sin[2*Pi*x],u[x,0]==0}
BC={u[0,t]==0,u[1,t]==0}
Sol1=NDSolve[{PDE1,IC,BC},u,{x,0,1},{t,0,10},MaxStepSize->S,PrecisionGoal->2];
f1=u/.First[Sol1];
```

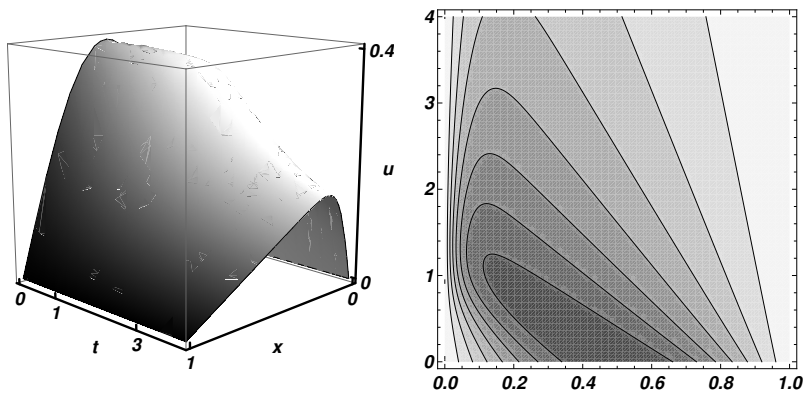

Figure 40.5. Numerical solution of the nonlinear diffusion equation: the 3D and contour plots for $x \in [0,1]$ and $t \in [0,4]$.

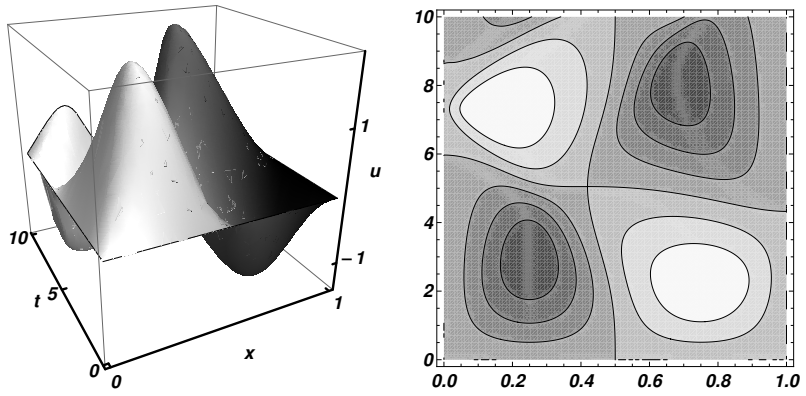

Figure 40.6. Numerical solution of the nonlinear wave equation: the 3D and contour plots for $x \in [0,1]$ and $t \in [0,10]$.

```
Map[Length,InterpolatingFunctionCoordinates[f1]]
{nN=3, L1={Red,Blue,Green}, L2={0.1,1/2,1}}
Do[G[i]=Plot[Evaluate[f1[x,L2[[i]]]],{x,0,1},
        PlotStyle->{L1[[i]],Thickness[0.01]},PlotPoints->NP*2,
        PlotRange->{All,{-1,1}}],{i,1,nN}];
Show[Table[G[i],{i,1,nN}]]
NumVals1=f1[1/2,Pi]; Print[NumVals1];
G3D=Plot3D[Evaluate[u[x,t]/.Sol1],{x,0,1},{t,0,10},
    ColorFunction->Function[{x,y},Hue[x]],BoxRatios->1,ViewPoint->{-1,-2,1}];
GCP=ContourPlot[Evaluate[u[x,t]/.Sol1],{x,0,1},{t,0,10},
    ColorFunction->Hue,ImageSize->300,PlotPoints->100];
GraphicsRow[{G3D,GCP}]
Animate[Plot[Evaluate[u[x,t]/.Sol1],{x,0,1},PlotRange->{-2,2},
        PlotStyle->{Blue,Thickness[0.01]},PlotPoints->NP],
        {t,0,10},AnimationRate->0.5]
```

40.5.2. Constructing Finite-Difference Approximations

Now we show the helpful role of computer algebra systems for generating and applying various finite-difference approximations for constructing numerical solutions of linear and nonlinear PDEs.

To approximate linear and nonlinear PDEs by finite differences, we have to generate a *mesh* (or a grid) in a domain $\mathcal{D}$; e.g., $\mathcal{D} = \{a < x < b, c < t < d\}$. The mesh can be of various types, e.g., rectangular, along the characteristics, polar, etc. We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable is $u(x,t)$ in the given PDE.

We write h and k for the line spacings and define the *mesh points* as follows: $X_i = a + ih$ and $T_j = c + jk$ ($i = 0,\ldots,NX$, $j = 0,\ldots,NT$) so that $h = (b-a)/NX$ and $k = (d-c)/NT$. We calculate approximations to the solution at these mesh points; these approximate points will be denoted by $U_{i,j} \approx u(X_i, T_j)$. We approximate the derivatives in a given equation by finite differences (of various types) and then solve the resulting difference equations.

Example 40.32. *Nonlinear parabolic equations. Burgers equation. Explicit method.* Consider the initial-boundary value problem for the Burgers equation

$$u_t = \nu u_{xx} + uu_x; \quad u(x,0) = \sin(\pi x), \quad u(0,t) = 0, \quad u(1,t) = 0$$

in the domain $\{0 \le x \le 1,\ 0 < t < T\}$, where $T = 0.4$ and $\nu = 0.009$.

Let us generate the rectangular mesh: $X = a + ih$ and $T = jk$ ($i = 0,\ldots,NX$, $j = 0,\ldots,NT$, $h = (b-a)/NX$, $k = T/NT$). We denote the approximate solution at the mesh point (i,j) by $U_{i,j}$.

In the *forward difference method*, the second derivative u_{xx} is replaced by the central difference approximation (CDA), and the first derivatives u_t and u_x are replaced by the forward difference approximation (FWDA). The FD scheme for the nonlinear Burgers equation has the form

$$U_{i,j+1} = U_{i,j} + r(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) + (k/h)U_{i,j}(U_{i+1,j} + U_{i,j}),$$

where $r = \nu k/h^2$. In this explicit FD scheme, the unknown value $U_{i,j+1}$ (on $j+1$ step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the j th step). We construct the approximate numerical solution of the initial-boundary value problem by applying the forward difference method as follows:

```
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,1},{0,1.05}},Joined->True];
{nu=0.009, NX=100, NT=100, a=0, b=1, L=1., T=0.4,
 h=(b-a)/NX//N, k=T/NT//N, r=nu*k/h^2}
f[x_]:=Sin[Pi*x/L]//N; Table[X[i]=a+i*h,{i,0,NX}];
IC=Table[U[i,0]->f[X[i]],{i,0,NX}];
BC={Table[U[a,j]->0,{j,0,NT}], Table[U[NX,j]->0,{j,0,NT}]};
IBC=Flatten[{IC,BC}]
FD[i_,j_]:=U[i,j]+r*(U[i+1,j]-2*U[i,j]+U[i-1,j])+k/h*U[i,j]*(U[i+1,j]-U[i,j]);
Do[U[i,j+1]=FD[i,j]/.IBC,{j,0,NT},{i,1,NX-1}];
G[j_]:=ListPlot[Table[{X[i],U[i,j]/.IBC},{i,0,NX}],
       PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
Grs=Evaluate[Table[G[j],{j,0,NT}]]; ListAnimate[Grs]
```

Example 40.33. *Inviscid Burgers equation. Explicit method.* Now consider a particular case of the Burgers equation, the inviscid Burgers equation; i.e., $u_t + uu_x = 0$. This equation is of hyperbolic type. The initial-boundary value problem for this equation has the form

$$u_t + uu_x = 0; \quad u(x,0) = \arctan(4x) + 2, \quad u(a,t) = 0$$

in the domain $\{a \le x \le b,\ 0 < t < T\}$, where $a = -4$, $b = 6$, and $T = 0.4$. Note that this equation can be rewritten in the *conservation form*, i.e., as the physical conservation law of the form $u_t + F_x(u) = 0$, where $F(u) = \frac{1}{2}u^2$. For the discretization of the problems, we use this form of the Burgers equation.

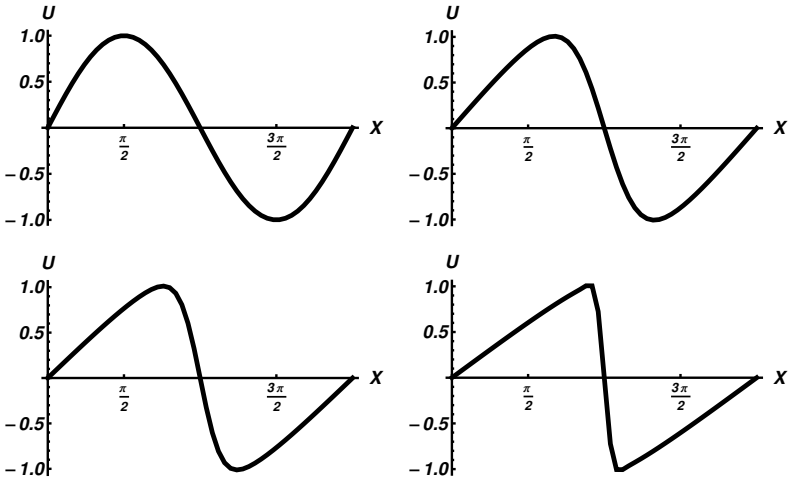

Figure 40.7. Inviscid Burgers equation: evolution of the wave profile for $t = 0, 0.56, 0.84, 1.4$.

Let us generate the rectangular mesh: $X = a + ih$ and $T = jk$ ($i = 0, \ldots, NX$, $j = 0, \ldots, NT$, $h = (b-a)/NX$, $k = T/NT$). We denote the approximate solution at the mesh point (i, j) by $U_{i,j}$.

In the *Lax method*, the derivative u_t is replaced by the forward difference approximation (FWDA), and the derivative $F_x(u)$ is replaced by the central difference approximation (CDA) (together with the corresponding substitution). The FD scheme for the nonlinear diffusion equation has the form

$$U_{i,j+1} = \frac{1}{2}\left(U(i+1,j) + U(i-1,j)\right) - \frac{1}{4}(k/h)\left(U^2(i+1,j) - U^2(i-1,j)\right).$$

In this explicit FD scheme, the unknown value $U_{i,j+1}$ (on $j+1$ step) is determined from the three known values $U_{i-1,j}$, $U_{i,j}$, and $U_{i+1,j}$ (at the j th step). This FD scheme is stable for $k/h < 1/|u|$; i.e., the stability condition depends on the solution. We construct the approximate numerical solution of the initial-boundary value problem by applying the Lax method as follows:

```
SetOptions[ListPlot,ImageSize->300,PlotRange->{All,All},Joined->True];
{NX=50,NT=9,a=-4,b=6,T=0.4,h=(b-a)/NX//N,k=T/NT//N}
f[x_]:=ArcTan[4*x]+2//N; Table[X[i]=a+i*h,{i,0,NX}];
IC=Table[U[i,0]->f[X[i]],{i,0,NX}]; BC={Table[U[a,j]->0,{j,0,NT}]};
IBC=Flatten[{IC,BC}]
FD[i_,j_]:=0.5*(U[i+1,j]+U[i-1,j])-k/(2*h)*(U[i+1,j]^2/2-U[i-1,j]^2/2);
Do[U[i,j+1]=FD[i,j]/.IBC,{j,0,NT},{i,1,NX-1}];
{L1={1,3,5,7,9},NL1=Length[L1]}
Do[G[j]=ListPlot[Table[{X[i],U[i,L1[[j]]]/.IBC},{i,0,NX}],
   PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}],{j,1,NL1}];
GraphicsGrid[{{G[1],G[2]},{G[3],G[4],G[5]}}]
Show[Table[G[i],{i,1,NL1}],Frame->True,Axes->False]
```

In particular, for the initial-boundary value problem $u_t + uu_x = 0$, $u(x,0) = \sin(x)$, $u(0,t) = 0$ in the domain $\mathcal{D} = \{0 \le x \le 2\pi,\ 0 < t < 1.4\}$, the evolution of the wave profile as t increases (leading to a breakdown of the differentiability of the solution) is shown in Fig. 40.7. The numerical solution can be constructed as follows:

```
SetOptions[ListPlot,ImageSize->300,PlotRange->{All,All},Joined->True];
{NX=50, NT=10, a=0, b=N[2*Pi], T=1.4, h=(b-a)/NX//N, k=T/NT//N}
f[x_]:=Sin[x]//N; Table[X[i]=a+i*h,{i,0,NX}];
```

```
IC=Table[U[i,0]->f[X[i]],{i,0,NX}]; BC={Table[U[a,j]->0,{j,0,NT}]};
IBC=Flatten[{IC,BC}]
FD[i_,j_]:=0.5*(U[i+1,j]+U[i-1,j])-k/(2*h)*(U[i+1,j]^2/2-U[i-1,j]^2/2);
Do[U[i,j+1]=FD[i,j]/.IBC,{j,0,NT},{i,1,NX-1}]; LT={0,4,6,10};
Do[G[j]=ListPlot[Table[{X[i],U[i,LT[[j]]]/.IBC},{i,0,NX}],
        PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}],{j,1,4}];
GraphicsGrid[{{G[1],G[2]},{G[3],G[4]}}]
```

Example 40.34. *Nonlinear wave equation. Explicit method.* Consider the initial-boundary value problem for the nonlinear wave equation describing the motion of a fixed string,

$$u_{tt} = \frac{1}{c^2}\,u_{xx} + e^u; \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = 0, \quad u(\tfrac{1}{2},t) = 0$$

in the domain $\{0 \le x \le L,\ 0 \le t \le T\}$, where $f(x) = 0$, $g(x) = \sin(4\pi x)$, $T = 0.5$, and $c = 1/(4\pi)$.

In the *explicit central difference method*, each second derivative is replaced by the central difference approximation (CDA). The FD scheme for the nonlinear wave equation has the form

$$U_{i,j+1} = 2(1-r)U_{i,j} + r(U_{i+1,j} + U_{i-1,j}) - U_{i,j-1} + \exp(U(i,j))(ck)^2,$$

where $r = (ck/h)^2$. In this FD scheme, we have one unknown value $U_{i,j+1}$ that depends explicitly on the four known values $U_{i,j}$, $U_{i+1,j}$, $U_{i-1,j}$, and $U_{i,j-1}$ at the previous time steps j and $j-1$. To start the process, we have to know the values of U on the time steps $j = 0$ and $j = 1$. Thus, we can define the initial conditions at these time steps: $U_{i,0} = f(X_i)$, $U(X_i,0)_t \approx (U_{i,1} - U_{i,0})/k = g(X_i)$, and $U_{i,1} = f(X_i) + kg(X_i)$.

We construct the approximate numerical solution of the initial-boundary value problem by applying the explicit finite difference method and plot the numerical solution as follows:

```
SetOptions[ListPlot,ImageSize->500,PlotRange->{{0,0.5},{-0.5,0.5}},Joined->True];
{L=0.5, T=2., NX=50, NT=50, c=0.25, h=L/NX, k=T/NT, r=(c*k/h)^2}
f[x_]:=0; g[x_]:=Sin[4*Pi*x]//N;
IC={Table[U[i,0]->f[i*h],{i,1,NX-1}],
    Table[U[i,1]->f[i*h]+k*g[i*h],{i,1,NX-1}]};
BC={Table[U[0,j]->0,{j,0,NT}],Table[U[NX,j]->0,{j,0,NT}]};
IBC=Flatten[{IC,BC}]
FD[i_,j_]:=2*(1-r)*U[i,j]+r*(U[i+1,j]+U[i-1,j])-U[i,j-1]+Exp[U[i,j]]*(c*k)^2;
Do[U[i,j+1]=FD[i,j]/.IBC,{j,1,NT-1},{i,1,NX-1}];
G[j_]:=ListPlot[Table[{i*h,U[i,j]/.IBC},{i,0,NX}],
       PlotStyle->{Blue,Thickness[0.01]},AxesLabel->{"X","U"}];
Grs=Evaluate[Table[G[j],{j,0,NT}]]; ListAnimate[Grs]
```

Example 40.35. *Nonlinear Poisson equation. Boundary value problem.* Consider the boundary value problem for the two-dimensional nonlinear Poisson equation

$$u_{xx} + u_{yy} = \sin u; \quad u(x,c) = 0, \quad u(x,d) = 0, \quad u(a,y) = 0, \quad u(b,y) = \cos y$$

describing a potential field $u(x,y)$ in a bounded rectangular domain $\{a \le x \le b,\ c \le y \le d\}$, where $a = 0$, $b = \pi$, $c = 0$, and $d = 2\pi$.

Let us generate the rectangular mesh: $x = a + ih$ and $y = c + jk$ ($i = 0,\ldots,NX$, $j = 0,\ldots,NY$, $h = (b-a)/NX$, $k = (d-c)/NY$). We denote the approximate solution at the mesh point (i,j) by $U_{i,j}$. The second derivatives in the Poisson equation can be replaced by the central difference approximation (CDA). The FD scheme has the form

$$2(1+r)U_{i,j} - U_{i+1,j} - U_{i-1,j} - rU_{i,j+1} - rU_{i,j-1} = h^2 \sin(U_{i,j}),$$

where $r = (h/k)^2$.

We construct an approximate numerical solution of the boundary value problem by applying the above explicit finite difference scheme as follows:

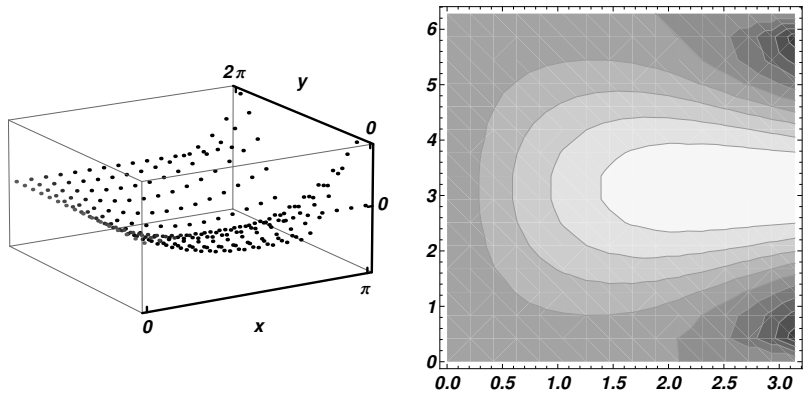

Figure 40.8. Numerical solution of the nonlinear Poisson equation: the 3D and contour plots inside the domain $\mathcal{D}$.

```
SetAttributes[{U,x,y},NHoldAll];
{a=0,b=Pi,c=0,d=2*Pi,NX=15,NY=15,nd=15,h=(b-a)/NX,k=(d-c)/NY,r=(h/k)^2}
X=Table[x[i]->a+i*h,{i,0,NX}];
Y=Table[y[j]->c+j*k,{j,0,NY}];
FD[i_,j_]:=N[2*(1+r)*U[i,j]-U[i+1,j]-U[i-1,j]-r*U[i,j+1]-r*U[i,j-1]-h^2*Sin[U[i,j]]];
F1[i_]:=0; F2[i_]:=0; F3[j_]:=0; F4[j_]:=Cos[j*k];
BC=Flatten[{Table[U[i,0]->F1[i],{i,0,NX}],Table[U[i,NY]->F2[i],{i,0,NX}],
  Table[U[0,j]->F3[j],{j,0,NY}],Table[U[NX,j]->F4[j],{j,0,NY}]}]
Eqs=Flatten[Table[FD[i,j]==0,{i,1,NX-1},{j,1,NY-1}]]; Eqs1=Flatten[Eqs/.BC];
vars=Flatten[Table[{U[i,j],0},{i,1,NX-1},{j,1,NY-1}],1]
Sol=FindRoot[Eqs1,vars]
Points=Table[{x[i],y[j],U[i,j]},{i,0,NX},{j,0,NY}];
Points1=Flatten[N[Points/.X/.Y/. BC/.Sol,nd],1];
G3D=ListPointPlot3D[Points1,BoxRatios->{2,2,1},PlotStyle->PointSize[0.01],
  ViewPoint->{-9,-15,5},PlotRange->All];
GCP=ListContourPlot[Points1,PlotRange->All];
GraphicsRow[{G3D,GCP},ImageSize->800]
```

The numerical solution of this boundary value problem inside the domain $\mathcal{D}$ (the 3D and contour plots) is shown in Fig. 40.8.

⊙ *References for Section* 40.5: R. J. LeVeque (2007), S. Larsson, and V. Thomée (2008).

40.6. Analytical-Numerical Solutions

In this section, we show the very helpful role of computer algebra systems for deriving numerical methods and testing the results obtained. By way of example, we consider an analytical-numerical approach to constructing an approximate numerical solution of the problem describing the nonlinear standing wave motion on the free surface of a fluid in *Eulerian coordinates* [see Birkhoff (1950)]. We compare this numerical solution with the asymptotic solution of this problem obtained in Eulerian and *Lagrangian coordinates* [for details, see Sekerzh-Zenkovich (1947), Shingareva and Lizárraga-Celaya (2007)].

40.6.1. Analytical Derivation of Numerical Methods

By way of example, we construct a *spectral collocation method* for solving an initial-boundary value problem for two-dimensional nonlinear standing waves of finite amplitude on the free surface of a fluid. The higher-order asymptotic solution of this nonlinear problem in Lagrangian coordinates with a weak capillarity was obtained using computer algebra systems [see Shingareva and Lizárraga-Celaya (2004), Shingareva and Lizárraga-Celaya (2007)].

There are two ways for describing the fluid motion: the Eulerian approach (in which the coordinates are fixed in the reference frame of the observer) and the Lagrangian approach (in which the coordinates are fixed in the reference frame of the moving fluid). In this section, we follow the Eulerian approach for constructing numerical solutions and compare the results with the asymptotic solution obtained in Lagrangian and Eulerian coordinates.

First, we describe the statement of the problem in Eulerian coordinates and the analytical construction of numerical methods, and then we describe its implementation in Mathematica for deriving formulas of the method, obtaining the numerical results, and for comparing the numerical solution with the asymptotic solution.

Statement of the problem. Consider the two-dimensional inviscid potential flow of a fluid bounded below by a horizontal bottom and above by a free surface. We assume the fluid motion to be periodic in time t and in the horizontal direction x. For simplicity, we consider a special case of infinite-depth pure gravity surface water waves (i.e., the fluid depth is infinite, and the surface tension constant is zero). The horizontal size of the domain L is given.

We define Cartesian coordinates according to the requirements: the fluid motion is symmetric with respect to the vertical plane $x = 0$; the free surface is $y = \eta(x,t)$; the line $y = 0$ corresponds to the mean water level; at $t = 0$ the fluid is at rest; and the infinite depth is defined as $y = -h = -\infty$.

Investigating this 2D nonlinear standing wave problem and assuming that the flow is irrotational, we have to solve the following dimensionless *Euler equations* for a fluid with initial and boundary conditions in the domain $\mathcal{D} = \{0 \le x \le \pi, -\infty \le y \le C\eta(x,t)\}$ for the wave profile $\eta(x,t)$, the velocity potential $\phi(x,y,t)$, and the angular frequency ω of the nonlinear standing wave:

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in } \{0 \le x \le \pi, -\infty \le y \le C\eta(x,t)\};$$
$$eta + \omega\phi_t + \tfrac{1}{2}C[(\phi_x)^2 + (\phi_y)^2] = 0, \phi_y - \omega\eta_t - C\phi_x\eta_x = 0 \quad \text{at } y = C\eta(x,t);$$
$$\phi_x = 0 \quad \text{at } x = 0, \pi; \quad \phi_y = 0 \quad \text{at } y = -h;$$
$$\int_0^{\pi} \eta(x,t)\,dx = 0,\ \nabla\phi(x,y,t+2\pi) = \nabla\phi(x,y,t),\ \eta(0,0) - \eta(\pi,0) = 2. \qquad (40.6.1.1)$$

Description of the method. Analytical derivation of formulas of the method. We study periodic solutions (in x and t) of the standing wave problem.

First step. To solve the *exact nonlinear problem* numerically, let us represent the un-

known functions $\eta(x,t)$ and $\phi(x,y,t)$ in the form of series:

$$\eta(x,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nk}\cos(nx)\cos[(m-1)t], \tag{40.6.1.2}$$

$$\phi(x,y,t)=A_0t+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{nm}\sin(mt)\cos[(n-1)x]\cosh[(n-1)(y+h)], \tag{40.6.1.3}$$

where A_0, $A_{11},\ldots$, $B_{11},\ldots$, are the unknown coefficients. Note that these forms of the functions $\eta(x,t)$ and $\phi(x,y,t)$ allow us to make the first, fourth, fifth, sixth, and seventh equations in the governing system (40.6.1.1) identities. Additionally, it can be shown that according to the symmetry and periodicity of the problem, some unknown coefficients are zero; i.e., $A_{nm}=0$, $B_{nm}=0$ if $n+m$ is even. Therefore, we have simplified our problem; i.e., instead of solving eight equations we have to find (for a given value of the amplitude C) the unknown nonzero coefficients A_0, A_{nm}, and B_{nm} and the nonlinear frequency ω satisfying the following three equations:

$$\begin{aligned}&\eta+\omega\phi_t+\tfrac{1}{2}C[(\phi_x)^2+(\phi_y)^2]=0,\quad \phi_y-\omega\eta_t-C\phi_x\eta_x=0 \quad \text{at } y=C\eta(x,t);\\ &\eta(0,0)-\eta(\pi,0)=2.\end{aligned} \tag{40.6.1.4}$$

We solve this problem approximately by replacing the infinite series by finite sums of the form

$$\eta(x,t)=\sum_{n=1}^{N-1}\sum_{m=1}^{N}B_{nk}\cos(nx)\cos[(m-1)t], \tag{40.6.1.5}$$

$$\phi(x,y,t)=A_0t+\sum_{n=1}^{N}\sum_{m=1}^{N-1}A_{nm}\sin(mt)\cos[(n-1)x]\cosh[(n-1)(y+h)]. \tag{40.6.1.6}$$

If N and $n+m$ are even, then $A_{nm}=0$, $B_{nm}=0$ and we can determine that the number of nonzero coefficients A_{nm} and B_{nm} is $N(N-1)$. Therefore, for a given value of the amplitude C, the total number of unknown coefficients (A_{nm}, B_{nm}, A_0, and ω) is $N(N-1)+2$. These unknown coefficients have to satisfy the governing system (40.6.1.4). We analytically derive the governing system (40.6.1.4) as follows:

```
{nN=4, S=0, tr1=y->C*eta, tr2={x->0,t->0}, tr3={x->0,t->Pi}}
Do[If[OddQ[n+m],S=S+B[n,m]*Cos[n*x]*Cos[(m-1)*t]],{m,1,nN},{n,1,nN-1}];
{eta=S, S=0}
Do[If[OddQ[n+m],S=S+A[n,m]*Cos[(n-1)*x]*Cosh[(n-1)*(y+h)]*Sin[m*t]],
   {m,1,nN-1},{n,1,nN}];
phi=A0*t+S
eq1=(eta+omega*D[phi,t]+1/2*C*((D[phi,x])^2+(D[phi,y])^2))/.tr1
eq2=(D[phi,y]-omega*D[eta,t]-C*D[phi,x]*D[eta,x])/.tr1
{eq3=2-(eta/.tr2)+(eta/.tr3), eq4=eq1/.tr3}
```

Second step. Now let us find these unknown coefficients by the collocation method, i.e., by satisfying the governing system (40.6.1.4). Therefore, we define the mesh (X_j,T_i) ($j=1,\ldots,N-1$, $i=1,\ldots,N/2$) in $\mathcal{D}$, in which we choose the mesh points as follows:

$$X_j=\begin{cases}\pi(j-1)/(N-2) & \text{if } N\neq 4,\\ \pi(j-\frac{1}{2})/(N-1) & \text{if } N=4,\end{cases}\qquad T_i=\pi(i-\tfrac{1}{2})/N,$$

where N is even (e.g., $N = 4,6,8,\ldots$). By substituting the truncated series (40.6.1.5) and (40.6.1.6) into the governing system (40.6.1.4) obtained at the first step at the mesh points, we can obtain $N(N-1)+1$ nonlinear transcendental equations; we can obtain the last equation by substituting the initial values $x = 0$, $t = 0$ into the first equation of the governing system (40.6.1.4). Therefore, we analytically construct the system of $N(N-1)+2$ nonlinear transcendental equations for $N(N-1)+2$ unknown coefficients A_{nm}, B_{nm}, A_0, and ω as follows:

```
tr4[j_,i_]:={x->X[j],t->T[i]};
Do[T[i]=Pi/nN*(i-1/2); Print["T","[",i,"]=",T[i]];
   Do[If[nN==4, X[j]=Pi/3*(j-1/2), X[j]=Pi/(nN-2)*(j-1)];
      Eq1[i,j]=eq1/.tr4[j,i]; Eq2[i,j]=eq2/.tr4[j,i];
      Print["Eq1","[",i,",",j,"]=",Eq1[i,j]];
      Print["Eq2","[",i,",",j,"]=",Eq2[i,j]],
{j,1,nN-1}],{i,1,nN/2}];
{L1={}, L2={}}
Do[If[OddQ[n+m],L1=Append[L1,A[n,m]]],{m,1,nN-1},{n,1,nN}];
Do[If[OddQ[n+m],L2=Append[L2,B[n,m]]],{m,1,nN},{n,1,nN-1}];
{LA=L1, LB=L2, k=1}
Do[F[k]=Eq1[i,j]; F[k+nN/2*(nN-1)]=Eq2[i,j]; k=k+1,{j,1,nN-1},{i,1,nN/2}];
{F[nN*(nN-1)+1]=eq3, F[nN*(nN-1)+2]=eq4, LF={}}
Do[LF=Append[LF,F[i]],{i,1,nN*(nN-1)+2}]; LFforNum=LF;
```

We have analytically constructed a system of $N(N-1)+2$ nonlinear transcendental equations for $N(N-1)+2$ unknown coefficients A_{nm}, B_{nm}, A_0, and ω. In our case, $N = 4$ and the number of equations is 14. The two above-described steps of the solution process can be performed for various even values of N, e.g., $N = 6,8$, which correspond, respectively, to 32 and 58 equations. Then we solve this system numerically in Mathematica.

40.6.2. Constructing Numerical Solutions

The system of nonlinear equations can be solved numerically; e.g., we solve it by Newton's iteration method in Mathematica. We apply the spectral collocation method constructed above for calculating standing waves for various values of the amplitude C. In particular, we are interested in determining the nonlinear frequency and wave profiles of standing waves. The process of constructing the numerical solution is as follows: Newton's iterations are performed starting from a small value of C which corresponds to the solution of the given linearized problem. It is known that the convergence of Newton's method depends on an appropriate initial guess. As follows from Bryant and Stiassnie (1994), there exist infinitely many solutions of this problem. But we restrict ourselves to the construction of an approximate numerical *one-mode solution*, in which the only main harmonic dominates, $\cos x \cos t$ or $\cos x \sin t \cosh(y+h)$.

Therefore, to specify an appropriate initial guess, i.e., to enter into a neighborhood of a point in the space $\mathbb{R}^{N(N-1)+2}$ that corresponds to a neighborhood of the point of the linear approximation, it is important to specify the coefficients A_{21} (of the main harmonic $\phi(x,y,t)$), B_{12} (of the main harmonic $\eta(x,t)$), and ω for all values of N ($N = 4,6,8,\ldots$) according to the following formulas: $A_{21} = -\omega/\sinh h$, $B_{12} = 1$, $\omega = \sqrt{\tanh h}$. These formulas can be obtained as a particular one-mode solution, $\eta(x,t) = \cos x \cos t$, $\phi(x,y,t) = -(\omega/\sinh h)\sin t \cos x \cosh(y+h)$, $\omega^2 = \tanh h$, of the corresponding linearized

problem $\eta + \omega\phi_t = 0$, $\phi_y - \omega\eta_t = 0$ at $y = 0$. The other coefficients A_{nm} and B_{nm}, i.e., the coordinates of a point in the space $\mathbb{R}^{N(N-1)+2}$, can be specified as small values (or zeros) with respect to A_{21} and B_{12}. We define the module `SolOmega0` for the first calculation with specifying an appropriate initial guess as follows:

```
SolOmega0[H_,c_,LEqs_,LLA_,LLB_]:=Module[
 {param,param1,LF1,IVA,IVB,IVA1,IVB1,IVA0,IVals,Sol0,OmegaNum0},
 param={C->c,h->H}; param1=omega0->Sqrt[Tanh[H]];
 LF1=LEqs/.param;
 IVA=Complement[LLA,{A[2,1]}];
 IVB=Complement[LLB,{B[1,2]}];
 IVA1=Table[{IVA[[i]],0.},{i,1,Length[IVA]}];
 IVB1=Table[{IVB[[i]],0.},{i,1,Length[IVB]}];
 IVA0={{A0,0.}};
 IVals=Union[{{A[2,1],-omega0/Sinh[H]},{B[1,2],1.},
       {omega,omega0}}/.param1/.param,IVA1,IVB1,IVA0];
 Sol0=FindRoot[LF1,IVals];
 OmegaNum0=Select[Sol0,MemberQ[#,omega]&][[1,2]]/omega0/.param1/.param;
 Return[{OmegaNum0,Sol0}]
];
```

Also, we define the module `SolOmega` for the subsequent calculations using the solutions obtained at the previous step as an appropriate initial guess as follows:

```
SolOmega[H_,c_,LEqs_,IVals_]:=Module[
 {param,param1,LF1,Sol,OmegaNum},
 param={C->c,h->H}; param1=omega0->Sqrt[Tanh[H]];
 LF1=LEqs/.param;   Sol=FindRoot[LF1,IVals];
 OmegaNum=Select[Sol,MemberQ[#,omega]&][[1,2]]/omega0/.param1/.param;
 Return[{OmegaNum,Sol}]
];
```

The general procedure of constructing the subsequent numerical solutions consists of the following steps:

1°. Find a solution by applying Newton's method (`SolOmega0`) for a small value of C after a few Newton's iterations.

2°. Consider the solution obtained at the previous step as an initial guess, gradually change the wave amplitude, and apply Newton's method (`SolOmega`).

3°. Repeat step 2° until the wave amplitude has the maximum desired value.

```
SetOptions[ListPlot,ImageSize->500,Joined->True,
           PlotRange->{{0.980,All},{0.01,All}},TicksStyle->Directive[Blue,9],
           Ticks->{Table[0.980+i*0.002,{i,0,10}],Automatic}];
{Cn=10,C0=0.07,Cf=0.39,Ch=(Cf-C0)/Cn}
{LAmpl=Table[C0+i*Ch,{i,0,Cn}], LNum={}, nLAmpl=Length[LAmpl]}
Do[If[i==1,{Sol[i]=SolOmega0[3.,LAmpl[[i]],LFforNum,LA,LB],
            LNum=Append[LNum,Sol[i][[1]]]},
           {SolP[i]=Table[{Sol[i-1][[2]][[k,1]],
            Sol[i-1][[2]][[k,2]]},{k,1,Length[Sol[1][[2]]]}];
            Sol[i]=SolOmega[3.,LAmpl[[i]],LFforNum,SolP[i]],
            LNum=Append[LNum,Sol[i][[1]]]}],
{i,1,nLAmpl}];
```

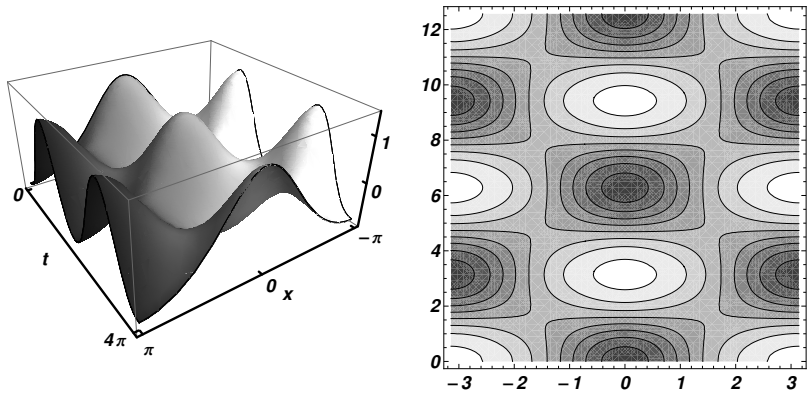

Figure 40.9. Numerical solution of the nonlinear standing waves problem: the 3D and contour plots for $x \in [-\pi, \pi]$ and $t \in [0, 4\pi]$.

```
LOmegaNum=LNum
PNum=Table[{LOmegaNum[[i]],LAmpl[[i]]},{i,1,Length[LOmegaNum]}];
GNum=ListPlot[PNum,PlotStyle->{Green,Thickness[0.01]}];
Show[GNum]
```

40.6.3. Comparison of Asymptotic and Numerical Solutions

The asymptotic solution of the problem on capillary-gravity waves in an infinite depth-fluid in Lagrangian variables has been obtained by Shingareva and Lizárraga-Celaya (2007), and we write out the frequency-amplitude dependence up to the 3rd order:

$$\frac{\omega}{\omega_0} = 1 + \varepsilon^2 \left[\frac{81\delta^3 + 36\delta^2 + 27\delta - 8}{64(3\delta + 1)(1 - 3\delta)} \right] + O(\varepsilon^3),$$

where ω_0 is the frequency of linear standing waves with the dispersion relation $\omega_0^2 = g\kappa(1 + T_z)$. Here the dimensionless surface tension is $T_z = \kappa^2 ST/(\rho g)$. Note that we follow the notation of Concus [see Concus (1962)], $\delta = \dfrac{T_z}{1 + T_z}$, for a more concise presentation of the result. Consider the special case in which the surface tension is neglected [see Andreev (1992)]. Therefore, $\delta = 0$, and $\omega/\omega_0 = 1 - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3)$. This result is equal to the previous results obtained by Shingareva (1995) in Lagrangian variables and equal to the analytic solution obtained by Rayleigh (1915), Tadjbakhsh and Keller (1960), Penney and Price (1952), and Aoki (1980), Okamura (1997) in Eulerian variables. The coincidence of the results has been explained by Shingareva and Lizárraga-Celaya (2007).

We compare the numerical solution obtained in Eulerian variables with the asymptotic solutions obtained in Lagrangian and Eulerian variables (up to the third order) and visualize the results as follows:

```
{trH={h->3.},
  Omega2=(81*deltaˆ3+36*deltaˆ2+27*delta-8)/(64*(3*delta+1)*(1-3*delta)),
  omega0=Sqrt[Tanh[h]]/.trH//N}
LAsym={};
```

```
Do[LAsym=Append[LAsym,1+Omega2*epsilon^2/omega0/.
  {delta->0,epsilon->LAmpl[[i]]}],{i,1,Length[LAmpl]}];
LOmegaAsym=LAsym
PAsym=Table[{LOmegaAsym[[i]],LAmpl[[i]]},{i,1,Length[LOmegaAsym]}]
GAsym=ListPlot[PAsym,PlotStyle->{Blue,Thickness[0.005]}];
LPP={};
Do[LPP=Append[LPP,1-1/8*AA^2/.{AA->LAmpl[[i]]}],{i,1,Length[LAmpl]}];
LOmegaPP=LPP
PPP=Table[{LPP[[i]],LAmpl[[i]]},{i,1,Length[LOmegaPP]}]
GPP=ListPlot[PPP,PlotStyle->{Magenta,Thickness[0.005]}];
{Show[{GNum,GAsym,GPP}],Show[{GNum,GAsym}],Show[{GNum,GPP}]}
```

Comment. To compare the numerical and analytical results up to the third approximation, it is necessary to choose $N = 4$ in the numerical procedure. Also, we choose the value $h = 3$, which corresponds to the problem on standing waves in an infinite-depth fluid.

Finally, we construct the surface profiles, $y = \eta(x,t)$, by using the approximate numerical solution obtained up to the third approximation. In particular, the 3D and contour plots for $x \in [-\pi,\pi]$, $t \in [0,4\pi]$ are shown in Fig. 40.9.

```
{Sol[11], C11=LAmpl[[11]]}
Prof11F[x1_,t1_]:=(eta/.Sol[11][[2]])/.{x->x1,t->t1};
Prof11F[x,t]
G3D=Plot3D[Prof11F[x,t],{x,-Pi,Pi},{t,0,4*Pi},
    PlotRange->{-1,1.5},PlotStyle->{LightBlue,Thickness[0.01]}];
GCP=ContourPlot[Prof11F[x,t],{x,-Pi,Pi},{t,0,4*Pi},
    PlotRange->{-1,1.5}]; GraphicsRow[{G3D,GCP},ImageSize->500]
Animate[Plot[Prof11F[x,t],{x,-Pi,Pi},PlotRange->{-1,1.5},
        PlotStyle->{Blue,Thickness[0.01]},PlotPoints->100],{t,0,4*Pi}]
```

⊙ *References for Section* 40.6: Lord Rayleigh (1915), G. Birkhoff (1950), W. G. Penney and A. T. Price (1952), I. Tadjbakhsh and J. B. Keller (1960), P. Concus (1962), H. Aoki (1980), V. K. Andreev (1992), P. J. Bryant and M. Stiassnie (1994), I. K. Shingareva (1995), I. K. Shingareva and C. Lizárraga-Celaya (2007).

Chapter 41
Nonlinear Partial Differential Equations with MATLAB

41.1. Introduction

In the previous two chapters, we paid special attention to analytical solutions of nonlinear PDEs and systems of PDEs owing to the availability of computer algebra systems Maple and Mathematica in modern mathematics. However, the functions and data in nonlinear PDE problems are frequently defined at discrete points and the nonlinear equation is too complicated, so that it is not possible to construct analytical solutions. Therefore, we have to study and develop numerical approximation methods for nonlinear PDEs [e.g., see Crank and Nicolson (1947), Larsson and Thomée (2008), Lax (1968), LeVeque (2007)].

Nowadays, for this purpose one can use computers and supercomputers extensively applying convenient and powerful computational software, e.g., an interactive programming environment for scientific computing, MATLAB, which provides integrated numeric computation and graphics visualization in a high-level programming language. Additionally, MATLAB excellent graphics capabilities can help one understand the results and the solution properties.

In this chapter, we turn our attention to numerical methods for solving nonlinear partial differential equations. Following the most important ideas and methods, we apply and develop numerical methods to obtain numerical and graphical solutions for studying nonlinear partial differential equations. We compute numerical solutions via MATLAB predefined functions (which are an implementation of known methods for solving partial differential equations) and develop new procedures for constructing numerical solutions (e.g., by applying the method of characteristics and finite-difference approximations) with the aid of MATLAB.

⊙ *References for Section* 41.1: J. Crank and P. Nicolson (1947), P. D. Lax (1968), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008).

41.2. Brief Introduction to MATLAB

MATLAB (short for "matrix laboratory") is not a general purpose programming language like Maple and Mathematica.

MATLAB is an interactive programming environment that provides powerful high-performance numeric computing, excellent graphics visualization, symbolic computing capabilities, and capabilities for writing new software programs using a high-level programming language.

The *Symbolic Math Toolbox* (Ver. ≥ 4.9), based on the muPAD symbolic kernel, provides symbolic computations and variable-precision arithmetic. Earlier versions of the *Symbolic Math Toolbox* are based on the Maple symbolic kernel.

Simulink (short for "simulation and link"), also included in MATLAB, offers modeling, simulation, and analysis of dynamical systems (e.g., signal processing, control, communications, etc.) under a *graphical user interface* (GUI) environment.

The first concept of MATLAB and original version (written in Fortran) was developed by Prof. Cleve Moler at the University of New Mexico in the late 1970s, for providing his students with a simple interactive access (without having to learn Fortran) to LINPACK and EISPACK software.* Over the next several years, this original version of MATLAB spread within the applied mathematics community. In early 1983, Jack Little (an engineer), together with Cleve Moler and Steve Bangert, developed a professional version of MATLAB (written in C and integrated with graphics). The company MathWorks was created in 1984 and headquartered in Natick, Massachusetts, to continue its development.

The most important features of MATLAB are as follows: interactive user interface; a combination of comprehensive mathematical and graphics functions with a powerful high-level language in an easy-to-use environment; fast numerical computation and visualization, especially for performing matrix operations [e.g., see Higham (2008)]; great flexibility in data manipulation; symbolic computing capabilities via the Symbolic Math Toolbox (Ver. < 4.9 or Ver. ≥ 4.9), based on the Maple or muPAD symbolic kernel, respectively; easy usability; the basic data element is an array that does not require dimensioning; a large library of functions for a wide range of applications; it is easy to incorporate new user-defined capabilities (toolboxes consist of M-files and are written for specific applications); understandable and available for almost all operating systems; powerful programming language, intuitive and concise syntax, and easy debugging; Simulink, as an integral part of MATLAB, provides modeling, simulation, and analysis of dynamical systems; free resources (e.g., MathWorks website `www.mathworks.com`, MathWorks Education website `www.mathworks.com/education`, MATLAB group `comp.soft-sys.matlab`, etc.).

MATLAB consists of five parts: *Development Environment*, a set of tools that facilitate using MATLAB functions and files (e.g., graphical user interfaces and the workspace); *Mathematical Function Library*, a vast collection of computational algorithms; the MATLAB *language*, a high-level matrix/array language (with flow control statements, functions, data structures, input/output, and object-oriented programming features); the MATLAB *graphics system*, which includes high-level functions (for 2D/3D data visualization, image processing, animation, etc.) and low-level functions (for fully customizing the appearance of graphics and constructing complete graphical user interfaces); *Application Program Interface* (API), a library for writing C and Fortran programs that interact with MATLAB.

Basic concepts. The *prompt symbol* >> serves for typing a MATLAB function; typing a statement and pressing `Return` or `Enter` at the end of the function serves for evaluating the MATLAB function, displaying the result, and inserting a new prompt; while the semicolon (;) symbol at the end of the function serves only for evaluating the MATLAB function and not displaying any result.

*LINPACK and EISPACK are a collection of Fortran subroutines, developed by Cleve Moler and several his colleagues, for solving linear equations and eigenvalue problems, respectively.

In MATLAB, it is not possible to move the cursor to the desired line (as in Maple and Mathematica) but, for sufficiently simple problems, corrections can be made by pressing the up- and down-arrow keys for scrolling through the list of functions (recently used) and then left- and right-arrow keys for changing the text. Also, corrections can be made using copy/paste of the previous lines located in the Command Window or Command History.

The *previous result* (during a session) can be referred to with the variable ans (the last result). MATLAB prints the *answer* and assigns the value to ans, which can be used for further calculations.

MATLAB has many forms of help: a complete *online help system* with tutorials and reference information for all the functions; the command line help system, which can be accessed by using the Help menu, by pressing F1, by selecting Help->Demos, and by entering Help and selecting Functions->Alphabetical List or Index, Search, MATLAB->Mathematics; or by typing helpbrowser, lookfor (e.g., lookfor plot); by typing help FunctionName, doc FunctionName, etc.

In MATLAB (Ver. 7), a new feature for correctly typing function names has been added. It is possible to type only the first few letters of the function and then to press the TAB key (to observe all available functions and complete typing the function).

MATLAB *desktop* appears, containing tools (graphical user interfaces) for managing files, variables, and applications. The default configuration of desktop includes various tools, e.g., Command Window, Command History, Current Directory, Workspace, Find Files (for more details, see demo MATLAB desktop), etc. It is possible to modify the arrangement of tools and documents.

For a new problem, it is best to begin with the statement clear all for cleaning all variables from MATLAB's memory. All examples and problems in the book assume that they begin with clear all.

MATLAB program can be typed at the prompt, >> or, alternatively (e.g., for more complicated problems), by creating an *M-file* (with .m extension) using MATLAB *editor* (or using another text editor). MATLAB editor is invoked by typing edit at the prompt.

M-files are files that contain code in the MATLAB language. There are two kinds of M-files: *script M-files* (which do not accept input arguments or return output data) and *function M-files* (which can accept input arguments and return output arguments). In the process of working with various M-files, it is necessary to define the path, which can be done by selecting File->Set Path->Add Folder or via the cd function.

The *structure* of a MATLAB program or source code is as follows: the *main program* or *script* and the necessary *user-defined functions*. The execution starts by typing the file name of the main program.

Incorrect response. If you get no response or an incorrect response, you may have entered or executed the function incorrectly. Do correct the function or interrupt the computation by entering debug mode and setting breakpoints: select Debug->Open M-files when Debugging and Debug->Stop if Errors/Warnings on the Desktop menu. Also, it is possible to detect erroneous or unexpected behavior in a program with the aid of MATLAB functions, e.g., break, warning, and error.

Palettes can be used, e.g., for building or editing graphs (Figure Palette), displaying the names of the GUI components (Component Palette), etc.

MATLAB *graphical user interface development environment* (GUIDE) provides a set of tools for creating graphical user interfaces (GUIs). These tools greatly simplify the process of constructing GUIs, e.g., layout the GUI components (panels, buttons, menus, etc.), and program the GUI.

MATLAB consists of a family of add-on *toolboxes*, which are collections of MATLAB functions (M-files) that extend the MATLAB environment to solve particular classes of problems. Toolboxes can be *standard* and *specialized* (see `Contents` in `Help`). Nowadays, a vast number of specialized toolboxes are available. MATLAB can be augmented by a number of toolboxes consisting of M-files and written for specific applications.

41.2.1. MATLAB Language

MATLAB language is a high-level procedural dynamic and imperative programming language (similar to Fortran 77, C, and C++), with powerful matrix/array operations, control statements, functions, data structures, input/output, and object-oriented programming features. On the other hand, MATLAB language is an interpreted language, similar to Maple and Mathematica [e.g., see Shingareva and Lizárraga-Celaya (2009)]; i.e., the instructions are translated into machine language and executed in real time (one at a time). MATLAB language allows programming-in-the-small (coding or creating programs for performing small-scale tasks) and programming-in-the-large (creating complete large and complex application programs). It supports a large collection of data structures or MATLAB classes and operations among these classes.

In linear algebra, there exist two different types of operations with vectors/matrices: operations based on the mathematical structure of vector spaces and element-by-element operations on vectors/matrices as in data arrays. This difference can be made in the name of the operation or the name of the data structure. In MATLAB, separate operations are defined (for matrix and array manipulation), but the data structures `array` and `vector/matrix` are the same. But, for example, in Maple the situation is opposite: the operations are the same, but the data structures are different.

Arithmetic operators: scalar operators (`+ - * / ^`), matrix multiplication/power (`* ^`), array multiplication/power (`.* .^`), matrix left/right division (`\ /`), and array division (`./`).

Logical operators: and (`&`), or (`|`), exclusive or (`xor`), not (`~`).

Relational operators: less/greater than (`< >`), less/greater than or equal to (`<= >=`), equal/not equal (`== ~=`).

A variable name is a character string of letters, digits, and underscores, such that it begins with a letter and its length is bounded by `N=namelengthmax` (e.g., $N = 63$). Punctuation marks are not allowed (see `genvarname` function). Variable declaration is not necessary in MATLAB, but all variables must be given initial values, e.g., `a12_new=9`. A variable can change in the calculation process, e.g., from integer to real (and vice versa).

MATLAB is case sensitive, and there is a difference between lowercase and uppercase letters, e.g., `pi` and `Pi`.

Various reserved keywords, symbols, names, and functions, for example, reserved keywords and function names, cannot be used as variable names (see `isvarname`, `which -all`, `isreserved`, `iskeyword`).

A string variable is enclosed by single quotes and belongs to the `char` class (e.g., `x='string'`), and the function `sin(x)` is invalid. Strings can be used with converting, formatting, and parsing functions (e.g., see `cellstr`, `char`, `sprintf`, `fprintf`, `strfind`, `findstr`).

MATLAB provides three basic types of variables: *local variables*, *global variables*, and *persistent variables*.

The operator "set equal to" (=). A variable in MATLAB (in contrast to Maple and Mathematica) cannot be "free" (with no assigned value) and must be assigned any initial value by the operator "set equal to" (=).

The difference between the operators "set equal to" (=) and "equal" (==). The operator `var=val` is used to assign `val` to the variable `var`, and the operator `val1==val2` compares two values, e.g., `A=3; B=3; A==B`.

Statements are input instructions from the keyboard that are executed by MATLAB (e.g., `for i=1:N s=s+i*2; end`). A MATLAB statement may begin at any position in a line and may continue indefinitely in the same line, or may continue in the next line, by typing three dots (...) at the end of the current line. White spaces between words in a statement are ignored; a number cannot be split into two pieces separated by a space.

The statement separator semicolon (;). The result of a statement followed with a semicolon (;) will not be displayed. If the semicolon is omitted, the results will be printed on the screen, e.g., `x=-pi:pi/3:pi;` and `x=-pi:pi/3:pi`.

Multiple statements in a line: two or more statements may be written in the same line if they are separated with semicolons.

Comments can be included with the percentage sign `%` and all characters following it up to the end of a line. Comments at the start of a code have a special significance: they are used by MATLAB to provide the entry for the help manual for a particular script. The block comment operators, `%{ %}`, can be used for writing comments that require more than one line.

An expression is a valid statement and is formed as a combination of numbers, variables, operators, and functions. The arithmetic operators have different precedences (increasing precedence `+ - * / ^`). Precedence is altered by parentheses (expressions within parentheses are evaluated before expressions outside parentheses).

A boolean or *logical expression* is formed with *logical* and *relational operators*, e.g., `x>0`. Logical expressions are used in `if`, `switch`, and `while` statements. The logical values, *true* and *false*, are represented by numerical values, *1* and *0*, respectively.

A regular expression is a string of characters that defines a *pattern*; e.g., `'Math?e\w*'`. Regular and dynamic expressions can be used to search text for a group of words that matches the pattern (e.g., for parsing or replacing a subset of characters within text).

MATLAB is sensitive to types of brackets and quotes (see `help paren`, `help punct`).

Types of brackets:

square brackets, `[]`, for constructing vectors and matrices, `A1=[1 2 3]`, `A2=[1, 2, 3]`, `A3=[1, 2; 4, 5]`, for multiple assignment statements, for example, `A4=[1, 5; 2, 6]` and `[L,U]=lu(A4)`.

parentheses, `()`, for grouping expressions, `(5+9)*3`, for delimiting the arguments of functions, `sin(5)`, for vector and matrix elements, `A1(2)`, `A3(1,1)`, `A2([1 2])`; in logical expressions, `A1(A1>2)`.

curly brackets, { }, for working with cell arrays, `C1={int8(3) 2.59 'A'}`, `C1{1}`, `X(2,1)={[1 3; 4 6]}`.

dot-parentheses, `.( )`, for working with a structure via a dynamic field name, `S.F1=1; S.F2=2; F='F1'; val1=S.(F)`.

Quotes:

forward quotes, (`' '`), for creating strings, e.g., `T='the name=7;' k=5; disp('the value of k is'); disp(k)`,

a single forward quote and dot single forward quote, (`' .'`), for matrix transposition (the complex conjugate/nonconjugate transpose of a matrix), `A1=[1+i,i;-i,1-i]; A1'; A.'`.

Types of numbers. Numbers are stored (by default) as double-precision floating point (class `double`). To operate with integers, it is necessary to convert from double to the integer type (e.g., classes `int8`, `int16`, `int32`), `x=int16(12.3)`, `str='MATLAB'`, `int8(str)`. Mathematical operations that involve integers and floating-point numbers result in an integer data type. Real numbers can be stored as *double-precision floating point* (by default) or *single-precision floating point*, e.g., `x1=3.25`, `x2=single(x1)`, `x3=double(x2)` (for details, see `whos`, `isfloat`, `class`). Complex numbers can be created as `z1=1+2*i`, `z2=complex(1,2)`. Rational numbers can be formed by setting the `format` to `rational`, e.g., `x=3.25; format rational x format`. To check the current format setting, we type `get(0,'format')`.

Predefined constants: symbols for definitions of commonly used mathematical constants, e.g., `true`, `false`, `pi`, `i`, `j`, `Inf`, `inf`, `NaN` (not a number), `exp(1)`, the Euler constant γ, `-psi(1)`, `eps`.

In MATLAB, there are *predefined functions* and *user-defined functions*.
Predefined functions are divided into *built-in functions* and *library functions*.
Built-in functions are precompiled executable programs and run much more efficiently (see `help elfun`, `help elmat`).
Library functions are stored as M-files (in the libraries or toolboxes), which are available in readable form (see `which`, `type`, `exist`). MATLAB can be complemented with locally user-developed M-files and toolboxes.

Many functions are overloaded (i.e., have an additional implementation of an existing function) so that they handle different classes (e.g., `which -all plot`).

Numerous *special functions* are defined, e.g., `help bessel`, `help specfun`.

User-defined functions can be created as M-files (see `help 'function'`) or as anonymous functions.

User-defined function written in an M-file (with the extension `.m`) must contain only one function. It is best to have the same name for the *function name* and the *file name*. The process of creating functions is as follows: create and save an M-file using a text editor, then call the function in the main program (or in `Command Window`).

Functions written in M-files have the forms:
`function OArg=FunName(IArg); FunBody;` or
`function [OArg1,OArg2,...]=FunName(IArg1,IArg2,...); FunBody;`
where `OArg` and `IArg` are output arguments and input arguments, respectively.
For example, the function $y = \sin x$ is defined as `function f=SinFun(x); f=sin(x);`

Evaluation of functions: `FunName(Args)`.

For example, for the sine function we have `cd('c:/mypath'); SinFun(pi/2); type SinFun`.

Anonymous functions serve for creating simple functions without storing functions to files. Anonymous functions can be constructed either in the `Command Window` or in any function or script, e.g., the function $f(x) = \sin x$ is defined as `f=@(x) sin(x); f(pi/2)`.

A *function handle*, `@`, is one of the standard MATLAB data types that provides calling functions indirectly, e.g., to call a subfunction when outside the file that defines that function (see class `function_handle`) .

Nested functions are allowed in MATLAB; i.e., one or more functions or *subfunctions* within another function can be defined in MATLAB. In this case, the `end` statements are necessary.

In MATLAB language there are the following *control structures*: the selection structures `if`, `switch`, `try` and the repetition structures `for`, `while`.

MATLAB does not have a *module system* in the traditional form: it has a system based on storing *scripts* and *functions* in M-files and placing them into *directories* (see `cd` function for changing the current directory, `help ..`).

MATLAB *data structures* or *classes*, vectors, matrices, and arrays are used for representing more complicated data. There are 15 fundamental classes, which are in the form of a matrix or array: `double`, `single`, `int8`, `uint8`, `int16`, `uint16`, `int32`, `uint32`, `int64`, `uint64`, `char`, `logical`, `function_handle`, `struct`, and `cell`. The numerical values are represented (by default) as floating-point double precision data (`float double`). It is possible to construct various *composite data types* (e.g., sequences, lists, sets, tables, etc.) using the classes `struct` and `cell`.

Vectors are ordered lists of numbers separated by commas or spaces inside `[ ]`; no dimensioning is required. But *vector and array indices* can only be positive and nonzero. The notation `X=[1:0.1:9]` stands for a vector of numbers from 1 to 9 in increments of 0.1 (see `help colon`).

Matrices are rectangular arrays of numbers (row/column vectors are special cases of matrices).

⊙ *References for Section* 41.2: N. J. Higham (2008), I. K. Shingareva and C. Lizárraga-Celaya (2009).

41.3. Numerical Solutions Via Predefined Functions

In this section, we consider the construction of numerical and graphical solutions of various initial-boundary value problems using predefined MATLAB functions and default methods, e.g., for scalar nonlinear PDEs in one space dimension (the Burgers and Fisher equations), systems of nonlinear PDEs in one space dimension (the FitzHugh–Nagumo equations), scalar nonlinear elliptic PDEs in two space dimensions (the nonlinear Poisson equation), and systems of nonlinear PDEs in two space dimensions (the system of nonlinear Poisson equations).

Additionally, we construct numerical and graphical solutions of nonlinear problems defined on a more complicated geometry, which can consist of various solid objects, e.g., for the nonlinear Poisson–Boltzmann equation defined on an irregular domain [see Harries (1998)].

41.3.1. Scalar Nonlinear PDEs in One Space Dimension

First, consider the predefined MATLAB functions with the aid of which we can obtain approximate numerical solutions solving various linear and nonlinear PDE problems. Applying the predefined function `pdepe`, which stands for parabolic-elliptic partial differential equations, we can numerically solve initial-boundary value problems for a class of nonlinear parabolic PDEs in one space variable x and time t.

Consider a single nonlinear PDE in one space dimension. In this case, the predefined function `pdepe` allows us to solve initial-boundary value problems for a single parabolic PDE in 1D (see `help pdepe`).

These PDEs involve an unknown function u that depends on a scalar space variable x and a scalar time variable t. In MATLAB notation, a general class of parabolic PDEs defined in the domain $\mathcal{D} = \{a \le x \le b,\ t_0 \le t \le t_f\}$ is represented in the form

$$c(x,t,u,u_x)u_t = x^{-m}\partial_x\big(x^m f(x,t,u,u_x)\big) + s(x,t,u,u_x), \tag{41.3.1.1}$$

where a, b, t_0, t_f are given constants, $f(x,t,u,u_x)$ is a flux, $s(x,t,u,u_x)$ is a source term, $m \in \{0,1,2\}$ (which corresponds to slab, cylindrical, or spherical symmetry, respectively), and $c(x,t,u,u_x) > 0$.*

The initial condition at the initial time $t = t_0$ has the form

$$u(x,t_0) = u_0(x) \tag{41.3.1.2}$$

and the boundary conditions at $x = a$, $x = b$ (for $t_0 \le t \le t_f$) have the form

$$p(a,t,u) + q(a,t)f(a,t,u,u_x) = 0,\quad p(b,t,u) + q(b,t)f(b,t,u,u_x) = 0. \tag{41.3.1.3}$$

One can impose the Dirichlet, Neumann, Robin, or periodic boundary conditions.

