

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Maxwell's equations  
in differential form

# Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

- Charges are sources of the electric field
- The magnetic field has no sources (no monopoles)
- Time-dependent magnetic fields generate electric fields
- Time-dependent electric fields and currents generate magnetic fields

Vacuum permeability  $\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{N}}{\text{A}^2}$

Vacuum permittivity  $\epsilon_0 = 8.854... \cdot 10^{-12} \frac{\text{As}}{\text{Vm}}$

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

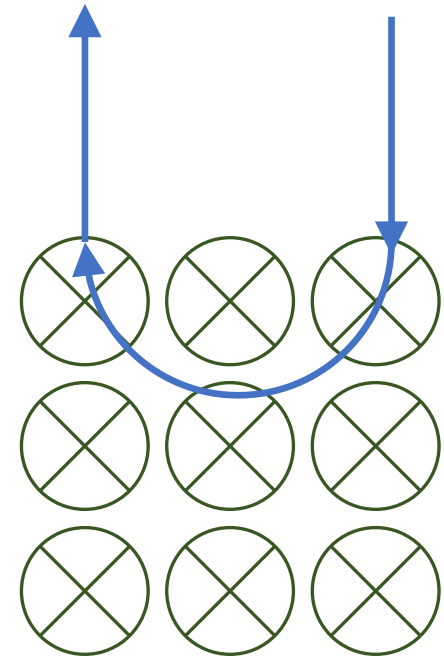
Motivating Maxwell's  
equations by  
symmetry arguments

# Maxwell's equations

Quantity		Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$
Time	$t, \frac{\partial}{\partial t}$	asymmetric -	symmetric +
Position	$r, \nabla$	symmetric +	asymmetric -
Velocity	$\dot{r}, \frac{\partial r}{\partial t}$	asymmetric -	asymmetric -
Acceleration	$\ddot{r}, \frac{\partial^2 r}{\partial t^2}$	symmetric +	asymmetric -
Force	$F$	symmetric +	asymmetric -
Charge density	$\rho$	symmetric +	symmetric +
Current density	$j = \rho v$	asymmetric -	asymmetric -

# Maxwell's equations

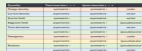
Quantity		Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Charge density	$\rho$	symmetric +	symmetric +	scalar
Current density	$j = \rho v$	asymmetric -	asymmetric -	vector
Electric field	$E$	symmetric +	asymmetric -	vector
Magnetic field	$B$	asymmetric -	symmetric +	(pseudo)vector
Time derivative	$\frac{\partial E}{\partial t}$	asymmetric -	asymmetric -	vector
	$\frac{\partial B}{\partial t}$	symmetric +	symmetric +	(pseudo)vector
Divergence	$\nabla \cdot E$	symmetric +	symmetric +	scalar
	$\nabla \cdot B$	asymmetric -	asymmetric -	(pseudo)scalar
Rotation	$\nabla \times E$	symmetric +	symmetric +	(pseudo)vector
	$\nabla \times B$	asymmetric -	asymmetric -	vector



$B$

$$F = qE + qv \times B$$

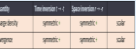
# Maxwell's equations

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Charge density $\rho$	symmetric +	symmetric +	scalar
Current density $\mathbf{j} = \rho \mathbf{v}$	asymmetric -	asymmetric -	vector
Electric field $\mathbf{E}$	symmetric +	asymmetric -	vector
Magnetic field $\mathbf{B}$	asymmetric -	symmetric +	(pseudo)vector
Time derivative $\frac{\partial \mathbf{E}}{\partial t}$	asymmetric -	asymmetric -	vector
$\frac{\partial \mathbf{B}}{\partial t}$	symmetric +	symmetric +	(pseudo)vector
Divergence $\nabla \cdot \mathbf{E}$	symmetric +	symmetric +	scalar
$\nabla \cdot \mathbf{B}$	asymmetric -	asymmetric -	(pseudo)scalar
Rotation 	symmetric +	symmetric +	(pseudo)vector
$\nabla \times \mathbf{B}$	asymmetric -	asymmetric -	vector

# Maxwell's equations

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Charge density $\rho$	symmetric +	symmetric +	scalar
Divergence $\nabla \cdot \mathbf{E}$	symmetric +	symmetric +	scalar

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Time derivative $\frac{\partial \mathbf{B}}{\partial t}$	symmetric +	symmetric +	(pseudo)vector
Rotation $\nabla \times \mathbf{E}$	symmetric +	symmetric +	(pseudo)vector