```
                sol=pdepe(m,@PDEfun,@ICfun,@BCfun,xMesh,tSpan,ops)
uk=sol(j,:,k);              [uOut,DuOutDx]=pdeval(m,xMesh,uk,xOut)
```

Description:

`PDEfun`, `ICfun`, and `BCfun` are function handles.

`[c,f,s]=PDEfun(x,t,u,DuDx)` evaluates the quantities defining the PDE.

`u0=ICfun(x)` evaluates the initial conditions.

`[pL,qL,pR,qR]=BCfun(xL,uL,xR,uR,t)` evaluates the boundary conditions at time t.

`uk` approximates component k of the solution at time `tSpan(j)` and mesh points `xMesh`.

`pdeval` evaluates the function u and u_x derivative `DukDx` at the array of points `xOut` at points that are not in `xMesh`; it stores them in `uOut` and `DuOutDx`, respectively.

*If $c(x,t,u,ux) = 0$ (which corresponds to elliptic equations in the case of systems of parabolic and elliptic equations, see Section 41.3.2.), then a single elliptic PDE is transformed into ODE.

Implementing the *method of lines* (in a general setting), the predefined function `pdepe` converts the PDEs to ODEs using a second-order accurate *spatial discretization* [for details, see Skeel and Berzins (1990), Schiesser and Griffiths (2009), Lee and Schiesser (2004)] (i.e., by replacing only the spatial derivatives with finite differences) based on a fixed set of nodes or a mesh `xMesh` (represented as an array), where `xMesh(1)=a`, `xMesh(end)=b`, and `xMesh(i)<xMesh(i+1)` (for i=2, ..., `end-1`). Then the resulting ODEs are integrated (with a stiff ODE solver `ode15s`) to obtain approximate solutions at times `tSpan`, where `tSpan(1)=t_0`, `tSpan(end)=t_f`, and `tSpan(i)<tSpan(i+1)` (for i=2, ..., `end-1`).

The numerical solution obtained is represented as a three-dimensional array `sol`, where `sol(:,:,k)` approximates component k of the solution, `sol(i,:,k)` approximates component k of the solution at time `tSpan(i)` and mesh points `xMesh(:)`, and `sol(i,j,k)` approximates component k of the solution at time `tSpan(i)` and the mesh point `xMesh(j)`.

Numerical solutions can be used for obtaining visualizations (solution surface, solution profile, or animation), e.g., using `plot` and `surf` (in more detail, see `help graphics`).

For more advanced applications, it is best to specify *options* that change the default integration parameters error control and step size (for details, see `odeset`, `RelTol`, `AbsTol`, `NormControl`, `InitialStep`, `MaxStep`).

Example 41.1. *Burgers equation.* Let us find numerical and graphical solutions of the following initial-boundary value problem for the Burgers equation:

$$u_t = \nu u_{xx} + u u_x, \quad u(x,0) = A\sin(\pi x), \quad u(0,t) = 0, \quad u(1,t) = 0,$$

in the domain $\mathcal{D} = \{0 \le x \le 1,\ 0 \le t \le T\}$ with initial amplitude $A = 0.4$ and kinematic viscosity $\nu = 0.009$.

First, let us rewrite the Burgers equation with initial and boundary conditions in the form (41.3.1.1)–(41.3.1.3). We create three separate function M-files for each set of functions, i.e., `PDE1.m` for the functions c, f, and s, `IC1.m` for the function $u_0(x)$, and `BC1.m` for the functions p and q. The `pdepe` function will combine these M-files and construct a solution to the problem.

For the Burgers equation, we have $m = 0$, $c(x,t,u,u_x) = 1$, $f(x,t,u,u_x) = \nu u_x$, and $s(x,t,u,u_x) = u u_x$; for the initial condition, we have $u_0(x) = A\sin(\pi x/L)$; and for the boundary conditions, we have $p(0,t,u) = u$, $q(0,t) = 0$, $p(1,t,u) = u$, and $q(1,t) = 0$. These functions are specified in the function M-files as follows:

```
function [c,f,s]=PDE1(x,t,u,DuDx)
   nu=0.009; c=1; f=nu*DuDx; s=u*DuDx;
function [pL,qL,pR,qR]=BC1(xL,uL,xR,uR,t)
   pL=uL; qL=0; pR=uR; qR=0;
function u0=IC1(x)
   A=0.4; L=1; u0=A*sin(pi*x/L);
```

Then, for constructing numerical and graphical solutions of this initial-boundary value problem for the Burgers equation, we compose the script M-file `SolPDE1` as follows:

```
clear all; close all; echo on; format long;
m=0; a=0; b=1; Nx=101; t0=0; tf=40; Nt=161;
x=linspace(a,b,Nx); t=linspace(t0,tf,Nt);
sol=pdepe(m,@PDE1,@IC1,@BC1,x,t); u=sol(:,:,1);
n=2:10; m=1:10;  u(m,n)'
figure(1);
  surf(x,t,u); rotate3d on; title('Burgers equation. Surface plot of solution');
  xlabel('Distance x'); ylabel('Time t');
figure(2);
  plot(x,u(1,:),'k-',x,u(3,:),'k--',x,u(5,:),'k:','LineWidth',2);
  set(gca,'FontSize',14); set(gca,'FontName','Arial'); set(gca,'LineWidth',1);
```

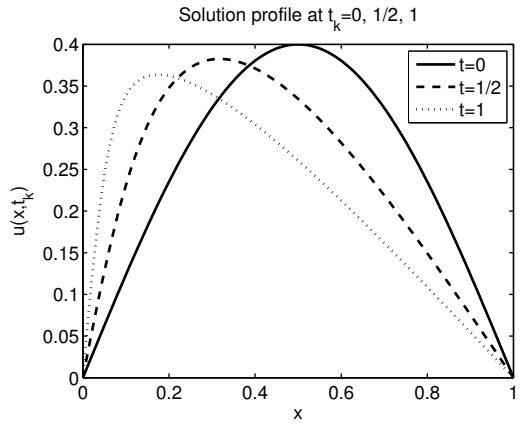

Figure 41.1. Surface profiles of the solution of the Burgers equation at $t_k = 0, \frac{1}{2}, 1$.

```
  title('Solution profile at t_k=0, 1/2, 1');
  xlabel('x'); ylabel('u(x,t_k)'); legend('t=0','t=1/2','t=1');
figure(3);
  G=plot(x,u(1,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u(k,:)); pause(0.7); end
  title('Animation of the solution'); xlabel('x'); ylabel('u(x,t_k)');
echo off
```

In this script M-file, we choose a grid of x and t values (`linsolve` functions), solve the nonlinear equation (`pdepe` function), and extract the solution u from `sol` as `u=sol(:,:,1)`. Then we print some numerical values of the solution, plot it as a surface (with interactive rotation, the `rotate3d` function), plot the surface profile at $t_k = 0, 1/2, 1$, and produce animations of the surface profile for $t \in [0, 40]$.

Before working with these new files that are located in a new working directory in the `Command Window`, we can make this directory the current working directory in MATLAB by using `cd` function, for example `cd('c:/npde')`, or by selecting this directory in the `Desktop->Current Directory`, or by adding it to the path of directories that MATLAB searches to find files `File->SetPath->AddFolder`, or by using `addpath('c:/npde')`. Finally, we run the file `SolPDE1` in the `Command Window`.

The text and numerical values display in the same `Command Window`, but graphics appears in a separate graphics windows (`Figure1`, `Figure2`, and `Figure3`). Three surface profiles of the solution obtained at times $t_k = 0, \frac{1}{2}, 1$ are shown in Fig. 41.1.

Example 41.2. *Fisher equation.* Consider the nonlinear Fisher equation of the form

$$u_t = u_{xx} + u(1-u).$$

This nonlinear evolution equation was introduced by Fisher in 1936 for studying the wave propagation of an advantageous gene in a population.

We find numerical and graphical solutions of the Fisher equation subject to the following boundary and initial conditions:

$$u_t = u_{xx} + u(1-u), \quad \lim_{x\to-\infty} u(x,t) = 1, \quad \lim_{x\to\infty} u(x,t) = 0, \quad u(x,0) = \begin{cases} 1, & x < a, \\ f(x), & a < x < b, \\ 0, & x > b, \end{cases}$$

where a and b are arbitrary parameters, $f(x)$ is an arbitrary function, e.g., we choose $a = -1$, $b = 1$, $f(x) = \cos(x)$, and $\mathcal{D} = \{-50 \le x \le 50,\ 0 \le t \le 5\}$.

As before, we rewrite the Fisher equation with initial and boundary conditions in the form (41.3.1.1)–(41.3.1.3), create three separate function M-files, `PDE2.m` for the functions c, f, and s, `IC2.m` for the function $u_0(x)$, and `BC2.m` for the functions p and q.

For the Fisher equation, we have $m = 0$, $c(x,t,u,u_x) = 1$, $f(x,t,u,u_x) = u_x$, and $s(x,t,u,u_x) = u(1-u)$.

For the boundary conditions, we have $p(0,t,u) = u - 1$, $q(0,t) = 0$, $p(1,t,u) = u$, $q(1,t) = 0$, and discontinuous initial data $u_0(x)$ defined above.

These functions are specified in the function M-files as follows:

```
function [c,f,s]=PDE2(x,t,u,DuDx)
   c=1; f=DuDx; s=u*(1-u);
function [pL,qL,pR,qR]=BC2(xL,uL,xR,uR,t)
   pL=uL-1; qL=0; pR=uR; qR=0;
function u0=IC2(x)
   a=-1; b=1;
   if x<a u0=1; elseif x>a && x<b u0=cos(x); else u0=0; end
```

Then, for constructing numerical and graphical solutions of this initial boundary value problem for the Fisher equation, we compose the script M-file `SolPDE2` as follows:

```
clear all; close all; echo on; format long;
m=0; a=-50; b=50; Nx=101; t0=0; tf=10; Nt=101;
x=linspace(a,b,Nx); t=linspace(t0,tf,Nt);
sol=pdepe(m,@PDE2,@IC2,@BC2,x,t); u=sol(:,:,1);
figure(1);
  surf(x,t,u);  rotate3d on;
  title('Fisher equation. Surface plot of solution');
  xlabel('Distance x'); ylabel('Time t');
figure(2);
  plot(x,u(3,:),'k-',x,u(71,:),'k--',x,u(101,:),'k:','LineWidth',2);
  set(gca,'FontSize',14); set(gca,'FontName','Arial');
  set(gca,'LineWidth',1);
  title('Solution profile at t_k=0.2, 7, 10')
  xlabel('x'); ylabel('u(x,t_k)'); legend('t=0.2','t=7','t=10');
figure(3);
  G=plot(x,u(1,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u(k,:)); pause(0.7); end
  title('Animation of the solution'); xlabel('x'); ylabel('u(x,t_k)');
echo off
```

We choose a grid of x and t values (`linsolve` functions), solve the nonlinear equation (`pdepe` function), and extract the solution u from `sol` as `u=sol(:,:,1)`. Then we plot the solution as a surface (with interactive rotation, `rotate3d` function), plot the surface profile at $t_k = 0.2, 7, 10$, and produce animations of the surface profile for $t \in [0, 10]$.

Kolmogorov, Petrovsky, and Piskunov (1937), in their study of nonlinear systems of this type (with applications to combustion and wave propagation in biology and chemistry), have proved that for the bounded initial data of the form, $0 \le u(x,0) \le 1$ ($x \in \mathbb{R}$), the solution of the Fisher equation is also bounded; i.e., $0 \le u(x,t) \le 1$ ($x \in \mathbb{R}$) for all x, t. They also showed that for the discontinuous initial data (described above) the solution of the Fisher equation propagates to the right as a shock- or kink-like traveling wave. Three surface profiles of the solution obtained at times $t_k = 0.2, 7, 10$ are shown in Fig. 41.2.

41.3.2. Systems of Nonlinear PDEs in One Space Dimension

Now we consider systems of nonlinear partial differential equations in one space dimension. In this case, the predefined function `pdepe` allows us to solve initial boundary value problems for systems of parabolic-elliptic PDEs in 1D; there must be at least one parabolic

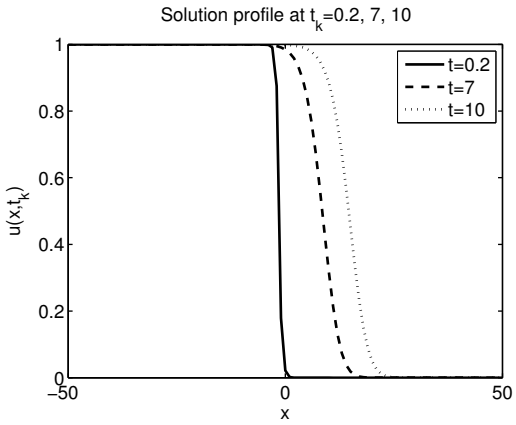

Figure 41.2. Surface profile evolution of the Fisher equation for $t_k = 0.2, 7, 10$.

equation in the system (see `help pdepe`). The class of parabolic-elliptic PDEs, defined in $\mathcal{D} = \{a \le x \le b,\ t_0 \le t \le t_f\}$, to which the function `pdepe` can be applied, has the form

$$C(x,t,\mathbf{u},\mathbf{u}_x)\mathbf{u}_t = x^{-m}\partial_x\big(x^m \mathbf{f}(x,t,\mathbf{u},\mathbf{u}_x)\big) + \mathbf{s}(x,t,\mathbf{u},\mathbf{u}_x), \tag{41.3.2.1}$$

where $\mathbf{u}$ is a vector-valued unknown function that depends on a scalar space variable x and a scalar time variable t; the flux function $\mathbf{f}$ and the source function $\mathbf{s}$ are vector-valued functions; the integer $m \in \{0,1,2\}$ corresponds to slab, cylindrical, and spherical symmetry, respectively; the function C is a diagonal matrix whose diagonal entries are zero or positive (which corresponds to elliptic or parabolic equations, respectively).*

Initial condition at $t = t_0$ and for $a \le x \le b$ and a given function $\mathbf{u}_0$ is defined as follows:

$$\mathbf{u}(x,t_0) = \mathbf{u}_0(x). \tag{41.3.2.2}$$

The boundary conditions at $x = a$ and $x = b$ and for $t_0 \le t \le t_f$ have the form

$$\mathbf{p}(a,t,\mathbf{u}) + \mathbf{q}(a,t)\mathbf{f}(a,t,\mathbf{u},\mathbf{u}_x) = \mathbf{0}, \quad \mathbf{p}(b,t,\mathbf{u}) + \mathbf{q}(b,t)\mathbf{f}(b,t,\mathbf{u},\mathbf{u}_x) = \mathbf{0}, \tag{41.3.2.3}$$

where $\mathbf{p}$ and $\mathbf{q}$ are given vector-valued functions.

Since the predefined function `pdepe` implements a second-order spatial discretization method based on the `xMesh` values, it follows that the choice of `xMesh` is important and can affect the accuracy and cost of the numerical solution (e.g., it is best to define closely spaced `xMesh` points for domains where the solution can vary rapidly with respect to x). The time points in $[t_0, t_f]$ at which the solution is obtained are given in the vector `tSpan`, where `tSpan(1)=t_0`, `tSpan(end)=t_f`, and `tSpan(i)<tSpan(i+1)` (for `i=2,...,end-1`). Since the time integration in `pdepe` is performed by the stiff ODE solver `ode15s`, the actual time step values are chosen dynamically and do not affect the accuracy and cost.

Example 41.3. *FitzHugh–Nagumo equations.* Consider systems, called the FitzHugh–Nagumo type equations [see Murray (1993)], that arise in mathematical biology and model the nerve impulse propagation along an axon. One example is the nonlinear reaction–diffusion system of the form

$$u_t = u_{xx} + u(\alpha - u)(u-1) - v,$$
$$v_t = \beta u,$$

*According to this notation, there must be at least one parabolic equation.

where function $u(x,t)$ is the membrane potential, $v(x,t)$ is a phenomenological recovery variable, $0 < \alpha < 1$, and $\beta > 0$.

We find numerical and graphical solutions of this nonlinear system subject to the following boundary and initial conditions:

$$u_x(a,t) = 0, \quad u_x(b,t) = 0, \quad v_x(a,t) = 0, \quad v_x(b,t) = 0,$$
$$u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} \qquad v(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

To apply the `pdepe` function, we denote (u,v) by (u_1,u_2) and rewrite the nonlinear system with initial and boundary conditions in the form (41.3.2.1)–(41.3.2.3) as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\partial}{\partial x} \begin{pmatrix} u_{1x} \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(\alpha - u_1)(u_1 - 1) \\ \beta u_1 \end{pmatrix}. \tag{41.3.2.4}$$

Then we choose $\mathcal{D} = \{-100 \le x \le 100,\ 0 \le t \le 50\}$, $\alpha = 0.1$, and $\beta = 0.01$. Creating three separate function M-files, `SYS1.m`, `ICSYS1.m`, and `BCSYS1.m`, we specify all parameters and functions as follows:

```
function [c,f,s]=SYS1(x,t,u,DuDx)
  alpha=0.1; beta=0.01; c=[1;1]; f=[DuDx;0];
  s=[u(1)*(alpha-u(1))*(u(1)-1); beta*u(1)];
function u0=ICSYS1(x)
  if x<0 u0(1)=1; else u0(1)=0; end;
  if x<0 u0(2)=0; else u0(2)=1; end; u0=[u0(1),u0(2)];
function [pL,qL,pR,qR]=BCSYS1(xL,uL,xR,uR,t)
  pL=[0;0]; qL=[1;1]; pR=[0;0]; qR=[1;1];
```

Then, for constructing numerical and graphical solutions of this initial-boundary value problem, we compose a script M-file `SolSYS1` as follows:

```
clear all; close all; echo on; format long;
m=0; x=linspace(-100,100,201); t=linspace(0,50,101);
sol=pdepe(m,@SYS11,@ICSYS11,@BCSYS11,x,t);
u1=sol(:,:,1); u2=sol(:,:,2);
figure; subplot(2,1,1); colormap(gray);
  surf(x,t,u1);  rotate3d on; view(20,45);
  set(gca,'FontSize',14); set(gca,'FontName','Arial');
  set(gca,'LineWidth',1); title('u_1(x,t)'); xlabel('x');ylabel('t');
  subplot(2,1,2); surf(x,t,u2);  rotate3d on;
  set(gca,'FontSize',14); set(gca,'FontName','Arial');
  set(gca,'LineWidth',1); title('u_2(x,t)'); xlabel('x');ylabel('t');
figure; subplot(2,1,1);
  plot(x,u1(2,:),'k-',x,u1(81,:),'k--',x,u1(Nt,:),'k:','LineWidth',2);
  title('Solution profile at t_k=0.5,40,50');
  xlabel('x'); ylabel('u1(x,t_k)'); legend('t=0.5','t=40','t=50');
  subplot(2,1,2);
  plot(x,u2(2,:),'k-',x,u2(81,:),'k--',x,u2(Nt,:),'k:','LineWidth',2);
  title('Solution profile at t_k=0.5,40,50');
  xlabel('x'); ylabel('u2(x,t_k)'); legend('t=0.5','t=40','t=50');
figure;  subplot(2,1,1); G=plot(x,u1(2,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u1(k,:)); pause(0.2); end
  title('Animation of u_1(x,t_k)'); xlabel('x'); ylabel('u1(x,t_k)');
subplot(2,1,2); G=plot(x,u2(2,:),'erase','xor');
  for k=2:length(t) set(G,'xdata',x,'ydata',u2(k,:)); pause(0.2); end
  title('Animation of u_2(x,t_k)'); xlabel('x'); ylabel('u2(x,t_k)');
  echo off
```

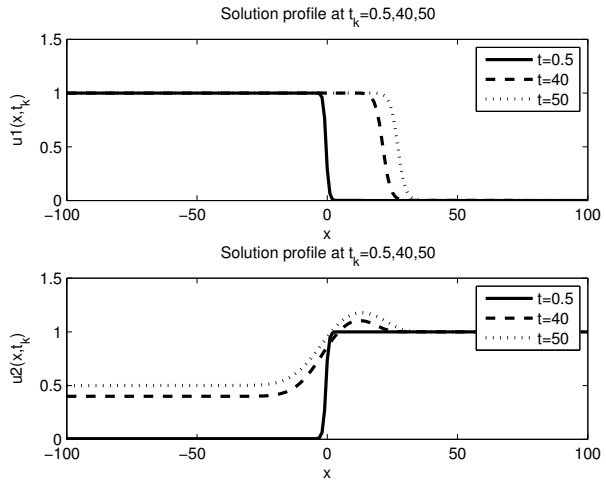

Figure 41.3. Surface profiles of the solution at $t_k = 0.5, 40, 50$.

In this case, we extract the solution u from `sol` as `u1=sol(:,:,1)`, `u2=sol(:,:,2)`. Then we plot the solution as a surface (with interactively rotation), plot the surface profile at $t_k = 0.5, 40, 50$, and produce animations of the surface profile for $t \in [0, 50]$. Three surface profiles of the solution obtained at times $t_k = 0.5, 40, 50$ are shown in Fig. 41.3.

41.3.3. Nonlinear Elliptic PDEs in Two Space Dimensions

In MATLAB, partial differential equations or their systems in two space dimensions (2D) and time, which arise in a wide variety of phenomena in all branches of science and engineering [e.g., see Schiesser (1994)], can be solved with the aid of the `PDE Toolbox`, which allows us to define a PDE problem (geometry, boundary conditions, and PDE or system of PDEs), numerically solve the problem (generate meshes, discretize the equations, and find approximation to the solution), and visualize the results. The PDEs are solved using the *Finite Element Method* (FEM).

The `PDE Toolbox` allows one to solve nonlinear elliptic equations and systems. Nonlinear parabolic and hyperbolic PDEs can be solved by using other programs (e.g., Multiphysics*) or by writing individual programs (e.g., for the method of lines). In some cases, it is possible to solve nonlinear PDEs in the `PDE Toolbox` applying the universal iteration process.

`pdetool`	`Draw`	`Boundary`	`PDE`	`Mesh`	`Solve Plot`
`pdeinit`	`pdecirc`	`pdeellip`	`pdepoly`	`pderect`	`pdebound`
`assemb`	`initmesh`	`refinemesh`	`assempde`	`pdenonlin`	`parabolic`
`hyperbolic`	`pdeeig`	`pdesurf`	`pdeplot`	`pdemesh`	`pdecont`

Description:

`pdetool`, starting the `PDE Toolbox` that invokes the *graphical user interface* (GUI).

*Multiphysics is the numerical environment, which has bidirectional interface with MATLAB and can solve nonlinear PDEs in 1D, 2D, and 3D and time.

`Draw`, `Boundary`, `PDE`, `Mesh`, `Solve`, `Plot`, different icons (or modes) in the GUI-menu corresponding to different stages of the solution process.

`pdeinit`, `pderect`, `pdebound`, `initmesh`, `assempde`, `pdenonlin`, `pdeplot`, etc., functions (that correspond to different modes of the GUI) for writing programs.

For starting the `PDE Toolbox`, it is necessary to type `pdetool` in the `Command Window`. This invokes the graphical user interface (GUI), which is appropriate for simple or standard PDE problems. For more complicated problems and full control over the numerical process, it is best to write programs and call them from the prompt or M-files.

The `PDE Toolbox` allows solving the following four types of partial differential equations (defined on a bounded domain $\mathcal{D} \in \mathbb{R}^2$):

$$\begin{aligned} -\nabla\cdot(c\nabla u)+au &= f, && \text{elliptic equations.}\\ bu_t-\nabla\cdot(c\nabla u)+au &= f, && \text{parabolic equations.}\\ bu_{tt}-\nabla\cdot(c\nabla u)+au &= f, && \text{hyperbolic equations.}\\ -\nabla\cdot(c\nabla u)+au &= \lambda bu, && \text{eigenvalue problems.} \end{aligned}$$

Here c is the scalar coefficient, λ is an unknown eigenvalue, and a, b, f, and u are complex-valued functions defined on $\mathcal{D}$. For nonlinear PDEs, the coefficients c, a, and f can depend on the unknown solution u. For hyperbolic and parabolic PDEs, the coefficients can depend on time.

In this section, we consider a scalar nonlinear elliptic PDE in two space dimensions. Since the Finite Element Method yields a nonlinear system of the form $K(U)U = F(U)$, iterative methods can be applied for solving the nonlinear system. The damped Gauss–Newton method with the Armijo–Goldstein line search strategy (for improving convergence from bad initial guesses) and an adaptive mesh refinement algorithm are implemented for solving elliptic problems via the `pdenonlin` function.

The `PDE Toolbox` allows one to define the following boundary conditions:

$$hu = r \ \text{ on } \ \partial\mathcal{D}, \qquad \vec{n}\cdot(c\nabla u)+qu = g \ \text{ on } \ \partial\mathcal{D},$$

which, respectively, are referred to as the Dirichlet (or essential) boundary conditions, generalized Neumann (or natural) boundary conditions,* and the mixed boundary conditions (a combination of Dirichlet and generalized Neumann condition). Here $\vec{n}$ is the outward unit normal, and g, q, h, and r are complex-valued functions defined on the boundary $\partial\mathcal{D}$. For nonlinear PDEs, the coefficients g, q, h, and r can depend on the unknown solution u. For eigenvalue problems, $g = 0$ and $r = 0$, and for hyperbolic and parabolic PDEs, the coefficients can depend on time.

Let us briefly describe the process of solution for simple problems via the graphical user interface (GUI). There are different icons (or modes) in the GUI-menu that correspond to different stages of the solution process:

*In some contexts, the generalized Neumann boundary conditions are also referred to as the Robin boundary conditions.

`Draw`, for defining the domain $\mathcal{D}$ and the geometry (using the constructive solid geometry (CSG) model paradigm), e.g., by combining various solid objects (rectangle, circle, etc.) using set formulas.

`Boundary`, for specifying the boundary conditions (e.g., different types of boundary conditions on different boundary segments).

`PDE`, for interactively specifying the type of PDE and the coefficients c, a, f, and b (e.g., specifying the coefficients for each subdomain independently).

`Mesh`, for generating and plotting meshes and controlling the parameters of the automated mesh generator.

`Solve`, for invoking and controlling the nonlinear and adaptive solvers (for elliptic PDEs); for specifying the initial conditions and the times for which the solution should be constructed (for parabolic/hyperbolic PDEs); for specifying the interval in which to search for eigenvalues (for eigenvalue problems).

`Plot`, for visualizing numerical solutions (inside the GUI and in separate figures); for simultaneously plotting three different solution properties (using color, height, and vector field plots); for plotting surfaces, meshes, contours, and arrows (quivers); for producing animations of solutions (for parabolic/hyperbolic PDEs).

Comment. After solving a problem, it is possible to refine the mesh (returning to `Mesh`) and then solve the problem again and again. Also this can be made by using an adaptive mesh refiner and solver.

Example 41.4. *Nonlinear Poisson equation.* Consider the boundary value problem for the two-dimensional nonlinear Poisson equation

$$u_{xx}+u_{yy}=\sin(u), \quad u(x,0)=0, \quad u(x,6)=0, \quad u(0,y)=0, \quad u(3,y)=\cos(y),$$

describing a potential field $u(x,y)$ in a bounded domain $\mathcal{D}=\{0\le x\le 3,\ 0\le y\le 6\}$.

First, we solve this problem with the aid of the graphical user interface (GUI).

1°. We draw the domain $\mathcal{D}=\{0\le x\le 3,\ 0\le y\le 6\}$. Selecting `Options->Grid`, we generate the grid. Then, selecting `Options->Axes Limits`, we change the size of the default domain; i.e., we put the new size $[-14]$ for x-axis and $[-18]$ for y-axis, `Apply`, and `Close`. Selecting `rectangle`-icon, we draw a rectangle with the aid of a mouse (click-drag); its name `R1` is introduced by MATLAB. The exact coordinates of the cursor appear in the top right of the window. To check and correct our coordinates, we double-click on `R1` and enter the exact coordinates.

2°. We specify the boundary conditions. If we select `Boundary->Boundary Mode`, the boundary will appear as a red arrow, indicating the type of condition (red, blue, and green for Dirichlet, Neumann, and mixed conditions) and the direction. Then, by double-clicking on the bottom horizontal line, we set the first boundary condition $u(x,0)=0$ by introducing $h=1$ and $r=0$ in the corresponding menu (since the Dirichlet boundary conditions have the form $hu=r$). Next, we specify the remaining three boundary conditions. Then, by selecting `File->Save As`, we save this domain as M-file `Domain1Poisson.m`.

3°. Selecting `PDE->PDE Specification`, we define the type of PDE: `elliptic` (the type of the nonlinear Poisson equation), $c=-1$, $a=0$, and $f(u)=\sin(u)$.

4°. Selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh. Then, by selecting a `triangle-inside-a-triangle` icon, we can refine the initial mesh.

5°. By selecting `Solve->Parameters`, we specify some parameters for solving the nonlinear partial differential equation: `nonlinear solver`, the `nonlinear tolerance`, and `Jacobian` (fixed). Then, selecting `Solve->Solve PDE`, we obtain the numerical solution. By default, the solution appears as a color-scale contour plot.

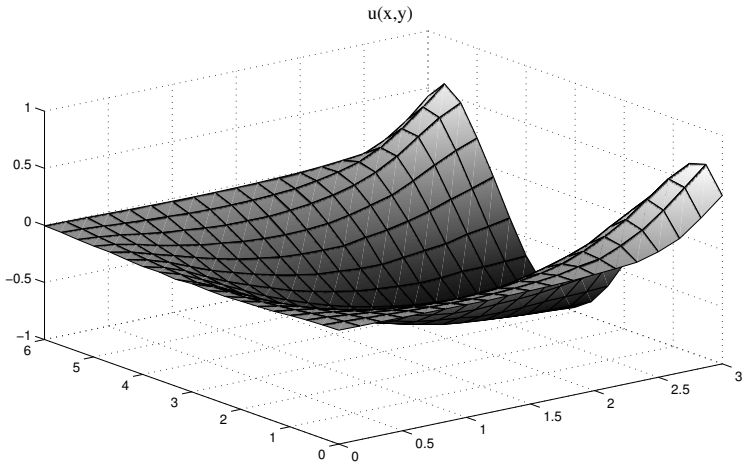

Figure 41.4. Surface plot of the solution of the nonlinear Poisson equation.

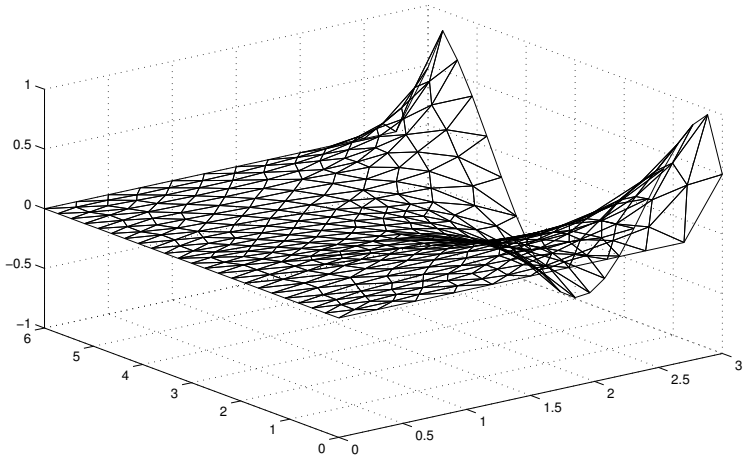

Figure 41.5. Mesh plot for the solution of the nonlinear Poisson equation.

6°. Selecting `Plot->Parameters`, we can set various options for generating a figure, e.g., `Height (3-D plot)`, `Plot in x-y grid`, `Color`, `colormap` (`gray`), and `Show Mesh`. Then, by selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained (in the MATLAB graphics editor) and create the final plot shown in Fig. 41.4.

Comment. The MATLAB graphics editor has various options for creating an appropriate figure (e.g., interactive rotation, `Tools->Rotate 3D`).

In addition, we show how to solve this problem in another way, i.e., by writing an M-file and by applying the *export facilities* of the GUI. This can be very useful when solving complicated problems.

1°. We type `pdetool` and open the M-file (created recently) `Domain1Poisson.m`. We have created the domain $\mathcal{D}$ and defined boundary conditions using the graphical user interface (GUI). Then, we export the *decomposed geometry* (i.e., the domain $\mathcal{D}$) and the *boundary conditions* to the workspace by selecting `Boundary->Export Decomposed Geometry, Boundary Conditions`. By default, the decomposed geometry

and boundary conditions are described by g and b, respectively. In our case, g is `Decomposed Geometry Matrix` and b is `Boundary Condition Matrix`.

2°. We create the triangular mesh using the GUI. Then, selecting `Mesh->Export Mesh`, we export the mesh to the workspace. By default, the mesh is described by p, e, and t. In our case, p is `Point Matrix`, e is `Edge Matrix`, and t is `Triangle Matrix`.

3°. Now we can solve the nonlinear Poisson equation with the following M-file `Elliptic2D1.m`:

```
echo on; format long;
g, b, p, e, t
c=-1; a=0; f='sin(u)'; tol=1e-3;
[p, e, t]=refinemesh(g, p, e, t);
u=pdenonlin(b, p, e, t, c, a, f, 'Tol',tol)
pdeplot(p, e, t, 'zdata', u, 'mesh', 'on')
echo off
```

In this script M-file, we use the export variables, g, b, p, e, t, the PDE coefficients, $c = -1$, $a = 0$, $f(u) = \sin(u)$, and the tolerance, tol=$1e-3$, for solving PDE with the aid of `pdenonlin` function. Also, we refine the mesh, and the new mesh is returned using the same matrix variables. Finally, we draw the solution and the mesh, which are shown in Fig. 41.5.

Comment. The M-file `Domain1Poisson.m` can be observed by `type Domain1Poisson` and modified using MATLAB editor.

41.3.4. Systems of Nonlinear Elliptic PDEs in Two Space Dimensions

Now consider systems of N nonlinear elliptic PDEs in two space dimensions (2D) and time over the domain $\mathcal{D}$, i.e., systems of the form

$$-\nabla \cdot (c(\mathbf{u}) \otimes \nabla \mathbf{u}) + a(\mathbf{u})\mathbf{u} = \mathbf{f}(\mathbf{u}),$$

where $c(\mathbf{u})$ is an $N \times N \times 2 \times 2$ tensor, $a(\mathbf{u})$ is an $N \times N$ matrix, $\mathbf{f}(\mathbf{u})$ and $\mathbf{u}$ are column vectors of length N, respectively, describing flux and unknown functions. In the MATLAB notation, $\nabla \cdot (c(\mathbf{u}) \otimes \nabla \mathbf{u})$ means the $N \times 1$ matrix with $(i,1)$-element

$$\sum_{j=1}^{N} \left(\frac{\partial c_{ij11}(u_j)}{\partial x} \frac{\partial u_j}{\partial x} + \frac{\partial c_{ij12}(u_j)}{\partial x} \frac{\partial u_j}{\partial y} + \frac{\partial c_{ij21}(u_j)}{\partial y} \frac{\partial u_j}{\partial x} + \frac{\partial c_{ij22}(u_j)}{\partial y} \frac{\partial u_j}{\partial y} \right).$$

In general, the boundary conditions are mixed (i.e., for each point on the boundary a combination of Dirichlet and generalized Neumann conditions) and have the form

$$h\mathbf{u} = \mathbf{r}, \quad \vec{n} \cdot (c \otimes \nabla \mathbf{u}) + q\mathbf{u} = \mathbf{g} + h'\boldsymbol{\mu},$$

where h is a matrix $M \times N$ ($M \geq 0$) for M Dirichlet conditions, $h'\boldsymbol{\mu}$ is a source in the generalized Neumann condition, and $\boldsymbol{\mu}$ are the Lagrange multipliers. The h, $\mathbf{r}$, q, and $\mathbf{g}$ are the data to be specified. In the MATLAB notation, $\vec{n} \cdot (c \otimes \nabla \mathbf{u})$ means the $N \times 1$ matrix with $(i,1)$-element

$$\sum_{j=1}^{N} \left(\cos\alpha\, c_{ij11} \frac{\partial u_j}{\partial x} + \cos\alpha\, c_{ij12} \frac{\partial u_j}{\partial y} + \sin\alpha\, c_{ij21} \frac{\partial u_j}{\partial x} + \sin\alpha\, c_{ij22} \frac{\partial u_j}{\partial y} \right),$$

where the outward normal vector on the boundary is $\vec{n} = (\cos\alpha, \sin\alpha)$.

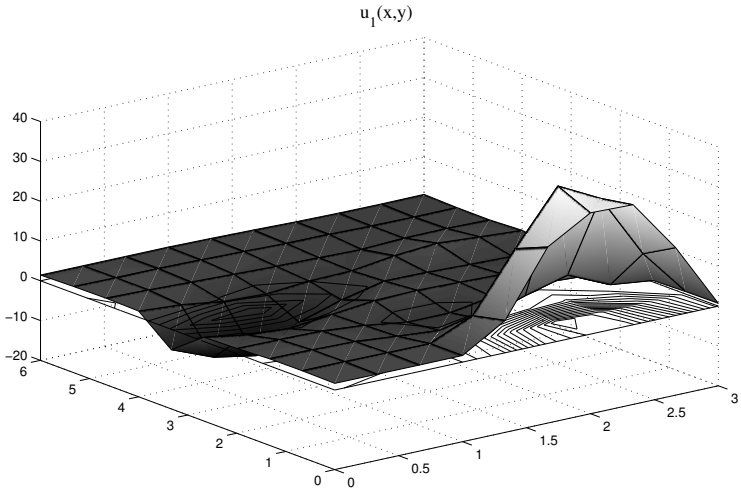

Figure 41.6. Surface plot with contour lines of the solution $u_1(x,y)$.

A generalized Neumann boundary condition is defined for $M = 0$, a Dirichlet boundary condition is defined for $M = N$, and a mixed boundary condition is defined for $0 < M < N$. These systems can be solved with the aid of the `PDE Toolbox`.

Let us consider the special case of the two-dimensional system of the form

$$\begin{aligned} -\nabla\cdot(c_{11}\nabla u_1) - \nabla\cdot(c_{12}\nabla u_2) + a_{11}u_1 + a_{12}u_2 = f_1,\\ -\nabla\cdot(c_{21}\nabla u_1) - \nabla\cdot(c_{22}\nabla u_2) + a_{21}u_1 + a_{22}u_2 = f_2. \end{aligned} \tag{41.3.4.1}$$

The Dirichlet boundary conditions are

$$h_{11}u_1 + h_{12}u_2 = r_1, \quad h_{21}u_1 + h_{22}u_2 = r_2. \tag{41.3.4.2}$$

The generalized Neumann boundary conditions are

$$\begin{aligned} \vec{n}\cdot(c_{11}\nabla u_1) + \vec{n}\cdot(c_{12}\nabla u_2) + q_{11}u_1 + q_{12}u_2 = g_1,\\ \vec{n}\cdot(c_{21}\nabla u_1) + \vec{n}\cdot(c_{22}\nabla u_2) + q_{21}u_1 + q_{22}u_2 = g_2. \end{aligned}$$

The mixed boundary conditions are

$$\begin{aligned} &h_{11}u_1 + h_{12}u_2 = r_1,\\ &\vec{n}\cdot(c_{11}\nabla u_1) + \vec{n}\cdot(c_{12}\nabla u_2) + q_{11}u_1 + q_{12}u_2 = g_1 + h_{11}\mu,\\ &\vec{n}\cdot(c_{21}\nabla u_1) + \vec{n}\cdot(c_{22}\nabla u_2) + q_{21}u_1 + q_{22}u_2 = g_2 + h_{12}\mu. \end{aligned}$$

Example 41.5. *System of nonlinear elliptic equations.* Consider the boundary value problem for two nonlinear Poisson equations in two space variables:

$$\begin{aligned} &(u_1)_{xx} + (u_1)_{yy} = -\sin(u_2^2)\cos(u_1^2),\\ &(u_2)_{xx} + (u_2)_{yy} = -\sin(u_2^2)\cos(u_1^2),\\ &u_1(x,0) = 0, \quad u_1(x,6) = 0, \quad u_1(0,y) = 0, \quad u_1(3,y) = \cos(y),\\ &u_2(x,0) = 0, \quad u_2(x,6) = 0, \quad u_2(0,y) = 0, \quad u_2(3,y) = \sin(y), \end{aligned}$$

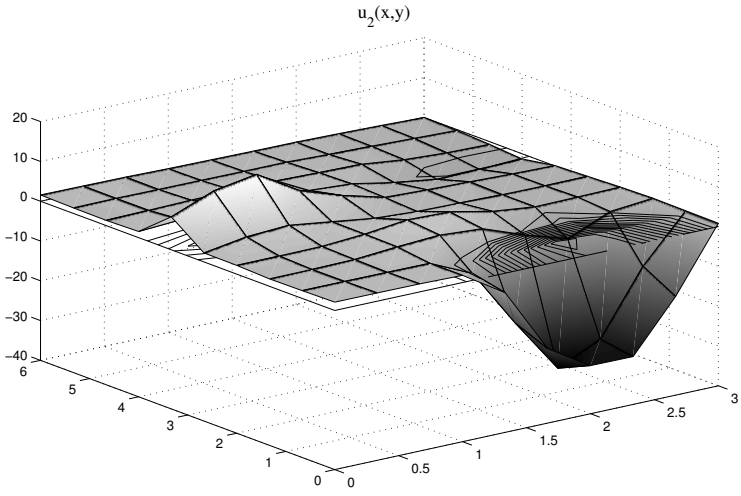

Figure 41.7. Surface plot with contour lines of the solution $u_2(x,y)$.

describing a potential field $\mathbf{u}(x,y)$ in a bounded domain $\mathcal{D} = \{0 \le x \le 3,\ 0 \le y \le 6\}$. We solve this problem with the aid of the GUI.

1°. We draw the domain $\mathcal{D} = \{0 \le x \le 3,\ 0 \le y \le 6\}$ (as in the previous example). Then, selecting `Options->Application->Generic System`, we indicate that we work with a *generic system*, i.e., a system of two equations (it is possible to work with systems of arbitrary dimension by writing programs).

2°. We specify the boundary conditions (as in the previous example), but for a system of two equations. When selecting `Boundary->Boundary Mode`, the boundaries and their directions will appear as *red arrows*. According to equations (41.3.4.2), we set the Dirichlet boundary conditions (e.g., $u_1(x,0) = 0$ by introducing $h_{11} = h_{12} = h_{21} = h_{22} = 0$ and $r_1 = r_2 = 0$). Selecting `File->Save As`, we save this domain as M-file `Domain1PoissonSys.m`.

3°. Selecting `PDE->PDE Specification`, we define the type of the PDE system: `elliptic`, and introduce, according to the form of the system (41.3.4.1), the coefficients of the PDE system: $c_{11} = c_{121} = c_{21} = c_{22} = -1$, $a_{11} = a_{12} = a_{21} = a_{22} = 0$, $f_1 = f_2 = -\sin(u_2^2)\cos(u_1^2)$, or, according to the MATLAB syntax, `-sin(u(2).^2).*cos(u(1).^2)` (with the array operations `.*` and `.^`).

4°. Selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh.

5°. Selecting `Solve->Parameters`, we specify the parameters for solving nonlinear PDE: `nonlinear solver`, the `nonlinear tolerance` (`1e-4`), `Jacobian` (`lumped`), and `norm` (`inf`). Then, selecting `Solve->Solve PDE`, we obtain the numerical solution. By default, the first solution $u_1(x,y)$ appears (or `u`, in MATLAB notation). Selecting `Plot->Parameters->Property` (`v`), we plot the second solution $u_2(x,y)$.

6°. Then, selecting `Plot->Parameters` and setting various options (`Height (3-D plot)`, `Plot in x-y grid`, `Contour`, `Color`, `colormap` (`gray`), `Show Mesh`), we create final figures shown in Figs. 41.6 and 41.7.

41.3.5. Nonlinear Elliptic PDEs. Geometrical Models

`PDE Toolbox` allows us to describe and work with a more complicated geometry, which can consist of various solid objects. Various geometrical models can be constructed interactively (using `pdetool` GUI) or via M-files (using `pdegeom`, `pdebound`).

We can devise a *Constructive Solid Geometry* (CSG) model of the geometry by drawing solid objects that can overlap. By default, each object is assigned a unique name (e.g., `C1`, `R1`), which can be changed. There are four types of solid objects: *circle*, *polygon*, *rectangle*,

and *ellipse*, which represent the set of points inside an object. It is possible to move, rotate, cut, and paste among the selected objects.

The CSG models are described by the *Geometry Description Matrix* `gd`. The current `gd` matrix can be viewed in the main workspace by selecting the `Draw->Export Geometry Description, Set Formula, Labels`. Each column in the `gd` matrix corresponds to an object in the CSG model.

By default, the resulting geometrical model is the union of all objects. However, the solid objects can be combined by typing a *set formula* (which is displayed in the GUI and can be changed, e.g., `R1-C1-C2`). The resulting geometrical model is the set of points for which the set formula evaluates to true.

A set formula `sf` is expressed with the set of variables (that correspond to the name of each object) and the operators `+ * -` (that correspond to the set operations union, intersection, and set difference, respectively).

Example 41.6. *Nonlinear Poisson–Boltzmann equation.* Consider the following boundary value problem for the two-dimensional nonlinear Poisson–Boltzmann equation defined on an irregular domain $\mathcal{D} = R_1 - \Omega$:

$$u_{xx} + u_{yy} = \kappa \sinh(u) - g(x,y), \tag{41.3.5.1}$$

$$u(x,-1) = x^2 + 1 + e^x \cos(1), \quad u(x,1) = x^2 + 1 + e^x \cos(1), \tag{41.3.5.2}$$

$$u_x(-1,y) = 0, \quad u_y(1,y) = 0, \tag{41.3.5.3}$$

$$u(-0.541,-0.173) = 0.896, \quad u(0.392,0.538) = 1.713, \tag{41.3.5.4}$$

where κ is a positive constant (e.g., $\kappa = 1$), $g(x,y) = 4 - \sinh(x^2 + y^2 + e^x \cos(y))$, R_1 is a rectangle, $R_1 = [-1,-1] \times [-1,-1]$ with the boundary ∂R_1, and Ω is an arbitrary multi-connected domain within R_1 with boundary $\partial\Omega$; in our case, the interior region of the two ellipses E_1 and E_2, defined by the equations: $(x + 0.35)^2/0.25^2 + (y + 0.35)^2/0.3^2 - 1 = 0$ and $(x - 0.35)^2/0.2^2 + (y - 0.35)^2/0.2^2 - 1 = 0$. The nonlinear Poisson–Boltzmann equation is defined outside Ω but within the rectangle R_1. Along the boundary $\partial\Omega$, a Dirichlet boundary condition (41.3.5.4) is defined.

According to different applications, different boundary conditions can be imposed along the boundary of the rectangle R_1; e.g., we set the Dirichlet boundary conditions (41.3.5.2) and the Neumann boundary conditions (41.3.5.3).

The nonlinear Poisson–Boltzmann equation (41.3.5.1) has been used in many applications, e.g., models describing bio-molecular processes and electrostatic interactions between colloidal particles [see e.g., Bowen and Williams (2002), S. L. Carnie, D. Y. C. Chan, and J. Stankovich (1994), Z. Li, C. V. Pao, and Z. Qiao (2007)].

We solve this problem with the aid of the graphical user interface in the `Generic Scalar` mode (by selecting `Options->Application->Generic Scalar`).

1°. Selecting `Options->Grid` and `Options->Axes Limits`, we generate the grid. We draw the rectangle `R1`, $[-1,-1] \times [-1,-1]$, and the two ellipses, `E1` and `E2`, defined with the parameters `X-center`, -0.35, `Y-center`, -0.35, `A-semiaxes`, 0.25, `B-semiaxes`, 0.3, `Rotation`, 0 (for `E1`), and `X-center`, 0.35, `Y-center`, 0.35, `A-semi axes`, 0.2, `B-semiaxes`, 0.2, and `Rotation`, 0 (for `E2`). Then we draw the domain $\mathcal{D} = R_1 - \Omega = R_1 - E_1 - E_2$ by typing the set formula `R1-E1-E2`.

2°. We specify the boundary conditions. If we select `Boundary->Boundary Mode`, the boundaries will appear as red arrows, indicating the type of conditions (i.e., Dirichlet conditions) and the direction. Then, by double-clicking on the bottom horizontal line, we set the first boundary condition in (41.3.5.2), $u(x,-1) = x^2 + 1 + e^x \cos(1)$, by introducing `h=1` and `r=x.^2+1+exp(x).*cos(1)` in the corresponding menu. In similar manner, we specify the remaining Dirichlet boundary conditions on the upper horizontal line, by typing `h=1` and `r=x.^2+1+exp(x).*cos(1)`, and on the two ellipses `E1` and `E2`, by typing, respectively, `h=1` and `r=0.896` and `h=1` and `r=1.713`. Then we specify the two remaining Neumann boundary conditions by typing `q=0` and `g=0`. Finally, by selecting `File->Save As`, we save this domain as M-file `PoissonBoltzmann.m`.

3°. Selecting `PDE->PDE Specification`, we define the type of PDE:
`elliptic`, `c=-1.0`, `a=0.0`, `f=sinh(u)-4.0-sinh(x.^2+y.^2+exp(x).*cos(y))`.

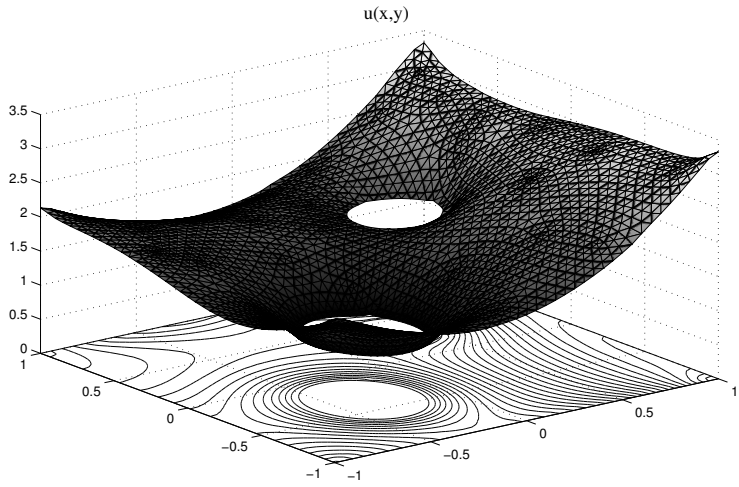

Figure 41.8. Mesh plot with the contour lines of the solution of the Poisson–Boltzmann equation.

4°. Selecting `Mesh->Initialize Mesh`, we generate a triangular finite element mesh. Then, selecting `Mesh->Refine Mesh`, we can refine the initial mesh (e.g., two times), and selecting `Mesh->Jiggle Mesh`, we can jiggle the mesh (for improving the triangle quality).

5°. Selecting `Solve->Parameters`, we specify some parameters for solving nonlinear PDE: `nonlinear solver`, the `nonlinear tolerance`, `1e-9`, and `Jacobian`, `fixed`, `norm`, `inf`. Then selecting `Solve->Solve PDE`, we obtain the numerical solution. By default, the solution appears as a color-scale contour plot.

6°. Selecting `Plot->Parameters`, we can set various options for generating a figure, e.g., `Height (3-D plot)`, `Contour`, `Color`, `colormap` (gray), `Show Mesh`. Then, selecting the mesh and `Edit->Current Object Properties`, we can edit the plot obtained and create the final plot shown in Fig. 41.8.

⊙ *References for Section* 41.3: A. Kolmogorov, I. Petrovsky, and N. Piskunov (1937), R. D. Skeel and M. Berzins (1990), J. D. Murray (1993), S. L. Carnie, D. Y. C. Chan, and J. Stankovich (1994), W. E. Schiesser (1994), D. Harries (1998), W. R. Bowen and P. M. Williams (2002), H. J. Lee and W. E. Schiesser (2004), Z. Li, C. V. Pao, and Z. Qiao (2007), W. E. Schiesser and G. W. Griffiths (2009).

41.4. Solving Cauchy Problems. Method of Characteristics

In this section, we consider the method of characteristics, which allows us to reduce a PDE to a system of ODEs along which the given PDE with some initial data (the Cauchy data) is integrable. Once the system of ODEs is found, it can be solved along the characteristic curves and transformed into a general solution for the original PDE [e.g., see Schiesser and Griffiths (2009)].

Example 41.7. *Method of characteristics. Classical Cauchy problem.* Consider the initial value problem

$$u_t + t^2 u u_x = 7, \quad u(x,0) = x.$$

By applying the method of characteristics, we can obtain

$$\frac{dU(t)}{dt} = 7, \quad U(t) = 7t + X_0, \quad \frac{dX(t)}{dt} = t^2 U(t), \quad \frac{dX(t)}{dt} = t^2(7t + X_0),$$

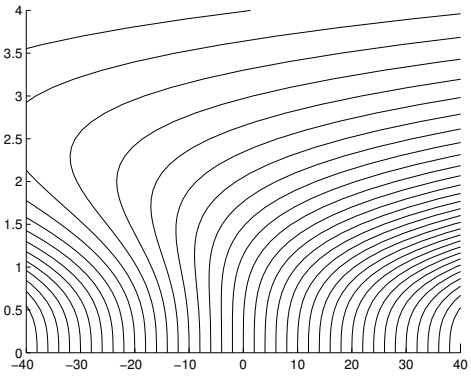

Figure 41.9. Characteristic curves for $u_t + t^2 u u_x = 7$.

and the characteristic curves are defined parametrically as $X(t) = \frac{7}{4}t^4 + \frac{1}{3}X_0 t^3 + X_0$, where $t \in [t_0, t_f]$. Also we can find the solution of this Cauchy problem, $u(x,t) = 7t - \dfrac{3}{4}\dfrac{7t^4 - 4x}{t^3+3}$, and plot the characteristic curves; see Fig. 41.9.

```
clear all; close all; t=0:0.1:4; hold on;
figure(1)
for x0=-40:2:40
 ChCur=(7/4).*t.^4+(1/3).*x0.*t.^3+x0; plot(ChCur,t,'b-');
end
hold off; set(gca,'XLim',[-40 40],'YLim',[0 4]);
```

Example 41.8. *Inviscid Burgers equation. Shock waves and rarefaction waves.* Consider the initial value problem for the inviscid Burgers equation*:

$$u_t + u u_x = 0, \quad u(x,0) = f(x).$$

This equation, studied first by Poisson and Riemann in the 19th century [see Whitham (1974)], and its generalizations are very important for modeling various physical and chemical processes (e.g., in gas dynamics, traffic flow, flood waves in rivers, chemical reactions, etc.).

By applying the method of characteristics, we obtain

$$\frac{dU(t)}{dt} = 0, \quad U(t) = f(X_0), \quad \frac{dX(t)}{dt} = U(t), \quad \frac{dX(t)}{dt} = f(X_0),$$

and the characteristic curves are defined parametrically as $X(t) = f(X_0)t + X_0$, where $t \in [t_0, t_f]$. We consider a more general case, when $f(x)$ is a piecewise nonlinear function. Two special cases are possible:

1°. Let the initial data $f(x)$ be a decreasing piecewise nonlinear function, $f'(x) < 0$, for example,

$$f(x) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{1}{3}x^2(4-x), & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$$

In this case, the characteristic lines will intersect each other for $t > 0$ (see the first plot in Fig. 41.10), and $u(x,t)$ is a multi-valued solution from the mathematical point of view (see the first plot in Fig. 41.11) and makes

*In scientific literature, this equation can appear under other names, e.g., the Hopf equation or the nonlinear transport equation.

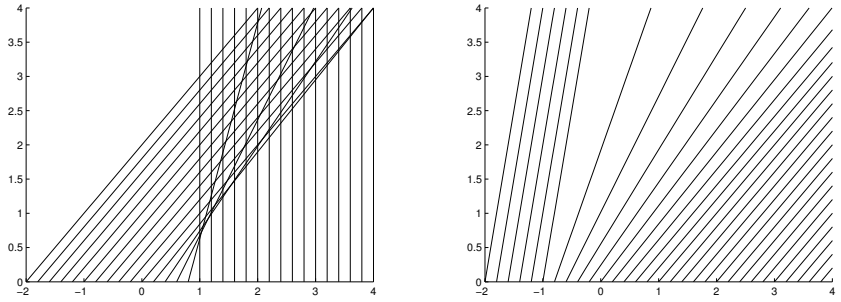

Figure 41.10. Burgers equation: characteristic curves for shock and rarefaction waves.

no sense from the physical viewpoint. There exists a critical time, $t = t_{cr} > 0$, when the first two characteristic lines intersect each other and the profile of the solution becomes vertical; i.e., $u_x(x_{cr},t) \to \infty$ as $t \to t_{cr}$. It can be shown that the critical time is $t_{cr} = \min\{-1/f'(x)\}$, where $f'(x) < 0$. In our case $t_{cr} = 9/16$. Physically, such solutions represent the formation of *shock waves*. Mathematically, this solution can be continued beyond $t_{cr} = 9/16$ as a *weak solution* with a discontinuity (see the problem below).

```
clear all; close all; hold on; a=-2; b=4; Hx=0.2;
t0=0; tf=4; Ht=0.1; t=t0:Ht:tf; x=a:Hx:b;
Nt=length(t); Nx=length(x);
figure(1);
 for x0=a:Hx:b ChCur=BurgersFunShock(x0).*t+x0; plot(ChCur,t,'k-'); end
 set(gca,'XLim',[a b],'YLim',[t0 tf]); hold off;
figure(2);
 x=a:Hx:b; n=length(t);
 f=1*(x<=0)+((x<=1)-(x<=0)).*(1-x.^2.*(4-x)/3)+0*(x>=1);
 plot(x,f,'k-')
 set(gca,'XLim',[a b],'YLim',[-0.5 1.5]); hold on;
 for m=1:Nt  plot(x+t(m)*f,f,'k-'); end
function f=BurgersFunShock(x)
 if x<=0; f=1;  elseif x>=0 && x<=1; f=1-x.^2.*(4-x)/3; else f=0; end
```

2°. Let the initial data $f(x)$ be an increasing piecewise nonlinear function, and let $f'(x) \geq 0$, e.g.,

$$f(x) = \begin{cases} 0.2, & x \leq -1, \\ 1 + \frac{1}{10}(x-1)^2(x-1), & -1 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

In this case, the characteristic lines never intersect each other for $t \geq 0$ (see the second plot in Fig. 41.10) and the solution is well defined for $t \geq 0$ (see the second plot in Fig. 41.11). Physically, such solutions represent *rarefaction waves*.

```
clear all; close all; hold on;
a=-2; b=4; Hx=0.2; t0=0; tf=4; Ht=0.2;
t=t0:Ht:tf; x=a:Hx:b;
Nt=length(t); Nx=length(x);
figure(1);
 for x0=a:Hx:b ChCur=BurgersFunRar(x0).*t+x0; plot(ChCur,t,'k-'); end
 set(gca,'XLim',[a b],'YLim',[t0 tf]); hold off;
figure(2);
```

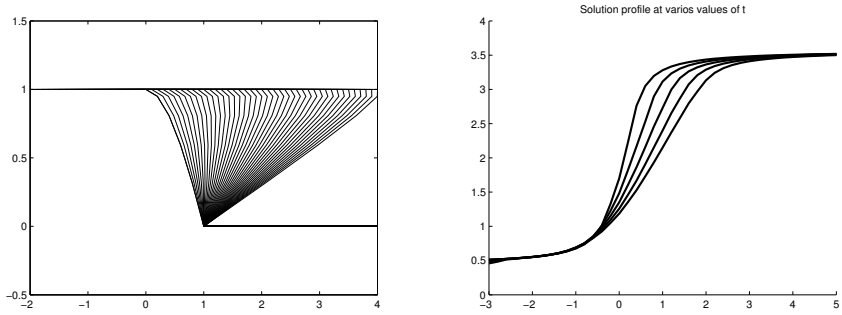

Figure 41.11. Burgers equation: shock and rarefaction waves.

```
 x=a:Hx:b; n=length(t);
 f=0.2*(x<-1)+((x<1.01)-(x<-1)).*(1+(x-1).^2.*(x-1)/10)+1*(x>1);
 plot(x,f,'k-')
 set(gca,'XLim',[a b],'YLim',[t0 1.5]); hold on;
 for m=1:Nt  plot(x+t(m)*f,f,'k-'); end
function f=BurgersFunRar(x)
 if x<=-1 f=0.2; elseif x>=-1 && x<=1 f=1+(x-1).^2.*(x-1)/10; else f=1; end
```

Example 41.9. *Inviscid Burgers equation. Shock waves and weak solutions.* Consider the same initial-value problem for the inviscid Burgers equation:

$$u_t + uu_x = 0, \quad u(x,0) = f(x).$$

We choose the initial data $f(x)$ as a decreasing piecewise nonlinear function, e.g.,

$$f(x) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{1}{4}x^2(3-x), & 0 \le x \le 2, \\ 0, & x > 2. \end{cases}$$

By applying the method of characteristics for a short time ($t < t_{cr}$), we can obtain a smooth solution. The previous example of inviscid Burgers equation shows that the characteristics may intersect with each other (after $t = t_{cr}$) and the solution becomes multiple-valued. Therefore, the classical solution exists only for $t < t_{cr}$.

Let us continue the solution for $t > t_{cr}$ as a weak solution with discontinuity (when the profiles are vertical). By applying the *Rankine–Hugoniot condition*

$$s = \frac{F(u_L) - F(u_R)}{u_L - u_R} = \frac{1}{2}(u_L + u_R),$$

where $F = \frac{1}{2}u^2$, and by allowing the solution to be discontinuous (shock), we can determine the weak solution as follows:

$$u(x,t) = \begin{cases} u^L(x,t), & x < x(t), \\ u^R(x,t), & x > x(t), \end{cases}$$

where $x(t)$ is the shock trajectory. In this case, the vertical segment of the profile indicates the formation of a shock at time t_i, and there exists a time $t_{st} > t_{cr}$ when the weak solution becomes a "step wave" (see Fig. 41.12).

```
clear all; close all;
a=-2; b=4; k=10; hx=0.1; t0=0; tf=5; ht=0.2;
t=t0:ht:tf; x=a:hx:b;
Nt=length(t); Nx=length(x);
f=1*(x<0)+((x<2.001)-(x<0)).*(1-x.^2.*(3-x)/4)+0*(x>2);
```

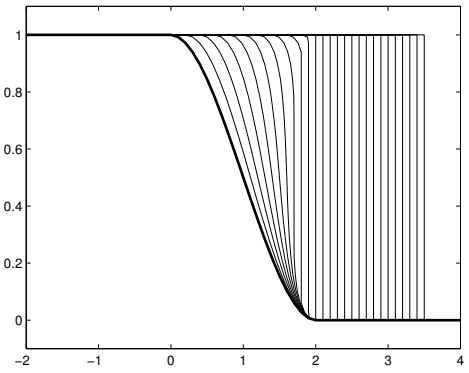

Figure 41.12. Burgers equation: formation of shock waves.

```
xSh=1; s=(max(f)+min(f))/2;
for i=1:Nt
 for j=1:Nx
  if (x(j)+t(i)*f(j)-(xSh+s*t(i)))/(x(j)-1)>0 U(i,j)=x(j)+t(i)*f(j);
  else U(i,j)=xSh+s*t(i); end
 end
end
fMax=max(f); fMin=min(f); delta=(fMax-fMin)/k;
plot(x,f,'k-','LineWidth',2);
axis([a,b,fMin-delta,fMax+delta]); hold on;
for i=2:Nt plot(U(i,:),f,'k-'); end
hold off
```

⊙ *References for Section* 41.4: G. B. Whitham (1974), W. E. Schiesser and G. W. Griffiths (2009).

41.5. Constructing Finite-Difference Approximations

Now we show the helpful role of MATLAB for applying and developing various finite-difference approximations to construct numerical solutions of initial value problems and initial-boundary value problems for nonlinear PDEs (e.g., parabolic and hyperbolic equations).

To approximate nonlinear PDEs by finite differences, we have to generate a mesh in a domain $\mathcal{D}$; e.g., $\mathcal{D} = \{x_L \le x \le x_R, t_0 \le t \le t_f\}$.

We assume (for simplicity) that the sets of lines of the mesh are equally spaced and the dependent variable is $u(x,t)$ in the given PDE. We write h and k for the line spacings and define the mesh points as follows:

$$x_j = x_L + (j-1)h, \quad t_n = t_0 + (n-1)k,$$

where $j = 1,\ldots,Nx+1$, $n = 1,\ldots,Nt+1$, $h = (x_R - x_L)/Nx$, and $k = (t_f - t_0)/Nt$.

We calculate approximations to the solution at these mesh points; these approximate points will be denoted by $u_{j,n} \approx u(x_j,t_n)$. We approximate the derivatives in the given equation by finite differences (of various types) and then solve the resulting difference equations.

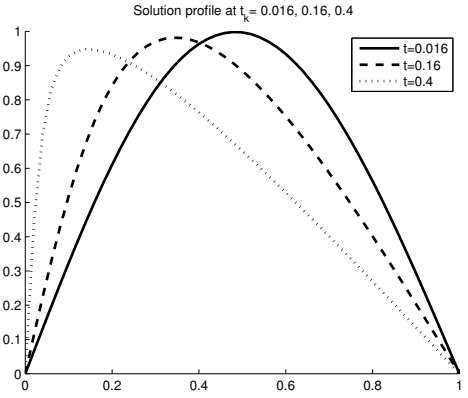

Figure 41.13. Burgers equation: solution profiles at $t_k = 0.016, 0.16, 0.4$.

Example 41.10. *Burgers equation.* Consider the initial boundary value problem for the Burgers equation

$$u_t = \nu u_{xx} + uu_x, \quad u(x,0) = \sin(\pi x), \quad u(x_L,t) = 0, \quad u(x_R,t) = 0$$

in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$, where $x_L = 0$, $x_R = 1$, $t_0 = 0$, $t_f = 0.4$, $\nu = 0.009$.

Let us generate the rectangular mesh

$$x_j = x_L + (j-1)h, \quad t_n = nk,$$

where $j = 1,\ldots,Nx+1$, $n = 1,\ldots,Nt$, $h = (x_R - x_L)/Nx$, and $k = t_f/Nt$. We denote the approximate solution of $u(x,t)$ at the mesh point (j,n) as $u_{j,n}$.

In *forward difference method*, the second derivative u_{xx} is replaced with the central difference approximation (CDA), and the first derivatives u_t and u_x are replaced by the forward difference approximation (FWDA). The finite difference scheme for the nonlinear Burgers equation has the form:

$$u_{j,n+1} = u_{j,n} + r(u_{j+1,n} - 2u_{j,n} + u_{j-1,n}) + (k/h)u_{j,n}(u_{j+1,n} + u_{j,n}),$$

where $r = \nu k/h^2$.

In this *explicit finite difference scheme*, the unknown value $u_{j,n+1}$ (at the $n+1$ step) is determined from three known values $u_{j-1,n}$, $u_{j,n}$, and $u_{j+1,n}$ (at the n th step).

We construct an approximate numerical solution of the initial-boundary value problem by applying the forward difference method and plot (see Fig. 41.13) the solution profiles at different times (for example, $t_k = 0.016, 0.16, 0.4$) as follows:

```
clear all; close all; nu=0.009; L=1; xL=0; xR=1; Nx=50; t0=0; tf=0.4; Nt=25;
h=(xR-xL)/Nx; k=(tf-t0)/Nt; r=nu*k/h^2;
x=xL:h:xR; u=zeros(Nx+1,Nt); f=sin(pi*x/L);
for n=1:Nt  t=n*k;
  if n==1
    for j=2:Nx u(j,n)=f(j)+r*(f(j+1)-2*f(j)+f(j-1))+k/h*f(j)*(f(j+1)-f(j)); end;
    u(1,n)=0; u(Nx+1,n)=0;
  else
    for j=2:Nx
     u(j,n)=u(j,n-1)+r*(u(j+1,n-1)-2*u(j,n-1)+u(j-1,n-1))+...
            k/h*u(j,n-1)*(u(j+1,n-1)-u(j,n-1));
    end;
    u(1,n)=0; u(Nx+1,n)=0;
  end
end; T=1*k:k:Nt*k;
```

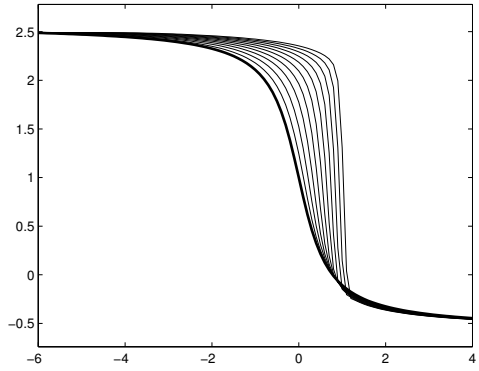

Figure 41.14. Numerical solution of inviscid Burgers equation.

```
figure(1);
  colormap(gray); surf(x,T,u'); xlabel('x'); ylabel('y'); zlabel('u');
  title('Numerical solution of viscous Burgers equation');
figure(2);
  hold on; plot(x,u(:,1),'k-',x,u(:,10),'k--',x,u(:,Nt),'k:','LineWidth',2);
  title('Solution profile at t_k= 0.016, 0.16, 0.4');
  legend('t=0.016','t=0.16','t=0.4');
figure(3);
  G=plot(x,u(:,1),'LineWidth',3,'erase','xor');
  for j=2:Nt set(G,'xdata',x,'ydata',u(:,j)); pause(0.1); end
  title('Animation of solution'); xlabel('x'); ylabel('u(x,t_k)');
```

Example 41.11. *Inviscid Burgers equation. Initial value problem.* Now consider the inviscid Burgers equation

$$u_t + uu_x = 0.$$

This equation is of hyperbolic type. Let us solve the initial value problem

$$u_t + c(u)u_x = 0, \quad u(x,0) = -\arctan(ax) + b,$$

in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$, where $c(u) = u$, $a = 2$, $b = 1$, $t_0 = 0$, and $t_f = 1$.

Note that this equation can be rewritten in the *conservation form*, i.e., as the physical conservation law of the form $u_t + F_x(u) = 0$, where $F(u) = \frac{1}{2}u^2$, $F'(u) = c(u) \ge 0$ (for all values of u). For the discretization of the problems, we use this form of the Burgers equation.

Let us generate the rectangular mesh: $x_j = a + (j-1)h$ and $t_n = nk$ ($j = 1,\ldots,Nx-1$, $n = 1,\ldots,Nt$, $h = (x_R - x_L)/Nx$, $k = t_f/Nt$). We denote the approximate solution at the mesh point (j,n) by $u_{j,n}$. We replace the derivative $F_x(u)$ by a backward difference approximation (BWDA) in the quantity $F(u)$. The finite-difference scheme for this nonlinear equation has the form

$$u_{j,n+1} = u_{j,n} - \frac{1}{r}\left(F(u_{j,n}) - F(u_{j-1,n})\right).$$

In this *upwind explicit finite difference scheme*, the unknown value $u_{j,n+1}$ (on $n+1$ step) is determined from the two known values $u_{j,n}$ and $u_{j-1,n}$ (at the n th step). This finite-difference scheme is valid for $c(u) \ge 0$, and if $c(u)$ changes sign, it is necessary to apply another method, e.g., Lax–Friedrichs, Lax–Wendroff, and Godunov methods [for details, see LeVeque (2007), Lax (1968), Lapidus and Pinder (1999), Larsson and Thomée (2008), Morton and Mayers (1995), Strikwerda (2004), Thomas (1995)].

We construct the approximate numerical solution of the initial value problem by applying the upwind method for the nonlinear conservation law as follows:

```
clear all; close all;
xL=-6; xR=4; Nx=100; h=(xR-xL)/Nx;
Nt=zeros(10); Nt(:)=10; K=length(Nt); k=0.01; r=h/k;
x=xL:h:xR; Nu=10; a=2; b=1; u=-atan(a*x)+b;
uMax=max(u); uMin=min(u); delta=(uMax-uMin)/Nu; CFC=uMax/r
plot(x,u,'k-','LineWidth',2);
axis([xL,xR,uMin-delta,uMax+delta]); j=[2:Nx+1]; hold on
for m=1:K
  for n=1:Nt(m) u(j)=u(j)-(F(u(j))-F(u(j-1)))/r; end;
  plot(x,u,'k-');
end; hold off
function value=F (u)
  value=0.5*u.^2;
```

Comment. In this finite-difference scheme, the spatial step h and the time step k must be chosen so that the *Courant–Friedrichs–Levy condition* (CFL) is satisfied, i.e., $c_{\max}/r \le 1$ with $c_{\max} = \max|c(u)| = \max|c(f(x))|$. In our case, the CFL constant is equal to 0.2488. In this program, `t1=Nt(1)*k`, ..., `tf=(Nt(1)*k)*10`.

Example 41.12. *Inviscid Burgers equation.* Now consider the following initial-boundary value problem for the inviscid Burgers equation:

$$u_t + uu_x = 0, \quad u(x,0) = \arctan(4x) + 2, \quad u(a,t) = 0$$

in the domain $\{a \le x \le b,\ 0 < t < T\}$, where $a = -4$, $b = 6$, and $T = 0.4$.

Since this equation can be rewritten in the *conservation form*, i.e., as the physical conservation law of the form $u_t + f_x(u) = 0$ (where $f(u) = \frac{1}{2}u^2$), we use this form for the discretization of the problem.

Let us generate the rectangular mesh: $X = a + ih$, $T = jk$ ($i = 0, \ldots, Nx$, $j = 0, \ldots, Nt$, $h = (b-a)/Nx$, $k = T/Nt$). We denote the approximate solution at the mesh point (i, j) by $U_{i,j}$.

In the *Lax method*, the derivative u_t is replaced by the forward difference approximation (FWDA), and the derivative $f_x(u)$, by the central difference approximation (CDA) (together with the corresponding substitution). The finite-difference scheme for the nonlinear diffusion equation has the form

$$u_{j,n+1} = \frac{1}{2}\left(u(j+1,n) + u(j-1,n)\right) - \frac{1}{4}(k/h)\left(u^2(j+1,n) - u^2(j-1,n)\right).$$

In this *explicit finite difference scheme*, the unknown value $u_{j,n+1}$ (on $n+1$ st step) is determined from three known values $u_{j-1,n}$, $u_{j,n}$, and $u_{j+1,n}$ (at the nth step). This finite-difference scheme is stable for $k/h < 1/|u|$; i.e., the stability condition depends on the solution. We construct an approximate numerical solution of the initial boundary value problem by applying the Lax method as follows:

```
clear all; close all;
T=0.4; a=-4; b=6; Nx=50; Nt=9; h=(b-a)/Nx; k=T/Nt;
x=a:h:b; u=zeros(Nx+1,Nt); f=atan(4*x)+2;
for n=1:Nt
  t=n*k;
  if n==1
    for j=2:Nx
      u(j,n)=0.5*(f(j+1)+f(j-1))-k/(2*h)*(f(j+1)^2/2-f(j-1)^2/2);
    end;
    u(1,n)=0;
  else
    for j=2:Nx
      u(j,n)=0.5*(u(j+1,n-1)+u(j-1,n-1))-k/(2*h)*(u(j+1,n-1)^2/2-u(j-1,n-1)^2/2);
    end;
    u(1,n)=0;
  end
end;
```

```
T=1*k:k:Nt*k;
figure(1);
  colormap(gray); surf(x,T,u'); xlabel('x'); ylabel('y'); zlabel('u');
  title('Numerical solution of inviscid Burgers equation');
figure(2);
  hold on; plot(x,u(:,1),'k-',x,u(:,3),'k-',x,u(:,5),'k-',x,u(:,7),'k-',...
               x,u(:,Nt),'k-','LineWidth',2);
  set(gca,'XLim',[-3 5],'YLim',[0 4]);
  title('Solution profile at varios values of t');
```

Example 41.13. *Nonlinear wave equation.* Consider the initial-boundary value problem for the nonlinear wave equation describing the motion of a fixed string,

$$u_{tt} = \frac{1}{c^2}\,u_{xx} + e^u, \quad u(x,0) = 0, \quad u_t(x,0) = \sin(4\pi x), \quad u(x_L,t) = 0, \quad u(x_R,t) = 0$$

in the domain $\mathcal{D} = \{x_L \le x \le x_R,\ t_0 \le t \le t_f\}$. We choose $x_L = 0$, $x_R = 0.5$, $t_0 = 0$, $t_f = 0.5$, and $c = 1/(4\pi)$.

In *explicit central difference method*, each second derivative is replaced by the CDA. The finite-difference scheme for the nonlinear wave equation has the form

$$u_{j,n+1} = 2(1-r)u_{j,n} + r(u_{j+1,n} + u_{j-1,n}) - u_{j,n-1} + \exp(u(j,n))(ck)^2,$$

where $r = (ck/h)^2$. In this finite difference scheme, we have one unknown value $u_{j,n+1}$ that depends explicitly on the four known values $u_{j,n}, u_{j+1,n}, u_{j-1,n}, u_{j,n-1}$ at the previous time steps n and $n-1$. To start the process, we have to know the values of u at the time steps $n = 0$ and $n = 1$. Thus, we can define the initial conditions at these time steps:

$$u_{j,0} = f(x_j) \quad \text{and} \quad u(x_j,0)_t \approx (u_{j,1} - u_{j,0})/k = g(x_j), \quad u_{j,1} = f(x_j) + kg(x_j).$$

We construct an approximate numerical solution of the initial-boundary value problem by applying the explicit finite difference method and plot the numerical solution in $\mathcal{D}$ as follows:

```
clear all; close all;
c=0.25; xL=0; xR=0.5; Nx=25; t0=0; tf=2; Nt=40; h=(xR-xL)/Nx;
k=(tf-t0)/Nt; r=(c*k/h)^2; x=xL:h:xR; u=zeros(Nx+1,Nt); f=0; g=sin(4*pi*x);
for n=1:Nt
  t=n*k;
  if n==1
    for j=2:Nx u(j,n)=f; end;
    u(1,n)=0;
    u(Nx+1,n)=0;
  elseif n==2
    for j=2:Nx u(j,n)=f+k*g(j); end;
    u(1,n)=0;
    u(Nx+1,n)=0;
  else
    for j=2:Nx
      u(j,n)=2*(1-r)*u(j,n-1)+r*(u(j+1,n-1)+u(j-1,n-1))-u(j,n-2)+...
             exp(u(j,n-1))*(c*k)^2;
    end;
    u(1,n)=0;
    u(Nx+1,n)=0;
  end
end;
T=1*k:k:Nt*k;
figure(1);
  colormap(gray); surf(x,T,u'); xlabel('x'); ylabel('y'); zlabel('u');
  title('Numerical solution of nonlinear wave equation');
```

```
figure(2);
  hold on; plot(x,u(:,1),'k-',x,u(:,10),'k--',x,u(:,Nt),'k:','LineWidth',2);
  title('Solution profile at t_k= 0.05, 0.5, 2');
  legend('t=0.05','t=0.5','t=2');
figure(3);
  G=plot(x,u(:,1),'LineWidth',3,'erase','xor');
  for n=1:Nt uMax=max(u(:,j)); uMin=min(u(:,j)); end
  Nu=2; delta=(uMax-uMin)/Nu; axis([xL,xR,uMin-delta,uMax+delta]);
  for j=2:Nt set(G,'xdata',x,'ydata',u(:,j)); pause(0.1); end
  title('Animation of solution'); xlabel('x'); ylabel('u(x,t_k)');
```

⊙ *References for Section* 41.5: P. D. Lax (1968), K. W. Morton and D. F. Mayers (1995), J. W. Thomas (1995), L. Lapidus and G. F. Pinder (1999), L. Strikwerda (2004), R. J. LeVeque (2007), S. Larsson and V. Thomée (2008).

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Supplements

Painlevé Transcendents Functional Equations

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Chapter 42
Painlevé Transcendents

42.1. Preliminary Remarks. Singular Points of Solutions

1°. Singular points of solutions to ordinary differential equations can be *fixed* or *movable*. The coordinates of fixed singular points remain the same for different solutions of an equation.* The coordinates of movable singular points vary depending on the particular solution selected (i.e., they depend on the initial conditions).

Listed below are simple examples of first-order ordinary differential equations and their solutions having movable singularities:

Equation	*Solution*	*Solution's singularity type*
$y'_z = -y^2$	$y = 1/(z - z_0)$	movable pole
$y'_z = 1/y$	$y = 2\sqrt{z - z_0}$	algebraic branch point
$y'_z = e^{-y}$	$y = \ln(z - z_0)$	logarithmic branch point
$y'_z = -y \ln^2 y$	$y = \exp[1/(z - z_0)]$	essential singularity

Algebraic branch points, logarithmic branch points, and essential singularities are called *movable critical points*.

2°. The Painlevé equations arise from the classification of the following second-order differential equations over the complex plane:

$$y''_{zz} = R(z, y, y'_z),$$

where $R = R(z, y, w)$ is a function rational in y and w and analytic in z. It was shown by P. Painlevé (1897–1902) and B. Gambier (1910) that all equations of this type whose solutions do not have movable critical points (but are allowed to have fixed singular points and movable poles) can be reduced to 50 classes of equations. Moreover, 44 classes out of them are integrable by quadrature or admit reduction of order. The remaining 6 equations are irreducible; these are known as the Painlevé equations and their solutions are known as Painlevé transcendents. Very often the six irreducible Painlevé equations are also referred to as the Painlevé transcendents.

The canonical forms of the Painlevé transcendents are given below in Sections 42.2 through 42.7. Solutions of the first, second, and fourth Painlevé transcendents have movable poles (no fixed singular points). Solutions of the third and fifth Painlevé transcendents have two fixed logarithmic branch points, $z = 0$ and $z = \infty$. Solutions of the sixth Painlevé transcendent have three fixed logarithmic branch points, $z = 0$, $z = 1$, and $z = \infty$.

It is significant that the Painlevé equations often arise in mathematical physics.

* Solutions of linear ordinary differential equations can only have fixed singular points, and their positions are determined by the singularities of the equation coefficients.

42.2. First Painlevé Transcendent

1°. The first Painlevé transcendent has the form

$$y''_{zz} = 6y^2 + z. \tag{1}$$

The solutions of the first Painlevé transcendent are single-valued functions of z that have finitely many poles in a finite portion of the plane. The function $y(z)$ is therefore meromorphic in the finite portion of the plane and can be represented as the ratio of two integral functions: $y(z) = [(\xi'_z)^2 - \xi\xi''_{zz}]/\xi^2$, with the integral function $\xi(z)$ determined by the equation

$$\xi\xi''''_{zzzz} - 4\xi'_z\xi'''_{zzz} + 3(\xi''_{zz})^2 + z\xi^2 = 0.$$

Its solutions corresponding to specific initial conditions of equation (1) can be represented by a Taylor series. In particular, if $y(0) = 0$ and $y'_z(0) = 1$, then

$$\xi(z) = 1 + \frac{1}{2}z + \frac{1}{8}z^2 - \frac{7}{8}z^3 + \sum_{k=4}^{\infty} \xi_k z^k.$$

2°. For any solution, the point of infinity $z = \infty$ is a condensation point of movable poles of this solution; furthermore, the solutions of equation (1) can be represented, in the vicinity of a movable pole z_{p}, as the series

$$y = \frac{1}{(z - z_{\mathrm{p}})^2} + \sum_{n=2}^{\infty} a_n (z - z_{\mathrm{p}})^n,$$

$$a_2 = -\tfrac{1}{10}z_{\mathrm{p}}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = C, \quad a_5 = 0, \quad a_6 = \tfrac{1}{300}z_{\mathrm{p}}^2,$$

where z_{p} and C are arbitrary constants; the coefficients a_j ($j \geq 7$) are uniquely determined in terms of z_{p} and C. Apart from second-order poles, solutions of equation (1) do not have other movable singularities.

3°. In a neighborhood of a fixed point $z = z_0$, the solution of the Cauchy problem for the first Painlevé transcendent (1) can be represented by the Taylor series:

$$y = A + B(z - z_0) + \tfrac{1}{2}(6A^2 + z_0)(z - z_0)^2 + \tfrac{1}{6}(12AB + 1)(z - z_0)^3 + \tfrac{1}{2}(6A^3 + B^2 + Az_0)(z - z_0)^4 + \cdots,$$

where A and B are initial data of the Cauchy problem, so that $y|_{z=z_0} = A$ and $y'_z|_{z=z_0} = B$.

Remark. The solutions of the Cauchy problems for the second and fourth Painlevé transcendents can be expressed likewise (fixed singular points should be excluded from consideration for the remaining Painlevé transcendents).

4°. For large values of $|z|$, the following asymptotic formula holds:

$$y \sim z^{1/2}\wp\big(\tfrac{4}{5}z^{5/4} - a;\ 12,\ b\big),$$

where the Weierstrass elliptic function $\wp(\zeta; 12, b)$ is defined implicitly by the integral

$$\zeta = \int \frac{d\wp}{\sqrt{4\wp^3 - 12\wp - b}};$$

a and b are some constants. For $b = \pm 8$, the asymptotic solution is expressed in terms of elementary functions.

The first Painlevé transcendent (1) is invariant under scaling of variables: $z = \lambda\bar{z}$, $y = \lambda^3\bar{y}$, where $\lambda^5 = 1$.

42.3. Second Painlevé Transcendent

1°. The second Painlevé transcendent has the form

$$y''_{zz} = 2y^3 + zy + \alpha. \tag{2}$$

The solutions of the second Painlevé transcendent are single-valued meromorphic functions of z. Equation (2) can only have simple poles as singular points.

The solutions of equation (2) can be represented, in the vicinity of a movable pole z_{p}, in terms of the series:

$$y = \frac{m}{z - z_{\mathrm{p}}} + \sum_{n=1}^{\infty} b_n (z - z_{\mathrm{p}})^n,$$

$$b_1 = -\tfrac{1}{6} m z_{\mathrm{p}}, \quad b_2 = -\tfrac{1}{4}(m + \alpha), \quad b_3 = C, \quad b_4 = \tfrac{1}{72} z_{\mathrm{p}}(m + 3\alpha),$$

$$b_5 = \tfrac{1}{3024}\left[(27 + 81\alpha^2 - 2z_{\mathrm{p}}^3)m + 108\alpha - 216 C z_{\mathrm{p}}\right],$$

where $m = \pm 1$, z_{p} and C are arbitrary constants, and the coefficients b_n ($n \geq 6$) are uniquely determined in terms of z_{p} and C.

2°. For a fixed α, denote the solution by $y(z, \alpha)$. Then the relation

$$y(z, -\alpha) = -y(z, \alpha) \tag{3}$$

holds, while the solutions $y(z, \alpha)$ and $y(z, \alpha - 1)$ are related by the Bäcklund transformations

$$\begin{aligned} y(z, \alpha - 1) &= -y(z, \alpha) + \frac{2\alpha - 1}{2y'_z(z, \alpha) - 2y^2(z, \alpha) - z}, \\ y(z, \alpha) &= -y(z, \alpha - 1) - \frac{2\alpha - 1}{2y'_z(z, \alpha - 1) + 2y^2(z, \alpha - 1) + z}. \end{aligned} \tag{4}$$

Therefore, in order to study the general solution of (2) with arbitrary α, it suffices to construct the solution for all α out of the band $0 \leq \operatorname{Re} \alpha < \frac{1}{2}$.

Three solutions corresponding to α and $\alpha \pm 1$ are related by the rational formulas:

$$y_{\alpha+1} = -\frac{(y_{\alpha-1} + y_\alpha)(4y_\alpha^3 + 2zy_\alpha + 2\alpha + 1) + (2\alpha - 1)y_\alpha}{2(y_{\alpha-1} + y_\alpha)(2y_\alpha^2 + z) + 2\alpha - 1},$$

where y_α stands for $y(z, \alpha)$.

The solutions $y(z, \alpha)$ and $y(z, -\alpha - 1)$ are related by the Bäcklund transformations

$$\begin{aligned} y(z, -\alpha - 1) &= y(z, \alpha) + \frac{2\alpha + 1}{2y'_z(z, \alpha) + 2y^2(z, \alpha) + z}, \\ y(z, \alpha) &= y(z, -\alpha - 1) - \frac{2\alpha + 1}{2y'_z(z, -\alpha - 1) + 2y^2(z, -\alpha - 1) + z}. \end{aligned}$$

3°. For $\alpha = 0$, equation (2) has the trivial solution $y = 0$. Taking into account this fact and relations (3) and (4), we find that the second Painlevé transcendent with $\alpha = \pm 1, \pm 2, \ldots$ has rational particular solutions:

$$y(z, \pm 1) = \mp \frac{1}{z}, \quad y(z, \pm 2) = \pm\left(\frac{1}{z} - \frac{3z^2}{z^3 + 4}\right), \quad \ldots.$$

Any rational solution of equation (2) has the form

$$y(z) = \frac{1}{z}\frac{P(z^3)}{Q(z^3)},$$

where P and Q are some polynomials.

For $\alpha = \frac{1}{2}$, equation (2) admits the one-parameter family of solutions

$$y\left(z, \tfrac{1}{2}\right) = -\frac{w'_z}{w}, \qquad \text{where} \quad w = \sqrt{z}\left[C_1 J_{1/3}\left(\tfrac{\sqrt{2}}{3}z^{3/2}\right) + C_2 Y_{1/3}\left(\tfrac{\sqrt{2}}{3}z^{3/2}\right)\right]. \tag{5}$$

(Here, the function w is a solution of the second-order linear ordinary differential equation $w''_{zz} + \frac{1}{2}zw = 0$.) It follows from (3)–(5) that the second Painlevé transcendent for all $\alpha = n + \frac{1}{2}$ with $n = 0, \pm1, \pm2, \ldots$ has a one-parameter family of solutions that can be expressed in terms of Bessel functions.

42.4. Third Painlevé Transcendent

1°. The third Painlevé transcendent has the form

$$y''_{zz} = \frac{(y'_z)^2}{y} - \frac{y'_z}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}. \tag{6}$$

In terms of the new independent variable ζ defined by $z = e^\zeta$, the solutions of the transformed equation will be single-valued functions of ζ.

Any solution of the Riccati equation

$$y'_z = ky^2 + \frac{\alpha - k}{kz}y + c, \tag{7}$$

where $k^2 = \gamma$, $c^2 = -\delta$, and $k\beta + c(\alpha - 2k) = 0$, is a solution of (6). Substituting $z = \lambda\tau$ and $y = -u'_z/(ku)$ with $\lambda^2 = 1/(kc)$ into (7), we obtain a linear equation,

$$u''_{\tau\tau} + \frac{k - \alpha}{k\tau}u'_\tau + u = 0,$$

whose general solution is expressed in terms of Bessel functions:

$$u = \tau^{\frac{\alpha}{2k}}\left[C_1 J_{\frac{\alpha}{2k}}(\tau) + C_2 Y_{\frac{\alpha}{2k}}(\tau)\right].$$

2°. In some special cases, (6) can be integrated by quadrature. Let us rewrite (6) in the form of integro-differential relations in two ways:

$$\left(\frac{y'_\zeta}{y}\right)^2 + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^\zeta = 2\int\left[\left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + \left(\frac{\beta}{y} - \alpha y\right)e^\zeta\right]d\zeta; \tag{8}$$

$$\frac{y'_\zeta}{y} = \int\left[\left(\frac{\delta}{y^2} + \gamma y^2\right)e^{2\zeta} + \left(\frac{\beta}{y} + \alpha y\right)e^\zeta\right]d\zeta, \qquad z = e^\zeta. \tag{9}$$

It is obvious from (8) that for $\alpha = \beta = \gamma = \delta = 0$, the general solution has the form $y = C_1 z^{C_2}$.

Adding (9) multiplied by 2 to (8), we obtain

$$\left(\frac{y'_\zeta}{y}\right)^2 + 2\frac{y'_\zeta}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^{\zeta} = 4\int \frac{\delta e^{2\zeta} + \beta e^{\zeta} y}{y^2}\, d\zeta. \tag{10}$$

Subtracting (9) multiplied by 2 from (8) yields

$$\left(\frac{y'_\zeta}{y}\right)^2 - 2\frac{y'_\zeta}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^{\zeta} = -4\int (\gamma e^{2\zeta} y^2 + \alpha e^{\zeta} y)\, d\zeta. \tag{11}$$

Substituting $\delta = \beta = 0$ into (10) and $\gamma = \alpha = 0$ into (11), we arrive at

$$\left(\frac{y'_\zeta}{y}\right)^2 + 2\frac{y'_\zeta}{y} - 2\alpha y e^{\zeta} - \gamma y^2 e^{2\zeta} = C_1, \tag{12}$$

$$\left(\frac{y'_\zeta}{y}\right)^2 - 2\frac{y'_\zeta}{y} + \frac{\delta}{y^2}e^{2\zeta} + \frac{2\beta}{y}e^{\zeta} = C_2. \tag{13}$$

Equations (12) and (13) are integrable in terms of elementary functions. Substituting $y = e^{-\zeta}/v$ into (12), we obtain an autonomous equation:

$$(v'_\zeta)^2 = 2\alpha v + \gamma + (1 + C_1)v^2. \tag{14}$$

As a result, we get:

$$y = \begin{cases} \dfrac{2\alpha}{z(\alpha^2 \ln^2 z + 2\alpha C \ln z + C^2 - \gamma)} & \text{if } C_1 = -1 \text{ and } \beta = \delta = 0; \\ \dfrac{1}{z\left(\sqrt{\gamma}\ln z + C\right)} & \text{if } C_1 = -1 \text{ and } \alpha = \beta = \delta = 0; \\ \dfrac{z^{m-1}}{C_2 z^{2m} + K_1 z^m + K_2} & \text{if } C_1 \neq -1 \text{ and } \beta = \delta = 0, \end{cases}$$

where

$$C_2 \neq 0, \quad K_1 = -\frac{\alpha}{C_1 + 1}, \quad K_2 = \frac{\alpha^2 - \gamma(1 + C_1)}{4C_2(1 + C_1)^2}, \quad m^2 = 1 + C_1.$$

Accordingly, (13) is reduced to (14) with the substitution $y = ve^{\zeta}$.

If $\beta = -\alpha$ and $\delta = -\gamma$, the substitution $y = e^{-iw}$ brings (6) to the following form: $w''_{zz} + \frac{1}{z}w'_z = \frac{2\alpha}{z}\sin w + 2\gamma \sin 2w$.

3°. If the conditions $a^2 = \gamma$ and $c^2 = -\delta$ hold, provided that $\gamma \neq 0$, then any solution of the equation

$$w' = aw^2 + \frac{\alpha - a}{az}w + c \tag{15}$$

will also solve equation (6). In (15), setting $w = -u'/(au)$ and $z = \lambda\tau$, with $\lambda^2 = (ac)^{-1}$ and $c \neq 0$, yields the linear equation

$$u'' + \frac{a - \alpha}{a\tau}u' + u = 0,$$

whose general solution is expressed in terms of Bessel functions:

$$u = \tau^{\alpha/2a}\left[C_1 J_{\alpha/2a}(\tau) + C_2 J_{-\alpha/2a}(\tau)\right],$$

where $J_{\alpha/2a}$ is the cylinder function of the first kind ($\alpha/2a$ is noninteger). If $\alpha/2a$ is an integer, then $J_{-\alpha/2a}$ must be replaced with the cylinder function of the second kind $Y_{\alpha/2a}$.

In special cases, the general solution of equation (15) can be an elementary function. Such is the case of $\alpha = a(1 + 2n)$ with integer n; we have

$$u = \tau^{1+2n}\left[C_1\frac{d^n}{(\tau d\tau)^n}\left(\frac{\cos\tau}{\tau}\right) + C_2\frac{d^n}{(\tau d\tau)^n}\left(\frac{\sin\tau}{\tau}\right)\right].$$

If $c = 0$ and $\alpha \neq 0$, the general solution of equation (15) is expressed as

$$w = \frac{z^{(\alpha-a)/a}}{C - a^2 z^{\alpha/a}/\alpha},$$

where C is an arbitrary constant.

4°. Consider the special case

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(y^2 + \beta) - \frac{1}{w} \tag{16}$$

of equation (6). If $y(z, \beta)$ solves equation (16) for some fixed value of β, then the function

$$y(z, \beta_1) = y^{-2}[(\varepsilon - \beta)y + z - \varepsilon z y'] \tag{17}$$

will also solve this equation for $\beta_1 = \beta - 2\varepsilon$ with $\varepsilon^2 = 1$.

Substituting $z = \tau^3$ into (16) yields the equation

$$y'' = \frac{(y')^2}{y} - \frac{y'}{\tau} + 9\tau^2\left(y^2 + \beta - \frac{\tau^3}{y}\right), \tag{18}$$

which has rational solutions for $\beta = 2k$ with integer k. In particular, $y = \lambda\tau$ with $\lambda^3 = 1$ for $\beta = 0$. Using (17), one obtains

$$y_1 = \frac{2\varepsilon\lambda + 3\tau^2}{3\lambda^2\tau} \quad \text{for} \quad \beta = -2\varepsilon, \qquad y_2 = \frac{9\lambda\tau^5 + 24\varepsilon\lambda^2\tau^3 + 20\tau}{(3\tau^2 + 2\varepsilon\lambda)^2} \quad \text{for} \quad \beta = -4\varepsilon, \qquad \ldots.$$

5°. Let us define a transformation of equation (6) as follows:

$$T : y \to \sigma_1 y^{k_1}, \quad z \to \sigma_2 z^{k_2}, \quad \sigma_j, k_j \neq 0. \tag{19}$$

By setting $k_1 = k_2 = 1$ and selecting σ_1 and σ_2 appropriately, one can fix two nonzero parameters in (6). Suppose $y = \varphi(z, \alpha, \beta, \gamma, \delta)$ is a solution of equation (6). Then the transformations

$$T_1(\sigma_1, \sigma_2) : \varphi \to \sigma_1\varphi(\sigma_2 z, \alpha\sigma_1^{-1}\sigma_2, \beta\sigma_1\sigma_2, \gamma\sigma_1^{-2}\sigma_2^2, \delta\sigma_1^2\sigma_2^2),$$
$$T_2 : \varphi \to 1/\varphi(z, -\beta, -\alpha, -\delta, -\gamma),$$
$$T_3 : \varphi(z, \alpha, \beta, 0, 0) \to \sqrt{\varphi(z^2, 0, 0, 2\alpha, 2\beta)}$$

will generate solutions of equation (6).

If, in equation (6), one performs transformation (19) with $k_1 = k_2 = 1$, $\sigma_1 = 1/\sigma_2\sqrt{\gamma}$, and $\sigma_2 = (-1/\gamma\delta)^{1/4}$, then one obtains

$$y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}(\alpha y^2 + \beta) + y^3 - \frac{1}{y} \tag{20}$$

for one value of the root. To summarize, without loss of generality, one can treat equation (20) instead of (6), provided that $\gamma\delta \neq 0$. It has a rational solution if and only if $\alpha + \beta\varepsilon = 4n$ with $\varepsilon^2 = 1$ and integer n.

Similarly, if $\alpha + \beta\varepsilon = 4n + 2$ with $\varepsilon^2 = 1$ and integer n, equation (20) admits a one-parameter family of solutions expressible in terms of Bessel functions.

42.5. Fourth Painlevé Transcendent

1°. The fourth Painlevé transcendent has the form

$$y''_{zz} = \frac{(y'_z)^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y}. \tag{21}$$

The solutions of the fourth Painlevé transcendent are single-valued functions of z. All poles of the meromorphic function $y(z)$ are simple. The solution of equation (21) can be represented as the ratio of integral functions, $y(z) = v(z)/u(z)$, satisfying the system

$$uu'' - (u')^2 + v^2 + 2zuv = 0,$$
$$vv'' - (v')^2 + 2vu' - 2\beta u^2 = 0.$$

This system does not involve the parameter α—it arises as an arbitrary constant when the system is integrated.

The Laurent-series expansion of the solution of (21) in the vicinity of a movable pole z_{p} is given by:

$$y = \frac{m}{z - z_{\mathrm{p}}} - z_{\mathrm{p}} - \frac{m}{3}(z_{\mathrm{p}}^2 + 2\alpha - 4m)(z - z_{\mathrm{p}}) + C(z - z_{\mathrm{p}})^2 + \sum_{j=3}^{\infty} a_j (z - z_{\mathrm{p}})^j,$$

where $m = \pm 1$, z_{p} and C are arbitrary constants, and the coefficients a_j ($j \geq 3$) are uniquely defined in terms of α, β, z_{p}, and C.

2°. If the condition $\beta + 2(1 + \alpha s)^2 = 0$, where $s = \pm 1$, holds, then every solution of the Riccati equation

$$y'_z = sy^2 + 2szy - 2(1 + \alpha s)$$

is simultaneously a solution of the fourth Painlevé transcendent (21).

Equation (21) is invariant under the transformation $y = \lambda\bar{y}$, $z = \lambda\bar{z}$, $\alpha = \bar{\alpha}\lambda^2$, $\beta = \bar{\beta}$, where $\lambda^4 = 1$.

3°. Two solutions of (21) corresponding to different values of the parameters α and β are related to each other by the Bäcklund transformations:

$$\widetilde{y} = \frac{1}{2sy}\left(y'_z - q - 2szy - sy^2\right), \qquad q^2 = -2\beta, \quad 2\beta = -\left(\widetilde{\alpha}s - 1 - \tfrac{1}{2}p\right)^2;$$
$$y = -\frac{1}{2s\widetilde{y}}\left(\widetilde{y}'_z - p + 2sz\widetilde{y} + s\widetilde{y}^2\right), \qquad p^2 = -2\widetilde{\beta}, \quad 4\alpha = -2s - 2\widetilde{\alpha} - 3sp,$$

where $y = y(z, \alpha, \beta)$, $\widetilde{y} = \widetilde{y}(z, \widetilde{\alpha}, \widetilde{\beta})$, and $s = \pm 1$.

By applying these transformations to statement 2°, one finds that, with $\beta = 2(\alpha s + 2n - 1)^2$ or $\beta = -2n^2$ and integer n, equation (21) admits one-parameter families of solutions expressible in terms of Weber–Hermite functions.

4°. For some values of the parameters, equation (21) has rational solutions:

$$\begin{aligned} &y \equiv 0 \quad \text{for} \quad \beta = 0, \\ &y = -2z \quad \text{for} \quad \alpha = 0, \ \beta = -2, \\ &y = -2z/3 \quad \text{for} \quad \alpha = 0, \ \beta = -2/9. \end{aligned} \tag{22}$$

Using the transformations of Item 3°, one can also find rational solutions for other values of the parameters. For example, the second solution in (22) generates a variety of rational solutions for

$$\alpha = n, \quad \beta = -2(1 + 2m - n)^2 \qquad (m \text{ and } n \text{ are integer}), \tag{23}$$

which include, in particular, solutions of the form

$$y = -\sqrt{s}\, H_n'(\tau)/H_n(\tau) \qquad \text{for} \quad \alpha = -s(1+n), \ \ \beta = -2n^2, \ \ z = \sqrt{s}\tau,$$
$$y = -2z + i\sqrt{s}\, H_n'(\tau)/H_n(\tau) \quad \text{for} \quad \alpha = ns, \ \ \beta = -2(1+n)^2, \ \ z = i\sqrt{s}\tau,$$

where $H_n(\tau)$ are Chebyshev–Hermite polynomials; $s^2 = 1$.

The third solution in (22) generates a variety of rational solutions for

$$\alpha = n, \quad \beta = -\tfrac{2}{9}(6m - 3n + s)^2 \qquad (m \text{ and } n \text{ are integer}). \tag{24}$$

Rational solutions of equation (21) only exist for parameters (23) and (24), with a single rational solution (without regard for the first solution in (22)) corresponding to a fixed pair of such values α and β.

42.6. Fifth Painlevé Transcendent

1°. The fifth Painlevé transcendent has the form

$$y_{zz}'' = \frac{3y-1}{2y(y-1)}(y_z')^2 - \frac{y_z'}{z} + \frac{(y-1)^2}{z^2}\Big(\alpha y + \frac{\beta}{y}\Big) + \gamma\frac{y}{z} + \frac{\delta y(y+1)}{y-1}. \tag{25}$$

In terms of the new independent variable $z = e^\zeta$, the solutions are single-valued functions of ζ.

Solutions of the fifth Painlevé transcendent (25) corresponding to different values of parameters are related by:

$$y(z,\alpha,\beta,\gamma,\delta) = y(-z,\alpha,\beta,-\gamma,\delta),$$
$$y(z,\alpha,\beta,\gamma,\delta) = \frac{1}{y(z,-\beta,-\alpha,-\gamma,\delta)}.$$

2°. By setting $z = e^t$ in (25), one obtains

$$y_{tt}'' = \frac{3y-1}{2y(y-1)}(y_t')^2 + (y-1)^2\Big(\alpha y + \frac{\beta}{y}\Big) + \gamma y e^t + \frac{\delta y(y+1)}{y-1}e^{2t}. \tag{26}$$

If $\gamma = \delta = 0$, (26) is reduced, by integration, to a first-order autonomous equation:

$$y_t' = (y-1)\sqrt{2\alpha y^2 + Cy - 2\beta},$$

which is readily integrable by quadrature.

If the condition

$$\gamma = \sqrt{-2\delta}\,\big(1 + \sqrt{-2\beta} - \sqrt{2\alpha}\,\big)$$

holds, any solution of the Riccati equation