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Current density $\mathbf{j} = \rho \mathbf{v}$	asymmetric -	asymmetric -	vector
Time derivative $\frac{\partial \mathbf{E}}{\partial t}$	asymmetric -	asymmetric -	vector
Rotation 	asymmetric -	asymmetric -	vector

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$	
Divergence $\nabla \cdot \mathbf{B}$	asymmetric -	asymmetric -	(pseudo)scalar

# Maxwell's equations

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$		Unit
Charge density $\rho$	symmetric +	symmetric +	scalar	As/m <sup>3</sup>
Divergence $\nabla \cdot E$	symmetric +	symmetric +	scalar	V/m <sup>2</sup>

$$\nabla \cdot E = \frac{1}{\epsilon_0} \rho$$

$$[1/\epsilon_0] = \text{Vm/As}$$

Quantity	Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$		Unit
Time derivative $\frac{\partial B}{\partial t}$	symmetric +	symmetric +	(pseudo)vector	T/s = V/m <sup>2</sup>
Rotation $\nabla \times E$	symmetric +	symmetric +	(pseudo)vector	V/m <sup>2</sup>

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

# Maxwell's equations in vacuum

Quantity		Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$		Unit
Current density	$j = \rho v$	asymmetric -	asymmetric -	vector	A/m <sup>2</sup>
Time derivative	$\frac{\partial E}{\partial t}$	asymmetric -	asymmetric -	vector	V/ms
Rotation	$\nabla \times B$	asymmetric -	asymmetric -	vector	T/m = Vs/m <sup>3</sup>

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 j$$

$$[\mu_0] = \text{N/A}^2 = \text{Vs/mA}$$

$$[\epsilon_0 \mu_0] = \text{s}^2/\text{m}^2$$

Quantity		Time inversion $t \rightarrow -t$	Space inversion $r \rightarrow -r$		Unit
Divergence	$\nabla \cdot B$	asymmetric -	asymmetric -	(pseudo)scalar	T/m = Vs/m <sup>3</sup>

$$\nabla \cdot B = 0$$

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Maxwell's equations  
in integral formulation

# Maxwell's equations

Differential formulation

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

Integral formulation

$$\epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV$$

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0$$

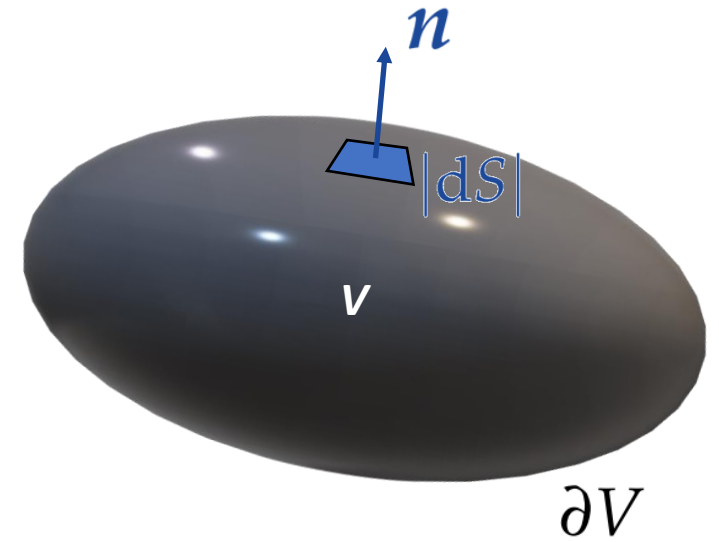
$$\oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \int_A \mathbf{j} \cdot d\mathbf{S} + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

# Gauss's law

$$\int_V^{(3D)} (\nabla \cdot \mathbf{a}) dV = \oint_{\partial V}^{(2D)} \mathbf{a} \cdot d\mathbf{S}$$

$$d\mathbf{S} = |dS| \cdot \mathbf{n}$$



# Maxwell's equations

Differential

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho$$

Integral

$$\epsilon_0 \int_V (\nabla \cdot \mathbf{E}) dV = \int_V \rho dV$$

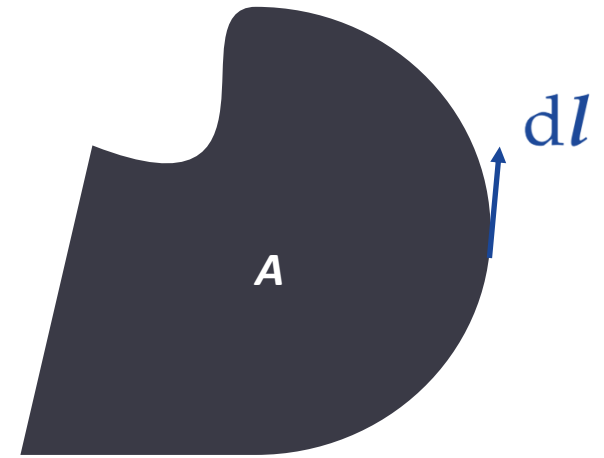
$$\int_V (\nabla \cdot \mathbf{B}) dV = 0$$

$$\epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV = Q_v$$

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0$$

# Stoke's law

$$\int_A^{(2D)} (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \oint_{\partial A}^{(1D)} \mathbf{a} \cdot d\mathbf{l}$$



# Maxwell's equations

Differential

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Integral

$$\frac{1}{\mu_0} \int_A (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \int_A \mathbf{j} \cdot d\mathbf{S} + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = I + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$$\int_A (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = - \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = - \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

# Maxwell's equations

Differential formulation

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

Integral formulation

$$\epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV$$

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \int_A \mathbf{j} \cdot d\mathbf{S} + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Discussion of  
Maxwell's equations

# Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

- Charges are sources of the electric field

$$\nabla \cdot \mathbf{B} = 0$$

- The magnetic field has no sources (no monopoles)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Time-dependent magnetic fields generate electric fields

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

- Time-dependent electric fields and currents generate magnetic fields

# Maxwell's equations

Differential formulation

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

Integral formulation

$$\epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV$$

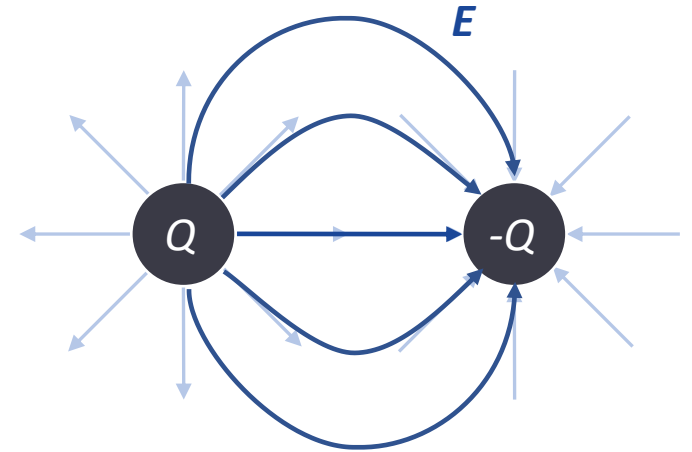
$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = - \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

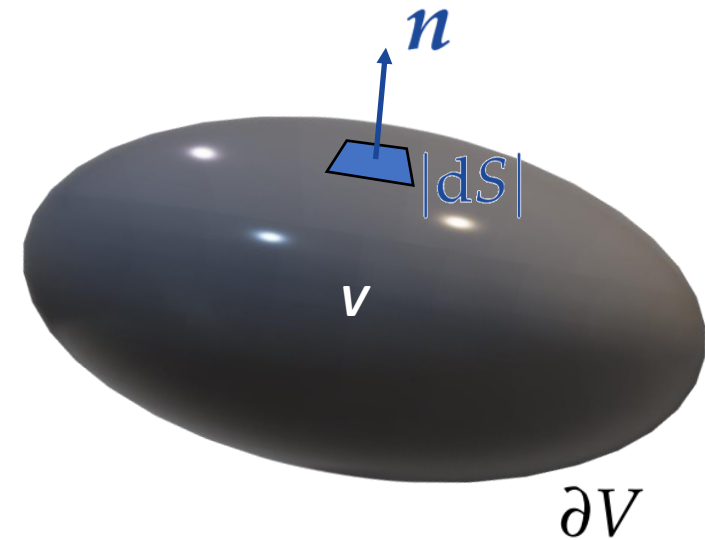
$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \int_A \mathbf{j} \cdot d\mathbf{S} + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

# Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad \epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV = Q_v$$



- Charges are sources of the electric field
- The electric flux through a closed surface is determined by the enclosed charge