$$zy_z' = \sqrt{2\alpha}\,y^2 + \big(\sqrt{-2\delta}\,z - \sqrt{2\alpha} - \sqrt{-2\beta}\,\big)y + \sqrt{-2\beta} \tag{27}$$

is simultaneously a solution of the fifth Painlevé transcendent (25). Equation (21) can be reduced to a degenerate hypergeometric equation.

3°. Suppose $u = u(z, \alpha_1, \beta_1)$ is a solution of equation (20) such that

$$R(z) \equiv u' - \varepsilon u^2 = (\alpha_1 \varepsilon - 1)\frac{u}{z} + 1 \not\equiv 0 \qquad (\varepsilon^2 = 1).$$

Then the function $y(\tau) = 1 - 2/R\big(\sqrt{2\tau}\,\big)$ will solve equation (25), written in terms of $y(\tau)$, provided that

$$\alpha = \frac{1}{32}(\beta_1 - \alpha_1\varepsilon + 2)^2, \quad \beta = -\frac{1}{32}(\beta_1 + \alpha_1\varepsilon - 2)^2, \quad \gamma = -\varepsilon, \quad \delta = 0.$$

42.7. Sixth Painlevé Transcendent

1°. The sixth Painlevé transcendent has the form

$$y''_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y'_z)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y'_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right]. \qquad (28)$$

In (28), the points $z = 0$, $z = 1$, and $z = \infty$ are fixed logarithmic branch points.

Painlevé found two integrable cases of the equation. First, if $\alpha = \beta = \gamma = \delta = 0$, the general solution of (28) has the form:

$$y = E(C_1\omega_1 + C_2\omega_2,\ z),$$

where $E(u, z)$ is the elliptic function defined by the integral

$$u = \int_0^E \frac{dy}{\sqrt{y(y-1)(y-z)}}, \qquad (29)$$

with periods $2\omega_1$ and $2\omega_2$, which are functions of z. Secondly, if $\alpha = \beta = \gamma = 0$ and $\delta = \frac{1}{2}$, the general solution of (28) is expressed as:

$$y = E(w + C_1\omega_1 + C_2\omega_2,\ z),$$

where $w \not\equiv 0$ is any particular solution of the linear equation

$$w''_{zz} - \frac{2z-1}{z(z-1)}w'_z + \frac{1}{4z(z-1)}w = 0$$

and $E(u, z)$ is the elliptic function defined by formula (29).

2°. Solutions of the sixth Painlevé transcendent (28) corresponding to different values of parameters are related by:

$$y(z,\ -\beta,\ -\alpha,\ \gamma,\ \delta) = \frac{1}{y\big(\frac{1}{z},\ \alpha,\ \beta,\ \gamma,\ \delta\big)},$$

$$y(z,\ -\beta,\ -\gamma,\ \alpha,\ \delta) = 1 - \frac{1}{y\big(\frac{1}{1-z},\ \alpha,\ \beta,\ \gamma,\ \delta\big)},$$

$$y\big(z,\ -\beta,\ -\alpha,\ -\delta + \tfrac{1}{2},\ -\gamma + \tfrac{1}{2}\big) = \frac{z}{y(z,\ \alpha,\ \beta,\ \gamma,\ \delta)}.$$

The successive application of these relations yields 24 equations of the form (28) with different values of the parameters, related by known transformations.

3°. Every solution of the Riccati equation

$$y'_z = \frac{\sqrt{2\alpha}}{z(z-1)}y^2 + \frac{\lambda z+\mu}{z(z-1)}y + \frac{\sqrt{-2\beta}}{z-1},$$

where

$$\lambda = \frac{\sqrt{2\alpha}-(\alpha+\beta+\gamma+\delta)}{\sqrt{2\alpha}-\sqrt{-2\beta}-1}, \qquad \mu = \frac{\sqrt{-2\beta}-(\alpha+\beta-\gamma-\delta)}{\sqrt{2\alpha}-\sqrt{-2\beta}-1},$$

is simultaneously a solution of (28) if $\sqrt{2\alpha}-\sqrt{-2\beta}\neq 1$ and the condition

$$2\sqrt{2\alpha}\,(3\beta-\alpha+\gamma-\delta)+2\sqrt{-2\beta}\,(3\alpha-\beta-\gamma+\delta)+4\sqrt{-\alpha\beta}\,(\beta-\alpha+\gamma-\delta-1)$$
$$+(\alpha+\beta+\gamma+\delta)^2+2(\alpha-\beta-\gamma-4\alpha\beta-2\alpha\gamma-2\beta\delta)=0$$

is satisfied (one should take the value of $\sqrt{-\alpha\beta}$ that coincides with $\sqrt{\alpha}\sqrt{-\beta}$).

4°. Suppose $y(z,\alpha,\beta,\gamma,\delta)$ is a solution of equation (28) such that

$$M(f)=\left(J+\frac{k(z+1)}{z-1}f\right)\left(\frac{z-1}{2}y^2-\frac{z-1}{2z^2}(y-1)(y-z)\right)$$
$$-\frac{z(z-1)}{2}\left(J-\frac{k(z+1)}{z-1}f\right)-2kzyf\not\equiv 0, \qquad k\neq 0,$$

where

$$f = z\frac{y'}{y}+\frac{\lambda-k-1}{2(z-1)}y+\frac{\lambda+k+1}{2(z-1)}\frac{1}{y}-\frac{\lambda}{2}\frac{z+1}{z-1}-\frac{1}{2}-\frac{\mu}{4},$$

with $J=f^2+\frac{1}{2}\mu f+\nu$, $k=\sqrt{-2\beta}-\sqrt{2\alpha}-1$, $\lambda=\sqrt{2\alpha}+\sqrt{-2\beta}$, $\mu=4(\frac{1}{2}-\gamma-\delta)/k$, and $\nu=2\delta-1+\frac{1}{4}(\frac{1}{2}\mu+k)^2$. Then the function

$$y_1 = y+\frac{2kf[(z+1)y-2z]}{-2z(z-1)f'+(z-1)J-(z+1)kf}$$

will solve equation (28), provided that

$$\alpha_1=\tfrac{1}{2}\big(\sqrt{-2\beta}-1\big)^2, \quad \beta_1=-\tfrac{1}{2}\big(\sqrt{2\alpha}+1\big)^2, \quad \gamma_1=\gamma, \quad \delta_1=\delta.$$

5°. Equation (28) is integrable in terms of elliptic functions and their derivatives if $\alpha=\frac{1}{2}(m_1+1)^2$, $\beta=-\frac{1}{2}(2m_2-m_1)^2$, $\gamma=\frac{1}{2}(2n_2-n_1)^2$, and $\delta=\frac{1}{2}(1-n_1^2)$, where n_j and m_j are integers.

6°. Equation (28) is integrable in terms of superpositions of elliptic and hypergeometric functions and their derivatives if $\alpha=\frac{1}{2}m_1^2$, $\beta=-\frac{1}{2}(2m_2-m_1)^2$, $\gamma=\frac{1}{2}(2n_2-n_1)^2$, $\delta=\frac{1}{2}(1-n_1^2)$, where n_j and m_j are integers.

42.8. Examples of Solutions to Nonlinear Equations in Terms of Painlevé Transcendents

1°. Consider the equation

$$w_{tt}-w_{xx}+\varepsilon(w^2+w_{xx})_{xx}=0, \qquad \varepsilon^2=1. \tag{30}$$

With $\varepsilon = -1$, it is a Boussinesq equation. With $\varepsilon = 1$, it is a consequence of the Kadomtsev–Petviashvili equation (two-dimensional Korteweg–de Vries equation)

$$(u_\tau + 2uu_\eta + u_{\eta\eta\eta})_\eta + \alpha^2 u_{\rho\rho} = 0, \tag{31}$$

obtained by setting

$$x = \eta - \frac{\rho}{3\alpha} - \frac{10}{9}\tau, \quad t = \frac{\rho}{\alpha} + \frac{2}{3}\tau, \quad w = u.$$

In equation (30), let us set $w(x,t) = w(z)$ with $z = x - \beta t$. Integrating the equation twice with respect to z, one obtains

$$w'' + w^2 + \varepsilon w(\beta^2 - 1) = C_1 z + C_2, \tag{32}$$

where C_1 and C_2 are arbitrary constants. If $C_1 = 0$ in (32), the equation is integrable in terms of Weierstrass elliptic functions.

Suppose $C_1 \neq 0$. Let us set $z = \lambda\xi + \mu$ and $w(\xi) = ay(\xi) + b$ with $\lambda = -(6/C_1)^{1/5}$, $\mu = C_1^{-1}(b^2 + \varepsilon b\beta^2 - b\varepsilon - C_2)$, $a = C_1\lambda^3$, and $b = \frac{1}{2}\varepsilon(1 - \beta^2)$. In this case, the function $y(\xi)$ satisfies the first Painlevé transcendent (1).

2°. By setting $2w + 4\varepsilon t^2 = v$ and $x + t^2 = \xi$ in (30) and integrating the resulting equation with respect to ξ, one arrives at the following equation for $v(\xi)$:

$$v'''_{\xi\xi\xi} + vv'_\xi - \varepsilon v'_\xi + 2\varepsilon v - 8\xi = K_1, \tag{33}$$

where K_1 is the constant of integration. Substituting $\varepsilon v = 1 - 12y - 4\varepsilon z$, $z = m\xi + n$, $m^2 = \varepsilon$, and $n = \varepsilon(K_1 m^{-1} - 2)/8$ in (33), multiplying by y, and integrating with respect to z, one obtains the equation

$$yy'' = \frac{1}{2}(y')^2 + 4y^3 + 2zy^2 + K_2,$$

where K_2 is the constant of integration.

If $K_2 = 0$, the function $u(z)$ determined by $y = u^2(z)$ satisfies equation (2) with $\alpha = 0$. If $K_2 \neq 0$, then by setting $y = \lambda v_1$, $z = \mu\xi$, $\mu^3 = -1/2$, and $\lambda^2 = -2\mu^2 K_2$, one arrives at the following equation for $v_1(\xi)$:

$$v_1 v_1'' = \frac{(v_1')^2}{2} + 4pv_1^3 - \xi v_1^2 - \frac{1}{2}, \quad p = \lambda\mu^2 \neq 0.$$

Its general solution is expressed as

$$v_1 = \frac{u'_\xi + u^2 + \frac{1}{2}\xi}{2p},$$

where $u(\xi)$ is the solution of the second Painlevé transcendent (2) with $\alpha = -2\lambda\mu^2 - 1/2$.

3°. In the Korteweg–de Vries equation

$$3w_t - 6ww_x + w_{xxx} = 0, \tag{34}$$

let us set $w = -\frac{2}{9}\beta\delta^{-1}t - \frac{1}{3}v(\eta)$ and $\eta = \delta x - \frac{2}{3}\gamma t - \frac{2}{9}\beta t^2$ with $\beta\delta \neq 0$ to obtain the following equation for $v(\eta)$:

$$v'''_{\eta\eta\eta} + 2\delta^{-2}vv'_\eta - 2\gamma\delta^{-3}v'_\eta + 2\beta\delta^{-4} = 0. \tag{35}$$

Equation (35) has the first integral

$$v''_{\eta\eta} + \delta^{-2}v^2 - 2\gamma\delta^{-3} + 2\beta\delta^{-4}\eta = K, \tag{36}$$

where K is the constant of integration. Equation (36) is reduced to the canonical form (1) with respect to the new function $y(z)$ by the substitution

$$v = -6\left(\frac{\beta}{3\delta}\right)^{2/5} y + \frac{\gamma}{\delta}, \qquad \eta = \left(\frac{3\delta^6}{\beta}\right)^{1/5} z + \frac{\gamma^2}{2\beta} + \frac{K\delta^4}{2\beta}.$$

4°. Suppose $y(z)$ is a solution of equation (2). Then the function

$$w(x,t) = t^{-2/3}[y'_z + y^2(z)], \qquad z = xt^{-1/3}, \tag{37}$$

will solve the Korteweg–de Vries equation (34). Using (37), one can find special classes of solutions to the Korteweg–de Vries equation (34). For example, rational solutions of equation (2) generate rational solutions of equation (34) that decay as $|x| \to +\infty$. In particular, rational solutions of equation (2) with $\alpha = 0, \pm 1, \pm 2$ generate the following solutions:

$$w_1 = 0, \qquad w_2 = \frac{2}{x^2}, \qquad w_3 = \frac{6x(x^3 - 8t)}{(x^3 - 4t)^2}.$$

In a similar way, one can obtain one-parameter families of solutions to equation (34) that represent rational functions expressible in terms of Bessel functions and their derivatives.

5°. Consider the equation

$$\frac{\partial^2 w}{\partial x \partial t} = \nu \sin w + \mu x t \sin 2w.$$

With $\nu = 1$ and $\mu = 0$, it is the sine-Gordon equation. For $\nu = 0$ and $\mu \neq 0$, the equation can also be reduced to the sine-Gordon equation by the substitution $w = w_1/2$, $x_1 = x^2$, $t_1 = \mu t^2/2$. In what follows, we assume that $\mu = 2$ without loss of generality:

$$\frac{\partial^2 w}{\partial x \partial t} = \nu \sin w + 2x t \sin 2w. \tag{38}$$

Suppose $w = w(z)$ with $z = xt$ is an arbitrary solution of equation (38). Then the function $y = \exp(-iw)$ will solve the equation

$$\frac{d^2y}{dz^2} = \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{\nu}{2z}(y^2 - 1) + y^3 - \frac{1}{y}, \tag{39}$$

which is a special case of equation (20).

Suppose $y = y(z, \nu)$ is a solution of equation (39) for a fixed value of ν, such that

$$R \equiv y' - y^2 + \frac{1 - \nu/2}{z} y + 1 \not\equiv 0.$$

Then the function

$$y_1(z, \nu_1) = \frac{2zR(R-2)}{2zR' + \nu R - 2\nu + 4}$$

will solve equation (39) for $\nu_1 = 4 - \nu$. This results can be transferred to equation (38) as well: suppose $w = w(z, \nu)$ with $z = xt$ is a solution of equation (38) for a fixed ν, such that

$$\Phi \equiv \Big[-w' + 2\sin w + \frac{i}{z}\Big(\frac{\nu}{2} - 1\Big)\Big] i\exp(-iw) \not\equiv 0.$$

Then the function

$$w_1(z, \nu_1) = i\ln\frac{2z\Phi\,(\Phi - 2)}{2z\Phi' + \nu\Phi - 2\nu + 4} \tag{40}$$

will solve equation (38) for $\nu_1 = 4 - \nu$ and $xt = z$. It follows that solutions of equation (38) with $z = xt$, $\nu = 4n$, and integer n are expressed in terms of solutions to the sine-Gordon equation. For all values of $\nu = 4n$ with integer n, equation (39) has rational solutions, which generate exact solutions of equation (38). These can be obtained from, for example, the solution $w = \pi k/2$ with integer k and $\nu = 0$ using formula (40). After the first step, one gets

$$w(z, 4) = i\ln\frac{2z - i}{2iz - 1}, \qquad z = xt.$$

Since for $\nu = 4n+2$ with integer n, equation (39) has a one-parameter family of solutions, a similar results can be obtained for equation (38) as well. For example, if $n = 0$ (or $\nu = 2$), one obtains the solution

$$w = \arccos\frac{C + \exp(4z)}{C - \exp(4z)}, \qquad z = xt.$$

For other values of n, families of solutions can be found using the above transformations and the relation $w(z, \nu) = w(-z, -\nu)$.

6°. Consider the equation

$$\frac{\partial^2 w}{\partial x\partial t} = \frac{\sec^2(w/2)}{xt}\Big(a\cot^3\frac{w}{2} + b\tan\frac{w}{2}\Big) + c\sin w + d\,xt\sin 2w. \tag{41}$$

If $a = b = d = 0$ and $c = 1$, it becomes the sine-Gordon equation.

Suppose $w = w(\eta)$ with $\eta = xt$ is an arbitrary solution of equation (41). Then the function

$$y(z) = -\cot^2\frac{w}{2}, \qquad z = \eta,$$

will solve equation (25) if $\alpha = -a$, $\beta = -b$, $\gamma = -2c$, and $\delta = -4d$. Since equation (25) is integrable for $\gamma = \delta = 0$, equation (41) with $xt = \eta$ and $d = c = 0$ has the general solution

$$\int \cot\frac{w}{2}\Big(C_1\cot^2\frac{w}{2} - a\cot^4\frac{w}{2} + b\Big)^{-1/2} dw = C_2 + \sqrt{2}\ln\eta,$$

where C_1 and C_2 are arbitrary constants.

Suppose $w = w(\eta) \neq \pi k$ with $\eta = xt$ and integer k is a solution of equation (41) for fixed a, b, $c^2 = 1/4$, and $d = 0$, with $\sqrt{-2a} \neq 1$. Then the function

$$w_1 = 2\,\mathrm{arccot}\,\sqrt{-1 + 2M_1^2/\Phi_1}$$

with

$$M_1 = w'\eta\cot\frac{w}{2}\sin^{-2}\frac{w}{2} - \sqrt{-2a}\cot^4\frac{w}{2} - \big(\sqrt{-2a} + \sqrt{2b}\,\big)\cot^2\frac{w}{2} - \sqrt{2b},$$

$$\Phi_1 \equiv -2\eta\cot^2\frac{w}{2}\,M_1' + M_1^2 - \Big[2\eta\cot\frac{w}{2}\sin^{-2}\frac{w}{2}\,w' + 2\cot^2\frac{w}{2}\big(\sqrt{-2a} + \sqrt{2b} - 2\big)\Big]M_1 - 4c\eta\cot^4\frac{w}{2},$$

will solve equation (41) for

$$a_1 = -\tfrac{1}{2}\big(\sqrt{-2a} - 1\big), \qquad b_1 = \tfrac{1}{2}\big(\sqrt{2b} + 1\big), \qquad c_1 = c, \qquad d = 0.$$

Suppose $w = w(\eta)$ with $\eta = xt$ is a solution of equation (41) for fixed a, b, c, and $d = 1/8$, such that

$$R = w'\eta \frac{\cot\frac{w}{2}}{\sin^2\frac{w}{2}} - \lambda\cot^4\frac{w}{2} - (\lambda - \mu + \eta)\cot^2\frac{w}{2} + \mu \not\equiv 0$$

with $\lambda^2 = -2a$ and $\mu^2 = 2b$. Then the function

$$w_1 = 2\operatorname{arccot}\left(-1 - \frac{2\eta}{R}\cot^2\frac{w}{2}\right)^{1/2}$$

will solve equation (41) for

$$a_1 = -\tfrac{1}{8}(1 - 2c - \lambda - \mu)^2, \qquad b_1 = \tfrac{1}{8}(1 + 2c - \lambda - \mu)^2, \qquad c_1 = -\tfrac{1}{2}(\mu - \lambda), \qquad d_1 = \tfrac{1}{8}.$$

For example, equation (41) with $a + b = 0$ with $c = 0$ has the solution $w = \pi/2$. By setting $\lambda = \sqrt{-2a}$, $\mu = \sqrt{2b}$, and $\lambda = -\mu$, one finds the following solution of equation (41):

$$w_1 = 2\operatorname{arccot}\left(\frac{\eta + 4\mu}{\eta - 4\mu}\right)^{1/2}, \qquad \eta = xt,$$

with $a_1 = -1/8$, $b_1 = 1/8$, $c_1 = -\mu$, and $d_1 = 1/8$.

⊙ *References for Chapter* 42: P. Painlevé (1900), B. Gambier (1910), G. M. Murphy (1960), E. L. Ince (1964), A. S. Fokas and M. J. Ablowitz (1982), V. I. Gromak and N. A. Lukashevich (1990), R. Conte (1999), A. R. Chowdhury (2000), A. D. Polyanin and V. F. Zaitsev (2003).

Chapter 43
Functional Equations

43.1. Method of Differentiation in a Parameter

43.1.1. Classes of Equations. Description of the Method

Consider linear functional equations of the form

$$w(x,t) = \theta(x,t,a)\, w\big(\varphi(x,t,a),\ \psi(x,t,a)\big), \tag{43.1.1.1}$$

where x and t are independent variables, $w = w(x,t)$ is the function to be found, $\theta = \theta(x,t,a)$, $\varphi = \varphi(x,t,a)$, $\psi = \psi(x,t,a)$ are given functions, and a is a free parameter that can take any value (on some interval). Assume that for a particular $a = a_0$, we have

$$\theta(x,t,a_0) = 1, \quad \varphi(x,t,a_0) = x, \quad \psi(x,t,a_0) = t, \tag{43.1.1.2}$$

i.e., for $a = a_0$ the functional equation (43.1.1.1) turns into identity.

Let us expand (43.1.1.1) in powers of the small parameter a in a neighborhood of a_0, taking into account (43.1.1.2), and then divide the resulting equation by $a - a_0$ and pass to the limit for $a \to a_0$. As a result, we obtain a first-order linear partial differential equation for the function w:

$$\varphi_a^\circ(x,t)\frac{\partial w}{\partial x} + \psi_a^\circ(x,t)\frac{\partial w}{\partial t} + \theta_a^\circ(x,t)w = 0, \tag{43.1.1.3}$$

where we have used the following notation:

$$\varphi_a^\circ(x,t) = \frac{\partial \varphi}{\partial a}\Big|_{a=a_0}, \quad \psi_a^\circ(x,t) = \frac{\partial \psi}{\partial a}\Big|_{a=a_0}, \quad \theta_a^\circ(x,t) = \frac{\partial \theta}{\partial a}\Big|_{a=a_0}.$$

In order to solve equation (43.1.1.3), one should consider the corresponding system of equations for characteristics (see Section 23.1):

$$\frac{dx}{\varphi_a^\circ(x,t)} = \frac{dt}{\psi_a^\circ(x,t)} = -\frac{dw}{\theta_a^\circ(x,t)w}. \tag{43.1.1.4}$$

Let

$$u_1(x,t) = C_1, \quad u_2(x,t,w) = C_2 \tag{43.1.1.5}$$

be independent integrals of the characteristic system (43.1.1.4). Then the general solution of equation (43.1.1.3) has the form

$$u_2(x,t,w) = F\big(u_1(x,t)\big), \tag{43.1.1.6}$$

where $F(z)$ is an arbitrary function. The function w should be expressed from (43.1.1.6) and substituted into the original equation (43.1.1.1) for verification [there is a possibility of redundant solutions; it is also possible that a solution of the partial differential equation (43.1.1.3) is not a solution of the functional equation (43.1.1.1); see Example 43.3 below].

Remark 1. Equation (43.1.1.3) can be obtained from (43.1.1.1) by differentiation in the parameter a, after which one should take $a = a_0$.

Remark 2. It is convenient to choose the second integral in (43.1.1.5) to be a linear function of w, i.e., $u_2(x,t,w) = \xi(x,t)w$, and rewrite (43.1.1.6) as a relation resolved with respect to w.

43.1.2. Examples of Solutions of Some Specific Functional Equations

Example 43.1. Self-similar solutions, which often occur in mathematical physics, may be defined as solutions invariant with respect to the scaling transformation, i.e., solutions satisfying the functional equation

$$w(x,t) = a^k w(a^m x, a^n t), \tag{43.1.2.1}$$

where k, m, n are given constants, and $a > 0$ is an arbitrary constant.

Equation (43.1.2.1) turns into identity for $a = 1$. Differentiating (43.1.2.1) in a and taking $a = 1$, we come to the first-order partial differential equation

$$mx\frac{\partial w}{\partial x} + nt\frac{\partial w}{\partial t} + kw = 0. \tag{43.1.2.2}$$

The first integrals of the corresponding characteristic system of ordinary differential equations

$$\frac{dx}{mx} = \frac{dt}{nt} = -\frac{dw}{kw}$$

can be written in the form

$$xt^{-m/n} = C_1, \quad t^{k/n}w = C_2 \qquad (n \neq 0).$$

Therefore, the general solution of the partial differential equation (43.1.2.2) has the form

$$w(x,t) = t^{-k/n}F(z), \quad z = xt^{-m/n}, \tag{43.1.2.3}$$

where $F(z)$ is an arbitrary function. Direct verification shows that expression (43.1.2.3) is a solution of the functional equation (43.1.2.1).

Example 43.2. Consider the functional equation

$$w(x,t) = a^k w(a^m x, t + \beta \ln a), \tag{43.1.2.4}$$

where k, m, and β are given constants, and $a > 0$ is an arbitrary constant.

Equation (43.1.2.4) turns into identity for $a = 1$. Differentiating (43.1.2.4) in a and taking $a = 1$, we come to the first-order partial differential equation

$$mx\frac{\partial w}{\partial x} + \beta\frac{\partial w}{\partial t} + kw = 0. \tag{43.1.2.5}$$

The corresponding characteristic system of ordinary differential equations

$$\frac{dx}{mx} = \frac{dt}{\beta} = -\frac{dw}{kw}$$

admits the first integrals

$$x\exp(-mt/\beta) = C_1, \quad w\exp(kt/\beta) = C_2.$$

Therefore, the general solution of the partial differential equation (43.1.2.5) has the form

$$w(x,t) = \exp(-kt/\beta)F(z), \quad z = x\exp(-mt/\beta), \tag{43.1.2.6}$$

where $F(z)$ is an arbitrary function. Direct verification shows that (43.1.2.6) is a solution of the functional equation (43.1.2.4).

Example 43.3. Now consider the functional equation

$$w(x,t) = a^k w\big(x + (1-a)t,\, a^n t\big), \tag{43.1.2.7}$$

where $a > 0$ is arbitrary and n is a constant.

Equation (43.1.2.7) turns into identity for $a = 1$. Differentiating (43.1.2.7) in a and taking $a = 1$, we come to the first-order partial differential equation

$$-t\frac{\partial w}{\partial x} + nt\frac{\partial w}{\partial t} + kw = 0. \tag{43.1.2.8}$$

The corresponding characteristic system

$$-\frac{dx}{t} = \frac{dt}{nt} = -\frac{dw}{kw}$$

has the first integrals

$$t + nx = C_1, \quad wt^{k/n} = C_2.$$

Therefore, the general solution of the partial differential equation (43.1.2.8) has the form

$$w(x,t) = t^{-k/n} F(nx + t), \tag{43.1.2.9}$$

where $F(z)$ is an arbitrary function.

Substituting (43.1.2.9) into the original equation (43.1.2.7) and dividing the result by $t^{-k/n}$, we obtain

$$F(nx + t) = F(nx + \sigma t), \qquad \sigma = (1 - a)n + a^n. \tag{43.1.2.10}$$

Hence, for $F(z) \neq \text{const}$ we have $\sigma = 1$ or

$$(1 - a)n + a^n = 1. \tag{43.1.2.11}$$

Since (43.1.2.10) must hold for all $a > 0$, it follows that (43.1.2.11), too, must hold for all $a > 0$. This can take place only if

$$n = 1.$$

In this case, the solution of equation (43.1.2.7) is given by the following formula [see (43.1.2.9) for $n = 1$]:

$$w(x,t) = t^{-k} F(x + t), \tag{43.1.2.12}$$

where $F(z)$ is an arbitrary function.

If $n \neq 1$, then equation (43.1.2.7) admits only a degenerate solution $w(x,t) = Ct^{-k/n}$, where C is an arbitrary constant [the degenerate solution corresponds to $F = \text{const}$ in (43.1.2.10)].

43.2. Method of Differentiation in Independent Variables

43.2.1. Preliminary Remarks

1°. In some situations, differentiation in independent variables can be used to eliminate some arguments of the functional equation under consideration and reduce it to an ordinary differential equation (see Example 43.4 below). The solution obtained in this way should be then inserted into the original equation in order to get rid of redundant integration constants, which may appear due to the differentiation.

2°. In some situations, differentiation in independent variables should be combined with the multiplication (division) of the equation and the results of its differentiation by suitable functions. Sometimes it is useful to take a logarithm of the equation or the results of its transformation (see Examples 43.5 and 43.6 below).

3°. In some situations, differentiation of a functional equation in independent variables allows us to eliminate some arguments and reduce the equation to a simpler functional equation whose solution is known (see Subsection 43.5).

43.2.2. Examples of Solutions of Some Specific Functional Equations

Example 43.4. Consider *Pexider's equation*

$$f(x) + g(y) = h(x + y), \tag{43.2.2.1}$$

where $f(x)$, $g(y)$, $h(z)$ are the functions to be found.

Differentiating the functional equation (43.2.2.1) in x and y, we come to the ordinary differential equation $h''_{zz}(z) = 0$, where $z = x + y$. Its solution is the linear function

$$h(z) = az + b. \tag{43.2.2.2}$$

Substituting this expression into (43.2.2.1), we obtain

$$f(x) + g(y) = ax + ay + b.$$

Separating the variables, we find the functions f and g:

$$\begin{aligned} f(x) &= ax + b + c, \\ g(y) &= ay - c. \end{aligned} \tag{43.2.2.3}$$