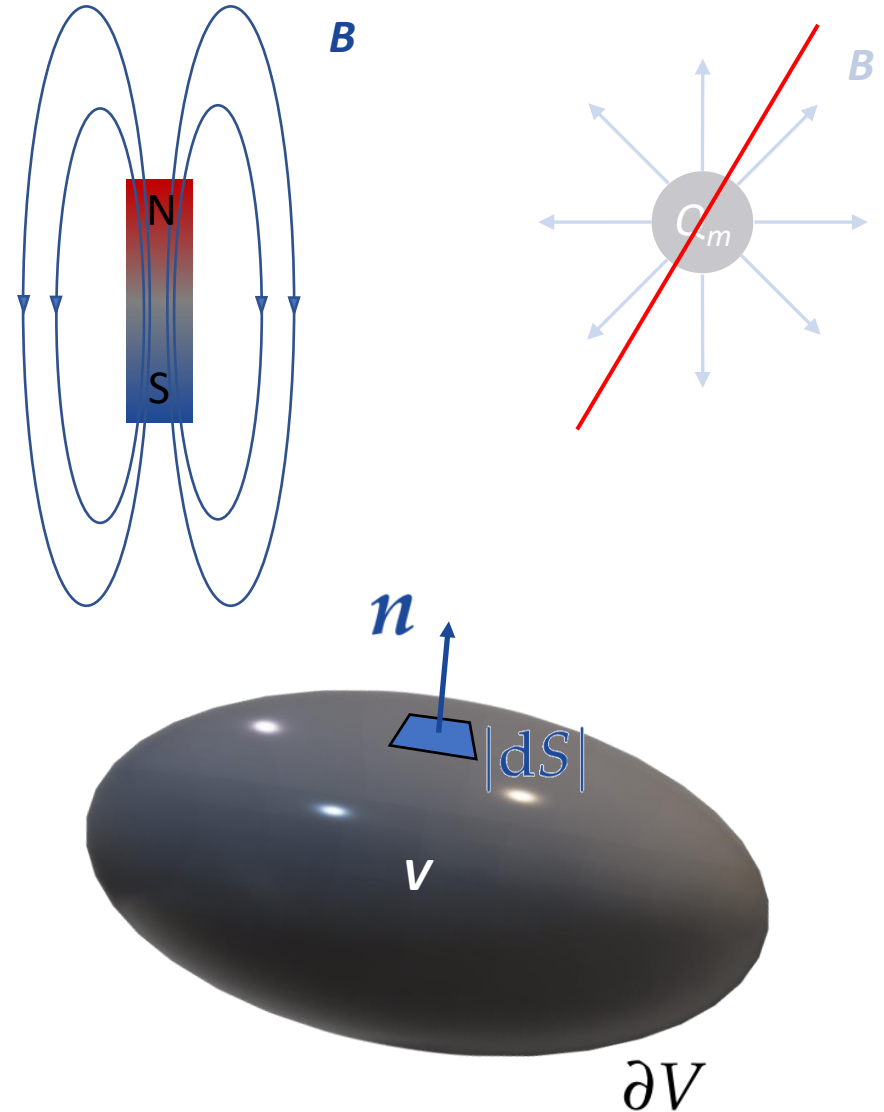


# Gauss's law for magnetic fields

$$\nabla \cdot \mathbf{B} = 0$$

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{S} = 0$$

- The magnetic field has no sources (monopoles do not exist)
- The magnetic flux vanishes when integrated over every closed surface

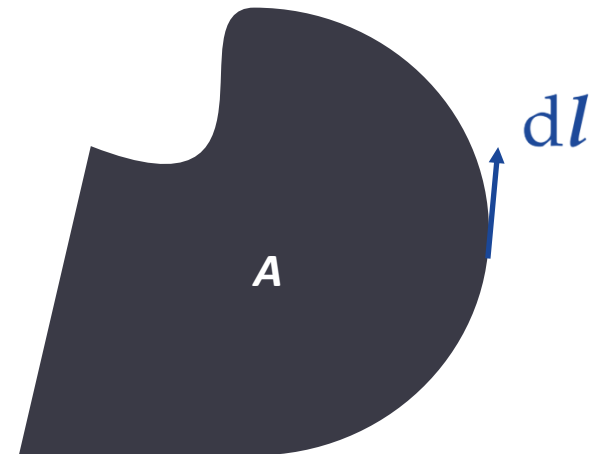


# Induction law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

- Time-dependent changes in the magnetic field generate electric fields
- An electric field is generated by a change of the magnetic flux (for time-independent  $A$ )

$$\phi = \int_A \mathbf{B} \cdot d\mathbf{S} \quad \oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\dot{\phi}$$

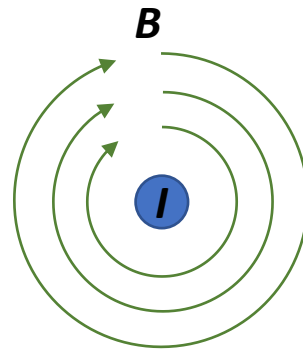


# Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$

$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \int_A \mathbf{j} \cdot d\mathbf{S} + \epsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

1825: Ørsted, Ampère



~~$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$~~

*Violates continuity equation*

~~$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$~~

~~$$\mu_0 \nabla \cdot \mathbf{J} = 0 \quad \text{Missing term ...}$$~~

~~$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$~~

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Coulomb's law

# Coulomb's law

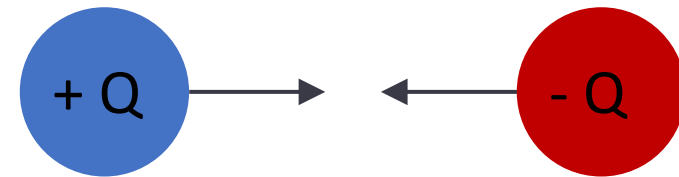
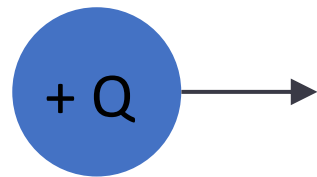
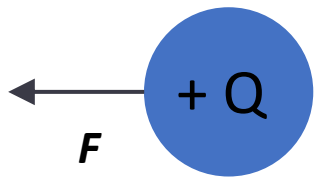
**1769:** Robinson

$$F \propto r^{-2.06}$$

**1780s:** Coulomb

$$F = k_e \frac{|q_1 q_2|}{r^2}$$

$$k_e = \frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$$



# Coulomb's law

- Maxwell  $\epsilon_0 \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_V \rho dV = Q_v$

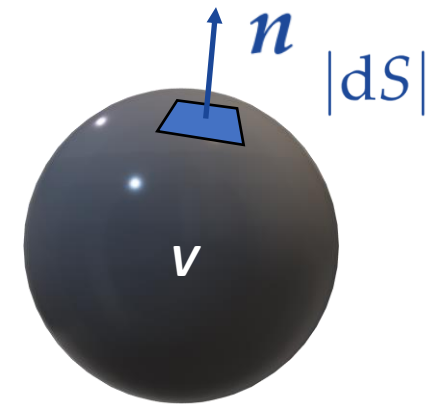
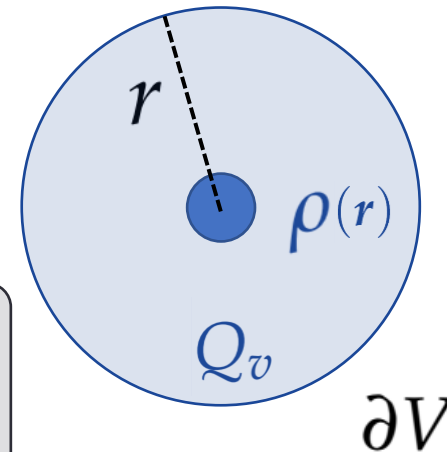
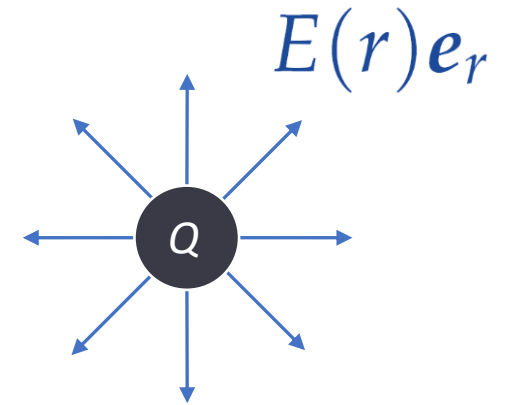
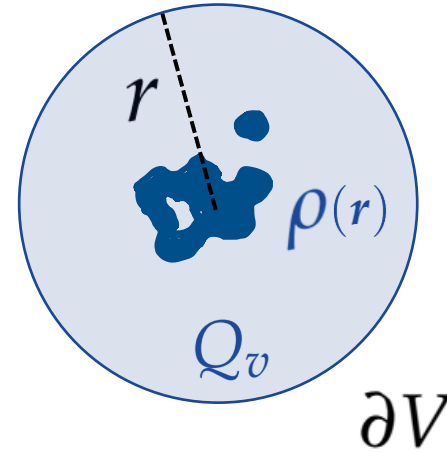
$$\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r \quad \mathbf{E} \parallel d\mathbf{S}$$

$$\epsilon_0 E(r) \oint_{\partial V} dS = Q_v$$

$$\epsilon_0 E(r) 4\pi r^2 = Q_v$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q_v}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q_v}{r^2} \mathbf{e}_r$$