Thus, the solution of Pexider's equation (43.2.2.1) is given by the formulas (43.2.2.2), (43.2.2.3), where a, b, c are arbitrary constants.

Example 43.5. Consider the nonlinear functional equation

$$f\big(\sqrt{x^2+y^2}\,\big) = af(x)f(y). \tag{43.2.2.4}$$

For $a = 1$ it is the *Gauss's equation*.

We first take the logarithm of equation (43.2.2.4) and then differentiate with respect to x and y. We get zero on the right-hand side and the expression

$$\frac{\partial^2}{\partial x\partial y}\left[\ln f\big(\sqrt{x^2+y^2}\,\big)\right] = \frac{\partial}{\partial y}\left[\frac{x}{z}(\ln f)'_z\right] = \frac{xy}{z}\left[\frac{(\ln f)'_z}{z}\right]'_z,$$
$$z = \sqrt{x^2+y^2}, \quad f = f(z),$$

on the left-hand side. This leads to the ordinary differential equation $[(\ln f)'_z/z]'_z = 0$, whose general solution is expressed as

$$f(z) = C_1\exp(C_2z^2), \tag{43.2.2.5}$$

where C_1 and C_2 are arbitrary constants. Substituting (43.2.2.5) in (43.2.2.4), one finds that $C_2 = 1/a$. As a result, one obtains a solution to equation (43.2.2.4) in the form

$$f(z) = \frac{1}{a}\exp(Cz^2),$$

where $C = C_2$ is an arbitrary constant. In addition, the function $f(z) \equiv 0$ is also a solution.

Example 43.6. Consider the nonlinear functional equation

$$f(x+y) = f(x) + f(y) + af(x)f(y), \qquad a \neq 0, \tag{43.2.2.6}$$

which occurs in the theory of probability with $a = -1$.

Differentiating both sides of this equation in x and y, we get

$$f''_{zz}(z) = af'_x(x)f'_y(y), \tag{43.2.2.7}$$

where $z = x + y$. Taking the logarithm of both sides of equation (43.2.2.7) and differentiating the resulting relation in x and y, we come to the ordinary differential equation

$$[\ln f''_{zz}(z)]''_{zz} = 0. \tag{43.2.2.8}$$

Integrating (43.2.2.8) in z twice, we get

$$f''_{zz}(z) = C_1\exp(C_2z), \tag{43.2.2.9}$$

where C_1 and C_2 are arbitrary constants. Substituting (43.2.2.9) into (43.2.2.7), we obtain the equation

$$C_1\exp[C_2(x+y)] = af'_x(x)f'_y(y),$$

which admits separation of variables. Integration yields

$$f(x) = A\exp(C_2x) + B, \qquad A = \pm\frac{1}{C_2}\sqrt{\frac{C_1}{a}}. \tag{43.2.2.10}$$

Substituting (43.2.2.10) into the original equation (43.2.2.6), we find the values of the constants: $A = -B = 1/a$ and $C_2 = \beta$ is an arbitrary constant. As a result, we obtain the desired solution

$$f(x) = \frac{1}{a}\big(e^{\beta x} - 1\big).$$

43.3. Method of Argument Elimination by Test Functions

43.3.1. Classes of Equations. Description of the Method

Consider linear functional equations of the form

$$w(x,t) = \theta(x,t,a)\,w\big(\varphi(x,t,a),\ \psi(x,t,a)\big), \tag{43.3.1.1}$$

where x and t are independent variables, $w = w(x, t)$ is the function to be found, $\theta = \theta(x, t, a)$, $\varphi = \varphi(x, t, a)$, $\psi = \psi(x, t, a)$ are given functions, and a is a free parameter, which can take any value (on some interval).

Instead of equation (43.3.1.1), consider an auxiliary more general functional equation

$$w(x, t) = \theta(x, t, \xi)\, w\big(\varphi(x, t, \xi),\ \psi(x, t, \xi)\big), \tag{43.3.1.2}$$

where $\xi = \xi(x, t)$ is an arbitrary function.

Basic idea: If an exact solution of equation (43.3.1.2) can somehow be obtained, this function will also be a solution of the original equation (43.3.1.1) [since (43.3.1.1) is a special case of equation (43.3.1.2) with $\xi = a$].

In view that the function $\xi = \xi(x, t)$ can be arbitrary, let us first take a test function so that it satisfies the condition

$$\psi(x, t, \xi) = b, \tag{43.3.1.3}$$

where b is a constant (usually, it is convenient to take $b = 1$ or $b = 0$). Resolving (43.3.1.3) with respect to ξ and substituting the test function $\xi = \xi(x, t)$ thus obtained into (43.3.1.2), we have

$$w(x, t) = \theta(x, t, \xi(x, t))\, \Phi\big(\varphi(x, t, \xi(x, t))\big), \tag{43.3.1.4}$$

where $\Phi(\varphi) \equiv w(\varphi, b)$.

Expression (43.3.1.4) is crucial for the construction of an exact solution of the original functional equation: this expression should be substituted into (43.3.1.2) and one should find out for which functions $\Phi(\varphi)$ it is indeed a solution of the equation for arbitrary $\xi = \xi(x, t)$ (in this connection, some constraints on the structure of the determining functions θ, φ, ψ may appear).

Remark 1. Expression (43.3.1.4) may be substituted directly into the original equation (43.3.1.1).

Remark 2. Condition (43.3.1.3) corresponds to the elimination of the second argument (it is replaced by a constant) in the right-hand side of equation (43.3.1.2).

Remark 3. Instead of (43.3.1.3), a similar condition $\varphi(x, t, \xi) = b$ can be used for choosing the test function $\xi = \xi(x, t)$.

43.3.2. Examples of Solutions of Specific Functional Equations

Example 43.7. Consider the functional equation

$$w(x, t) = a^k w(a^m x, a^n t) \tag{43.3.2.1}$$

(k, m, n are given constants, $a > 0$ is an arbitrary constant), which is a special case of equation (43.3.1.1) for $\theta(x, t, a) = a^k$, $\varphi(x, t, a) = a^m x$, $\psi(x, t, a) = a^n t$.

Following the scheme described in Subsection 43.3.1, let us use the auxiliary equation

$$w(x, t) = \xi^k w(\xi^m x, \xi^n t) \tag{43.3.2.2}$$

and the test function ξ defined, according to (43.3.1.3), from the condition

$$\xi^n t = 1 \qquad (b = 1). \tag{43.3.2.3}$$

Hence, we find that $\xi = t^{-1/n}$. Substituting this expression into (43.3.2.2), we get

$$w(x, t) = t^{-k/n} \Phi(t^{-m/n} x), \tag{43.3.2.4}$$

where $\Phi(\varphi) \equiv w(\varphi, 1)$.

It is easy to show by direct verification that (43.3.2.4) is a solution of the functional equation (43.3.2.1) for an arbitrary function Φ and coincides (to within notation) with solution (43.1.2.3) obtained by the method of differentiation in a parameter.

Remark. Instead of 1 on the right-hand side of (43.3.2.3) we can take any constant $b \neq 0$ and obtain the same result (to within notation of the arbitrary function Φ).

Example 43.8. Consider the functional equation

$$w(x,t) = a^k w(a^m x, t + \beta \ln a) \tag{43.3.2.5}$$

(k, m, β are given constants, $a > 0$ is an arbitrary constant), which is a special case of equation (43.3.1.1) for $\theta(x,t,a) = a^k$, $\varphi(x,t,a) = a^m x$, and $\psi(x,t,a) = t + \beta \ln a$.

Following the above scheme, consider a more general auxiliary equation

$$w(x,t) = \xi^k w(\xi^m x, t + \beta \ln \xi). \tag{43.3.2.6}$$

The test function ξ is found from the condition

$$t + \beta \ln \xi = 0 \qquad (b = 0).$$

We have $\xi = \exp(-t/\beta)$. Substituting this expression into (43.3.2.6), we get

$$w(x,t) = e^{-kt/\beta} \Phi(x e^{-mt/\beta}), \tag{43.3.2.7}$$

where $\Phi(\varphi) \equiv w(\varphi, 0)$. Direct verification shows that (43.3.2.7) is a solution of the functional equation (43.3.2.5) for an arbitrary function Φ and coincides with solution (43.1.2.6) obtained by the method of differentiation in a parameter.

Example 43.9. Now consider the functional equation

$$w(x,t) = a^k w\big(x + (1-a)t,\ a^n t\big) \tag{43.3.2.8}$$

($a > 0$ is arbitrary, n is a constant), which is a special case of equation (43.3.1.1) for $\theta(x,t,a) = a^k$, $\varphi(x,t,a) = x + (1-a)t$, and $\psi(x,t,a) = a^n t$.

Following the scheme described above, consider the auxiliary equation

$$w(x,t) = \xi^k w\big(x + (1-\xi)t,\ \xi^n t\big) \tag{43.3.2.9}$$

and define the test function ξ from the condition (43.3.2.3), according to (43.3.1.3). We have $\xi = t^{-1/n}$. Substituting this expression into (43.3.2.9), we get

$$w(x,t) = t^{-k/n} \Phi(z), \qquad z = x + t - t^{(n-1)/n}, \tag{43.3.2.10}$$

where $\Phi(\varphi) \equiv w(\varphi, 1)$.

Substituting (43.3.2.10) into the original equation (43.3.2.8) and dividing the result by $t^{-k/n}$, we find that

$$\Phi\big(x + t - t^{(n-1)/n}\big) = \Phi\big(x + (1 - a + a^n)t - a^{n-1} t^{(n-1)/n}\big). \tag{43.3.2.11}$$

Since this relation must hold for all $a > 0$, there are two possibilities:

1) n is arbitrary, $\Phi = C = \text{const}$;
2) $n = 1$, Φ is arbitrary.

In the second case, which corresponds to $n = 1$ in the functional equation (43.3.2.8), its solution can be written in the form

$$w(x,t) = t^{-k} F(x + t), \tag{43.3.2.12}$$

where $F(z)$ is an arbitrary function, $F(z) = \Phi(z - 1)$. We see that expression (43.3.2.12) coincides with solution (43.1.2.12) obtained by the method of differentiation in a parameter.

Remark 1. The results of solving specific functional equations obtained in Section 43.3 by the elimination of an argument coincide with those obtained for the same equations in Section 43.1 by the method of differentiation in a parameter. However, it should be observed that the intermediate results, when solving equation (43.3.2.8) by these methods, may not coincide [cf. (43.3.2.11) and (43.1.2.10)].

Remark 2. The method of elimination of an argument is much simpler than that of differentiation in a parameter, since the former only requires to solve algebraic (transcendental) equations of the form (43.4.4.3) with respect to ξ and does not require solutions of the corresponding first-order partial differential equations (see Section 43.1).

43.4. Nonlinear Functional Equations Reducible to Bilinear Equations

43.4.1. Bilinear Functional Equations

1°. A binomial bilinear functional equation has the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) = 0, \tag{43.4.1.1}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ $(n = 1, 2)$ are unknown functions of different arguments. In this section, it is assumed that $f_n \not\equiv 0$, $g_n \not\equiv 0$.

Separating the variables in (43.4.1.1), we find the solution:

$$f_1 = Af_2, \quad g_2 = -Ag_1, \tag{43.4.1.2}$$

where A is an arbitrary constant. The functions on the right-hand sides in (43.4.1.2) are assumed arbitrary.

2°. The trinomial bilinear functional equation

$$f_1(x)g_1(y) + f_2(x)g_2(y) + f_3(x)g_3(y) = 0, \tag{43.4.1.3}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ $(n = 1, 2, 3)$ are unknown functions, has two solutions:

$$\begin{aligned} &f_1 = A_1f_3, \quad f_2 = A_2f_3, \quad g_3 = -A_1g_1 - A_2g_2; \\ &g_1 = A_1g_3, \quad g_2 = A_2g_3, \quad f_3 = -A_1f_1 - A_2f_2, \end{aligned} \tag{43.4.1.4}$$

where A_1 and A_2 are arbitrary constants. The functions on the right-hand sides of the equations in (43.4.1.4) are assumed arbitrary.

3°. The quadrinomial functional equation

$$f_1(x)g_1(y) + f_2(x)g_2(y) + f_3(x)g_3(y) + f_4(x)g_4(y) = 0, \tag{43.4.1.5}$$

where all f_i are functions of the same argument and all g_i are functions of another argument, has a solution

$$\begin{aligned} &f_1 = A_1f_3 + A_2f_4, \quad f_2 = A_3f_3 + A_4f_4, \\ &g_3 = -A_1g_1 - A_3g_2, \quad g_4 = -A_2g_1 - A_4g_2 \end{aligned} \tag{43.4.1.6}$$

depending on four arbitrary constants $A_1, \ldots, A_4$. The functions on the right-hand sides of the solutions in (43.4.1.6) are assumed arbitrary.

Equation (43.4.1.5) has two other solutions:

$$\begin{aligned} &f_1 = A_1f_4, \quad f_2 = A_2f_4, \quad f_3 = A_3f_4, \quad g_4 = -A_1g_1 - A_2g_2 - A_3g_3; \\ &g_1 = A_1g_4, \quad g_2 = A_2g_4, \quad g_3 = A_3g_4, \quad f_4 = -A_1f_1 - A_2f_2 - A_3f_3 \end{aligned} \tag{43.4.1.7}$$

involving three arbitrary constants.

4°. Consider a bilinear functional equation of the general form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_k(x)g_k(y) = 0, \tag{43.4.1.8}$$

where $f_n = f_n(x)$ and $g_n = g_n(y)$ are unknown functions $(n = 1, \ldots, k)$.

It can be shown that the bilinear functional equation (43.4.1.8) has $k - 1$ different solutions:

$$\begin{aligned} f_i(x) &= A_{i,1}f_{m+1}(x) + A_{i,2}f_{m+2}(x) + \cdots + A_{i,k-m}f_k(x), \quad && i = 1, \ldots, m; \\ g_{m+j}(y) &= -A_{1,j}g_1(y) - A_{2,j}g_2(y) - \cdots - A_{m,j}g_m(y), && j = 1, \ldots, k-m; \\ & m = 1, 2, \ldots, k-1, \end{aligned} \tag{43.4.1.9}$$

where the $A_{i,j}$ are arbitrary constants. The functions $f_{m+1}(x), \ldots, f_k(x)$, $g_1(y), \ldots, g_m(y)$ on the right-hand sides of solutions (43.4.1.9) can be chosen arbitrarily. It is obvious that for a fixed m, solution (43.4.1.9) contains $m(k-m)$ arbitrary constants.

Remark 1. Formulas (43.4.1.9) imply that equation (43.4.1.8) may hold only if the functions f_n (and g_n) are linearly dependent.

Remark 2. The bilinear functional equation (43.4.1.8) and its solutions (43.4.1.9) play an important role in the methods of generalized and functional separation of variables for nonlinear PDEs (see Chapters 29 and 30).

43.4.2. Functional-Differential Equations Reducible to a Bilinear Equation

Consider a nonlinear functional-differential equation of the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_k(x)g_k(y) = 0, \tag{43.4.2.1}$$

where $f_i(x)$ and $g_i(x)$ are given function of the form

$$\begin{aligned} f_i(x) &\equiv F_i\big(x, \varphi_1, \varphi_1', \varphi_1'', \ldots, \varphi_n, \varphi_n', \varphi_n''\big), \qquad \varphi_p = \varphi_p(x); \\ g_i(y) &\equiv G_i\big(y, \psi_1, \psi_1', \psi_1'', \ldots, \psi_m, \psi_m', \psi_m''\big), \quad \psi_q = \psi_q(y). \end{aligned} \tag{43.4.2.2}$$

The problem is to find the functions $\varphi_p = \varphi_p(x)$ and $\psi_q = \psi_q(y)$ depending on different variables. Here, for simplicity, we consider an equation that contains only second-order derivatives; in the general case, the right-hand sides of (43.4.2.2) may contain higher-order derivatives of $\varphi_p = \varphi_p(x)$ and $\psi_q = \psi_q(y)$.

The functional-differential equation (43.4.2.1)–(43.4.2.2) is solved by the method of splitting (see Section 29.5). On the first stage, we treat (43.4.2.1) as a purely functional equation that depends on two variables x and y, where $f_i = f_i(x)$ and $g_i = g_i(y)$ are unknown quantities. The solutions of this equation are described by (43.4.1.9). On the second stage, we successively substitute the functions $f_i(x)$ and $g_i(y)$ from (43.4.2.2) into all solutions (43.4.1.9) to obtain systems of ordinary differential equations for the unknown functions $\varphi_p(x)$ and $\psi_q(y)$. Solving these systems, we get solutions of the functional-differential equation (43.4.2.1)–(43.4.2.2).

Remark. The method of splitting will be used in Subsection 43.4.3 for the construction of solutions of some nonlinear functional equations.

43.4.3. Nonlinear Functional Equations Containing the Complex Argument

Here, we discuss some nonlinear functional equations with two variables. Such equations often arise when the method of functional separation of variables is used for finding solutions of nonlinear equations of mathematical physics.

1°. Consider a functional equation of the form

$$f(t) + g(x) + h(x)Q(z) + R(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t). \tag{43.4.3.1}$$

Here, one of the two functions $f(t)$ and $\psi(t)$ is prescribed and the other is assumed unknown; also one of the functions $g(x)$ and $\varphi(x)$ is prescribed and the other is unknown, and the functions $h(x)$, $Q(z)$, and $R(z)$ are assumed unknown.*

Differentiating equation (43.4.3.1) with respect to x, we obtain the two-argument equation

$$g_x' + h_x'Q + h\varphi_x'Q_z' + \varphi_x'R_z' = 0. \tag{43.4.3.2}$$

* In similar equations with a composite argument, it is assumed that $\varphi(x) \not\equiv \text{const}$ and $\psi(y) \not\equiv \text{const}$.

Such equations were discussed in Subsection 43.4.2; their solutions are found with the help of (43.4.1.6) and (43.4.1.7). Hence, we obtain the following system of ordinary differential equations [see formulas (43.4.1.6)]:

$$\begin{aligned} g'_x &= A_1 h\varphi'_x + A_2\varphi'_x, \\ h'_x &= A_3 h\varphi'_x + A_4\varphi'_x, \\ Q'_z &= -A_1 - A_3 Q, \\ R'_z &= -A_2 - A_4 Q, \end{aligned} \tag{43.4.3.3}$$

where $A_1, \ldots, A_4$ are arbitrary constants. Integrating the system of ODEs (43.4.3.3) and substituting the resulting solutions into the original functional equation, one obtains the following results.

Case 1. If $A_3 = 0$ in (43.4.3.3), then the corresponding solution of the functional equation is given by

$$\begin{aligned} f &= -\tfrac{1}{2}A_1A_4\psi^2 + (A_1B_1 + A_2 + A_4B_3)\psi - B_2 - B_1B_3 - B_4, \\ g &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2, \\ h &= A_4\varphi + B_1, \\ Q &= -A_1 z + B_3, \\ R &= \tfrac{1}{2}A_1A_4 z^2 - (A_2 + A_4B_3)z + B_4, \end{aligned} \tag{43.4.3.4}$$

where A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 2. If $A_3 \neq 0$ in (43.4.3.3), then the corresponding solution of the functional equation is

$$\begin{aligned} f &= -B_1B_3e^{-A_3\psi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\psi - B_2 - B_4 - \frac{A_1A_4}{A_3^2}, \\ g &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2, \\ h &= B_1e^{A_3\varphi} - \frac{A_4}{A_3}, \\ Q &= B_3e^{-A_3 z} - \frac{A_1}{A_3}, \\ R &= \frac{A_4B_3}{A_3}e^{-A_3 z} + \left(\frac{A_1A_4}{A_3} - A_2\right)z + B_4, \end{aligned} \tag{43.4.3.5}$$

where A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 3. In addition, the functional equation has two degenerate solutions [formulas (43.4.1.7) are used]:

$$f = A_1\psi + B_1, \quad g = A_1\varphi + B_2, \quad h = A_2, \quad R = -A_1 z - A_2 Q - B_1 - B_2, \tag{43.4.3.6}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants, and

$$f = A_1\psi + B_1, \quad g = A_1\varphi + A_2 h + B_2, \quad Q = -A_2, \quad R = -A_1 z - B_1 - B_2, \tag{43.4.3.7}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $h = h(x)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants.

2°. Consider a functional equation of the form

$$f(t) + g(x)Q(z) + h(x)R(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t). \tag{43.4.3.8}$$

Differentiating (43.4.3.8) in x, we get the two-argument functional-differential equation

$$g'_x Q + g\varphi'_x Q'_z + h'_x R + h\varphi'_x R'_z = 0, \tag{43.4.3.9}$$

which coincides with equation (43.4.1.5), up to notation.

Nondegenerate case. Equation (43.4.3.9) can be solved with the help of formulas (43.4.1.6)–(43.4.1.7). In this way, we arrive at the system of ordinary differential equations

$$\begin{aligned} g'_x &= (A_1 g + A_2 h)\varphi'_x, \\ h'_x &= (A_3 g + A_4 h)\varphi'_x, \\ Q'_z &= -A_1 Q - A_3 R, \\ R'_z &= -A_2 Q - A_4 R, \end{aligned} \tag{43.4.3.10}$$

where $A_1, \ldots, A_4$ are arbitrary constants.

The solution of equation (43.4.3.10) is given by

$$\begin{aligned} g(x) &= A_2 B_1 e^{k_1\varphi} + A_2 B_2 e^{k_2\varphi}, \\ h(x) &= (k_1 - A_1)B_1 e^{k_1\varphi} + (k_2 - A_1)B_2 e^{k_2\varphi}, \\ Q(z) &= A_3 B_3 e^{-k_1 z} + A_3 B_4 e^{-k_2 z}, \\ R(z) &= (k_1 - A_1)B_3 e^{-k_1 z} + (k_2 - A_1)B_4 e^{-k_2 z}, \end{aligned} \tag{43.4.3.11}$$

where $B_1, \ldots, B_4$ are arbitrary constants and k_1 and k_2 are the roots of the quadratic equation

$$(k - A_1)(k - A_4) - A_2 A_3 = 0. \tag{43.4.3.12}$$

In the degenerate case $k_1 = k_2$, the terms $e^{k_2\varphi}$ and $e^{-k_2 z}$ in (43.4.3.11) should be replaced with $\varphi e^{k_1\varphi}$ and $z e^{-k_1 z}$, respectively. In the case of purely imaginary or complex roots, one should separate the real (or imaginary) part of the roots in solution (43.4.3.11).

On substituting (43.4.3.11) into the original functional equation, one obtains conditions for the free coefficients and identifies the function $f(t)$, namely,

$$\begin{aligned} &B_2 = B_4 = 0 \quad \Longrightarrow \quad f(t) = [A_2 A_3 + (k_1 - A_1)^2]B_1 B_3 e^{-k_1\psi}, \\ &B_1 = B_3 = 0 \quad \Longrightarrow \quad f(t) = [A_2 A_3 + (k_2 - A_1)^2]B_2 B_4 e^{-k_2\psi}, \\ &A_1 = 0 \quad \Longrightarrow \quad f(t) = (A_2 A_3 + k_1^2)B_1 B_3 e^{-k_1\psi} + (A_2 A_3 + k_2^2)B_2 B_4 e^{-k_2\psi}. \end{aligned} \tag{43.4.3.13}$$

Solution (43.4.3.11), (43.4.3.13) involves arbitrary functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$.

Degenerate case. In addition, the functional equation has two degenerate solutions [formulas (43.4.1.7) are used],

$$f = B_1 B_2 e^{A_1\psi}, \quad g = A_2 B_1 e^{-A_1\varphi}, \quad h = B_1 e^{-A_1\varphi}, \quad R = -B_2 e^{A_1 z} - A_2 Q,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions; A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = B_1 B_2 e^{A_1\psi}, \quad h = -B_1 e^{-A_1\varphi} - A_2 g, \quad Q = A_2 B_2 e^{A_1 z}, \quad R = B_2 e^{A_1 z},$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $g = g(x)$ are arbitrary functions; and A_1, A_2, B_1, and B_2 are arbitrary constants.

3°. Consider a more general functional equation of the form

$$f(t) + g_1(x)Q_1(z) + \cdots + g_n(x)Q_n(z) = 0, \qquad \text{where} \quad z = \varphi(x) + \psi(t). \tag{43.4.3.14}$$

By differentiation in x, this equation can be reduced to a functional differential equation, which may be regarded as a bilinear functional equation of the form (43.4.1.8). Using formulas (43.4.1.9) for the construction of its solution, one can first obtain a system of ODEs and then find solutions of the original equation (43.4.3.14).

4°. Consider a functional equation of the form

$$\begin{gathered} f_1(t)g_1(x) + \cdots + f_m(t)g_m(x) + h_1(x)Q_1(z) + \cdots + h_n(x)Q_n(z) = 0, \\ z = \varphi(x) + \psi(t). \end{gathered} \tag{43.4.3.15}$$

Assume that $g_m(x) \not\equiv 0$. Dividing equation (43.4.3.15) by $g_m(x)$ and differentiating the result in x, we come to an equation of the form

$$f_1(t)\bar{g}_1(x) + \cdots + f_{m-1}(t)\bar{g}_{m-1}(x) + \sum_{i=1}^{2n} s_i(x)R_i(z) = 0$$

with a smaller number of functions $f_i(t)$. Proceeding in this way, we can eliminate all functions $f_i(t)$ and obtain a functional-differential equation with two variables of the form (43.4.2.1)–(43.4.2.2), which can be reduced to the standard bilinear functional equation by the method of splitting.

5°. Consider a functional equation of the form

$$\begin{gathered} [\alpha_1(t)\theta(x) + \beta_1(t)]R_1(z) + \cdots + [\alpha_n(t)\theta(x) + \beta_n(t)]R_n(z) = 0, \\ z = \varphi(t)\theta(x) + \psi(t). \end{gathered} \tag{43.4.3.16}$$

Passing in (43.4.3.16) from the variables x and t to new variables z and t [the function θ is replaced by $(z - \psi)/\varphi$], we come to the bilinear equation of the form (43.4.1.8):

$$\sum_{i=1}^{n} \alpha_i(t)zR_i(z) + \sum_{i=1}^{n} [\varphi(t)\beta_i(t) - \psi(t)\alpha_i(t)]R_i(z) = 0.$$

Remark. Instead of the expressions $\alpha_i(t)\theta(x) + \beta_i(t)$ in (43.4.3.16) linearly depending on the function $\theta(x)$, one can consider polynomials of $\theta(x)$ with coefficients depending on t.

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