$\partial V$

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Fifth law of  
electrodynamics:  
The Lorentz force

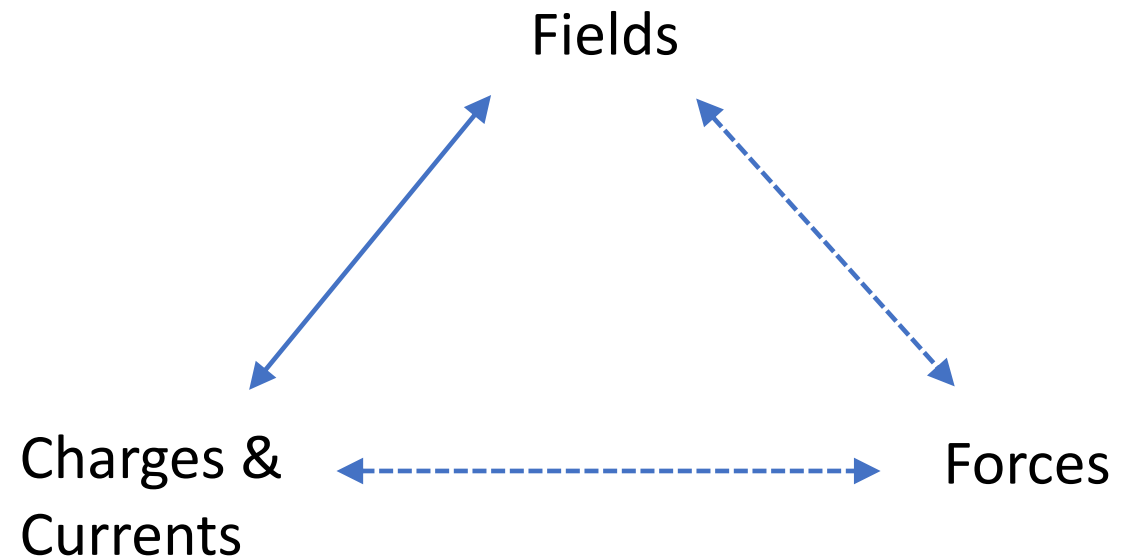
# Lorentz force

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$



Lorentz force  $F = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

Explains **how** the fields act on currents and charges  
→ makes them physically relevant

# Lorentz force

$$\text{Lorentz force} \quad \mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

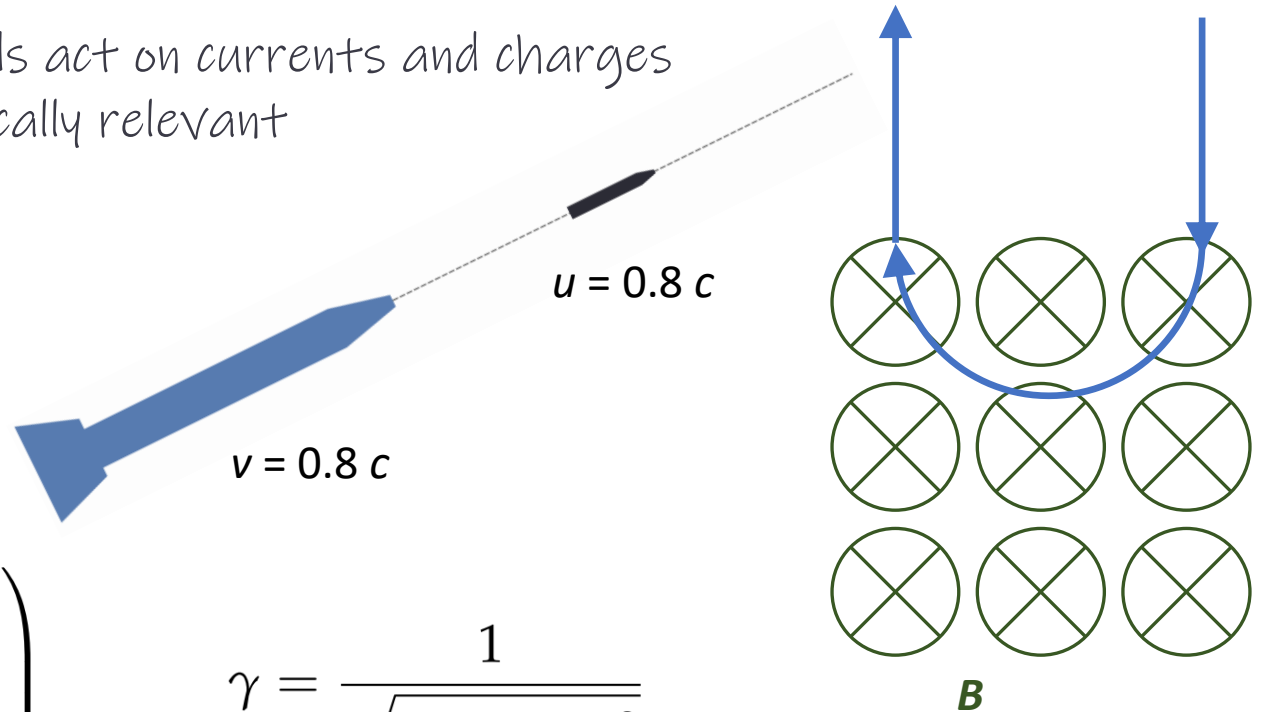
Explains **how** the fields act on currents and charges  
→ makes them physically relevant

- **Special relativity:**

Electric fields in one frame of reference are magnetic fields in another frame of reference (space contraction)

$$\mathbf{F} = q \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -\gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$



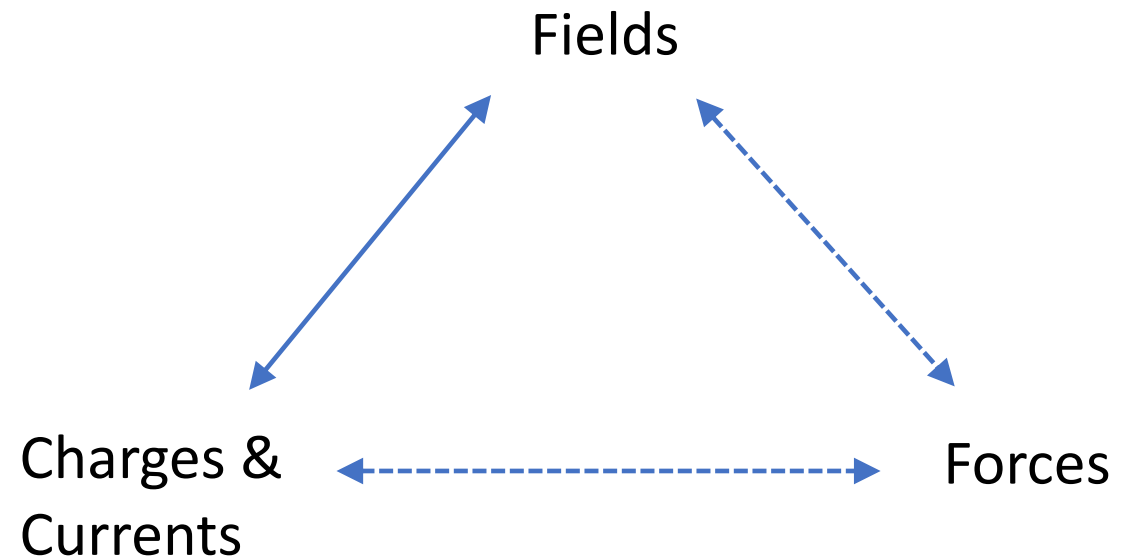
# Lorentz force

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$



Lorentz force  $F = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

Explains **how** the fields act on currents and charges  
→ makes them physically relevant

# Electrodynamics

SECTION

Maxwell's equations

LECTURE

Field energy

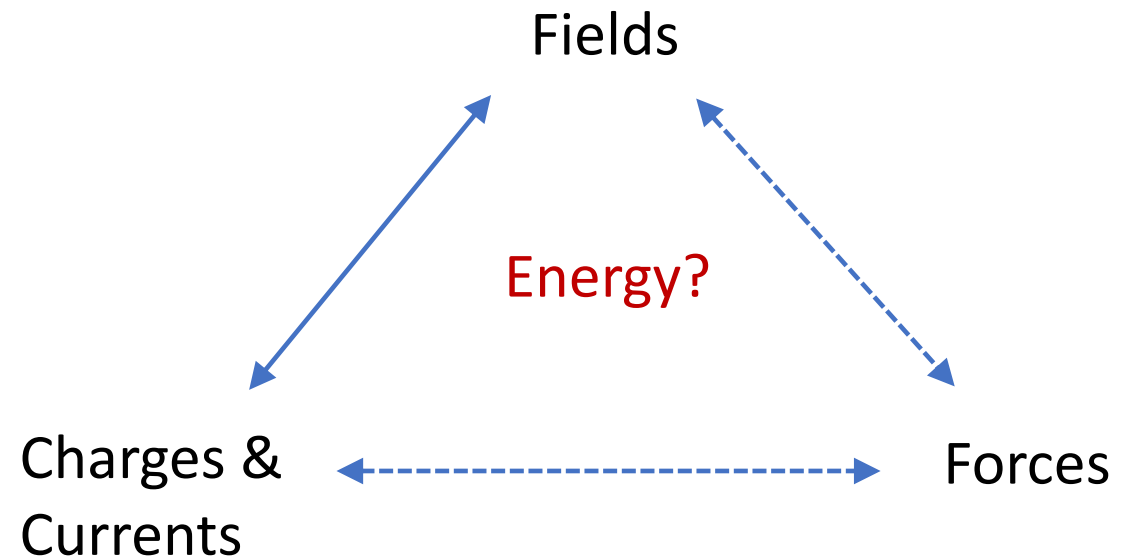
# Lorentz force

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

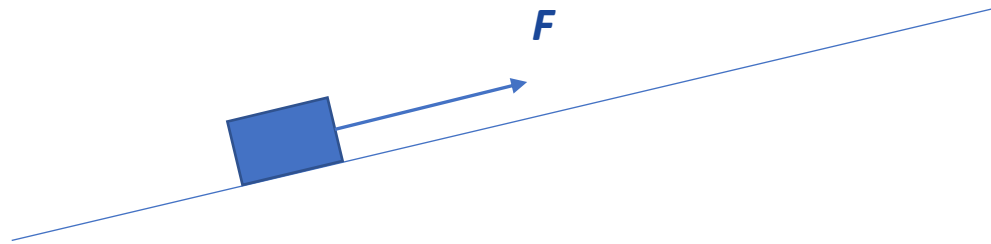
$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$$



Lorentz force  $F = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

Explains **how** the fields act on currents and charges

# Field energy



- Power  
(energy generation / consumption rate)

$$W = \frac{E}{t} = -\mathbf{F} \cdot \mathbf{v}$$

$$W = -Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = -Q\mathbf{E} \cdot \mathbf{v} \quad \dots \text{No power from magnetic fields}$$

- Power density  
(energy generation density)

$$\nu = -\rho \mathbf{E} \cdot \mathbf{v}$$

$$\nu = -\mathbf{j} \cdot \mathbf{E}$$

# Field energy

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j} + \epsilon_0 \dot{\mathbf{E}}$$

$$-\mathbf{j} = \epsilon_0 \dot{\mathbf{E}} - \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

$$-\mathbf{j} \cdot \mathbf{E} = \left( \epsilon_0 \dot{\mathbf{E}} - \frac{1}{\mu_0} \nabla \times \mathbf{B} \right) \cdot \mathbf{E}$$

$$-\mathbf{j} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left( \epsilon_0 \frac{\mathbf{E}^2}{2} \right) - \frac{1}{\mu_0} \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \mathbf{E})$$

$$-\mathbf{j} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left( \epsilon_0 \frac{\mathbf{E}^2}{2} + \frac{1}{2\mu_0} \mathbf{B}^2 \right) + \nabla \cdot \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

$$\nu = \frac{\partial}{\partial t} w + \nabla \cdot \mathbf{S}$$

Continuity equation for energy density

# Field energy

$$-\mathbf{j} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left( \epsilon_0 \frac{\mathbf{E}^2}{2} + \frac{1}{2\mu_0} \mathbf{B}^2 \right) + \nabla \cdot \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

$$\nu = \frac{\partial}{\partial t} w + \nabla \cdot \mathbf{S}$$

Continuity equation for energy density

- Power density  
(energy generation density)

$$\nu = -\mathbf{j} \cdot \mathbf{E}$$

- Energy density

$$w = \epsilon_0 \frac{\mathbf{E}^2}{2} + \frac{1}{2\mu_0} \mathbf{B}^2$$

- Poynting vector  
(energy-current density)

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$